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UPSC CSE 2023 Mathematics Paper 1 – Solutions

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1a) Let $V_1 = (2,-1,3,2), V_2 = (-1,1,1,-3)$ and $V_3 = (1,1,9,-5)$ be three vectors of the space $\mathbb{R}^4$. Does $(3,-1,0,-1) \in \text{span}\,\{V_1, V_2, V_3\}$? Justify your answer.

Method 1 : $(3,-1,0,-1) \in \text{span}(v_1\ v_2, v_3) \Rightarrow$

$$(3,-1,0,-1) = l(2,-1,3,2)+m(-1,1,1,-3)+n(1,1,9,-5)$$

& $l, m, n \rightarrow$ All not zero

$2l-m+n=3$ — ①

$-l+m+n=-1$ — ②

$3l+m+9n=0$ — ③

$2l-3m-5n=-1$ — ④

① + ② $\Rightarrow$ $l+2n=2$ — ⑤

① + ③ $\Rightarrow$ $5l+10n=3$

$l+2n=3/5$ — ⑥

Clearly $\left.\begin{array}{l} l+2n=2 \\ l+2n=3/5 \end{array}\right]$ Not possible to get any solution in real R

No soln in real R $\Rightarrow$ Does not span

* There may be soln in Complex space

Method 2 :
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 2 & -1 & 3 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & 9 & -5 \\ 3 & -1 & 0 & -1 \end{pmatrix}$$

Do Row operations such that get row 4 ie R_4 to $(0,0,0,0)$ wh

Getting $R_4 \rightarrow (0,0,0,0)$ means R_4 can be expressed as $(R_1 \cup R_2 \cup R_3)$

$R_1 \rightarrow R_1 - 2R_3$
$R_2 \rightarrow R_2 + R_4$
$R_4 \rightarrow R_4 - 3R_3$

$$\begin{pmatrix} 0 & -3 & -15 & 12 \\ 0 & 2 & 10 & -8 \\ 1 & 1 & 9 & -5 \\ 0 & -4 & -27 & 14 \end{pmatrix}$$

$R_4 \rightarrow R_4 + 2R_2$

$$\begin{array}{c} \\ \\ \\ R_4 \rightarrow \\ \\ \end{array}\begin{pmatrix} 0 & -3 & -15 & 12 \\ 0 & 2 & 10 & -8 \\ 1 & 1 & 9 & -5 \\ 0 & 0 & -7 & -2 \end{pmatrix}$$
$$\begin{array}{cc} \uparrow & \uparrow \\ c_1 & c_2 \end{array}$$

Without disturbing 0,0 in Row-4, its not possible to do row operations for R_4 to get $(-7,-2)$ to $(0,0)$

we cannot proceed further.

R_4 cannot be expressed as R_1, R_2, R_3

So does not span

1b) Find the rank and nullity of the linear transformation:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$

Nullity: $T(x,y,z) = (0,0,0)$

$$\left.\begin{aligned} x+z&=0 \\ x+y+2z&=0 \\ 2x+y+3z&=0 \end{aligned}\right]$$

Let $x = l \Rightarrow x+z=0 \Rightarrow z=-l$

Put in $x+y+2z=0$

$l+y-2l=0 \Rightarrow y=l$

Put in $2x+y+3z=0 \Rightarrow 2l+l-3l=0$ Satisfy

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ l \\ -l \end{pmatrix} = l \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rightarrow$$

Nullity $\rightarrow 1$

Null Space $(1,1,-1)$

Rank + Nullity = dim R^3 $\rightarrow$ rank $+ 1 = 3$

$\Rightarrow$ rank $= 2$

Method 2 for Rank:

$e_1 = (1,0,0)$

$e_2 = (0,1,0)$

$e_3 = (0,0,1)$

$T(e_1) = (1,1,2)$

$T(e_2) = (0,1,1)$

$T(e_3) = (1,2,3)$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} R_3 \to R_3 - R_1 \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Range space is $(1,1,2)$ $(0,1,1)$

Rank $= 2$

1c) Find the values of p and q for $\lim_{x\to 0}\frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1

$$I = \underset{x\to 0}{Lt}\ \frac{x(1+p\cos x)-q\sin x}{x^3} \quad \left(\frac{0}{0}\right) \text{ L.H rule}$$

$$= \underset{x\to 0}{Lt}\ \frac{1+p\cos x - px\sin x - q\cos x}{3x^2} \quad \left(\frac{1+p-q}{0}\right)$$

To proceed further, use LH Rule so $1+p-q=0$ must satisfy

$$= \underset{x\to 0}{Lt}\ \frac{1+(p-q)\cos x - px\sin x}{3x^2} \quad \text{use LH Rule}$$

$$= \underset{x\to 0}{Lt}\ \frac{-(p-q)\sin x - p\sin x - px\cos x}{6x} \quad \left(\frac{0}{0}\right) \text{ Use LH}$$

$$= \underset{x\to 0}{Lt}\ \frac{-(p-q)\cos x - 2p\cos x + px\sin x}{6} = \frac{-3p+q}{6}$$

$$= 1 \text{ (Given)} \Rightarrow \left.\begin{array}{l} -3p+q=6 \text{ — ①} \\ 1+p-q=0 \to ② \end{array}\right] \text{Add } p=\frac{-5}{2} \quad q=-3/2$$

Solve $p=-\frac{5}{2}$ $q=-3/2$

Method-2 : If Qn is complicated & Time taking

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$p\cos x = p - \frac{x^2}{2!}p + \frac{x^4}{4!}p - \frac{x^6}{6!}p$$

$$x(1+p\cos x) = x + px - \frac{x^3}{2!}p + \frac{x^5}{4!}p - \frac{x^7}{6!}p$$

$$q\sin x = qx - q\frac{x^3}{3!} + \frac{qx^5}{5!} - \frac{qx^7}{7!}$$

$$x(1+p\cos x) - q\sin x = x(1+p-q) + x^3\left(\frac{-p}{2}+\frac{q}{6}\right) + x^5(\ ) + x^7(\ )$$

$$\frac{x(1+p\cos x)-q\sin x}{x^3} = \frac{1-p+q}{x^2} + \underbrace{\left(\frac{-p}{2}+\frac{q}{6}\right)}_{\text{equal 1 given}} + \underbrace{x^2(\ )}_{0} + x^4(\ ) \to 0$$

$x \to 0 \Rightarrow$ Value $\to 1$

$\llcorner$ Must be 0 $\Rightarrow \left.\begin{array}{l} 1-p+q=0 \\ \frac{-p}{2}+\frac{q}{6}=1 \end{array}\right]$ Solve $p=-5/2$ $q=-3/2$

1d) Examine the convergence of the integral $\int_0^1 \frac{\log x}{1+x} dx$

Check at $x=0$, $x=1$

$x=1$ is not point of discontinuity

$$\frac{\log 1}{1+1} \rightarrow \frac{\log 1}{2} \rightarrow 0 \quad \text{Finite value}$$

$x=0$ is point of discontinuity

$$f(x) = \frac{\log x}{1+x} \quad \text{Let } g(x) = \cancel{x} = \frac{1}{x^\alpha}$$

$\alpha > 0$
and $\alpha < 1$
$0 < \alpha < 1$

$$\underset{x \to 0^+}{Lt} \frac{f(x)}{g(x)} = \underset{x \to 0^+}{Lt} \frac{x^\alpha \log x}{1+x}$$

$$\underset{x \to 0}{Lt} \; x^\alpha \log x \rightarrow 0 \quad \text{for } 0 < \alpha < 1$$

By comparision test $f(x)$ and $g(x)$ converge together

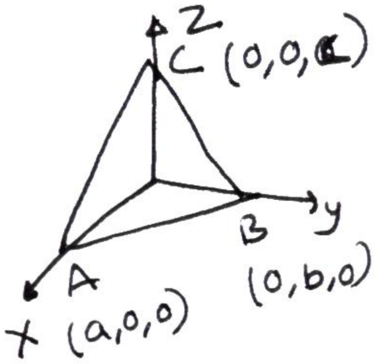
$$\int_0^1 g(x)\, dx = \int_0^1 \frac{1}{x^\alpha} dx \quad \text{Converge for } 0 < \alpha < 1$$

$$\& \; \underset{x \to 0}{Lt} \frac{f(x)}{g(x)} \rightarrow 0$$

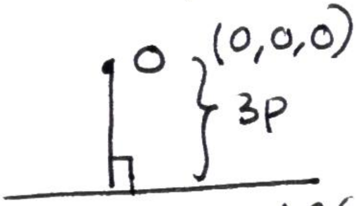
So $\int_0^1 f(x)$ Converge

1e) A variable plane which is at a constant distance $3p$ from the origin O cuts the axes in the points A, B, C respectively. Show that the locus of the centroid of the tetrahedron $OABC$ is

$$9\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\frac{16}{p^2}$$



Plane eqn ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



ABC plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$

$$3p=\left|\frac{0\cdot\frac{1}{a}+0\cdot\frac{1}{b}+0\cdot\frac{1}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right| \Rightarrow 9p^2=\frac{1}{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}$$

$$\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2=\frac{1}{9p^2}$$

Centroid $O(0,0,0)$ $A(a,0,0)$ $B(0,b,0)$ $C(0,0,c)$

is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$

Centroid locus : $\frac{a}{4}\to x$ $\frac{b}{4}\to y$ $\frac{c}{4}\to z$

$a\to 4x$ $b\to 4y$ $c\to 4z$

Put in $\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2=\frac{1}{9p^2}$

$$\frac{1}{4^2x^2}+\frac{1}{4^2y^2}+\frac{1}{4^2z^2}=\frac{1}{9p^2}$$

$$9\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\frac{16}{p^2}$$

2a) **If the matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis (1,0,0), (0,1,0), (0,0,1) is** $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ **then find the matrix of T relative to the basis $\{(1,1,1),(0,1,1),(0,0,1)\}$.**

