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Velocity and Acceleration Along Radial and Transverse Directions

1.1 INTRODUCTION

A particle is said to be in *motion* relative to surroundings if it changes its position at different times relative to these objects. If the particle is free to move in a straight line we say that the motion is rectilinear and particle has only one degree of freedom. If the particle is free to move in a plane along any curve we say that the motion is in a plane and the particle has two degree of freedom. If the particle is free to move in space we say that the particle has three degrees of freedom.

Here we shall discuss the motion of a particle when it has two degrees of freedom, *i.e.,* the motion of the particle is in a plane.

To determine the position of a moving point in a plane we must know any of the following :

(a) its co-ordinates with respect to two axes fixed in the plane (say x and y axes).

OR

(b) its distance from a point and angular distance from a straight line, both fixed in the plane (say pole and initial line in case of polar coordinates).

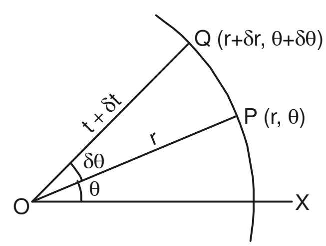
OR

(c) its arcual distance from a point fixed on its path and the angle, its direction of motion makes with a straight line fixed in the plane (say s and ψ).

We shall discuss all the three case separately.

1.2 ANGULAR VELOCITY

If a point P moves in a plane and if O be a fixed point and OX be a fixed line throguh O in the plane, then the angular velocity of P about O (or of the line OP) is the rate of change of the angle XOP.



Let P be the position of the moving particle at any time t and let $\angle POX = \theta$. Let Q be the position of the particle at time $(t + \delta t)$ and let $\angle QOX = \theta + \delta\theta$.

Clearly, in time δt, the angle turned through by the particle about $O = \delta\theta$.

$\therefore$ Average rate of changing of the angle about $O = \dfrac{\delta\theta}{\delta t}$

$\therefore$ **The angular velocity of the point *P***

$$\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta}$$

1.3 ANGULAR ACCELERATION

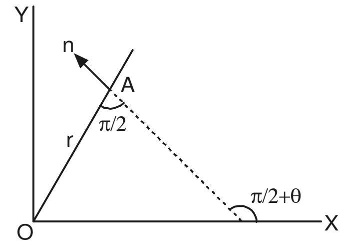
Angular acceleration of the point P is the rate of change or increase of its angular velocity. Hence, we have

$$\textbf{Angular acceleration} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}.$$

Units. Since the angle is measured in radians, hence the unit of angular velocity is radians per second.

The unit of angular acceleration will be radian per second square.

1.4 RATE OF CHANGE OF UNIT VECTOR



Let $\hat{r}$ denote the unit vector $\overrightarrow{OA}$ such that $OA = 1$ and $\angle AOX = \theta$, where OX and OY are mutually perpendicular fixed lines in the plane.

Let **i, j** be the unit vectors along OX and OY respectively. Then

$$\hat{\mathbf{r}} = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}$$

$$\therefore \quad \frac{d}{dt}(\hat{\mathbf{r}}) = (-\sin\theta)\frac{d\theta}{dt}\,\mathbf{i} + (\cos\theta)\frac{d\theta}{dt}\,\mathbf{j}$$

$$= [(-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}]\frac{d\theta}{dt} \qquad \text{...(i)}$$

Also let $\hat{\mathbf{n}}$ is an unit vector perpendicualr to OA. Then it will make an angle $\frac{\pi}{2} + \theta$ with OX. Hence we will have

$$\hat{\mathbf{n}} = \left\{\cos\left(\frac{\pi}{2} + \theta\right)\right\}\mathbf{i} + \left\{\sin\left(\frac{\pi}{2} + \theta\right)\right\}\mathbf{j}$$

$$= (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we get

$$\frac{d}{dt}(\hat{\mathbf{r}}) = \frac{d\theta}{dt}(\hat{\mathbf{n}}) \qquad \text{...(iii)}$$

Here $\hat{\mathbf{n}}$ is in the sense in which θ increases.

Also from (ii),

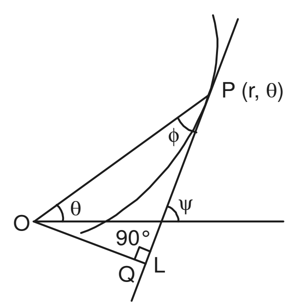
$$\frac{d}{dt}(\hat{\mathbf{n}}) = (-\cos\theta)\frac{d\theta}{dt}\,\mathbf{i} + (-\sin\theta)\frac{d\theta}{dt}\,\mathbf{j}$$

$$= -[(\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}]\frac{d\theta}{dt}$$

$\therefore$ From (i), we get

$$\frac{d}{dt}(\hat{\mathbf{n}}) = -\left(\frac{d\theta}{dt}\right)\hat{\mathbf{r}} \qquad \text{...(iv)}$$

1.5 RELATION BETWEEN ANGULAR AND LINEAR VECTORS



If v be the velocity of a point P moving in a plane curve and (r, θ) its polar co-ordinates referred to fixed point O in the plane, then the angular velocity of P about O is equal to vp/r^2, where p is the perpendicular drawn from O to the tangent at P.

Let radius vector OP makes an angle ϕ with the tangent at P to the given curve.

i.e., $\angle OPL = \phi$

Let $OL = p$ be the perpendicular from O to tangent at P. Then from ΔPOL, we have

$$p = r\sin\phi$$

Also, we know that

$$r\frac{d\theta}{ds} = \sin\phi \qquad \text{or} \qquad \frac{d\theta}{ds} = \frac{\sin\phi}{r} = \frac{p}{r^2} \qquad \text{...(i)}$$

$[\because\ p = r\sin\phi]$

$\therefore$ Angular velocity of P

$$= \frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{p}{r^2}\cdot v$$

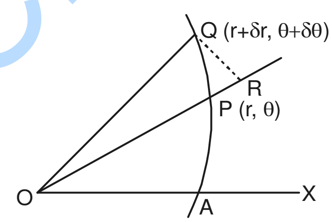
$\therefore$ **Angular velocity of** $P = \dfrac{vp}{r^2}$

1.6 RADIAL AND TRANSVERSE VELOCITIES AND ACCELERATIONS (POLAR COORDIANTES)

If a particle moves in a plane curve and if at timer t the position of the particle be at (r, θ), referred to O as pole and OX as initial line, then the resolved part of velocity at P along the radius vector OP *in the sense of r increasing is called the radial velocity* and the resolved part of the velocity at P along a line through P but at right angles to OP *in the sense in which* θ *increases* is called the transverse velocity. Similarly *radial and transverse accelerations* are defined.

Radial and Transverse Velocities

Let the particles be moving along the curve APQ and it is at P and Q at time t and $t+\delta t$ respectively. The polar co-ordinates of P and Q with respect to O as pole and OX as initial line are (r, θ) and $(r+\delta r, \theta+\delta\theta)$ respectively.



