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Analytic Functions

15.0 PRELIMINARIES

A complex number z is of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ is called the imaginary unit.

x is called the real part of z and is denoted as Re z.

y is called the imaginary part of z and is denoted as Im z.

Thus, $x = \text{Re } z$, $y = \text{Im } z$.

1. Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, written as $z_1 = z_2$, if and only if $x_1 = x_2$ and $y_1 = y_2$.

Note Given two complex numbers z_1 and z_2, we can only say $z_1 = z_2$ or $z_1 \neq z_2$. We cannot say $z_1 < z_2$ because there is no order relation in the field of complex numbers as in the field of real numbers.

The set of all complex numbers is denoted by C.

2. **Complex conjugate**

   If $z = x + iy$ is any complex number, then its conjugate $\bar{z} = x - iy$

   We can easily prove the following properties:

   1. $z = \bar{z}$ if and only if z is real
   2. $\bar{\bar{z}} = z$
   3. $z + \bar{z} = 2\,\text{Re}\,z$
   4. $z - \bar{z} = 2i\,\text{Im}\,z$
   5. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
   6. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
   7. $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$
   8. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

3. **Modulus of a complex number**

   If $z = x + iy$ is a complex number, then its modulus is $|z| = \sqrt{x^2 + y^2}$.

   $|z|$ is a non-negative real number.

   1. $z\bar{z} = |z|^2$, $|\bar{z}| = |z|$
   2. $|z_1 z_2| = |z_1||z_2|$
   3. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
   4. $|z| \geq |\text{Re}\,z| \geq \text{Re}\,z$
   5. $|z| \geq |\text{Im}\,z| \geq \text{Im}\,z$
   6. $|z_1 + z_2| \leq |z_1| + |z_2|$. This is called triangle inequality.

4. **Geometric representation of complex numbers**

   Any complex number $a + ib$ can be represented by a point P (a, b) in the xy-plane w.r.to rectangular coordinate axes.

   Any number of the form $a + i0 = a$ is a real number and it is represented by the point $(a, 0)$ which lies on the x-axis. So, the x-axis is called the real axis.

   Any number of the form $0 + ib = ib$ is purely imaginary and it is represented by the point $(0, b)$ which lies on the y-axis. So, the y-axis is called the imaginary axis. Origin represents the complex number $0 + i0 = 0$.

The plane in which points represent complex numbers is called the complex plane or Argand plane or Argand diagram.

5. **Vector Form**

If P represents a complex number z in the Argand diagram then $\overrightarrow{OP} = z$. We refer to z as the point z or vector z. If P and Q represent the complex numbers z_1 and z_2 in the Argand diagram, then $\overrightarrow{OP} = z_1$ and $\overrightarrow{OQ} = z_2$.

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = z_2 - z_1.$$

$$\therefore \quad PQ = |z_2 - z_1|$$

So, the distance between the points z_1 and z_2 is $|z_2 - z_1|$.

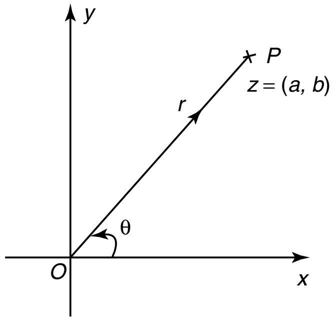


Fig. 15.1

6. **Polar form of complex number**

Let P represent the complex number $z = a + ib$ in the Argand diagram. Then P is (a, b)

If $OP = r$, $X\hat{O}P = \theta$ (as in Fig. 4.1) then (r, θ) are the polar coordinates of P and $a = r\cos\theta$, $b = r\sin\theta$.

$$\therefore \quad r = \sqrt{a^2 + b^2} = |z|$$

So, r is the modulus of the complex number z and θ is called the argument or amplitude of z and θ is measured in radians. θ is given by $\tan\theta = \frac{b}{a}$.

The principal value of argument of z is the value of θ satisfying $-\pi < \theta \le \pi$

The principal value of $\theta = \tan^{-1}\frac{b}{a}$ if $a > 0$. That is, θ is in the 1st or 4th quadrant.

$= \tan^{-1}\frac{b}{a} + \pi$, if $a < 0$, $b > 0$. That is, θ is in the 2nd quadrant.

$= \tan^{-1}\frac{b}{a} - \pi$ if $a < 0$, $b < 0$. That is, θ is in the 3rd quadrant.

$z = a + ib$ can be written in polar form as $z = r(\cos\theta + i\sin\theta)$

Euler's formula: For any real θ, $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore$ any complex number z can be written as $z = re^{i\theta}$ and it is called the exponential form of z.

15.1 FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a complex variable.

Definition 15.1 Let S be a set of complex numbers. A function f from S to C is a rule that assigns to each z in S a unique complex number w in C (as in Fig. 4.2)

The number w is called the value of f at z and is denoted by $f(z)$.

Thus, $w = f(z)$. S is called the domain of the function f and f is called a complex valued function of a complex variable. Such a function is simply referred to as **"function of a complex variable"**.

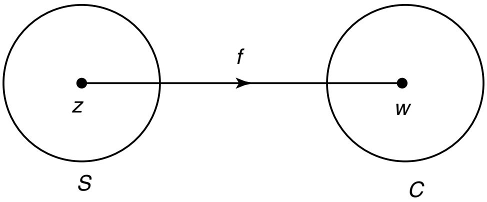


Fig. 15.2

The set of all values of f is called the **range of** f.

Note

1. If A is a set of real numbers, then a function f from A into C is called a complex valued function of real variable.
2. We also have real-valued function of a complex variable.
   (e.g.) If $z = x + iy$, x and y real variables, then
   $f(z) = |z|^2 = x^2 + y^2$ is a real valued function of a complex variable.
   If $w = u + iv$ is the value of f at $z = x + iy$, then $u + iv = f(x + iy)$.
   Each of the real numbers u and v depends on the real variables x and y and so $f(z)$ can be expressed in terms of a pair of real valued functions of the real variables x and y.
   Thus, $\quad f(z) = u(x, y) + iv(x, y)$
   For example if $z = x + iy$, and if $f(z) = z^2$, then
   $$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy = u(x,y) + iv(x,y)$$
   where $\quad u(x,y) = x^2 - y^2$ and $v = 2xy$.
3. A generalisation of the concept of function is a rule that assigns more than one value to a point z in the domain. Set theoretically, such associations are not functions. By abuse of language these associations are known as multiple-valued functions in complex function theory. When multiple-valued functions are studied, we take one of the possible values at each point of domain, in a systematic way, and construct a single valued function from the multiple-valued function. In contrast, a function is known as single valued function.

   For example: $f(z) = z^{\frac{1}{2}}$ is multiple valued because
   $$f(z) = \pm\sqrt{r}e^{i\theta/2}, \; -\pi < \theta \le \pi, \text{ Putting } z = re^{i\theta}$$
   If we choose the positive sign value $\sqrt{r}$ and write $f(z) = \sqrt{r}e^{\frac{i\theta}{2}}$, $r > 0$, $-\pi < \theta \le \pi$, then $f(z)$ is single valued.

15.1.1 Geometrical Representation of Complex Function or Mapping

We know the graph of a real continuous function $y = f(x)$ is a curve in the xy-plane.

The graph of the real continuous function $z = f(x, y)$ is a surface in 3-dimensional space.

If $w = f(z)$ is a function of a complex variable z, then $u + iv = f(x + iy)$ where x, y, u, v are 4 real variables. Hence, a four dimensional space is required to represent this function graphically. Since it is not possible to exhibit a 4-dimensional space, we choose 2 two-dimensional spaces or planes for z and w variables. The plane in which $z = x + iy$ is plotted is called the z-plane or xy-plane and the plane in which the corresponding $w = u + iv$ is plotted is called the w-plane or uv plane. When a function f is exhibited in this way, it is often referred to as a **mapping** or **transformation** and w is the image of z under f.

The terms translation, rotation and reflection are used to convey the dominant geometric characteristics of certain mappings.

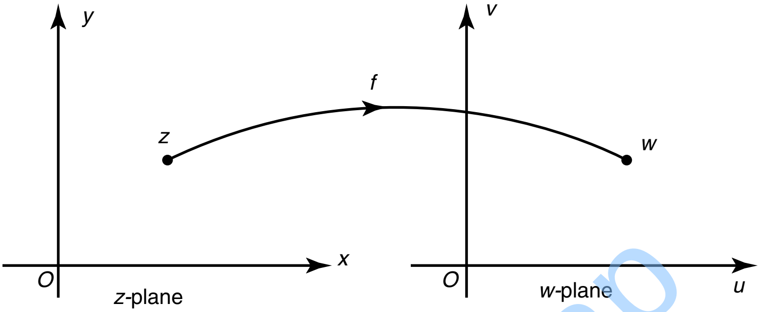


Fig. 15.3

15.1.2 Extended Complex Number System

The complex number system C is the set $\{x + iy / x, y \in R\}$. By extended complex number system, we mean the set C of complex numbers together with a symbol ∞, which satisfies the following properties.

1. If $z \in C$, then $z + \infty = \infty$, $z - \infty = \infty$ and $\frac{z}{\infty} = 0$.
2. If z is non-zero complex number, then $z \cdot \infty = \infty$, $\frac{z}{0} = \infty$.
3. $\infty + \infty = \infty$, $\infty \cdot \infty = \infty$
4. $\frac{\infty}{z} = \infty$ if $z \in C$.

The extended complex number system is $C \cup \{\infty\}$ and when represented in a plane geometrically it is called the extended complex plane and ∞ is known as the point at infinity in this plane.

To visualise point at infinity, we consider the unit sphere with centre at $z = 0$ and the complex plane $\boldsymbol{\pi}$ is a diametral plane.

Let P represent the complex number z in the complex plane $\boldsymbol{\pi}$.

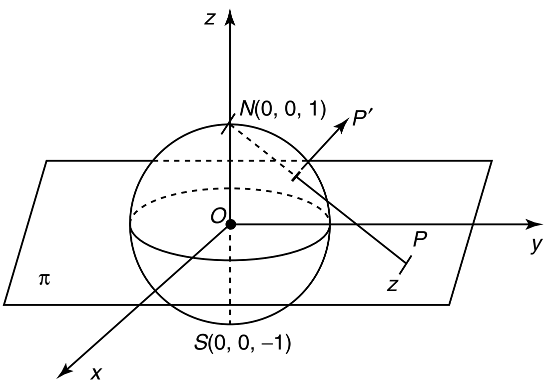


Fig. 15.4

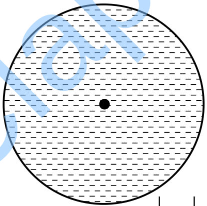
Let the line joining P and north pole N of the sphere meet the surface of the sphere at P'. Thus, to each complex number z, there corresponds only one point P' on the sphere. Conversely for each point P' on the surface of the sphere, other than N, there corresponds only one point z in the plane π. But there is no point in π which corresponds to N. By defining point at infinity ∞ corresponds to N, we obtain a one to one correspondence between points of the sphere and the points of the extended complex plane. This correspondence is a central projection with centre of projection $N(0, 0, 1)$.

The sphere is known as **Riemann sphere** and the correspondence is called a **Stereographic projection**.

Note The extended real number system contains **two symbols ∞ and $-\infty$ with R**. But the extended complex number system contains **one symbol ∞ with C**.

15.1.3 Neighbourhood of a Point and Region

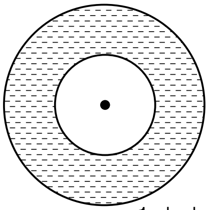
1. By a neighbourhood of a point z_0 in the complex plane we mean the set of all points of the complex plane inside the circle with centre z_0. i.e., the interior of the circle $|z - z_0| = r$. The interior is an open circular disk.
   A δ-neighbourhood of z_0 is the open circular disk $|z - z_0| < \delta$.
   A deleted δ-neighbourhood of z_0 is $0 < |z - z_0| < \delta$
   i.e., the open circular disk punctured at z_0 is the deleted neighbourhood



$|z| < 1$

Fig. 15.5

2. **A set S is open**, if it contains none of its boundary points. e.g. $|z| < 1$.
3. An open set S is **connected** if every pair of points z_1 and z_2 in it can be joined by a polygonal line consisting of finite number of line segments joined end to end.
   For example, the set $|z| < 1$ is connected
   and the set $1 < |z| < 2$ is connected.
   The region $1 < |z| < 2$ is an open annulus as shown in Fig. 15.6



$1 < |z| < 2$

Fig. 15.6

4. An open connected set is called a **domain**.
   e.g. Any neighbourhood of a point z_0 is a domain
   i.e., any open disk is a domain.
5. A domain together with some, none or all of its boundary points is called a **region**.
6. A set is closed if its complement is open.
   e.g. $|z| \le 1$ is a closed set because its complement $|z| > 1$ is open.

15.2 LIMIT OF A FUNCTION

Definition 15.2 Let f be a function defined in some neighbourhood of z_0, except possibly at z_0. We say a complex number w_0 is the **limit of $f(z)$** as z tends to z_0 if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \text{ for } |z - z_0| < \delta.$$

Symbolically, we write $\lim_{z \to z_0} f(z) = w_0$.

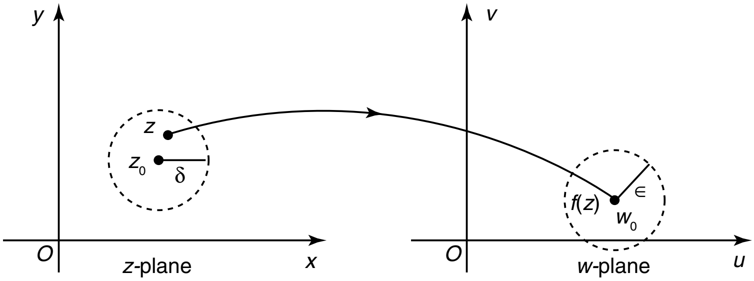


Fig. 15.7

Note

1. When the limit exists it is unique, in whatever direction, z approaches z_0.
2. Suppose $f(z) = u(x, y) + iv(x, y)$ is defined in a neighbourhood of $z_0 = x_0 + iy_0$.

   Then $\lim_{z \to z_0} f(z) = u_0 + iv_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

3. $\lim_{z\to\infty} f(z) = w$ if $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w$, $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0}\left(\frac{1}{f(z)}\right) = 0$

   and $\lim_{z\to\infty} f(z) = \infty$ if $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$

15.2.1 Continuity of a Function

Definition 15.3 Let f be a function defined in a neighbourhood of z_0 (including z_0). f is continuous at the point z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

This means that for continuity at a point.

limiting value = function value at the point.

A function f is continuous in a region R of the complex plane if f is continuous at each point of R. If $f(z) = u(x,y) + iv(x,y)$ and if $f(z_0) = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = f(z_0) \qquad \text{if and only if}$$

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

15.2.2 Derivative of $f(z)$

Definition 15.4 Let f be a function defined in a neighbourhood of z_0. The derivative of f at z_0 is $f'(z_0) = \lim_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, if the limit exists.

When $f'(z_0)$ exists, we say the function f is differentiable at z_0.
A function is differential in a region R if it is differentiable at every point of the region R.

Note

1. Put $z - z_0 = \Delta z \quad \therefore$ as $z \to z_0,\ \Delta z \to 0$

   then $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

   If $w = f(z)$, then $$\Delta w = f(z + \Delta z) - f(z)$$

   $\therefore$ the derivative at any point z is $f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$

2. If a function f is differentiable at z_0, then it is continuous at z_0. But the converse is not true. i.e., if f is continuous at z_0, then it need not be differentiable at z_0.

EXAMPLE

Consider the function $f(z) = \bar{z} = x - iy$, where $z = x + iy$.
It can be seen that $f(z)$ is continuous at $z = 0$, but not differentiable at $z = 0$.
Here $u(x, y) = x$ and $v(x, y) = -y$

$$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{x\to 0} x = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} v(x,y) = \lim_{y\to 0}(-y) = 0$$

$$\therefore \quad \lim_{z\to 0} f(z) = 0 = f(0)$$

$\therefore f(z)$ is continuous at $z = 0$.

Now $$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\bar{z} - 0}{z - 0} = \lim_{z\to 0} \frac{\bar{z}}{z}$$

Choose the path $z \to 0$ along $y = mx$

then $$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x - mx}{x + imx} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{1 - im}{1 + im} = \frac{1 - im}{1 + im}$$

which varies with m. So, the limit is not unique.

Hence, $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist. $\quad \therefore f(z)$ is not differentiable at $z = 0$.

15.2.3 Differentiation Formulae

If $f(z)$ and $g(z)$ are differentiable at z, then

1. $\frac{d}{dz}(cf(z)) = cf'(z)$, where c is a constant.
2. $\frac{d}{dz}(f(z) \pm g(z)) = f'(z) \pm g'(z)$
3. $\frac{d}{dz}(f(z)g(z)) = f(z)g'(z) + g(z)f'(z)$
4. $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ if $g(z) \neq 0$
5. $\frac{d}{dz}[f(g(z))] = f'[g(z)] \cdot g'(z)$
6. $\frac{d}{dz}(z^n) = nz^{n-1}$

15.3 ANALYTIC FUNCTION

The concept of analytic function is the core of complex analysis. Unlike differentiable functions, analytic functions have many additional properties.

Definition 15.5

A complex function $f(z)$ is said to be **analytic** at a point z_0, if $f(z)$ is differentiable at z_0 and at every point of some neighbourhood of z_0.

A function is analytic in a domain D if it is analytic at each point of D.

Note An analytic function is also known as **regular function or holomorphic function**.

15.3.1 Necessary and Sufficient Condition for $f(z)$ to be Analytic

Theorem 15.1 The necessary and sufficient conditions for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain D are

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in the domain D.

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ i.e., $u_x = v_y$ and $u_y = -v_x$.

The second condition $u_x = v_y$ and $u_y = -v_x$ are known as **Cauchy-Riemann equations** or **briefly C-R equations.**

Proof **Necessary condition**

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D then $f'(z)$ exists at any point z in D.

$$\therefore \qquad f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$

We know that when $f'(z)$ exists, it is unique.

i.e., it is independent of the path along which $\Delta z \to 0$

Let $z = x + iy$, then $\Delta z = \Delta x + i\Delta y$. As $\Delta z \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$.

$$\therefore \qquad f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{ \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)]}{\Delta x + i\Delta y} + \frac{i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y} \right\} \qquad (1)$$

We shall find the limit $\Delta z \to 0$ along two paths.

Let Δz be real so that $\Delta y = 0$ and $\Delta z = \Delta x$, so $\Delta z \to 0 \Rightarrow \Delta x \to 0$ i.e., the path is parallel to x-axis.

$$\therefore (1) \Rightarrow \qquad f'(z) = \lim_{\Delta x \to 0} \left[\frac{[u(x+\Delta x, y) - u(x,y)]}{\Delta x} + i \frac{[v(x+\Delta x, y) - v(x,y)]}{\Delta x} \right]$$

$$\Rightarrow \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2}$$

Let Δz be purely imaginary so that $\Delta x = 0$ and $\Delta z = i\Delta y$.

So, $\Delta z \to 0 \Rightarrow \Delta y \to 0$. i.e., the path is parallel to the y-axis.

$$\therefore \ (1) \Rightarrow \qquad f'(z) = \lim_{\Delta y \to 0}\left\{\frac{[u(x, y+\Delta y) - u(x,y)]}{i\Delta y} + i\frac{[v(x,y+\Delta y) - v(x,y)]}{i\Delta y}\right\}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{3}$$

Since $f'(z)$ is unique, from (2) and (3), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore \qquad \boldsymbol{u_x = v_y} \quad \textbf{and} \quad \boldsymbol{u_y = -v_x}$$ ■

Sufficient condition

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function with continuous partial derivatives i.e., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at each point of a domain D and satisfy C-R equations, then $f(z)$ is analytic.

Proof $f(z)$ satisfies C-R equations.

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

We use Taylor's series expansion for a function of two variables.

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

Now $\qquad f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)\right]$$

[omitting second and higher degrees of Δx, Δy]

$$= u(x,y) + iv(x,y) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

$$\Rightarrow \quad f(z+\Delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y \qquad \text{[from (1)]}$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)i\Delta y$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta z$$

$$\therefore \quad \frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\therefore \quad \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Since $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}$ exist at any point of D and are continuous, $\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$ exist in D.

$\Rightarrow f'(z)$ exists in D and $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ in D.

So, derivative exists at every point and in a neighbourhood of it.

$\therefore$ $f(z)$ is analytic in D. ■

Note

1. To prove a function is analytic in a domain, it is enough we show that it is differentiable at each point of the domain.
2. When we say a function is analytic, it is to be understood that it is analytic in a domain D.

15.3.2 C-R Equations in Polar Form

The polar form is derived under the assumption $z \neq 0$.
We know the transformation of cartesian to polar coordinates are

$$x = r\cos\theta,\ y = r\sin\theta$$

$$\therefore \quad z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$f(z) = u+iv = f(re^{i\theta}) \qquad (1)$$

Differentiating (1) partially w.r.to r and θ we get

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta})\cdot e^{i\theta} \qquad (2)$$

and

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\cdot ire^{i\theta}$$

$$\Rightarrow \quad \frac{1}{ir}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}=f'(re^{i\theta})\cdot e^{i\theta} \quad \Rightarrow -\frac{i}{r}\cdot\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}=f'(re^{i\theta})e^{i\theta} \qquad (3)$$

From (2) and (3), we have

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=\frac{-i}{r}\frac{\partial u}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

which are the C-R equations in polar form.

Definition 15.6 Entire Function

A complex function f is said to be an entire function if it is analytic in the entire complex plane (finite plane).

For example: A polynomial function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_n \neq 0$ is an entire function, since it is differentiable everywhere in the plane.

WORKED EXAMPLES

EXAMPLE 1

Show that $f(z) = \dfrac{1}{z-1}$ is analytic at $z = 1 + i$.

Solution.

Given $$f(z)=\frac{1}{z-1}.$$

We have to show that u_x, u_y, v_x, v_y are continuous in some neighbourhood of $z = 1 + i$ and the C-R equations are satisfied in this neighbourhood. $z = 1 + i \Rightarrow x = 1, y = 1$

$$f(z)=\frac{1}{z-1}=\frac{1}{x+iy-1}=\frac{1}{(x-1)+iy}=\frac{(x-1)-iy}{(x-1)^2+y^2}$$

$$\therefore \quad u(x,y)=\frac{x-1}{(x-1)^2+y^2} \quad \text{and} \quad v(x,y)=\frac{-y}{(x-1)^2+y^2}$$

Since $u(x, y)$ and $v(x, y)$ are rational functions of the real variables x and y and are defined at (1, 1) and in a neighbourhood of (1, 1), u_x, u_y, v_x, v_y exist and are continuous.

$$u_x=\frac{[(x-1)^2+y^2]\cdot 1-(x-1)2(x-1)}{[(x-1)^2+y^2]^2}=\frac{y^2-(x-1)^2}{[(x-1)^2+y^2]^2}$$

$$u_y=(x-1)\cdot\frac{-1}{[(x-1)^2+y^2]^2}\cdot 2y=\frac{-2y(x-1)}{[(x-1)^2+y^2]^2}$$

$$v_x=\frac{-y\cdot(-1)\cdot 2(x-1)}{[(x-1)^2+y^2]^2}=\frac{2y(x-1)}{[(x-1)^2+y^2]^2}$$

$$v_y=\frac{-[((x-1)^2+y^2)\cdot 1-y\cdot 2y)]}{[(x-1)^2+y^2]^2}=\frac{y^2-(x-1)^2}{[(x-1)^2+y^2]^2}$$

$$\therefore \quad u_x=v_y \quad \text{and} \quad u_y=-v_x$$

So, C-R equations are satisfied. $\therefore$ $f(z)$ is analytic.

EXAMPLE 2

If $f(z)$ is analytic at a point, then $cf(z)$ is analytic at that point for any constant $c \neq 0$.

Solution.

Let $f(z)$ be the analytic at the point $z_0 = x_0 + iy_0$ and $f(z) = u + iv$
$\therefore$ C.R equations are satisfied at z_0.

$\therefore \quad u_x = v_y$ and $u_y = -v_x$ at this point z_0.

Let $\quad g(z) = cf(z) = c(u + iv) = cu + icv = U + iV$

where $\quad U = cu$ and $V = cv$

$U_x = c\,u_x = c\,v_y$ and $V_x = c\,v_x = -cu_y$

$U_y = c\,u_y$ and $V_y = c\,v_y$

$\therefore \quad U_x = V_y$ and $U_y = -V_x$.

Here U and V satisfy C-R equations.
U_x, U_y, V_x, V_y are continuous, since u_x, u_y, v_x, v_y are continuous.

Hence, $cf(z)$ is analytic.

EXAMPLE 3

If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.

Solution.

