Avinash Choudhary
avichoudhary237@gmail.com
8604122783

144/250

SuccessClap

Best Coaching for UPSC MATHEMATICS

TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2022

FULL LENGTH TEST -1 PAPER 1

Time Allowed: Three Hours **Maximum Marks: 250**

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions:

There are **EIGHT** questions divided in **TWO SECTIONS**

Candidate must attempt **FIVE** questions in all.

Question Nos. **1** and **5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from one section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question – cum – Answer (QCA) Booklet in the space provided. No marks will be given for answers written in a medium other than the authorized one.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless and otherwise indicated, symbols and notations carry their usual standard meaning.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question – cum – Answer Booklet must be clearly struck off.

Section A

1a) Let $A, B \in M_n(\mathbb{C})$. Show that if $AB = 0$, then $r(A) + r(B) \leq n$ (10)

$A, B \in M_n(C)$, $AB = 0$

1b) A function f is defined by

$$f(x) = x^p \cos(1/x), x \neq 0; f(0) = 0.$$

What conditions should be imposed on p that f may be
(i) continuous at $x = 0$
(ii) differentiable at $x = 0$? (10)

(i) $f(x) = \begin{cases} x^p \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

f(x) cont at $x = 0$

so $\lim_{x \to 0} x^p \cos\frac{1}{x} = 0$

for this to hold p should be greater than zero,

so $p > 0$

2

(ii) f diff at $x = 0$

hence conti. also, so $p > 0$

Now for diff.

$f'(x) = \begin{cases} p x^{p-1} \cos\frac{1}{x} + \frac{x^p}{x^2} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(x) = \begin{cases} p x^{p-1} \cos\frac{1}{x} + x^{p-2} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

so $\lim_{x \to 0} \left(\left(p x^{p-1} \cos\frac{1}{x}\right) + \left(x^{p-2} \sin\frac{1}{x}\right) \right) = 0$

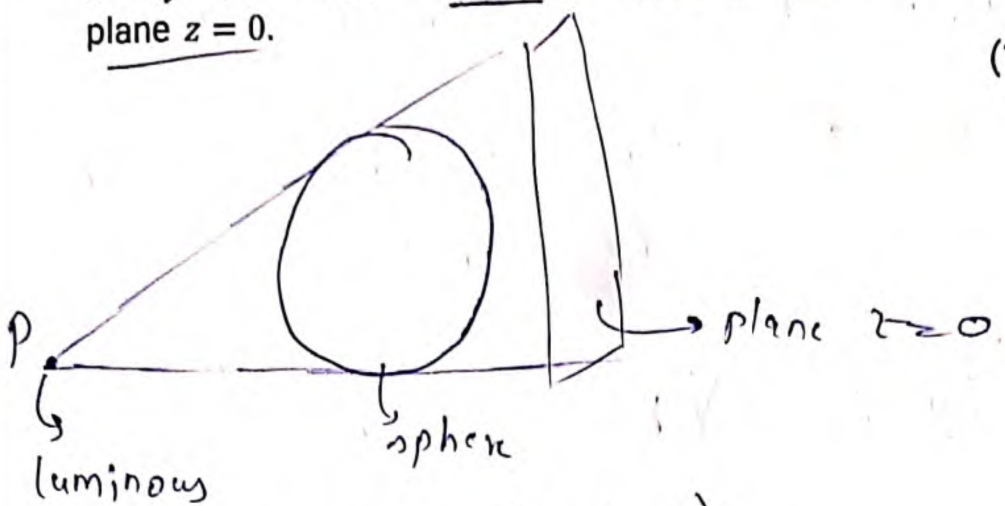
$p > 1$ for Lt to exist, $p > 2$ for Lt to exist

Scanned with CamScanner

so for differentiability at $x=0$

the required condition is $\boxed{p > 2}$

1c) Find the locus of a luminous point which moves so that the sphere $x^2 + y^2 + z^2 - 2az = 0$ casts a parabolic shadow on the plane $z = 0$. (10)



let $P = (\alpha, \beta, \gamma)$

then, eqⁿ of ~~pair of tangents~~ circumscribing cone from P to the sphere is

$$SS_1 = T^2$$

$$(x^2+y^2+z^2-2az)(\alpha^2+\beta^2+\gamma^2-2a\gamma) = (x\alpha + y\beta + z\gamma - a(z+\gamma))^2 \quad \text{--- (1)}$$

Now given that, when this cone meets the plane $z=0$, the resultant is a parabola, so put $z=0$ in (1)

$$(x^2+y^2)(\alpha^2+\beta^2+\gamma^2-2a\gamma) = (x\alpha+y\beta-a\gamma)^2$$

$$x^2\alpha^2 + x^2(\beta^2+\gamma^2-2a\gamma) + y^2\beta^2 + y^2(\alpha^2+\gamma^2-2a\gamma)$$
$$= x^2\alpha^2 + y^2\beta^2 + a^2\gamma^2 + 2xy\alpha\beta - 2ax\alpha\gamma - 2ay\beta\gamma$$

$$x^2(\beta^2+\gamma^2-2a\gamma) + y^2(\alpha^2+\gamma^2-2a\gamma) - 2xy\alpha\beta + 2ax\alpha\gamma$$
$$+ 2ay\beta\gamma - a^2\gamma^2 = 0$$

Scanned with CamScanner

for this to a parabola

$$(\alpha\beta)^2 = \left((\beta^2+\gamma^2)-2a\gamma\right)\left(\alpha^2+\gamma^2-2a\gamma\right)$$

$$\alpha^2\beta^2 = \beta^2\alpha^2 + \beta^2\gamma^2 - 2a\beta^2\gamma + \gamma^2\alpha^2 + \gamma^4 - 2a\gamma^3$$
$$- 2a\gamma\alpha^2 - 2a\gamma^3 + 4a\gamma^2$$

(8a) $$(y^2+z^2-2az)(x^2+z^2-2az) = x^2y^2$$ ✓ 8

is the required locus.

1d) Determine the dimension and basis for the following subspaces of R^3 and R^4.