Draw a perpendicular QR on OP from Q.

Obviously the displacements of the moving point along and perpendicular to the radius vector OP are PR and QR respectively during interval δt.

Radial velocity at P

$$= \lim_{\delta t\to 0}\frac{\text{displacement along } OP}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{PR}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{OR - OP}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{(r+\delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{(r+\delta r)\left[1 - \frac{(\delta\theta)^2}{2!} + \ldots\right] - r}{\delta t} \quad \text{expanding } \cos\delta\theta.$$

$$= \lim_{\delta t\to 0}\frac{(r+\delta r).1 - r}{\delta t}, \quad \text{neglecting higher powers of } \delta\theta.$$

$$= \lim_{\delta t\to 0}\frac{\delta r}{\delta t} = \frac{\mathbf{dr}}{\mathbf{dr}} = \dot{\mathbf{r}}, \quad \text{in the direction of } OP$$

Transverse velocity

$$= \lim_{\delta t\to 0}\frac{\text{displacement perpendicular to } OP}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{RQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\sin\delta\theta}{\delta t}$$

$$= \frac{\lim_{\delta t \to 0} (r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^2}{3\,!} + \ldots\right]}{\delta t}$$

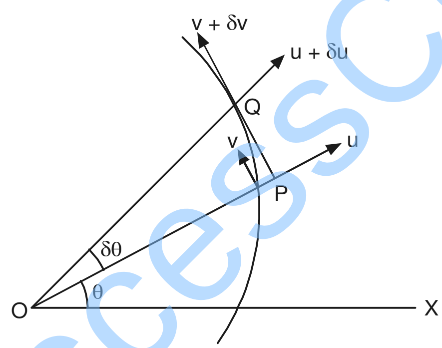
$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\,\delta\theta}{\delta t}, \text{ neglecting higher power of } \delta\theta$$

$$= \lim_{\delta t \to 0} \frac{r\,\delta\theta}{\delta t}, \text{ neglecting } \delta r\,.\,\delta\theta \text{ as it is very small.}$$

$$= \mathbf{r}\frac{\mathbf{d\theta}}{\mathbf{dt}} = \mathbf{r}\dot{\theta} \text{ in the sense of } \dot{\theta} \text{ increasing.}$$

Radial and Transverse Accelerations

Let the velocities along and perpendicular to radius vector at P and Q are $u, v, u+\delta u$ and $v+\delta v$ respectively. Then



Radial acceleration at P along the direction of r increasing

$$= \lim_{\delta t \to 0} \frac{\text{change of velocity along } OP \text{ during an interval } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\cos\delta\theta - (v+\delta v)\sin\delta\theta - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\,.\,1 - (v+\delta v)\,\delta\theta - u}{\delta t}$$

Expanding $\cos\delta\theta$ and $\sin\delta\theta$ and neglecting higher powers of $\delta\theta$.

$$= \lim_{\delta t \to 0} \frac{\delta u - v\,\delta\theta}{\delta t}$$

$$= \frac{du}{dt} - v\frac{d\theta}{dt}$$

Since $u =$ radial velocity $= \frac{dr}{dt}$ and $v =$ transverse velocity $= r\frac{d\theta}{dt}$

$\therefore$ Radial acceelration at $P = \frac{d}{dt}\left(\frac{dr}{dt}\right) - \left(r\frac{d\theta}{dt}\right)\left(\frac{d\theta}{dt}\right)$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$$

And transverse acceleration at *P*

$$= \lim_{\delta t \to 0} \frac{\text{change in velocity perrpendicular to } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{[(u+\delta u)\sin\delta\theta + (v+\delta v)\sin(90^\circ - \delta\theta)] - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta\theta)(\delta\theta) + (v+\delta v)(1) - v}{\delta t}, \text{ expanding } \sin\delta\theta, \cos\delta\theta$$

and neglecting higher powers of $\delta\theta$

$$= \lim_{\delta t \to 0} \frac{u\,\delta\theta + \delta v}{dt} = u\frac{d\theta}{dt} + \frac{dv}{dt}, \text{ where } u = \frac{dr}{dt};\ v = \frac{r\,d\theta}{dt}$$

$$= \frac{dr}{dt}\cdot\frac{d\theta}{dt} + \frac{d}{dt}\left(r\cdot\frac{d\theta}{dt}\right) = \frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\cdot\frac{dr}{dt}$$

$$= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} = \frac{1\,d}{r\,dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1\,d}{r\,dt}(r^2\dot{\theta}),$$

in the sense in which θ increases.

EXAMPELS

1. *A particle describes the curve $r = ae^{m\theta}$ with a constant velocity. Find the components of velocity and acceleration along the radius vector and perpendicular to it.*

Sol. The given path is

$$r = ae^{m\theta} \quad \text{...(i)}$$

Also velocity of particle $= \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}$

$= \text{constant} = v$ (say)

or $$\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = v^2 \quad \text{...(ii)}$$

From (i), we get

$$\frac{dr}{dt} = ame^{m\theta}\cdot\frac{d\theta}{dt} = mr\frac{d\theta}{dt}$$

or $$r\frac{d\theta}{dt} = \left(\frac{1}{m}\right)\frac{dr}{dt} \quad \text{...(iii)}$$

$\therefore$ From (ii) and (iii), we get

$$\left(\frac{dr}{dt}\right)^2 + \left(\frac{1}{m}\cdot\frac{dr}{dt}\right)^2 = v^2$$

or $$\left(\frac{dr}{dt}\right)^2\left[1 + \left(\frac{1}{m}\right)^2\right] = v^2$$

or $$\left(\frac{dr}{dt}\right)^2 = \frac{v^2m^2}{m^2+1}$$

or $$\frac{dr}{dt} = \frac{vm}{\sqrt{(m^2+1)}} \quad \text{...(iv)}$$

$\therefore$ From (iii), we get

$$r\frac{d\theta}{dt} = \frac{v}{\sqrt{(m^2+1)}} \quad \text{...(v)}$$

Components of velocity along and perpendicular to the radius vector are given by (iv) and (v).
From (iv) on differentiating, we get

$$\frac{d^2r}{dt^2} = 0$$

$\therefore$ Component of acceleration along radius vector

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -r\left(\frac{d\theta}{dt}\right)^2$$

$$= \left(-\frac{1}{r}\right)\left(\frac{v}{\sqrt{m^2+1}}\right)^2$$

$$= \frac{v^2}{r(m^2+1)}$$

and transverse component of acceleration

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$= \frac{1}{r}\frac{d}{dt}\left(\frac{vr}{\sqrt{(m^2+1)}}\right)$$

$$= \frac{v}{r\sqrt{(m^2+1)}}\cdot\frac{dr}{dt} = \frac{v^2 m}{r(m^2+1)}$$

2. *If the curve is an equiangular spiral* $r = ae^{\theta\cot\alpha}$ *and if the radius vector to the particle has constant angular velocity, show that the resultant acceleration of the particle makes an angle* 2α *with the radius vector and is of magnitude* v^2/r, *when v is the speed of the particle.*

