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Best Coaching for UPSC MATHEMATICS

160/250

TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2022

FULL LENGTH TEST -3 PAPER 1

Time Allowed: Three Hours **Maximum Marks: 250**

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions:

There are **EIGHT** questions divided in **TWO SECTIONS**

Candidate must attempt **FIVE** questions in all.

Question Nos. **1** and **5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from one section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question – cum – Answer (QCA) Booklet in the space provided. No marks will be given for answers written in a medium other than the authorized one.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless and otherwise indicated, symbols and notations carry their usual standard meaning.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question – cum – Answer Booklet must be clearly struck off.

SECTION – A

1. (a) If A, B are square matrices each of order n and I is the corresponding unit matrix, show that the equation AB–BA=I can never hold. [10]

~~(a) (i) $A \neq 0$ is an invertible matrix~~

~~then let $AB - BA = I$ be true~~

~~$\Rightarrow AB = BA + I$~~

~~$AB = BA + A^{-1}A$~~

~~$AB = (B + A^{-1})A$~~

wkt for a square matrices

$\text{trace}(AB) = \text{trace}(BA)$

let $AB - BA = I$ hold.

then taking trace on both side

$\text{trace}(AB - BA) = \text{trace}(I)$

$\text{trace}(AB) - \text{trace}(BA) = n$

$0 = n$

Not possible

so $AB - BA = I$ can never hold

why Trace AB = Trace BA ??

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1. (b) If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices. **[10]**

let A, B be the 2 non-zero square matrices, then

$AB = 0$

1. (c) If f(x) be real value and differentiable on **R** and $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$, then f(x) = tan (x f'(0)). **[10]**

put $x=y=0$

$$f(0)=\frac{2f(0)}{1-f^2(0)} \Rightarrow \text{(i) } f(0)=0$$

(or)

$$1-f^2(0)=2$$

$$f^2(0)=-1$$ (Not possible as f(x) is real valued)

$\therefore$ $\boxed{f(0)=0}$

Now, wkt

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-0}{h}$$

$$f'(0)=\lim_{h\to 0}\frac{f(h)}{h} \longrightarrow (1)$$

Now, also

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(x)+f(h)}{1-f(x)f(h)}-f(x)}{h}$$ (by the given relation)

$$=\lim_{h\to 0}\frac{\cancel{f(x)}+f(h)-\cancel{f(x)}+f^2(x)f(h)}{h(1-f(x)f(h))}$$

$$f'(x) = \lim_{h \to 0} \frac{f(h)\left(1+f^2(x)\right)}{h\left(1-f(x).f(h)\right)}$$

$$= \left(\frac{1+f^2(x)}{(1-0)}\right) . \lim_{h \to 0} \frac{f(h)}{h} \quad (\because f(0)=0)$$

so $f'(x) = (1+f^2(x)) . f'(0) \quad (\because \text{using } ①)$

$$\frac{d f(x)}{dx} = (1+f^2(x)) . f'(0)$$

$$\int \frac{d f(x)}{1+f^2(x)} = \int f'(0) \, dx$$

$$(\tan^{-1} f(x)) = f'(0) . x + c$$

Now at $x=0$, $f(x)=0$

so $0 = 0 + c \Rightarrow c = 0.$

$\therefore$ so $\tan^{-1}(f(x)) = f'(0) . x$

(or) $$\boxed{f(x) = \tan(x . f'(0))}$$

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1. (d) If $z=(x+y)\phi(y/x)$, where ϕ is any arbitrary function.

Prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$. [10]

$$z = (x+y)\phi(y/x)$$

$$= x\left(1+\frac{y}{x}\right)\phi(y/x) = x\,\psi(y/x)$$

where $\psi(y/x) = \left(1+\frac{y}{x}\right)\phi\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x\,\psi(y/x)\right)$$

$$= x\frac{\partial \psi(y/x)}{\partial x} + \psi(y/x)\cdot 1$$

$$= x\,\psi'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) + \psi(y/x)$$

$$= \frac{-y}{x}\psi'\left(\frac{y}{x}\right) + \psi(y/x) \longrightarrow (1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x\,\psi(y/x)\right) = x\,\psi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$= \psi'\left(\frac{y}{x}\right) \longrightarrow (2)$$

using (1), (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{-y}{x}\psi'\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)\right) + y\left(\psi'\left(\frac{y}{x}\right)\right)$$

$$= \cancel{-y\psi'\left(\frac{y}{x}\right)} + x\,\psi\left(\frac{y}{x}\right) + \cancel{y\,\psi'\left(\frac{y}{x}\right)}$$

$$= x\,\psi\left(\frac{y}{x}\right)$$

$$= z$$

H.P.

