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UPSC CSE Mathematics: Previous Year Questions: Real Analysis

2025

1) Examine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is absolutely or conditionally convergent.
2) Define Cauchy sequence and prove that every convergent sequence of real numbers is a Cauchy sequence. What is the importance of Cauchy condition?
3) Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.
4) Prove that every continuous function is Riemann integrable.

2024

1) Test the convergence of $\int_0^2 \frac{\log x}{\sqrt{(2-x)}} dx$.
2) Using Cauchy's general principle of convergence, examine the convergence of the sequence $\langle f_n \rangle$, where $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$.
3) Consider the series $\sum_{n=1}^{\infty} U_n(x), 0 \le x \le 1$, the sum of whose first n terms is given by $S_n(x) = \frac{1}{2n^2} \log(1 + n^4x^2), x \in [0,1]$. Show that the given series can be differentiated term-by-term, though $\sum_{n=1}^{\infty} U_n'(x)$, does not converge uniformly on $[0,1]$.
4) Find the upper and lower Riemann integrals for the function f defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} (1-x^2)^{1/2}, \text{ if } x \text{ is rational.} \\ (1-x), \text{ if } x \text{ is irrational.} \end{cases}$$

Hence, show that f is not Riemann integrable on $[0,1]$.

2023

1) Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3 \cdot 5 \ldots (2n-1)}{2.4.6 \ldots (2n)} \cdot \frac{x^{2n+1}}{(2n+1)}, x > 0$
2) Using the method of Lagrange's multipliers, find the minimum and maximum distances of the point $P(2,6,3)$ from the sphere $x^2 + y^2 + z^2 = 4$.
3) Prove that the oscillation of a real-valued bounded function f defined on $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$.

2022

1) Test the convergence of $\int_0^\infty \frac{\cos x}{1+x^2} dx$.
2) Let $f(x) = x^2$ on $[0,k], k > 0$. Show that f is Riemann integrable on closed interval $[0,k]$ and $\int_0^k f dx = \frac{k^3}{3}$
3) Find the maximum and minimum values of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, when $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Interpret the result geometrically.
4) Test for convergence or divergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \cdots \ (x > 0)$

2021

1) Test the uniform convergence of the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$ on $[0,1]$
2) If a function f is monotonic in the interval $[a,b]$, then prove that f is Riemann integrable in $[a,b]$.
3) Find the maximum and minimum values of $f(x) = x^3 - 9x^2 + 26x - 24$ for $0 \leq x \leq 1$.
4) Find the stationary values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically

2020

1) Prove that the sequence (a_n) satisfying the condition $|a_{n+1} - a_n| \leq \alpha|a_n - a_{n-1}| \quad 0 \leq \alpha \leq 1$ for all-natural numbers $0 \leq \alpha \leq 1$ is a Cauchy sequence.
2) Prove that the function $f(x) = \sin\ x^2$ is not uniformly continuous on the interval $[0, \infty[$
3) If $u = \tan^{-1} \frac{x^3+y^3}{x-y}, x \neq y$ then show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial u} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2\ u)\sin\ 2u$
4) Show that $\int_0^{\pi/2} \frac{\sin^2\ x}{\sin\ x + \cos\ x} dx = \frac{1}{\sqrt{2}} \log_e\ (1 + \sqrt{2})$

2019

1) Show that the function $f(x,y) = \begin{cases} \frac{x^2-y^2}{x-y} & (x,y) \neq (1,-1) \quad (1,1) \\ 0 & (x,y) = (1,1) \quad (1,-1) \end{cases}$ is continuous and differentiable at $(1,-1)$
2) Evaluate $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} dx, a \geq 0, a \neq 1$
3) Using differentials, find an approximate value of $f(4.1, 4.9)$ where $f(x,y) = (x^3 + x^2y)^{\frac{1}{2}}$
4) Discuss uniform convergence of $f_n(x) = \frac{nx}{1+n^2x^2}, \forall x \in \mathbb{R}(-\infty, \infty) \quad n = 1,2,3, \ldots$
5) Find the maximum value of the $f(x,y,z) = x^2y^2z^2$ subject to the subsidiary condition $x^2 + y^2 + z^2 = c^2, (x,y,z \geq 0)$
6) Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{\ln\ x} dx$

2018

1) Prove the inequality: $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$

2) Find the range of $p(> 0)$ for which the series

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots, a > 0$$

(i) absolutely convergent and (ii) conditionally convergent.