Sol. The curve is given by

$$r = ae^{\theta\cot\alpha} \quad \text{...(i)}$$

Also, we are given that angular velocity

$$\frac{d\theta}{dt} = \text{constant} = \omega \text{ (say)} \quad \text{...(ii)}$$

From (i), we get

$$\frac{dr}{dt} = a\cot\alpha\,.\,e^{\theta\cot\alpha}\frac{d\theta}{dt} = r\omega\cot\alpha, \text{ from (i) and (ii)}$$

Again differentaiting, we get

$$\frac{d^2r}{dt^2} = \frac{dr}{dt}\omega\cot\alpha = r\omega^2\cot^2\alpha$$

$\therefore$ Radial acceleration

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = (r\omega^2\cot^2\alpha) - r\omega^2$$

$$= r\omega^2(\cot^2\alpha - 1) \quad \text{...(iii)}$$

and transverse acceleration

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}(r^2\omega) = \frac{1}{r}\,.\,2r\omega\frac{dr}{dt} = 2\omega\frac{dr}{dt}$$

$$= 2\omega\,(r\omega \cot\alpha) = 2r\omega^2 \cot\alpha \qquad \text{...(iv)}$$

If β be the angle which the resultant acceleration makes with the radius vector, then

$$\tan\alpha = \frac{\text{transverse acceleration}}{\text{radial acceleration}}$$

$$= \frac{2r\omega^2 \cot\alpha}{r\omega^2 (\cot^2\alpha - 1)}, \text{ from (iii) and (iv)}$$

$$= \frac{(2\cos\alpha/\sin\alpha)\sin^2\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos 2\alpha}$$

$$= \frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha$$

or $\qquad \beta = 2\alpha$

Also, speed of the particle

$$= \sqrt{\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right]}$$

$$= \sqrt{[\{(r\omega\cot\alpha)^2 + (r\omega)^2\}]} = r\omega \operatorname{cosec}\alpha = v \text{ (given)}$$

Again from (iii) and (iv), resultant acceleration.

$$= \sqrt{[\{(\text{radial acceleration})^2 + (\text{transverse acceleration})^2\}]}$$

$$= \sqrt{[\{r^2\omega^2(\cot^2\alpha - 1)\}^2 + (2r\omega^2\cot\alpha)^2]} = \sqrt{[(r^2\omega^4 \operatorname{cosec}^4\alpha)]}$$

$$= r\omega^2 \operatorname{cosec}^2\alpha = v^2/r, \text{ from (iv)}$$

3. *The velocity of a particle along and perpendicular to the radius vector are* λr *and* $\mu\theta$. *Find the path and show that the acceleration along and perpendicular to the radius vector are* $\lambda^2 r - \mu^2\theta^2/r$ *and* $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$ *respectively.*

Sol. Given that

$$\frac{dr}{dt} = \lambda r \qquad \text{...(i)}$$

and $\qquad r\frac{d\theta}{dt} = \mu\theta \qquad \text{...(ii)}$

Differentiating (i), we get

$$\frac{d^2r}{dt^2} = \lambda\frac{dr}{dt} = \lambda^2 r \qquad \text{...(iii)}$$

Dividing (i) by (ii), we get

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\theta} \quad \text{or} \quad \frac{\mu}{\lambda}\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Integrating, we get

$$-\frac{\mu}{\lambda r} = \log\theta - \log c, \text{ where } c, \text{ is constant of integration}$$

or $\qquad \dfrac{\mu}{\lambda r} = \log\left(\dfrac{\theta}{c}\right) \quad \text{or} \quad \theta = ce^{-\mu/\lambda r}$

This is the equation of path.

Now radial acceleration

$$= \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \lambda^2 r - r\,(\mu\theta / r)^2, \text{ from (ii) andf (iii)}$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

and transverse acceleration.

$$= \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{\mu\theta}{r}\right) \qquad \text{from (ii)}$$

$$= \frac{1}{r}\frac{d}{dt}(\mu\, r\theta) = \frac{\mu}{r}\left[r\frac{d\theta}{dt} + \theta\frac{dr}{dt}\right]$$

$$= \left(\frac{\mu}{r}\right)(\mu\theta + \theta\,.\,\lambda r) \qquad \text{from (i) and (ii)}$$

$$= \mu\left(\frac{\mu}{r} + \lambda\right).$$

4. *If the radial and transverse velocity of a particle are always proportional to each other, then*

(a) Show that the path is an equiangular spiral.

(b) If in addition, the radial and transverse accelerations are always proportional to each other, show that the velocity of the particle varies as some power of the radius vector.

Sol. (a) We are given that

$$\frac{dr}{dt} = k\left(r\frac{d\theta}{dt}\right) \qquad \text{...(i)}$$

or
$$\frac{1}{r}\,dr = k\,d\theta$$

Integrating, $\log r - \log c = k\,\theta$

where $\log c$ is constant of integration

or $\log\,(r/c) = k\theta$ or $r = ce^{k\theta}$,

which is an equiangular spiral.

(b) If in addition to (i), we have

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \lambda\,.\,\frac{1}{r}\left(r^2\frac{d\theta}{dt}\right)$$

or
$$\frac{d^2 r}{dt^2} - r\left\{\frac{1}{kr}\frac{dr}{dt}\right\}^2 = \frac{\lambda}{r}\frac{d}{dt}\left\{r^2\frac{1}{kr}\frac{dr}{dt}\right\}, \text{ from (i)}$$

or
$$\frac{d^2 r}{dt^2} - \frac{1}{k^2 r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{kr}\frac{d}{dt}\left[r\frac{dr}{dt}\right]$$

$$= \frac{\lambda}{kr}\left[r\frac{d^2 r}{dt^2} + \frac{dr}{dt}\,.\,\frac{dr}{dt}\right]$$

or
$$\left(1 - \frac{\lambda}{k}\right)\frac{d^2 r}{dt^2} = \frac{1}{r}\left(\frac{1}{k^2} + \frac{\lambda}{k}\right)\left(\frac{dr}{dt}\right)^2$$

or
$$A\frac{d^2 r}{dt^2} = \frac{B}{r}\left(\frac{dr}{dt}\right)^2$$

where A and B are constants.

