



# SuccessClap

## Best Coaching for UPSC Mathematics

*Note: Solutions of all the questions are available in sequence in Crash Course. There is very less time for Mains exam and it is not possible to solve all problems, hence it is advisable to go through the SOLUTIONS (Quick glance) of all these important questions at once. All topics, Important questions and important PYQ questions are covered.*

### Important Questions for UPSC Maths 2026

#### CALCULUS and REAL ANALYSIS

Topics Covered:

- 1) Mean Value Theorem
- 2) Max/Min Lagrange Method
- 3) Reiman Integrals
- 4) Sequences
- 5) Series
- 6) Uniform Convergence
- 7) Functions of Several Variable
- 8) Improper Integrals
- 9) Partial Differentiation, Euler, Total Differentiation
- 10) Definite Integral as Sum
- 11) Beta Gamma Function
- 12) Jacobiabs
- 13) Differentiation under Integral Sign
- 14) Multiple Integrals
- 15) Indeterminants
- 16) Asymptotes
- 17) Continuity
- 18) Differentiability
- 19) Max/Min One Variable
- 20) Max/Min Two variable
- 21) Length of Curves
- 22) Areas
- 23) Volumes
- 24) Surface Area
- 25) Curve Tracing

**01-MEAN VALUE THEOREM**

- 1) Show that between any two roots of  $e^x \cos x = 1$ , there exists at least one root of  $e^x \sin x - 1 = 0$ .
- 2) Prove that if  $a_0, a_1, \dots, a_n$  are real numbers such that  $\frac{a_0}{(n+1)} + \frac{a_1}{n} + \dots + \frac{a_{(n-1)}}{2} + a_n = 0$   
Then there exists at least one real number  $x$  between 0 and 1 such that  $a_0 x^n + a_1 x^{(n-1)} + \dots + a_n = 0$ .
- 3) Prove that between any two real roots of  $e^x \sin x = x$ , is at least one root of  $\cos x + \sin x = e^{-1}$ .
- 4) If  $p(x)$  is polynomial and  $k \in \mathbb{R}$ , prove that between any two real roots of  $p(x)=0$ , there is a root of  $p'(x)+kp(x)=0$ .
- 5) Prove that if  $P$  be any polynomial and  $P'$  the derivative of  $P$ , then between any two consecutive zeros of  $P'$ , there lies at the most of zero of  $P$ .
- 6) Prove that if  $f$  be defined for all real  $x$  such that  $|f(x) - f(y)| < (x - y)^2$  for all real  $x$  and  $y$ , then  $f$  is constant.
- 7) Show  $\frac{v-u}{1+v^2} < \tan^{-1} u < \frac{v-u}{1+u^2}$ , if  $0 < u < v$ , and deduce that  $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$ .
- 8) Let the function  $f$  be continuous in  $[a, b]$  and derivable in  $[a, b]$ . show that there exists a number  $c$  in  $[a, b]$  such that  $2c[f(a) - f(b)] = f'(c)[a^2 - b^2]$ .
- 9) If  $f'$  and  $g'$  exist for all  $x \in [a, b]$  and if  $g'(x) \neq 0 \forall x \in [a, b]$ , then prove that for some  $c \in [a, b]$   $\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$
- 10) Prove that  $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$ , if  $0 < x < 1$ .
- 11) Applying Lagrange's mean value theorem to the function defined by  $f(x) = \log(1+x)$  for all  $x > 0$  show that  $0 < [\log(1+x)]^{-1} - x^{-1} < 1$  whenever  $x > 0$ .
- 12) Prove that  $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ , if  $x > 0$ .
- 13) If  $0 < x < 1$ , show that  $2x < \log \frac{1+x}{1-x} < 2x \left(1 + \frac{1}{3} \cdot \frac{x^2}{(1-x^2)}\right)$ .
- 14) Prove that  $\tan x > x$ , whenever  $0 < x < \frac{\pi}{2}$ .
- 15) Show that  $\frac{2}{\pi} < \frac{\sin x}{x} < 1, 0 < x < \frac{\pi}{2}$ .
- 16) Prove that  $\frac{\tan x}{x} > \frac{x}{\sin x}$ , whenever  $0 < x < \frac{\pi}{2}$ .
- 17) Let  $g(x) = f(x) + f(1-x)$  and  $f''(x) > 0, \forall x \in (0,1)$ . Find the intervals of increase and decrease of  $g(x)$ .
- 18) Use Cauchy's mean value theorem to evaluate  $\lim_{x \rightarrow 1} \left[ \frac{\cos \frac{1}{2} \pi x}{\log \frac{1}{x}} \right]$

- 19) Show that the number  $\theta$  which occurs in the Taylor's theorem with Lagrange's form of remainder after  $n$  terms approaches the limit  $\frac{1}{(n+1)}$  as  $h$  approaches zero provided that  $f^{(n+1)}(x)$  is continuous and different from zero at  $x=0$ .
- 20) If  $\phi(x) = f(x) + f(1-x)$  and  $f''(x) < 0$  for all  $x \in [0,1]$ , show that  $\phi$  increases in  $[0,1/2]$  and decrease in  $[1/2,1]$ . Hence or otherwise prove that  $\pi < \frac{\sin \pi x}{x(1-x)} \leq 4$  when  $0 < x < 1$ .

## 02- MAX/MIN- LAGRANGE METHOD

- 1) Show that the maximum and minimum of radii vectors of the section of the surface  $(x^2 + y^2 + z^2)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ , by the plane  $\lambda x + \mu y + \nu z = 0$  are given by the equation  $\frac{a^2 \lambda^2}{1-a^2 r^2} + \frac{b^2 \mu^2}{1-b^2 r^2} + \frac{c^2 \nu^2}{1-c^2 r^2} = 0$
- 2) Find the maximum and minimum values of  $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ , when  $lx+my+nz=0$  and  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ . Interpret result geometrically.
- 3) Prove that of all rectangular parallelepiped of the same volume, the cube has the least surface.
- 4) Prove that if  $x+y+z = l$ ,  $ayz + bcx + cxy$  has an extreme value equal to  $\frac{abc}{(2bc+2ca+2ab-a^2-b^2-c^2)}$ .
- 5) Prove also that if  $a, b, c$  are all positive and  $c$  lies between  $a+b-2\sqrt{ab}$  and  $a+b+2\sqrt{ab}$  this value is true maximum and that if  $a, b, c$  are all negative and  $c$  lies between  $a+b \pm 2\sqrt{ab}$ , it is true minimum.
- 6) If  $F(a) = \mu \neq 0, F'(a) = 0$  and  $x, y, z$  satisfy relation  $F(x) F(y) F(z) = \mu^3$ . Prove that the function  $\phi(x) + \phi(y) + \phi(z)$  has a maximum when  $x=y=z=a$ , provided  $\phi'(a) \left\{ \frac{F''(a)}{F'(a)} - \frac{F'(a)}{F(a)} \right\} > \phi''(a)$ .
- 7) Given  $u = 5xyz/(x+2y+4z)$ . Find the values of  $x, y, z$  for which  $u$  is maximum subject to the condition  $xyz = 8$ .
- 8) Find points where  $u = ax^p + by^q + cz^r$  has extreme values subject to the condition  $x^l + y^m + z^n = k$
- 9) Show that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  is  $8abc/3\sqrt{3}$ .
- 10) Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

- 11) A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.
- 12) If two variables  $x$  and  $y$  are connected by the relation  $ax^3+by^2 = ab$ , show that the maximum and minimum values of the function  $x^2+y^2 +xy$  will be the values of  $u$  given by the equation  $4(u-a) (u-b) = ab$ .
- 13) Determine the max. value of  $OP$ ,  $O$  being at the origin of coordinates where  $P$  describes the curve  $x^2 + y^2+ 2z^2 = 5, x +2y +z = 5$ .
- 14) If  $x,y, z$  are subject to the condition  $ax+by +cz =1$ , show that  $x^3 + y^3 +z^3 - 3xyz$  has an extreme value  $0$  which is a maximum or a minimum according as  $a+b+c < \text{or} > 0$ .