(i) the plane $3x - 2y + 5z = 0$

(ii) the line $x = 2t, y = -t, z = 4t$

(iii) all vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$

(10)

(i) $3x - 2y + 5z = 0$

$y = \frac{3x + 5z}{2}$

so $(x, y, z) = (x, \frac{3x+5z}{2}, z)$

$= x(1, \frac{3}{2}, 0) + z(0, \frac{5}{2}, 1)$ ✓

so $\boxed{\dim = 2}$

basis $= \{(1, \frac{3}{2}, 0), (0, \frac{5}{2}, 1)\}$

(ii) $x = 2t, \ y = -t, \ z = 4t$

so $(x, y, z, t) = (2t, -t, 4t, t)$

$= t(2, -1, 4, 1)$

so $\boxed{\dim = 1}$

basis $= \{(2, -1, 4, 1)\}$ ✓

(iii) $d = a + b, \ c = a - b$

so $(a, b, c, d) = (a, b, a-b, a+b)$

$= a(1, 0, 1, 1) + b(0, 1, -1, 1)$

8

Scanned with CamScanner

∴ $\boxed{\dim = 2}$

basis = $\{(1,0,1,1),(0,1,-1,1)\}$.

1e) Find the length of the shortest distance between the $z = axis$ and the line $x + y + 2z - 3 = 0 = 2x + 3y + 4z - 4$.

(10)

z axis $\equiv \frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

any pt on z axis $\equiv (0, 0, \lambda)$

plane containing the above line

P: $(x + y + 2z - 3) + \mu(2x + 3y + 4z - 4) = 0$

$(2\mu + 1)x + (3\mu + 1)y + (4\mu + 2)z - (4\mu + 3) = 0$

for S.D. plane P is || to z-axis

so $0.(2\mu + 1) + 0(3\mu + 1) + 1(4\mu + 2) = 0$

$4\mu + 2 = 0$

$\mu = -1/2$

so $P_1 : -\frac{y}{2} - 1 = 0$

$P_1 : y + 2 = 0$

⊥ distance from $(0, 0, \lambda)$ to P_1

$\frac{|0 + 0 + 0 + 2|}{1}$

so S.D = 2

8

2a) (a) Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4. Find the eigen values and eigen vectors of A

(b) Determine the eigen values of A^{-1}.

(15)

Consider

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)((2-\lambda)(3-\lambda)-2) - 0 - 1(2-2(2-\lambda)) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+6-2) - (2-4+2\lambda) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+4) + 2(1-\lambda) = 0$$

$$(1-\lambda)(\lambda^2-5\lambda+6) = 0$$

$$(1-\lambda)(\lambda-3)(\lambda-2) = 0$$

so eigen values of A are 1, 2, 3 ✓

Now for $\lambda = 1$.

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{-x_1 = 0}$, $x_1 + x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = -x_3}$

$$\text{so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

so for $\lambda = 1$, eigen vector $= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Now for $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad x_1 + x_3 = 0$$

$$\boxed{x_3 = -x_1}$$

also $2x_1 + 2x_2 + x_3 = 0 \Rightarrow 2x_1 + 2x_2 - x_1 = 0$

$$\boxed{x_2 = -x_1/2} \quad \text{so} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1/2 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1/2 \\ -1 \end{pmatrix}$$

so for ev $\lambda = 2$, eigen vector $= \begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix}$

Now for $\lambda = 3$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad -2x_1 - x_3 = 0 \Rightarrow \boxed{x_3 = -2x_1}$$

$$2x_1 + 2x_2 = 0. \quad \boxed{x_2 = -x_1}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \\ -2x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

so for ev $\lambda = 3$, eigen vector $= \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

so $P = [ev_1, ev_2, ev_3]$

$$\boxed{P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1/2 & -1 \\ -1 & -1 & -2 \end{bmatrix}}$$

(b) $AX = \lambda X$

$A^{-1}AX = \lambda A^{-1}X$ (mul with A^{-1}).

$\lambda^{-1}X = A^{-1}X$

so eigen values of A are $\left(1, \frac{1}{2}, \frac{1}{3}\right)$

12

2b) If A is such a matrix that $A^3 = 2I$, show that B is invertible, where

$$B = A^2 - 2A + 2I$$

(10)

$B = A^2 - 2A + 2I$

$BA = (A^2 - 2A + 2I)A$

$= A^3 - 2A^2 + 2A$

$= -2A^2 + 2A + 2I$

$BA = 2(-A^2 + A + I)$

↳ (1)

and

$BA^2 = (A^2 - 2A + 2I)A^2$

$= A^4 - 2A^3 + 2A^2$

$= 2IA - 2.2I + 2A^2$

$BA^2 = 2(A^2 + A - 2I)$

↳ (2)

2c) A function f is twice derivable and satisfies for $x > a$ the inequalities

$$|f(x)| < A, |f'(x)| < B,$$

where A and B are constants.

Prove that for x >a, $|f'(x)| < 2\sqrt{(AB)}$

(10)

f : twice derivable

for $x > a$ $|f(x)| < A$ (A, B - constants)

$|f'(x)| < B$

2d) Show that the normals from $(x', y'. z')$ to the paraboloid $ax^2 + by^2 = 2cz$ lie on the cone

$$\frac{x'}{x - x'} - \frac{y'}{y - y'} + c\frac{(1/a - 1/b)}{z - z'} = 0.$$

(15)

let the normal and paraboloid meet at (α, β, γ)

then dr's of normal $(a\alpha, b\beta, -c)$

~~eqn N: Normal~~

eqn of Normal $\Rightarrow \frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c} = \lambda$

Now this line passes through foot of perpendicular, ie, (x', y', z') so

$\frac{x'-\alpha}{a\alpha} = \frac{y'-\beta}{b\beta} = \frac{z'-\gamma}{-c} = \lambda$ (i) $\frac{x'-\alpha}{pa\alpha} = \frac{y'-\beta}{pb\beta} = \frac{z'-\gamma}{-cp}$

$\hookrightarrow \alpha = \frac{x'}{1+a\lambda}, \beta = \frac{y'}{1+b\beta}, \gamma = z' + c\lambda$ | which is also $\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n}$

so $l = pa\alpha$, | $m = pb\beta$ | $n = -cp$

(ii) $l = \frac{pax'}{1+a\lambda}$ | $m = \frac{pby'}{1+b\lambda}$ | $n = -cp$

(iii) $1+a\lambda = \frac{pax'}{l}$ | $1+b\lambda = \frac{pby'}{m}$ | $1 = \frac{-cp}{n}$