or
$$\frac{d^2 r}{dt^2} = \frac{\mu}{r}\left(\frac{dr}{dt}\right)^2, \text{ where } \mu = B/A.$$

or $$\frac{d^2r/dt^2}{(dr/dt)} = \frac{\mu}{r}\frac{dr}{dt}$$

Integrating,

$$\log\left(\frac{dr}{dt}\right) = \mu \log r + \log D, \text{ where } D \text{ is constant}$$

or $$\frac{dr}{dt} = D.r^{\mu}$$

Substituting the value of $\frac{dr}{dt}$ in (i), we get

$$D.r^{\mu} = kr\frac{d\theta}{dt}$$

or $$r\frac{d\theta}{dt} = \alpha.r^{\mu}, \text{ where } \alpha = D/k.$$

$\therefore$ The resultant velocity

$$= \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2} = \sqrt{(r^{\mu}.D)^2 + (\alpha r^{\mu})^2}$$

$$= r^{\mu}\sqrt{(D^2 + \alpha^2)} \qquad \therefore \quad \text{velocity} \propto r^{\mu}.$$

5. *A small bead slides with constant speed v on a smooth wire in the shape of the cardioid* $r = a(1 + \cos\theta)$. *Show that the value of* $\frac{d\theta}{dt}$ *is* $\left(\frac{v}{2a}\right)\sec\frac{\theta}{2}$ *and that the radial component of the acceleration is constant.*

Sol. The path of the bead is

$$r = a(1 + \cos\theta) \qquad \text{...(i)}$$

$$\therefore \quad \frac{dr}{dt} = a(-\sin\theta)\frac{d\theta}{dt} \qquad \text{...(ii)}$$

and $$\frac{d^2r}{dt^2} = (-a\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (a\sin\theta)\frac{d^2\theta}{dt^2} \qquad \text{...(iii)}$$

$\therefore$ Speed of the bead

$$= \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}$$

$$= \sqrt{\left[(-a\sin\theta)\frac{d\theta}{dt}\right]^2 + \left\{a(1+\cos\theta)\frac{d\theta}{dt}\right\}^2}$$

$$= a\sqrt{[\sin^2\theta + (1+\cos\theta)^2]}\,\frac{d\theta}{dt}$$

$$= a\sqrt{2(1+\cos\theta)}\,\frac{d\theta}{dt}$$

$$= \left[2a\cos\frac{\theta}{2}\right]\frac{d\theta}{dt} = v \text{ (given)}$$

i.e., $$\frac{d\theta}{dt} = \left(\frac{v}{2a}\right)\sec\left(\frac{\theta}{2}\right) \qquad \text{...(iv)}$$

Now, radial acceleration of the bead

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \left[(-a\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (a\sin\theta)\frac{d^2\theta}{dt^2} \right] - a(1+\cos\theta)\left(\frac{d\theta}{dt}\right)^2,$$

from (i) and (iii)

$$= -a(1+2\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (a\sin\theta)\frac{d^2\theta}{dt^2},$$

$$= -a(1+2\cos\theta)\cdot\left(\frac{v}{2a}\sec\frac{\theta}{2}\right)^2 - a\sin\theta\left[\frac{v}{2a}\frac{1}{2}\sec\frac{\theta}{2}\tan\frac{\theta}{2}\frac{d\theta}{dt}\right],$$

from (iv)

$$= -a(1+2\cos\theta)\cdot\left(\frac{v}{2a}\sec\frac{\theta}{2}\right)^2 - \frac{1}{2}a\tan\frac{\theta}{2}\sin\theta\left(\frac{v}{2a}\sec\frac{\theta}{2}\right)^2$$

$$= -a\left(\frac{v^2}{4a^2}\right)\sec^2\frac{\theta}{2}\left[(1+2\cos\theta)+\frac{1}{2}\tan\frac{\theta}{2}\sin\theta\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\sec^2\frac{\theta}{2}\left\{(1+2\cos\theta)+\sin^2\frac{\theta}{2}\right\}$$

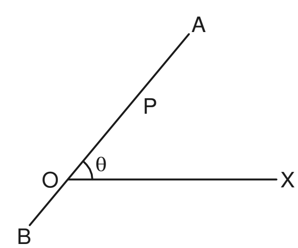
$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\sec^2\frac{\theta}{2}\left[(1+2\cos\theta)+\frac{1}{2}(1-\cos\theta)\right]$$

$$= -\frac{1}{4}\left(\frac{v^2}{a}\right)\sec^2\frac{\theta}{2}\left[\frac{3}{2}(1+\cos\theta)\right]$$

$$= -\frac{3}{4}\left(\frac{v^2}{a}\right)\left(\sec^2\frac{\theta}{2}\right)\left(\cos^2\frac{\theta}{2}\right)$$

$$= -\left(\frac{3}{4}\right)\frac{v^2}{a} = \text{constant}.$$

6. *One end of a rod describes a plane curve and the rod always passes through a fixed point in the plane of the curve. If the angular velocity of the rod is constant, show that the transverse acceleration of every point of the rod is the same at the same instant. What curve must the end describe to make this acceleration the same at every instant ?*



Sol. Let the co-ordinates of the end A be (r, θ) referred to fixed point O as pole.

where P is any point on the rod AB. Then $OP = r - d$.

The transverse acceleration at P

$$= \frac{1}{(r-d)}\frac{d}{dt}\left[(r-d)^2 \cdot \frac{d\theta}{dt}\right]$$

Also angular velocity of the rod

$$= \frac{d\theta}{dt} = \text{constant} = \omega \text{ (say)}$$

$\therefore$ The transverse acceleration of $P = \dfrac{1}{(r-d)}\dfrac{d}{dt}[(r-d)^2\omega]$

$$\frac{1}{(r-d)}2(r-d)\,\omega\cdot\frac{dr}{dt} = 2\omega\frac{dr}{dt}.$$

This is free from d and hence will be same for every point of the rod at the same instant.
If this acceleration be same at every instant also, then

$$2\omega \frac{dr}{dt} = \text{constant} \quad \text{or} \quad \frac{dr}{dt} = \lambda \text{ (say)}$$

$\therefore$ We have $\dfrac{dr}{dt} = \lambda$ and $\dfrac{d\theta}{dt} = \omega$

Dividing these two, we get

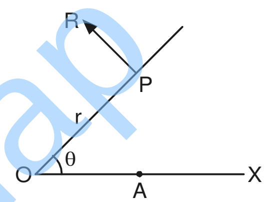
$$\frac{dr}{d\theta} = \frac{\lambda}{\omega} = k \text{ say} \quad \text{or} \quad dr = k\, d\theta$$

Integraing, we get $r = k\theta + c.$

This is the required equation of the curve.

7. *A straight smooth tube revolves with angular velocity* ω *in a horizontal plane about one extremity which is fixed, if at zero time a particle inside it be at a distance a from a fixed end and moving with velocity V along the tube, show that its distance at time t is*

$$a \cosh \omega t + \left(\frac{V}{\omega}\right) \sinh \omega t$$



Sol. Let initially OX be the position of the tube and A that of the particle inside it.

Let P be the position of the particle at any time t.

Let $OP = r$ and $\angle POX = \theta$.