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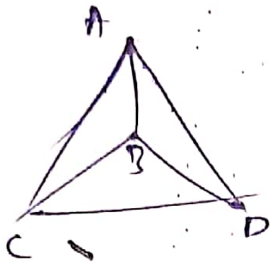
1. (e) Find the equation of the sphere circumscribing the tetrahedron whose faces are

$$\frac{y}{b}+\frac{z}{c}=0, \frac{z}{c}+\frac{x}{a}=0, \frac{x}{a}+\frac{y}{b}=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$ **[10]**



let BCD be the plane

$P: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

BC : intersection of $P, \frac{x}{a}+\frac{y}{b}=0$

CD : —— " —— , $\frac{z}{c}+\frac{x}{a}=0$

DB : —— " —— , $\frac{y}{b}+\frac{z}{c}=0$

So BC: $\frac{x}{a}=\frac{y}{-b}=\frac{z-c}{0}$

CD: $\frac{x}{a}=\frac{y-b}{0}=\frac{z}{-c}$

DB: $\frac{x-a}{0}=\frac{y}{b}=\frac{z}{-c}$

$\Rightarrow$ gives

$B \equiv (a, -b, c)$

$C \equiv (-a, b, c)$

$D \equiv (a, b, -c)$

and A is the intersection of planes

$\frac{y}{b}+\frac{z}{c}=0; \frac{z}{c}+\frac{x}{a}=0, \frac{x}{a}+\frac{y}{b}=0$

which is $A \equiv (0,0,0)$

let the eqn of sphere circumscribing the tetrahedron is

$S: x^2+y^2+z^2+2ux+2vy+2wz+d=0$

it passes through A, B, C, D

satisfying $A(0,0,0) \equiv \boxed{d=0}$

— " — $B(a,-b,c) \Rightarrow a^2+b^2+c^2+2ua-2vb+2wc=0 \rightarrow (1)$

— " — $C(-a,b,c) = a^2+b^2+c^2-2au+2bv+2wc=0 \rightarrow (2)$

— " — $D(a,b,-c) = a^2+b^2+c^2+2au+2bv-2wc=0 \rightarrow (3)$

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from (1) - (2)

$44a - 4vb = 0 \Rightarrow \boxed{v = \frac{ua}{b}} \rightarrow (4)$

ly (3) - (1)

$4wc - 4au = 0 \Rightarrow \boxed{w = \frac{au}{c}} \rightarrow (5)$

puttin v, w in (1)

$a^2+b^2+c^2 + 2au - 2\frac{ua}{b} \cdot b + 2 \cdot \frac{au}{c} \cdot c = 0$

$2au = -(a^2+b^2+c^2)$

$u = -\left(\frac{a^2+b^2+c^2}{2a}\right)$

so $v = -\frac{(a^2+b^2+c^2)}{2b}$

$w = -\left(\frac{a^2+b^2+c^2}{2c}\right)$

so S: $x^2+y^2+z^2 - \left(\frac{a^2+b^2+c^2}{a}\right)x - \frac{(a^2+b^2+c^2)}{b}y - \frac{(a^2+b^2+c^2)}{c}z = 0$

or $\boxed{x^2+y^2+z^2 - (a^2+b^2+c^2)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right) = 0}$

is the required eqn.

2\. (a) Show that the vectors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. **[15]**

(i) over the field of real no..

$V = a(X_1) + bX_2 + cX_3 \qquad a, b, c \in \mathbb{R}$

~~Consider the Matrix~~ $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ ~~row ops using real nos only~~

$$\begin{bmatrix} \cancel{1} & \cancel{1+i} & \cancel{i} \\ \cancel{i} & \cancel{-i} & \cancel{1-i} \\ \cancel{0} & \cancel{1-2i} & \cancel{2-i} \end{bmatrix}$$

$$V = a(1, 1+i, i) + b(i, -i, 1-i) + c(0, 1-2i, 2-i)$$

$$= (a + bi,\ (a+c) + i(a-b-2c),\ (b+2c) + i(a-b-c))$$

Vectors $X_1\ X_2\ X_3$ are LD if for some $a, b, c \in \mathbb{R} - \{0,0,0\}$, $V = 0$

so let $V = 0 \Rightarrow$

$$\left.\begin{aligned} a + bi &= 0 \\ (a+c) + i(a-b-2c) &= 0 \\ b + 2c + i(a-b-c) &= 0 \end{aligned}\right] \rightarrow \begin{aligned} a &= 0 \\ b &= 0 \\ a + c &= 0 \\ a - b - 2c &= 0 \\ b + 2c &= 0 \\ a - b - c &= 0 \end{aligned}$$

gives $a = b = c = 0$.

Hence vectors X_1, X_2, X_3 are linearly independent over field of real nos.

(ii) Over field of complex nos.

Consider the matrix $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

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$$\begin{bmatrix} 1 & 1+i & i \\ i & -i & 1-i \\ 0 & 1-2i & 2-i \end{bmatrix} \xrightarrow{R_2 \to R_2 - iR_1} \begin{bmatrix} 1 & 1+i & i \\ 0 & -2i+1 & 1-i+1 \\ 0 & 1-2i & 2-i \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1+i & i \\ 1 & 1-2i & 2-i \\ 0 & 0 & 0 \end{bmatrix}$$

rank < 3 so the vectors are linearly dependent over the field of complex nos.