### 03-REIMAN INTEGRALS

- 1) Show that  $f(x) = \sin x$  is integrable on  $[0, \pi/2]$  and  $\int_0^{\pi/2} \sin x \, dx = 1$ .
- 2) If  $f(x)$  be defined on  $[0,2]$  as follows,

$$f(x) = x + x^2, \text{ when } x \text{ is rational}$$

$$= x^2 + x^3, \text{ when } x \text{ is irrational}$$

Then evaluate the upper and lower Riemann integrals of  $f$  over  $[0,2]$  and show that  $f$  is not R-integrable over  $[0,2]$ .

- 3) Find the upper and lower Riemann integrals for the functions  $f$  defined on  $[0,1]$  as

$$f(x) = (1 - x^2)^{1/2}, \text{ if } x \text{ is rational}$$

$$\text{And } f(x) = 1 - x, \text{ if } x \text{ is irrational}$$

Hence show that  $f$  is not Riemann integrals on  $[0,1]$ .

- 4) Show that

$$a) \lim_{n \rightarrow \infty} \left[ \frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n} \right] = \frac{3}{8}$$

$$b) \lim_{n \rightarrow \infty} \left[ \frac{n^n}{n!} \right]^{1/n} = e$$

- 5) Every continuous function is integrable. (VERY IMP QN)
- 6) A bounded function  $f$  is integrable in  $[a, b]$ , if the set of its points of discontinuity is finite.
- 7) A bounded function  $f$  is integrable in  $[a, b]$ , if the set of its points of discontinuity has a finite number of limit points.
- 8) If  $f$  is monotonic in  $[a, b]$ , then it is integrable in  $[a, b]$ .

9) A function  $f$  is defined in  $[0, 1]$  as follows:  $f(x) = 1/q$  when  $x$  is any non-zero rational number  $p/q$  in its lowest terms, and  $f(x) = 0$ , when  $x$  is irrational or 0. Show that  $f$  is integrable in  $[0, 1]$  and the value of the integral is 0.

10) If the function  $f$  be defined on  $[a, b]$  as follows:

$$f(x) = 1/q^2 \text{ when } x = p/q,$$

$$= 1/q^3, \text{ when } x = \sqrt{p/q},$$

Where  $p$  and  $q$  are relatively prime integers and  $f(x) = 0$  for all other values of  $x$ , then show that the function  $f$  is Riemann-integrable on  $[a, b]$ . (Checkout Video Solution in Youtube SuccessClap)

11) If  $f$  is non-integrable continuous function on  $[a, b]$  such that  $\int_0^a f(x) dx > 0$ . Then show that  $f(x) = 0 \forall x \in [a, b]$ .

12) Show that the function  $f$  defined as follows:  $f(x) = \frac{1}{2^n}$  when  $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$ : ( $n = 0, 1, 2, 3 \dots \dots$ ),  
 $f(0) = 0$

Is integrable in  $[0, 1]$ , although it has an infinite number of points of discontinuities. Also show that  $\int_0^1 f(x) dx = \frac{2}{3}$ .

13) Show that  $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2} dx \leq \frac{2}{\pi}$ .

14) If a function  $f$  is continuous in  $[0, 1]$ , show that  $\lim_{n \rightarrow \infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx = \frac{\pi}{2} f(0)$ .

15) If  $f$  be defined in  $[0, 1]$  by the condition  $f(x) = (-1)^{r-1}$ , when  $\frac{1}{r+1} < x < \frac{1}{r}$ , ( $r = 1, 2, 3, \dots$ )  $f(0) = 0$ . Show that  $\int_0^1 f dx = \log 4 - 1$ .

16) Show that the function  $f$  defined on  $[0, 1]$  by the conditions  $f(x) = 2rx$  when  $\frac{1}{r+1} < x < \frac{1}{r}$ ,  $r = 1, 2, 3, \dots$ s integrable over  $[0, 1]$  and that  $\int_0^1 f(x) dx = \frac{\pi^2}{6}$ .

17) A function  $f$  is defined on  $[0, 1]$  by  $f(x) = 1/n$  for  $1/n \geq x > 1/(n+1)$ ,  $n = 1, 2, 3, \dots$ . Prove that  $f$  is integrable over  $[0, 1]$  and evaluate  $\int_0^1 f(x) dx$ .

18) A function  $f$  is defined on  $[0, 1]$  as follows:  $f(x) = \frac{1}{a^{r-1}}$ , when  $\frac{1}{a^r} < x < \frac{1}{a^{r-1}}$  ( $r = 1, 2, 3 \dots$ )  
 $f(0) = 0$ , where  $a$  is an integer greater than 2.

Show that  $\int_0^1 f dx$ . Exists and is equal to  $\frac{a}{a+1}$ .

19) Integrate on  $[0, 2]$  the function  $f(x) = x [x]$ , where  $[x]$  denotes the greatest integer not greater than  $x$ .

- 20) Evaluate  $\int_0^2 f(x) dx$  where  $f(x) = 0$ , when  $x = \frac{n}{n+1}, \frac{n+1}{n}$  ( $n = 1, 2, 3, \dots$ )  $f(x) = 1$ , elsewhere.  
Is  $f$  integrable in  $[0, 2]$ ? Examine for continuity the function  $f$  so defined at  $x=1$ .