3) Show that if a function f defined on an open interval (a, b) of $\mathbb{R}$ is convex, then f is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous.

4) Suppose $\mathbb{R}$ be the set of all real numbers and $f: \mathbb{R} \to \mathbb{R}$ is a function such that the following equations hold for all , $y \in \mathbb{R}$: (i) $f(x + y) = f(x) + f(y)$ (ii) $f(xy) = f(x)f(y)$ Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$, or, $f(x) = x$.

2017

1) Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}, n = 1,2,3, \dots$ show that the sequence $x_1, x_2, x_3..$ is convergent.

2) Find the Supremum and the Infimum of $\frac{x}{\sin x}$ on the interval $\left(0, \frac{\pi}{2}\right]$.

3) Let $f(t) = \int_0^t [x]dx$ where $[x]$ denote the largest integer less than or equal to x

   a) Determine all the real numbers t at which f is differentiable.

   b) Determine all the real numbers t at which f is continues but not differentiable.

4) Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100

2016

1) For that the function $f: (0, \infty) \to R$ given by $(x) = x^2 \sin \frac{1}{x}, 0 < x < \infty$. Show that there is a differentiable function $g: R \to R$ that extends f

2) Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:

$x_1 = \frac{1}{2}, y_1 = 1, x_n = \sqrt{x_{n-1} y_{n-1}}, n = 2,3,4, \dots$and $\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right), n = 2,3,4, \dots.$

Prove that $x_{n-1} < x_n < y_n < y_{n-1}, n = 2,3,4, \dots$ and deduce that both the sequence converges to the same limit l where $\frac{1}{2} < l < 1$

3) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ conditionally convergent (if you use any theorem (s) to show it then you must give a proof of that theorem(s).

4) Find the relative maximum minimum values of the function

$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

5) Let $f: R \to R$ be a continuous function such $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are finite. Prove that is uniformly continuous on $\mathbb{R}$

2015

1) Test for convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2+1}\right)$

2) Is the function $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n} \\ 0 & x = 0 \end{cases}$ Riemann Integrable? If yes, obtain the value of $\int_0^1 f(x)dx$

3) Test the series of functions $\sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$ for uniform convergence

4) Find the absolute maximum and minimum values of the function

$f(x, y) = x^2 + 3y^2 - y$ over the region $x^2 + 2y^2 \le 1$

2014

1) Test the convergence of the improper integral $\int_1^{\infty} \frac{dx}{x^2(1+e^{-x})}$

2) Integrate $\int_0^1 f(x)dx$, , where $f(x) = \begin{cases} 2x\sin\frac{1}{x}\cos\frac{1}{x}, & x \in [0,1] \\ 0 & x = 0 \end{cases}$

3) Obtain $\frac{\partial^2 f(0,0)}{\partial x\, \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y\, \partial x}$ for the function $f(x, y) = \begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & , (x, y) \ne (0,0) \\ 0 & , (x, y) \ne (0,0) \end{cases}$

Also, discuss the continuity $\frac{\partial^2 f}{\partial x\, \partial y}$ and $\frac{\partial^2 f}{\partial y\, \partial x}$ of at $(0,0)$

4) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multipliers.

2013

1) Let $f(x) = \begin{cases} \frac{x^2}{2} + 4 & \text{if } x \ge 0 \\ \frac{-x^2}{2} + 2 & \text{if } x < 0 \end{cases}$ Is f Riemann integrable in the interval $[-1,2]$? Why? Does there exist a function g such that $g'(x) = f(x)$? Justify your answer.

2) Show that the series $\sum_1^{\infty} \frac{(-1)^{n-1}}{n+x^2}$, is uniformly convergent but not absolutely for all real values of x

3) Show that every open subset of R is countable union of disjoint open intervals

4) Let $[x]$ denote the integer part of the real number x, i.e., if $n \le x < n + 1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in the function in $[-1,2]$? If not, explain why. If it is integrable, compute $\int_{-1}^{2} ([x]^2 + 3)dx$

5) Let $f(x,y) = y^2 + 4xy + 3x^2 + x^3 + 1$. At what points will $f(x,y)$ have a maximum or minimum?