Now so

$\frac{1}{a}(1+a\lambda) - \frac{1}{b}(1+b\lambda) + \left(\frac{1}{b} - \frac{1}{a}\right)\cdot 1$

$= \frac{1}{a}\left(\frac{pax'}{l}\right) - \frac{1}{b}\frac{pby'}{m} + \left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{-cp}{n}\right)$

$0 = \frac{px'}{l} - \frac{py'}{m} + \frac{c(1/a - 1/b)p}{n}$ Now replace l by $x-x'$, m by $y-y'$, n by $z-z'$

we get

$$\frac{x'}{x-x'} - \frac{y'}{y-y'} + \frac{c(1/a - 1/b)}{z-z'} = 0$$

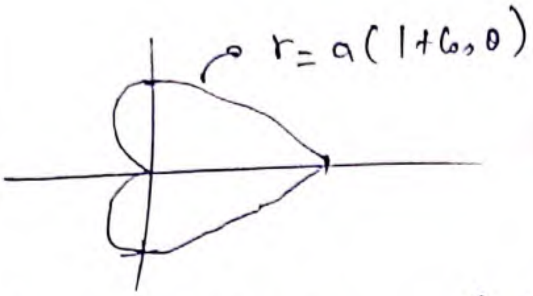
is the required cone eqn.

12

Section B

5a) Evaluate by Green's theorem $\oint_C(-x^2y\,dx + xy^2\,dy)$ where C is the cardioid $r = a(1+\cos\theta)$.

(10)



by Green's theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$\left.\begin{array}{l} P = -x^2y \\ Q = xy^2 \end{array}\right] \rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2$$

$$\therefore\ I = \iint_S (x^2+y^2)\,dx\,dy$$

converting into polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta$$

$$I = \iint_S r^2 \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a(1+\cos\theta)} r^3\,dr\,d\theta = \int_0^{2\pi} \frac{a^4(1+\cos\theta)^4}{4}\,d\theta$$

$$= \frac{a^4}{4}\int_0^{2\pi} (1+\cos^2\theta + 2\cos\theta)^2\,d\theta$$

$$= \frac{a^4}{4}\int_0^{2\pi}\left(1 + \cos^4\theta + 6\cos^2\theta + 4\cos\theta + 4\cos^3\theta\right)d\theta$$

$$= \frac{a^4}{4}\left(2\pi + 4^2\int_0^{\frac{\pi}{2}}\left(\cos^4\theta + 6\cos^2\theta\right)d\theta\right)$$

$$= \frac{a^4}{4}\left(2\pi + 4\left(\frac{1}{2}\cdot\frac{\Gamma(5/2)\,\Gamma(1/2)}{\Gamma(3)} + 6\,\frac{1}{2}\,\frac{\Gamma(3/2)\,\Gamma(1/2)}{\Gamma(2)}\right)\right)$$

$$= \frac{a^4}{2}\left(\pi + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\pi}{2} + \frac{6\cdot\frac{1}{2}\cdot\pi}{1}\right)$$

$$= \frac{a^4}{2}\left(\pi + \frac{3\pi}{8} + 3\pi\right)$$

$$= \frac{a^4}{2}\left(4\pi + \frac{3\pi}{8}\right)$$

$$I = \frac{35\pi a^4}{16}$$

④

5b) Solve $(D^2+1)^2 = 24x\cos x$ given that $y = Dy = D^2y = 0$ and $D^3y = 12$ when $x = 0$. (10)

C.F.: $(m^2+1)^2 = 0 \Rightarrow (m+i)^2(m-i)^2 = 0$

$m = i, i, -i, -i$, (or) $\pm i, \pm i$

$y_c = (c_1 + c_2 x)\sin x + (c_3 + c_4 x)\cos x$

$$y_p = \frac{1}{(D^2+1)^2} 24x\cos x = 24\left(\frac{1}{(D^2+1)^2} x\cos x\right)$$

$$= 24\left(x\cdot\frac{1}{(D^2+1)^2}\cos x - \frac{2(D^2+1)\cdot 2D}{(D^2+1)^4}\cos x\right)$$

$$= 24\left(x\cdot\frac{1}{(D^2+1)^2}\cos x - \frac{4D}{(D^2+1)^3}\cos x\right)$$

$$= 24\left(x\cdot\frac{1}{(D^2+1)^2}\cos x + \frac{4\sin x}{(D^2+1)^3}\right) \to (1)$$

Now

$$\frac{\cos x}{(D^2+1)^2} = real\left(\frac{e^{ix}}{(D^2+1)^2}\right) = real\left(e^{ix}\cdot\frac{1}{((D+i)^2+1)^2}\right)$$

$$= real\left(e^{ix}\cdot\frac{1}{(D^2+2iD)^2}\right) = real\left(e^{ix}\cdot\frac{1}{(2iD)^2}\left(1+\frac{D}{2i}\right)^{-2}\cdot 1\right)$$

$$= real\left(\frac{e^{ix}}{(2i)^2}\cdot\frac{1}{D^2}\left(1 - \frac{2D}{2i} + \frac{3D^2}{(2i)^2}\cdots\right)\cdot 1\right)$$

$$= real\left(\frac{e^{ix}}{2i}\cdot\frac{1}{D}(1)\right) = real\left(\frac{e^{ix}}{2i}\cdot x\right)$$

$$= real\left(-\frac{1}{2}(i)x(\cos x + i\sin x)\right)$$

$$= real\left(-\frac{1}{2}(-x\sin x + ix\cos x)\right)$$

~~$\text{real}\left(\frac{x\sin x}{2} - j\frac{x\cos x}{2}\right)$~~ $= \frac{\text{real}}{-4}\left(e^{jx}\cdot\frac{x^2}{2}\right)$