### 04 SEQUENCES

- 1) Show that sequence  $\langle r^n \rangle$  converges to zero if  $|r| < 1$ .
- 2) (Cauchy's first theorem on limits) If  $\lim_{n \rightarrow \infty} f_n = l$ , then  $\lim_{n \rightarrow \infty} \frac{f_1 + f_2 + \dots + f_n}{n} = l$ .
- 3) Show  $\lim_{n \rightarrow \infty} \frac{1}{n} (1 + 2^{1/2} + 3^{1/3} + \dots + n^{1/n}) = 1$
- 4) Cesario's Theorem If  $\lim f_n = l$  and  $\lim g_n = l'$ , then  $\lim \frac{f_1 g_n + f_2 g_{(n-1)} + \dots + f_n g_1}{n} = ll'$ .
- 5) If  $\langle f_n \rangle$  is a sequence of positive numbers such that  $f_n = \frac{1}{2}(f_{n-1} + f_{n-2})$ , for all  $n \geq 3$ , Then show that  $\langle f_n \rangle$  converges to  $\frac{f_1 + 2f_2}{3}$ .
- 6) If  $\langle S_n \rangle$  is a bounded sequence such that  $S_1 = a > 0$ ,  $s_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}}$ ,  $b > a, \forall n \geq 1$ , Then show that the sequence  $\langle S_n \rangle$  is an increasing sequence and  $\lim S_n = b$ .
- 7) If  $a_{n+1} = \sqrt{k + a_n}$ , where  $k, a_1$  are positive. Prove that the sequence  $\langle a_n \rangle$  is increasing or decreasing according as  $a_1$  is less then or greater than the positive root of the equation  $x^2 = x + k$  and has, in either case, this root as its limit.
- 8) If  $x_1, y_1$  are two positive unequal numbers and  $x_n = \frac{(x_{n-1} + y_{n-1})}{2}$  and  $y_n = \sqrt{(x_{n-1} \cdot y_{n-1})}$ ,  $\forall n \geq 2$ .  
Prove that the sequence  $\langle x_n \rangle$  and  $\langle y_n \rangle$  are monotonic and they converge to the same limit.
- 9) Show  $\lim_{n \rightarrow \infty} \left[ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right] = 0$ .
- 10) Prove that  $\lim_{n \rightarrow \infty} \left( \frac{(3n)!}{(n!)^3} \right)^{1/n} = 27$ .
- 11) A sequence  $\langle a_n \rangle$  defined as follows:  $a_1 = 1, a_{n+1} = \frac{4+3a_n}{3+2a_n}, n \geq 1$ .  
Show that  $\langle a_n \rangle$  converges and find its limit.
- 12) A sequence  $\langle a_n \rangle$  defined as follows:  $a_1 > 0, a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right), n \geq 1$ . Show that  $\langle a_n \rangle$  is a bounded monotonic sequence. Find its limit.
- 13) A sequence  $\langle S_n \rangle$  is defined by  $S_1 = 1, S_{n+1} = \left( \frac{3+S_n^2}{2} \right)^{1/2}$   
Show that  $\langle S_n \rangle$  is bounded monotonically increasing sequence and converges to  $\sqrt{3}$ .

- 14) Show that the sequence  $\langle a_n \rangle$  defined by  $a_{n+1} = 1 - \sqrt{1 - a_n} \forall n \geq 1$  and  $a_n < 1$ .  
Convergence to 0.
- 15) If  $\langle a_n \rangle$  be a sequence of positive numbers such that  $a_n = \sqrt{a_{n-1}a_{n-2}}$  for  $n > 2$ , then show that the sequence converges to  $(a_1a_2^2)^{1/3}$ .

### 05 SERIES

- 1) Examine for convergence of infinite series
- a)  $\sum(\sqrt{n^4 + 1} - \sqrt{n^4 - 1})$
- b)  $\sum(n^3 + 1)^{\frac{1}{3}} - n$
- 2) If  $\alpha > 0, \beta > 0$ , test the convergence of  $1 + \frac{\alpha+1}{\beta+1} + \frac{\alpha+1(2\alpha+1)}{\beta+1(2\beta+1)} + \frac{\alpha+1(2\alpha+1)(3\alpha+1)}{\beta+1(2\beta+1)(3\beta+1)} + \dots$
- 3) Test the convergence of the series  $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$
- 4) Test for the convergence of the following:
- a)  $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots$
- b)  $\frac{a}{b} + \frac{a(a+1)}{b(b+a)} + \frac{a(a+1)(a+2)}{b(b+a)(b+2)} + \dots$
- 5) Test for convergence the following series
- a)  $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^p + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^p + \dots$
- b)  $x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \dots$
- 6) Using integral test, show that the series  $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$  is convergent if  $p > 1$  and divergent if  $0 < p < 1$
- 7) Examine the series for convergence  $\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \frac{(\log 4)^2}{4^2} + \dots + \frac{(\log n)^2}{n^2} + \dots$
- 8) Test convergence the series  $\sum_{n=1}^{\infty} \frac{1}{3^{n+x}}$ , for all positive values of x.
- 9) Test for convergence of series whose nth term is  $\frac{(n!)^2}{(2n)!} x^n$ .
- 10) Test for convergence of  $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^p} \cdot p > 0$ .
- 11) Test for convergence of series:  $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n\alpha}{n^3}$ ,  $\alpha$  being real.
- 12) Test for convergence and absolute convergence the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{\log(n+1)} = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$$

13) Test for convergence and absolute convergence the series:

$$\frac{1}{2(\log 2)^p} - \frac{1}{3(\log 3)^2} + \frac{1}{4(\log 4)^p} - \dots (p > 0)$$

### 06 UNIFORM CONVERGENCE

1) Show that the sequence  $\{f_n\}$ , where  $f_n(x) = \frac{nx}{1+n^2x^2}$ ,  $x \in \mathbb{R}$  is point-wise convergent but is not uniformly convergent in any interval containing zero.

2) Show that the sequence  $\{f_n\}$ ,  $f_n(x) = x^n$ , is uniformly convergent in  $[0, k]$ ,  $k < 1$  and only point-wise convergent in  $[0, 1]$ .

3) Show that the sequence  $\{f_n\}$ , where  $f_n(x) = \frac{x}{1+nx^2}$ , converges uniformly on  $\mathbb{R}$ .

4) Show that the sequence  $\{f_n\}$ , where  $f_n(x) = nx e^{-nx^2}$  is not uniformly convergent on  $[0, 1]$ .

5) Show that the sequence  $\{f_n\}$ , where  $f_n(x) = \frac{n^2x}{1+n^3x^2}$ , is not uniformly convergent on  $[0, 1]$ .

6) Show that the series  $\sum_1^\infty \frac{x}{(1+n^2x^2)}$  is not uniformly convergent on  $[0, 1]$  while it is so on  $[\frac{1}{2}, 1]$ .

7) Show that the series  $\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \dots$

is uniformly convergent in  $]k, \infty[$ , where  $k$  is a positive number. Show that the series is not uniformly convergent in  $[0, \infty]$ .

8) Show that the sequence  $\{f_n\}$ ,  $f_n(x) = nx e^{-nx^2}$  converges but not uniformly to  $f$ , where  $f(x) = 0$  for all  $x \in [0, 1]$  and that  $\lim_{n \rightarrow \infty} \int_0^1 f_n dx \neq \int_0^1 f dx$ .

9) Given  $f(x) = \sum_{n=1}^\infty \frac{1}{n^2+n^4x^2}$ , justify the validity of  $f(x) = -2x \sum_{n=1}^\infty \frac{1}{n^2(1+nx^2)^2}$ ,

or

Show that the series  $\sum_{n=1}^\infty \frac{1}{n^2+n^4x^2}$ , is uniformly convergent for all  $x$ .

10) Show that  $\sum_{n=1}^\infty \frac{1}{1+n^2+n^4x^2}$ , converges uniformly for all values of  $x$ .

11) Show that the sequence  $\{f_n\}$ , where  $f_n(x) = \frac{1}{n^2} \log(1 + n^2x^2)$ , is uniformly convergent on  $[0, 1]$ .

12) Discuss the uniform convergence of  $1 + \frac{e^{-2\alpha}}{2^2-1} - \frac{e^{-4\alpha}}{4^2-1} + \frac{e^{-8\alpha}}{6^2-1} - \dots$

For all real  $x \geq 0$ .