2012

1) Let, $f_n(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x}, & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n}, \\ 0, & \text{if } x > \frac{1}{n} \end{cases}$

Show that $f_n(x)$ converges to a continuous function but not uniformly.

2) Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent

3) Let $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$ Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$.

4) Find the minimum distance of the line given by the planes $3x + 4y + 5z = 7$ and $-z = 9$, from the origin, by the method of Lagrange's multipliers.

5) Let $f(x)$ be differentiable on $[0,1]$ such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x)dx = 1$.Prove that $\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$

6) Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify

2011

1) Let $S = (0,1)$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \le 1$ (in). Is f uniformly continuous on ? Justify your answer.

2) Let $f_n(x) = nx(1-x)^n, x \in [0,1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0,1]$

3) Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2 + 8xy + 7y^2 = 225$

4) Show that the series for which the sum of first n terms $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$ cannot be differentiated term-by-term at $x = 0$. What happens at $\neq 0$?

5) Show that if $S(x) = \sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$, then its derivative $S'(x) = -2x\sum_{n=1}^{\infty}\frac{1}{n^2(1+nx^2)^2}$, for all x

2010

1) Discuss the convergence of the sequence $\{x_n\}$ where $X_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{8}$

2) Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$ Show that the sequence converges to $\left(\frac{1+\sqrt{17}}{2}\right)$

3) Define the function $f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$. Find $f'(x)$. Is $f'(x)$ continuous at $=0$? Justify your answer.

4) Consider the series $\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^2}$. Find the values of x for which it is convergent and also the sum function. Is the converse uniform? Justify your answer.

5) Let $f_n(x)=x^n$ on $-1<x\leq 1$ for $n=1,2$ ..... Find the limit function. Is the convergence uniform? Justify your answer.

2009

1) State Roll's theorem. Use it to prove that between two roots of $e^x\cos\ x=1$ there will be a root of $e^x\sin\ x=1$

2) Let $f(x)\begin{cases}\frac{|x|}{2}+1 & \text{if } x<1\\ \frac{x}{2}+1 & \text{if } 1\leq x<2\\ -\frac{|x|}{2}+1 & \text{if } 2\leq x\end{cases}$ What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify yours answer.

3) Show that the series $\left(\frac{1}{3}\right)^2+\left(\frac{1.4}{3.6}\right)^2+\cdots..+\left(\frac{1.4.7......(3n-2)}{3.6.9...........3n}\right)^2+\cdots$ ... converges

4) Show that if $f:[a,b]\to R$ is a continuous function then $f([a,b])=[c,d]$ form some real numbers c and d, $c\leq d$

5) Show that: $\lim_{x\to 1}\sum_{n=1}^{\infty}\frac{n^2x^2}{n^4+x^4}=\sum_{n=1}^{\infty}\frac{n^2}{n^4+1}$ Justify all steps of your answer by quoting the theorems you are using

6) Show that a bounded infinite subset R must have a limit point

2008

1) For $x>0$, show $\frac{x}{1+x}<log\ (1+x)<x$

2) Let $T=\left\{\frac{1}{n},n\in N\right\}\cup\left\{1+\frac{3}{2n},n\in N\right\}\cup\left\{6-\frac{1}{3n},n\in N\right\}$. Find derived set T of T. Also find Supremum of T and greatest number of T.

3) If $f:R\to R$ is continuous and $f(x+y)=f(x)+f(y)$, for all $x,y\in R$ then show that $f(x)=xf(1)$ for all $x\in R$.

4) Discuss the convergence of the series $\frac{x}{2}+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots$ ..., $x>0$.

5) Show that the series $\sum\frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod_2^{\infty}\left(1+\frac{1}{n^2-1}\right)$

6) Let f be a continuous function on $[0,1]$. Using first Mean Value theorem on Integration, prove that $\lim_{n\to\infty}\int_0^1\frac{nf(x)}{1+n^2x^2}dx=\frac{\pi}{2}f(0)$

7) Prove that the sets $A=[0,1]$, $B=(0,1)$ are equivalent sets.

8) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}, x \in \left(0, \frac{\pi}{2}\right)$

2007

1) Show that the function given by $f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exists at $(0,0)$

2) Using Lagrange's mean value theorem, show that $|\cos\, b - \cos\, a| \leq |b - a|$

3) Given a positive real number a and any natural number n, prove that there exists one and only one positive real number ξ such that $\xi^n = a$

4) Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$

5) Rearrange the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ to converge to 1

2006

1) Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$

2) Prove that the function f defined by $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ is nowhere continuous.

3) A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove there is at least one value $\xi, a < \zeta < b$ for which $f''(\zeta) < 0$.

4) Show that the function given by $f(x,y) = \begin{cases} \frac{x^3+2y^2}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$ (i) is continuous at $(0,0)$ (ii) possesses partial derivative $f_x(0,0)$ and $f_y(0,0)$

5) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

2005

1) If u, v, w are the roots of the equation in λ and $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$, evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

2) Evaluate $\iiint \ln(x+y+z)\,dxdydz$ The integral being extended over all positive values of x, y, z such that $x + y + z \leq 1$

3) If f' and g' exist for every $x \in [a,b]$ and if $g'(x)$ does not vanish anywhere (a,b), show that there exists c in (a,b) such that $\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$

4) Show that $\int_0^{\infty} e^{-t} t^{n-1} dt$ is an improper integral which converges for $n > 0$

2004

1) Show that the function $f(x)$ defined as: $f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} \le x \le \frac{1}{2^n}, n = 0,1,2, \ldots$ and $f(0) = 0$ is integrable in [0,1], although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = \frac{2}{3}$

2) Show that the function $f(x)$ defined on by: $f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational} \end{cases}$ is continuous only at $x = 0$

3) If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c.

4) Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$

5) Let $f(x) \ge g(x)$ for every x in $[a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$ then prove that $\int_a^b f(x)dx > \int_a^b g(x)dx$ and hence show that $-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}$

2003

1) Let a be a positive real number and $\{x_n\}$ sequence of rational numbers such that $\lim_{n\to\infty} x_n = 0$. Show that $\lim_{n\to\infty} a x_n = 1$

2) If a continuous function of x satisfies the functional equation $f(x + y) = f(x) + f(y)$ then show that $f(x) = \alpha x$ where α is a constant.

3) Show that the maximum value of $x^2y^2z^2$ subject to condition $x^2 + y^2 + z^2 = c^2$ is $\frac{c^6}{27}$. Interpret the result

4) The axes of two equal cylinders intersect at right angles. If a be their radius, then find the volume common to the cylinder by the method of multiple integrals.

5) Show that $\int_0^\infty \frac{dx}{1+x^2 \sin^2 x}$ is divergent

2002

1) Prove that the integral $\int_0^\infty x^{m-1} e^{-x} dx$ is convergent if and only if $m > 0$.

2) Find all the positive values of a for which the series $\sum_{n=1}^\infty \frac{(an)^n}{n!}$ converges.

3) Test uniform convergence of the series $\sum_{n=1}^\infty \frac{\sin nx}{n^p}$, where $p > 0$

4) Obtain the maxima and minima of $x^2 + y^2 + z^2 - yz - zx - xy$ subject to condition $x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0$

5) A solid hemisphere H of radius' a ' has density ρ depending on the distance R from the center of and is given by $\rho = k(2a - R)$ where k is a constant. Find the mass of the hemisphere by the method of multiple integrals

2001

1) Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x} dx$ exists if and only if $m < n + 1$

2) If $\lim_{n\to\infty} a_n = l$, then prove that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots a_n}{n} = l$,

3) A function f is defined in the interval (a, b) as follows

$f(x) = \begin{cases} \frac{1}{q^2} \text{ when } x = \frac{p}{q} \\ \frac{1}{q^3} \text{ when } x = \sqrt{\frac{p}{q}} \end{cases}$ where p, q relatively prime integers. $f(x) = 0$ for all other values of x. Is f Riemann integrable? Justify your answer.

4) Show that $U = xy + yz + zx$ has a maximum value when the three variables are connected by the relation $ax + by + cz = 1$ and a, b, c are positive constants satisfying the condition $2(ab + bc + ca) > (a^2 + b^2 + c^2)$

5) Evaluate $\iiint (ax^2 + by^2 + cz^2)dxdydz$ taken throughout the region $x^2 + y^2 + z^2 \leq R^2$