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UPSC CSE Mathematics: Previous Year Questions: Complex Analysis

2025

1) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for $1 < |z| < 3$.

2) Use the method of contour integration to prove that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}$.

3) Evaluate the integral $\oint_C \frac{e^z}{z^2(z+1)^3} \; dz, C: |z| = 2$

2024

1) If $\mathrm{w} = \mathrm{f(z)}$ is an analytic function of z, then show that $\left(\frac{\partial^2}{\partial \mathbf{x}^2} + \frac{\partial^2}{\partial \mathbf{y}^2}\right) \log |\mathrm{f}'(\mathrm{z})| = 0$

2) If ϕ and ψ are functions of x and y satisfying Laplace equation, then show that $f(z) = p + iq, i = \sqrt{-1}$ is an analytic function, where $\mathrm{p} = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial \mathrm{x}}$ and $\mathrm{q} = \frac{\partial \phi}{\partial \mathrm{x}} + \frac{\partial \psi}{\partial y}$.

3) Find the function which is analytic inside and on the circle C : $\mathrm{z} = \mathrm{e}^{\mathrm{i}\theta}, 0 \le \theta \le 2\pi$ and has the value $\frac{(\mathrm{a}^2-1)\cos\,\theta + \mathrm{i}(\mathrm{a}^2+1)\sin\,\theta}{\mathrm{a}^4 - 2\mathrm{a}^2\cos\,2\theta + 1}$, on the circumference of C , where $\mathrm{a}^2 > 1$.

4) Locate the poles and their order for the function $f(z) = \frac{1}{z(\sin\,\pi z)\left(z+\frac{1}{2}\right)}$. Also, find the residue of $f(z)$ at these poles.

2023

1) State the sufficient conditions for a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be analytic in its domain. Hence, show that $f(z) = \log\, z$ is analytic in its domain and find $\frac{df}{dz}$.

2) Evaluate using $\int_0^{2\pi} \frac{\cos\,2\theta}{5+4\cos\,\theta} d\theta$ contour integration

3) Prove that $u(x,y) = e^x(x\cos\, y - y\sin\, y)$ is harmonic. Find its conjugate harmonic function $v(x,y)$ and express the corresponding analytic function $f(z)$ in terms of z.

4) Classify the singular point $z = 0$ of the function $f(z) = \frac{e^z}{z-\sin\, z}$ and obtain the principal part of its Laurent series expansion.

2022

1) If $f(z) = u + iv$ is an analytic function of z, and $u - v = \frac{\cos\, x + \sin\, x - e^{-y}}{2\cos\, x - e^y - e^{-y}}$, then find $f(z)$ subject to the condition $f\left(\frac{\pi}{2}\right) = 0$.

2) Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for the regions

(i) $0 < |z-1| < 2$ and (ii) $0 < |z-3| < 2$.

3) Apply the calculus of residues to evaluate $\int_{-\infty}^{\infty} \frac{\cos xdx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$.

4) Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i| = 2$

2021

1) Let $c: [0,1] \to \mathbb{C}$ be the curve, where $c(t) = e^{4\pi it}, 0 \le t \le 1$. Evaluate the contour integral $\int_c \frac{dz}{2z^2-5z+2}$

2) Find the Laurent series expansion of $f(z) = \frac{z^2-z+1}{z(z^2-3z+2)}$ in the powers of $(z+1)$ in the region $|z+1| > 3$.

3) Let f be an entire function whose Taylor series expansion with centre $z = 0$ has infinitely many terms. Show that $z = 0$ is an essential singularity of $f\left(\frac{1}{z}\right)$.

4) Using contour integration, evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin xdx}{x(x^2+a^2)}, a > 0$.

2020

1) Evaluate the integral $\int_C (z^2 + 3z)dz$ counterclockwise from $(2,0)$ to $(0,2)$ along the curve C where C is the circle $|z| = 2$

2) Using contour integration, evaluate the integral $\int_0^{2\pi} \frac{1}{3+2\sin\,\theta} d\theta$

3) If $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\,\theta, r \neq 0$ then an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$

2019

1) Suppose $f(z)$ is Analytical function on a domain D in $\mathbb{C}$ and satisfies the equation. $f(z) = (\mathrm{Re}\ f(z))^2, z \in D$ Show that $f(z)$ is constant in D

2) Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytical and non-zero at z_0.

Moreover $\underset{z=z_0}{\mathrm{Res}}\, f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, m \ge 1$

3) Evaluate the integral $\int_c \mathrm{Re}\,(z^2)dz$ from 0 to $2+4i$ along the curse C where C is a parabola $y = x^2$

4) Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{(e^z-1)}$ above the point $z = 0$ valid in the region $0 \le |z| \le 2\pi$

2018

1) Prove that the function: $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

2) Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when

   a) $|z| < 1$

   b) $1 < |z| < 2$

   c) $|z| > 2$

3) Show by applying the residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0.$

2017

1) Determine all entire functions $f(z)$ such that 0 is removable singularity of $f\left(\frac{1}{z}\right)$.

2) Using contour integral method, proves that $\int_0^\infty \frac{x\sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$.

3) Let $f = u + iv$ be analytic function on the unit disc $D = \{z \in C: |z| < 1\}$.
   Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ at all points of D.