- 13) Show that the series  $\sum \frac{\cos n\theta}{n^2}$ ,  $\sum \frac{\sin n\theta}{n^2}$  Converge uniformly for all values of  $p > 0$  in  $[\alpha, 2\pi - \alpha]$ , where  $0 \leq \alpha \leq \pi$ .
- 14) Show that the sequence  $\langle f_n \rangle$ , where  $f_n(x) = nx(1-x)^n$  is not uniformly convergent on  $[0,1]$ .
- 15) Show that the sequence  $\langle f_n \rangle$  where  $f_n(x) = \tan^{-1} nx$  is not uniformly convergent on  $[0,1]$ .
- 16) Show that the series  $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$  is uniformly convergent for all real  $x$  if  $p + q > 2$ .
- 17) Show that the series  $\sum \frac{(-1)^{n-1}}{n} |x|^n$  is uniformly convergent in  $[-1,1]$ .
- 18) Prove that  $\sum a_n n^{-x}$  is uniformly convergent on  $[0,1]$  if  $\sum a_n$  converges uniformly in  $[0,1]$ .
- 19) Discuss the uniform convergence of
- $$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \dots \text{ on } [0,1]$$
- $$x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \dots$$
- 20) Show that  $\sum \frac{1}{n^p + n^q x^2}$ ,  $p > 1$  is uniformly convergent for all values of  $x$  and can be differentiated term by term if  $q < 3p - 2$ .

### 07 FUNCTIONS of SEVERAL VARIABLE

- 1) Prove that  $f_{xy}(0,0) \neq f_{yx}(0,0)$  for the function  $f$  given by  $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ ;  $(x,y) \neq (0,0)$   
 $f(0,0) = 0$ .
- 2) Show that  $f_{xy}(0,0) \neq f_{yx}(0,0)$ . where  $f(x,y) = 0$  if  $xy = 0$ ,  $f(x,y) = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$ , if  $xy \neq 0$ .
- 3) If  $f(x,y) = y \sin \frac{1}{x} + x \sin \frac{1}{y}$  where  $x \neq 0, y \neq 0$ . Then prove that  $f(x,y) \rightarrow 0$  as  $(x,y) \rightarrow (0,0)$
- 4) Show that the following function is discontinuous at  $(0,0)$
- $$f(x,y) = \begin{cases} \frac{x^3 + y^3}{x - y} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$
- 5) Let  $f(x,y)$  be a function, defined by
- $$f(x,y) = x \sin \frac{1}{x} + y \sin \frac{1}{y}, \quad x \neq 0, y \neq 0$$
- $$f(0,y) = y \sin \frac{1}{y} \quad y \neq 0,$$

$$f(x, 0) = x \sin \frac{1}{x} \quad x \neq 0,$$

$$f(0,0) = 0,$$

Examine existence of  $f_x$  and  $f_{yx}$  at  $x = 0, y = 0$

6) Let  $f(x, y) = \frac{x^2 y}{x^4 + y^2}$ , for  $x \neq 0, y \neq 0$  and  $f(0,0) = 0$

Show that the partial derivatives  $f_x, f_y$  exist everywhere in region  $-1 \leq x \leq 1, -1 \leq y \leq 1$ , although  $f(x, y)$  is discontinuous in  $(x, y)$  at the origin.

7) Let  $f(x, y)$  be a function defined by

$$f(x, y) = \begin{cases} xy \frac{(x^2 - y^2)}{x^2 + y^2} & , \text{ for } (x, y) \neq (0, 0) \\ 0 & , \text{ for } (x, y) = (0, 0) \end{cases}$$

Prove that

- $f, f_x, f_y$  are continuous in  $(x, y)$ ,
- $f_{xy}$  and  $f_{yx}$  exist at every point  $(x, y)$  and are continuous except at  $(0, 0)$
- $f_{xy}(0, 0) = 1$  and  $f_{yx}(0, 0) = -1$ .

8) Let  $f(x, y) = \begin{cases} x^2 y^2 \cos \frac{1}{x} & , \text{ for all values of } y \text{ so long as } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$

Show that

- $f_{xy} = f_{yx}$  at all points  $(x, y)$ .
- Neither  $f_{xy}$  nor  $f_{yx}$  is continuous in  $x$  at  $x = 0$  if  $y \neq 0$ .
- $f_{xy}$  and  $f_{yx}$  are continuous in  $(x, y)$ , together at the origin.

9) Examine the continuity and differentiability of the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

10) Let  $f(x, y)$  be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2) \log(x^2 + y^2) & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Prove that  $f_{xy}$  and  $f_{yx}$  are continuous at  $(0, 0)$  but  $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ .

11) Discuss the differentiability of  $f$  defined by

$$f(x, y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}$$

12) Show that the function

$$f(x, y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} & , x, y \neq 0 \\ x^2 \sin \frac{1}{x} & , x \neq 0, y = 0 \\ y^2 \sin \frac{1}{y} & , x = y = 0 \end{cases}$$

and  $f(0,0) = 0$ , is differentiable at the origin.

13) Prove or disprove:  $f_{xy}(0,0) = f_{yx}(0,0)$ , where

$$f(x, y) = \frac{x^2y + xy^2 \sin(x - y)}{x^2 + y^2}, (x, y) \neq (0, 0)$$

### 08 IMPROPER INTEGRALS

Quickly checkout different substitutions and methods to solve.

1) Examine the convergence of:

a)  $\int_0^{\frac{1}{2}\pi} \frac{\sin x}{x^n} dx,$

b)  $\int_0^1 x^{a-1} e^{-x} dx,$

c)  $\int_0^1 \frac{x^n}{1+x} dx,$

d)  $\int_0^\pi \frac{\sqrt{x}}{\sin x} dx,$

e)  $\int_0^\infty \frac{\sin^2 x}{x^2} dx,$

f)  $\int_0^\infty \frac{x^{m-1}}{1+x} dx,$

g)  $\int_1^\infty \frac{x^{a-1} \log x}{1+x} dx.$

2) Show  $\int_0^\pi x^m \operatorname{cosec}^n x dx$  exists if  $n < (m + 1)$

3) Show that the integral  $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$  exists iff  $n < m + 1$ .

4) Examine the convergence of  $\int_1^2 \frac{\sqrt{x}}{\log x} dx.$

5) Show  $\int_0^1 x^{m-1} (1-x)^{n-1} dx$  exists iff  $m > 0$  and  $n > 0$ .

6) Discuss convergence of  $\int_0^\infty \left( \frac{x^{2m}}{1+x^{2n}} \right) (m > 0, n > 0).$