From given info $T(e_1) = 1.e_1 + -1.e_2 + 0.e_3 = (1,-1,0)$

$T(e_2) = 1.e_1 + 2e_2 + 1.e_3 = (1,2,1)$

$T(e_3) = 2e_1 + 1.e_2 + 3e_3 = (2,1,3)$

$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$

↳ why? $(x,y,z) = xe_1 + ye_2 + ze_3$

$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$

$T(x,y,z) = x(1,-1;0) + y(1,2,1) + z(2,1,3)$

$= (x+y+2z, -x+2y+z, y+3z)$

$= l\alpha_1 + m\alpha_2 + n\alpha_3$

$\alpha_1 = (1,1,1)$
$\alpha_2 = (0,1,1)$
$\alpha_3 = (0,0,1)$

$= l(1,1,1) + m(0,1,1) + n(0,0,1)$

$= (l, l+m, l+m+n)$

$l = x+y+2z$ —①
$l+m = -x+2y+z$ —②
$l+m+n = y+3z$ —③

③ - ② $\Rightarrow n = y+3z+x-2y-z$
$= x-y+2z$

② - ① $\Rightarrow m = -x+2y+z-x-y-2z$
$= -2x+y-z$

$T(x,y,z) = (x+y+2z)\alpha_1 + (-2x+y-z)\alpha_2 + (x-y+2z)\alpha_3$

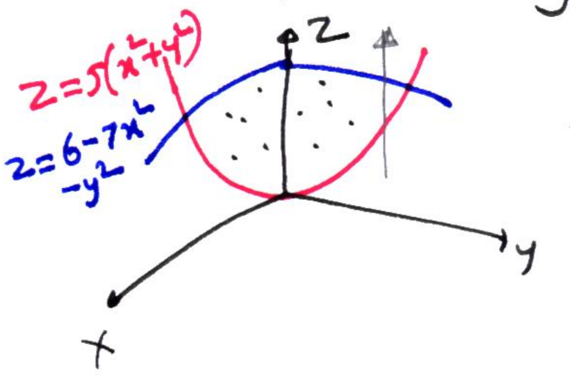
$T(\alpha_1) = T(1,1,1) = 4\alpha_1 - 2\alpha_2 + 2\alpha_3$
$T(\alpha_2) = T(0,1,1) = 3\alpha_1 + 0\alpha_2 + 1\alpha_3$
$T(\alpha_3) = T(0,0,1) = 2\alpha_1 - 1.\alpha_2 + 2\alpha_3$

$$T_\alpha = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

2b) Evaluate the triple integral which gives the volume of the solid enclosed between the two paraboloids $Z = 5(x^2 + y^2)$ and $Z = 6 - 7x^2 - y^2$.

Intersection $z = 5x^2+5y^2 = 6-7x^2-y^2$

$\Rightarrow 12x^2+6y^2=6 \Rightarrow \underline{2x^2+y^2=1}$



$z \rightarrow 5x^2+5y^2$ to $6-7x^2-y^2$

$y \rightarrow -\sqrt{1-2x^2}$ to $\sqrt{1-2x^2}$

$x \rightarrow -1/\sqrt{2}$ to $1/\sqrt{2}$

$dV = dz\,dy\,dz$

$$V = \int_{x=-1/\sqrt{2}}^{1/\sqrt{2}} \int_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \underbrace{\int_{z=5x^2+5y^2}^{6-7x^2-y^2} dz}\, dy\, dx$$

$$(6-7x^2-y^2)-(5x^2+5y^2) = 6-12x^2-6y^2$$

$$\left[\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6-12x^2-6y^2)\,dy\right]$$

Even

$$2\int_0^{\sqrt{1-2x^2}} (6-12x^2-6y^2)\,dy$$

$$= 2\left[6y-12x^2y-\frac{6y^3}{3}\right]_0^{\sqrt{1-2x^2}} = 12\left[\sqrt{1-2x^2} - 2x^2\sqrt{1-2x^2} - \frac{1}{3}(1-2x^2)^{3/2}\right]$$

d

$$V=12\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[(1-2x^2)\sqrt{1-2x^2} - \frac{1}{3}(1-2x^2)^{3/2}\right]dx$$

$1-\frac{1}{3}=\frac{2}{3}$

$$= 12 \times \frac{2}{3} \times 2 \times \int_0^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx$$

Even →

$$= 16\int_0^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx$$

$x = \frac{1}{\sqrt{2}}\sin\theta$ $\quad 1-x^2=\cos^2\theta$

$\downarrow$ $\quad dx = \frac{1}{\sqrt{2}}\cos\theta\,d\theta$

$x=\frac{1}{\sqrt{2}} \rightarrow \theta=\pi/2$

$x=0 \rightarrow \theta=0$

$$\int_0^{\pi/2}\cos^4\theta = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=16\int_0^{\pi/2} \cos^3\theta \cdot \frac{1}{\sqrt{2}}\cos\theta\,d\theta$$

$$=\frac{16}{\sqrt{2}}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{3\pi}{\sqrt{2}}$$

2c(i) Show that the equation $2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$ represents an elliptic paraboloid. Also find its principal axis and principal planes.

$$2x^2+3y^2-8x+6y-12z+11=0$$

$$\hookrightarrow (2x^2-8x)+(3y^2+6y)-12z+11=0$$

$$2(x^2-4x)+3(y^2+2y)-12z+11=0$$

$$2((x-2)^2-4)+3((y+1)^2-1)-12z+11=0$$

$$2(x-2)^2+3(y+1)^2-12z=0$$

$$\begin{array}{r} -8 \\ -3 \\ +11 \\ \hline 0 \end{array}$$

$$12z=2(x-2)^2+3(y+1)^2$$

$$12Z=2X^2+3Y^2$$

$X=x-2$

$Y=y+1$

$\hookrightarrow$ Elliptical paraboloid format $Z=z$

→ Principal axis is $z=0$ ie z-axis

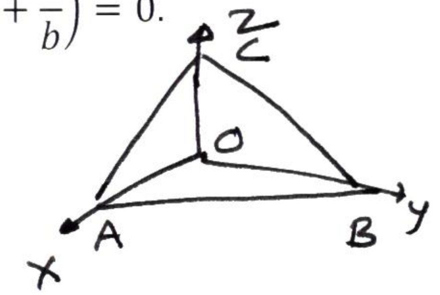
→ Principal plane $X=0$ $Y=0$

$\Rightarrow x-2=0,\ y+1=0$

$x=2,\ y=-1$

2c(ii) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0.$$



Plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

A(a,0,0) B(0,b,0)
C(0,0,c) O(0,0,0)

Sphere thru O, A, B, C is $x^2+y^2+z^2-ax-by-cz=0$

Plane meets sphere in circle ABC

Circle is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$
+
$x^2+y^2+z^2-ax-by-cz=0$

Cone thru origin ⇒ Homogenous

↳ Make it homogeneous

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$x^2+y^2+z^2-\left(x^2+y^2+z^2+yz\left(\frac{b}{c}+\frac{c}{b}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)\right.$$

$$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0$$

3a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(i) Verify the Cayley-Hamilton theorem for the matrix A.