Since the particle is moving with constant velocity V along the tube, *i.e.*, along the radius vector, hence the radial acceleration of the particle will be zero throughout the motion. Hence,

$$\frac{d^2 r}{dt} - r\left(\frac{d\theta}{dt}\right)^2 = 0 \qquad \text{...(i)}$$

Its solution is

$$r = c_1 \cosh \omega t + c_2 \sinh \omega t \qquad \text{...(ii)}$$

where c_1 and c_2 are constant and $\dfrac{d\theta}{dt} = \omega$.

Initially $t = 0$ and $r = a$,

$\therefore$ from (ii) $c_2 = a.$

Differentiating (ii)

$$\frac{dr}{dt} = c_1 \omega \sinh \omega t + c_2 \omega \cosh \omega t.$$

Initially $t = 0, \dfrac{dr}{dt} = V.$

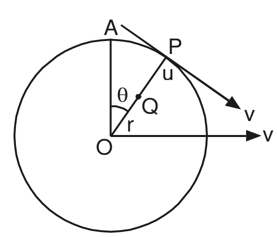
Hence $V = c_2 \omega$ or $c_2 = V / \omega$

$\therefore$ From (ii) we get

$$r = a \cosh \omega t + (V/\omega) \sinh \omega t$$

This gives the distance of the particle from O at time t.

8. *An insect crawls at a constant rate u along the spoke of a cart wheel of radius a, the cart moving with velocity v. Find the acceelrations along and perpendicular to the spoke.*



Sol. Let the initial position of spoke be OA and that of insect be O. At any time t, let the position of spoke be OP and that of insect be Q.

Let $\angle AOP = \theta$ and $OQ = r$.

Since the insect is crawling at a constant rate u along the spoke, we have

$$r = ut \qquad \text{...(i)}$$

The cart is moving with constant velocity v, hence the acceleration of the point O is zero and

$\frac{d\theta}{dt} = \frac{v}{OP} = \frac{v}{a}$ (where a is the radius of wheel)

$\therefore$ Acceleration of the insect along OP

= radial acceleration

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= -0 - r\left(\frac{v}{a}\right)^2 = \frac{-rv^2}{a} \qquad \left[\because \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{du}{dt} = 0\right]$$

Acceleration perpendicular to spoke.

= Transverse acceleration

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{v}{a}\right)$$

$$= \frac{2}{a}v\frac{dr}{dt} = \frac{2uv}{a} \qquad \because \frac{dv}{dt} = 0.$$

EXERCISES

1. A point P describes an equiangular spiral $r = ae^{\theta \cot \alpha}$ with constant angular velocity about the pole O. Find its acceleration and show that its direction makes the same angle with the tangent at P as the radius vector OP makes with the tangent.

2. A particle describe an equiangular spiral $r = ae^{\theta}$ in such a manner that its acceleration has no radial component. Prove that its angular velocity is constant and that the magnitude of the velocity and acceleration is each proportional to r.

3. If the angular velcotiy of a point moving in a plane curve be constant about a fixed origin, show that its transverse acceleration varies as its radial velocity.

4. If the radial and transverse velocities of a particle are always equal, then prove that the particle describes an equiangular spiral.

5. A particle moves in a circular path of radius a, so that its angular velocity about a fixed point in the circumference is constant and equal to ω. Show that the resultant acceleration of the particle at evey point is constant magnitude $4a\omega^2$.

6. A point P describes, with a constant angular velocity about O, the equiangular spiral $r = ae^{\theta}$, O being the pole of the spiral. Obtain the radial and transverse acceleration of P.

[**Ans.** $0, 2\omega^2 r$]

7. A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Show that the transverse acceleration varies as the fifth power of cosec θ.

8. A particle P describes a curve with constant velocity and its angualr velocty about a given fixed point O varies inversely as its distance from O. Show that the curve is an equiangular spiral.

9. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the equation to the path is $\frac{\lambda}{\theta} = \frac{\mu}{2r} + c$ and the components of accelerations are $2\lambda^2 r^3 - \mu^2\frac{\theta^4}{r}$ and $\lambda\mu r\theta^2 + 2\mu^2\frac{\theta^3}{r}$.

10. A point P describes, with a constant angular velocity about O, the equiangular spiral $r = e^{\theta}, O$ being the pole of the spiral. Obtain the radial and transverse accelerations of P.

[**Ans.** $0, 2\omega^2 r$]

11. A ring which can slide on a thin long smooth rod rests at a distance d from one end O. The rod is then set revolving uniformly about O in a horizontal plane. Show that in space the ring describes the curve $r = d \cos h\theta$.

12. Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the other is perpendicular to radius OP drawn from a fixed point O, is a conic section and the other is perpendicular to radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is u / v.

13. The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector. Prove that these components are

$$-\frac{1}{r \sin\theta}\frac{d}{dt}(r^2\dot{\theta}) \text{ and } \frac{\cot\theta}{r}\cdot\frac{d}{dt}(r^2\theta) + \ddot{r} - r\dot{\theta}^2$$

14. A point moves on a parabola $2a = r(1 + \cos\theta)$ in such a manner that the component of velocity at right angles to the radius vector from the focus is constant. Show that acceleration of the point is constant in magnitude.

15. A point describes a circle of radius a with a uniform speed v, show that the radial and transverse accelerations are $-(v^2/a)\cos\theta$ and $-(v^2/a)\sin\theta$ if a diameter is taken as initial line and one end of its diameter as pole.

16. If a rod which always passes through the origin rotates with uniform angular velocity ω, while one end describes the curve $r = a + be^{\theta}$, show that the radial acceleration of any point of the rod is the same at every instant, and the radial velocity is the same at every point at a given instant.

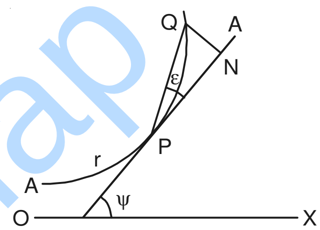
❑ ❑ ❑

Tangential and Normal Velocities and Accelerations

2.1 TANGENTIAL AND NORMAL VELOCITIES AND ACCELERATIONS

Tangential and normal velocities and accelerations are defined as the resolved parts of the velocities and accelerations along the tangent and normal respectively.

Tangential and Nurmal Velocities



Let a moving point moves along the curve APQ. At time t, let it comes to a point P and at time $t + \delta t$ at point Q.