- 7) Test for convergence  $\int_0^1 x^p \left(\log \frac{1}{x}\right)^p dx$ .
- 8) Show that  $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$  is convergent.
- 9) Show  $\int_0^1 x^{n-1} \log x dx$  converges iff  $n > 0$ .
- 10) Show  $\int_1^{\infty} \left(\frac{\sin x}{x^p}\right) dx$  is convergent for  $p > 0$ .
- 11) Test for convergence  $\int_1^{\infty} (\sin x^p) dx$
- 12) Show that  $\int_0^{\infty} e^{-ax} \left(\frac{\sin x}{x}\right) dx$ . ( $a > 0$ ) is convergent.
- 13) Test the convergence of  $\int_0^{\infty} e^{-a^2 x^2} \cos bx dx$ .
- 14) Test the convergence of  $\int_0^{\infty} \left(\frac{\sin x^m}{x^n}\right) dx$ .
- 15) Discuss the convergence of  $\int_2^{\infty} \left(\frac{\cos x}{\log x}\right) dx$ .
- 16) Discuss the convergence of  $\int_0^{\infty} \left(\frac{x^p \sin^2 x}{1+x^2}\right) dx$ .
- 17) Test the convergence  $\int_0^{\infty} \left(\frac{\sin(x+x^2)}{x^n}\right) dx$ .
- 18) Show that the improper integral  $\int_0^{\infty} \left(\frac{\sin x}{x}\right) dx$  is not absolutely convergent.
- 19) Show that  $\int_0^{\infty} \left(\frac{x dx}{1+x^4 \cos^2 x}\right) dx$  is divergent.
- 20) Show that  $\int_0^{\infty} \left(\frac{x dx}{1+x^4 \sin^2 x}\right) dx$  is convergent.
- 21) Show that  $\int_0^{\infty} \left(\frac{\cos x}{\log x}\right) dx$  is not absolutely convergent.
- 22) Test the convergence of  $\int_0^1 \left(\frac{\sin(1/x)}{\sqrt{x}}\right) dx$ .
- 23) Show that  $\int_0^{\pi/2} \cos 2nx \log \sin x dx$  Converges.
- 24) Test the convergence of  $\int_0^{\infty} (1 - e^{-x}) \frac{\cos x}{x^2} dx$ , when  $a > 0$ .
- 25) Prove that  $\int_a^{\infty} \frac{\cos \alpha x - \cos \beta x}{x} dx$ , is convergent.
- 26) Show that the integral  $\int_0^{\infty} e^{-\alpha x} \frac{\sin x}{x} dx$ , is convergent when  $\alpha \geq 0$ .
- 27) Show that  $\int_a^{\infty} (1 - e^{-x}) \frac{\cos x}{x} dx$ , is convergent.

### 09 PARTIAL DIFFERENTIATION, Euler, Total Differentiation

- 1) If  $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$ .

2) Find the value of  $\frac{1}{a^2} \frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2} \frac{\partial^2 z}{\partial y^2}$  when  $a^2 x^2 + b^2 y^2 - c^2 z^2 = 0$ .

3) If  $1/u = \sqrt{(x^2 + y^2 + z^2)}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -u$ , and  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ .

4) If  $x = r \cos \theta, y = r \sin \theta$ , prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} \left[ \left( \frac{\partial r}{\partial x} \right)^2 + \left( \frac{\partial r}{\partial y} \right)^2 \right]$$

$$\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left( \frac{\partial^2 r}{\partial x \partial y} \right)^2.$$

$$\left( \frac{\partial r}{\partial x} \right)^2 + \left( \frac{\partial r}{\partial y} \right)^2 = 1$$

5) If  $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ , prove that  $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

6) If  $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ , prove that  $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\right)$

7) If  $x^x y^y z^z = c$  show that at  $x = y = z$ ,  $\frac{\partial^2 z}{\partial x \partial y} = -(x \log e x)^{-1}$ .

8) If  $\theta = t^n e^{-r^2/4t}$ , what value of  $n$  will make  $\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) = \frac{\partial \theta}{\partial t}$

9) Euler's theorem. If  $u$  is a homogeneous function of  $x$  and  $y$  of degree  $n$ , then  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$ .

10) If  $u$  be a homogeneous function of  $x$  and  $y$  of degree  $n$ , show that

$$x \left( \frac{\partial^2 u}{\partial x^2} \right) + y \left( \frac{\partial^2 u}{\partial x \partial y} \right) = (n-1) \left( \frac{\partial u}{\partial x} \right), \text{ and } x \left( \frac{\partial^2 u}{\partial x \partial y} \right) + y \left( \frac{\partial^2 u}{\partial y^2} \right) = (n-1) \left( \frac{\partial u}{\partial y} \right).$$

$$\text{Hence deduce that } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

11) If  $u = \cot^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$ , show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{4} \sin 2u = 0$ .

12) If  $u = \tan^{-1} \frac{x^3+y^3}{x-y}$ ,  $x \neq y$  show that

$$(i) x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u. \quad (ii) x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u) \sin 2u$$

13) If  $z = x^m f(y/x) + x^n g(x/y)$ , prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1) \left( x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right).$$

14) If  $V = \log_e \sin \left\{ \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}} \right\}$ , find the value of  $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z}$  when  $x=0, y=1, z=2$ .

15) If  $u = \sin^{-1} \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$  show that  $\frac{\partial u}{\partial x} = -\frac{y}{x} \frac{\partial u}{\partial y}$ .

16) If  $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ , prove that  $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$ .

17) If  $f(x,y,z)$  is a homogeneous function of the  $n^{\text{th}}$  degree in  $x,y,z$  prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + 2yz \frac{\partial^2 f}{\partial y \partial z} + 2zx \frac{\partial^2 f}{\partial z \partial x} + 2xy \frac{\partial^2 f}{\partial x \partial y} = n(n-1)f(x,y,z)$$

18) If  $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ , prove that  $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ .

19) If  $u = \log(x^3+y^3+z^3 - 3xyz)$  show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$$

### 10 DEFINITE INTEGRAL AS SUM

1) Show that  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{4n^2-r^2}} = \frac{\pi}{6}$ .

2) Find the value of  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n^3+1} + \frac{4}{n^3+8} + \frac{9}{n^3+27} + \dots + \frac{n^2}{2n^3} \right]$

3) Show that  $\lim_{n \rightarrow \infty} \left[ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{1/n} = \frac{4}{e}$ .

4) Show that  $\lim_{n \rightarrow \infty} \left[ \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right)^2 \left(1 + \frac{3^2}{n^2}\right)^2 \dots \left(1 + \frac{n^2}{n^2}\right)^n \right]^{2/n^2} = \frac{4}{e}$ .

### 11 BETA GAMMA FUNCTIONS

1) Show that  $\Gamma(1/2) = \sqrt{\pi}$ .

2) If  $n$  is a positive integer and  $m > -1$  prove that  $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ .

3) Show  $\Gamma(n)\Gamma(1-n) = \pi/\sin n\pi$ .  $0 < n < 1$

4) To show that  $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ .

5) Prove that

a)  $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{2}$ .

b)  $\int_0^{\pi/2} \tan^n x dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$ ,  $-1 < n < 1$ .

6) Show  $I = \int_0^{\pi/2} \sqrt{\sin \theta} d\theta$ .  $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \pi$ .

7) Legendre-Duplication Formula.  $\Gamma(n)\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \Gamma(2n)$ ,  $n > 0$ .

8) Prove that for  $0 < n < 1$   $\Gamma(n)\Gamma\left(\frac{1-n}{2}\right) = \frac{\sqrt{\pi} \Gamma(n/2)}{2^{1-n} \cos(n\pi/2)}$ .

9) Find the value of  $\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right)$  where  $n$  is an integer.

10) Evaluate  $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ .

11) Show that  $\int_0^1 \frac{x^2}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$ .

- 12) Evaluate  $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ .
- 13) Evaluate  $\int_a^b (x-a)^p(b-x)^q dx$ . Where p and q are +ve integers.
- 14) Show  $\int_0^\infty \cos(bz^{1/n}) dz = \frac{1}{b^{1/n}} \Gamma(n+1) \cos \frac{n\pi}{2}$ .
- 15) Show that if l, m, n, are all positive

$$\iiint x^{l-1} \cdot y^{m-1} \cdot z^{n-1} dx \cdot dy \cdot dz = \frac{a^l \cdot b^m \cdot c^n \Gamma(l/2)}{8} \frac{\Gamma(m/2) \Gamma(n/2)}{\Gamma\left(\frac{l+m+n+2}{2}\right)}$$

Where the multiple integral is taken throughout the part of the ellipsoid

$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ , Which lies in the +ve octant.