(ii) Show that $A^n = A^{n-2} + A^2 - I$ for $n \geqslant 3$, where I is the identity matrix of order 3. Hence, find A^{40}.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-1) = 0$$

$$(\lambda^2-1)(\lambda-1) = \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Cayley Thm gives $A^3 - A^2 - A + 1 = 0$

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A^3 = A^2 . A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$A^3 - A^2 - A + 1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Satisfied Cayley Hamilton Thm

(ii) Use Induction for $A^n = A^{n-2} + A^2 - 1$

$n = 3 \Rightarrow A^3 = A + A^2 - 1 \Rightarrow A^3 - A^2 - A + 1 = 0$

Let true for $n = k$ $\quad A^k = A^{k-2} + A^2 - I$

For $n = k+1$ $\quad A^{k+1} = A^k . A = (A^{k-2} + A^2 - I) A$

$$= A^{k-1} + \underline{A^3 - A}$$

$$A^3 - A = A^2 - I$$

$$= A^{k-1} + A^2 - 1$$

Proved for $k+1$ So True for all $n \geqslant 3$

(iii) $A^{40} = A^{38} + A^2 - I$

$= A^{36} + 2(A^2 - I)$

$= A^{34} + 3(A^2 - I)$

$= A^2 + P(A^2 - I)$

$= A^2 + 19(A^2 - I)$

$= 20A^2 - 19I \quad = \begin{pmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 20 & 0 & 1 \end{pmatrix}$

Pattern is ~~$A^n = A^2 (n-1)(A^2-1)$~~

$\frac{38-36}{2} + 1 = 2$

$\frac{38-34}{2} + 1 = 3$

$\frac{38-2}{2} + 1 = 19$

$P = 19$

3b) Justify whether (0,0) is an extreme point for the function
$f(x, y) = 2x^4 - 3x^2y + y^2$.

$$f(x,y) = 2x^4 - 3x^2y + y^2$$

$$f_x = 8x^3 - 6xy \qquad f_{xx} = 24x^2 - 6y \qquad f_{xy} = -6x$$

$$f_y = -3x^2 + 2y \qquad f_{yy} = -6x$$

$$f_x(0,0) = 0 = f_y(0,0) = f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0)$$

$$f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0$$

$$I = f(x,y) - f(0,0) = 2x^4 - 3x^2y + y^2$$

$$= (x^2 - y)(2x^2 - y)$$

$I > 0$ if $y < 0$ or $\left.\begin{matrix} x^2 > y \\ \& \\ 2x^2 > y \end{matrix}\right] \Rightarrow x^2 > y$

$\downarrow$ (+) × (+)

Min value

$\downarrow$

Also $x^2 > y > 0$

positive vale

$I < 0 \Rightarrow$ If $x^2 - y > 0$ then $2x^2 - y = x^2 + x^2 - y$

Not possible for (+) × (−) Always > 0

Let $x^2 - y < 0$ & $2x^2 - y > 0$

$$x^2 < y \qquad 2x^2 > y \Rightarrow x^2 > \frac{y}{2}$$

$$y > x^2 \qquad x^2 > \frac{y}{2}$$

$$y > x^2 > \frac{y}{2} > 0$$

I does not hae same sign near the orign.
Hence f has neither max nor min at origin

3c) Find the equation of the sphere through the circle $x^2+y^2+z^2-4x-6y+2z-16=0; 3x+y+3z-4=0$ in the following two cases.

(i) the point $(1,0,-3)$ lies on the sphere.

(ii) the given circle is a great circle of the sphere.

Eqn of sphere is $S+\lambda L=0$

$$(x^2+y^2+z^2-4x-6y+\overset{2}{\blacksquare}z-16)+\lambda(3x+y+3z-4)=0$$

a) $(1,0,-3)$ lie on sphere $\Rightarrow$ put value $1+0+9-4-0-6-16+\lambda(3-9-4)=0$

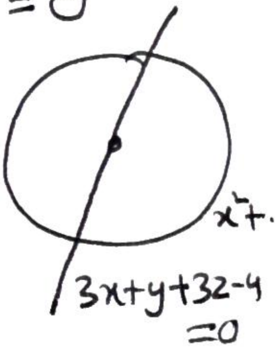
$$-16+\lambda(-10)=0 \Rightarrow \lambda=\frac{-16}{10}=\frac{-8}{5}$$

$$x^2+y^2+z^2+x\left(-4-\frac{24}{5}\right)+y\left(-6-\frac{8}{5}\right)+z\left(2-\frac{24}{5}\right)-16+\frac{32}{5}=0$$

$$x^2+y^2+z^2-\frac{44}{5}x-\frac{38}{5}y-\frac{14}{5}z-\frac{48}{5}=0$$ Soln

16
5 3
80
32
48

(b) $x^2+y^2+z^2+x(3\lambda-4)+y(\lambda-6)+z(3\lambda+2)-16-4\lambda=0$



Centre is $\left(\frac{-(3\lambda-4)}{2}, \frac{-(\lambda-6)}{2}, \frac{-(3\lambda+2)}{2}\right)$

Centre lie on plane

$$3\left(\frac{-(3\lambda-4)}{2}\right)+\frac{-(\lambda-6)}{2}+3\left(\frac{-(3\lambda+2)}{2}\right)=4$$

$$3(4-3\lambda)+6-\lambda-9\lambda-6=8$$

$$12-9\lambda+6-\lambda-9\lambda-6=8$$

$$-19\lambda=-4 \Rightarrow \lambda=\frac{4}{19}$$ ✓

$$x^2+y^2+z^2-\frac{64}{19}x-\frac{110}{19}y+\frac{50z}{12}-\frac{320}{19}=0$$

$\frac{12}{19}-4=\frac{-64}{19}$

$\frac{4}{19}-6=\frac{-110}{19}$

$\frac{12}{19}+2=\frac{50}{19}$

$16+\frac{16}{19}=\frac{320}{19}$

4a) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

by reducing it to row-reduced echelon form.

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{pmatrix} \xrightarrow{R_4 \to 5R_4}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -5 & 10 & -5 \end{pmatrix} \xrightarrow{R_4 \to R_4 + R_2} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & 0 & 9 & -9 \end{pmatrix} \begin{matrix} R_4 \to \frac{R_4}{9} \\ R_2 \to R_2 \times 3 \\ R_3 \to R_3 \times 5 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 15 & -3 & -12 \\ 0 & -15 & 25 & -10 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 15 & -3 & -12 \\ 0 & 0 & 22 & -22 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} R_3 \to \frac{R_3}{22} \\ R_2 \to \frac{R_2}{3} \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \underset{\substack{R_2 \to R_2 + R_3 \\ R_1 \to R_1 + R_3}}{\xrightarrow{R_4 \to R_4 - R_3}} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_2 \to \frac{R_2}{5}$$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_1 \to R_1 - 2R_2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Completely

$$\exists \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \underline{\text{Rank} = 3}$$

Check out Curves - PDF shared on SuccessClap Telegram Channel

Same qn present

4b) Trace the curve $y^2(x^2 - 1) = 2x - 1$.

The curve is symmetrical about the x-axis.

It does not pass through the origin.

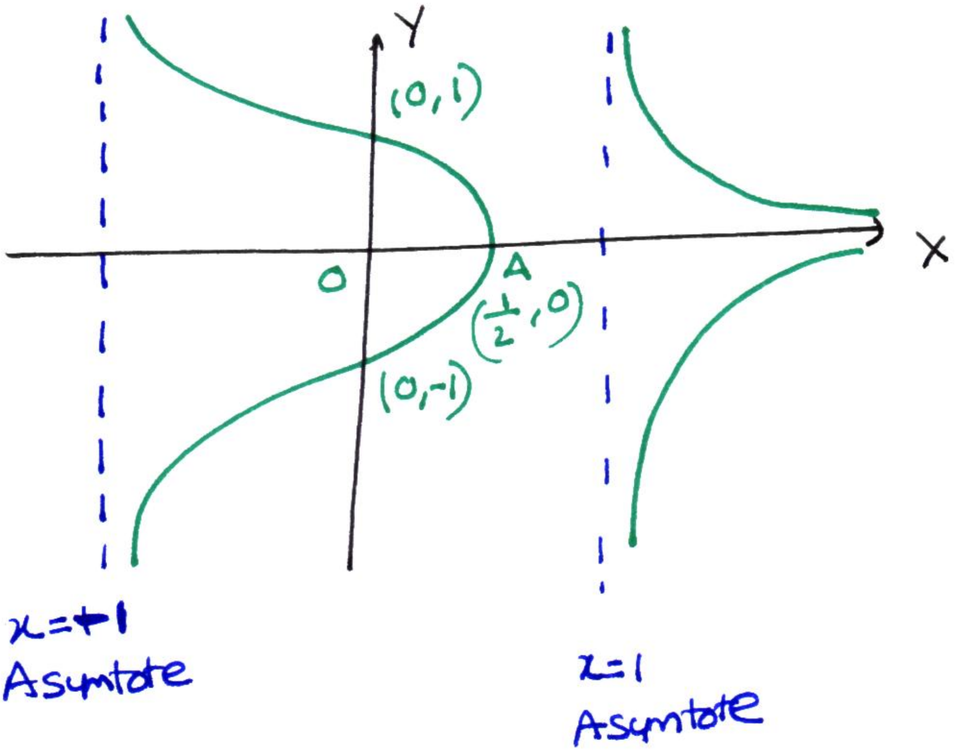
The curve meets the x-axis at the point $A(1/2,0)$ and the y-axis at the points $B(0,1)$ and $C(0,-1)$.