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2. (b) Show that the transformation $T(ax^2 + bx + c) = 2ax + b$ of $P_2 \to P_1$ is linear. Find the image of $3x^2 - 2x + 1$. Determine another element of P_2 that has the same image. **[17]**

$$T(ax^2 + bx + c) = 2ax + b$$

~~let $X_1 : ax_1^2 + bx_1 + c \Rightarrow T(X_1) = 2ax_1 + b$~~

~~$X_2 : ax_2^2 + bx_2 + c \Rightarrow T(X_2) = 2ax_2 + b$~~

Now ~~T is linear if~~

~~$T(lX_1 + mX_2) = lT(X_1) + mT(X_2)$~~

~~so $T(lX_1 + mX_2) = T(l(ax_1^2 + bx_1 + c) + m(ax_2^2 + bx_2 + c))$~~

~~$= T(\ldots)(lx_1^2 + mx_2^2)$~~

let $V_1 = a_1x^2 + b_1x + c_1 \Rightarrow T(V_1) = 2a_1x + b_1$

$V_2 = a_2x^2 + b_2x + c_2 \Rightarrow T(V_2) = 2a_2x + b_2$

Now T is linear if

$$T(lV_1 + mV_2) = lT(V_1) + mT(V_2)$$

Now $T(lV_1 + mV_2) = T(l(a_1x^2 + b_1x + c_1) + m(a_2x^2 + b_2x + c_2))$

$$= T((la_1 + ma_2)x^2 + (lb_1 + mb_2)x + (lc_1 + mc_2))$$

$$= 2(la_1 + ma_2)x + lb_1 + mb_2$$

$$= 2la_1x + 2ma_2x + lb_1 + mb_2$$

$$= l(2a_1x + b_1) + m(2a_2x + b_2)$$

$$= lT(V_1) + mT(V_2)$$

Hence T is linear.

$T(3x^2-2x+1)$ Here $a=3$
$b=-2$
$c=1$

$$T(3x^2-2x+1) = 2\times 3\,x + (-2)$$
$$= 6x-2$$

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All the eq^n of the form $3x^2-2x+c'$ where $c' \in \mathbb{R}$ has the same image $= 6x-2$

2. (c) Let P_n denote the vector space of all real polynomials of degree atmost n and $T : P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)dt$, $p(x) \in P_2$. Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. **[18]**

$$T(p(x)) = \int_0^x p(t)dt$$

$$\text{base}(P_2) = \{1, x, x^2\}$$

$$\text{base}(P_3) = \{1, x, 1+x^2, 1+x^3\}$$

Let us

$$T(1) = \int_0^x 1\, dx$$

$$= x$$

$$T(1) = 0\cdot 1 + 1\cdot x + 0\cdot(1+x^2) + 0\cdot(1+x^3) \to (1)$$

$$T(x) = \int_0^x x\, dx$$

$$= \frac{x^2}{2}$$

$$= \left(\frac{-1}{2}\right)\cdot 1 + 0\cdot x + \frac{1}{2}\cdot(1+x^2) + 0\cdot(1+x^3) \to (2)$$

$$T(x^2) = \int_0^x x^2\, dx$$

$$= \frac{x^3}{3}$$

$$= \left(\frac{-1}{3}\right)\cdot 1 + 0\cdot x + 0\cdot(1+x^2) + \frac{1}{3}(1+x^3) \to (3)$$

$$\therefore M_{P_2 \to P_3} = \begin{bmatrix} 0 & -1/2 & -1/3 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

is the required matrix.

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For nullspace

$$T(p(x)) = 0$$

$$\int_0^x p(t)\,dt = 0$$

possible, only when $p(t) = 0$

ie $p(x) = 0$

so

$$\boxed{N(T) = \{0\}}$$

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SECTION – B

5. (a) Find the orthogonal trajectories of family of curves $r^2 = a^2 \cos 2\theta$ **[10]**

diff wrt θ

$$2r\frac{dr}{d\theta} = -(a^2 \sin 2\theta)\cdot 2$$

$$r\frac{dr}{d\theta} = -a^2 \sin 2\theta$$

from the original eqn $a^2 = \frac{r^2}{\cos 2\theta}$

so $$r\frac{dr}{d\theta} = -\frac{r^2}{\cos 2\theta} \sin 2\theta$$

$$\frac{dr}{d\theta} = -r\tan 2\theta$$

Now for orthogonal trajectories, replace

$$\frac{dr}{d\theta} \xrightarrow{\text{by}} -r^2\frac{dr}{d\theta}$$

$\Rightarrow$ $-r^2\frac{dr}{d\theta} = -r\tan 2\theta$

$-r\,dr = -\tan 2\theta\, d\theta$

$\int r\,dr = \int \tan 2\theta\, d\theta$

$\frac{r^2}{2} = \frac{1}{2}\ln|\sec 2\theta| + c'$

$$\boxed{r^2 = \ln(\sec 2\theta) + c}$$

is the required eqn, where c is any arbitrary constant.

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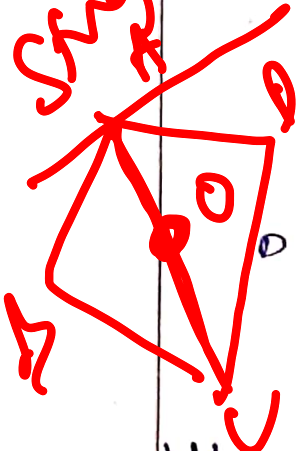
$-T\delta(AC) + W\delta(AB) = 0$

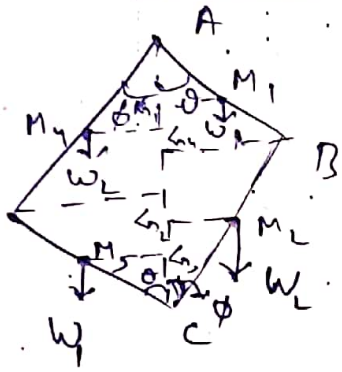
C.G must pass thru AC

Just write this

String AC

5. (b) Four uniform rods are freely jointed at their extremities and form a parallelogram *ABCD*, which is suspended by the joint *A*, and is kept in shape by a string *AC*. Prove that the tension of the string is equal to half the weight of all the four rods. **[10]**





Let $AB = DC = 2a \; (W_1)$

& $BC = AD = 2b \; (W_2)$

Let $L_{n_1}, L_{n_2}, L_{n_3}, L_{n_4}$ be the projection of M_1, M_2, M_3, M_4 on the line AC respectively.