- 16) Evaluate  $\iiint x^\alpha \cdot y^\beta \cdot z^\gamma (1-x-y-z)^\lambda dx dy dz$  over the interior of the tetrahedron formed by the coordinate planes and the plane  $x+y+z < 1$ .
- 17) Evaluate  $\iiint \sqrt{\{a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2\}} dx dy dz$  taken through ellipsoid  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .
- 18) Prove  $\iint_D e^{-x^2-y^2} dx dy = \frac{\pi}{2} (1 - e^{-R^2})$ , Where D is the region defined by  $x \geq 0, y \geq 0$ , and  $x^2 + y^2 \leq R^2$
- 19) Evaluate  $\iiint_R (x+y+z+1)^2 dx dy dz$ , Where R is the region defined by  $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$ .

## 12 JACOBIANS

- 1) Find the Jacobian of  $y_1, y_2, y_3, \dots, y_n$ , being given  
 $y_1 = x_1(1-x_2), y_2 = x_1x_2(1-x_3), \dots, y_{n-1} = x_1x_2 \dots x_{n-1}(1-x_n), y_n = x_1x_2x_3 \dots x_n$ .
- 2) If  $u^3+v+w = x+y^2+z^2, u+v^3+w = x^2+y+z^2, u+v+w^3 = x^2+y^2+z$ ,  
 Prove that  $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$
- 3) If  $x+y+z = u, y+z = uv, z = uvw$ , show that  $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$
- 4) If  $u^3 = xyz, 1/v = 1/x + 1/y + 1/z, w^2 = x^2 + y^2 + z^2$ , prove that  $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w(yz+zx+xy)}$
- 5) If  $u^3 + v^3 + w^3 = x+y+z, u^2 + v^2 + w^2 = x^3 + y^3 + z^3, u+v+w = x^2 + y^2 + z^2$ , then prove that  
 $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$
- 6) If  $\lambda, \mu, \nu$  are the roots of the equation in k,  $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$ , prove that  $\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}$

7) The roots of the equation is  $\lambda, (\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$  are  $u, v, w$ . Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2 \frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

8) If  $f(0) = 0$  and  $f'(x) = \frac{1}{1+x^2}$ , prove without using the method of integration, that

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

### 13 DIFFERENTIATION UNDER INTEGRAL SIGN

1) Assuming the validity of differentiation under the integral sign, show that

$$\int_0^{\pi/2} \frac{\log(1 + y \sin^2 x)}{\sin^2 x} dx = \pi(\sqrt{1+y} - 1), \quad \text{where } y > -1.$$

2) Assuming the validity of differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), \quad a \geq 0.$$

Also find the value of integral if  $a < 0$ .

3) If  $|a| < 1$ , prove that  $\int_0^{\pi} \log(1 + a \cos x) dx = \pi \log(1/2 + \sqrt{1-a^2}/2)$ .

4) Assuming the validity of differentiation under the integral sign, show that

$$\int_0^1 \log\left(\frac{1+ax}{1-ax}\right) \frac{dx}{x\sqrt{1-x^2}} = \pi \sin^{-1} a.$$

5) Show that  $\int_0^{\infty} \frac{\tan^{-1} ax \tan^{-1} \beta x}{x^2} dx = \frac{\pi}{2} \log\left\{\frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha \beta^\beta}\right\}, \alpha > 0, \beta > 0.$

6) Show that  $\int_0^{\pi/2} \sin \theta \cos^{-1}(\cos \alpha \cdot \operatorname{cosec} \theta) d\theta = \frac{\pi}{2}(1 - \cos \theta).$

7) Using differentiation under integral sign, show that  $\int_0^1 \frac{x^y - 1}{\log x} dx = \log(1+y).$

### 14 MULTIPLE INTEGRAL

1) Change the order of integration in the double integral  $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dx dy$ . And hence find the value.

2) Prove that  $I = \int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y) dx dy = \frac{\pi}{2} \left(\frac{\pi}{2} - 1\right).$

3) Transform to polar co-ordinates and integrate  $\iint \sqrt{\frac{(1-x^2-y^2)}{(1+x^2+y^2)}} dx dy$ , the integral being extending over positive values of  $x$  and  $y$  subject to  $x^2 + y^2 \leq 1$ .

- 4) By using the transformation  $x + y = u, y = uv$ , prove that  $\int_0^{\pi/2} xy(1-x-y)^{1/2} dx dy$  taken over the area of the triangle bounded by the lines  $x = 0, y = 0, x + y = 1$  is  $\frac{2\pi}{105}$ .
- 5) Transform the integral  $I = \iiint (x + y + z)^n xyz dx dy dz$  taking over the volume bounded by  $x = 0, y = 0, z = 0, x + y + z = 1$ , substituting  $u = x + y + z, x + y = uv, y = uvw$  and hence evaluate its value.
- 6) Show that  $\iiint \frac{dx dy dz}{(x+y+z+1)^3} = \frac{1}{2} \left( \log 2 - \frac{5}{8} \right)$  throughout the volume bounded by the coordinate planes and the plane  $x + y + z = 1$ .

### 15 INDETERMINANTS

- 1) Evaluate  $\lim_{x \rightarrow 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$
- 2) Determine the values of p and q for which  $\lim_{x \rightarrow 0} \frac{x(1+p \cos x) - q \sin x}{x^3}$  exists and equals 1
- 3) Evaluate  $\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$
- 4) Evaluate  $\lim_{x \rightarrow 0} \left( \cot^2 x - \frac{1}{x^2} \right)$
- 5) Evaluate  $\lim_{x \rightarrow a} \left( 2 - \frac{x}{a} \right)^{\tan \left( \frac{\pi x}{2a} \right)}$
- 6) Evaluate  $\lim_{x \rightarrow 0} \frac{e^x - e^{x \cos x}}{x - \sin x}$
- 7) Evaluate  $\lim_{x \rightarrow 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$
- 8) Evaluate  $\lim_{x \rightarrow 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$
- 9) Evaluate  $\lim_{x \rightarrow +\infty} \frac{a^{1/x} - b^{1/x}}{\log \left\{ \frac{x}{x-1} \right\}}$
- 10) Evaluate  $\lim_{x \rightarrow 0} x^m (\log x)^n$ , where m, n are positive integers.

### 16 ASYMPTOTES

- 1) Form the equation of a curve which has  $x = 0, y = 0, y = x$  and  $y = -x$  four asymptotes and which passes through the point (a,b) and cuts its asymptotes again in eight points lying upon the circle  $x^2 + y^2 = a^2$ .

- 2) Find the equation of the cubic which has the same asymptotes as the curve  $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$  and which touches the axis of y at the origin and passes through the point (3,2).
- 3) Find the equation of the cubic which has the same asymptotes as the curve  $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$  and which passes through the points (0,0), (2,0) and (0,2).
- 4) Show that the four asymptotes of the  $(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$  cut the curve in eight points which lie on the circle  $x^2 + y^2 = 1$ .
- 5) Find the asymptotes of the curve  $4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$  and show that they pass through the points of intersection of the curve with ellipse  $x^2 + 4y^2 = 4$ .
- 6) Show that the asymptotes of the curve  $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$  form a square through two of whose angular points the curve passes.

**Find asymptote of the curve**

- 7)  $(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy = 1$ .