The asymptotes parallel to the y-axis are $x^2 - 1 = 0$ or $x = \pm 1$ and that parallel to the x-axis is $y = 0$ (i.e., x-axis).

It may be seen that $y^2 < 0$ in the region $1 < x < 1$ [take, for example, $x = 3$ in $y^2 = (2x - 1/(x^2 - 1)]$.

Thus the curve does not lie in the region $1 < x < 1$.

Hence the graph of the curve is as shown in.



4c) Prove that the locus of a line which meets the lines $y = mx, z = c; y = -mx, z = -c$ and the circle $x^2 + y^2 = a^2, z = 0$ is $c^2m^2(cy - mzx)^2 + c^2(yz - cmx)^2 = a^2m^2(z^2 - c^2)^2$

The given lines are $y-mx=0,\ z-c=0$ — ①
$y+mx=0\ ,\ z+c=0$ — ②
& circle $x^2+y^2=a^2\ ;\ z=0$ — ③

Any line intersecting (1) & ② is
$y-mx-k_1(z-c)=0,\quad y+mx-k_2(z+c)=0$ — ④
If it meets the circle (3), we have to eliminate x,y,z from (3) & ④. Putting $z=0 \Rightarrow \left.\begin{matrix} y-mx+k_1c=0 \\ y+mx-k_2c=0 \end{matrix}\right]$

$$\frac{y}{mck_2-mck_1} = \frac{x}{ck_1+ck_2} = \frac{1}{m+m} \Rightarrow x = \frac{c(k_1+k_2)}{2m}$$
$$y = \frac{c(k_2-k_1)}{2}$$

Put x,y in (3) $\Rightarrow \dfrac{c^2(k_1+k_2)^2}{4m^2} + \dfrac{c^2(k_1-k_2)^2}{4} = a^2$

$\Rightarrow c^2(k_1+k_2)^2 + c^2m^2(k_2-k_1)^2 = 4m^2a^2$ — ⑤

To find locus we have to eliminate k_1, k_2 frm ④ & ⑤

Substitute $k_1 = \dfrac{y-mc}{z-c}\quad k_2 = \dfrac{y+mx}{z+c}$

we get $c^2\left[\dfrac{y-mx}{z-c} + \dfrac{y+mx}{z+c}\right]^2 + c^2m^2\left[\dfrac{y+mx}{z+c} - \dfrac{y-mx}{z-c}\right]^2 = 4a^2m^2$

or $c^2\left[\dfrac{yz-mxz+cy-cmx+yz+mxz-cy-cmx}{z^2-c^2}\right]^2$
$+c^2m^2\left[\dfrac{yz+mxz-cy-cmx-(yz-mxz+cy-cmx)}{z^2-c^2}\right]^2 = 4a^2m^2$

or $c^2(2yz-2cmx)^2 + c^2m^2(2mxz-2cy)^2 = 4a^2m^4(z^2-c^2)^2$

$4c^2(yz-cmx)^2 + 4c^2m^2(mxz-cy)^2 = 4a^2m^2(z^2-c^2)^2$

$c^2m^2(cy-mxz)^2 + c^2(yz-cmx)^2 = a^2m^2(z^2-c^2)^2$

5a) Obtain the solution of the initial-value problem $\frac{dy}{dx} - 2xy = 2,\ y(0) = 1$ in the form $y = e^{x^2}[1 + \sqrt{\pi}\operatorname{erf}(x)]$

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

Such functions are mentioned in Laplace Transform
erf, $\sin(t) = \int_0^t \frac{\sin u}{u} du$, cost, Laguerre polynomial
Dirac, exponential

$\frac{dy}{dx} - 2xy = 2$ $\quad y(0) = 1$

$P = -2x \quad Q = 2 \quad IF = e^{\int P dx} = e^{\int -2x dx} = e^{-x^2}$

Soln $y(IF) = \int Q(IF) + C$

$$ye^{-x^2} = \int 2e^{-x^2} dx + C$$

$x=0$ $y=1$ $\Rightarrow$ $1 \times 1 = 0 + C$ $\Rightarrow$ $C = 1$

$$y = e^{x^2}\left[1 + \int 2e^{-x^2} dx\right]$$

$$= e^{x^2}\left[1 + \sqrt{\pi} \int \frac{2}{\sqrt{\pi}} e^{-x^2} dx\right]$$

$$= e^{x^2}\left[1 + \sqrt{\pi} \operatorname{erf}(x)\right]$$

5b) Given that $L\{f(t);p\} = F(p)$.
Show that $\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty F(x)dx$. Hence evaluate the integral $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$.

By Division theorem

If $L\{f(t)\} = F(P)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)dp$

Given $L\{f(t)\} = F(P)$

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty e^{-pt}\frac{f(t)}{t}dt \quad \text{(laplace Defn)}$$

$$= \int_p^\infty F(p)dp \quad \text{(Division Thm)}$$

$$\int_0^\infty e^{-pt}\frac{f(t)}{t}dt = \int_p^\infty F(p)dp = \int_p^\infty F(x)dx$$

$$= \int_p^0 F(x)dx + \int_0^\infty F(x)dx$$

$$= -\int_0^p F(x)dx + \int_0^\infty F(x)dx$$

Take Limit on both sides as $p \to 0^+$ $\quad e^{-0t} \to 1$ $\quad \int_0^p F(x)dx \to 0$

$$\int_0^\infty \frac{f(t)dt}{t} = \int_0^\infty F(x)dx \quad \underline{\text{Proved}}$$

$f(t) = e^{-t} - e^{-3t}$ $\quad L\{f(t)\} = L(e^{-t}) - L(e^{-3t})$

$$= \frac{1}{p+1} - \frac{1}{p+3} = F(p)$$

$$\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty F(P)dp = \int_0^\infty F(x)dx$$

$$= \int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx = \cancel{\int_0^\infty} \log(x+1) - \log(x+3)\Big|_0^\infty$$

$$= \log\left(\frac{x+1}{x+3}\right)\Big|_0^\infty = \log\frac{1+\frac{1}{x}}{1+\frac{3}{x}}\Bigg|_{x\to\infty} - \log\frac{1}{3}$$

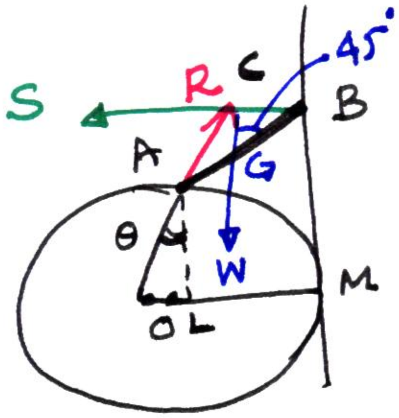
$$= \underline{\log 3}$$

5c) A cylinder of radius 'a' touches a vertical wall along a generating line. Axis of the cylinder is fixed horizontally. A uniform flat beam of length 'l' and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of $45°$ with the vertical. If frictional forces are neglected, then show that

$$\frac{a}{l} = \frac{\sqrt{5}+5}{4\sqrt{2}}$$

Also, find the reactions of the cylinder and wall.

Beam equilibrium
AB → Beam
under 3 forces
a) Weight vertical down thru G midpt
b) Normal reactn R at A
c) Normal reactn S at B



All Must meet in a point (C)

$\angle CGB = 45°$ given $\quad \angle BCG = 90°$

Let $\angle ACG = \theta \quad AG = GB = \frac{l}{2} \quad AB = l$

(m-n) Theorem ABC $\left(\frac{l}{2}+\frac{l}{2}\right)\cot 45 = \frac{l}{2}\cot\theta - \frac{l}{2}\cos 90°$

$\cot\theta = 2 \Rightarrow \cos\theta = \frac{2}{\sqrt{5}} \quad \sin\theta = \frac{1}{\sqrt{5}}$

Lami Thm at C $\quad \frac{R}{\sin 90} = \frac{S}{\sin(180-\theta)} = \frac{W}{\sin(90+\theta)}$

$$\left.\begin{aligned} R &= \frac{W}{\cos\theta} = \frac{\sqrt{5}}{2}W \\ S &= \frac{W\sin\theta}{\cos\theta} = \frac{1}{2}W \end{aligned}\right]$$

$OM = OL + LM = OL + AN = OA\sin\theta + AB\sin 45°$

$a = a\frac{1}{\sqrt{5}} + l\cdot\frac{1}{\sqrt{2}} = \frac{a}{\sqrt{5}} + \frac{\sqrt{2}\,l}{2}$

$a\left(1-\frac{1}{\sqrt{5}}\right) = \frac{\sqrt{2}\,l}{2} \Rightarrow a\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right) = \frac{\sqrt{2}\,l}{2}$

$\frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{\sqrt{5}+1 \times \sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$

$\frac{a}{l} = \frac{\sqrt{2}}{2}\frac{\sqrt{5}}{\sqrt{5}-1} = \frac{\sqrt{2}}{2}\sqrt{5}\frac{(\sqrt{5}+1)}{4} = \frac{\sqrt{5}+5}{4\sqrt{2}}$ ✓

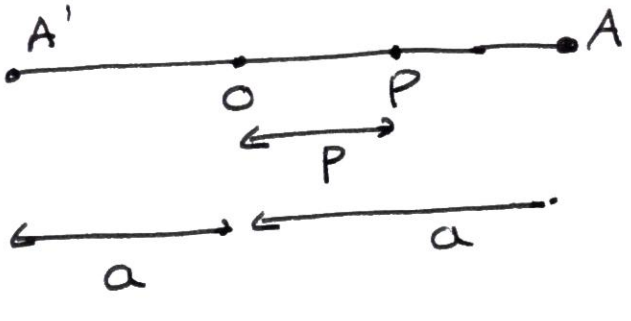
Wall reactn $= S = \frac{W}{2}$

5d) A particle is moving under Simple Harmonic Motion of period T about a centre O. It passes through the point P with velocity v along the direction OP and $OP = p$. Find the time that elapses before the particle returns to the point P. What will be the value of p when the elapsed time is $\frac{T}{2}$?