Given $f(z) = u + iv$ is analytic in a domain D.

Let $\quad f(z) = u + iv$

We know that if $f(z)$ is analytic, then $cf(z)$ is analytic for any constant $c \neq 0$ [by example 2]
Take $c = i$, then $if(z)$ is analytic.
But $\quad if(z) = i(u + iv) = iu + i^2v = -v + iu$

$\therefore -v + iu$ is analytic.
Take $c = -i$, then $-if(z)$ is analytic.

But $\quad -if(z) = -i(u + iv) = -iu - i^2v = v - iu.$

$\therefore v - iu$ is analytic.
Hence, if $u + iv$ is analytic, then $-v + iu$ and $v - iu$ are analytic.

EXAMPLE 4

If $f(z)$ and $\overline{f(z)}$ are analytic functions prove that $f(z)$ is a constant.

Solution.

Let $\quad f(z) = u + iv \quad \therefore \overline{f(z)} = \overline{u + iv} = u - iv$

Given $f(z)$ is analytic $\quad \therefore$ it satisfies C-R equations.

$\therefore \quad u_x = v_y \quad \text{and} \quad u_y = -v_x \qquad (1)$

Given $\overline{f(z)}$ is also analytic. $\therefore$ it satisfies C-R equations.

$\therefore \quad u_x = -v_y \quad \text{and} \quad u_y = v_x \qquad (2)$

From (1) and (2), $2u_x = 0$ and $2u_y = 0$

$\Rightarrow \quad u_x = 0$ and $u_y = 0$

$\therefore$ u is a constant $\Rightarrow u = c_1$

Also $v_x = 0$ and $v_y = 0$

$\therefore$ v is a constant $\Rightarrow v = c_2$

$\therefore$ $f(z) = u + iv \Rightarrow f(z) = c_1 + ic_2$ is a constant.

EXAMPLE 5

Find the analytic region of $f(z) = (x - y)^2 + 2i(x + y)$.

Solution.

Given $f(z) = (x - y)^2 + 2i(x + y)$

$\Rightarrow \quad u + iv = (x - y)^2 + 2i(x + y)$

$\therefore \quad u = (x - y)^2$ and $v = 2(x + y)$

$\therefore \quad u_x = 2(x - y)$ and $v_x = 2$

$u_y = -2(x - y)$ and $v_y = 2$

Since u and v are polynomials, their partial derivatives are continuous everywhere.
For analyticity, it should satisfy C-R equations, $u_x = v_y$, $u_y = -v_x$

$\therefore \quad u_x = v_y$ and $u_y = -v_x$

$\Rightarrow \quad 2(x - y) = 2$ and $-2(x - y) = -2$

$\Rightarrow \quad x - y = 1$ and $x - y = 1$

$\therefore$ for points on $x - y = 1$, C-R equations are satisfied.
Hence, $f(z)$ is analytic for points on $x - y = 1$.

EXAMPLE 6

Prove that an analytic function with constant modulus is constant.

Solution.

Let $f(z) = u + iv$ be the analytic function.

Given $|f(z)|$ is constant.

$\therefore \quad |u + iv| = \sqrt{u^2 + v^2}$ is constant. $\therefore u^2 + v^2 = C$

where C is a constant. (1)

Differentiating (1) partially w.r.to x and y, we get

$$2u\,u_x + 2v\,v_x = 0 \quad \text{and} \quad 2u\,u_y + 2v\,v_y = 0$$

$$\Rightarrow \quad u\,u_x + v\,v_x = 0 \quad (2) \quad \text{and} \quad u\,u_y + v\,v_y = 0 \quad (3)$$

But given $f(z)$ is analytic.

$\therefore$ it satisfies C-R equations. $\therefore u_x = v_y$ and $u_y = -v_x$

(2) and (3) becomes $uu_x + vv_x = 0$ and $-uv_x + vu_x = 0$

$$\Rightarrow \quad uu_x + vv_x = 0 \quad (4) \quad \text{and} \quad vu_x - uv_x = 0 \quad (5)$$

Treating (4) and (5) homogeneous linear equations in u_x, v_x

we have $$D = \begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -u^2 - v^2 = -(u^2 + v^2)$$

If $D \neq 0$, then $u_x = 0$, $v_x = 0$ is the only solution.

Using C-R equations, we get $v_y = 0$, $u_y = 0$

Thus, $u_x = 0,\ u_y = 0 \Rightarrow u = c_1$, a constant.

and $v_x = 0,\ v_y = 0 \Rightarrow v = c_2$, a constant.

$\therefore$ $f(z) = u + iv = c_1 + ic_2$ is a constant.

If $D = 0$, then $u^2 + v^2 = 0 \Rightarrow u = 0, v = 0$

$\therefore$ $f(z) = 0$, which is a constant.

$\therefore$ $f(z)$ is always a constant.

EXAMPLE 7

Show that an analytic function with constant imaginary part is constant.

Solution.

Let $f(z) = u + iv$ be an analytic function, where v is a constant and let $v = c_2$.

$\therefore$ $v_x = 0$ and $v_y = 0$.

Given $f(z)$ is analytic. $\therefore$ it satisfies C-R equations

$\therefore$ $u_x = v_y$ and $u_y = -v_x \Rightarrow u_x = 0$ and $u_y = 0 \Rightarrow u$ is a constant and let $u = c_1$.

$\therefore$ $f(z) = u + iv = c_1 + ic_2 \Rightarrow f(z) =$ constant.

EXAMPLE 8

Test whether the following functions are analytic or not.

1. $f(z) = z^2$
2. $w = \sin z$
3. $f(z) = \bar{z}$
4. $f(z) = |z|^2$
5. $f(z) = 2xy + i(x^2 - y^2)$

Solution.

1. Given $f(z) = z^2 \Rightarrow u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

$\therefore$ $u = x^2 - y^2$ and $v = 2xy$

$\therefore$ $u_x = 2x$ and $v_x = 2y$

$u_y = -2y$ and $v_y = 2x$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Hence, C-R equations are satisfied for all x and y and the partial derivatives being polynomial in x, y are continuous everywhere in the complex plane.
$\therefore$ $f(z)$ is analytic in the entire plane.
So, it is an entire function.

2. Given $w = \sin z$

$\therefore$ $u + iv = \sin(x + iy)$

$= \sin x \cos(iy) + \cos x \sin(iy) = \sin x \cosh y + i \cos x \sinh y$

$\therefore$ $u = \sin x \cosh y$ and $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ and $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ and $v_y = \cos x \cosh y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Hence, C-R equations are satisfied at all the points and the partial derivatives are continuous at all the points.
$\therefore$ the function is analytic at all the points. So, it is an entire function.

3. Given $f(z) = \bar{z}$

If $z = x + iy$, then $\bar{z} = x - iy$ $\therefore u + iv = x - iy$

$\therefore$ $u = x$ and $v = -y$

$u_x = 1$ and $v_x = 0$

$u_y = 0$ and $v_y = -1$

$\therefore$ $u_x \neq v_y$ and $u_y = -v_x$ for all x, y $\therefore$ C-R equations are not satisfied anywhere.
Hence, $f(z)$ is not analytic anywhere.

4. Given $f(z) = |z|^2$

If $z = x + iy$, then $|z|^2 = x^2 + y^2$

$\therefore$ $u + iv = x^2 + y^2$

$\therefore$ $u = x^2 + y^2 \Rightarrow u_x = 2x$ and $u_y = 2y$

and $v = 0 \Rightarrow v_x = 0$ and $v_y = 0$

We see $u_x = v_y \Rightarrow x = 0$ and $u_y = -v_x \Rightarrow y = 0$

So, C-R equations are satisfied at (0, 0) and C-R equations are not satisfied for $(x, y) \neq (0, 0)$.
Hence, $f(z)$ not analytic for all z including $z = 0$.

5. Given $f(z) = 2xy + i(x^2 - y^2) \Rightarrow u + iv = 2xy + i(x^2 - y^2)$

$$\therefore \quad u = 2xy \quad \text{and} \quad v = x^2 - y^2$$

$$u_x = 2y \quad \text{and} \quad v_x = 2x$$

$$u_y = 2x \quad \text{and} \quad v_y = -2y$$

$$\therefore \quad u_x \neq v_y \quad \text{and} \quad u_y \neq -v_x$$

Hence, C-R equations are not satisfied except (0, 0).
Hence, $f(z)$ is not analytic at any point.

EXAMPLE 9

Show that the function defined by $f(z) = \sqrt{|xy|}$ is not analytic at origin, although C-R equations are satisfied.

Solution.

Given $f(z) = \sqrt{|xy|}$, where $z = x + iy$

$$\Rightarrow \quad u + iv = \sqrt{|xy|} \Rightarrow u = \sqrt{|xy|},\ v = 0$$

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Similarly,

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow u_x = v_y \text{ and } u_y = -v_x$$

Hence, the C-R equations are satisfied at (0, 0).

Now

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{\sqrt{|xy|}-0}{x+iy} = \lim_{z\to 0}\frac{\sqrt{|xy|}}{x+iy}$$

If $z \to 0$ along the straight line $y = mx$, then $x \to 0, y \to 0$.

$$\therefore \quad f'(0) = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x+imx} = \lim_{x\to 0}\frac{\sqrt{|m|}x}{(1+im)x} = \lim_{x\to 0}\frac{\sqrt{|m|}}{1+im} = \frac{\sqrt{|m|}}{1+im}$$

Since the limit depends on m, for different paths, we have different limits. So, limit is not unique and hence $f'(0)$ does not exist. Thus, $f(z)$ is not analytic at $z = 0$, even though C-R equations are satisfied at the origin.

EXAMPLE 10

If $w = f(z)$ is analytic prove that $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$, where $z = x + iy$ and prove that $\dfrac{\partial^2 w}{\partial z\,\partial \bar{z}} = 0$.

Solution.

Given $\quad w = f(z)$ is analytic.

Let $w = u + iv$, then u, v satisfy the C-R equations. $\therefore\ u_x = v_y$ and $u_y = -v_x$ (1)

Now $\quad \dfrac{dw}{dz} = f'(z) = u_x + iv_x = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}(u+iv) = \dfrac{\partial w}{\partial x}$

Also $\quad \dfrac{dw}{dz} = f'(z) = u_x + iv_x$

$= v_y - iu_y$ [Using (1)]

$= -i(u_y + iv_y) = -i\left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right) = -i\dfrac{\partial}{\partial y}(u+iv) = -i\dfrac{\partial w}{\partial y}$

$\therefore \quad \dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$

Since $\quad z = x + iy, \quad \bar{z} = x - iy$

$\therefore \quad z + \bar{z} = 2x \quad \Rightarrow \quad x = \dfrac{z+\bar{z}}{2}$ and $\quad z - \bar{z} = 2iy \quad \Rightarrow \quad y = \dfrac{z-\bar{z}}{2i}$

$\therefore \quad \dfrac{\partial x}{\partial \bar{z}} = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial \bar{z}} = -\dfrac{1}{2i}$

Now $u(x, y)$ and $v(x, y)$ can be considered as functions of z and $\bar{z}$.

$$\therefore \quad \frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} + i\left[\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right]$$

$$= \frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(\frac{-1}{2i}\right) + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \frac{1}{2}(0) + \frac{i}{2}(0) \qquad \text{[using C-R equations]}$$

$$\therefore \quad \frac{\partial w}{\partial \bar{z}} = 0 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial z\,\partial \bar{z}} = 0$$

Note

1. $\dfrac{\partial w}{\partial \bar{z}} = 0 \Rightarrow w$ is independent of $\bar{z}$.

   $\dfrac{\partial w}{\partial \bar{z}} = 0$ or $\dfrac{\partial f}{\partial \bar{z}} = 0$ is called the complex form of the C.R equations of $f(z)$.

2. Infact, we have proved the result, every analytic function $f(z)$ is independent of $\bar{z}$.

EXAMPLE 11

If $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0, \end{cases}$

Prove that $f(z)$ is continuous and the C-R equations are satisfied at $z = 0$, yet $f'(0)$ does not exist.

Solution.

Given $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}, \qquad z \neq 0$$

If $f(z) = u + iv$, then $$u = \frac{x^3 - y^3}{x^2+y^2} \quad \text{and} \quad v = \frac{x^3+y^3}{x^2+y^2}$$

Since $z \neq 0$, $x \neq 0$ or $y \neq 0$
$\therefore$ u, v are rational functions of x and y with non-zero denominators.
So, u and v are continuous and hence $f(z)$ is continuous for $z \neq 0$.
To test the continuity at $z = 0$, we shall transform to polar coordinates.

$\therefore$ $$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$$

then $$u = r(\cos^3\theta - \sin^3\theta) \quad \text{and} \quad v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$\therefore$ $$\lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

and $$\lim_{z\to 0} v = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

$\therefore$ $$\lim_{z\to 0} f(z) = \lim_{z\to 0} u + i\lim_{z\to 0} v = 0 = f(0)$$

$\therefore$ $f(z)$ is continuous at $z = 0$.
Hence, $f(z)$ is continuous for all values of z.
Now we shall verify C-R equations at (0, 0)

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = \lim_{x\to 0} 1 = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0}\frac{0-y}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0}\frac{y-0}{y} = 1$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$\therefore$ C-R equations are satisfied at (0, 0).

Now $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{x^3 - y^3 + i \cdot (x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$

Let $z \to 0$ along $y = x$, then $x \to 0$, $(y \to 0)$.

$$\therefore \quad f'(0) = \lim_{x \to 0} \frac{2ix^3}{2x^2(1+i)x} = \frac{2i}{1+i} \tag{1}$$

Now let $z \to 0$ along the x-axis (i.e., $y = 0$), then $f'(0) = \lim_{x \to 0} \dfrac{x^3 + ix^3}{x^3} = 1 + i$ (2)

Since limits (1) and (2) are different, the limit does not exist.
$\therefore$ $f'(0)$ does not exist.

EXAMPLE 12

Find the values of a and b such that the function $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.

Solution.

Given $\quad f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$

$\Rightarrow \quad u + iv = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$

$\therefore \quad u = x^2 + ay^2 - 2xy$ and $v = bx^2 - y^2 + 2xy$

$\Rightarrow \quad u_x = 2x - 2y$ and $v_x = 2bx + 2y$

$u_y = 2ay - 2x$ and $v_y = -2y + 2x$

Since $f(z)$ is analytic in a domain D, it satisfies C-R equations in D.

$\therefore \quad u_x = v_y$ and $u_y = -v_x$ in D.

$\therefore \quad 2x - 2y = -2y + 2x$, which is true for all x and y in D.

and $\quad 2ay - 2x = -[2bx + 2y]$

$\Rightarrow \quad 2ay - 2x = -2bx - 2y \quad \forall\, x, y \in$ D.

$\therefore$ comparing the coefficients of x and y on both sides,

$$2a = -2 \quad \Rightarrow \quad a = -1 \qquad \text{and} \quad -2b = -2 \quad \Rightarrow \quad b = 1$$

Now $\quad f'(z) = \dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}$

$= 2x - 2y + i(2x + 2y)$

$= 2(x + i^2 y) + 2i(x + y) \qquad [\because b = 1]$

$= 2[(x + iy) + i(x + iy)] = 2(1 + i)(x + iy) = 2(1 + i)z$

EXAMPLE 13

Find a such that the function $f(z) = r^2\cos 2\theta + ir^2\sin a\theta$ is analytic.

Solution.
Given $f(z) = r^2\cos 2\theta + ir^2 \sin a\theta$ is analytic in a domain D.

$$\therefore \quad u + iv = r^2 \cos 2\theta + ir^2 \sin a\theta$$

$$\therefore \quad u = r^2 \cos 2\theta \quad \text{and} \quad v = r^2 \sin a\theta$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = 2r \cos 2\theta \quad \text{and} \quad \frac{\partial v}{\partial r} = 2r \sin a\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = ar^2 \cos a\theta$$

Since $f(z)$ is analytic, it satisfies the C-R equations in polar coordinates.

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \Rightarrow \quad 2r \cos 2\theta = ar \cos a\theta \qquad (1)$$

and

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\Rightarrow \quad -2r^2 \sin 2\theta = -2r^2 \sin a\theta \quad \Rightarrow \quad r^2 \sin 2\theta = r^2 \sin a\theta \qquad (2)$$

These two equations are true for all r and θ, if $a = 2$. $\therefore\ a = 2$

EXERCISE 15.1

1. If $f(z)$ is analytic in a domain D and $f'(z) = 0$ for all $z \in D$, then show that $f(z)$ is a constant.
2. Prove that an analytic function with constant real part is constant.
3. Test the following functions are analytic or not.
   (i) $f(z) = z|z|$
   (ii) $f(z) = 2xy + i(x^2 - y^2)$
   (iii) $f(z) = \log_e z$
   (iv) $f(z) = z^3$ [put $z = re^{i\theta}$]
   (v) $f(z) = e^z$
   (vi) $f(z) = xy + iy$
   (vii) $f(z) = z^2 + z$
   (viii) $f(z) = e^x(\cos y - i\sin y)$
4. Examine the nature of the function $f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$

   in a region including origin.
5. Is $f(z) = z^n$ analytic? Justify.
6. If $f(z) = u + iv$ is analytic in a domain D, then prove that $f(z)$ is constant if $\arg f(z)$ is constant.

   [**Hint:** $\arg f(z) = \tan^{-1}\dfrac{v}{u}$ is constant $\Rightarrow \dfrac{v}{u}$ is constant $= c \Rightarrow v = cu$]
7. Shown that $f(z) = x(x + iy)$ is differentiable at origin, but not analytic there.

8. Show that $f(z) = e^x(\cos y + i \sin y)$ is analytic in the finite plane. Find its derivative.
9. Show that $f(z) = e^{-y}(\cos x + i \sin x)$ is differentiable every where in the finite plane and $f'(z) = f(z)$.
10. Show that $e^y(\cos x + i \sin x)$ is nowhere differentiable.
11. Show that $f(z) = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$, $r > 0$, $0 < \theta < 2\pi$ is analytic. Find $f'(z)$.
12. Verify whether $w = (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy)$ is an analytic function of $z = x + iy$.
13. If $u = x^2 - y^2$, $v = -\dfrac{y}{x^2 + y^2}$ then prove that $u + iv$ is not an analytic function.
14. Determine P such that the function $f(z) = \frac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}\frac{Px}{y}$ be an analytic function.

ANSWERS TO EXERCISE 15.1

3. (i) not analytic (ii) not analytic
   (iii) analytic, except $z = 0$ (iv) analytic (v) analytic
   (vi) not analytic (vii) analytic (viii) not analytic
4. C-R equation are satisfied at $z = 0$, but $f'(0)$ does not exist.
5. Analytic
12. w is an analytic function of z.
14. $P = -1$

15.4 HARMONIC FUNCTIONS AND PROPERTIES OF ANALYTIC FUNCTION

Definition 15.7 A real function ϕ of two variables x and y is said to be harmonic in a domain D if it has continuous second order partial derivatives and satisfies the Laplace equation $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$.

Note Harmonic functions play an important role in applied mathematics. For example the temperature $T(x, y)$ in thin plates lying in the xy-plane are harmonic. The practical importance of complex analysis in engineering mathematics results from the fact that both the real and imaginary parts of an analytic function satisfy Laplace's equation which is the most important equation in physics, electrostatics, fluid flow, heat conduction and so on.

Property 1 **If $f(z) = u + iv$ is analytic in a domain D, then u and v are harmonic in D.**

Proof Given $f(z) = u + iv$ is analytic. Then its component functions u and v have continuous first order partial derivatives and satisfy the C-R equations in D.

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

i.e., $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (1) and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ (2)

Differentiating (1) w.r.to x and (2) w.r.to y we get

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$ (3) and $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x}$ (4)

Since u_x, u_y, v_x, v_y are continuous, the mixed second derivatives are equal.

$$\therefore \qquad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u is harmonic.
Now differentiating (1) w.r.to y and (2) w.r.to x we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\therefore \qquad \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v is harmonic. ■

Note The theory of harmonic functions is called potential theory.

Property 2 If $f(z) = u + iv$ is an analytic function, then the level curves $u(x,y) = c_1$ and $v(x,y) = c_2$ form an orthogonal system of curves.

Proof Given $f(z) = u + iv$ is analytic in a domain D.
$\therefore$ u and v have continuous partial derivatives and satisfy C-R equations.

$$\therefore \qquad u_x = v_y \text{ and } u_y = -v_x \text{ in } D. \qquad (1)$$

Let $u(x, y) = C'$ and $v(x, y) = C''$ be two members of the given families intersecting at $P(x_0, y_0)$.

Then $du = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y}$

The slope of the tangent at the point $P(x_0, y_0)$ to the curve $u(x, y) = C'$ is

$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y}$$

Similarly, the slope of the tangent at the point $P(x_0, y_0)$ to the curve $v(x, y) = C''$ is

$$m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}$$

Now
$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right) \cdot \left(-\frac{v_x}{v_y}\right)$$

$$= \frac{u_x}{u_y} \cdot \frac{v_x}{v_y} = \frac{u_x}{-v_x} \cdot \frac{v_x}{u_x} = -1 \qquad \text{[using C-R equations (1)]}$$

$\therefore$ the curves cut orthogonally.
Hence, the two systems of curves are orthogonal. ■

Note

1. The level curves u = constant are called equipotential lines. In the application of fluid flow for a given flow under suitable assumptions there exists an analytic function.

   $f(z) = u(x, y) + iv(x, y)$ is called the complex potential of the flow such that the curves $v(x, y) = c_1$ are the stream lines and the curves $u(x, y) = c_2$ are the equipotential lines. So, the function v is called stream function and the function u is called the velocity potential.

   In heat flow problems the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are known as isothermals and heat flow lines respectively.
2. Property (1) says if $f(z) = u + iv$ is analytic then u and v are harmonic functions. However, for any two harmonic functions u and v, $u + iv$ need not be analytic.

   For example, consider $u = x$, $v = -y$, then $u_x = 1$, $\quad v_x = 0$

$$u_y = 0, \qquad v_y = -1$$
$$u_{xx} = 0, \qquad u_{yy} = 0$$

$\therefore$ $$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0$$

i.e., u and v are harmonic functions.
But $u_x \neq v_y$ and so, C–R equations are not satisfied and hence $u + iv$ is not analytic.

Definition 15.8 If two harmonic functions u and v satisfy the C-R equations in a domain D, then they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **conjugate harmonic function or harmonic conjugate function of u in D**.

Note that the word "conjugate" here is different from the one used in defining $\bar{z}$.

15.4.1 Construction of an Analytic Function Whose Real or Imaginary Part is Given Milne-Thomson Method

Let $\quad f(z) = u(x, y) + iv(x, y)$ (1)

Since $\quad z = x + iy, \bar{z} = x - iy$, then $x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$

$\therefore$ $$f(z) = u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

Considering this as a formal identity in the two independent variables $z, \bar{z}$, putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \qquad (2)$$

$\therefore$ (2) is the same as (1), if we replace x by z and y by 0.
This is valid for any function of the form $f(x + iy)$.

This method provides an elegant method of finding an analytical function $f(z)$ when its real part or imaginary part is given. It is due to **Milne-Thomson**.

1. Let $u(x, y)$ be the given real part of an analytic function $f(z)$.
   We have to find $f(z)$ and its imaginary part $v(x, y)$.

   Since $u(x, y)$ is given, find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$

Since $f(z)$ is analytic $f'(z) = u_x + iv_x$

$$= u_x - iu_y = u_x(x, y) - iu_y(x, y) \qquad [\because u_y = -v_x, \text{ C–R equations}]$$

By Milne-Thomson method,

$$f'(z) = u_x(z, 0) - iu_y(z, 0) \qquad [\text{replacing } x \text{ by } z \text{ and } y \text{ by } 0]$$

$$\therefore \quad f(z) = \int [u_x(z,0) - iu_y(z,0)]dz + c$$

where c is an arbitrary complex constant of integration.
Then separating real and imaginary parts, we find $v(x, y)$.

2. Suppose the imaginary part $v(x, y)$ is given, find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

$$f'(z) = u_x + iv_x = v_y + iv_x = v_y(x,y) + iv_x(x,y) \qquad [\text{by C-R equations } u_x = v_y]$$

$$\Rightarrow \quad f'(z) = v_y(z,0) + iv_x(z,0)$$

By Milne-Thomson method,

$$f(z) = \int [v_y(z,0) + iv_x(z,0)]dz + c \qquad [\text{replacing } x \text{ by } z \text{ and } y \text{ by } 0]$$

Then we find $u(x, y)$ by equating real parts.