**17 CONTINUITY**

- 1) Discuss the continuity of the function  $f(x) = \log_{n \rightarrow \infty} \frac{\log(2+x) - x^{2n} \sin x}{(1+x^{2n})}$  at  $x=1$ .
- 2) Given

$$f(x) = \begin{cases} (\cos x - \sin x)^{\operatorname{cosec} x} & , -\frac{\pi}{2} < x < 0 \\ a & , x = 0 \\ \frac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}}}{ae^{\frac{2}{x}} + be^{\frac{3}{x}}} & , 0 < x < \frac{\pi}{2} \end{cases}$$

If  $f(x)$  is continuous at  $x=0$ , find a and b.

- 3) Examine the function  $f(x) = (x - a) \cdot \sin \left\{ \frac{1}{(x-a)} \right\}$  When  $x \neq a$  and  $f(a) = 0$  for continuity at  $x=a$ .
- 4) Let  $f(x) = \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} + \dots \infty$  Is  $f(x)$  is continuous at the origin? Give reasons for your answer.
- 5) If  $f(x) = -x$  for  $x \leq 0$  and  $f(x) = +x$  for  $x \geq 0$ , prove that  $f(x)$  is continuous but not differentiable at  $x=0$ .
- 6) If  $f(x) = x^2 \sin \left( \frac{1}{x} \right)$  for  $x \neq 0$  and  $f(0) = 0$ , show that  $f(x)$  is continuous and differentiable at  $x=0$ .
- 7) If  $f(x) = e^{\frac{-1}{x^2}} \sin \frac{1}{x}$  for  $x \neq 0$  and  $f(0) = 0$ , test the differentiability of  $f(x)$  at  $x=0$ .

- 8) If  $f(x) = \frac{x e^{\frac{1}{x}}}{1 + e^x}$  for  $x \neq 0$  and  $f(0) = 0$ , show that  $f(x)$  is continuous at  $x=0$ , but  $f'(0)$  does not exist.
- 9) Let  $f(x) = x \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^x + e^{-x}}$ ,  $x \neq 0$ ;  $f(0) = 0$  Show that  $f(x)$  is continuous but not derivable at  $x=0$ .
- 10) Prove that the function  $f$  defined on  $\mathbf{R}^+$  as  $f(x) = \sin \frac{1}{x}$ ,  $\forall x > 0$  is continuous but not uniformly continuous on  $\mathbf{R}^+$ .
- 11) Let a function  $f; \mathbf{R} \rightarrow \mathbf{R}$  satisfy the equation  $f(x + y) = f(x) + f(y)$ ,  $\forall x, y \in \mathbf{R}$  show that if  $f$  is continuous at the point  $x=a$ , then it is continuous for all  $x \in \mathbf{R}$ .  
If  $f$  is continuous then  $f(x) = kx$  for some constant  $k$ .
- 12) If  $f; \mathbf{R} \rightarrow \mathbf{R}$  is continuous function and satisfies the relation  $f(x + y) = f(x)f(y)$ ,  $\forall x, y \in \mathbf{R}$  then either  $f(x) = 0 \forall x \in \mathbf{R}$  Or there exists an  $a > 0$  such that  $f(x) = a^x$ ,  $\forall x \in \mathbf{R}$ .
- 13) Show that the function  $f$  defined by
- $$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$
- is continuous only at  $x=0$ .
- 14) Show that the function  $f$  defined by
- $$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ 0, & \text{when } x \text{ is rational} \end{cases}$$
- is continuous only at  $x=0$ .
- 15) Let  $f$  be a function defined on  $[0,1]$  by;
- $$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are positive integers} \\ & \text{having no common factor.} \end{cases}$$
- Prove that  $f$  is continuous at each irrational point and discontinuous at each rational point.
- 16) Show that the function  $f$  defined by
- $$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational} \\ (1 - x), & \text{if } x \text{ is rational} \end{cases}$$
- is continuous only at  $x = \frac{1}{2}$ .
- 17) Prove that the function  $h(x) = \sqrt{\sin x}$  is continuous on  $[0, \pi]$ .
- 18) Is the function  $f(x) = \frac{x}{x+1}$  uniformly continuous for  $x \in [0,2]$ ? Justify your answer?
- 19) Show that the function  $f$  defined by  $f(x) = x^3$  is uniformly continuous in the interval  $[0,3]$ .

- 20) Show that  $\sin x$  is uniformly continuous on  $[0, \infty]$ .
- 21) Prove that  $f(x) = \sin x^2$  is not uniformly continuous on  $[0, \infty[$ .
- 22) Show that  $f(x) = \sin \frac{1}{x}$  is not uniformly continuous on  $]0, \infty[$ .

### 18 DIFFERENTIABILITY

- 1) If  $f(x) = x^2 \sin (1/x)$  when  $x \neq 0$  and  $f(0)$ , show that  $f$  is derivable for every value of  $x$  but the derivative is not continuous for  $x = 0$
- 2) If  $f(x) = \left\{ x \exp \left[ - \left( \frac{1}{|x|} + \frac{1}{x} \right) \right], x \neq 0, \right.$  then test whether  $f(x)$  is continuous at  $x = 0$   
 $f(x)$  is differentiable at  $x = 0$
- 3) Let  $f(x)$  be an even function. If  $f'(0)$  exists, find its value.
- 4) Determine the set of all points where the function  $f(x) = x / (1 + |x|)$  is differentiable.
- 5) Let  $f(x) = x \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$ ,  $x \neq 0$ , find  $f(0) = 0$ . Show that  $f(x)$  is continuous but not derivable at  $x = 0$
- 6) Let  $f(x) = e^{-1/x^2} \sin(1/x)$  when  $x \neq 0$  and  $f(0) = 0$ . Show that at every point  $f$  has a differential coefficient and this is continuous at  $x=0$ .
- 7) Examine the derivability of the function  $f$  defined by  $f(x) = x^m \sin 1/x$ , if  $x \neq 0$   $0$ , if  $x = 0$   
Determine  $m$  when  $f'$  is continuous at  $x = 0$ .

### 19 MAX MIN-Single Variable

- 1) Investigate for maxima and minima, of  $f(x) = \int_1^x \{ 2(t-1)(t-2)^3 + 3(t-1)^2 + (t-2)^2 \} dt$
- 2) Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius  $r$ .
- 3) A cone is circumscribed to a sphere of radius  $r$ , show that when the volume of the cone is least its altitude is  $4r$  and its semi-vertical angle is  $\sin^{-1} 1/3$ .
- 4) Find the coordinates of the point on the parabola  $y=x^2$  which is nearest to the point  $(3,0)$ .
- 5) Show that the semi-vertical angle of the cone of maximum volume and of given slant height is  $\tan^{-1} \sqrt{2}$
- 6) A thin closed rectangular box is to have one rectangular edge  $n$  times the length of another edge and the volume of the box is given to be  $S$ . prove that the least surface  $S$  is given by  $nS^3 = 54(n+1)^2 V^2$ .