$= \frac{\text{real}}{-8}\left(x^2\cos x + x^2\sin x\, j\right)$

~~$= \frac{x\sin x}{2}$~~ $= \frac{-x^2\cos x}{8}$

and

$$\frac{\sin x}{(D^2+1)^3} = \text{img}\left(\frac{e^{jx}}{(D^2+1)^3}\right)$$

$$= \text{img}\left(e^{jx}\cdot\frac{1}{((D+j)^2+1)^3}\right) = \text{img}\left(e^{jx}\cdot\frac{1}{(D^2+2jD)^3}\right)$$

$$= \text{img}\left(\frac{e^{jx}}{(2jD)^3}\left(1+\frac{D}{2j}\right)^{-3}\cdot 1\right)$$

$$= \text{img}\left(e^{jx}\cdot\frac{1}{-8jD^3}\cdot 1\right) = \text{img}\left(\frac{e^{jx}}{-8j}\cdot\frac{x^3}{6}\right)$$

$$= \frac{1}{48}\,\text{img}\left(j x^3(\cos x + j\sin x)\right)$$

$$= \frac{x^3\cos x}{48}$$

putting in (1)

$$y_p = 24\left(x\cdot\left(\frac{-x^2\cos x}{8}\right) + 4\cdot\frac{x^3\cos x}{48}\right)$$

$$= x^3(-3\cos x + 2\cos x) = -x^3\cos x.$$

$\therefore\ y = (C_1 + C_2 x)\sin x + (C_3 + C_4 x)\cos x - x^3\cos x$

Now $y(0) = 0 \Rightarrow \boxed{C_3 = 0}$

$y'(0) = 0 \Rightarrow C_1\cos x + C_2\sin x + C_2 x\cos x + C_4\cos x - C_4 x\sin x$

$- 3x^2\cos x + x^3\sin x = 0 \Rightarrow \boxed{C_1 + C_4 = 0}$

$y''(0) = 0 \Rightarrow C_2\cos x + C_2\cos x - C_2\sin x\cdot x - C_4\sin x - C_4 x\cos x$

$- 6x\cos x + 3x^2\sin x + 3x^2\sin x + x^3\cos x = 0 \Rightarrow \boxed{C_2 = 0}$

$\therefore\ y = C_1(\sin x - x\cos x) - x^3\cos x \Rightarrow y'''(0) = 12 \Rightarrow \boxed{C_1 = 9}$

$$\boxed{y = 9(\sin x - x\cos x) - x^3\cos x}$$

6

Scanned with CamScanner

5c) A particle is moving with central acceleration $\mu(r^5 - c^4 r)$ being projected from an apse at a distance c with velocity $c^3(2\mu/3)^{1/2}$, show that its path is the curve $x^4 + y^4 = c^4$.

(10)

$P = \mu(r^5 - c^4 r)$ when $u = \frac{1}{r}$

$P = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)$ $du = \frac{-1}{r^2} dr$

eq^n of motion

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2}$$

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu}{u^3}\left(\frac{1}{u^4} - c^4\right)$$

mul. both sides by $2\frac{du}{d\theta}$ & integrate w.r.t $d\theta$

$$h^2\left[2u\frac{du}{d\theta} + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right]d\theta = 2\mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)\frac{du}{d\theta}d\theta$$

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[\frac{-1}{6u^6} + \frac{c^4}{2u^2}\right]$$

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left[\frac{c^4}{u^2} - \frac{1}{3u^6}\right] \rightarrow (1)$$

Now at apse, $u = \frac{1}{c}$, $\frac{du}{d\theta} = 0$

$$h^2\left(\frac{1}{c^2}\right) = \mu\left[c^6 - \frac{c^6}{3}\right] = \frac{2\mu c^6}{3}$$

$$h^2 = \frac{2\mu c^8}{3} \rightarrow \text{put in (1)}$$

$$\frac{2\mu c^8}{3}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{3c^4u^4 - 1}{3u^6}\right)$$

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{(3c^4u^4 - 1)}{2c^8u^6}$$

$$\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{3c^4}{r^4} - 1\right)\frac{r^6}{2c^8}$$

Scanned with CamScanner

$$\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]\frac{1}{r^4}=\frac{[3c^4-r^4]\,r^2}{2c^8}$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2=\frac{(3c^4-r^4)\cdot r^6}{2c^8}$$

$$\left(\frac{dr}{d\theta}\right)^2=\frac{(3c^4-r^4)\cdot r^6-2c^8r^2}{2c^8}$$

$$\frac{dr}{d\theta}=\frac{r}{\sqrt{2}c^4}\sqrt{(3c^4-r^4)\cdot r^4-2c^8}$$

$$\int\frac{dr}{r\sqrt{3c^4r^4-r^8-2c^8}}\qquad\int\sqrt{2}c^4\,d\theta$$

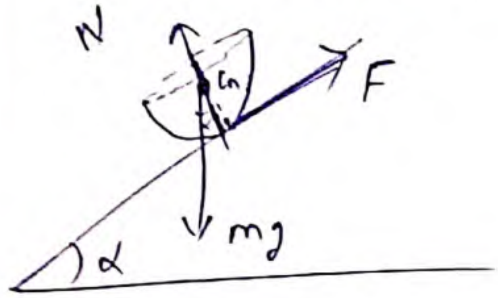
$\oint$ solving this integral ~~given~~ and using condition of apse $\left(r=c,\ v=c^3\left(\frac{2\mu}{5}\right)^{1/2}\right)$

gives $r^4(\sin^4\theta+\cos^4\theta)=c^4$

(or) $x^4+y^4=c^4$

5d) A solid hemisphere rests on a plane inclined to the horizon at angle $\alpha < \sin^{-1}\left(\frac{3}{8}\right)$ and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.

(10)



let mass of solid hemi = m

Cm = CoM of solid hemi

F = friction

equating forces

$N = mg\cos\alpha$

$F = mg\sin\alpha$

5e) Apply the method of variation of parameters to solve $y_2 - y = 2/(1+e^x)$ (10)

$$(y'' - y) = \frac{2}{1+e^x}$$

for CF: $(D^2-1)y = 0 \Rightarrow m = \pm 1$

$$y = c_1 e^x + c_2 e^{-x}$$

let $u = e^x$, $v = e^{-x}$

then $uv' - vu' = -e^x \cdot e^{-x} - e^{-x} \cdot e^x$
$= -2$

then $y_p = Au + Bv$

where

$$A = -\int \frac{vR\,dx}{uv' - vu'}$$
$$= -\int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{-2}$$
$$= \int \frac{1}{e^x(1+e^x)} dx$$
$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$
$$= -e^{-x} + \log(e^x+1) - \log e^x$$
$$= -e^{-x} + \log(e^x+1) - x$$

$$B = \int \frac{uR}{uv' - vu'} dx$$
$$= \int \frac{e^x \cdot \frac{2}{1+e^x}}{-2} dx$$
$$= -\int \frac{e^x}{1+e^x} dx$$
$$= -\log(1+e^x)$$

so $y_p = e^x\left(-e^{-x} - x + \log(e^x+1)\right) + e^{-x}\left(-\log(1+e^x)\right)$