Working rule: Milne-Thomson method

1. **If real part $u(x, y)$ is given, to find $f(z) = u + iv$**
   Step 1: Find $u_x(x, y)$, $u_y(x, y)$
   Step 2: Find $u_x(z, 0)$, $u_y(z, 0)$ [replacing x by z and y by 0]
   Step 3: $f'(z) = u_x(z, 0) - iu_y(z, 0)$
   Step 4: $f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + c$

2. **If the imaginary part $v(x, y)$ is given, to find $f(z) = u + iv$.**
   Step 1: Find $v_x(x, y)$, $v_y(x, y)$
   Step 2: Find $v_x(z, 0)$, $v_y(z, 0)$
   Step 3: $f'(z) = v_y(z, 0) + iv_x(z, 0)$
   Step 4: $f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + c$

We shall now obtain the complex form of Laplace equation.
Let $u(x, y)$ be a harmonic function.

Since $z = x + iy$ and $\bar{z} = x - iy$, we have $x = \frac{z+\bar{z}}{2}; y = \frac{z-\bar{z}}{2i}$

Hence, u is ultimately a function of z and $\bar{z}$.

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}(u_x - iu_y)$$

$$\therefore \quad \frac{\partial^2 u}{\partial \bar{z}\partial z} = \frac{\partial}{\partial \bar{z}}\left[\frac{1}{2}(u_x - iu_y)\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}(u_x - iu_y)\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}(u_x - iu_y)\cdot\frac{\partial y}{\partial \bar{z}}\right]$$

$$= \frac{1}{2}\left[(u_{xx} - iu_{yx})\frac{1}{2} + (u_{xy} - iu_{yy})\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}[(u_{xx} - iu_{yx}) + i(u_{xy} - iu_{yy})] \qquad \left[\because -\frac{1}{i} = i\right]$$

$$= \frac{1}{4}[u_{xx} - iu_{xy} + iu_{xy} + u_{yy}] = \frac{1}{4}[u_{xx} + u_{yy}] \qquad [\because u_{xy} = u_{yx}]$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} \qquad (1)$$

Since u is harmonic, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial \bar{z}\partial z} = 0$

This is the **complex form** of Laplace equation.

From (1), we get the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}$

WORKED EXAMPLES

EXAMPLE 1

If $f(z)$ is analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Solution.

Given $f(z)$ is analytic and let $f(z) = u + iv$.

Then u and v have continuous partial derivatives and they satisfy C-R equations.

$$\therefore \quad u_x = v_y \text{ and } u_y = -v_x \quad \text{and} \quad f'(z) = u_x + iv_x$$

$$\therefore \quad |f'(z)|^2 = u_x^2 + v_x^2 \qquad (1)$$

Since u and v are harmonic functions, we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}$$

$$\text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial \bar{z}\partial z}|f(z)|^2 = 4\frac{\partial^2}{\partial \bar{z}\partial z}(f(z)\cdot\overline{f(z)})$$

Since $f(z)$ is an analytic function, it is independent of $\bar{z}$.

i.e., $f(z)$ is a function of z only.

Similarly, its conjugate $\overline{f(z)}$ is analytic function of $\bar{z}$ only.

So, we can denote $\overline{f(z)}$ by $\bar{f}(\bar{z})$ and write $\overline{f(x+iy)} = \bar{f}(x-iy)$

$$\therefore \qquad \text{L.H.S.} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}(f(z)\bar{f}(\bar{z})) = 4\frac{\partial}{\partial \bar{z}}[\bar{f}(\bar{z})]\frac{\partial}{\partial z}[f(z)]$$

$$= 4\bar{f}'(\bar{z})\cdot f'(z) = 4\overline{f'(z)}\cdot f'(z)$$

$$= 4|f'(z)|^2 = \text{R.H.S.} \qquad [\because z\bar{z} = |z|^2]$$

EXAMPLE 2

If $f(z)$ is analytic, then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(|f(z)|^p) = p^2|f(z)|^{p-2}\cdot|f'(z)|^2.$$

Solution.

Let $\qquad f(z) = u + iv.$

Since $f(z)$ is analytic, u and v are harmonic functions.

$$\therefore \qquad \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z}$$

$$\therefore \qquad \text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(|f(z)|^p) = 4\frac{\partial^2}{\partial \bar{z}\partial z}|f(z)|^p$$

$$= 4\cdot\frac{\partial^2}{\partial \bar{z}\partial z}\left[|f(z)|^2\right]^{\frac{p}{2}}$$

$$= 4\frac{\partial^2}{\partial \bar{z}\partial z}[f(z)\overline{f(z)}]^{\frac{p}{2}}$$

$$= 4\frac{\partial^2}{\partial \bar{z}\partial z}\left\{[f(z)]^{\frac{p}{2}}\cdot[\bar{f}(\bar{z})]^{\frac{p}{2}}\right\}$$

$$= 4\frac{\partial}{\partial \bar{z}}[\bar{f}(\bar{z})]^{\frac{p}{2}}\frac{\partial}{\partial z}(f(z))^{\frac{p}{2}}$$

$$= 4\frac{p}{2}[\bar{f}(\bar{z})]^{\frac{p}{2}-1}\cdot\bar{f}'(\bar{z})\cdot\frac{p}{2}[f(z)]^{\frac{p}{2}-1}f'(z)$$

$$= p^2[f(z)\cdot\bar{f}(\bar{z})]^{\frac{p}{2}-1}\cdot f'(z)\cdot\bar{f}'(\bar{z})$$

$$= p^2\left[|f(z)|^2\right]^{\frac{p-2}{2}}\cdot|f'(z)|^2$$

$$= p^2|f(z)|^{p-2}\cdot|f'(z)|^2 = \text{R.H.S.}$$

EXAMPLE 3

If $f(z)$ is an analytic function of z, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\operatorname{Re} f(z)|^2 = 2|f'(z)|^2$.

Solution.

Let $f(z) = u + iv$. Given $f(z)$ is analytic.

Since Re $f(z) = u$, we have to prove

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2|f'(z)|^2.$$

We have $f'(z) = u_x + iv_x \quad \therefore \quad |f'(z)|^2 = u_x^2 + v_x^2$

$$\text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}$$

$$= \frac{\partial}{\partial x}(2u\cdot u_x)+\frac{\partial}{\partial y}(2u\cdot u_y)$$

$$= 2\{u\cdot u_{xx}+u_x^2\}+2\{u\cdot u_{yy}+u_y^2\}$$

$$= 2u\{u_{xx}+u_{yy}\}+2(u_x^2+u_y^2)$$

Since u is harmonic, $u_{xx}+u_{yy}=0$

$\therefore$ L.H.S. $= 2(u_x^2+u_y^2) = 2(u_x^2+v_x^2)$ $\quad [\because u_y = -v_x$, C-R equation]

$$= 2|f'(z)|^2 = \text{R.H.S.}$$

EXAMPLE 4

If $f(z)$ is an analytic function, then prove that $\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$.

Solution.

Let $f(z) = u + iv$.

Given $f(z)$ is analytic, then u, v are harmonic functions.

$$\therefore \quad |f(z)| = \sqrt{u^2+v^2}$$

$$\therefore \quad \frac{\partial}{\partial x}(|f(z)|) = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right) = \frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\therefore \quad \left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{(uu_x+vv_x)^2}{u^2+v^2}$$

Similarly,
$$\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{(uu_y+vv_y)^2}{u^2+v^2}$$

$$\therefore \quad \left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{(uu_x + vv_x)^2 + (uu_y + vv_y)^2}{u^2 + v^2}$$

$$= \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2 + v^2}$$

$$= \frac{u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) + 2uv(u_xv_x + u_yv_y)}{u^2 + v^2}$$

$$= \frac{u^2(u_x^2 + v_x^2) + v^2(v_x^2 + u_x^2) + 2uv(u_yv_x + (-v_x)v_y)}{u^2 + v^2}$$

$[\because u_y = -v_x,\ u_x = v_y, \text{C.R equations}]$

$$= \frac{u^2|f'(z)|^2 + v^2|f'(z)|^2 + 2uv[v_yv_x - v_x \cdot v_y]}{u^2 + v^2}$$

$$= \frac{(u^2 + v^2)|f'(z)|^2 + 0}{u^2 + v^2} = |f'(z)|^2$$

EXAMPLE 5

Prove that the function $u = e^x(x \cos y - y \sin y)$ satisfies Laplace's equation and find the corresponding analytic function $f(z) = u + iv$.

Solution.

Given $\quad u = e^x(x \cos y - y \sin y)$ is the real part u of $f(z) = u + iv$

$\therefore \quad u_x = e^x[\cos y] + (x \cos y - y \sin y)e^x = e^x[\cos y + x \cos y - y \sin y)$

$\therefore \quad u_{xx} = e^x[\cos y] + [\cos y + x \cos y - y \sin y]e^x = e^x[2\cos y + x \cos y - y \sin y]$

$u_y = e^x[-x \sin y - (y \cdot \cos y + \sin y)]$

$u_{yy} = e^x[-x \cos y - (-y \sin y + \cos y) - \cos y] = e^x[-x \cos y + y \sin y - 2\cos y]$

$\therefore \quad u_{xx} + u_{yy} = e^x[2\cos y + x \cos y - y \sin y] + e^x[-x \cos y + y \sin y - 2\cos y]$

$= e^x(0) = 0$

$\therefore$ u satisfies Laplace's equation and so u is a harmonic function.

Now replacing x by z and y by 0, we get

$u_x(z, 0) = e^z[\cos 0 + z \cos 0 - 0] = e^z(1 + z)$

$u_y(z, 0) = e^z(0) = 0$

By Milne–Thomson method,

$$f'(z) = u_x(z, 0) - iu_y(z, 0) = (1 + z)e^z - i0 = (1 + z)e^z$$

Integrating, $\quad f(z) = \int (1 + z)e^z dz = (1 + z)e^z - 1 \cdot e^z + c = ze^z + c$

EXAMPLE 6

If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find the corresponding analytic function $f(z) = u + iv$.

Solution.

Given $$u = \frac{\sin 2x}{\cosh 2y + \cos 2x} \text{ is the real part } u \text{ of } f(z) = u + iv$$

$$\therefore \quad u_x = \frac{(\cosh 2y + \cos 2x)\cdot 2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cdot\cosh 2y\cdot\cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cdot\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y\cdot\cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

$$u_y = \frac{(\cosh 2y + \cos 2x)\cdot 0 - \sin 2x\cdot 2\sinh 2y}{(\cosh 2y + \cos 2x)^2} = \frac{-2\sin 2x\cdot\sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

Replacing x by z and y by 0, we get

$$u_x(z,0) = \frac{2\cosh 0\cdot\cos 2z + 2}{(\cosh 0 + \cos 2z)^2} \qquad [\cosh 0 = 1]$$

$$= \frac{2(1+\cos 2z)}{(1+\cos 2z)^2} = \frac{2}{1+\cos 2z} = \frac{2}{2\cos^2 z} = \sec^2 z$$

$$u_y(z,0) = \frac{-2\sin 2z\cdot\sinh 0}{(\cosh 0 + \cos 2z)^2} = 0. \qquad [\because \sinh 0 = 0]$$

By Milne–Thomson method,

$$f'(z) = u_x(z,0) - iu_y(z,0) = \sec^2 z - 0$$

$$\therefore \quad f(z) = \int \sec^2 z\,dz = \tan z + c$$

EXAMPLE 7

Determine the analytic function $f(z) = u + iv$ such that $u - v = e^x(\cos y - \sin y)$.

Solution.

Given $\quad f(z) = u + iv \quad \therefore \quad if(z) = iu - v$

Adding, $\quad (1 + i)f(z) = u - v + i(u + v)$

Put $U = u - v$, $V = u + v$ and $F(z) = (1 + i)f(z) \quad \therefore \quad F(z) = U + iV$

Since $f(z)$ is analytic, $F(z)$ is analytic.

Given $U = u - v = e^x(\cos y - \sin y)$

$\therefore$ $U_x = e^x(\cos y - \sin y)$ and $U_y = e^x(-\sin y - \cos y)$

Replacing x by z and y by 0 we get,

$$\therefore \quad U_x(z, 0) = e^z(\cos 0 - \sin 0) = e^z$$

$$U_y(z, 0) = e^z(-\sin 0 - \cos 0) = -e^z$$

By Milne–Thomson method,

$$F'(z) = U_x(z, 0) - iU_y(z, 0) = e^z + ie^z = (1+i)e^z$$

Integrating, $F(z) = (1 + i)\int e^z\, dz$

$\Rightarrow$ $F(z) = (1 + i)e^z + c$

$\Rightarrow$ $(1 + i)f(z) = (1 + i)e^z + c \quad \Rightarrow \quad f(z) = e^z + c'$, where $c' = \dfrac{c}{1+i}$

EXAMPLE 8

If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u + iv$ is an analytic function of z, find $f(z)$ in terms of z.

Solution.

Given $f(z) = u + iv$ $\quad\therefore\quad if(z) = iu - v$

Adding, $(1 + i)f(z) = u - v + i(u + v)$

Put $U = u - v, \quad V = u + v, \quad F(z) = (1 + i)f(z) \quad \therefore \quad F(z) = U + iV$

Since $f(z)$ is analytic, $F(z)$ is analytic.

Also given $$u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$

$$\therefore \quad V = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\therefore \quad V_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2[\cosh 2y\cos 2x - (\cos^2 2x + \sin^2 2x)]}{[\cosh 2y - \cos 2x]^2} = \frac{2(\cosh 2y\cos 2x - 1)}{[\cosh 2y - \cos 2x]^2}$$

$$V_y = \sin 2x \cdot \frac{(-1)}{(\cosh 2y - \cos 2x)^2} 2\sinh 2y = -\frac{2\sinh 2y\sin 2x}{(\cosh 2y - \cos 2x)^2}$$

Replacing x by z and y by 0, we get

$$V_x(z, 0) = \frac{2(\cosh 0\cos 2z - 1)}{(\cosh 0 - \cos 2z)^2} = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} \qquad [\because \cosh 0 = 1]$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = -\frac{2}{1-\cos 2z} = -\frac{2}{2\sin^2 z} = -\text{cosec}^2 z$$

$$V_y(z,0) = \frac{-2\sinh 0 \sin 2z}{(\cosh 0 - \cos 2z)^2} = 0 \qquad [\because \sinh 0 = 0]$$

Since imaginary part is given, by Milne–Thomson method,

$$F'(z) = V_y(z,0) + iV_x(z,0) = 0 + i(-\text{cosec}^2 z) = -i\text{cosec}^2 z$$

Integrating, $$F(z) = -i\int \text{cosec}^2 z\, dz = -i(-\cot z) + c$$

$\Rightarrow$ $$(1+i)f(z) = i\cot z + c$$

$\Rightarrow$ $$f(z) = \frac{i}{1+i}\cot z + \frac{c}{1+i}$$

$$= \frac{i(1-i)}{2}\cot z + c_1 = \frac{1+i}{2}\cot z + c_1, \qquad \text{where } c_1 = \frac{c}{1+i}$$

EXAMPLE 9

Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$.

Solution.

Given $$2u + 3v = e^x(\cos y - \sin y) \qquad (1)$$

and $$f(z) = u + iv$$

Consider $$3f(z) = 3u + i3v \qquad \text{and} \qquad i2f(z) = 2iu - 2v$$

Adding, $$(3+2i)f(z) = (3u - 2v) + i(2u + 3v)$$

Put $$U = 3u - 2v, \quad V = 2u + 3v, \quad F(z) = (3 + 2i)f(z)$$

$\therefore$ $$F(z) = U + iV$$

Since $f(z)$ is analytic, $F(z)$ is analytic and intergrating part is

$$V = 2u + 3v = e^x(\cos y - \sin y) \qquad \text{[using (1)]}$$

$\therefore$ $$V_x = e^x(\cos y - \sin y) \quad \text{and} \quad V_y = e^x(-\sin y - \cos y)$$

Replacing x by z and y by 0, we get

$\therefore$ $$V_x(z,0) = e^z(\cos 0 - \sin 0) = e^z \quad \text{and} \quad V_y(z,0) = e^z(-\sin 0 - \cos 0) = -e^z$$

Since imaginary part is given, by Milne–Thomson method,

$$F'(z) = V_y(z,0) + iV_x(z,0)$$

$\Rightarrow \qquad F'(z) = -e^z + ie^z = (-1+i)e^z$

Integrating, $\qquad F(z) = (-1+i)\int e^z\, dz$

$\Rightarrow \qquad (3+2i)f(z) = (-1+i)e^z + c$

$\Rightarrow \qquad f(z) = \dfrac{(-1+i)e^z}{3+2i} + \dfrac{c}{3+2i}$

$$= \frac{(3-2i)(-1+i)e^z}{9+4} + c_1 = \frac{(-1+5i)e^z}{13} + c_1, \qquad c_1 = \frac{c}{3+2i}$$

EXERCISE 15.2

1. Show that the following functions are harmonic.

   (i) $u = 2x - x^3 + 3xy^2$ (ii) $v(x,y) = -\dfrac{y}{x^2+y^2}$ (iii) $u = 3x^2y - y^3$

   (iv) $u = \log_e \sqrt{x^2+y^2}$ (v) $v = \log[(x-1)^2 + (y-2)^2]$

2. If $w = \phi + i\psi$ represents the complex potential of an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ determine ϕ.
3. Find the analytic function $f(z) = u + iv$ if $u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$.
4. Find the analytic function $w = u + iv$ given that $v = e^{-x}\{x\cos y + y\sin y\}$ and $w(0) = 1$.
5. Determine the analytic function $f(z) = u + iv$ whose real part $u = e^{2x}(x\cos 2y - y\sin 2y)$.
6. Determine the analytic function $f(z) = u + iv$ if $v = \log(x^2+y^2) + x - 2y$.
7. Show that the function $u = \sin x\cosh y + 2\cos x\sinh y + x^2 - y^2 + 4xy$ is harmonic and find the analytic function $f(z) = u + iv$.
8. Find the analytic function $f(z) = u + iv$, given that $2u + v = e^{2x}\{(2x+y)\cos 2y + (x-2y)\sin 2y\}$.
9. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and, $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \sin x}$ and $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$, then find $f(z)$.
10. If $f(z) = u + iv$ is an analytic function of z, then find $f(z)$ if $2u + v = e^x(\cos y - \sin y)$.
11. If $w = f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$.
12. If $f(z)$ is analytic, then prove that $\nabla^2 \log_e |f(z)| = 0$.
13. Determine the analytic function $u + iv$ whose real part $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.
14. Prove that the function $v = e^{-x}(x\cos y + y\sin y)$ is harmonic and determine the corresponding analytic function $f(z) = u + iv$.
15. Show that the function $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

16. If $u = \log(x^2 + y^2)$, then find v and $f(z)$ such that $f(z) = u + iv$ is analytic.
17. Find the analytic function $w = u + iv$ given $v = e^{-2xy} \cdot \sin(x^2 - y^2)$.
18. If $w = u + iv$ is an analytic function and $v = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ find u.
19. Find the analytic function $f(z) = u + iv$ if $u - v = (x - y)(x^2 + 4xy + y^2)$.
20. If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$, find $f(z)$ given that $f\left(\dfrac{\pi}{2}\right) = 0$.
21. Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16xy$.

ANSWERS TO EXERCISE 15.2

2. $\phi = -2xy + \dfrac{y}{x^2 + y^2} + c$
3. $f(z) = z^2 \cdot e^{-z} + c$
4. $w = ize^{-z} + 1$
5. $f(z) = ze^{2z} + c$
6. $f(z) = (i - 2)z + 2i\log z + c$
7. $f(z) = (1 - 2i)(\sin z + z^2) + c$
8. $f(z) = ze^{2z} + c$
9. $f(z) = \cot \dfrac{z}{2} + \dfrac{1 - i}{2}$
10. $f(z) = \left(\dfrac{1 + 3i}{5}\right)e^z + c$
13. $z^3 + 3z^2 + c$
14. $ize^{-z} + c$
15. $v = 4xy - x^3 + 3xy^2 + c_2$
16. $v = 2\tan^{-1}\dfrac{y}{x} + c_2$
17. $f(z) = e^{iz^2} + c$
18. $u = -2xy + \dfrac{y}{x^2 + y^2} + c_1$
19. $f(z) = -iz^3 + c_1$
20. $f(z) = \dfrac{1}{2}\left(1 - \cot \dfrac{z}{2}\right)$
21. $f(z) = (1 - 2i)z^2 + c'$.

15.5 CONFORMAL MAPPING

In this section we study mappings $w = f(z)$ which map curves and regions from one complex plane to the other complex plane. We will discuss how arcs and regions in z-plane are transformed to the w-plane by some elementary functions and bilinear functions.

Conformal mappings transform curves and domains from one complex plane to the other with regard to size and orientation. Conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shape. Smaller portions of these domains and arcs are conformally mapped by analytic functions to well-known domains and arcs and then studied.

We know that the circle $x^2 + y^2 = 1$ in the xy-plane can be written parametrically as $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

In the complex plane we write this as

$$z(t) = \cos t + i \sin t,\ 0 \le t \le 2\pi.$$

Definition 15.9 Arc

An **arc** in the complex plane is a set of points $z = (x, y)$, if $x = x(t)$ and $y = y(t)$ are continuous functions of t in the interval $[a, b]$.

Complex Integration

16.0 INTRODUCTION

Integrals are extremely important in the study of functions of a complex variable mainly for two reasons. Some properties of analytic functions can be proved by complex integration easily. For instance, the existence of higher derivatives of analytic functions. Secondly in applications real integrals occur which cannot be evaluated by usual methods, but can be evaluated by complex integration.

We know that definite integral of a real function is defined on an interval of the real line. But integral of a complex valued function of a complex variable is defined on a curve or arc in the complex plane. A complex definite integral is called a (complex) line integral.

Definition 16.1 Contour

A **contour** is a piecewise smooth path consisting of finite number of smooth arcs joined end to end. An arc is given by an equation $z(t) = x(t) + iy(t)$, $a \le t \le b$, where $x(t)$ and $y(t)$ are continuous functions of t.

16.1 CONTOUR INTEGRAL

Definition 16.2 If $f(z)$ is a function of a complex variable z which is defined on a given arc or curve C in the complex plane, then the complex line integral is written as $\int_C f(z)dz$.

If the equation of C is $z(t) = x(t) + iy(t)$, $a \le t \le b$ and C is the contour from $z_1 = z(a)$ to $z_2 = z(b)$, then we write $\int_{z_1}^{z_2} f(z)dz$.

If $f(z)$ is piecewise continuous on C then $f[z(t)]$ is piecewise continuous on $[a, b]$. We define the line integral or contour integral of f along C as

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\; z'(t)dt \qquad (1)$$

If $f(z) = u(x, y) + iv(x, y)$, then

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$
$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

16.1.1 Properties of Contour Integrals

1. $\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$, where z_0 is a constant.
2. $\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

3. $\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$

4. $\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ where C is broken up into C_1 and C_2

WORKED EXAMPLES

EXAMPLE 1

Evaluate $\int_C f(z)\,dz$ where $f(z) = y - x - i3x^2$ from $z = 0$ to $z = 1 + i$ along the path (i) from (0, 0) to $A(1, 0)$ and to $B(1, 1)$, (ii) $y = x$.

Solution.

Given $f(z) = y - x - i3x^2$

(i) $\int_C f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz$

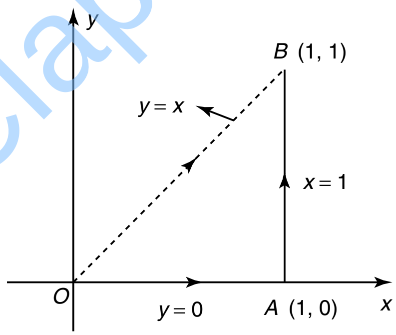


Fig. 16.1

on OA: $y = 0, z = x \quad \therefore \quad dz = dx$

and $f(z) = -x - i3x^2$

$$\therefore \int_{OA} f(z)\,dz = \int_0^1 (-x - i3x^2)\,dx$$

$$= -\left[\frac{x^2}{2} + i\frac{3x^3}{3}\right]_0^1 = -\left[\frac{1}{2} + i\right]$$

on AB: $x = 1$ and $z = 1 + iy \quad \therefore \quad dz = i\,dy$ and y varies from 0 to 1

$$\therefore \int_{AB} f(z)\,dz = \int_0^1 (y - 1 - 3i)\,i\,dy$$

$$= i\left[\frac{y^2}{2} - (1+3i)y\right]_0^1 = i\left[\frac{1}{2} - (1+3i)\right] = i\left[-\frac{1}{2} - 3i\right] = \frac{-i}{2} + 3$$

$$\therefore \int_C f(z)\,dz = -\frac{1}{2} - i - \frac{i}{2} + 3 = \frac{5-3i}{2}$$

(ii) **on C:** $y = x \quad \therefore \quad z = x + ix = (1 + i)x$ and $f(z) = x - x - i3x^2 = -i3x^2$.