Then

$$AL_{n_1} = AM_1 \cos\theta = a\cos\theta.$$

$$AL_{n_2} = AB\cos\theta + BC\cos\phi - CM_2\cos\phi$$

$$= 2a\cos\theta + 2b\cos\phi - b\cos\phi$$

$$= 2a\cos\theta + b\cos\phi$$

$$AL_{n_3} = AB\cos\theta + BC\cos\phi - CM_3\cos\theta$$

$$= a\cos\theta + 2b\cos\phi.$$

$$\boxed{\begin{aligned} AC &= AB\cos\theta + BC\cos\phi \\ &= 2a\cos\theta + 2b\cos\phi \end{aligned}}$$

$$AL_{n_4} = AM_4\cos\phi = a\cos\phi$$

Now if we virtually displace the system by small angle, then

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Virtual Work done $= 0$ | String's tension $= -T\delta AC$, VW

$\therefore \omega_1(\delta A \angle n_1) + \omega_2\, \delta(A\angle n_2) + \omega_3\, \delta(A\angle n_3) + \omega_4\, \delta(A\angle n_4)$
$- T\,\delta(AC) = 0$

$\omega_1(-a\sin\theta\, d\theta) + \omega_2(-2a\sin\theta\, d\theta - b\cos\phi\, d\phi) + \omega_3(-a\sin\phi\, d\phi)$
$+\omega_4(-a\sin\theta d\theta - 2b\sin\phi d\phi) + T(2a\sin\theta d\theta + 2b\sin\phi d\phi) = 0$

$2T(a\sin\theta\, d\theta + b\sin\phi\, d\phi) = (\omega_1 + \omega_2)(2)(a\sin\theta\, d\theta + b\sin\phi d\phi)$

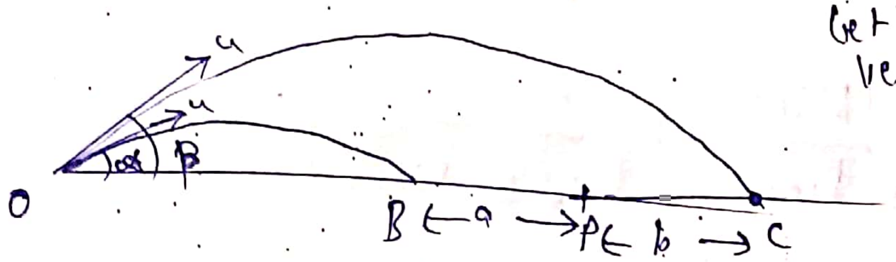
$T = \omega_1 + \omega_2 = \frac{1}{2}(2\omega_1 + 2\omega_2) = \frac{1}{2}(\text{Total wt})$

$$\boxed{T = \frac{1}{2}W}$$

H.P

5. (c) A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is $\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}$ **[10]**

Let the initial velocity $= u$



let $OP = R$, then $OB = R - a$
$OC = R + b$

Now $R - a = \frac{u^2 \sin 2\alpha}{g} \rightarrow (1)$

& $R + b = \frac{u^2 \sin 2\beta}{g} \rightarrow (2)$

Now, ~~for~~ let elevation be θ for reaching P

then $R = \dfrac{u^2 \sin 2\theta}{g} \rightarrow (3)$

divide (1)/(2)

$$\frac{R-a}{R+b} = \frac{\sin 2\alpha}{\sin 2\beta}$$

$$R\sin 2\beta - a\sin 2\beta = R\sin 2\alpha + b\sin 2\alpha$$

$$R = \frac{a\sin 2\beta + b\sin 2\alpha}{a\sin 2\beta - \sin 2\alpha}$$

also (1) - (2)

$$b+a = \frac{u^2}{g}(\sin 2\beta - \sin 2\alpha)$$

$$\text{so } \frac{u^2}{g} = \frac{b+a}{(\sin 2\beta - \sin 2\alpha)}$$

Now, putting value of R & $\frac{u^2}{g}$ in (3)

$$\frac{a\sin 2\beta + b\sin 2\alpha}{\cancel{\sin 2\beta - \sin 2\alpha}} = \left(\frac{(b+a)}{\cancel{\sin 2\beta - \sin 2\alpha}}\right) \cdot \sin 2\theta$$

$$\sin 2\theta = \frac{a\sin 2\beta + b\sin 2\alpha}{a+b}$$

$$2\theta = \sin^{-1}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right)$$

$$\boxed{\theta = \frac{1}{2}\sin^{-1}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right)}$$

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5. (d) Verify Stoke's theorem for $\mathbf{F} = -y^3\,\mathbf{i} + x^3\mathbf{j}$, where S is the circular disc $x^2 + y^2 \le 1$, $z = 0$. **[10]**