8

$$y_p = -1 - xe^x + \log(1+e^x)(e^x - e^{-x})$$

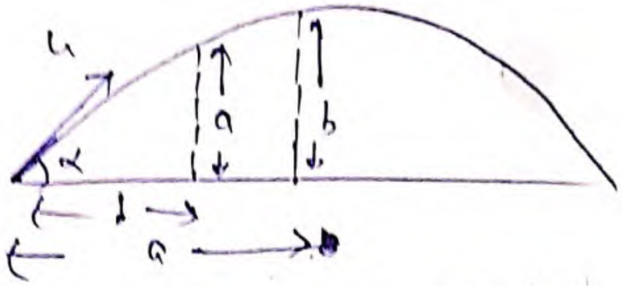
$\therefore$

$$y = c_1 e^x + c_2 e^{-x} - 1 - xe^x + \log(1+e^x)(e^x - e^{-x})$$

Scanned with CamScanner

6a) A ball is projected so as just to clear two walls the first of height a at a distance b from the point of projection and the second of height b at a distance a from the point of projection. Show that the range on the horizontal plane is $(a^2 + ab + b^2)/(a + b)$, and that the angle of projection exceeds $\tan^{-1} 3$.

(10)



For projectile

$x = u\cos\alpha\, t$

$t = \frac{x}{u\cos\alpha}$

$y = u\sin\alpha\, t - \frac{1}{2} g t^2$

$y = x\tan\alpha - \frac{1}{2} g \cdot \frac{x^2}{u^2\cos^2\alpha}$

$R = \frac{u^2 \sin 2\alpha}{g}$

Now this eq satisfies (b, a) & (a, b)

putting (b, a)

$b = a\tan\alpha - \frac{g a^2}{2u^2\cos^2\alpha} = a\tan\alpha\left(1 - \frac{a}{R}\right) \rightarrow (1)$

and (a, b)

$a = b\tan\alpha - \frac{g b^2}{2u^2\cos^2\alpha} = b\tan\alpha\left(1 - \frac{b}{R}\right) - (2)$

~~$a = \left(a\tan\alpha - \frac{g a^2}{2u^2\cos^2\alpha}\right)\tan\alpha - \frac{g\left(a\tan\alpha - \frac{g a^2}{2u^2\cos^2\alpha}\right)^2}{2u^2\cos^2\alpha}$~~

~~$2u^2\cos^2\alpha \cdot a = a\left(u^2\sin 2\alpha - ag\right)\tan\alpha - \frac{a^2 g\left(u^2\sin 2\alpha - ga\right)^2}{4u^4\cos^4\alpha}$~~

~~so $a = a\tan\alpha\left(1 - \frac{a}{R}\right)\tan\alpha\left(1 - \frac{a\tan\alpha\left(1 - \frac{a}{R}\right)}{R}\right)$~~

divide eq (1) by (2)

$$\frac{b}{a} = \frac{a\tan\alpha\left(1-\frac{a}{R}\right)}{b\tan\alpha\left(1-\frac{b}{R}\right)}$$

$$b^2\left(1-\frac{b}{R}\right) = a^2\left(1-\frac{a}{R}\right)$$

$$b^2 - \frac{b^3}{R} = a^2 - \frac{a^3}{R}$$

$$\frac{a^3-b^3}{R} = a^2-b^2$$

$$R = \frac{a^3-b^3}{a^2-b^2} = \frac{(a-b)(a^2+b^2+ab)}{(a-b)(a+b)}$$

so $\boxed{R = (a^2+ab+b^2)/(a+b)}$ ✓

putting it in (1)

$$b = a\tan\alpha\left(1 - \frac{a(a+b)}{(a^2+b^2+ab)}\right)$$

$$b = a\tan\alpha\left(\frac{a^2+ab+b^2-a^2-ab}{a^2+ab+b^2}\right)$$

$$b = \frac{a\tan\alpha\,(b^2)}{a^2+b^2+ab} \Rightarrow \tan\alpha = \frac{a^2+b^2+ab}{ab}$$

$$\tan\alpha = \frac{(a-b)^2+3ab}{ab}$$

so $\tan\alpha \geq \frac{0+3ab}{ab} \Rightarrow \tan\alpha \geq 3$ ✓

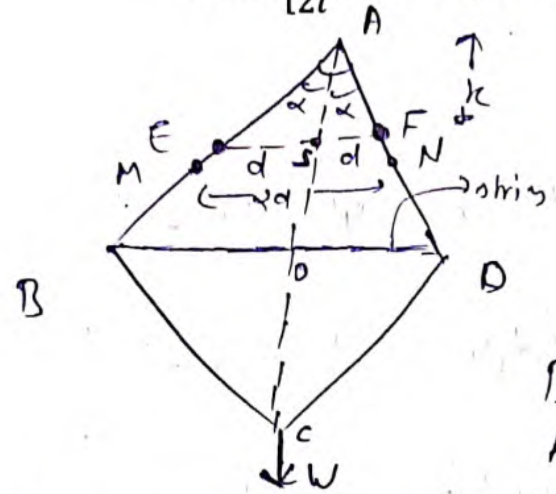
so $\boxed{\alpha \geq \tan^{-1}(3)}$ ✓

(8)

6b) ABCD is a rhombus formed with four rods each oflength land of weight wjoined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and Dare joined by a string. If the distance of the pegs apart is $2d$ and the angle at A is 2α, show that the tension in the string is

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\text{cosec}^3\,\alpha-(W+2w)\right]$$