$\therefore \quad dz = (1 + i)dx$ and x varies from 0 to 1.

$$\therefore \int_C f(z)\,dz = \int_0^1 (-i3x^2)(1+i)\,dx = -i(1+i)\cdot 3\left[\frac{x^3}{3}\right]_0^1 = -i(1+i) = 1 - i$$

EXAMPLE 2

Evaluate $\int_C z^2\,dz$ where C is the arc from $A(1, 1)$ to $B(2, 4)$ along (i) $y = x^2$, (ii) $y = 3x - 2$.

Solution.

$$\int_C z^2 dz = \int_C (x+iy)^2 (dx+idy)$$

$$= \int_C [(x^2-y^2)+i2xy][dx+idy]$$

(i) **Along $y = x^2$**

$\therefore \quad dy = 2xdx$

$dz = dx + i\,dy = dx + i\,2xdx = (1 + i\,2x)dx$

and $\quad x$ varies from 1 to 2

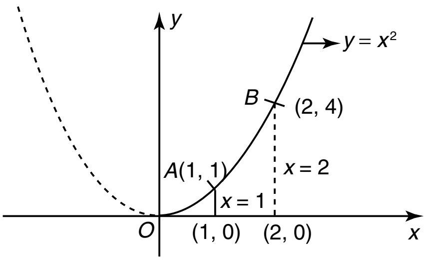


Fig. 16.2

$$\therefore \quad \int_C z^2 dz = \int_1^2 [x^2 - x^4 + i2x\cdot x^2][1+i2x]dx$$

$$= \int_1^2 [x^2 - x^4 - 4x^4 + i(2x^3 - 2x^5 + 2x^3)]dx$$

$$= \int_1^2 [x^2 - 5x^4]dx + i\int_1^2 (4x^3 - 2x^5)dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^5}{5}\right]_1^2 + i\left[4\frac{x^4}{4} - 2\cdot\frac{x^6}{6}\right]_1^2$$

$$= \frac{1}{3}(2^3-1^3) - (2^5-1) + i\left[2^4 - 1^4 - \frac{1}{3}(2^6-1^6)\right] = \frac{7}{3} - 31 + i[15-21] = -\left[\frac{86}{3} + 6i\right]$$

(ii) **Along $y = 3x - 2$** $\quad \therefore \quad dy = 3dx$

$dz = dx + idy = (1 + i3)dx$

and x varies from 1 to 2.

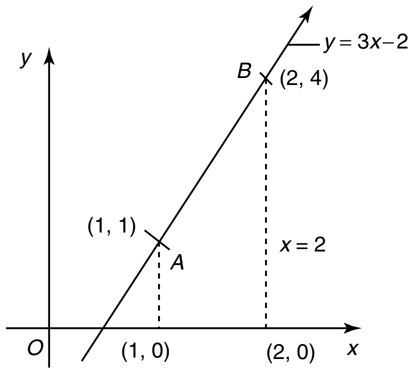


Fig. 16.3

$$\therefore \int_C z^2 dz = \int_1^2 [x^2 - (3x-2)^2 + 2ix(3x-2)][1+3i]dx$$

$$= \int_1^2 [-8x^2 + 12x - 4 + i(6x^2 - 4x)](1+3i)dx$$

$$= (1+3i)\left[\int_1^2 (-8x^2+12x-4)dx + i\int_1^2 (6x^2-4x)dx\right]$$

$$= (1+3i)\left\{\left[-8\frac{x^3}{3} + 12\frac{x^2}{2} - 4x\right]_1^2 + i\left[\frac{6x^3}{3} - \frac{4x^2}{2}\right]_1^2\right\}$$

$$= (1+3i)\left\{\left[-\frac{8}{3}(2^3-1) + 6(2^2-1) - 4(2-1)\right] + i[2(2^3-1) - 2(2^2-1)]\right\}$$

$$= (1+3i)\left[-\frac{56}{3}+18-4+i(14-6)\right]$$

$$= (1+3i)\left[-\frac{56}{3}+14+i8\right]$$

$$= (1+3i)\left[-\frac{14}{3}+i8\right] = -\frac{14}{3}-24+i(8-14) = -\left[\frac{86}{3}+6i\right]$$

Note that the value of $\int_C f(z)\,dz$ along the different paths are same, because $f(z)$ is analytic [Refer note in page 6]

16.1.2 Simply Connected and Multiply Connected Domains

Definition 16.3 A domain D in the complex plane is called **simply connected** if every simple closed curve which lies in D can be shrunk to a point without leaving D.

Example: Interior of a circle and ellipse are simply connected domains.
A domain D which is not simply connected is called **multiply connected**.

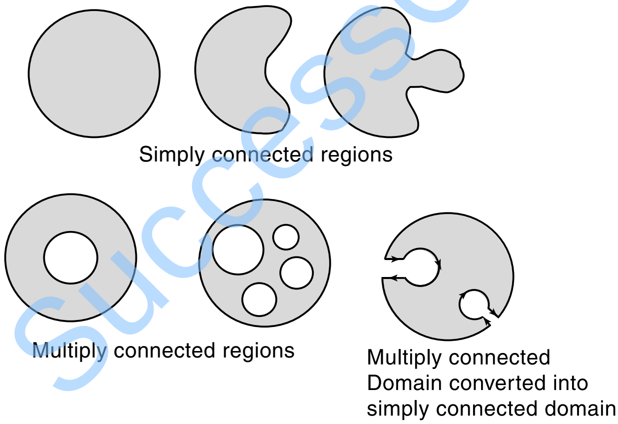


Fig. 16.4

16.2 CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

Statement If $f(z)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C, then

$$\oint_C f(z)\,dz = 0$$

Proof Let $f(z) = u(x, y) + iv(x, y)$

and $z = x + iy \qquad \therefore\ dz = dx + idy.$

Then
$$\int_C f(z)\,dz = \oint_C (u+iv)(dx+idy) = \oint_C (udx - v\,dy) + i\oint_C (v\,dx + udy)$$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous.

Hence, by Green's theorem

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$\therefore$
$$\oint_C f(z)\,dz = \iint_R (-v_x - u_y)\,dxdy + i\iint_R (u_x - v_y)\,dxdy$$

where R is the region bounded by C.
Since $f(z)$ is analytic, u and v satisfy C-R equations.

$\therefore$
$$u_x = v_y \text{ and } u_y = -v_x$$

$\therefore$
$$\oint_C f(z)\,dz = \iint_R (u_y - u_y)dxdy + i\iint_R (u_x - u_x)\,dy = 0 + i0 = 0$$ ■

The French Mathematician E. Goursat proved the above theorem without the condition of continuity of $f'(z)$.

So, the modified statement due to Goursat is known as **Cauchy-Goursat theorem**, which is given below.

16.2.1 Cauchy-Goursat Integral Theorem

If $f(z)$ is analytic at all points inside and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$

Note Cauchy's integral theorem proved for a simply connected region can be extended to multiply connected regions.

Corollary If $f(z)$ is analytic in a domain D and if P and Q are two points in D and if C_1 and C_2 be the two different paths joining P and Q, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Proof Given $f(z)$ is analytic in D.
C_1 and C_2 are two paths joining the points P and Q in D.

By Cauchy's theorem, $\int_{PRQEP} f(z)\,dz = 0$

$$\Rightarrow \quad \int_{PRQ} f(z)\,dz + \int_{QEP} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

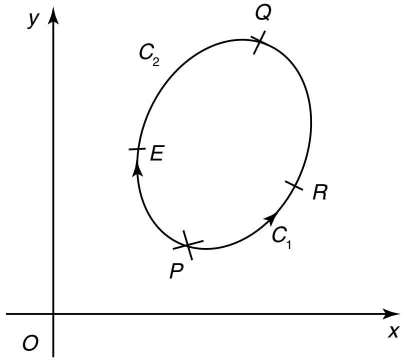


Fig. 16.5

$$\Rightarrow \quad \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0 \quad \Rightarrow \quad \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$ ■

Note The theorem says that if $f(z)$ is analytic in a domain D, then $\int_C f(z)\,dz$ does not dependent on the path when the end points are same.

Definition 16.4 Singular Points

If a function f fails to be a analytic at a point z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is a singular point of f.

Example: For $f(z) = \dfrac{1}{z}$, $z = 0$ is a singular point.

16.3 CAUCHY'S INTEGRAL FORMULA

Statement Let $f(z)$ be an analytic function inside and on a simple closed contour C, taken in the positive sense. If a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$$

Proof

Since $f(z)$ is analytic on and inside C, $\dfrac{f(z)}{z-a}$ is analytic except at $z = a$

i.e., a is a singular point of $\dfrac{f(z)}{z-a}$.

Hence, we draw a small circle C_1 with centre at a and radius ρ (and omit the point a). C_1 is interior to C.

Since $\dfrac{f(z)}{z-a}$ is analytic in the closed region consisting of the contour C and C_1 and all points between them, by Cauchy's integral formula to the multiply connected region, we get

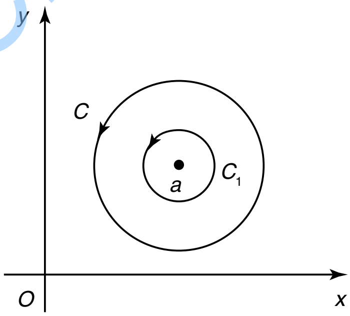


Fig. 16.6

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz$$

Since $|z-a| = \rho$, $z - a = \rho e^{i\theta} \Rightarrow z = a + \rho e^{i\theta} \Rightarrow dz = i\rho e^{i\theta}\,d\theta$

$$\therefore \int_{C_1} \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta}) i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}} = i\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

Since ρ is small, taking $\rho \to 0$, we get

$$\int_C \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a)\,d\theta = if(a)[\theta]_0^{2\pi} = if(a)\cdot 2\pi = 2\pi i f(a)$$

$$\Rightarrow \qquad f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \qquad \blacksquare$$

Note Cauchy's integral formula tells us that if a function $f(z)$ is analytic within and on a simple closed contour C, then the value of f at an interior point of C is completely determined by the values of f on C. When the sense of a curve is not specified, we take the anticlockwise sense as the positive sense.

16.3.1 Cauchy's Integral Formula for Derivatives

If $f(z)$ is analytic inside and on a simple closed curve C, then Cauchy's integral formula is

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Differentiating w.r.to a,

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz, \quad f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz, \ldots, f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

i.e., $$\int_C \frac{f(z)dz}{(z-a)^2} = 2\pi i f'(a), \quad \int_C \frac{f(z)dz}{(z-a)^3} = \frac{2\pi i}{2!}f''(a), \ldots, \int_C \frac{f(z)dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!}f^{(n)}(a)$$

WORKED EXAMPLES

EXAMPLE 1

Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$**, where** C **is** $|z| = \frac{3}{2}$**.**

Solution.

Cauchy's integral formula is $\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$ (1)

Given $I = \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$

$\therefore$ $z = 1, z = 2$ are the singularities.

C is the circle $|z| = \frac{3}{2}$, with centre (0, 0) and radius $= \frac{3}{2}$

If $z = 1$, then $|z| = |1| = 1 < \frac{3}{2}$ $\quad\therefore$ $z = 1$ lies inside C

If $z = 2$, then $|z| = |2| = 2 > \frac{3}{2}$ $\quad\therefore$ $z = 2$ lies outside C.

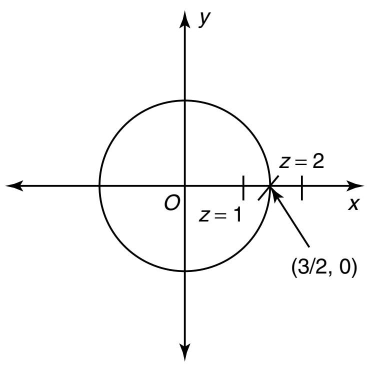


Fig. 16.7

Now $$I = \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = \int_C \frac{\frac{\cos \pi z^2}{z-2}}{z-1}dz$$

Here $a = 1$ and $f(z) = \dfrac{\cos \pi z^2}{z-2}$ is analytic inside and on C.

$$\therefore \qquad f(a) = f(1) = \frac{\cos \pi}{1-2} = 1$$

$$\therefore \text{ by (1),} \qquad I = 2\pi i f(a) = 2\pi i$$

EXAMPLE 2

Evaluate $\displaystyle\int_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle (i) $|z+1+i| = 2$ (ii) $|z+1-i| = 2$.

Solution.

Cauchy's integral formula is

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a) \qquad (1)$$

The given integral is $\quad I = \displaystyle\int_C \frac{z+4}{z^2+2z+5}dz$

The singular points are given by $z^2 + 2z + 5 = 0 \Rightarrow \quad z = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$\therefore$ $z_1 = -1 + 2i$ and $z_2 = -1 - 2i$ are the singular points.

$\therefore$ $z^2 + 2z + 5 = (z - z_1)(z - z_2) = [z - (-1+2i)][z - (-1-2i)]$

(i) Given C is the circle $|z+1+i| = 2$

$\Rightarrow \qquad |z - (-1-i)| = 2$, with centre $P(-1, -1)$ and radius $= 2$

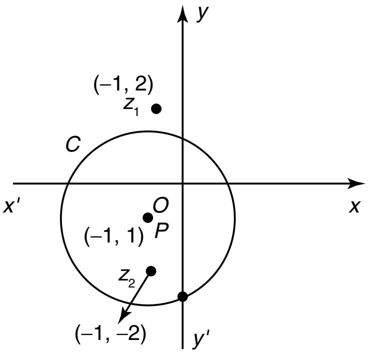


Fig. 16.8

If $z_1 = -1 + 2i$, then $|z+1+i| = |-1+2i+1+i|$

$$= |3i| = 3 > 2$$

$\therefore$ $z_1 = -1 + 2i$ lies outside C.

If $z_2 = -1 - 2i$, then $|z+1+i| = |-1-2i+1+i|$

$$= |-i| = 1 < 2$$

$\therefore \quad z_2$ lies inside C

$$\text{Now } I = \int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{z+4}{(z-z_1)(z-z_2)}dz = \int_C \frac{\frac{z+4}{z-z_1}}{z-z_2}dz$$

Here $a = -1 - 2i$ and $f(z) = \dfrac{z+4}{z-z_1} = \dfrac{z+4}{z-(-1+2i)}$ is analytic inside and on C.

$\therefore$ $$f(a) = f(-1-2i) = \frac{-1-2i+4}{-1-2i-(-1+2i)} = -\frac{(3-2i)}{4i}$$

$\therefore$ by (1), $$I = 2\pi i f(a) = 2\pi i\left[-\frac{(3-2i)}{4i}\right] = \frac{\pi}{2}(2i-3)$$

(ii) C is the circle $|z+1-i| = 2$

$\Rightarrow$ $|z-(-1+i)| = 2$, with centre $P(-1, 1)$ and radius $= 2$.

We have $z_1 = -1+2i$, and $z_2 = -1-2i$

If $z_1 = -1+2i$, then $|z+1-i| = |-1+2i+1-i|$
$= |i| = 1 < 2$

$\therefore$ z_1 lies inside C

If $z_2 = -1-2i$, then $|z+1-i| = |-1-2i+1-i|$
$= |-3i| = 3 > 2$

$\therefore$ z_2 lies outside C

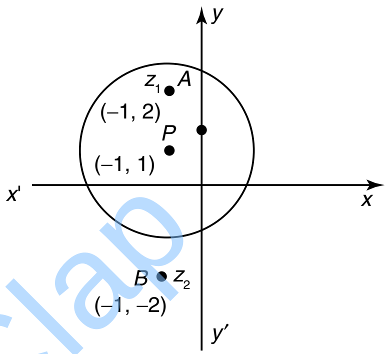


Fig. 16.9

$\therefore$ $$I = \int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{z+4}{(z-z_1)(z-z_2)}dz = \int_C \frac{\frac{z+4}{z-z_2}}{z-z_1}dz$$

Here $a = -1+2i$ and $$f(z) = \frac{z+4}{z-z_2} = \frac{z+4}{z-(-1-2i)}$$

is analytic inside and on C.

$\therefore$ $$f(a) = f(-1+2i) = \frac{-1+2i+4}{-1+2i-(-1-2i)} = \frac{3+2i}{4i}$$

$\therefore$ by (1), $$I = 2\pi i f(a) = 2\pi i\frac{3+2i}{4i} = \frac{\pi}{2}(3+2i)$$

EXAMPLE 3

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$, where C is $|z| = 3$.

Solution.

Cauchy's integral formula is $\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$ (1)

Given integral is $$I = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$$

The singular points are $z = 1$ and $z = 2$

The circle is $|z| = 3$ with centre $(0, 0)$ and radius $= 3$

If $z = 1$, then $|z| = |1| = 1 < 3$
and if $z = 2$ then $|z| = |2| = 2 < 3$
$\therefore \; z = 1, z = 2$ lie inside C.
Here $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.

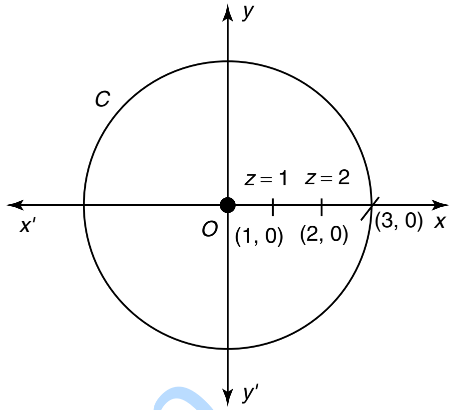


Fig. 16.10

Let $\dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$\therefore \qquad 1 = A(z-2) + B(z-1)$

Put $z = 1$. $\quad \therefore\; 1 = A(1-2) \quad \Rightarrow \quad A = -1$

Put $z = 2$. $\quad \therefore\; 1 = B(2-1) \quad \Rightarrow \quad B = 1$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$\therefore \quad I = \int_C -\frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$\therefore$ by (1),
$$\begin{aligned} I &= -2\pi i f(1) + 2\pi i f(2) \\ &= -2\pi i[\sin \pi + \cos \pi] + 2\pi i[\sin 4\pi + \cos 4\pi] \\ &= -2\pi i(0-1) + 2\pi i(0+1) = 2\pi i + 2\pi i = 4\pi i \end{aligned}$$

EXAMPLE 4

By Cauchy's integral formula, evaluate $\displaystyle\int_C \frac{z+1}{z^4 - 4z^3 + 4z^2} dz$, where C is the circle $|z - 2 - i| = 2$.

Solution.

Given $\quad I = \displaystyle\int_C \frac{z+1}{z^4 - 4z^3 + 4z^2} dz$

$$= \int_C \frac{z+1}{z^2(z^2 - 4z + 4)} dz = \int_C \frac{z+1}{z^2(z-2)^2} dz$$

$\therefore$ singular points are $z = 0$ and $z = 2$.
C is the circle $|z - 2 - i| = 2$ with centre (2, 1)
and radius = 2

If $\quad z = 0$, then $|z - 2 - i| = |0 - 2 - i| = \sqrt{5} > 2$

$\therefore \quad z = 0$, lies outside C

If $z = 2$, then $|z - 2 - i| \; |2 - 2 - i| = 1 < 2$

$\therefore \quad z = 2$, lies inside C

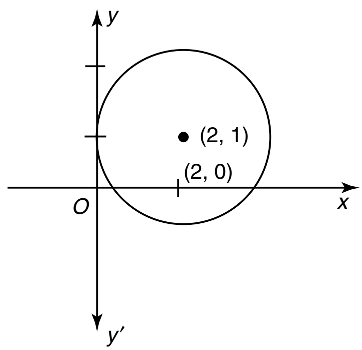


Fig. 16.11

$$\therefore \quad I = \int_C \frac{z+1}{z^2(z-2)^2} dz = \int_C \frac{\frac{z+1}{z^2}}{(z-2)^2} dz$$

By Cauchy's integral formula for derivative, $\int_C \frac{f(z)}{(z-a)^2} = 2\pi i f'(a)$ (1)

Here $f(z) = \frac{z+1}{z^2}$ and $a = 2$

$\therefore$ $f'(z) = \frac{z^2 \cdot 1 - (z+1)2z}{z^4} = \frac{-z^2 - 2z}{z^4} = -\frac{z+2}{z^3}$

$\therefore$ $f'(a) = f'(2) = -\frac{2+2}{8} = -\frac{1}{2}$

$\therefore$ by (1), $I = 2\pi i f'(2) = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$

EXAMPLE 5

If $f(a) = \int_C \frac{3z^2 + 7z + 1}{z - a} dz$, where C is $|z| = 2$, find $f(3)$, $f(1)$, $f'(1-i)$, $f''(1-i)$.

Solution.

Given $f(a) = \int_C \frac{3z^2 + 7z + 1}{z - a} dz = \frac{1}{2\pi i}\int_C \frac{2\pi i(3z^2 + 7z + 1)}{z - a} dz$

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} dz$$

Here $f(z) = 2\pi i(3z^2 + 7z + 1)$ and C is the circle $|z| = 2$ with centre (0, 0) and radius = 2

(i) If $z = 3$, then $|z| = |3| = 3 > 2$ $\therefore$ $z = 3$ lies outside C

$\therefore$ $\frac{f(z)}{z-3}$ is analytic inside and on C. $\therefore$ $\int_C \frac{f(z)}{z-3} dz = 0 \Rightarrow f(3) = 0$

(ii) Now $z = 1$ lies inside C, since $|z| = |1| = 1 < 3$

$\therefore$ $f(1) = 2\pi i(3 \cdot 1 + 7 \cdot 1 + 1) = 22\pi i$

(iii) We have $f(z) = 2\pi i(3z^2 + 7z + 1)$

$\therefore$ $f'(z) = 2\pi i(6z + 7)$, $f''(z) = 2\pi i \times 6 = 12\pi i$

If $z = 1 - i$, then $|z| = |1 - i| = \sqrt{2} < 3$ $\therefore$ $(1 - i)$ lies inside C

$\therefore$ $f'(1-i) = 2\pi i(6\cdot(1-i)+7) = 2\pi i(13-6i) = 2\pi(6+13i)$

and $f''(1-i) = 12\pi i$

EXAMPLE 6

Using Cauchy's integral formula evaluate $\int_C \frac{7z-1}{z^2-3z-4}dz$, where C is the ellipse $x^2+4y^2=4$.

Solution.

Let $$I = \int_C \frac{7z-1}{z^2-3z-4}dz,$$

where C is the ellipse $x^2+4y^2=4 \Rightarrow \frac{x^2}{4}+y^2=1$

Singular points are given by $z^2-3z-4=0 \Rightarrow (z-4)(z+1)=0 \Rightarrow z=-1, 4$

If $z=-1$, then $x+iy=-1 \Rightarrow x=-1, y=0$

$$\therefore \quad \frac{x^2}{4}+y^2=\frac{1}{4}+0=\frac{1}{4}<1$$

So, $z=-1$ lies inside the ellipse.

If $z=4$, then $x+iy=4 \Rightarrow x=4, y=0$

$$\therefore \quad \frac{x^2}{4}+y^2=\frac{4^2}{4}+0=4>1$$

So, $z=4$ lies outside the ellipse.

$$\therefore \quad I=\int_C \frac{7z-1}{(z-4)(z+1)}dz=\int_C \frac{\left(\frac{7z-1}{z-4}\right)}{z+1}dz=\int_C \frac{f(z)}{z+1}dz$$

where $f(z)=\frac{7z-1}{z-4}$ is analytic inside and on C and $a=-1$

$\therefore$ Cauchy's integral formula is $\int_C \frac{f(z)}{z-a}dz=2\pi i f(a)=2\pi i f(-1)$

$$\therefore \quad I=2\pi i f(-1)=2\pi i\left(\frac{7(-1)-1}{-1-4}\right)=2\pi i\cdot\frac{8}{5}=\frac{16}{5}\pi i$$

EXERCISE 16.1

Evaluate the following integrals using Cauchy's integral formula.

1. $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$, where C is the circle $|z|=3$.
2. $\int_C \frac{z+1}{(z-1)(z-3)}dz$, where C is $|z|=2$.
3. $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)(z-3)}dz$, where C is $|z|=4$.
4. $\int_C \frac{z^2}{(z+1)^2}dz$, where C is $|z-i|=1$.
5. $\int_C \frac{dz}{(z^2+4)^2}$, where C is $|z-i|=2$.
6. $\int_C \frac{zdz}{(z-1)(z-2)}$, where C is $|z-2|=\frac{1}{2}$.

7. $\int_C \frac{z^2+1}{(z-1)(z-2)}dz$, where C is $|z|=3$.

8. $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is $|z|=2$.

9. $\int_C \frac{e^z}{z^2+4}dz$, where C is $|z-i|=2$.

10. If $f(a)=\int_C \frac{4z^2+z+5}{z-a}dz$, where C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, find the values of $f(i), f'(-1)$ and $f''(-i)$.

11. $\int_C \frac{e^{iz}}{z^3}dz$ where C is $|z|=2$.

12. $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}dz$, where C is $|z|=3$.

13. $\int_C z^2 \cdot e^{\frac{1}{z}}dz$ where C is $|z|=1$.