Let the co-ordinates of P and Q be (s, ψ) and $(s + \delta s, \psi + \delta\psi)$ respectively. From Q, let us draw a perpendicular on the tangent at P which is QN and let the $\angle QPN = \varepsilon$. Join chord PQ, then

Tangential velocity at P

$$= \lim_{\delta t \to 0} \frac{\text{displacement along the tangent at } P \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PN}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(chord\ PQ) \,.\cos\varepsilon}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\text{are } PQ} \cdot \frac{\text{arc } PQ}{\delta t} \cos\varepsilon$$

$$= \lim_{\delta t \to 0} 1 . \frac{\delta s}{\delta t} . \cos\varepsilon \qquad \left[\because \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1 \text{ and arc } PQ = \delta s\right]$$

Now as Q tends to P, ε will be zero and $\cos\varepsilon$ tends to 1

$\therefore$ **tangentail velocity at** $P = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt}$

Similarly,

Normal velocity at P

$$= \lim_{\delta t \to 0} \frac{\text{displacement along the normal at } P \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{QN}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(\text{chord } PQ) \sin\varepsilon}{\delta t}$$

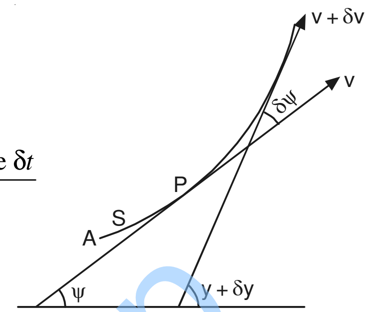
$$= \lim_{\delta t \to 0} \frac{\text{chord } PN}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\delta t} . \sin\varepsilon$$

Here as Q tends to P, ε tends to zero, and $\sin \varepsilon$ tends to zero.

$\therefore$ **Normal velocity at P = O**

Remark : Please note that in this case that total velocity is along the tangent.

Tangential and Normal Accelerations : Here let the velocity at P be v along the tangent which is making an angle ψ with the initial line and the velocity at Q be $v+\delta v$ along the tangent at Q, making an angle $\psi + \delta\psi$ with OX.



$\therefore$ **Tangential Acceleration at P**

$$= \lim_{\delta t \to 0} \frac{\text{Change in velocity along the tangent at } P \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\cos\delta\psi - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v).1 - v}{\delta t}$$

$[\because \cos\delta\psi = 1, \text{ neglecting higher powers of } \delta\psi]$

$$= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} = \ddot{s}$$

Also $$\frac{d^2 s}{dt^2} = \frac{d}{ds}\left(\frac{ds}{dt}\right)\frac{ds}{dt} = v\frac{dv}{ds}$$

Again, **Normal Acceleration at P**

$$= \lim_{\delta t \to 0} \frac{\text{change in velocity along the normal at } P \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(v+\delta v)\sin\delta\psi}{\delta t}$$

$$= \lim_{\delta t \to 0} (v+\delta v)\frac{\sin\delta\psi}{\delta\psi}\cdot\frac{\delta\psi}{\delta t}$$

$$= v\cdot\frac{d\psi}{dt}$$

$$= v\cdot\frac{d\psi}{dt}\cdot\frac{ds}{dt} = \frac{v^2}{\rho}$$

EXAMPLES

1. *A point describes the cycloid* $s = 4a\sin\psi$ *with uniform speed v. Find its acceleration at any point.*

Sol. The equation of the cycloid is

$$s = 4a\sin\psi$$

Also given that

$$\frac{ds}{dt} = v = \text{constant}$$

$$\therefore \qquad \frac{d^2 s}{dt^2} = 0$$

Hence tangential acceleration = 0

Differentiating eqn. (i) with respect to ψ, we get

$$\frac{ds}{d\psi} = 4a \cos \psi, \text{ or } \rho = 4a \cos \psi$$

$\therefore$ The resultant acceleration

$$= \sqrt{\left[\left\{\left(\frac{d^2 s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}\right]}$$

$$= \sqrt{\left[0 + \left(\frac{v^2}{4a \cos \psi}\right)^2\right]} = \frac{v^2}{4a\sqrt{(1 - \sin^2 \psi)}}$$

$$= \frac{v^2}{4a\sqrt{\left[\left(1 - \frac{s^2}{16a^2}\right)\right]}} = \frac{v^2}{\sqrt{[(16a^2 - s^2)]}}$$

2. *A point moves in a plane curve, so that its tangential and normal accelerations are equal and the angular velocity of the tangent is constant. Find the curve.*

Sol. Given that

$$v\frac{dv}{ds} = \frac{v^2}{\rho} \qquad \text{...(i)}$$

and $$\frac{d\psi}{dt} = \text{constant} = \omega \text{(say)}. \qquad \text{...(ii)}$$

From (i) $$\frac{dv}{ds} = \frac{v}{\rho} = \frac{v}{(ds/d\psi)} = v\frac{d\psi}{ds}$$

or $$dv = v\, d\psi$$

or $$\frac{dv}{v} = d\psi$$

Integrating, we get

$$\log v = \psi + \log c$$

where $\log c$ is the constant of integration.

or $$\log \frac{v}{c} = \psi \quad \text{or} \quad v = ce^{\psi}$$

or $$\frac{ds}{dt} = ce^{\psi} \quad \text{or} \quad \frac{ds}{d\psi}\frac{d\psi}{dt} = ce^{\psi}$$

or $$\frac{ds}{d\psi} = \frac{c}{\omega}e^{\psi}, \quad \text{from (i)}$$

$$ds = \frac{c}{\omega}e^{\psi}\, d\psi$$

Integrating, we get

$$s = \left(\frac{c}{\omega}\right)e^{\psi} + k$$

where k is a constant of integration.

or $$s = Ae^{\psi} + B.$$

where A and B are arbitrary constants. This is the required intrinsic equation of the curve.

3. *A particle is describibng a plane curve. If the tangential and normal accelerations are each constant throughout the motion, prove that the angle* ψ, *through which the direction of motion turns in time t is given by* $\psi = A \log(1 + Bt)$.

Sol. Given that

$$\frac{d^2s}{dt^2} = k \quad \text{..(i)}$$

and $$\frac{v^2}{\rho} = \lambda, \quad \text{...(ii)}$$

where k and λ are constants.
Integrating (i), we get

$$\frac{ds}{dt} = kt + c \quad \text{...(iii)}$$

where c is constant of integration.
From (ii), we get

$$\frac{v^2}{(ds/d\psi)} = \lambda \quad \text{or} \quad \frac{(ds/dt)^2}{(ds/d\psi)} = \lambda$$

or $$\frac{ds}{dt} \cdot \frac{d\psi}{dt} = \lambda$$

or $$(kt + c)\frac{d\psi}{dt} = \lambda, \text{ from (iii)}$$

or $$d\psi = \frac{\lambda}{(kt + c)} dt$$

Integrating, we get

$$\psi = \left(\frac{\lambda}{k}\right) \log(kt + c) + \log \mu,$$

where $\log \mu$ is a constant of integration.

Let $\psi = 0$, where $t = 0$, then

$$0 = (\lambda / k) \log c + \log \mu$$

$\therefore$ $$\psi = \frac{\lambda}{k}(kt + c) - \frac{\lambda}{k} \log c$$

$$= \frac{\lambda}{k} \log\left(\frac{kt + c}{c}\right)$$

or $$\psi = \left(\frac{\lambda}{k}\right) \log\left(1 + \frac{kt}{c}\right)$$

or $$\psi = A \log(1 + Bt),$$

where $A = \lambda / k$ and $B = k / c$.