(15)



length of rods = l

weight = w

E, F → pegs

$BD = 2l\sin\alpha$

$AO = l\cos\alpha$

let $AS = x$, then by similarity of triangles

$$\frac{x}{d} = \frac{AO}{OD} \Rightarrow \frac{x}{d} = \frac{l\cos\alpha}{l\sin\alpha} \Rightarrow x = \frac{d}{\tan\alpha}$$

so $OS = AO - AS = l\cos\alpha - \frac{d\cos\alpha}{\sin\alpha}$

$OS = l\cos\alpha - d\cot\alpha$

and $CS = OS + OC = l\cos\alpha - d\cot\alpha + l\cos\alpha$

$= 2l\cos\alpha - d\cot\alpha$

Now CoG of the 4 rods (without wt W) can be considered at O, so,
if we give a small angular displacement to the system. Then the virtual work is zero. So VW about line EF is
(let T be tension in string)

$$4W(\delta OS) + W(\delta CS) - T(\delta BD) = 0$$

$$4W\, d(l\cos\alpha - d\cot\alpha) + W(2l\cos\alpha - d\cot\alpha) - T\, d(2l\sin\alpha) = 0$$

$$4W(-l\sin\alpha + d\,\mathrm{cosec}^2\alpha)\,d\alpha + W(-2l\sin\alpha + d\,\mathrm{cosec}^2\alpha)\,d\alpha - 2Tl\cos\alpha\, d\alpha = 0$$

so $2Tl\cos\alpha = d\,\mathrm{cosec}^2\alpha\,(W + 4W) - 2l\sin\alpha(2W + W)$

$$T(2l\cos\alpha) = \sin\alpha\left(d(W+4W)\,\mathrm{cosec}^3\alpha - 2l(W+4W)\right)$$

$$T = \tan\alpha\left(\frac{d}{2l}(W+4W)\,\mathrm{cosec}^3\alpha - (W+4W)\right)$$

is the required tension.

12

6c) Use Laplace Solve $y'' - ty' + y = 1$, if $y(0) = 1, y'(0) = 2$. (15)

Let $L\{y\} = K$

~~then $L\{y'\} = p\,L\{y\} - y(0)$~~
~~$= p^2 - 1$~~

~~$L\{y''\} = p\,L\{y'\} - y'(0)$~~
~~$= p(p^2-1)$~~

then $L\{y'\} = p\,L\{y\} - y(0)$
$= pk - 1$

$L\{y''\} = p\,L\{y'\} - y'(0)$
$= p(pk-1) - 2$
$= p^2 k - p - 2$

takin Laplace of above ODE

$L\{y''\} - L\{ty'\} + L\{y\} = L\{1\}$

$p^2 k - p - 2 - (-1)\frac{d}{dp}(L\{y'\}) + k = \frac{1}{p}$

~~$p^2 k - p - 2 + k + k = \frac{1}{p}$~~

$(p^2+2)k = \frac{1}{p} + (p+2)$

$k = \frac{1}{p(p^2+2)} + \frac{p+2}{p^2+2}$

$k = \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+2}\right) + \frac{p}{p^2+1} + \frac{2}{p^2+2}$

$$L\{y\} = \frac{1}{2}\cdot\frac{1}{p} + \frac{1}{2}\cdot\frac{p}{p^2+2} + \frac{2}{p^2+2}$$

$$y = \frac{1}{2}L^{-1}\left\{\frac{1}{p}\right\} + \frac{1}{2}L^{-1}\left\{\frac{p}{p^2+2}\right\} + 2L^{-1}\left\{\frac{1}{p^2+2}\right\}$$

$$y = \frac{1}{2} + \frac{1}{2}\cos\sqrt{2}\,t + \frac{2}{\sqrt{2}}\cdot\sin\sqrt{2}\,t$$

$$\boxed{y = \frac{1}{2} + \frac{1}{2}\cos\sqrt{2}\,t + \sqrt{2}\sin\sqrt{2}\,t}$$

6d) Prove that $\nabla^2\left[\nabla\cdot\left(\frac{\bar{r}}{r^2}\right)\right] = 2r^{-4}$, where $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

(10)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \nabla\cdot\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot x\hat{i} + y\hat{j} + z\hat{k}$

$= 3$

also $r = \sqrt{x^2+y^2+z^2}$

so $\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$

$= \frac{x\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

or $\nabla r = \frac{\vec{r}}{r}$

Now $\nabla\cdot\frac{\vec{r}}{r^2} = \nabla\left(\frac{1}{r^2}\right)\cdot\vec{r} + \frac{1}{r^2}(\nabla\cdot\vec{r})$

$= -\frac{2}{r^3}\nabla(r)\cdot\vec{r} + \frac{1}{r^2}(3)$

$= -\frac{2}{r^3}\frac{\vec{r}}{r}\cdot\vec{r} + \frac{3}{r^2}$

$= -\frac{2r^2}{r^4} + \frac{3}{r^2} = \frac{1}{r^2}$

Now $\nabla^2\left(\frac{1}{r^2}\right) = \nabla\cdot\left(\nabla\left(\frac{1}{r^2}\right)\right) = \nabla\left(\frac{-2}{r^3}\nabla r\right)$

$= \nabla\cdot\left(\frac{-2\vec{r}}{r^4}\right)$

$= -2\left(\nabla\cdot\left(\frac{\vec{r}}{r^4}\right)\right)$

$= -2\left(\frac{1}{r^4}(\nabla\cdot\vec{r}) + \nabla\left(\frac{1}{r^4}\right)\cdot\vec{r}\right)$

$= -2\left(\frac{1}{r^4}(3) + -\frac{4}{r^5}\nabla r\cdot\vec{r}\right)$

$= -2\left(\frac{3}{r^4} - \frac{4}{r^5}\left(\frac{\vec{r}}{r}\cdot\vec{r}\right)\right)$

$$= -2\left(\frac{3}{r^4} - \frac{4}{r^6}\cdot r^2\right)$$

$$= -2\left(\frac{3}{r^4} - \frac{4}{r^4}\right)$$

$$= \frac{2}{r^4}$$

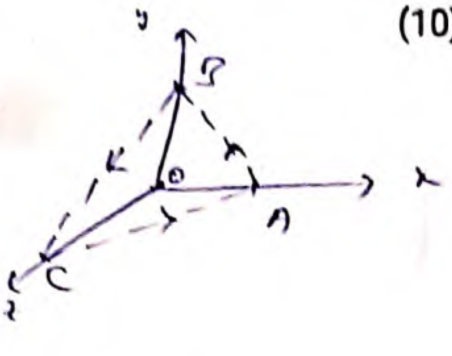
$$= 2r^{-4}$$

Hence proved

8a) Using Stokes' theorem, find the work done in moving a particle once around the perimeter of the triangle with vertices at (2,0,0), (0,3,0) and (0,0,6) under the force field $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$.