14. $\int_C \frac{e^z\,dz}{(z+2)(z+1)^2}$, where C is $|z|=3$.

15. $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is $|z|=2$.

16. $\int_C \frac{z-1}{(z+1)(z-2)^2}\,dz$, where C is $|z-i|=2$.

17. $\int_C \frac{z\,dz}{(z+1)^2(z-2)}$, where C is the circle $|z-2|=\frac{1}{2}$.

18. $\int_C \frac{dz}{z-2}$, where C is the circle whose centre is (2, 0) and radius 4.

19. $\int_C \frac{z+1}{z^2(z-2)}dz$, where C is the circle $|z-2-i|=2$.

20. $\int_C \frac{z}{(z-1)^3}dz$, where C is $|z|=2$, using Cauchy's integral formula.

21. $\int_C \frac{z^2+1}{z^2-1}dz$, where C is $|z-1|=1$.

22. $\int_C \frac{dz}{(z+1)^2(z-2)}$, where C is the circle $|z|=\frac{3}{2}$.

23. $\int_C \frac{z+3}{2z+5}dz$, where C is $|z|=3$.

ANSWERS TO EXERCISE 5.1

1. $2\pi i(e^4-e^2)$ 2. $2\pi i$ 3. $-4\pi i$ 4. $\frac{\pi}{2}$ 5. $\frac{\pi}{16}$ 6. $4\pi i$ 7. $6\pi i$

8. $\frac{8\pi}{3}ie^{-2}$ 9. $\frac{\pi}{2}e^{2i}$ 10. $2\pi(-1+i)$; $-14\pi i$, $16\pi i$ 11. $-\pi i$ 12. $4\pi(\pi+1)i$

13. $\frac{\pi i}{3}$ 14. $\frac{2\pi i}{e^2}$ 15. $\frac{8\pi i}{3e^2}$ 16. $-\frac{4}{9}\pi i$ 17. $-2\pi i$ 18. $2\pi i$ 19. $\frac{3\pi i}{2}$

20. 0 21. $2\pi i$ 22. $-\frac{2\pi i}{9}$ 23. $\frac{\pi i}{2}$

16.4 TAYLOR'S SERIES AND LAURENT'S SERIES

Complex power series are the most important and fundamental series in complex analysis because power series represent analytic functions. Conversely, every analytic function can be represented by a power series namely Taylor's series.

Power series methods are considered as efficient tools for solving various physical and engineering problems.

Definition 16.5 Power Series

A series of the form $\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ (1)

where z is a complex variable, a_0, a_1, a_2, … and a are complex constants, is called a power series in powers of $z - a$ or a power series about the point a.

a_0, a_1, a_2, … are called the **co-efficients** of the series and a is called the **centre** of the series.

If $a = 0$, then we get the particular power series in powers of z or power series about the origin

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

Note

1. The power series converges at all points inside a circle $|z-a| = R$ for some positive number R and diverges outside the circle. This circle is called the **circle of convergence** and its radius R is called the **radius of convergence**.
2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$, then $R = \frac{1}{l}$ is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(z-a)^n$.

   This formula is called **Hadmard's formula.**

   If $l = 0$, then $R = \infty$ and so the power series converges for all z in the finite plane.

   If $l = \infty$, then $R = 0$ and so the series converges only at the centre $z = a$.

16.4.1 Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at a and radius R, then at each point z inside the circle, $f(z)$ has the series representation

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \infty$$

Note

1. This equation means that the power series converges to $f(z)$ at each point z inside C. Further, this is the expansion of $f(z)$ into a Taylor's series about the point a.
2. Any function $f(z)$ that is known to be analytic at a point a must have a Taylor's series about a
3. If $a = 0$, we get the Taylor's series about the origin which is called the Maclaurin's series.

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots$$

We now list below the Maclaurin's series for some elementary functions.

1. $e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$ if $|z| < \infty$
2. $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots$ if $|z| < \infty$
3. $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$ if $|z| < \infty$
4. $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$ if $|z| < 1$
5. $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$ if $|z| < 1$
6. $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$ if $|z| < 1$
7. $\log_e(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \cdots$ if $|z| < 1$

16.4.2 Laurent's Series

Let C_1 and C_2 be concentric circles with centre a and radii r_1, r_2 $(r_1 > r_2)$.

Let $f(z)$ be analytic in the annular domain between the circles.

Then at each point of the annular domain $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}, \quad r_2 < |z-a| < r_1 \qquad (1)$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, \quad n = 0, 1, 2, \ldots$$

and $$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{1-n}} dw, \quad n = 1, 2, 3, \ldots$$

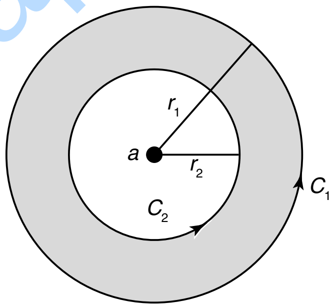


Fig. 16.12

Note

1. The series (1) is called Laurent's series about $z = a$.
   Laurent's series is a series with positive and negative integral powers of $(z - a)$. The part of the series with positive powers of $(z - a)$ is called the **analytic part** or **regular part** and the part with negative powers of $(z - a)$ is called the **principal part** of the Laurent's series.
2. Taylor's series of an analytic function is unique and Laurent's series of an analytic function $f(z)$ in its annular region is unique. So, we can find the series by any method, not necessarily using the formula for a_n and b_n given above. Binomial series is usually used.
3. If $f(z)$ is analytic at a point a we find Taylor's series expansion about a.
   If $f(z)$ is not analytic at a but is analytic in some neighbourhood of a, then we find Laurent's series expansion of $f(z)$ about a.

Important Remark If $f(z)$ has many isolated singular points then there are many annular regions $R_1, R_2 \ldots$ in which $f(z)$ is analytic and so there are many Laurent's series for $f(z)$ about a, one for each region.

The Laurent's series is usually taken as the one that converges near a.

WORKED EXAMPLES

EXAMPLE 1

Find the Taylor's series to represent the function $\dfrac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

Solution.

Let $$f(z)=\frac{z^2-1}{(z+2)(z+3)}$$

The singular points $z=-2, z=-3$ are outside $|z|<2$.
So, $f(z)$ is analytic in the open disk $|z|<2$ about 0.
We shall split $f(z)$ into partial fractions.
Since the degrees of Nr. and Dr. are same, we have,

$$\frac{z^2-1}{(z+2)(z+3)}=1+\frac{A}{z+2}+\frac{B}{z+3}$$

$\Rightarrow$ $$z^2-1=(z+3)(z+2)+A(z+3)+B(z+2)$$

Put $z=-2$, then $4-1=A(-2+3) \quad \Rightarrow A=3$

Put $z=-3$, then $9-1=B(-3+2) \quad \Rightarrow B=-8$

$\therefore$ $$\frac{z^2-1}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

Now $|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{3}\right|<\frac{2}{3}<1$.

$\therefore$ the Taylor's series is

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

EXAMPLE 2

Expand $f(z)=\sin z$ about $z=\dfrac{\pi}{4}$.

Solution.

Given $f(z)=\sin z$. It is analytic for all z

$\therefore$ $f(z)$ can be expanded as Taylor's series about $z=\dfrac{\pi}{4}$.

$\therefore$ $$f(z)=f\left(\frac{\pi}{4}\right)+\frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(z-\frac{\pi}{4}\right)+\frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z-\frac{\pi}{4}\right)^3+\cdots$$

We have $f(z) = \sin z,$ $\quad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ $f'(z) = \cos z,$ $\quad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z,$ $\quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z,$ $\quad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$$\therefore \quad f(z) = \frac{1}{\sqrt{2}} + \frac{1}{1!\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{2!\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!\sqrt{2}}\left(z - \frac{\pi}{4}\right)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(z - \frac{\pi}{4}\right)^2 + \cdots\right]$$

EXAMPLE 3

Expand $\frac{1}{z^2 - 3z + 2}$ in the region (i) $1 < |z| < 2$ (ii) $0 < |z - 1| < 2$ (iii) $|z| > 2$

Solution.

Let $$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)}$$

Let $$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$\therefore$ $1 = A(z-2) + B(z-1)$

Put $z = 1$, then $1 = A(1-2) \Rightarrow A = -1$

Put $z = 2$, then $1 = B(2-1) \Rightarrow B = 1$

$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

(i) If $1 < |z| < 2$, then z lies in the annular region about $z = 0$, where $f(z)$ is analytic. So, we expand as Laurent's series about $z = 0$

Now $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$ and $1 < |z| \Rightarrow \frac{1}{|z|} < 1$

So, the Laurent's series is

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{z}\frac{1}{\left(1 - \frac{1}{z}\right)} - \frac{1}{2}\frac{1}{\left(1 - \frac{z}{2}\right)}$$

$$= -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= -[z^{-1}+z^{-2}+z^{-3}+\cdots]-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\cdots\right]$$

(ii) Given $0<|z-1|<2$

This region is annular about $z=1$

So, the expansion of $f(z)$ in the region is Laurent's series about $z=1$

Put $t=z-1 \quad \Rightarrow \quad z=t+1 \quad \therefore \quad 0<|t|<2$

$\therefore$ the Laurent's series is

$$f(z) = -\frac{1}{z-1}+\frac{1}{z-2} = -\frac{1}{t+1-1}+\frac{1}{t+1-2}$$

$$= -\frac{1}{t}+\frac{1}{t-1}$$

$$= -\frac{1}{t}-(1-t)^{-1}$$

$$= -t^{-1}-[1+t+t^2+t^3+\cdots] \qquad [\because |t|<2,\ |t|<1 \text{ is true}]$$

$$= -(z-1)^{-1}-[1+(z-1)+(z-1)^2+\cdots]$$

(iii) Given $|z|>2$

This region is annular about $z=0$, where $f(z)$ is analytic.

So, the expansion is Laurent's series about $z=0$

Now $\qquad |z|>2 \Rightarrow \left|\frac{z}{2}\right|>1 \Rightarrow \left|\frac{2}{z}\right|<1$ and $\left|\frac{1}{z}\right|<\frac{1}{2}<1.$

$\therefore$ the Laurent's series is

$$f(z) = -\frac{1}{z-1}+\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$= -[z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots]+[z^{-1}+2z^{-2}+4z^{-3}+8z^{-4}+\cdots]$$

$$\therefore \qquad f(z) = z^{-2}+3z^{-3}+7z^{-4}+\cdots$$

EXAMPLE 4

Expand $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (i) $2<|z|<3$, (ii) $|z|>3$.

Solution.

Let $$f(z)=\frac{z^2-1}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3}$$ [Ref. Example 1]

(i) Given $2<|z|<3$

The region is annular about $z=0$ and $f(z)$ is analytic in this region.

So, the expansion is Laurent's series about $z=0$

Now $|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$ and $2<|z| \Rightarrow \frac{2}{|z|}<1$

$\therefore$ the Laurent's series is

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$=1+3[z^{-1}-2z^{-2}+4z^{-3}-8z^{-4}+\cdots]-\frac{8}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right]$$

(ii) Given $|z|>3$

This region is annular about $z=0$, where $f(z)$ is analytic.

Now $|z|>3 \Rightarrow \frac{1}{|z|}<\frac{1}{3} \Rightarrow \frac{2}{|z|}<\frac{2}{3}<1$ and $\frac{3}{|z|}<1$

$\therefore$ the Laurent's series is

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$=1+3[z^{-1}-2z^{-2}+4z^{-3}-8z^{-4}+\cdots]-8[z^{-1}-3z^{-2}+9z^{-3}-27z^{-4}+\cdots]$$

$$=1-5z^{-1}+18z^{-2}-60z^{-3}+192z^{-4}+\cdots$$

EXAMPLE 5

Find the Laurent's series expansion of $f(z)=\dfrac{7z-2}{z(z-2)(z+1)}$ in $1<|z-1|<3$.

Solution.

Given $$f(z) = \frac{7z-2}{z(z-2)(z+1)} \text{ and } 1 < |z+1| < 3$$

The singular points are $z = 0, z = 2, z = -1$ which lie outside the annular region $1 < |z+1| < 3$ about $z = -1$
So, we can expand $f(z)$ as a Laurent's series about $z = -1$ or in terms of $z + 1$.
We shall split $f(z)$ into partial fractions.

Let $$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ $$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 0$, then $-2 = A(-2)(1)$ $\Rightarrow$ $A = 1$
Put $z = 2$, then $14 - 2 = B \cdot 2(2+1)$ $\Rightarrow$ $B = 2$
Put $z = -1$, then $-7 - 2 = C(-1)(-1-2)$ $\Rightarrow$ $C = -3$

$\therefore$ $$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} + \frac{-3}{z+1}$$

We have $1 < |z+1| < 3$. put $t = z + 1$ $\Rightarrow z = t - 1$ $\therefore$ $1 < |t| < 3$

Now $1 < |t| \Rightarrow \frac{1}{|t|} < 1$ and $|t| < 3 \Rightarrow \frac{|t|}{3} < 1.$

$\therefore$ the Laurent's series is

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} = \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t}$$

$$= \frac{1}{t}\frac{1}{\left(1-\frac{1}{t}\right)} + \frac{2}{(-3)\left(1-\frac{t}{3}\right)} - \frac{3}{t}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t}$$

$$= \frac{1}{t}\left[1 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots\right] - \frac{2}{3}\left[1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots\right] - \frac{3}{t}$$

$$= -\frac{2}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots - \frac{2}{3}\left[1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \cdots\right]$$

$$= -2(z+1)^{-1} + (z+1)^{-2} + (z+1)^{-3} + \cdots$$
$$-\frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \frac{(z+1)^3}{27} + \cdots\right]$$

EXAMPLE 6

Find the Laurent's series of $f(z) = \dfrac{z}{(z^2+1)(z^2+4)}$ in $1 < |z| < 2$.

Solution.

Given $$f(z) = \frac{z}{(z^2+1)(z^2+4)} \text{ and } 1 < |z| < 2$$

Singular points are $z = \pm i,\ z = \pm 2i$, which lie outside $1 < |z| < 2$

$\therefore$ $f(z)$ is analytic in this annular region about $z = 0$

Let $$\frac{z}{(z^2+1)(z^2+4)} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+4}$$

$\therefore$ $$z = (Az+B)(z^2+4) + (Cz+D)(z^2+1)$$

Equating coefficients of z^3, $0 = A + C \Rightarrow A = -C$

Equating coefficients of z^2, $0 = B + D \Rightarrow D = -B$

Equating coefficients of z, $1 = 4A + C \Rightarrow 1 = 4A - A \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$

$\therefore$ $C = -\frac{1}{3}$

Put $z = 0$, then $0 = 4B + D \Rightarrow 4B - B = 0 \Rightarrow B = 0 \quad \therefore \quad D = 0$

$\therefore$ $$\frac{z}{(z^2+1)(z^2+4)} = \frac{z}{3(z^2+1)} - \frac{z}{3(z^2+4)}$$

$\therefore$ $$f(z) = \frac{z}{3(z^2+1)} - \frac{z}{3(z^2+4)}$$

Now $|z| < 2 \Rightarrow |z|^2 < 4 \Rightarrow \left|\frac{z^2}{4}\right| < 1$ $\qquad [\because |z|^2 = |z^2|]$

and $1 < |z| \Rightarrow \frac{1}{|z|} < 1 \Rightarrow \frac{1}{|z^2|} < 1$

$\therefore$ the Laurent's series is

$\therefore$ $$f(z) = \frac{z}{3}\cdot\frac{1}{z^2\left[1+\frac{1}{z^2}\right]} - \frac{z}{3}\cdot\frac{1}{4\left[1+\frac{z^2}{4}\right]}$$

$$= \frac{1}{3z}\left[1+\frac{1}{z^2}\right]^{-1} - \frac{z}{12}\left[1+\frac{z^2}{4}\right]^{-1}$$

$$= \frac{1}{3z}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right] - \frac{z}{12}\left[1-\frac{z^2}{4}+\frac{z^4}{16}-\frac{z^6}{64}+\cdots\right]$$

EXAMPLE 7

Find the Laurent's series of the following functions

(i) $f(z) = z^2 e^{\frac{1}{z}}$ **about** $z = 0$ **(ii)** $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ **about** $z = 1$

Solution.

(i) Given $f(z) = z^2 e^{\frac{1}{z}}$. $z = 0$ is a singular point

$\therefore$ $f(z)$ is analytic for $|z| > 0$ and the Laurent's series is

$$f(z) = z^2\left[1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!}\frac{1}{z^3}+\cdots\right]$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!}z^{-1} + \cdots$$

(ii) Given $f(z) = \dfrac{e^{2z}}{(z-1)^3}$. $z = 1$ is a singular point.

So, in the region $|z-1| > 0$, $f(z)$ is analytic and so $f(z)$ can be expanded as Laurent's series in the annular region about $z = 1$

Put $t = z - 1$. $\therefore$ $z = t + 1$

$$\therefore \quad f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(t+1)}}{t^3} = \frac{e^2}{t^3} e^{2t}$$

$$= \frac{e^2}{t^3}\left[1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \cdots\right]$$

$$= e^2\left[\frac{1}{t^3} + \frac{2}{t^2} + \frac{2}{t} + \frac{4}{3} + \frac{2}{3}t + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2}{3}(z-1) + \cdots\right]$$

EXERCISE 16.2

I. Find the Taylor's series for the following functions about the indicated point.

1. $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $z = 0$.
2. $f(z) = \dfrac{z-1}{z^2}$ about $z = 1$.
3. $f(z) = \dfrac{1}{z}$ about $z = 2$.
4. $f(z) = \dfrac{1}{z^2 - 4z + 3}$ about $z = 4$.
5. $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ when $|z| < 1$.
6. $f(x) = \dfrac{1}{z-2}$ at $z = 1$.
7. $f(x) = \dfrac{1}{z-3}$ valid in $|z| < 3$.

II. Expand the following functions as a Laurent's series.

1. $f(z) = \dfrac{z-1}{(z+2)(z+3)}$ valid in $2 < |z| < 3$.
2. $f(z) = \dfrac{z+3}{z(z^2 - z - 2)}$ valid in $1 < |z| < 2$.
3. $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in $1 < |z| < 2$.
4. $f(z) = \dfrac{z-1}{z^2}$ valid in $|z-1| > 1$.
5. $f(z) = \dfrac{1}{z(z-1)}$ for $0 < |z| < 1$ and $0 < |z-1| < 1$.
6. $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ valid in $2 < |z| < 3$.
7. $f(z) = \dfrac{z^2}{(z+2)(z-3)}$ valid in $2 < |z| < 3$.
8. $f(z) = \dfrac{1}{z(z^2 - 3z + 2)}$ valid in $1 < |z| < 2$.
9. $f(z) = \dfrac{12}{z(2-z)(1+z)}$ valid in (a) $0 < |z| < 1$, (b) $1 < |z| < 2$.
10. $f(x) = \dfrac{1}{z-3}$ valid in $|z| > 3$.

11. $f(z)=\dfrac{z}{(z-1)(z-3)}$ valid in the following region $0<|z-1|<2$.

12. $f(z)=\dfrac{1}{z(1-z)}$ valid in the region $|z+1|>2$.

13. $f(z)=\dfrac{1}{(z+1)(z+3)}$ valid for the regions $|z|>3$ and $1<|z|<3$.

ANSWERS TO EXERCISE 16.2

I

1. $-[1-z+z^2-z^3+\cdots]+\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right]$

2. $(z-1)-2(z-1)^2+3(z-1)^3-4(z-1)^4+\cdots$

3. $\frac{1}{2}-\frac{1}{2^2}(z-2)+\frac{1}{2^3}(z-2)^2-\cdots$

4. $\frac{1}{3}-\frac{4}{9}(z-4)+\frac{13}{27}(z-4)^2-\frac{40}{81}(z-4)^3+\cdots$

5. $\frac{1}{10}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right]+\frac{2-z}{5}(1-z^2+z^4-z^6+\cdots)$

6. $-[1+(z-1)+(z-1)^2+(z-1)^3+\cdots]$

7. $-\frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right]$

II

1. $-\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]+\frac{4}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\cdots\right]$

2. $-\frac{5}{12}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]-\frac{3}{2z}+\frac{2}{3z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$

3. $-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]-\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$

4. $\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots$

5. $\frac{1}{z}-[1+z+z^2+z^3+\cdots]$ and $\frac{1}{z-1}-[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]$

6. $1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$

7. $1-\frac{4}{5z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{3}{5}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$

8. $\frac{1}{2z}-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]-\frac{1}{4}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$

9. (a) $\frac{4}{z}[1-z+z^2-z^3+\cdots]+\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$

(b) $1+\frac{2}{z}+\frac{4}{z^2}-\frac{4}{z^3}+\cdots+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots$

10. $z^{-1}+3z^{-2}+3^2z^{-3}+\cdots$

11. $\frac{-(z-1)^{-1}}{2}-\frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\frac{(z-1)^3}{8}+\cdots\right]$

12. $-[(z+1)^{-2}+3(z+1)^{-3}+7(z+1)^{-4}+\cdots]$

13. $\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right]$

16.5 CLASSIFICATION OF SINGULARITIES

A point at which a complex function $f(z)$ is analytic is called a regular point or ordinary point of $f(z)$.

A point $z = a$ is a singular point of $f(z)$ if $f(z)$ is not analytic (or not even defined), but is analytic at some point in every deleted neighbourhood of a.

EXAMPLE

1. $f(z)=\frac{1}{z}$ has $z = 0$ as singular point, since $f(z)$ is not analytic at $z = 0$ but analytic at other points.

2. $f(z)=\frac{1}{z(z-1)}$ has $z = 0$, $z = 1$ as singular points, because $f(z)$ is not analytic at these points, but there are neighbourhoods where $f(z)$ is analytic.

   But $f(z)=z^2$ has no singular points, since it is analytic everywhere.

Definition 16.5 Isolated Singularity

A singular point $z = a$ is called an isolated singularity of the function $f(z)$ if there exists a neighbourhood of a in which there is no other singularity.

In example 2, we can find a circular disk with centre $z = 1$ and radius $r < 1$ in which there is no other singularity.

A singular point which is not isolated is called a **non-isolated singularity**.

For example, $f(z)=\frac{1}{\sin\frac{1}{z}}$ has $z=\frac{1}{n\pi}$, $n \in Z$ as singular points, while 0 is a non-isolated singular point because every deleted neighbourhood of 0 contains a singularity $\frac{1}{n\pi}$ for large n.

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For example, $f(z) = \dfrac{1}{\sin\frac{1}{z}}$ has $z = \dfrac{1}{n\pi}$, $n \in Z$

as singular points, while 0 is a non-isolated singular point because every deleted neighbourhood of 0 contains a singularity $\dfrac{1}{n\pi}$ for large n.

Note If a function has only finite number of singularities in a region, then they are isolated singularities.

Isolated singularities are further classified as (i) poles, (ii) essential singularities.

These classifications are made on the basis of the principal part of Laurent's series.

Definition 16.6 Pole

An isolated singular point ***a*** of $f(z)$ is said to be a **pole of order *m***, if there exists a positive integer m such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} \cdots = 0$ in the Laurent's series of $f(z)$ about a.

In other words, if the Laurent's series is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}, r_2 < |z-a| < r_1,$$ where $b_m \neq 0$, then the point ***a*** is called a **pole of order *m***.

If $m = 1$, then a is called a simple pole and if $m = 2$, then a is called a double pole or pole of order 2.

For example, $f(z) = \dfrac{1}{z(z-1)^2}$ has $z = 0$ as a simple pole and $z = 1$ as a pole of order 2.

Definition 16.7 Essential Singularity

An isolated singular point a is said to be an essential singularity of $f(z)$ if the principal part of Laurent's series of $f(z)$ about a contains infinitely many terms.

For example: $f(z) = e^{\frac{1}{z}}$ has $z = 0$ as essential singularity because

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \quad \text{if } 0 < |z| < \infty$$

$$= 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

has infinitely many terms in the principal part. We also say $z = 0$ is an isolated essential singularity.

An important note: Consider the expansion of

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \quad 1 < |z| < 3,$$ which is Laurent's series.

There are infinite number of negative powers of z. Yet $z = 0$ is not an essential singularity of $f(z)$. Why? The reason is the domain of convergence of $f(z)$ is $1 < |z| < 3$ which is not a deleted neighbourhood of 0.

It can be seen that it is the Laurent's series of the function $f(z) = \dfrac{2z}{(1-z)(z-3)}$ in the annular region $1 < |z| < 3$. So, it is important to consider the Laurent's series valid in the **immediate neighbourhood** of a singular point.

Definition 16.8 Removable Singularity

An isolated singular point $z = a$ of $f(z)$ is called a removable singularity of $f(z)$ if in some neighbourhood of a the Laurent's series expansion of $f(z)$ has no principal part.