Centre O is origin

Amplitude a

Lets measure time from A



$\hookrightarrow x = a\cos\omega t$, $T = \frac{2\pi}{\omega}$

$v^2 = \omega^2(a^2 - x^2) = \omega^2(a^2 - a^2\cos^2\omega t)$

$= \omega^2 a^2 \sin^2\omega t$ $\qquad 1 - \cos^2\theta = \sin^2\theta$

$v = \omega a \sin\omega t$

At P: → $x = p$ $\quad v = v$ (Given) $\Rightarrow p = a\cos\omega t_1$ -①

$v = \omega a \sin\omega t_1$ -②

↳ Time taken from A to P be t_1

Time taken from A to P = time from P to A

To reach point P time $= t_1 + t_1 = 2t_1$

$t = 2t_1$

$= 2\left(\frac{T}{2\pi}\tan^{-1}\frac{vT}{2\pi p}\right)$

$= \frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi p}$

$\frac{①}{②} \Rightarrow \frac{p}{v} = \frac{1}{\omega}\cot\omega t_1$

$\frac{v}{p\omega} = \tan\omega t_1$

$\frac{vT}{2\pi p} = \tan\omega t_1$ $\qquad T = \frac{2\pi}{\omega}$

$= \tan\frac{2\pi}{T}t_1$ $\qquad \omega = \frac{2\pi}{T}$

$t_1 = \frac{T}{2\pi}\tan^{-1}\frac{vT}{2\pi p}$

$\tan\frac{\pi}{2} = \infty$

Part 2: $t = \frac{T}{2}$

$\frac{T}{2} = \frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi p}$

$\tan\frac{\pi}{2} = \frac{vT}{2\pi p} = \infty \Rightarrow p = 0$

5e) If

$$\begin{aligned}\vec{a} &= \sin\,\theta\hat{\imath} + \cos\,\theta\hat{\jmath} + \theta\hat{k}\\ \vec{b} &= \cos\,\theta\hat{\imath} - \sin\,\theta\hat{\jmath} - 3\hat{k}\\ \vec{c} &= 2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}\end{aligned}$$

then find the values of the derivative of the vector function $\vec{a} \times (\vec{b} \times \vec{c})$ w.r.t. θ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

$a = (\sin\theta, \cos\theta, \theta) \quad b = (\cos\theta, -\sin\theta, -3) \quad c = (2,3,-3)$

$T = a \times (b \times c) = b(a.c) - c(a.b) \qquad l = a.c$

$= bl - cn \qquad n = a.b$

$\dot{T} = \dot{b}l + b\dot{l} - \dot{c}n - c\dot{n} \qquad \theta = \pi/2 \qquad \theta = \pi$

$\dot{b} = (-\sin\theta, -\cos\theta, 0) \qquad (-1,0,0) \qquad (0,1,0)$

$l = a.c = 2\sin\theta + 3\cos\theta - 3\theta \qquad (2 - \frac{3\pi}{2}) \qquad (-3-3\pi)$

$\dot{l} = (2\cos\theta - 3\sin\theta - 3) \qquad ((-6) \qquad (-5)$

$b = (\cos\theta, -\sin\theta, -3) \qquad (0,-1,-3) \qquad (-1,0,-3)$

$c = (2,3,-3) \qquad \dot{c} = (0,0,0)$

$n = a.b = \sin\theta\cos\theta - \sin\theta\cos\theta - 3\theta = -3\theta$

$\dot{n} = -3$

$\dot{T}(\theta = \pi/2) = (-1,0,0)(2 - \frac{3\pi}{2}) - 6(0,-1,-3) - (2,3,-3)(-3)$

$= (\frac{3\pi}{2} - 2 + 6, 6+9, +18-9)$

$= (\frac{3\pi}{2} + 4, 15, 9)$

$\dot{T}(\theta = \pi) = (0,1,0)(-3-3\pi) + (-1,0,-3)(-5) - (2,3,-3)(-3)$

$= 5+6, -3-3\pi+9, 15-9)$

$= (11, 6-3\pi, 6)$

Plz verify calculation again

6a) Solve the differential equation :

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos\ x.$$

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$$

$$(D-1)(D^2-2D+2) = 0 \Rightarrow D = 1, 1 \pm i$$

$$CF = c_1e^x + e^x(c_2\cos x + c_3\sin x)$$

$$PI_1 = \frac{1}{(D-1)(D^2-2D+2)}e^x = \frac{e^x}{(D-1)(1-2+2)} = \frac{e^x \cdot 1}{D-1}$$

$$= e^x\frac{1}{(D+1)-1}\cdot 1 = e^x\frac{1}{D}\cdot 1 = xe^x$$

$$PI_2 = \frac{1}{D^3-3D^2+4D-2}\cos x \qquad D^2 \to -1$$

$$D^3 \to D^2.D = -D$$

$$= \frac{1}{-D+3+4D-2}\cos x \qquad 3D^2 \to -3$$

$$= \frac{1}{1+3D}\cos x = \frac{1-3D}{(1+3D)(1-3D)}\cos x$$

$$= \frac{(1-3D)\cos x}{1-9D^2} \qquad D^2 \to -1$$

$$D\cos x = -\sin x$$

$$= \frac{1}{10}(1-3D)\cos x$$

$$= \frac{\cos x + 3\sin x}{10}$$

$$y = CF + PI_1 + PI_2$$

6b) When a particle is projected from a point O_1 on the sea level with a velocity v and angle of projection θ with the horizon in a vertical plane, its horizontal range is R_1. If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity v and same angle θ with the horizon, its horizontal range is R_2. Prove that $R_2 > R_1$ and $(R_2 - R_1): R_1$ is equal to $\frac{1}{2}\left\{\sqrt{\left(1+\frac{2gh}{v^2\sin^2\theta}\right)}-1\right\}:1$

(SuccessClap FULL LENGTH TEST 01 2023 Qn 5C)

Let O' be a point of the sea level. Let R_1 be its range when gun is fired from it. Then $OA = R_1 = (2u^2\sin\alpha\cos\alpha)/g$

Let O be a point at a height h above the sea level. Let R_2 be range $(= OB)$ on the sea level when the shot is fired from O. Referred to the horizontal and upward vertical lines OX and OY as coordinate axes, the equation of the path of this shot is $y = x\tan\alpha - (gx^2)/2u^2\cos^2\alpha$

Since $B(R_2, -h)$ lies on (2), we have

$$-h = R_2\tan\alpha - (gR_2^2)/2u^2\cos^2\alpha$$
$$gR_2^2 - (2u^2\sin\alpha\cos\alpha)\cdot R_2 - 2u^2h\cos^2\alpha = 0$$
$$\therefore\ R_2 = \left\{2u^2\sin\alpha\cos\alpha \pm (4u^4\sin^2\alpha\cos^2\alpha + 8u^2gh\cos^2\alpha)^{1/2}\right\}/2g$$
$$= \left\{2u^2\sin\alpha\cos\alpha + 2u^2\sin\alpha\cos\alpha(1 + 2gh/u^2\sin^2\alpha)^{1/2}\right\}/2g$$

$= \left\{R_1g + R_1g(1 + 2gh/u^2\sin^2\alpha)^{1/2}\right\}/2g,)$ [Since R_2 is + ve, we reject -ve sign]

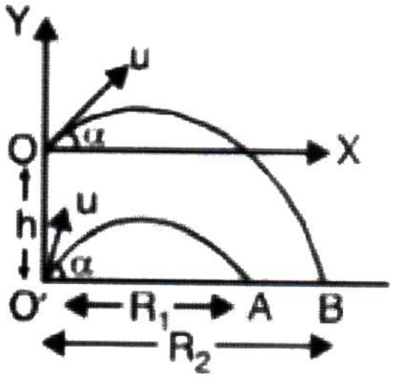
Thus, $R_2 = (R_1/2)\left\{1 + (1 + 2gh/u^2\sin^2\alpha)^{1/2}\right\}$

On subtracting R_1 from both sides, we get

$$\therefore\ R_2 - R_1 = (R_1/2)\left\{1 + (1 + 2gh/u^2\sin^2\alpha)^{1/2}\right\} - R_1$$

Hence the fraction by which R_1 increases is

$$(R_2 - R_1)/R_1 = (1/2)\left\{(1 + 2gh/u^2\sin^2\alpha)^{1/2} - 1\right\}.$$



6c) Evaluate the integral $\iint_S \left(3y^2z^2\hat{\imath} + 4z^2x^2\hat{\jmath} + z^2y^2\hat{k}\right)\cdot\hat{n}dS$, where S is the upper part of the surface $4x^2+4y^2+4z^2=1$ above the plane $z=0$ and bounded by the xy-plane. Hence, verify Gauss-Divergence theorem.