(10)

$$W = \int \vec{F}\cdot d\vec{r}$$



and by Stokes' theorem

$$\oint_c \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$$

Now, eqn of plane containing the triangle

i.e. $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

& $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = \hat{i}(1-(-1)) - \hat{j}(0-0) + \hat{k}(2-1) = 2\hat{i}+\hat{k}$

Also $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}}{\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}} = \frac{6(\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{6}\hat{k})}{\sqrt{14}}$

$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$

and if take projection of dS on xy plane

then $dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{14}}}$.

so $W = \iint_{S'} (2\hat{i}+\hat{k})\cdot\frac{(3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{14}}}$

Scanned with CamScanner

$$= \iint_{S'} 7 \, dx\, dy$$

$$= 7 \iint_{S'} dx\, dy$$

↳ area of the triangle OAB

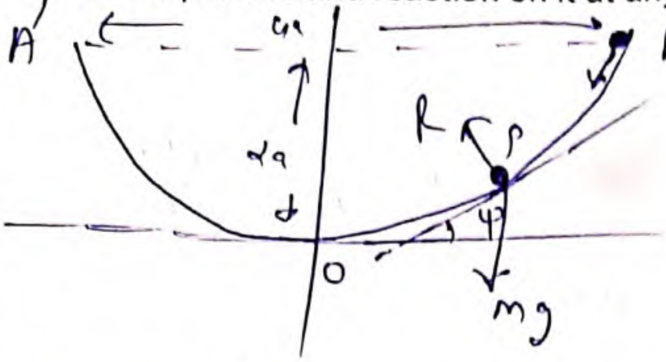
$$= 7 \times \frac{1}{2} \times 2 \times 3$$

$$W = 21 \text{ units}$$

✓ 8

8b) A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, starting from rest at the cusp. Find the velocity of the particle and reaction on it at any point of the cycloid.

(15)



let it start from cusp A and reaches P at any time t

Now at P, forces acting on the particle are mg downward, and Reaction force of cycloid

let s be the arc length OP

φ is the angle made by tangent at P

Then writing eqⁿ of motion,

$$\frac{m d^2 s}{dt^2} = -mg \sin \varphi \quad \rightarrow (1)$$

and

$$\frac{mv^2}{\rho} = R - mg \cos \psi \quad \rightarrow (2)$$

where ρ is the radius of curvature at P

Now, intrinsic eqⁿ of a cycloid is

$$s = 4a \sin \psi \quad \text{(or)} \quad s^2 = 8ay$$

(where 4a is the length AA')

so $\rho = \frac{ds}{d\psi} = 4a \cos \psi$

(at A
$s^2 = 8a \cdot 2a$
$s = 4a$)

so putting these in eqⁿ ① & ① gives

$$\frac{d^2s}{dt^2} = \frac{-gs}{4a}$$

$$\frac{mv^2}{4a\cos\varphi} = R - mg\cos\varphi \rightarrow ③$$

w.k.t ~~$v = \frac{d(\ldots)}{dt} = \frac{ds}{dt}$~~

so $$v\frac{dv}{ds} = \frac{-gs}{4a} \qquad \left(\because \frac{d^2s}{dt^2} = v\frac{dv}{ds}\right)$$

$$\int v\,dv = \int \frac{-gs}{4a}\,ds$$

$$\frac{v^2}{2} = \frac{-gs^2}{8a} + c$$

Now initially $s = 4a$, $v = 0$

so $$0 = \frac{-g \cdot 16a^2}{8a} + c \Rightarrow c = 2ag$$

$$\frac{v^2}{2} = \frac{-gs^2 + 16ga^2}{8a} \Rightarrow v^2 = \frac{g}{4a}(16a^2 - s^2)$$

so $$v = \sqrt{\frac{g}{4a}(16a^2 - s^2)}$$ ⓐ $$v = -\sqrt{\frac{g}{2a}(2a - y)}$$

where s is the arclength of an y is vertical distance of P from vertex.

also $$\frac{m\frac{g}{4a}(16a^2 - s^2)}{4a\cos\varphi} = R - mg\cos\varphi$$

so $$R = mg\left(\frac{(16a^2 - s^2)}{16a^2\cos\varphi} - \cos\varphi\right)$$

is the required reaction at any point of the cycloid.

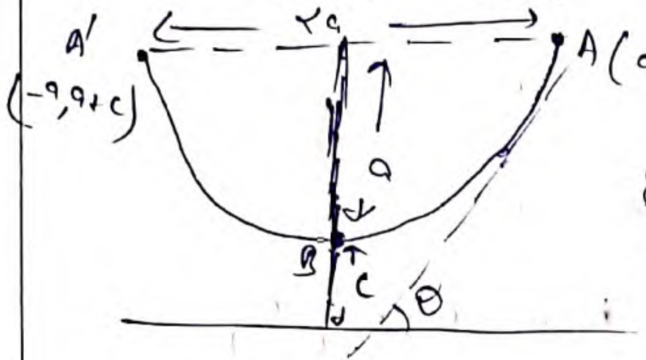
6

Scanned with CamScanner

8c) A heavy uniform chain of length $2l$ is suspended by its ends which are on the same horizantal level. The distance apart $2a$ of the ends is such that its lowest point of the chain is at a distance a vertically below the ends. Prove that if c be the distance of the lowest point from the directrix of the catenary. then

$$\frac{2a^2}{l^2-a^2} = \log\frac{l+a}{l-a} \text{ and } \tanh\frac{a}{c} = \frac{2al}{l^2+a^2}.$$

(15)



let s be the arc length from B (lowest pt of catenary)

then $arc(BA) = l$ ∵ (given length = 2l (arc AA'))