### 20 MAX/MIN Two Variables

- 1) Discuss the maximum or minimum values of  $u$  given by  $u = x^3 y^2(1-x-y)$
- 2) Find the maximum value of  $u$  where  $u = \frac{xyz}{(a+x)(x+y)(y+z)(z-b)}$
- 3) Find a point within a triangle such that the sum of the square of its distances from the three vertices is a minimum.
- 4) Find the maximum value of  $(ax + by + cz) e^{-\alpha^2 x^2 - \beta^2 y - \gamma z^2}$
- 5) Let  $f(x,y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ . Show that  $f(x,y)$  has neither a maximum nor a minimum at  $(0,0)$
- 6) Locate the stationery points of  $x^4 + y^4 - 2x^2 + 4xy - 2y^2$  and determine their nature.
- 7) Find points on  $z^2 = xy + 1$  nearest to the origin

### 21 LENGTH of CURVES

- 1) Find the length of arc of the curve  $x^3=y^2$  from  $x = 0$  to  $x=1$
- 2) Find the length of one loop of the curve  $3ay^2 = x(x-a)^2$
- 3) Find the length of the curve  $x^2(a^2 - x^2) = 8a^2y^2$
- 4) Find the length of the cardioid  $r = a(1 + \cos\theta)$ . Also show that the upper half is bisected by  $\theta = \frac{\pi}{3}$
- 5) Find the entire length of the asteroid  $x^{2/3} + y^{2/3} = a^{2/3}$  or  $x = a \cos^3\theta, y = a \sin^3\theta$
- 6) Find the length of the arc of the parabola  $y^2 = 4ax$  cut off by its latus rectum.
- 7) Find the length of the arc of the parabola  $x^2 = 4ay$  measured from the vertex to one extremity of the latus rectum.

### 22 AREAS

- 1) Find the area bounded by the curve  $y^2(2a-x) = x^3$  and its asymptote.
- 2) Find the area of a loop of the curve  $r = a \sin 3\theta$ .
- 3) Show that the area between the cardioids  $r = a(1+\cos\theta)$  and  $r = a(1-\cos\theta)$  is  $\frac{3\pi-8}{2} a^2$ .
- 4) Prove that the area of the loop of the curve  $x^3+y^3 = 3axy$  is  $3a^2/2$ .
- 5) Find the area of the parabola  $y^2 = 4ax$  bounded by its latus rectum.
- 6) Find the whole area of the curve  $x^2(x^2+y^2) = a^2(x^2-y^2)$ .
- 7) Find the area enclosed by the curve  $x^{3/3} + y^{3/3} = a^{3/3}$ .

- 8) Find the area included between the cycloid  $x = a(\theta + \sin\theta)$ ,  $y = a(1 - \cos\theta)$  and its base
- 9) Find the area enclosed between one arch of the cycloid  $x = a(\theta - \sin\theta)$ ,  $y = a(1 - \cos\theta)$  and its base.
- 10) Find the area included between the parabolas  $y^2 = 4ax$  and  $x^2 = 4ay$ .
- 11) Find the area of the total region bounded by the Lemniscate  $r^2 = a^2 \cos 2\theta$ .

### 23 VOLUMES

- 1) The axes of two right circular cylinders of the same radius 'a' intersect at right angles. Prove that the volume which is inside both the cylinder is  $\frac{16}{3}a^3$ .
- 2) Find the volume of the part of the cylinder  $x^2/a^2 + z^2/c^2 = 1$ , which lies between the planes  $y = 0$ ,  $y = mx$ .
- 3) Find the volumes cut from a sphere of radius a by a right circular cylinder with b as radius of the base and whose axis passes through the centre of the sphere.
- 4) Through a diameter of the upper base of a right cylinder of altitude 'a' and radius 'r' pass two planes which touch the lower base on opposite side. Find the volume of the cylinder included between the two planes.
- 5) Find the volume cut from the sphere  $x^2 + y^2 + z^2 = a^2$  by the cylinder  $x^2 + y^2 = ax$ .  
Or A sphere is cut by a right cylinder, the radius of whose base is half that of the sphere and one of whose edge passes through the centre of the sphere. Find the volume to both.
- 6) Find the volume enclosed by the surface  $(x/a)^{2n} + (y/b)^{2n} + (z/c)^{2n} = 1$  where n is integer.
- 7) A right cone has its vertex in the surface of the sphere and its axis coincident with the diameter of the sphere, passing through that point. Find the volume common to the cone and the sphere.
- 8) Find the volume of the solid bounded by the surface  $(x^2 + y^2 + z^2)^3 = 27a^3xyz$ .
- 9) The sphere  $x^2 + y^2 + z^2 = a^2$  is pierced by the cylinder  $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ , prove that the volume of the sphere that lies inside the cylinder is  $\frac{8}{3} \left( \frac{\pi}{4} + \frac{5}{3} - \frac{4\sqrt{2}}{3} \right) a^3$ .
- 10) Find the volume of the right circular cone formed by revolution of a right - angled triangle about a side which contains the right angle
- 11) Find the volume of the solid generated by revolving the curve  $x = a \cos^3 t$ ,  $y = a \sin^3 t$  (or  $x^{2/3} + y^{2/3} = a^{2/3}$ ) about the x - axis.

- 12) Find the volume of the solid obtained by revolving one arc of the cycloid  $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$  about x-axis.
- 13) Prove that the volume of the solid generated by the revolution of the curve  $y = a^3/(a^2+x^2)$  about its asymptote is  $\pi^2 a^3/2$ .
- 14) Find the volume of the solid obtained by revolving the cardioide  $r = a(1 + \cos \theta)$  about the initial line.

### 24 SURFACES

- 1) Find the area of the surface of the sphere  $x^2+y^2+z^2 = a^2$  which lies inside the cylinder  $x^2+y^2 = ay$ .
- 2) Find the surfaces of  $x^2 + z^2 = a^2$ , that lies inside the cylinder  $x^2+y^2 = a^2$ .
- 3) Find the surface of the solid generated by the revolution of the asteroid  $x = a \cos^3 t, y = a \sin^3 t$  about x- axis.
- 4) Find the surface area of the solid formed by the rotation of the arc of the cycloid  $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$  about x-axis.
- 5) Find the area of the surface of revolution formed by revolving the curve  $r = 2a \cos\theta$  about the initial line.
- 6) Find the surface area of the solid generated by revolving the curve  $r^2 = a^2 \cos 2\theta$  about the initial line.
- 7) Find the surface of the solid obtained by revolving the cardioid  $r = a(1 - \cos \theta)$  about the initial line.

### 25 Curve Tracing

- 1) Trace  $9ay^2 = x(x - 3a)^2$ .
- 2) Trace  $y(x^2 + 4a^2) = 8a^3$ .
- 3) Trace  $y^2(a + x) = x^2(3a - x)$ .
- 4) Trace  $y(1 - x^2) = x^2$ .
- 5) Trace the curve  $x^3 + y^3 = 3axy$ .
- 6) Trace the curve  $y^2(x^2 - 1) = 2x - 1$ .
- 7) Trace the curve  $y^2(2x - 1) = x(x - 1)$ .
- 8) Trace the curve  $y^2(x^2 - a^2) = a^2x^2$ .

- 9) Trace the curve  $y^3 = x(a^2 - x^2)$ .
- 10) Trace the curve  $y^2x = a(x^2 - a^2)$ .
- 11) Trace the curve  $a^2y = x(2a - x)(x - a)$ .
- 12) Trace the curve  $r = a \sin 3\theta$ .
- 13) Trace the curve  $r = \sin 2\theta$ .
- 14) Trace the curve  $r^2 = a^2 \cos 2\theta$ .
- 15) Trace the curve  $r = a(1 - \cos \theta)$ .