4. *A particle describes a curve (for which s and* ψ *vanish simultaneously) with uniform speed v. If the accleration at any point s be* $\dfrac{v^2 c}{s^2 + c^2}$, *find the intrinsic equation of the curve.*

Sol. Given that :

$$\frac{ds}{dt} = v \text{ (constant)},$$

so, $$\frac{d^2s}{dt^2} = 0$$

$\therefore$ Acceleration at any point

$$= \sqrt{\left(\frac{d^2 s}{dt^2}\right) + \left(\frac{v^2}{\rho}\right)^2} = \frac{v^2}{\rho}$$

Now, it is given that

Acceleration at any point $\dfrac{v^2}{\rho} = \dfrac{v^2 c}{s^2 + c^2}$

or $\dfrac{1}{\rho} = \dfrac{c}{s^2 + c^2}$ or $\dfrac{d\psi}{ds} = \dfrac{c}{s^2 + c^2}$

or $\dfrac{1}{c}\, d\psi = \dfrac{ds}{s^2 + c^2}$

Integrating, we get

$$\left(\frac{1}{c}\right)\psi + A = \frac{1}{c}\tan^{-1}\left(\frac{s}{c}\right)$$

where A is constant at integration.

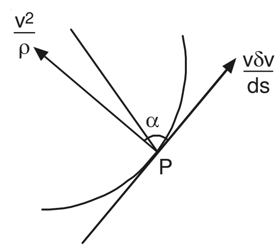
when $\psi = 0, s = 0$ (given), $\therefore$ $A = \tan^{-1} 0 = 0.$

Hence, $\left(\dfrac{1}{c}\right)\psi = \left(\dfrac{1}{c}\right)\tan^{-1}(s/c)$

or $s = c \tan \psi.$

5. *A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.*

Sol. Let at any instant particle be at P. The direction of tangential and normal acceleration at P are as shown in the figure. The resultant acceleration makes a constant angle α (say) with the tangent at P.



Hence

$$\tan \alpha = \frac{v^2/\rho}{(v\, dv/ds)}$$

or $$v = (\rho \tan \alpha)\frac{dv}{ds} = \tan \alpha \frac{ds}{d\psi} \cdot \frac{dv}{ds}$$

or $$\frac{1}{v}\, dv = \cot \alpha\, d\psi.$$

Integrating, we get

$$\log v = (\cot \alpha)\,\psi + \log c,$$

where $\log c$ is integration constant.

$\therefore$ $$v = ce^{\psi \cot \alpha} \quad \text{...(ii)}$$

Also given that acceleration is constant,

Therefore, we have

$$\sqrt{\left(\frac{v^2}{\rho}\right)^2 + \left(v\frac{dv}{ds}\right)^2} = k \quad \text{...(iii)}$$

$\therefore$ from (i) and (iii), we get

$$\tan^2 \alpha \left(v\frac{dv}{ds}\right)^2 + \left(v\frac{dv}{ds}\right)^2 = k^2$$

or $$\left(v\frac{dv}{ds}\right)^2 \sec^2\alpha = k^2$$

or $$v\frac{dv}{ds} = k\cos\alpha$$

or $$v\,dv = k\cos\alpha\,ds$$

Integrating, we get

$$v^2 = (2k\cos\alpha)\,s + c_1$$

where c_1 is constant of integration.

or $$v = \sqrt{[(2ks\cos\alpha + c_1)]}$$

$\therefore$ From (ii) and (iv), we get

$$\sqrt{[(2ks\cos\alpha + c_1)]} = ce^{\psi\cot\alpha}$$

or $$2ks\cos\alpha + c_1 = c^2 e^{2\psi\cot\alpha}$$

This is the intrinsic equation of the equiangular spiral.

6. *A particle moves in a catenary $s = c\tan\psi$. The direction of its accleration at any point makes equal angles with the tangent and the normal to the path at that point. If the speed at the vertex (where $\psi = 0$) be u, show that the velocity and acceleration at any other point ψ are ue^{ψ} and $\{\sqrt{(2/c)}\}\,u^2e^{2\psi}\cos^2\psi$.*

Sol. Since the direction of the resultant acceleration makes equal angles with the tangent and the normal hence the tangential and normal accelerations are equal.

i.e., $$v\frac{dv}{ds} = \frac{v^2}{\rho} \quad \text{...(i)}$$

or $$\frac{dv}{v} = \frac{1}{\rho}ds = \frac{d\psi}{ds}.ds = d\psi$$

Integrating, we have

$\log v = \psi + \log k$, where $\log k$ is constant of integration.

When $\psi = 0$, $v = u$, so we have

$\therefore$ $\log v = \psi + \log u$

or $$\log(v/u) = \psi \quad \text{or} \quad v = ue^{\psi} \quad \text{...(ii)}$$

This gives velocity at any point.

Now, the equation of catenary is

$$s = c\tan\psi$$

$\therefore$ radius of curvature $\rho = \dfrac{ds}{d\psi} = c\sec^2\psi$ ...(ii)

$\therefore$ Resultant acceleration

$$= \sqrt{\left\{\left(v\frac{dv}{ds}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}}$$

$$= \sqrt{\left\{\left(\frac{v^2}{\rho}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}} = \sqrt{2}\,\frac{v^2}{\rho}$$

$$= \sqrt{2}\,\frac{u^2e^{2\psi}}{c\sec^2\psi}, \text{ from (ii) and (iii)}$$

$$= (\sqrt{2}/c)\,u^2e^{2\psi}\cos^2\psi.$$

7. *A particle is moving in a parabla* $p^2 = ar$ *with uniform angular velocity about the focus, prove that its normal acceleration at any point is proportional to the radius of curvature of its path at that point.*

Sol. The equation of the parbala

$$p^2 = ar. \quad \text{...(i)}$$

Differentiating,

$$2p\frac{dp}{dr} = a \quad \text{or} \quad \frac{dp}{dr} = \frac{a}{2p}$$

$\therefore$ Radius of curvature $\rho = r\dfrac{dr}{dp} = \dfrac{r}{a/2p} = \dfrac{2pr}{a}$. ...(ii)

Angular velocity $= \dfrac{d\theta}{dt} =$ constant $= \omega$ (say)

Also, $$\frac{d\theta}{dt} = \frac{vp}{r^2} \quad \text{or} \quad \omega = \frac{vp}{r^2} \quad \text{...(iii)}$$

or $$v = \frac{\omega r^2}{p}.$$

$\therefore$ Normal acceleration $= \dfrac{v^2}{\rho}$ ...(iv)

$\therefore$ $$\frac{\text{Normal acceleration}}{\text{Radius of Curvature}} = \frac{v^2}{\rho^2} \quad \text{...(v)}$$

$$= \frac{\omega^2 r^4 / p^2}{4p^2 r^2 / a^2}, \text{ from (ii) and (v)}$$

$$= \frac{r^2\omega^2 a^2}{4p^4} = \frac{r^2\omega^2 a^2}{4a^2 r^2}, \text{ from (i)}$$

$$= \frac{\omega^2}{4} = \text{constant} = k \text{ (say)}$$

$\therefore$ normal acceleration α radius of curvature.