We know that for a catenary

$$y^2 = c^2 + s^2$$

∴ for pt A ⇒ $(a+c)^2 = c^2 + l^2$

$$a^2 + c^2 + 2ac = c^2 + l^2$$

$$a^2 + 2ac = l^2 \Rightarrow c = \frac{l^2 - a^2}{2a} \rightarrow (1)$$

also $y = c\sec\theta \Rightarrow a + c = c\sec\theta$

$$\Rightarrow \sec\theta = \frac{a+c}{c}$$

& $x = c\log(\tan\theta + \sec\theta)$

∴ $a = c\log\left(\sqrt{\left(\frac{a+c}{c}\right)^2 - 1} + \frac{a+c}{c}\right)$

$$= c\log\left(\frac{\sqrt{a^2+2ac} + a+c}{c}\right)$$

∴ from (1)

$$a = \frac{l^2-a^2}{2a}\log\left(\frac{l + a + \frac{l^2-a^2}{2a}}{\frac{l^2-a^2}{2a}}\right)$$

$$\frac{2a^2}{l^2-a^2} = \log\left(\frac{2al+2a^2+l^2-a^2}{l^2-a^2}\right)$$

$$\frac{2a^2}{l^2-a^2} = \log\left(\frac{(a+l)^2}{(l+a)(l-a)}\right)$$

Fin

$$\boxed{\frac{2a^2}{l^2-a^2} = \log\left(\frac{l+a}{l-a}\right)}$$

Hence proved.

Now, for a catenary wkt

$y = c\cosh\left(\frac{x}{c}\right)$, & $s = c\sinh\left(\frac{x}{c}\right)$

so $\frac{s}{y} = \tanh\left(\frac{x}{a}\right)$

Now for point A $(a, a+c)$ $(s = l)$

$$\tanh\left(\frac{a}{c}\right) = \frac{l}{a+c}$$

$$\tanh\left(\frac{a}{c}\right) = \frac{l}{a+\frac{l^2-a^2}{2a}} \quad \left(\because c = \frac{l^2-a^2}{2a}\right) \text{ from (1)}$$

$$\tanh\left(\frac{a}{c}\right) = \frac{l\cdot 2a}{2a^2+l^2-a^2}$$

which gives

$$\boxed{\tanh\left(\frac{a}{c}\right) = \frac{2al}{l^2+a^2}}$$

12

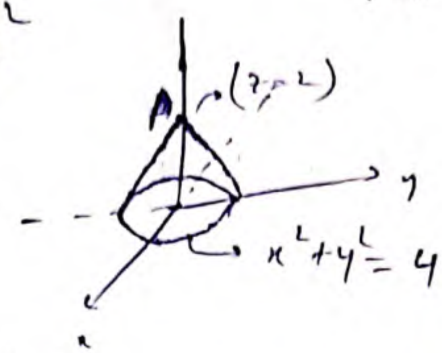
Scanned with CamScanner

8d) Evaluate $\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS$, where $\mathbf{A} = (x - z)\mathbf{i} + (x^3 + yz)\mathbf{j} - 3xy^2\mathbf{k}$ and S is the surface of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ above the xy-plane.

(10)

Conic $\Rightarrow$ $C: x^2+y^2 = (z-2)^2$

$z_{max} = 2$.



Now

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x^3+yz & -3xy^2 \end{vmatrix}$$

$$= \hat{i}(-6xy - y) - \hat{j}(-3y^2+1) + \hat{k}(3x^2-0)$$

$$= -(6xy+y)\hat{i} + (3y^2-1)\hat{j} + 3x^2\hat{k}$$

Also $\hat{n} = \frac{\nabla C}{|\nabla C|} = \frac{2x\hat{i} + 2y\hat{j} - 2(z-2)\hat{k}}{2\sqrt{x^2+y^2+(z-2)^2}}$

$$= \frac{x\hat{i} + y\hat{j} - (z-2)\hat{k}}{\sqrt{x^2+y^2+(z-2)^2}} = \frac{-(z-2)\hat{k} + x\hat{i} + y\hat{j}}{\sqrt{2}\sqrt{(z-2)^2}} \text{ (from cone)}$$

$$= \frac{(2-z)\hat{k} + x\hat{i} + y\hat{j}}{\sqrt{2}(2-z)} = \frac{1}{\sqrt{2}}\left(\frac{x\hat{i}+y\hat{j}}{2-z} + \hat{k}\right)$$

and also if we take projection of cone on the xy axis then $dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{2}}\cdot 1} = \sqrt{2}\,dx\,dy$

so $\iint_{S'} \left(-(6xy+y)\hat{i} + (3y^2-1)\hat{j} + 3x^2\hat{k}\right) \cdot \frac{1}{\sqrt{2}}\left(\frac{x\hat{i}+y\hat{j}}{2-z} + \hat{k}\right)(\sqrt{2}\,dx\,dy)$

$= \iint_{S'} \left(\frac{-(6xy+y)x + (3y^2-1)y}{(2-z)} + 3x^2\right) dx\,dy$ (where S' is the circle $x^2+y^2=4$, $z=0$)

$= \iint \left(\frac{-6x^2y - xy + 3y^3 - y + 6x^2}{2}\right) dx\,dy$

converting to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ | $r: 0$ to 2, $\theta = 0$ to 2π

$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \frac{(-6r^3\sin\theta\cos^2\theta - r^2\sin\theta\cos\theta + 3r^3\sin^3\theta - r\sin\theta + 6r^2\cos^2\theta)}{2} \cdot r\,dr\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\int_0^{2} \left(-6r^4\sin\theta\cos^2\theta - \frac{r^3\sin 2\theta}{2} + 3r^4\sin^3\theta - r^2\sin\theta + 6r^3\cos^2\theta\right) dr\,d\theta$

$= \frac{1}{2}\int_{\theta=0}^{2\pi}\left(\left(-\frac{6\cdot 2^5}{5}\sin\theta\cos^2\theta\right) - \frac{2^4}{2\cdot 4}\sin 2\theta + \frac{3\cdot 2^5}{5}\sin^3\theta - \frac{2^2}{3}\sin\theta + \frac{6\cdot 2^4}{4}\cos^2\theta\right) d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(-\frac{6\cdot 2^5}{5}\sin\theta\cos^2\theta + 24\cos^2\theta\right) d\theta$

$= 12\int_0^{2\pi}\cos^2\theta\,d\theta = 12\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta$

$= 6\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = 6\,[\theta]_0^{2\pi} = 12\pi$

so $\boxed{I = 12\pi}$

8