Stokes' Thm,

$$\iint_S (\nabla\times F)\cdot\hat{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$$

LHS:

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

$$= 3(x^2+y^2)\,\hat{k}$$

and $\hat{n} = \hat{k}$ $(\because z=0)$

so $(\nabla\times\vec{F})\cdot\hat{n} = 3(x^2+y^2)$

so $\iint_S (\nabla\times F)\cdot\hat{n}\, ds = \iint_S 3(x^2+y^2)\, dx\, dy$

Now let $x = r\cos\theta$, $y = r\sin\theta \Rightarrow dx\,dy = r\,dr\,d\theta$

where r: 0 to 1

θ: 0 to 2π

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2 \cdot r\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = 3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{3}{4}\cdot\int_0^{2\pi} d\theta$$

$$= \frac{3\pi}{2}$$

RHS: $\int \vec{F}\cdot d\vec{r} = \int_C (-y^3\hat{i} + x^3\hat{j})\cdot(dx\hat{i} + dy\hat{j})$ $(\hat{k}=0)$

$$= \int_C -y^3\, dx + x^3\, dy$$

Here $C: x^2+y^2=1 \Rightarrow$ (let) $x = \cos\theta$
$y = \sin\theta$
$\theta : 0$ to 2π

$$= \int_0^{2\pi} -\sin^3\theta \cdot -\sin\theta\, d\theta + \cos^3\theta \cdot \cos\theta\, d\theta$$

$$= \int_0^{2\pi}(\sin^4\theta + \cos^4\theta)\, d\theta = 4\int_0^{\pi/2}(\sin^4\theta + \cos^4\theta)\, d\theta$$

$$= 4\left(\frac{1}{2}\cdot\frac{\Gamma(5/2)\,\Gamma(1/2)}{\Gamma(3)} + \frac{1}{2}\,\frac{\Gamma(5/2)\,\Gamma(1/2)}{\Gamma(3)}\right)$$

$$= \frac{4\,\Gamma(5/2)\,\Gamma(1/2)}{\Gamma(3)} = \frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 1}$$

$$= \frac{3\pi}{2}$$

ie LHS = RHS

Hence Stokes' thm verified.

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5. (e) Given the space curve $x=t,\ y=t^2,\ z=\frac{2}{3}t^3$, find (*i*) the curvature κ. (*ii*) the torsion τ. **[10]**

$$x = t,\quad y = t^2;\quad z = \frac{2}{3}t^3$$

$$\vec{r} = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 2t^2\hat{k}$$

$$\frac{d^2r}{dt^2} = 2\hat{j} + 4t\hat{k}$$

$$\frac{d^3r}{dt^3} = 4\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2+4t^4}$$

$$= \sqrt{(2t^2+1)^2}$$

$$= (2t^2+1)$$

$$\left(\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = \hat{i}(8t^2 - 4t^2) - \hat{j}(4t-0) + \hat{k}(2-0)$$

$$= 4t^2\hat{i} + 4t\hat{j} + 2\hat{k}$$

$$= 2(2t^2\hat{i} + 2t\hat{j} + \hat{k})$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2r}{dt^2}\right| = 2\left(\sqrt{4t^4+4t^2+1}\right.$$

$$= 2\sqrt{(2t^2+1)^2}$$

$$= 2(2t^2+1).$$

$$\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] = \begin{vmatrix} 1 & 2t & 2t^2 \\ 0 & 2 & 4t \\ 0 & 0 & 4 \end{vmatrix} = 4\times 2 = \underline{8}.$$

so $\kappa = \dfrac{\left|\dfrac{dr}{dt} \times \dfrac{d^2r}{dt^2}\right|}{\left|\dfrac{dr}{dt}\right|^3} = \dfrac{2(2t^2+1)}{(2t^2+1)^3} = \dfrac{2}{(2t^2+1)^2}$

$\tau = \dfrac{\left[\dfrac{dr}{dt}, \dfrac{d^2r}{dt^2}, \dfrac{d^3r}{dt^3}\right]}{\left|\dfrac{dr}{dt} \times \dfrac{d^2r}{dt^2}\right|^2} = \dfrac{8}{4(2t^2+1)^2} = \dfrac{2}{(2t^2+1)^2}$

so $\boxed{\kappa = \tau = \dfrac{2}{(2t^2+1)^2}}$

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6. (a) Solve $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\log x \sin \log x + 1}{x}$ [15]

$x^2y'' - 3xy' + y = \dfrac{\log x \sin \log x + 1}{x}$

$(x^2D^2 - 3xD + 1)y$

Let $x = e^z$

then $x^2D^2 = D'(D'-1)$ $\boxed{D' = \dfrac{d}{dz}}$

$xD = D'$

$\therefore \left(D'(D'-1) - 3D' + 1\right)y = \dfrac{z \cdot \sin z + 1}{e^z}$

$(D'^2 - D' - 3D' + 1)y = \dfrac{z\sin z + 1}{e^z}$

$(D'^2 - 4D' + 1)y = \dfrac{z\sin z + 1}{e^z}$

for HE: $m^2 - 4m + 1 = 0 \Rightarrow m = 2 \pm \sqrt{3}$

$y_c = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z}$

for P.S: $y_p = \dfrac{(z\sin z + 1) \cdot e^{-z}}{(D'^2 - 4D' + 1)}$

$= e^{z} \dfrac{1}{(D'-1)^2 - 4(D'-1) + 1} (z\sin z + 1)$

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$$= e^{-t} \frac{1}{(D'^2 - 2D' + 1 - 4D' + 4 + 1)} (t \sin t + 1)$$