For example: $f(z) = \dfrac{\sin z}{z}, \qquad z \neq 0$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad 0 < |z| < \infty$$

It has no principal part. So, $z = 0$ is a removable singularity if $\frac{\sin z}{z}$ is not defined at $z = 0$

Suppose
$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

then the series converges to 1 at $z = 0$ and so $f(z)$ is analytic at $z = 0$.

Note From the above definitions, the following results follow.

1. A singularity $z = a$ is an essential singularity if there is no integer n such that
$$\lim_{z \to a}(z - a)^n f(z) = A \neq 0$$
2. A singularity $z = a$ is a pole of order m if $\lim_{z \to a}(z - a)^m f(z) = A \neq 0$
3. A singularity $z = a$ is a removable singularity of $f(z)$ if $\lim_{z \to a} f(z) =$ a finite number A.

Zero of order *m*

Let $f(z)$ be analytic at $z = z_0$. If $f(z_0) = 0$ then z_0 is called a zero of $f(z)$. If there is a positive integer m such that $f'(z_0) = 0, \quad f''(z_0) = 0, \quad f'''(z_0) = 0, \cdots, \quad f^{(m-1)}(z_0) = 0$ and $\boldsymbol{f^{(m)}(z_0) \neq 0}$,

then $f(z)$ is said to have a **zero of order m at z_0**.

Thus, $f(z) = (z - z_0)^m g(z), \; g(z_0) \neq 0$

Note If $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$, $q(z)$ are analytic at $z = a$ and $p(a) \neq 0$ and if a is a zero of order m for $q(z)$, then a is a pole of order m for $f(z)$.

16.6 RESIDUE

Definition 16.9

Let $z = a$ be an isolated singular point of $f(z)$. The coefficient b_1 of $(z - a)^{-1}$ in the Laurent's series expansion of $f(z)$ about a is called the **residue of $f(z)$** at $z = a$. We denote $b_1 = [\text{Res}\, f(z)]_{z=a}$ or $R(a)$.

16.6.1 Methods of Finding Residue

1. If $z = a$ is a simple pole, then $R(a) = \lim_{z \to a}(z - a)f(z)$
2. If $z = a$ is a pole of order m, then
$$R(a) = \frac{1}{(m-1)!}\lim_{z \to a}\left\{\frac{d^{m-1}}{dz^{m-1}}[(z - a)^m f(z)]\right\}$$
3. Let $f(z) = \frac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic functions at $z = a$. If $h(a) = 0$, $h'(a) \neq 0$ and $g(a) \neq 0$ are finite, then $z = a$ is a simple pole of $f(z)$ and $R(a) = \lim_{z \to a}\frac{g(z)}{h'(z)}$

Note

1. Residue is not defined for non-isolated singularity of $f(z)$.
2. The residue at an essential singularity of $f(z)$ is found out using Laurent's expansion of $f(z)$ directly.
3. Residue at a removable singularity of $f(z)$ is equal to zero.
4. The number of isolated singular points inside a simple closed curve is finite. This fact is used in Cauchy's residue theorem.

16.7 CAUCHY'S RESIDUE THEOREM

Statement If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ lying inside C, then

$$\int_C f(z)dz = 2\pi i[R(z_1)+R(z_2)+R(z_3)+\cdots+R(z_n)]$$

where the integral over C is taken in the anticlockwise sense.

Proof Since $z_1, z_2, \ldots, z_n$ are finite number of singular points lying inside the simple closed curve, they are isolated singular points.

So, we can find non overlapping neighbourhoods of these points which are small circles with these points as centres. Let $C_1, C_2, \ldots, C_n$ be positively oriented circles with centres at $z_1, z_2, z_3, \ldots, z_n$ respectively. So, $f(z)$ is analytic in the multiply connected region bounded by $C_1, C_2, \ldots, C_n$ and C.

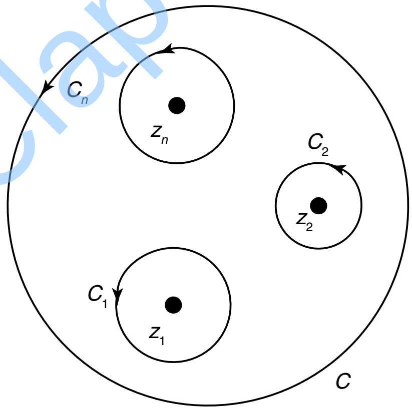

Fig. 16.13

Hence, by Cauchy's extension theorem to multiply connected region, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$$

But $$R(z_k) = [\text{Res} f(z)]_{z=z_k} = \frac{1}{2\pi i}\int_{C_k} f(z)dz$$

$\therefore$ $$\int_{C_k} f(z)dz = 2\pi i R(z_k)$$

$\therefore$ $$\int_C f(z)dz = 2\pi i[R(z_1)+R(z_2)+\ldots+R(z_n)]$$ ■

Note
1. Limit point of zeros is an isolated essential singularity.
2. Singularities of rational functions are poles.

Working Rule for Detecting the Type of Singularity

1. If $\lim_{z\to a} f(z)$ exists and is finite, then $z = a$ is removable singularity.

2. If $\lim_{z \to a} f(z) = \infty$, then $z = a$ is a pole of $f(z)$.
3. $\lim_{z \to a} f(z)$ does not exist, then $z = a$ is an essential singularity.

WORKED EXAMPLES

EXAMPLE 1

Discuss the nature of singularities of the following functions.

(i) $\dfrac{\sin z - z}{z^3}$ (ii) $\dfrac{z-2}{z^2}\sin\dfrac{1}{z-1}$ (iii) $\dfrac{e^{\frac{1}{z}}}{(z-a)^2}$

(iv) $\dfrac{e^z}{z^2+4}$ (v) $\dfrac{\cot \pi z}{(z-a)^3}$

Solution.

(i) Given
$$f(z) = \frac{\sin z - z}{z^3}$$
$$= \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots - z\right] = -\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$
$$\lim_{z \to 0} f(z) = -\frac{1}{3!} = -\frac{1}{6} = \text{finite}$$
∴ $z = 0$ is a removable singularity.

(or)

Since there is no negative powers of z, $z = 0$ is a removable singularity.

(ii) Given
$$f(z) = \frac{(z-2)\sin\left(\frac{1}{z-1}\right)}{z^2}$$
Poles of $f(z)$ are given by denominator of $f(z) = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$, twice

∴ $z = 0$ is a pole of order 2 of $f(z)$.

$z = 1$ is a singular point, as $f(z)$ is not defined when $z = 1$.

Zeros of $f(z)$ are given by the Numerator of $f(z) = 0$
$$\Rightarrow \quad (z-2)\sin\left(\frac{1}{z-1}\right) = 0 \Rightarrow z = 2 \text{ and } \sin\frac{1}{z-1} = 0$$
Now $\sin\dfrac{1}{z-1} = 0 \Rightarrow \dfrac{1}{z-1} = n\pi, \quad n \in Z$
$$\Rightarrow \quad z - 1 = \frac{1}{n\pi} \Rightarrow z = 1 + \frac{1}{n\pi}, \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$$
The limit of the zeros $1 + \dfrac{1}{n\pi}$ is 1.

∴ $z = 1$ is an isolated essential singularity.

(iii) Given $$f(z) = \frac{e^{\frac{1}{z}}}{(z-a)^2}$$

Poles of $f(z)$ are given by $(z-a)^2 = 0 \Rightarrow z = a, a$

$\therefore$ $z = a$ is a pole of order 2 of $f(z)$.

Now zeros of $f(z)$ are given by $e^{\frac{1}{z}} = 0$

$$\Rightarrow \quad 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \cdots = 0$$

There is no z in the complex number system which satisfies this.

So, $f(z)$ has no zeros.

$\therefore$ $$\lim_{z\to 0^-} f(z) = \infty$$

i.e., the limit does not exist.

Hence, $z = 0$ is an essential singular point.

(iv) Given $$f(z) = \frac{e^z}{z^2+4}$$

Poles of $f(z)$ are given by $z^2 + 4 = 0 \quad \Rightarrow \quad z = \pm 2i,$

which are simple poles.

(v) Given $$f(z) = \frac{\cot \pi z}{(z-a)^3} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^3}$$

Poles of $f(z)$ are given by $\sin \pi z\,(z-a)^3 = 0 \quad \Rightarrow \quad \sin \pi z = 0$ and $(z-a)^3 = 0$

$\therefore \sin \pi z = 0 \quad \Rightarrow \quad \pi z = n\pi, \quad \Rightarrow \quad z = n, \quad n \in Z,$

and $(z-a)^3 = 0 \quad \Rightarrow \quad z = a$ (thrice)

$\therefore$ $z = a$ is a pole of order 3 and $z = 0, \pm 1, \pm 2, \pm 3\ldots$ are all simple poles.

EXAMPLE 2

Calculate the residue of $f(z) = \dfrac{1-e^{2z}}{z^3}$.

Solution.

Given

$$f(z) = \frac{1-e^{2z}}{z^3}. = \frac{1-\left[1+\frac{2z}{1!}+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\cdots\right]}{z^3}$$

$$= -\left[\frac{2}{z^2}+\frac{2}{z}+\frac{8}{3!}+\frac{16z}{4!}+\cdots\right] = -\left[2z^{-2}+2z^{-1}+\frac{4}{3}+\frac{2}{3}z+\cdots\right]$$

As there are two negative powers of z, $z = 0$ is a pole of order 2 and $R(0)$ = coefficient of $z^{-1} = -2$.

EXAMPLE 3

Find the residue at $z = 0$ for (i) $\cot z$ (ii) $\operatorname{cosec}^2 z$ (iii) $\dfrac{1}{z^2 e^z}$.

Solution.

(i) Given $$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Poles are given by $\sin z = 0 \Rightarrow z = n\pi, n \in Z,$
which are simple poles.
If $n = 0$, $z = 0$, which is a simple pole.

$$\therefore \quad R(0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} z\frac{\cos z}{\sin z} = \lim_{z\to 0} \frac{\cos z}{\left[\frac{\sin z}{z}\right]} = \frac{1}{1} = 1$$

(ii) Given $$f(z) = \operatorname{cosec}^2 z = \frac{1}{\sin^2 z}$$

$$= \frac{1}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]^2} = \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right]^2}$$

$\therefore$ $z = 0$ is a pole of order 2.

$$\therefore \quad R(0) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}[z^2 f(z)]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[z^2 \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^2}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{1}{\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^2}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^{-2} = \lim_{z\to 0} -2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]^{-3}\left[-\frac{2z}{3!} + \cdots\right] = 0$$

(iii) Given $$f(z) = \frac{1}{z^2 e^z} = \frac{e^{-z}}{z^2} = \frac{1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots}{z^2}$$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \frac{z^2}{4!} + \cdots = z^{-2} - z^{-1} + \frac{1}{2!} - \frac{z}{3!} + \frac{z^2}{4!} - \cdots$$

As there are 2 terms with negative powers, $z = 0$ is a pole of order 2.

$\therefore$ $R(0) = \text{Coefficient of } z^{-1} = -1.$

Problems Using Residue Theorem

EXAMPLE 4

Evaluate $\int_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z - i| = 2$.

Solution.

Let $$f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

The poles of $f(z)$ are $z = -2i$, $2i$ which are poles of order 2.
C is the circle $|z - i| = 2$ with centre (0, 1) and radius $r = 2$.
Put $z = 2i$, then

$$|z - i| = |2i - i| = |i| = 1 < 2$$

$\therefore$ $z = 2i$ lies inside the circle C.

Put $z = -2i$, then

$$|z - i| = |-2i - i| = |-3i| = 3 > 2$$

$\therefore$ $z = -2i$ lies outside the circle C.

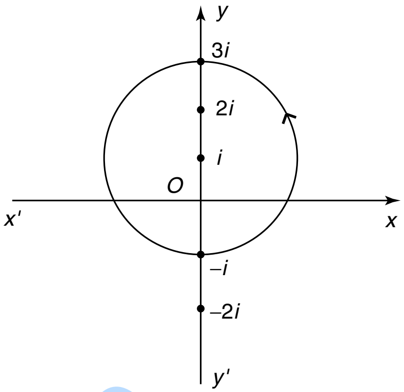

Fig. 16.14

Now $$R(2i) = \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{d}{dz}\left[(z-2i)^2 f(z)\right]$$

$$= \lim_{z\to 2i}\frac{d}{dz}\left[\frac{(z-2i)^2}{(z+2i)^2(z-2i)^2}\right]$$

$$= \lim_{z\to 2i}\frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right) = \lim_{z\to 2i}\left[-\frac{2}{(z+2i)^3}\right] = -\frac{2}{(4i)^3} = \frac{-2}{-64i} = \frac{1}{32i}$$

By Cauchy's residue theorem,

$$\int_C \frac{dz}{(z^2+4)^2} = 2\pi i R(2i) = 2\pi i \frac{1}{32i} = \frac{\pi}{16}$$

EXAMPLE 5

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z - i| = 2$.

Solution.

Given $$\int_C \frac{z-1}{(z+1)^2(z-2)}dz$$

Let $$f(z) = \frac{z-1}{(z+1)^2(z-2)}$$

The poles of $f(z)$ are given by $(z + 1)^2(z - 2) = 0$

$\Rightarrow$ $z = -1$, twice, $z = 2$

$\therefore$ $z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

C is the circle $|z - i| = 2$ with centre (0, 1) and radius $r = 2$

Put $z = -1$, then $|z - i| = |-1 - i| = \sqrt{2} < 2$
$\therefore$ $z = -1$ lies inside C

Put $z = 2$, then $|z - i| = |2 - i| = \sqrt{5} > 2$
$\therefore$ $z = 2$ lies outside C.

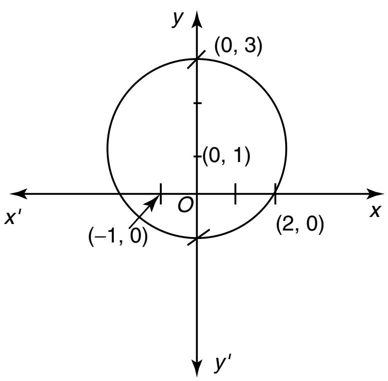

Fig. 16.15

$$\therefore \qquad R(-1) = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z\to -1}\frac{(z-2)\cdot 1-(z-1)\cdot 1}{(z-2)^2} = \lim_{z\to -1}\frac{-1}{(z-2)^2} = -\frac{1}{(-1-2)^2} = -\frac{1}{9}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i R(-1) = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

EXAMPLE 6

Evaluate $\displaystyle\int_C \frac{(\sin \pi z^2 + \cos \pi z^2)\,dz}{(z-1)^2(z-2)}$**, where** C **is** $|z| = 3$**.**

Solution.

Given $$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}dz$$

Let $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$$

The poles of $f(z)$ are given by

$$(z-1)^2(z-2) = 0 \quad \Rightarrow \quad z = 1, \text{ twice}, 2$$

$\therefore$ $z = 1$ is a pole of order 2 and $z = 2$ is a simple pole.
C is the circle $|z| = 3$
Clearly the poles lie inside C for $|1| = 1 < 3$ and $|2| = 2 < 3$

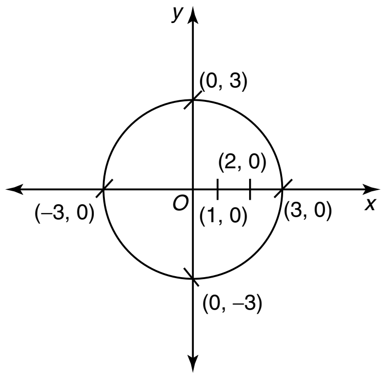

Fig. 16.16

$$\therefore \qquad R(1) = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{(z-1)^2[\sin \pi z^2 + \cos \pi z^2]}{(z-1)^2(z-2)}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]$$

$$= \lim_{z\to 1}\frac{(z-2)\{\cos \pi z^2(2\pi z) - \sin \pi z^2(2\pi z)\} - \{\sin \pi z^2 + \cos \pi z^2\}\cdot 1}{(z-2)^2}$$

$$= \frac{(1-2)[2\pi\cos \pi - 2\pi\sin \pi] - \{\sin \pi + \cos \pi\}}{(1-2)^2}$$

$$= (-1)(-2\pi - 0) - (0-1) = 2\pi + 1$$

and $$R(2) = \lim_{z \to 2}[(z-2)f(z)]$$

$$= \lim_{z \to 2}\left[\frac{(z-2)(\sin \pi z^2 + \cos \pi z^2)}{(z-1)^2(z-2)}\right]$$

$$= \lim_{z \to 2}\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = \frac{\sin 4\pi + \cos 4\pi}{(2-1)^2} = 1$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[R(1)+R(2)] = 2\pi i[2\pi + 1 + 1] = 4\pi i[\pi + 1]$$

EXAMPLE 7

Evaluate $\int_C \frac{\tan \frac{z}{2} dz}{(z-1-i)^2}$, where C is the boundary of the square whose sides are the lines $x = \pm 2$ and $y = \pm 2$.

Solution.

Given $$\int_C \frac{\tan \frac{z}{2} dz}{(z-1-i)^2}$$

Let $$f(z) = \frac{\tan \frac{z}{2}}{(z-1-i)^2}$$

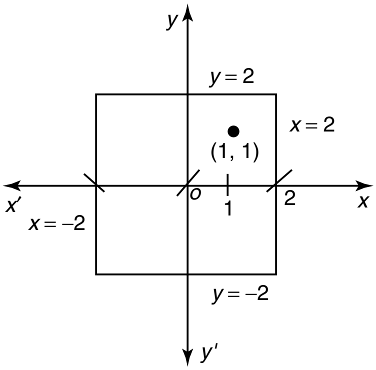

Fig. 16.17

The poles of $f(z)$ are given by $(z-1-i)^2 = 0$

$\Rightarrow$ $[z-(1+i)]^2 = 0$

$\Rightarrow$ $z = 1+i$, twice.

$\therefore$ $z = 1+i$ is a pole of order 2.

C is the square formed by $x = \pm 2$, $y = \pm 2$

$\therefore$ $z = 1+i$ lies inside.

$$R(1+i) = \frac{1}{(2-1)!}\lim_{z \to 1+i}\frac{d}{dz}[(z-1-i)^2 f(z)]$$

$$= \lim_{z \to (1+i)}\frac{d}{dz}\left[\tan\frac{z}{2}\right] = \lim_{z \to 1+i}\left[\sec^2\frac{z}{2}\cdot\frac{1}{2}\right] = \frac{1}{2}\sec^2\left(\frac{1+i}{2}\right)$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i R(1+i) = 2\pi i\frac{1}{2}\sec^2\left(\frac{1+i}{2}\right) = \pi i \sec^2\left(\frac{1+i}{2}\right)$$

EXAMPLE 8

Evaluate $\displaystyle\int_C \frac{z-1}{(z+1)^2\,(z+2)}\,dz$, **where** C **is the circle** $|z-i| = 2$ **using Cauchy's residue theorem.**

Solution.

Given $\displaystyle\int_C \frac{z-1}{(z+1)^2\,(z+2)}\,dz$

Let $\displaystyle f(z) = \frac{z-1}{(z+1)^2\,(z+2)}$

The poles of $f(z)$ are given by $(z+1)^2\,(z+2) = 0$

$\Rightarrow z = -1$, twice and $z = -2$.

$\therefore\ z = -1$ is a pole of order 2 and $z = -2$ is a simple pole.

C is the circle $|z-i| = 2$ with centre (0, 1) and radius $r = 2$.

Put $z = -1$, then $|-1-i| = \sqrt{2} < 2$ $\quad\therefore\ z = -1$ lies inside C.

Put $z = -2$, then $|-2-i| = \sqrt{5} > 2$ $\quad\therefore\ z = -2$ lies outside C.

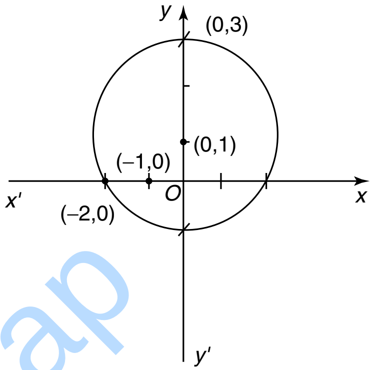

Fig. 16.18

$$R(-1) = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z-1}{(z+1)^2\,(z+2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z+2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z+2)\cdot 1-(z-1)\cdot 1}{(z+2)^2}\right] = \lim_{z\to -1}\left[\frac{3}{(z+2)^2}\right] = \frac{3}{(-1+2)^2} = 3.$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C \frac{z-1}{(z+1)^2\,(z+2)}\,dz = 2\pi i\,R(-1) = 2\pi i \times 3 = 6\pi i.$$

EXERCISE 16.3

I Find the nature of singularities of the following functions.

1. $\dfrac{1}{(z-5)^2(z^2-4)}$ 2. $\dfrac{\sin z}{z^5}$ 3. $\sin\dfrac{1}{z-2}$

4. $\dfrac{\tan z}{z}$ 5. $\dfrac{z}{e^z-1}$ 6. $e^{\frac{1}{z}}$

7. $\sin\left(\dfrac{1}{z+1}\right)$ 8. $\dfrac{1}{z(e^z-1)}$ 9. $\dfrac{1-e^z}{z^4}$ 10. $\dfrac{1-e^z}{1+e^z}$

II Find the residue of $f(z)$ at the singularity $z = 0$, where $f(z)$ is

1. $\dfrac{1+e^z}{\sin z + z\cos z}$ 2. $\dfrac{1}{z(e^z-1)}$ 3. $\dfrac{z-2}{z(z-1)}$ 4. $z\cos\dfrac{1}{z}$ 5. $\dfrac{1+e^z}{\sin z + z\cos z}$.

III Find the residue of $f(z)$ at the singular points where, $f(z)$ is

1. $\dfrac{1}{z^2+1}$ 2. $\dfrac{z}{(z-1)^2}$ 3. $\dfrac{z^3}{(z+a)^2}$ 4. $\dfrac{z+2}{(z+1)^2}$ 5. $\dfrac{z^2+1}{z^2-2z}$

6. Determine poles and their orders for the function $\dfrac{z+2}{(z+1)^2(z-2)}$. Find the residue at the poles.
7. Find the residue of $f(z) = \dfrac{\sin z}{1-z^4}$ at $z = i$.

IV Evaluate the following integrals:

1. $\displaystyle\int_C \frac{1-2z}{z(z-1)(z-2)}dz$ where C is $|z| = 1.5$.
2. $\displaystyle\int_C \frac{dz}{(z^2+1)(z^2-4)}$ where C is $|z| = \frac{3}{2}$.
3. $\displaystyle\int_C \frac{dz}{(z^2+4)^2}$ where C is $|z-i| = 2$.
4. $\displaystyle\int_C \frac{e^z}{z^2+1}dz$ where C is $|z| = 2$.
5. $\displaystyle\int_C \frac{z^2+2}{z^2+4}dz$ where C is $|z-i| = 2$.
6. $\displaystyle\int_C \frac{12z-7}{(z-1)^2(2z+3)}$ where C is $|z+i| = \sqrt{3}$.
7. $\displaystyle\int_C \frac{2z+3}{z(z-1)(z-2)}$ where C is $|z| = 2$.
8. $\displaystyle\int_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}dz$ where C is $|z-1| = 1$.
9. $\displaystyle\int_C \frac{e^z}{z^2+\pi^2}dz$ where C is $|z-i| = 3$.
10. $\displaystyle\int_C \frac{z^2+2}{(z+3)(z+2)^2}dz$ where C is the circle $|z| = 2$.
11. $\displaystyle\int_C \frac{z^2\,dz}{(z-1)^2(z+2)}$ where C is $|z| = 2.5$.
12. $\displaystyle\int_C z^2 e^{\frac{1}{z}}\,dz$ where C is the unit circle.
13. $\displaystyle\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$, where C is the circle $|z| = 4$ using Cauchy's residue theorem.
14. $\displaystyle\int_C \frac{12z-7}{(z^2-1)(2z+3)}dz$, where C is $|z-i| = \sqrt{3}$.
15. $\displaystyle\int_C \frac{z-3}{z^2+2z+5}dz$, where C is $|z+1-i| = 2$.
16. $\displaystyle\int_C \frac{z\sec z}{1-z^2}dz$, where C is the ellipse $4x^2 + 9y^2 = 9$.

17. $\int_C \frac{dz}{(z^2+1)(z^2-4)}$, where C is $|z| = \frac{3}{2}$.

18. $\int_C \frac{4z^2-4z+1}{(z-2)(z^2+4)}dz$, where C is the circle $|z| = 1$.

19. $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$, where C is the circle $|z| = 2$.