4) For a function $f: C \to C$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$ Let f be an entire function such that for some $n \geq 1, f^{(n)}\left(\frac{1}{k}\right) = 0$ for all $k = 1,2,3,..$ show that f is a polynomial.

2016

1) Is $v(x,y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim, if yes find its conjugate harmonic function and hence obtain the analytic function $u(x,y)$ whose real and imaginary parts are u and v respectively

2) Let $\gamma; [0,1] \to C$ be the curve $\gamma(t) = e^{2\pi it}, 0 \leq t \leq 1$ find giving justification the values of the contour integral $\int_\gamma \frac{dz}{4z^2-1}$

3) Prove that every power series represents an analytic function inside its circle of convergence

2015

1) Show that the function $v(x,y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z

2) Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$

3) State Cauchy's residue theorem. Using it, evaluate integral $\int_C \frac{e^z+1}{z(z+1)(z-i)^2} dz; C: |z| = 2$

2014

1) Prove that the function $f(z) = u + iv$, where $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0; f(0) = 0$ satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist.

2) Expand in Laurent series the function $f(z) = \frac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$.

3) Evaluate the integral $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues.

2013

1) Prove that if $be^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has n zeros in the unit circle.

2) Using Cauchy's residue theorem, evaluate the integral $I = \int_0^{\pi} \sin^4 \theta d\theta$

2012

1) Show that the function defined by $f(z) = \begin{cases} \frac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin.

2) Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4} dz$ where C is the circle $|z| = 2$

3) Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

   (a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$ (d) $|z| < 1$

4) Evaluate by contour integration $I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$ $a^2 < 1$

2011

1) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$,

   find $f(z)$ subject to the condition, $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$

2) If the function $f(z)$ is analytic and one valued in $|z - a| < R$, prove that for $0 < r < R$, $f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta} d\theta$, where $P(\theta)$ is the real part of $f(a + re^{i\theta})$

3) Evaluate by Contour integration, $\int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}}$

4) Find the Laurent series for the function $f(z) = \frac{1}{1-z^2}$ with centre $z = 1$

2010

1) Show that $u(x,y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x,y)$. Hence find the analytic function f for which $u(x,y)$ is the real part.

2) Evaluate the line integral $\oint_C f(z)dz$ where $f(z) = z^2$, C is the boundary of the triangle with vertices $A(0,0), B(1,0), C(1,2)$ in that order.

3) Find the image of the finite vertical strip $R: x = 5$ to $x = 9, -\pi \le y \le \pi$ of z-plane under exponential function

4) Find the Laurent series of the function $f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$ as $\sum_{n=-\infty}^{\infty} C_n z^n$ for $0, |z| < \infty$ where $C_n = \int_0^{\pi} \cos(n\phi - \lambda \sin\phi)d\phi \quad n = 0, \pm 1, \pm 2, \ldots$ with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}(-\pi \le \phi \le \pi)$ as contour in this region.

2009

1) Let $f(z) = \frac{a_0 + a_1 \ldots\ldots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots \ldots\ldots + b_n z^n}$, $b_n \neq 0$. Assume that the zeros of the denominator are simple. Show that the sum of the residues of $f(z)$ at its poles is equal to $\frac{a_{n-1}}{b_n}$.

2) If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$ show that: $\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$

2008

1) Find the residue of $\frac{\cot z \coth z}{z^3}$ at $z = 0$

2) Evaluate $\int_C \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2}\right] dz$ where C is the circle $|z| = 3$. State the theorems you use in evaluating above integral

2007

1) Prove that the function f defined by $f(z) = \begin{cases} \frac{z^5}{|z|^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not differentiable at $z = 0$

2) Evaluate (by using residue theorem) $\int_0^{2n} \frac{d\theta}{1 + 8\cos^2\theta}$

2006

1) With the aid of residues, evaluate $\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta, -1 < a < 1$

2) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$

2005

1) If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x(\cos\ y - \sin\ y)$, determined $f(z)$ in terms of z.

2) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for

   a) $1 < |z| < 3$

   b) $|z| < 3$ and

   c) $|z| < 1$

2004

1) If all zeros of a polynomial $P(z)$ lies in a half plane then show that zeros of the derivatives $P'(z)$ also lie in the same half plane.

2) Using contour integration evaluate $\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2} d\theta, 0 < p < 1$

2003

1) Use the method of contour integration to prove that $\int_0^{\pi} \frac{a d\theta}{a^2+\sin^2 \theta} = \frac{\pi}{\sqrt{1+a^2}} (a > 0)$

2002

1) Suppose that f and g are two analytic functions on the set ϕ of all complex numbers with $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for $n = 1,2,3, \dots$. Then show that $f(z) = g(z)$ for each z in ϕ

2) Show that, when $0 < |z-1| < 2$, that function $f(z) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of $(z-1)$ as $\frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$

3) Establish, by contour integration, $\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx = \frac{\pi}{2} e^{-a}$ where $a \geq 0$.

2001

1) Prove that the Riemann zeta function ζ defined by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for $\text{Re } z > 1$ and converges uniformly for $\text{Re } z \geq 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary small.

2) Find the Laurent series for the function $e^{\frac{1}{z}}$ in $0 < z < \infty$. Using this expansion, show that $\frac{1}{\pi} \int_0^{\pi} \exp(\cos\theta)\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}$ for $n = 1,2,3, \dots$

3) Show that $\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$