$$= e^{-t} \frac{1}{(D'^2 - 6D' + 6)} (t \sin t + 1) \rightarrow (1)$$

Now

$$\frac{t \sin t}{D'^2 - 6D' + 6} = t \cdot \frac{1}{(D'^2 - 6D' + 6)} \sin t - \frac{(2D' - 6)}{(D'^2 - 6D' + 6)^2} \sin t$$

$$= t \cdot \frac{\sin t}{(5 - 6D')} - 2\left(\frac{\cos t - 7\sin t}{(5 - 6D')^2}\right)$$

$$= \frac{t(5 + 6D')\sin t}{25 - 36D'^2} - 2\frac{(\cos t - 7\sin t)}{25 + 36D'^2 - 60D'}$$

$$= \frac{t(5\sin t + 6\cos t)}{25 + 36} + 2\frac{(\cos t - 7\sin t)}{(60D' + 11)}$$

$$= \frac{t(5\sin t + 6\cos t)}{61} + 2\frac{(60D' - 11)(\cos t - 7\sin t)}{-3600 - 121}$$

$$= \frac{t(5\sin t + 6\cos t)}{61} - 2\frac{(-60\sin t - 11\cos t - 160\cos t + 77\sin t)}{3721}$$

$$= \frac{1}{61}\left(t(5\sin t + 6\cos t) + \frac{2}{61}(191\cos t + 27\sin t)\right)$$

$$\frac{1}{D'^2 - 6D' + 6} = \frac{1}{6}\left(1 - \left(D' - \frac{D'^2}{6}\right)\right)^{-1} \cdot 1 = \frac{1}{6}$$

so $y_p = e^{-t}\left(\frac{1}{61}\left((t)(5\sin t + 6\cos t) + \frac{2}{61}(191\cos t + 27\sin t)\right) + \frac{1}{6}\right)$

where $t = \log x$

so $y = y_c + y_p$

$$= c_1 x^{2+\sqrt{5}} + c_2 x^{2-\sqrt{5}} + \frac{1}{61} \cdot \frac{\log x(5\sin\log x + 6\cos\log x)}{x}$$

$$+ \frac{2}{61 \times 61}\left(\frac{191\cos\log x + 27\sin\log x}{x}\right) + \frac{1}{6x}$$

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6. (b) Investigate $(p^2 + 1)(x - y)^2 = (x + yp)^2$ for signular solution and extraneous loci. [15]

$$(p^2+1)(x-y)^2 = (x+yp)^2$$

6. (c) Prove that $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$. **[07]**

$$L\left\{\frac{\cos at}{t} - \frac{\cos bt}{t}\right\} = L\left\{\frac{\cos at}{t}\right\} - L\left\{\frac{\cos bt}{t}\right\}$$

$$\lim_{t\to 0} \frac{\cos at - \cos bt}{t} = \lim_{t\to 0} \frac{b\sin bt - a\sin at}{1} = 0$$

(exists)

$$L\left\{\frac{1}{t} F(t)\right\} = \int_p^\infty f(x)\, dx$$

where $f(x) = L\{\cos at - \cos bt\}$

$$= \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2} = \frac{x}{x^2+a^2} - \frac{x}{x^2+b^2}$$

$$= x\left(\frac{x^2+b^2-x^2-a^2}{(x^2+a^2)(x^2+b^2)}\right) = \frac{x(b^2-a^2)}{(x^2+a^2)(x^2+b^2)}$$

$$L\left\{\frac{1}{t}(\cos at - \cos bt)\right\} = \int_p^\infty \frac{x(b^2-a^2)}{(x^2+a^2)(x^2+b^2)}\, dx$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$= \frac{1}{2}\int_{p^2}^\infty \frac{(b^2-a^2)}{(t+a^2)(t+b^2)}\, dt$$

$$= \frac{1}{2}\int_{p^2}^\infty \left(\frac{1}{t+a^2} - \frac{1}{t+b^2}\right) dt$$

$$= \frac{1}{2}\left[\log(t+a^2) - \log(t+b^2)\right]_{p^2}^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{t+a^2}{t+b^2}\right)\right]_{p^2}^{\infty}$$

$$= \frac{1}{2}\left[\lim_{t\to\infty} \log \frac{t\left(1+\frac{a^2}{t^2}\right)}{t\left(1+\frac{b^2}{t^2}\right)} - \log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\frac{p^2+b^2}{p^2+a^2}$$