ANSWERS TO EXERCISE 16.3

I
1. $z = 2, -2$ are simple poles and $z = 5$ is a pole of order 2.
2. $z = 0$ is a pole of order 4
3. $z = 2$ is an essential singularity
4. $z = 0$ is a removable singularity
5. $z = 0$ is a removable singularity, $z = 2n\pi i$ $(n \neq 0)$ are simple poles.
6. $z = 0$ is an essential singularity
7. $z = 1$ is an essential singularity
8. $z = 0$ is a double pole and $z = i2n\pi$, $n = \pm 1, \pm 2, \ldots$ are simple poles
9. $z = 0$ is a pole of order 3
10. $z = i(2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$ which are simple poles

II 1. 1, 2. $-\frac{1}{2}$, 3. 2, 4. -4 5. 1

III 1. $R(i) = \frac{1}{2i}$, $R(-i) = \frac{-1}{2i}$, 2. $R(1) = 1$, 3. $3a^2$ 4. 1

5. $R(0) = -\frac{1}{2}$, $R(2) = \frac{5}{2}$ 6. $R(-1) = -\frac{4}{9}$; $R(2) = \frac{4}{9}$ 7. $\frac{\sin i}{4i}$

IV 1. $3\pi i$, 2. 0, 3. $\frac{\pi}{16}$, 4. $\pi(e^i - e^{-i})$, 5. $\frac{3\pi}{8}$,

6. $4\pi i$, 7. $-7\pi i$, 8. $-2\pi i$, 9. -1, 10. $\frac{-7\pi i}{2}$

11. $2\pi i$ 12. $\frac{\pi i}{3}$ 13. $\frac{i}{\pi}$ 14. $20\pi i$ 15. $\pi(i-2)$

16. $-2\pi i \sec 1$ 17. 0 18. 0 19. $-\frac{5\pi i}{4}$

16.8 APPLICATION OF RESIDUE THEOREM TO EVALUATE REAL INTEGRALS

The applications of residue theorem include the evaluation of certain types of definite integrals, improper integrals and complicated real integrals occurring in real analysis and applied mathematics. Method of residues is used in inverse Laplace transforms. These real integrals are evaluated by expressing them in terms of integrals of complex functions over a suitable contour and evaluated using residue theorem. This process of evaluation of real integrals is called **contour integration**.

16.8.1 Type 1

Real definite integrals of the form $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$ where $F(\sin\theta, \cos\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

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Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$

$\Rightarrow$ $z = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$

$\therefore$ $z + \frac{1}{z} = 2\cos\theta$, $z - \frac{1}{z} = 2i\sin\theta$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Since $F(\sin\theta, \cos\theta)$ is a rational function in $\sin\theta$ and $\cos\theta$, we get a rational function of z, say $f(z)$.

$$\therefore \quad \frac{dz}{d\theta} = ie^{i\theta} = iz \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

As θ varies from 0 to 2π, z moves on the unit circle $|z| = 1$ in the anticlockwise sense.
So, the contour C is the unit circle $|z| = 1$

$$\therefore \quad \int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta = \oint_C f(z)\frac{dz}{iz}, \text{ where } C \text{ is } |z| = 1$$

WORKED EXAMPLES

EXAMPLE 1

Evaluate $\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$ by using contour integration.

Solution.

Let $$I = \int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$$

To evaluate this, consider the unit circle $|z| = 1$ as contour C

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$ $\therefore$ $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$

$$\therefore \quad I = \int_C \frac{\frac{dz}{iz}}{13 + 5\frac{(z^2 - 1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz + 5z^2 - 5}{2iz}} = 2\int_C \frac{dz}{5z^2 + 26iz - 5}$$

Let $f(z) = \frac{1}{5z^2 + 26iz - 5}$ $\therefore$ $I = 2\int_C f(z)dz$

The poles of $f(z)$ are given by $5z^2 + 26iz - 5 = 0$

$$\therefore \quad z = \frac{-26i \pm \sqrt{(26i)^2 - 4\cdot 5(-5)}}{10}$$

$$= \frac{-26i \pm \sqrt{-676+100}}{10}$$

$$= \frac{-26i \pm \sqrt{-576}}{10}$$

$$= \frac{-26i \pm 24i}{10} = -\frac{i}{5}, \quad -5i$$

which are simple poles.

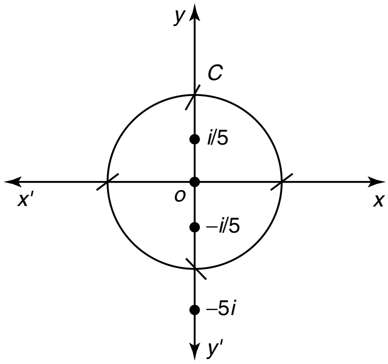

Fig. 16.19

Now $5z^2 + 26iz - 5 = 5\left(z + \frac{i}{5}\right)(z+5i)$

Since $z = -\frac{i}{5} \quad \Rightarrow \quad \left|-\frac{i}{5}\right| = \frac{1}{5} < 1,$

the pole $z = -\frac{i}{5}$ lies inside C.

and $\quad z = -5i \quad \Rightarrow \quad |-5i| = 5 > 1 \qquad \therefore$ the pole $z = -5i$ lies outside C.

Now $$R\left(-\frac{i}{5}\right) = \lim_{z \to -\frac{i}{5}} \left(z + \frac{i}{5}\right) f(z) = \lim_{z \to -\frac{i}{5}} \left[\frac{\left(z + \frac{i}{5}\right)}{5\left(z + \frac{i}{5}\right)[z+5i]}\right]$$

$$= \lim_{z \to -\frac{i}{5}} \left[\frac{1}{5(z+5i)}\right] = \frac{1}{5\left[-\frac{i}{5} + 5i\right]} = \frac{1}{24i}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i R\left(-\frac{i}{5}\right) = 2\pi i \times \frac{1}{24i} = \frac{\pi}{12}$$

$\therefore$ $$I = 2 \cdot \frac{\pi}{12} = \frac{\pi}{6}$$

EXAMPLE 2

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2p\sin\theta + p^2}$, $|p| < 1$.

Solution.

Let $$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\sin\theta + p^2}$$

Consider the unit circle $\quad |z| = 1$ as contour C

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$

$\therefore$ $$I = \int_C \frac{\frac{dz}{iz}}{1 - 2p\left(\frac{z^2-1}{2iz}\right) + p^2} = \int_C \frac{dz}{(1+p^2)iz - pz^2 + p} = -\int_C \frac{dz}{pz^2 - (1+p^2)iz - p}$$

Let $f(z) = \dfrac{1}{pz^2 - (1+p^2)iz - p}$ $\quad\therefore\quad I = -\int_C f(z)\,dz$

The poles of $f(z)$ are given by $pz^2 - (1+p^2)iz - p = 0$

$$\Rightarrow \quad z^2 - \left(\frac{1}{p} + p\right)iz - 1 = 0 \quad \Rightarrow \quad (z - ip)\left(z - \frac{i}{p}\right) = 0 \quad \Rightarrow \quad z = ip,\ \frac{i}{p}$$

which are simple poles.

$$\therefore \quad pz^2 - (1+p^2)iz - p = p\left(z - \frac{i}{p}\right)(z - ip)$$

Given $|p| < 1 \Rightarrow |ip| = |p| < 1 \quad \therefore \quad z = ip$ lies inside C

and $|p| < 1 \Rightarrow \dfrac{1}{|p|} > 1 \quad \therefore \quad \left|\dfrac{i}{p}\right| = \dfrac{1}{|p|} > 1 \quad \therefore \quad z = \dfrac{i}{p}$ lies outside C

Now
$$R(ip) = \lim_{z \to ip}(z - ip)f(z) = \lim_{z \to ip} \frac{(z - ip)}{p(z - ip)\left(z - \frac{i}{p}\right)}$$

$$= \lim_{z \to ip} \frac{1}{p\left(z - \frac{i}{p}\right)} = \frac{1}{p\left(ip - \frac{i}{p}\right)} = \frac{1}{i(p^2 - 1)}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i R(ip) = 2\pi i \frac{1}{i(p^2 - 1)} = \frac{2\pi}{p^2 - 1}$$

$$\therefore \quad I = -2\frac{\pi}{p^2 - 1} = \frac{2\pi}{1 - p^2}$$

EXAMPLE 3

Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$ using contour integration.

Solution.

Let
$$I = \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$$

Consider the unit circle $|z| = 1$ as contour C.

Put $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$ $\quad\therefore\quad d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{z + \frac{1}{z}}{2} = \dfrac{z^2 + 1}{2z}$

$z^3 = e^{i3\theta} = \cos 3\theta + i\sin 3\theta$ and $\cos 3\theta = R.P$ of z^3.

$\therefore \qquad I = R.P \text{ of } \int_C \dfrac{z^3 \dfrac{dz}{iz}}{5-4\left(\dfrac{z^2+1}{2z}\right)} = R.P \text{ of } \dfrac{1}{i}\int_C \dfrac{z^3\,dz}{5z-2z^2-2}$

$$= R.P \text{ of } \left(\frac{-1}{i}\right)\int_C \frac{z^3\,dz}{2z^2-5z+2}$$

Let $\qquad f(z) = \dfrac{z^3}{2z^2-5z+2} \quad \therefore \quad I = R.P \text{ of } \left(-\dfrac{1}{i}\right)\int_C f(z)\,dz$

The poles of $f(z)$ are given by

$$2z^2-5z+2=0 \quad \Rightarrow \quad 2z^2-4z-z+2=0 \quad \Rightarrow \quad 2z(z-2)-(z-2)=0$$

$\Rightarrow \quad (2z-1)(z-2)=0 \quad \Rightarrow \quad (2z-1)=0,\ z-2=0 \quad \Rightarrow \quad z=\dfrac{1}{2}, 2$

which are simple poles.

$\therefore \qquad 2z^2-5z+2 = 2\left(z-\dfrac{1}{2}\right)(z-2).$

C is the contour $|z|=1$

$\therefore \qquad z=\dfrac{1}{2} \quad \Rightarrow \quad |z| = \left|\dfrac{1}{2}\right| < 1 \qquad \therefore\ z=\dfrac{1}{2}$ lies inside C

and $\qquad z=2 \quad \Rightarrow \quad |z|=|2|>1 \qquad \therefore\ z=2$ lies outside C.

Now $\qquad R\left(\dfrac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\dfrac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}} \dfrac{\left(z-\dfrac{1}{2}\right)z^3}{2\left(z-\dfrac{1}{2}\right)(z-2)}.$

$$= \lim_{z\to\frac{1}{2}} \frac{z^3}{2(z-2)} = \frac{\frac{1}{8}}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i R\left(\frac{1}{2}\right) = 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

$\therefore \qquad I = R.P \text{ of } \left(-\dfrac{1}{i}\right)\left(-\dfrac{\pi i}{12}\right) = \dfrac{\pi}{12}$

EXAMPLE 4

Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta,\ a>b>0.$

Solution.

Let $$I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \int_0^{2\pi} \frac{(1-\cos 2\theta)}{2(a+b\cos\theta)}\,d\theta$$

Consider the unit circle $|z|=1$ as the contour C.

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$ $\therefore$ $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

$z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$ and $\cos 2\theta = R.P$ of z^2

$\therefore$ $1-\cos 2\theta = R.P$ of $(1-z^2)$

$$\therefore \quad I = R.P \text{ of } \int_C \frac{(1-z^2)\frac{dz}{iz}}{2\left(a+b\frac{z^2+1}{2z}\right)} = R.P \text{ of } \frac{1}{i}\int_C \frac{(1-z^2)\,dz}{2az+b(z^2+1)}$$

$$= R.P \text{ of } \frac{1}{i}\int_C \frac{(1-z^2)\,dz}{bz^2+2az+b}$$

Let $f(z) = \frac{1-z^2}{bz^2+2az+b}$ $\therefore$ $I = R.P$ of $\frac{1}{i}\int_C f(z)\,dz$

The poles of $f(z)$ are given by

$$bz^2+2az+b=0 \Rightarrow z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b} \qquad (1)$$

$$= \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b} = \frac{-a+\sqrt{a^2-b^2}}{b} \text{ or } \frac{-a-\sqrt{a^2-b^2}}{b}$$

Let $$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b} \quad \text{and} \quad z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$\therefore$ z_1 and z_2 are simple poles

Now $$bz^2+2az+b = b(z-z_1)(z-z_2)$$

$$\therefore \quad f(z) = \frac{1-z^2}{b(z-z_1)(z-z_2)}$$

The contour is $|z|=1$

Since $a>b>0$, $\frac{a}{b}>1 \Rightarrow \frac{a^2}{b^2}>1 \Rightarrow a^2>b^2 \Rightarrow a^2-b^2>0$

$$\therefore \quad |z_2| = \left|\frac{-a-\sqrt{a^2-b^2}}{b}\right|$$

$$= \left|\frac{a+\sqrt{a^2-b^2}}{b}\right| = \frac{a+\sqrt{a^2-b^2}}{b} > \frac{a}{b} > 1 \qquad \left[\because a+\sqrt{a^2-b^2} > a\right]$$

$\therefore$ z_2 lies outside C

Since $z_1 z_2 = 1$, $|z_1 z_2| = 1 \Rightarrow |z_1| = \dfrac{1}{|z_2|} < 1$ [product of the roots of the quadratic (1)]

$\therefore$ z_1 lies inside C.

$$R(z_1) = \lim_{z \to z_1} (z - z_1) f(z)$$

$$= \lim_{z \to z_1} \frac{(z - z_1)(1 - z^2)}{b(z - z_1)(z - z_2)}$$

$$= \lim_{z \to z_1} \frac{1 - z^2}{b(z - z_2)} = \frac{1 - z_1^2}{b(z_1 - z_2)}$$

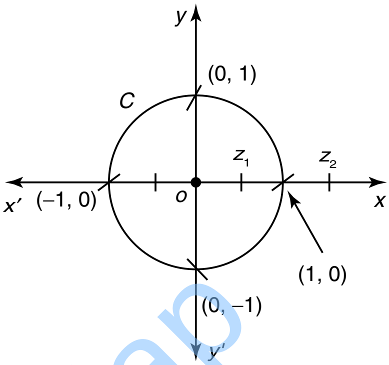

Fig. 16.20

Now $$1 - z_1^2 = 1 - \left[\frac{-a + \sqrt{a^2 - b^2}}{b}\right]^2$$

$$= 1 - \frac{\left(-a + \sqrt{a^2 - b^2}\right)^2}{b^2}$$

$$= \frac{b^2 - \left(a^2 + a^2 - b^2 - 2a\sqrt{a^2 - b^2}\right)}{b^2}$$

$$= \frac{2\left[b^2 - a^2 + a\sqrt{a^2 - b^2}\right]}{b^2} = \frac{2\sqrt{a^2 - b^2}\left[a - \sqrt{a^2 - b^2}\right]}{b^2}$$

and $$z_1 - z_2 = \frac{-a + \sqrt{a^2 - b^2}}{b} - \frac{\left[-a - \sqrt{a^2 - b^2}\right]}{b} = \frac{-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}}{b} = \frac{2\sqrt{a^2 - b^2}}{b}$$

$$\therefore \quad b(z_1 - z_2) = \frac{b \cdot 2\sqrt{a^2 - b^2}}{b} = 2\sqrt{a^2 - b^2}$$

$$\therefore \quad R(z_1) = \frac{2\sqrt{a^2 - b^2}\left(a - \sqrt{a^2 - b^2}\right)}{b^2 \cdot 2\sqrt{a^2 - b^2}} = \frac{a - \sqrt{a^2 - b^2}}{b^2}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i R(z_1) = 2\pi i \frac{\left(a - \sqrt{a^2 - b^2}\right)}{b^2} = \frac{2\pi i\left(a - \sqrt{a^2 - b^2}\right)}{b^2}$$

$$\therefore \quad I = R.P. \text{ of } \frac{1}{i} 2\pi i \frac{\left(a - \sqrt{a^2 - b^2}\right)}{b^2} = \frac{2\pi\left(a - \sqrt{a^2 - b^2}\right)}{b^2}.$$

EXAMPLE 5

Evaluate $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$, using contour integration.

Solution.

Let $$I = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$$

Consider the unit circle $|z| = 1$ as the contour C.

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$ $\therefore$ $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

$$\therefore \quad I = \int_C \frac{1+2\left(\frac{z^2+1}{2z}\right)}{5+4\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz} = \frac{1}{i}\int_C \frac{z+z^2+1}{z[5z+2z^2+2]}\,dz = \frac{1}{i}\int_C \frac{(z^2+z+1)}{z(2z^2+5z+2)}\,dz$$

Let $f(z) = \frac{z^2+z+1}{z(2z^2+5z+2)}$ $\therefore$ $I = \frac{1}{i}\int_C f(z)\,dz$

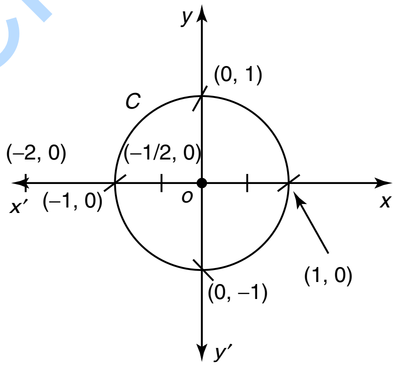

Fig. 16.21

The poles of $f(z)$ are given by

$$z(2z^2+5z+2) = 0$$

$$\Rightarrow \quad z = 0,\ 2z^2+5z+2 = 0$$

$$\Rightarrow \quad (2z+1)(z+2) = 0 \quad \Rightarrow \quad z = -\frac{1}{2},\ -2$$

$\therefore$ $z = 0, -\frac{1}{2}, -2$ are simple poles of $f(z)$

Since C is $|z| = 1$, $z = 0$ and $z = -\frac{1}{2}$ lie inside C and $z = -2$ lies outside C.

$$\therefore \quad R(0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z(z^2+z+1)}{z(2z^2+5z+2)} = \lim_{z\to 0}\frac{z^2+z+1}{2z^2+5z+2} = \frac{1}{2}$$

$$R\left(-\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z) = \lim_{z\to -\frac{1}{2}}\frac{\left(z+\frac{1}{z}\right)(z^2+z+1)}{2z\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}}\frac{z^2+z+1}{2z(z+2)} = \frac{\frac{1}{4}-\frac{1}{2}+1}{2\left(-\frac{1}{2}\right)\left(\frac{-1}{2}+2\right)} = \frac{\frac{3}{4}}{-\frac{3}{2}} = -\frac{1}{2}$$

$\therefore$ by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\left[R(0)+R\left(\frac{-1}{2}\right)\right] = 2\pi i\left[\frac{1}{2}-\frac{1}{2}\right] = 0$$

$\therefore$ $$I = \frac{1}{i}\times 0 = 0$$

EXAMPLE 6

Evaluate $\displaystyle\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$.

Solution.

Let $$I = \int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$$

If $$f(\theta) = \frac{1+2\cos\theta}{5+4\cos\theta},\ \text{then } f(2\pi-\theta) = \frac{1+2\cos(2\pi-\theta)}{5+4\cos(2\pi-\theta)} = \frac{1+2\cos\theta}{5+4\cos\theta} = f(\theta)$$

We know $\displaystyle\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$.

$\therefore$ $$\int_0^{a} f(x)dx = \frac{1}{2}\int_0^{2a} f(x)dx$$

$\therefore$ $$\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$$ [by example 5]

16.8.2 Type 2. Improper Integrals of Rational Functions

In real calculus, we consider integrals of the type $\int_{-\infty}^{\infty} f(x)dx$. For this integral, the interval of integration is not finite and hence it is an improper integral.

If $f(x)$ is continuous for all x, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx \qquad (1)$$

when both the limits exist on the right hand side. We then say that the improper integral converge.

We have another value of the integral called the Cauchy principal value, defined by

$$PV\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \qquad (2)$$

if the limit exists.

Note It can be proved that the convergence of integral in (1) implies that its converging value coincide with the Cauchy principal value. However, the converse is not true. That is the Cauchy principal value may exist even if the limits in (1) do not exist.

For example, if $f(x) = x$

Then $$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R} = \lim_{R\to\infty}\frac{1}{2}[R^2 - R^2] = 0$$

So, $$PV\int_{-\infty}^{\infty} f(x)\,dx = 0$$

But $$\lim_{R_2\to\infty}\int_{0}^{R_2} x\,dx = \lim_{R_2\to\infty}\left[\frac{x^2}{2}\right]_{0}^{R_2} = \lim_{R_2\to\infty}\frac{R^2}{2} = \infty$$

So, the integrals in (1) do not converge.

$\therefore$ $\int_{-\infty}^{\infty} f(x)\,dx$ is not convergent.

When $f(x)$ is even, the integrals (1) and (2) converge or diverge together and in this case

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx \qquad (3)$$

Usually, the integral on the R.H.S of (3) will be evaluated as the **Cauchy principal value**.

Now we state Cauchy's lemma which will be used in type 2 problems.

Cauchy's lemma If $f(z)$ is a continuous function such that $|zf(z)| \to 0$ as $|z| \to \infty$ on the upper semi-circle $S : |z| = R$ then $\int_S f(z)dz \to 0$ as $R \to \infty$

Type 2: Integrals of the form $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$, where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of $Q(x)$ is atleast 2 more than the degree of $P(x)$ and $Q(x)$ does not vanish for any real x.

To evaluate this integral, we consider $\int_C \frac{P(z)}{Q(z)}dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi-circle $S : |z| = R$, for large R, taken in the anticlockwise sense.

Let $$f(z) = \frac{P(z)}{Q(z)}. \quad \text{Then } \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz \qquad (1)$$

By Cauchy's residue theorem, we evaluate $\int_C f(z)\,dz$

and $$\int_C f(z)\,dz = 2\pi i \text{ [sum of the residues of } f(z)]$$

By Cauchy's lemma,

$$\int_S f(z)\,dz \to 0 \quad \text{as} \quad R \to \infty$$

$\therefore$ From (1) we get,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = 2\pi i \text{ [sum of the residues of } f(z)]$$

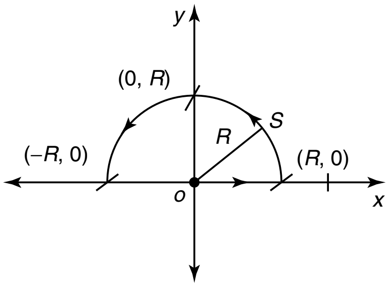

Fig. 16.22

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ [sum of the residues of } f(z)]$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{P(x)\,dx}{Q(x)} = 2\pi i \left[\text{sum of the residues of } \frac{P(z)}{Q(z)}\right]$$

WORKED EXAMPLES

EXAMPLE 7

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx.$

Solution.

Given $$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \int_{-\infty}^{\infty} \frac{(x^2 - x + 2)\,dx}{(x^2 + 1)(x^2 + 9)}$$

The integrand is a rational function with degree of Dr. two more than the degree of Nr. and Dr. $\neq 0$ for any real x.

Consider $\displaystyle\int_C \frac{z^2 - z + 2}{(z^2 + 1)(z^2 + 9)}\,dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi-circle $S : |z| = R$ taken in the anticlockwise sense and R is large.

Let $$f(z) = \frac{z^2 - z + 2}{(z^2 + 1)(z^2 + 9)}$$

Then $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz \qquad (1)$$

We shall evaluate $\displaystyle\int_C f(z)\,dz$ by residue theorem.

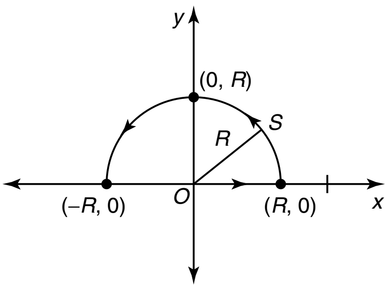

Fig. 16.23

The poles of $f(z)$ are given by

$(z^2 + 1)(z^2 + 9) = 0 \quad \Rightarrow \quad z = \pm i$ and $z = \pm 3i$,

which are simple poles.

But only $z = i$ and $z = 3i$ lie inside C.

Now $$R(i) = \lim_{z \to i} (z - i) f(z)$$

$$= \lim_{z \to i} \frac{(z - i)[z^2 - z + 2]}{(z + i)(z - i)(z^2 + 9)}$$

$$= \lim_{z \to i} \frac{z^2 - z + 2}{(z + i)(z^2 + 9)} = \frac{i^2 - i + 2}{2i(i^2 + 9)} = \frac{-1 - i + 2}{2i(-1 + 9)} = \frac{1 - i}{16i}$$

and $$R(3i) = \lim_{z\to 3i}(z-3i)f(z) = \lim_{z\to 3i}\frac{(z-3i)(z^2-z+2)}{(z^2+1)(z+3i)(z-3i)}$$

$$= \lim_{z\to 3i}\frac{z^2-z+2}{(z^2+1)(z+3i)} = \frac{(3i)^2-3i+2}{((3i)^2+1)(3i+3i)} = \frac{-9-3i+2}{(-9+1)6i} = \frac{7+3i}{48i}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i[R(i)+R(3i)] = 2\pi i\left[\frac{1-i}{16i}+\frac{7+3i}{48i}\right]$$

$$= \frac{2\pi[3(1-i)+7+3i]}{48} = \frac{\pi\times 10}{24} = \frac{5\pi}{12}$$

$\therefore$ from (1) we get,

$$\int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz = \frac{5\pi}{12}$$

But $$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z(z^2-z+2)}{(z^2+1)(z^2+9)} = 0$$

$\therefore$ by Cauchy's lemma $\int_S f(z)\,dz = 0$ as $R\to\infty$

$$\therefore \quad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{5\pi}{12} \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{5\pi}{12}$$

EXAMPLE 8

Evaluate $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$, $a>0, b>0$.