(Upper Surface Only)

(I) $F = 3y^2z^2\hat{i} + 4z^2x^2\hat{j} + z^2y^2\hat{k}$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{8(x\hat{i}+y\hat{j}+z\hat{k})}{8\sqrt{3}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}}$$

$$\nabla\varphi = \nabla(4x^2+4y^2+4z^2-1)$$
$$= 8(x\hat{i}+y\hat{j}+z\hat{k})$$
$$|\nabla\varphi| = 8\sqrt{3}$$

$$\hat{n}\cdot\hat{k} = \frac{z}{\sqrt{3}} \qquad dS = \frac{dx\,dy}{|n.k|} = \frac{dx\,dy}{\frac{z}{\sqrt{3}}}$$

$$F.n = \frac{3xy^2z^2 + 4x^2yz^2 + z^3y^2}{\sqrt{3}}$$

$$(F.n)\frac{dx\,dy}{(n.k)} = \left(3xy^2z + 4x^2yz + z^2y^2\right)dx\,dy$$

$$\int_{S'}(F.n)\frac{dx\,dy}{|n.k|} = \int_{S'}\left[(3xy^2+4x^2y)z + (z^2y^2)\right]dx\,dy$$

$$(3xy^2+4x^2y)z = \left(3r^3\cos\theta\sin^2\theta + 4r^3\cos^2\theta\sin\theta\right) \times \sqrt{\tfrac{1}{4}-r^2}$$

$$= r^3\sqrt{\tfrac{1}{4}-r^2}\left(3\cos\theta\sin^2\theta + 4\cos^2\theta\sin\theta\right)$$

$$\int_{S'}(3x^2y+4x^2y)z\,dx\,dy = \int[\quad]r\,dr\,d\theta$$

$$= \int_{r=0}^{r=1/2}\left(r^4\sqrt{\tfrac{1}{4}-r^2}\right)\int_{\theta=0}^{2\pi}\left(3\cos\theta\sin^2\theta + 4\cos^2\theta\sin\theta\right)d\theta$$

$$\downarrow$$

$$3\times\frac{\sin^3\theta}{2}\Big|_0^{2\pi} \qquad -\frac{4\cos^3\theta}{3}\Big|_0^{2\pi}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$S' = 4x^2+4y^2=1$

$x^2+y^2=\left(\frac{1}{2}\right)^2$

$z = \sqrt{\frac{1}{4}-(x^2+y^2)}$

Polarchge

$z = \sqrt{\left(\frac{1}{2}\right)^2 - r^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$r \to 0$ to $\frac{1}{2}$

$\theta \to 0$ to 2π

$dx\,dy = r\,dr\,d\theta$

Remaining $\int\limits_S (z^2y^2)\,dx\,dy$

$z=\sqrt{\frac{1}{4}-r^2}$

$y = \sin\theta . r$

$dxdy = rdrd\theta$

$$= \int_{r=0}^{1/2}\int_{\theta=0}^{2\pi}\left(\frac{1}{4}-r^2\right) r^2\sin^2\theta . rdrd\theta$$

$$= \int_0^{1/2}\left(\frac{r^3}{4}-r^5\right)dr \int_{\theta=0}^{2\pi}\sin^2\theta\,d\theta$$

$$\int_0^{2\pi}\sin^2\theta = 4\times\int_0^{\pi/2}\sin^2\theta$$

$$=4\times\frac{1}{2}\times\frac{\pi}{2}$$

$$\underline{=\pi}$$

$$\frac{r^4}{16}-\frac{r^6}{6}\Big|_0^{1/2}$$

$$\frac{1}{2^4\times 16}-\frac{1}{2^6\times 6} \qquad 16=4\times 4$$

$$\frac{1}{2^6.4}-\frac{1}{2^6.6} = \frac{1}{2^6}\left(\frac{6-4}{4.6}\right) = \frac{1}{2^6}.\frac{2}{4.6} = \frac{1}{64\times 12}$$

$$\underline{=\frac{1}{768}}$$

$$\int\limits_S (F.n)\,dS = \left(\frac{1}{768}\right)\times\pi = \frac{\pi}{768}$$

upper Surface

Part II : Gause Diverge $\int\limits_S (F.n)dS$ Total Surface $= \int(\nabla.F)dV$

Divergence for total Volume

$\nabla.F = 2zy^2$

Spherical coordinate

$(\nabla.F)dV = 2zy^2\,dV$

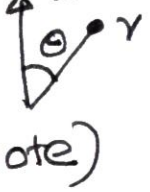
$x = r\sin\theta\cos\varphi$

$= 2(r\cos\theta)(r^2\sin^2\theta\sin^2\varphi)$

$y = r\sin\theta\sin\varphi$

$r^2\sin\theta\,dr\,d\theta\,d\varphi$

$z = r\cos\theta$



$= 2r^5(\cos\theta\sin^3\theta)(\sin^2\varphi)\,dr\,d\theta\,d\varphi$

$r \rightarrow 0$ to $1/2$

$\theta \rightarrow 0$ to $\pi/2$ (Note)

$\varphi \rightarrow 0$ to 2π

$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$$\int_V (\nabla\cdot F)\,dv = \int_{r=0}^{r=1/2} (2r^5)\,dr \int_{\theta=0}^{\pi/2} \cos\theta\sin^3\theta \int_{\varphi=0}^{2\pi} \sin^2\varphi$$

$$\frac{2r^6}{6}\Big|_0^{1/2} \qquad \frac{\sin^4\theta}{4}\Big|_0^{\pi/2} \qquad 4\times\int_0^{\pi/2} \sin^2\varphi$$

$$4\times\frac{1}{2}\times\frac{\pi}{2}$$

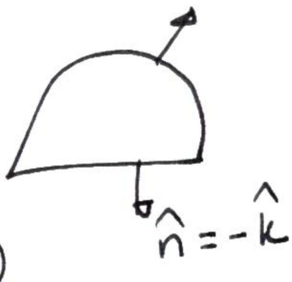
$$\frac{1}{64\times 3} \qquad \left(\frac{1}{4}\right) \qquad \left(\underline{\pi}\right)$$

$$= \left(\frac{1}{64\times 3}\right)\left(\frac{1}{4}\right)\times\pi = \frac{\pi}{768}$$

Prove Gauss Diverge

$$\int_V (\nabla\cdot F)\,dV = \int_{\text{upper Surface}} (F.n)\,ds + \int_{\text{Lower Surface}} F.n\,ds$$

we got $\int_{\text{upper}} (F.n)\,ds = \frac{\pi}{768}$



Lower Surface :

~~F.k)~~ $F.n = F.(-\hat{k}) = (-1)(F.k)$

$= -z^2y^2$

on this surface $z=0 \Rightarrow F.n=0$

$$\Rightarrow \int_{\text{Lower}} (F.n)\,dS = 0$$

$$\int_V (\nabla\cdot F)\,dV = \frac{\pi}{768} \qquad \int_{\substack{S\\ \text{upper}}} F.n\,ds = \frac{\pi}{768} \qquad \int_{\substack{S\\ \text{lower}}} F.n\,ds = 0$$

$$\int_V (\nabla\cdot F)\,dV = \int_{\substack{S\\ \text{total}}} (F.n)\,ds$$

7a(i) Find the solution of differential equation : $\frac{dy}{dx} = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$

Re arrage $(2xy^3+2)dx+(3x^2y^2+8e^{4y})dy=0$

$$M\,dx + N\,dy = 0$$

$\frac{\partial M}{\partial y} = 6xy^2$ $\quad \frac{\partial N}{\partial x} = 6xy^2$ $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$

$$\int\limits_{y=\text{cast}} (2xy^3+2)\,dx + \int\limits_{\text{remove } x} \left(\cancel{3x^2y^2}+8e^{4y}\right)dy$$

$$x^2y^3+2x+2e^{4y}+C$$

7a(ii) Reduce the equation $x^2p^2 + y(2x+y)p + y^2 = 0$ to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the equation and show that $y + 4x = 0$ is a singular solution of the differential equation