(or) $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

H.P

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6. (d) By using Laplace transform, solve $(D^3 - D^2 - D + 1) y = 8te^{-t}$ if $y = D^2 y = 0$; $Dy = 0$ when $t = 0$. **[13]**

let $L\{y\} = k$

then $L\{y'\} = p\,L\{y\} - y(0)$

$= pk \rightarrow (1)$

$L\{y''\} = p\,L\{y'\} - y'(0)$

$= p(pk) - 0$

$= p^2 k \rightarrow (2)$

$L\{y'''\} = p\,L\{y''\} - y''(0)$

$= p(p^2 k) - 0$

$= p^3 k \rightarrow (3)$

given ODE $(D^3 - D^2 - D + 1)\, y = 8te^{-t}$

taking Laplace both sides

$L\{(D^3 - D^2 - D + 1) y\} = 8\,L\{te^{-t}\} = 8(-1)\dfrac{d\,L\{e^{-t}\}}{dp}$

$(p^3 k - p^2 k - pk + k) = -8\dfrac{d}{dp}\left(\dfrac{1}{p+1}\right)$

$k(p^3 - p^2 - p + 1) = \dfrac{8}{(p+1)^2}$

$L\{y\} = \dfrac{8}{(p+1)^2(p^3 - p^2 - p + 1)} = \dfrac{8}{(p+1)^2(p^2(p-1) - 1(p-1))}$

$= \dfrac{8}{(p+1)^2(p^2-1)(p-1)} = \dfrac{8}{(p+1)^2(p+1)(p-1)^2} = \dfrac{8}{(p+1)^3(p-1)^2}$

$$L\{y\} = \frac{1}{(P+1)^3} + \frac{2}{(P+1)^2} + \frac{1}{(P-1)^2} - \frac{3}{2}\left(\frac{1}{P-1} - \frac{1}{P+1}\right)$$

taking Laplace inverse on both sides

$$y = L^{-1}\left\{\frac{1}{(P+1)^3}\right\} + L^{-1}\left\{\frac{2}{(P+1)^2}\right\} + L^{-1}\left\{\frac{1}{(P-1)^2}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{P-1} - \frac{1}{P+1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{(-1)^2 \frac{d^2 \frac{1}{(P+1)}}{dP}\right\} + 2L^{-1}\left\{(-1)^1 \frac{d\left(\frac{1}{P+1}\right)}{dP}\right\} + L^{-1}\left\{(-1)^1 \frac{d\frac{1}{(P-1)}}{dP}\right\} - \frac{3}{2}\left(e^{t} - e^{-t}\right)$$

$$\boxed{y = \frac{1}{2}t^2 e^{-t} + 2te^{-t} + te^{t} - \frac{3}{2}e^{t} + \frac{3}{2}e^{-t}}$$

is the required sol^n

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8. (a) Solve $(D^2 + D)\,y = t^2 + 2t$ where $y(0) = 4$, $y'(0) = -2$ by using Laplace transformation. [08]

let $L\{y\} = k$

then $L\{Dy\} = pL\{y\} - y(0)$

$= pk - 4$

" $L\{D^2y\} = pL\{Dy\} - y'(0)$

$= p(pk - 4) + 2$

$= p^2k - 4p + 2$

takin Laplace of ODE

$$L\{D^2y\} + L\{Dy\} = L\{t^2 + 2t\}$$

$$p^2k - 4p + 2 + pk - 4 = \frac{2!}{p^{2+1}} + \frac{2 \cdot 1!}{p^{1+1}}$$

$$k(p^2 + p) - 4p - 2 = \frac{2}{p^3} + \frac{2}{p^2}$$

$$k = 2\left(\frac{1}{p^3(p^2+p)} + \frac{1}{p^2(p^2+p)} + \frac{1}{p^2+p} + \frac{2p}{p^2+p}\right)$$

$$= 2\left(\frac{1}{p^4(p+1)} + \frac{1}{p^3(p+1)} + \frac{1}{p(p+1)} + \frac{2}{p+1}\right)$$

$$= 2\left(\frac{1}{p^4(p+1)}(1+p) + \left(\frac{1}{p} - \frac{1}{p+1}\right) + \frac{2}{p+1}\right)$$

$$= 2\left(\frac{1}{p^4} + \frac{1}{p} + \frac{1}{p+1}\right)$$

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taking inverse laplace on both sides

$$L^{-1}\{L\{y\}\} = 2L^{-1}\left\{\frac{1}{p^4} + \frac{1}{p} + \frac{1}{p+1}\right\}$$

$$y = 2\left(\frac{1}{3!}t^3 + 1 + e^{-t}\right)$$

$$\boxed{y = t^3/3 + 2 + 2e^{-t}}$$ is the required sol^n

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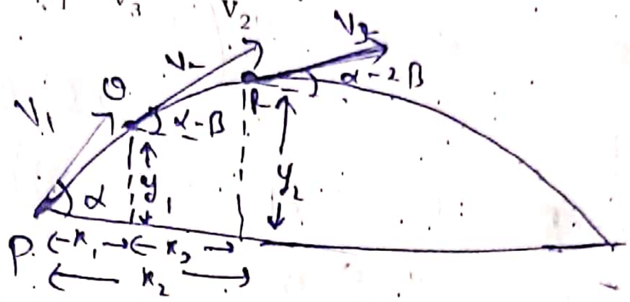
8. (b) If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are α, $\alpha - \beta$, $\alpha - 2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that $v_3 t_1 = v_1 t_2$

and $\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}$ **[12]**



let us take P as the initial point

time (PO) $= t_1$

time (OR) $= t_2$

In horizontal direction there is no force so no horizontal velocity equal

$$v_1 \cos\alpha = v_2 \cos(\alpha-\beta) = v_3 \cos(\alpha-2\beta) \rightarrow (1)$$

and gravity in vertically downwards, so

$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - g t_1 \rightarrow (2)$$

& $$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - g t_2 \rightarrow (3)$$