Solution.

Let $$f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)}$$

Then $f(-x) = f(x)$ $\qquad \therefore$ $f(x)$ is even function of x.

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^\infty f(x)\,dx \Rightarrow \int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

$f(x)$ is a rational function with degree of the Dr. 2 more than degree of the Nr. and the Dr. $\neq 0$ for any real x.

Consider $\displaystyle\int_C \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$, where C is the simple closed contour consisting of the real axis from $-R$ to R and the upper semi-circle $S: |z| = R$ taken in the anticlockwise sense and R is large.

Let $$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

Then $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz$$

We shall evaluate $\int_C f(z)\,dz$.

The poles of $f(z)$ are given by

$(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow \quad z = \pm ai,\ z = \pm bi$

which are simple poles.

Since $a > 0$, $b > 0$, $z = ai$ and $z = bi$ lie inside C.

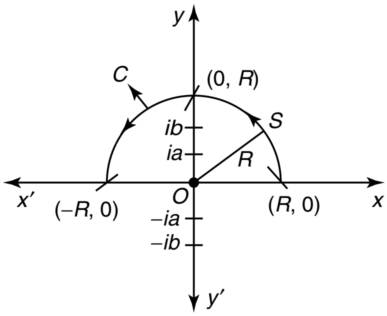

Fig. 16.24

Now $\quad R(ai) = \lim_{z\to ai}(z - ai)f(z)$

$$= \lim_{z\to ai}(z-ai)\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \lim_{z\to ai}\frac{z^2}{(z+ai)(z^2+b^2)}$$

$$= \frac{a^2 i^2}{(ai+ai)(a^2i^2+b^2)} = -\frac{a}{2i(b^2-a^2)}$$

Similarly, $\quad R(bi) = \dfrac{-b}{2i(a^2-b^2)} = \dfrac{b}{2i(b^2-a^2)}$

So, by Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i[R(ai)+R(bi)] = 2\pi i\left[-\frac{a}{2i(b^2-a^2)}+\frac{b}{2i(b^2-a^2)}\right] = \pi\left[\frac{b-a}{b^2-a^2}\right] = \frac{\pi}{b+a}$$

$\therefore$ from (1) we get, $\displaystyle\int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz = \frac{\pi}{b+a}$

Now $$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty}\frac{z\cdot z^2}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to\infty}\frac{1}{z\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{b^2}{z^2}\right)} = 0$$

So, by Cauchy's lemma, $\int_S f(z)\,dz \to 0$ as $R\to\infty$

$$\therefore \quad \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{\pi}{a+b} \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{b+a}$$

$$\therefore \quad \int_0^{\infty} f(x)\,dx = \frac{1}{2}\frac{\pi}{(a+b)}$$

EXAMPLE 9

Evaluate $\displaystyle\int_0^{\infty}\frac{dx}{(1+x^2)^2}$ using Contour integration.

Solution.

Let $$I = \int_0^{\infty}\frac{dx}{(1+x^2)^2} = \int_0^{\infty} f(x)\,dx$$

Here $f(x)=\frac{1}{(1+x^2)^2}$ is even function of x, since $f(-x)=f(x)$

Since $f(-x)=f(x)$

$\therefore \int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx \quad \Rightarrow \quad \int_0^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$

Now consider the integral $\int_C \frac{dz}{(1+z^2)^2}$, where C is the Contour consisting of the real axis from $-R$ to R and upper semi-circle $S:|z|=R$ taken with anticlockwise sense and R is large.

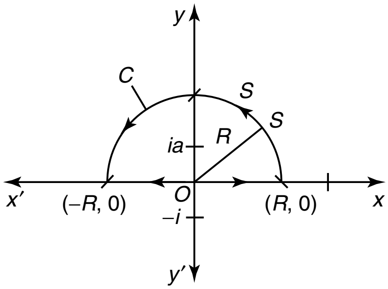

Fig. 16.25

Let $f(z)=\frac{1}{(1+z^2)^2}$

Then
$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz \qquad (1)$$

$\therefore$ the poles of $f(z)$ are given by $(1+z^2)^2=0 \quad \Rightarrow \quad (1+z^2)=0 \quad \Rightarrow \quad z=\pm i$, twice.
which are poles of order 2.
But only $z=i$ lie inside C.

$$\therefore \quad R(i)=\frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2 f(z)\right]$$
$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{[z-i]^2}{(z-i)^2(z+i)^2}\right]$$
$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]=\lim_{z\to i}\frac{-2}{(z+i)^3}=-\frac{2}{(i+i)^3}=\frac{2}{8i}=\frac{1}{4i}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i R(i) = 2\pi i\frac{1}{4i}=\frac{\pi}{2}$$

From (1), we get, $\int_{-R}^{R} f(x)dx + \int_S f(z)dz = \frac{\pi}{2}$

Now
$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z}{(1+z^2)^2} = \lim_{z\to\infty}\frac{z}{z^3\left(1+\frac{1}{z^2}\right)^2}=0$$

So, by Cauchy's lemma, $\int_S f(z)dz \to 0$ as $R\to\infty$.

$$\therefore \quad \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \frac{\pi}{2} \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2}$$

$$\therefore \quad \int_0^{\infty} f(x)dx = \frac{1}{2}\frac{\pi}{2}=\frac{\pi}{4}$$

16.8.3 Type 3

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)\sin mx}{Q(x)} dx$ and $\int_{-\infty}^{\infty} \frac{P(x)\cos mx}{Q(x)} dx,\ m > 0$, where $P(x)$ and $Q(x)$ are polynomials in x such that the degree of $Q(x)$ is greater than the degree of $P(x)$ and $Q(x)$ does not vanish for any real x. These integrals occur in connection with Fourier integrals – the sine and cosine transforms.

To evaluate the above integrals, we consider $\int_C f(z)e^{imz}\, dz$, where $f(z) = \frac{P(z)}{Q(z)}$ and C is the simple closed curve consisting of the real axis from $-R$ to R and upper semi-circle $S : |z| = R$ taken in the anticlockwise sense and R is large.

$$\therefore \qquad \int_C f(z)e^{imz}\, dz = \int_{-R}^{R} f(x)e^{imx}\, dx + \int_S f(z)e^{imz}\, dz \qquad (1)$$

We evaluate $\int_C f(z)e^{imz}\, dz$ by using Cauchy's residue theorem and substitute in (1).

By Jordan's lemma $\int_S f(z)e^{imz}\, dz \to 0$ as $R \to \infty$

and
$$\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{imx}\, dx = \text{value of } \int_C f(z)e^{imz}\, dz$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} f(x)e^{imx}\, dx = \text{value of } \int_C f(z)e^{imz}\, dz$$

Equating real and imaginary parts, we get the value of $\int_{-\infty}^{\infty} f(x)\cos mx\, dx$ and $\int_{-\infty}^{\infty} f(x)\sin mx\, dx$ respectively.

Jordan's lemma If $f(z)$ is a continuous function such that $|f(z)| \to 0$ uniformly as $|z| \to \infty$ then $\int_S f(z)e^{imz}\, dz \to 0$ as $R \to \infty$, where S is the upper semi-circle $|z| = R$

WORKED EXAMPLES

EXAMPLE 10

Evaluate $\int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx,\ a > 0,\ m > 0.$

Solution.

Let $I = \int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx \qquad \left[\because \frac{x \sin mx}{x^2 + a^2} \text{ is an even function of } x\right]$

Consider $\int_C \frac{ze^{imz}}{z^2 + a^2} dz$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi-circle $S : |z| = R$ taken in the anticlockwise sense and R is large.

Let $F(z) = \frac{ze^{imz}}{z^2 + a^2}$

$$\therefore \int_C F(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_S F(z)\,dz \qquad (1)$$

we shall evaluate $\int_C F(z)\,dz$.

The poles of $F(z)$ are given by $z^2 + a^2 = 0 \Rightarrow z = \pm ia$, which are simple poles.
But $z = ia$ lies inside C and $z = -ia$ lies outside C.

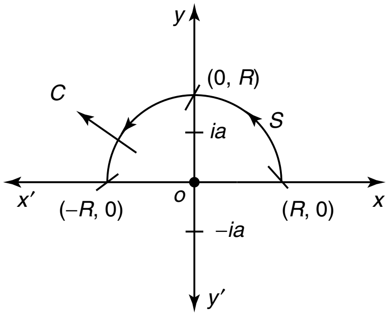

Fig. 16.26

$$\therefore R(ia) = \lim_{z\to ia}(z - ia)F(z)$$

$$= \lim_{z\to ia} \frac{(z-ia)ze^{imz}}{(z-ia)(z+ia)}$$

$$= \lim_{z\to ia} \frac{ze^{imz}}{(z+ia)} = \frac{iae^{i^2ma}}{ia+ia} = \frac{iae^{-ma}}{2ia} = \frac{e^{-ma}}{2}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C F(z)\,dz = 2\pi i R(ia) = 2\pi i \frac{e^{-ma}}{2} = \pi i e^{-ma}$$

$\therefore$ from (1) we get,

$$\int_{-R}^{R} F(x)\,dx + \int_S F(z)\,dz = \pi i e^{-ma}$$

Let $$f(z) = \frac{z}{z^2 + a^2}$$

Since $$|f(z)| = \frac{|z|}{|z^2 + a^2|} \to 0 \text{ as } |z| \to \infty,$$

by Jordan's lemma, $$\int_S f(z)e^{imz}\,dz = \int_S F(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\therefore \quad \lim_{R\to\infty} \int_{-R}^{R} F(x)\,dx = \pi i e^{-ma}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} F(x)\,dx = \pi i e^{-ma}$$

$$\Rightarrow \quad \int_0^{\infty} F(x)\,dx = \frac{\pi i e^{-ma}}{2} \quad \Rightarrow \quad \int_0^{\infty} \frac{xe^{imx}}{x^2+a^2}\,dx = \frac{\pi i e^{-ma}}{2}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{x(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \frac{\pi i e^{-ma}}{2}$$

Equating imaginary parts, we get,

$$\int_0^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2}.$$

EXAMPLE 11

Evaluate $\displaystyle\int_0^\infty \frac{\cos mx}{(1+x^2)^2}dx,\ m>0.$

Solution.

Let $\displaystyle I=\int_0^\infty \frac{\cos mx}{(1+x^2)^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos mx}{(1+x^2)^2}dx \qquad \left[\because \frac{\cos mx}{(1+x^2)^2} \text{ is an even function of } x\right]$

Consider $\displaystyle\int_C \frac{e^{imz}\,dz}{(1+z^2)^2}$, where C is the Contour consisting of the real axis from $-R$ to R and the upper semi-circle $S:|z|=R$ taken in the anticlockwise sense and R is large.

Let $$F(z)=\frac{e^{imz}}{(1+z^2)^2}$$

Then $$\int_C F(z)dz = \int_{-R}^{R} F(x)dx + \int_S F(z)dz \tag{1}$$

The poles of $F(z)$ are given by

$(1+z^2)^2=0 \Rightarrow 1+z^2=0 \Rightarrow z=\pm i,$

which are poles of order 2.
But $z=i$ lies inside C.

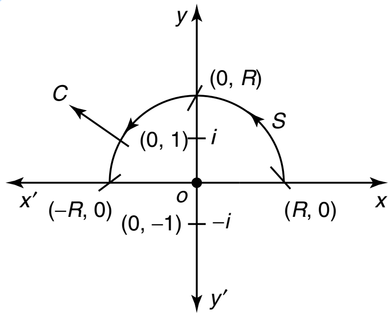

Fig. 16.27

$$\therefore \quad R(i)=\frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2F(z)]$$

$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2e^{imz}}{(z+i)^2(z-i)^2}\right]$$

$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{imz}}{(z+i)^2}\right]$$

$$=\lim_{z\to i}\frac{(z+i)^2\cdot e^{imz}\cdot im - e^{imz}\,2(z+i)}{(z+i)^4}$$

$$=\lim_{z\to i}\frac{[im(z+i)-2]e^{imz}}{(z+i)^3}=\frac{[(i+i)im-2]e^{mi^2}}{(i+i)^3}=\frac{(-2m-2)e^{-m}}{-8i}=\frac{(m+1)e^{-m}}{4i}$$

$\therefore$ by Cauchy's residue theorem,

$$\int_C F(z)dz = 2\pi i R(i) = 2\pi i\frac{(m+1)e^{-m}}{4i}=\frac{\pi(m+1)e^{-m}}{2}$$

$\therefore$ from (1) we get, $\displaystyle\int_{-R}^{R}F(x)dx+\int_S F(z)dz=\frac{\pi(m+1)e^{-m}}{2}$

Let $$f(z)=\frac{1}{(1+z^2)^2}.$$

Since $$|f(z)|=\left|\frac{1}{(1+z^2)^2}\right|\to 0 \text{ as } |z|\to\infty, \text{ by Jordan's lemma,}$$

$$\int_S f(z)e^{imz}\,dz=\int_S F(z)dz\to 0 \text{ as } R\to\infty$$

$$\therefore \quad \lim_{R\to\infty}\int_{-R}^{R} F(x)dx=\frac{\pi(m+1)e^{-m}}{2}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} F(x)dx=\frac{\pi(m+1)e^{-m}}{2}$$

$$\Rightarrow \quad \int_{0}^{\infty} F(x)dx=\frac{\pi(m+1)e^{-m}}{4} \quad \Rightarrow \quad \int_0^\infty \frac{e^{imx}\,dx}{(1+x^2)^2}=\frac{\pi(m+1)e^{-m}}{4}$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos mx+i\sin mx}{(1+x^2)^2}dx=\frac{\pi(m+1)e^{-m}}{4}$$

Equating real parts, we get, $\displaystyle\int_0^\infty \frac{\cos mx}{(1+x^2)^2}dx=\frac{\pi(m+1)e^{-m}}{4}$

EXAMPLE 12

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2+4x+5}$ using Contour integration.

Solution.

Let $$I=\int_{-\infty}^{\infty}\frac{\sin x\,dx}{x^2+4x+5}$$

Consider $\displaystyle\int_C \frac{e^{iz}\,dz}{z^2+4z+5}$, where C is the simple closed curve consisting of the real axis from $-R$ to R and the upper semi-circle $S:|z|=R$ taken in the antilock-wise sense and R is large.

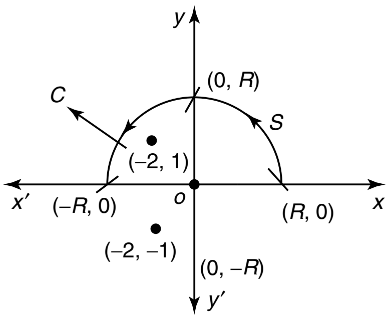

Fig. 16.28

Let $$F(z)=\frac{e^{iz}}{z^2+4z+5}$$

Then $$\int_C F(z)dz=\int_{-R}^{R}F(x)dx+\int_S F(z)dz$$

We shall evaluate $\displaystyle\int_C F(z)dz$.

The poles of $F(z)$ are given by $z^2 + 4z + 5 = 0$

$$\Rightarrow \qquad z = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm i2}{2} = -2 \pm i,$$

which are simple poles.

Then $\qquad z^2 + 4z + 5 = [z - (-2+i)][z - (-2-i)]$

$\therefore \qquad z = -2 + i$ lies inside C.

$$\therefore \qquad R(-2+i) = \lim_{z \to -2+i} \{[z - (-2+i)]F(z)\}$$

$$= \lim_{z \to -2+i} \frac{[z-(-2+i)]e^{iz}}{[z-(-2+i)][z-(-2-i)]}$$

$$= \lim_{z \to -2+i} \frac{e^{iz}}{z-(-2-i)} = \frac{e^{i(-2+i)}}{-2+i-(-2-i)} = \frac{e^{-1-2i}}{2i} = \frac{e^{-1}e^{-2i}}{2i} = \frac{e^{-2i}}{2ie}$$

By Cauchy's residue theorem,

$$\int_C F(z)\,dz = 2\pi i R(-2+i) = 2\pi i \frac{e^{-2i}}{2ie} = \frac{\pi e^{-2i}}{e}$$

$\therefore$ from (1) we get,

$$\int_{-R}^{R} F(x)\,dx + \int_S F(z)\,dz = \frac{\pi e^{-2i}}{e}$$

Let $\qquad f(z) = \dfrac{1}{z^2 + 4z + 5}$

Since $\qquad |f(z)| = \left| \dfrac{1}{z^2 + 4z + 5} \right| \to 0$ as $|z| \to \infty$, by Jordan's lemma,

$$\int_S f(z) e^{imz}\,dz = \int_S F(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\therefore \qquad \lim_{R \to \infty} \int_{-R}^{R} F(x)\,dx = \frac{\pi e^{-2i}}{e}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} F(x)\,dx = \frac{\pi e^{-2i}}{e} \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2 + 4x + 5} = \frac{\pi e^{-2i}}{e}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{(\cos x + i \sin x)\,dx}{x^2 + 4x + 5} = \frac{\pi[\cos 2 - i \sin 2]}{e}$$

Equating imaginary parts we get,

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2 + 4x + 5} = -\frac{\pi \sin 2}{e}.$$

Note Equating real parts we get, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2 + 4x + 5} = \frac{\pi \cos 2}{e}$.

EXERCISE 16.4

I. Using Contour integration evaluate the following integrals.

1. $\displaystyle\int_0^{2\pi}\frac{d\theta}{\sqrt{2-\cos\theta}}$
2. $\displaystyle\int_0^{2\pi}\frac{d\theta}{17-8\cos\theta}$
3. $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}$
4. $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta}$
5. $\displaystyle\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2}$ if $|p|<1$
6. $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta},\ a>b>0$
7. $\displaystyle\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2},\ a>b>0$
8. $\displaystyle\int_0^{2\pi}\frac{\sin\theta}{5+4\cos\theta}d\theta$
9. $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}d\theta$
10. $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{5-3\cos\theta}d\theta$
11. $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$
12. $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta,\ |a|<1$
13. $\displaystyle\int_0^{\pi}\frac{d\theta}{a^2+\sin^2\theta},\ a>0$

II. Using Contour integration evaluate the following integrals.

1. $\displaystyle\int_0^{\infty}\frac{x^2\,dx}{(x^2+9)(x^2+4)}$
2. $\displaystyle\int_0^{\infty}\frac{x^2\,dx}{x^4+1}$
3. $\displaystyle\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}\quad a>b>0$
4. $\displaystyle\int_0^{\infty}\frac{dx}{x^4+10x^2+9}$
5. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(1+x^2)^3}dx$
6. $\displaystyle\int_0^{\infty}\frac{dx}{(x^2+a^2)^3},\ a>0$
7. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}$
8. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}dx$
9. $\displaystyle\int_0^{\infty}\frac{dx}{x^4+1}$
10. $\displaystyle\int_0^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx$
11. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2+x+3}{x^4+5x^2+4}dx$
12. $\displaystyle\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}$
13. $\displaystyle\int_0^{\infty}\frac{x^2}{(x^2+1)^2}dx$
14. $\displaystyle\int_0^{\infty}\frac{x^2dx}{(x^2+9)(x^2+4)^2}$

III. Evaluate using contour integration

1. $\displaystyle\int_0^{\infty}\frac{\cos mx}{x^2+a^2}dx,\ a>0,\ m>0$
2. $\displaystyle\int_0^{\infty}\frac{\cos mx}{(x^2+a^2)^2}dx,\ a>0,\ m>0$
3. $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)},\ a>b>0$
4. $\displaystyle\int_0^{\infty}\frac{\cos x}{1+x^2}dx$
5. $\displaystyle\int_{-\infty}^{\infty}\frac{\cos 3x}{(x^2+1)^2}dx$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 + 2x + 2}$

7. $\displaystyle\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx,\ m >, a > 0$

8. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{x^2 + 1} dx, a > 0$

9. $\displaystyle\int_{0}^{\infty} \frac{x^2 \cos mx}{(x^2 + a^2)(x^2 + b^2)} dx,\ m > 0,\ a > b > 0$

ANSWERS TO EXERCISE 16.4

I.

1. 2π
2. $\dfrac{2\pi}{5}$
3. $\dfrac{2\pi}{3}$
4. $\dfrac{2\pi}{3}$
5. $\dfrac{2\pi}{1-p^2}$
6. $\dfrac{2\pi}{\sqrt{a^2-b^2}}$
7. $\dfrac{2\pi}{(a^2-b^2)^{3/2}}$
8. 0
9. $\dfrac{\pi}{5}$
10. $\dfrac{2\pi}{3}$
11. $\dfrac{2\pi}{\sqrt{3}}$
12. $\dfrac{2\pi a^2}{1-a^2}$
13. $\dfrac{\pi}{a\sqrt{a^2+1}}$

II.

1. $\dfrac{\pi}{10}$
2. $\dfrac{\pi}{2\sqrt{2}}$
3. $\dfrac{\pi}{2ab(a+b)}$
4. $\dfrac{\pi}{24}$
5. $\dfrac{\pi}{8}$
6. $\dfrac{3\pi}{16a^5}$
7. $\dfrac{\pi}{2}$
8. $\dfrac{\pi}{6}$
9. $\dfrac{\pi}{2\sqrt{2}}$
10. $\dfrac{\pi}{4}$
11. $\dfrac{5\pi}{6}$
12. $\dfrac{\pi}{3}$
13. $\dfrac{\pi}{4}$
14. $\dfrac{\pi}{200}$

III.

1. $\dfrac{\pi}{2a} e^{-ma}$
2. $\dfrac{\pi}{4a^3}(1+ma)e^{-ma}$
3. $\dfrac{\pi}{a^2-b^2}\left[\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right]$
4. $\dfrac{\pi}{e}$
5. $\dfrac{2\pi}{e^3}$
6. $\dfrac{\pi}{e}(\sin 1 + \cos 1)$
7. $\dfrac{\pi}{2} e^{-ma}$
8. $\dfrac{\pi}{2} e^{-a}$
9. $\dfrac{\pi[ae^{-ma} - be^{-mb}]}{2(a^2-b^2)}$

SHORT ANSWER QUESTIONS

1. **Evaluate $\displaystyle\int_C e^z\, dz$ where C is $|z| = 1$.**

2. **Evaluate $\displaystyle\int_C \frac{z^2+5}{z-3} dz$, where C is $|z| = 4$.**

3. **Evaluate $\displaystyle\oint_C \frac{z+2}{z} dz$, where C is the circle $|z| = 2$ in the z-plane.**

4. Evaluate $\oint_C \frac{z^2+1}{z^2-1}dz$, where C is the circle $|z-1|=1$.

5. Evaluate $\int_C \frac{dz}{z+4}$ where C is the circle $|z|=2$.

6. Evaluate $\int_C \frac{\cos \pi z}{z-1}dz$ if C is $|z|=2$.

7. Evalute $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$ where C is $|z|=3/2$.

8. Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1|=1$.

9. Evaluate $\int_C \frac{3z^2+7z+1}{z+1}dz$, where C is $|z|=\frac{1}{2}$.

10. Use Cauchy's integral formula to evaluate $\int_C \frac{1}{z^2-1}dz$, where C is the circle with centre 1 and radius = 1.

11. Evaluate $\int_C z^2 e^{1/z} dz$, where C is $|z|=1$.

12. Find the residue of $f(z)=\frac{4}{z^3(z-2)}$ at a simple pole.

13. Find the residue of $\frac{1-e^{2z}}{z^4}$ at $z=0$.

14. Find the singular point of $\frac{z+2}{(z+1)^2}$ and hence find its residue.

15. Determine the residue at the simple pole of $\frac{z+2}{(z+1)^2(z-2)}$.

16. Obtain the expansion of $\log_e(1+z)$ when $|z|<1$.

17. Write down the singularity of the function $\frac{1}{1-e^z}$.

18. Calculate the residue of $f(z)=\frac{e^{2z}}{(z+1)^2}$ at its pole.

19. If C is the circle $|z|=2$, evaluate $\int_C \tan z \, dz$.

20. If $f(z)=z\cos\frac{1}{z}$ find the residue at $z=0$.

21. Evaluate $\int_C z\sec z \, dz$ where C is $|z-\pi/2|=\frac{\pi}{2}$.