$y = u$

$dy = du$

$xy = V$

$dv = xdy + ydx$

$$\frac{dv}{du} = \frac{xdy + ydx}{dy} = x + y\frac{dx}{dy} \Rightarrow P = x + \frac{y}{p}$$

$P = \frac{dv}{du}$

$p = \frac{dy}{dx}$

$$\downarrow$$

$$x^2\left(\frac{y}{P-x}\right)^2 + \frac{(2x+y)\,y^2}{P-x} + y^2 = 0 \longleftarrow p = \frac{y}{P-x}$$

$$x^2 + (P-x)(2x+y) + (P-x)^2 = 0$$

$$x^2 + 2xP - 2x^2 + Py - xy + P^2 + x^2 - 2Px = 0$$

$$Py - xy + P^2 = 0$$

$xy = V$

$y = u$

$$\downarrow$$

$$V = uP + P^2 \longrightarrow \text{Clairaut Form} \xrightarrow{\text{Soln}} V = cu + c^2$$

$$xy = yc + c^2$$

Soln $\underline{xy = yc + c^2}$

$xy = yc + c^2$

Eqn can be written as $c^2 + yc - xy = 0$

c → Quadratic

c − Discriminat

$(b^2 - 4ac)$

$$y^2 - 4.1.(-xy) = 0$$

$$y^2 + 4xy = 0$$

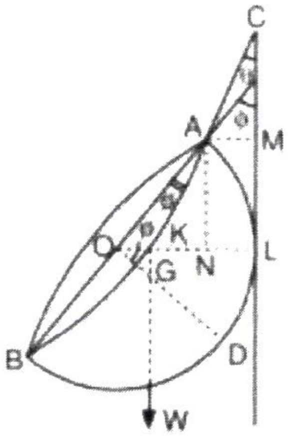
$$y(y + 4x) = 0$$

$y = 0$, $y + 4x = 0$ both satisfy main eqns

So both are singular

7b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ is the angle of inclination of the string with vertical and $\phi$is the angle of inclination of the plane base of the hemisphere to the vertical, then find the value of $(\tan\ \phi - \tan\ \theta)$.

(SuccessClap FULL LENGTH TEST 03 2023 Qn 6B)



Sol. AC is the string of length l (say). O is the centre of the base of the hemisphere of radius r (say), OD is the axis of the hemisphere on which its centre of gravity G lies, such that $OG = \frac{3}{8}r$. The weight W of the hemisphere acts at G. The distances are measured downwards from the fixed displacement such that θ changes into $\theta + \delta\theta$ and ϕ changes into $\phi + \delta\phi$ whereas the length of the string remains unaltered ($\therefore$ its tension will not do any work) then the equation of virtual work is

$$W\delta(\text{ depth of } G \text{ below } C) = 0$$

i.e. $\quad W\delta(CL + KG) = 0$

or $\quad \delta(CL + KG) = 0 \ \because W \neq 0$

Now $CM = CA\cos\ \theta = I\cos\ \theta; ML = AN = OA\cos\ \phi = r\cos\ \phi$

$\therefore CL = CM + ML = l\cos\ \theta + r\cos\ \phi$ and $KG = OG\sin\ \phi = \frac{3}{8}r\sin\ \phi$

$$\delta\left(l\cos\ \theta + r\cos\ \phi + \frac{3}{8}r\sin\ \phi\right) = 0$$

$$-l\sin\ \theta\delta\theta - r\sin\ \phi\delta\phi + \frac{3}{8}r\cos\ \phi\delta\phi = 0$$

$$l\sin\ \theta\delta\theta = \left(\frac{3}{8}\cos\ \phi - \sin\ \phi\right)r\delta\phi$$

Also $OL =$ radius of the hemisphere $= r$

Again $OL = ON + NL = ON + AM = r\sin\ \phi + l\sin\ \theta$

Differentiating, $0 = r\cos\ \phi\delta\phi + l\cos\ \theta\delta\theta, r,$ being constant.

$$l\cos\ \theta\delta\theta = -r\cos\ \phi\delta\phi$$

Dividing we get $\tan\ \theta = -\frac{3}{8} + \tan\ \phi$ **ANS: 3/8**

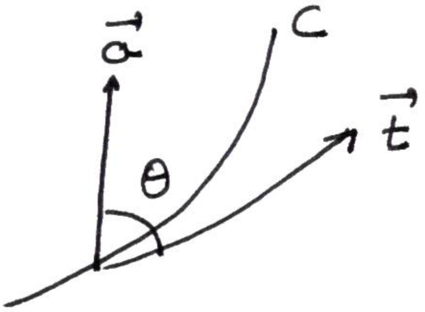
7c) If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\,\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction.

Let $\vec{a}$ be fixed line unit vector

C : Curve

$\vec{t}$ is tangent unit vector

$\theta \Rightarrow$ angle b/w $\vec{a}$ and $\vec{t}$



$\vec{t}\cdot\vec{a} = \cos\theta$ Both are unit vector

↳ Differentate w.r.to s $\quad t'\cdot a = 0 \quad t' = kn$

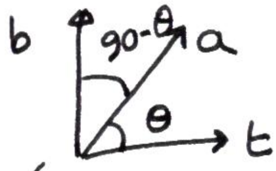
$kn\cdot a = 0$

Since $k \neq 0$ $\quad n\cdot a = 0$

⇒ show $\vec{a}$ is orthogal to n

⇓

$\vec{a}$ must lie in rectifying plane (plane containg t and b)



$a\cdot b = \cos(90-\theta)$
$= \sin\theta$

$a\cdot t = \cos\theta$

$n\cdot a = 0$

↓

differentate w.r.to s

$n'\cdot a = 0 \qquad n' = \tau b - kt$

$(\tau b - kt)\cdot a = 0 \Rightarrow \tau b\cdot a = kt\cdot a$

$\tau\sin\theta = k\cos\theta$

$$\frac{k}{\tau} = \tan\theta \qquad k = \frac{1}{\text{radius of curvatue}}$$

$$\frac{\text{Radius of Torsion}}{\text{Radius of Kurvatue}} = \tan\theta \qquad \tau = \frac{1}{\text{radius of Torsus}}$$

Part 2 :

Let $\frac{k}{\tau}$ = constant $\Rightarrow k = c\tau$ c is constant

we have $\underbrace{t' = kn = c\tau n \quad \text{Also } b' = -\tau n}$

$t' = -cb' \Rightarrow t' + cb' = 0 \Rightarrow \frac{d}{ds}(t+cb) \Rightarrow 0 \xrightarrow{\text{Integrate}} t + cb = a$

Take dot product with a on both side $\Rightarrow 1 = t\cdot a$

$t\cdot a$ = constant $\Rightarrow$ t makes constant angle θ w.r.to $\vec{a}$

8a) Solve the following initial value problem by using Laplace transform technique:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t)$$

$y(0) = 1, y'(0) = 0$ and $f(t)$ is a given function of t.

SuccessClap : Queston Bank : Laplace ODE
: Qn 29

Take laplace $L(y'') - 4L(y') + 3L(y) = L(f(t))$

$$s^2 L(y) - sy(0) - y'(0) - 4(sL(y) - y(0)] + 3L(y) = f(s)$$

$$L(f(t) = f(s) \Rightarrow f(t) = L^{-1}(f(s))$$

$$(s^2 - 4s + 3)L(y) - s + 4 = f(s)$$

$$L(y) = \frac{s-4+f(s)}{s^2-4s+3} = \frac{s-4}{(s-1)(s-3)} + \frac{f(s)}{(s-1)(s-3)}$$

$$L(y) = \frac{1}{2}\left[\frac{3}{s-1} - \frac{1}{s-3}\right] + \frac{1}{2}f(s)\left[\frac{1}{s-3} - \frac{1}{s-1}\right]$$

$$y = \frac{3}{2}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{2}L^{-1}(f(s)g(s) - \frac{1}{2}L^{-1}(f(s)h(s))$$

$$g(s) = \frac{1}{s-3} \qquad h(s) = \frac{1}{s-1}$$

$$G(t) = L^{-1}g(s) = e^{3t} \qquad H(t) = L^{-1}(h(s) = e^{t}$$

Convulation theorem

$$L^{-1}(f(s)g(s)) = \int_0^t F(u)G(t-u)\,du = \int_0^t F(u)e^{3(t-u)}\,du$$

$$= e^{3t}\int_0^t F(u)e^{-3u}\,du$$

$$L^{-1}(f(s)h(s) = \int_0^t F(u)H(t-u)\,du = \int_0^t F(u)e^{t-u)}\,du$$

$$= e^{t}\int_0^t F(u)e^{-u}\,du$$

$$y = \frac{1}{2}(3e^{t} - e^{3t}) + \frac{1}{2}e^{3t}\int_0^t F(u)e^{-3u}\,du$$

$$- \frac{1}{2}e^{t}\int_0^t F(u)e^{-u}\,du$$

8b) A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\frac{2\lambda}{3}c^3}$ and is moving with central acceleration $\lambda(r^5 - c^2 r)$. Find the path of motion of this particle. Will that be the curve $x^4 + y^4 = c^2$?