~~so $v_3 \sin(\alpha-2\beta) = v_1 \sin\alpha - g(t_1+t_2)$~~

~~as in (1) $\frac{v_1 \cos\alpha}{\cos(\alpha-\beta)} \sin(\alpha-\beta) = v_1 \sin\alpha - g t_1$~~

from (2) $$g = \left(v_1 \sin\alpha - v_2 \sin(\alpha-\beta)\right)/t_1$$

from (3) $$g = \left(v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)\right)/t_2$$

equate $\dfrac{V_1 \sin\alpha - V_2 \sin(\alpha-\beta)}{t_1} = \dfrac{V_2 \sin(\alpha-\beta) - V_3 \sin(\alpha-2\beta)}{t_2}$

using eqⁿ (1) replace V_2

$$t_2\left(V_1 \sin\alpha - \frac{V_1 \cos\alpha \sin(\alpha-\beta)}{\cos(\alpha-\beta)}\right) = t_1\left(\frac{V_3 \cos(\alpha-2\beta)\sin(\alpha-\beta)}{\cos(\alpha-\beta)} - V_3 \sin(\alpha-2\beta)\right)$$

$$t_2 V_1\left(\frac{\sin\alpha\cos(\alpha-\beta) - \cos\alpha \sin(\alpha-\beta)}{\cancel{\cos(\alpha-\beta)}}\right) = t_1 V_3\left(\frac{\cos(\alpha-2\beta)\sin(\alpha-\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)}{\cancel{\cos(\alpha-\beta)}}\right)$$

$$t_2 V_1 \sin(\alpha-(\alpha-\beta)) = t_1 V_3 \sin((\alpha-\beta)-(\alpha-2\beta))$$

$$t_2 V_1 \sin\beta = t_1 V_3 \sin\beta$$

$$\Rightarrow \boxed{t_2 V_1 = t_1 V_3}$$

Now by (1)

$$V_1 = \frac{V_2 \cos(\alpha-\beta)}{\cos\alpha} \Rightarrow \frac{1}{V_1} = \frac{1}{V_2}\frac{\cos\alpha}{\cos(\alpha-\beta)} \rightarrow (4)$$

$$V_3 = \frac{V_2 \cos(\alpha-\beta)}{\cos(\alpha-2\beta)} \Rightarrow \frac{1}{V_3} = \frac{1}{V_2}\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)} \rightarrow (5)$$

add (4), (5)

$$\frac{1}{V_1} + \frac{1}{V_3} = \frac{1}{V_2}\left(\frac{\cos\alpha + \cos(\alpha-2\beta)}{\cos(\alpha-\beta)}\right)$$

$$\frac{1}{V_1} + \frac{1}{V_3} = \frac{1}{V_2}\left(\frac{2\cos\left(\frac{\alpha+\alpha-2\beta}{2}\right)\cos\left(\frac{\alpha-(\alpha-2\beta)}{2}\right)}{\cos(\alpha-\beta)}\right)$$

$$\frac{1}{V_1} + \frac{1}{V_3} = \frac{1}{V_2}\cdot 2\,\frac{\cancel{\cos(\alpha-\beta)}\cdot\cos\beta}{\cancel{\cos(\alpha-\beta)}}$$

him $\boxed{\dfrac{1}{V_1} + \dfrac{1}{V_3} = \dfrac{2\cos\beta}{V_2}}$

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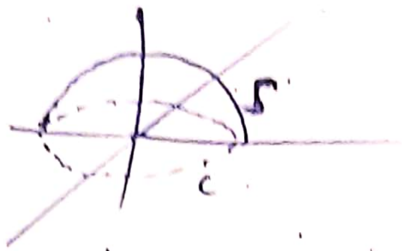
8. (c) (i) Derive an expression for $\nabla\Phi$ in orthogonal curvilinear coordinates.

Express (ii) $\nabla \times A$ and (iii) $\nabla^2\psi$ in spherical coordinates.

[6+6+6=18]

8. (d) If $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane. **[12]**



let C be the Circle $x^2+y^2=a^2$, $z=0$ in the xy plane

let $\Gamma = C + S$

then w.k.t by Gauss divergence then

$$\iint_\Gamma (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iiint \cancel{\nabla \cdot (\nabla \times \vec{F})}^{0}\, dv$$

(or) $$\iint_c (\nabla \times \vec{F}) \cdot \hat{n}\, ds + \iint_s (\nabla \times \vec{F}) \cdot \hat{n}\, ds = 0$$

so $$\iint_s (\nabla \times \vec{F}) \cdot \hat{n} \cdot ds = -\iint_c (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

here $\hat{n} = \hat{k}$, $ds = dx\,dy$

& $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix} = \hat{i}(-x+2x) - \hat{j}(-y) + \hat{k}(1-2z-1)$$

$$= x\hat{i} + y\hat{j} - 2z\hat{k}$$

$\therefore \iint_s (\nabla \times \vec{F}) \cdot \hat{n}\, ds$ (or) I $= -\iint_c (x\hat{i} + y\hat{j} - 2z\hat{k}) \cdot \hat{k}\; dx\,dy$

$= -\iint_c -2z\; dx\,dy$

$= 0$ $\quad (\because z = 0$ in xy plane$)$

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