(SuccessClap FULL LENGTH TEST 03 2023 Qn 7B)

Central force = Force per mass $= \lambda(r^5 - c^2 r)$

Diff eqn : $\cancel{v^2 = h^2}$ $h^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{F}{u^2} = \lambda\left(\frac{1}{u^7} - \frac{c^2}{u^3}\right)$

Put $r = \frac{1}{u}$

$\downarrow$

Multiplying both sides $\frac{2du}{d\theta}$ and integrating

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \lambda\left(\frac{-1}{3u^6} + \frac{c^2}{u^2}\right) + A$$

Apse $\left[\begin{array}{l} u = \frac{1}{r} = \frac{1}{\sqrt{c}} \quad \frac{du}{d\theta} = 0 \Rightarrow \\ v = \sqrt{\frac{2\lambda}{3}c^3} \end{array}\right.$

$$\underbrace{\frac{2\lambda}{3}c^3 = h^2}\left[\frac{1}{c} + 0\right] = \lambda\left(\frac{-c^3}{3} + c^3\right) + A$$

$$= \lambda\left(\frac{2}{3}c^3\right)$$

$$h^2 = \frac{2\lambda}{3}c^4$$

$$\Rightarrow A = 0$$

Put in eqn

$$\frac{2\lambda}{3}c^4\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(-\frac{1}{3u^6} + \frac{c^2}{u^2}\right)$$

$$c^4\left(\frac{du}{d\theta}\right)^2 + c^4u^2 = \frac{3}{2}\left(\frac{-1}{3u^6} + \frac{c^2}{u^2}\right)$$

$$= -\frac{1}{2u^6} + \frac{3}{2}\frac{c^2}{u^2}$$

$$c^4\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6} + \frac{3}{2}\frac{c^2}{u^2} - c^4u^2$$

$$= \frac{1}{u^6}\left(-\frac{1}{2} + \frac{3}{2}c^2u^4 - c^4u^8\right)$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^4u^8 - \frac{3}{2}c^2u^4\right)\right]$$

$$\downarrow$$

$$\left[\left(c^2u^4 - \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^2u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\frac{1}{16} - \left(c^2u^4 - \frac{3}{4}\right)^2\right] = \frac{1}{4}$$

$$c^4\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^2u^4 - \frac{3}{4}\right)^2\right]$$

$\downarrow$ root $\quad c^2u^3\frac{du}{d\theta} = \sqrt{\left(\frac{1}{4}\right)^2 - \left(c^2u^4 - \frac{3}{4}\right)^2}$

$$d\theta = \frac{c^2u^3\,du}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^2u^4 - \frac{3}{4}\right)^2}}$$

Put $z = c^2u^4 - 3/2$

$dz = 4c^2u^3du$

$\Rightarrow c^2u^3du = \frac{dz}{4}$

$$= \frac{dz}{4\sqrt{\left(\frac{1}{4}\right)^2 - z^2}}$$

Integrating

$$\theta = \frac{1}{4}\sin^{-1}\frac{z}{1/4} + C = \frac{1}{4}\sin^{-1}4z + c$$

$$= \frac{1}{4}\sin^{-1}(c^2u^4.4 - 3) + C$$

Initial $\theta = 0 \quad u = \frac{1}{\sqrt{c}}$

$$0 = \frac{1}{4}\sin^{-1}1 + C = \frac{\pi}{8} + C \Rightarrow C = -\frac{\pi}{8} \qquad \sin^{-1}1 = \frac{\pi}{2}$$

$$\theta + \frac{\pi}{8} = \frac{1}{4}\sin^{-1}(4c^2u^4 - 3)$$

$$\sin\left(4\theta + \frac{\pi}{2}\right) = 4c^2u^4 - 3$$

$\cos 4\theta = 4c^2u^4 - 3$

$1+\cos 4\theta = 2\cos^2 2\theta$

$3+\cancel{4}\cos 4\theta = 4c^2u^4$

$2+2\cos^2 2\theta = 4c^2u^4$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$1+\cos^2 2\theta = 2c^2u^4$

$1 = (\sin^2\theta + \cos^2\theta)$

$\downarrow \qquad \searrow$

$1 = (\sin^2\theta + \cos^2\theta)^2$

$(\sin^2\theta+\cos^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2$

$(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) + (\sin^4\theta + \cos^4\theta - 2\sin^2\theta\cos^2\theta)$

$= 2\sin^4\theta + \cos^4\theta.2 \quad = 2(\sin^4\theta + \cos^4\theta)$

$= 2c^2u^4$

$\sin^4\theta + \cos^4\theta = c^2u^4 = \frac{c^2}{r^4} \qquad u = \frac{1}{r}$

$(r\sin\theta)^4 + (r\cos\theta)^4 = c^2$

$x^2 + y^2 = c^2$

You can solve this problem in EXAM,
Only if you practise such questions
well in advance

8c) For a scalar point function ϕ and vector point function $\vec{f}$, prove the identity $$\nabla \cdot (\phi \vec{f}) = \nabla \phi \cdot \vec{f} + \phi(\nabla \cdot \vec{f}).$$
Also find the value of $\nabla \cdot \left(\frac{f(r)}{r}\vec{r}\right)$ and then verify stated identity.

$$\text{div } (\phi \mathbf{A}) = \nabla \cdot \left(\phi \mathbf{A} \cdot = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\phi \mathbf{A})\right.$$

$$= \mathbf{i} \cdot \frac{\partial}{\partial x}(\phi \mathbf{A}) + \mathbf{j} \cdot \frac{\partial}{\partial y}(\phi \mathbf{A}) + \mathbf{k} \cdot \frac{\partial}{\partial z}(\phi \mathbf{A})$$

$$= \Sigma\left\{\mathrm{i} \cdot \frac{\partial}{\partial x}(\phi \mathbf{A})\right\} = \Sigma\left\{\mathrm{i} \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{A} + \phi \frac{\partial \mathbf{A}}{\partial x}\right)\right\}$$

$$= \Sigma\left\{\mathbf{i} \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{A}\right)\right\} + \Sigma\left\{\mathbf{i} \cdot \left(\phi \frac{\partial \mathbf{A}}{\partial x}\right)\right\} \text{ A clo}$$

$$= \Sigma\left\{\left(\frac{\partial \phi}{\partial x}\right) \cdot \mathbf{A}\right\} + \Sigma\left\{\phi\left(\mathbf{i} \cdot \frac{\partial \mathbf{A}}{\partial x}\right)\right\} \; [Note \mathbf{a} \cdot (m\mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b})]$$

$$= \left\{\Sigma \frac{\partial \phi}{\partial x}\mathbf{i}\right\} \cdot \mathbf{A} + \phi \Sigma\left(\mathbf{i} \cdot \frac{\partial \mathbf{A}}{\partial x}\right) = (\nabla \phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A}).$$

$$\text{div}\left\{\frac{f(\boldsymbol{r})\mathbf{r}}{\boldsymbol{r}}\right\} = \text{div}\left\{\frac{f(\boldsymbol{r})}{\boldsymbol{r}}(x\mathbf{i} + y\mathbf{j} + \mathbf{zk})\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\} + \frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\} + \frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}$$

$$\text{Now } \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\} = \frac{f(r)}{r} + x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$$

$$= \frac{f(r)}{r} + x\left\{\frac{f'(r)}{r} - \frac{1}{r^2}f(r)\right\}\frac{x}{r} = \frac{f(r)}{r} + \frac{x^3}{r^2}f'(r) - \frac{x^2}{r^3}f(r).$$

$$\text{Similarly } \frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\} = \frac{f(r)}{r} + \frac{y^2}{r^2}f'(r) - \frac{y^2}{r^3}f(r)$$

$$\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\} = \frac{f(r)}{r} + \frac{z^2}{r^2}f'(r) - \frac{z^2}{r^3}f(r).$$

Putting these values in (1), we get

$$\text{div}\left\{\frac{f(r)\mathbf{r}\}}{r}\right\} = \frac{3}{r}f(r) + \frac{r^2}{r^2}f'(r) - \frac{r^2}{r^3}f(r)$$

$$= \frac{2}{r}f(r) + f'(r) = \frac{1}{r^2}\left[2rf(r) + r^2 f'(r)\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)].$$