EXERCISES

1. The rate of the change of direction of velocity of a particle moving in a cycloid is constant. Prove that acceleration must be constant in magnitude.
2. A particle describes a circle of radius r with a uniform speed v, show that its acceleration at any point of the path is v^2/r and is directed towards the centre of the circle.
3. Prove that acceleration of a point moving in a curve with unifrom speed is $\rho\psi^2$.
4. A particle describes cycloid with uniform speed. Prove that normal acceleration at any point varies inversely as the distance from the base of the cycloid.
5. A particle describes a plane curve with a constant speed and its acceleration is constant in magnitude. Prove that the path is circle.
6. If tangential and normal acceleration components of a particle be equal. Prove that its velocity varies as e^{ψ}.
7. The tangential acceleration of a particle moving along a circle of radius a is λ times the normal acceleration. If the speed at a certain time is u, prove that it will return to the same point after a time $\left(\dfrac{a}{\lambda u}\right)(1 - e^{-2\pi\lambda})$.

8. A point moves in a plane curve so that its tangential acceleration is constant and the magnitude of the tangential velocity and normal acceleration are in a constant ratio. Find the intrinsic equation of the curve.

9. The velocity of a point moving in a plane curve varies as the radius of curvature. Show that the direction of motion revolves with constant angular velocity.

$$\left[\textbf{Hint. } v = k\rho \text{ or } \frac{ds}{dt} = k\frac{ds}{d\psi} \text{ or } d\psi = k\,dt \text{ or } \frac{d\psi}{dt} = k = \text{const.}\right]$$

10. If the tangential and normal acceleration of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point is given by $\rho = (at + b)^2$.

11. A particle moves in a plane in such a manner that its tangential and normal accelerations are always equal and its velocity varies as exp $(\tan^{-1} s/c)$, s being the length of arc of the curve measured from a fixed point on the curve. Find the path. [**Ans.** $s = c \tan \psi$]

12. The direction of the acceleration of a particle moving in a cycloid makes with the normal an angle equal to that which the tangent to the cycloid at the point makes with tangent at the vertex and is in the same sense. Prove that the tangent at the point turns uniformly, and that the magnitude of the acceleration is constant.

13. One point describes the diameter AB of a circle with constant velocity and another the semi-circumference AB from rest with constant tangential acceleration. They start together from A and arrive together at B. Show that velocities at B are $\pi : 1$.

14. A particle, projected with a velocity u is acted on by a force which produces a constant acceleration f in the plane of the motion inclined at a constant angle α with the direction of motion. Obtian the intrinsic equation of the curve described and show that the particle will be moving in the opposite direction to that of projection at time

$$\frac{u}{\tan\alpha}(e^{\pi\cot\alpha} - 1).$$

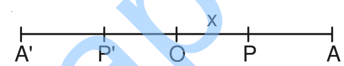
❑❑❑

Simple Harmonic Motion and Elastic Strings

3.1 SIMPLE HARMONIC MOTION

A particle is said to execute Simple Harmonic Motion, if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point. Simple Harmonic Motion is abbreviated as S.H.M.

Let O be the fixed point and the particle starts from rest from the point A such that $PA = a$. Let after time t the particle be at P, such that $OP = x$.



Then $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ will be the velocity and the acceleration of the particle at time t acting in the direction in which x increases.

Since in this case the acceleration is directed towards the fixed point O and is proportional to the distance of the particle from O, hence the equation of motion will be

$$m\frac{d^2x}{dt^2} = -\lambda x$$

or $$\frac{d^2x}{dt^2} = -\mu x, \text{ where } \mu = \left(\frac{\lambda}{m}\right) > 0$$

or $$v\frac{dv}{dx} = -\mu x. \quad \text{...(i)}$$

The differential equation (i) represents S.H.M. and is used generally as definition of S.H.M. If the equation of motion of a particle is of the form of (i) then we can at once say that the particle is executing S.H.M.

Integrating (i) with respect to x, we get

$$\frac{1}{2}v^2 = -\frac{1}{2}\mu x^2 + A,$$

where A is constant of integration. Initially,

at A, $x = a$ and $v = 0$

$$\therefore \quad 0 = -\frac{1}{2}\mu a^2 + A \quad \text{or} \quad A = \frac{1}{2}\mu a^2$$

$$\therefore \quad \frac{1}{2}v^2 = \frac{1}{2}\mu(a^2 - x^2).$$

or $$v^2 = \mu(a^2 - x^2) \quad \text{...(ii)}$$

or $$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{\{(a^2 - x^2)\}} \quad \text{...(iii)}$$

the negative sign is to be taken here as the particle is moving towards O, *i.e.*,; x decreases as t increases.

From (iii), we have

$$-\frac{dx}{\sqrt{\{(a^2-x^2)\}}}=\sqrt{\{(\mu)\}}\,t+B$$

Integrating, we get

$$\cos^{-1}(x/a)=\sqrt{\{(\mu)\}}\,t+B \qquad \text{...(iv)}$$

where B is a constant of integration

At A, $x=a$ and $t=0$, so from (iv), we get $B=0$, and hence

$$\cos^{-1}(x/a)=\sqrt{(\mu)}\,t$$

or

$$x=a\cos\{\sqrt{(\mu)}\,t\} \qquad \text{...(v)}$$

Equation (i), (iii) and (v) give the acceleration, velocity and the position of the particle at time t.

Nature of Simple Harmonic Motion

From (ii), $v=0$ at $x=\pm a$, thus if A' is a point on the other side of O such that $OA=OA'=a$, the particl comes to rest at A' also. When $x=0$, $v=\pm a\sqrt{\mu}$, *i.e.*, at O the velocity is $a\sqrt{\mu}$.

From (v) we have at $x=0$, $\cos\sqrt{(\mu)}\,t=0$ $\therefore$ $\sqrt{(\mu)}\,t=\frac{\pi}{2}$,

i.e., time from A to O is equal to $\frac{\pi}{2\sqrt{(\mu)}}$.

At $x=-a$,

$$\cos\sqrt{(\mu)}\,t=-1,$$

$\therefore$ $\sqrt{(\mu)}\,t=\pi$, *i.e.*, time from A to A' is $\frac{\pi}{\sqrt{\mu}}$ which is double the time from A to O.

At A, $\frac{d^2x}{dt^2}=-\mu a$ and due to this attraction the particle moves towards O and in time $\frac{\pi}{2\sqrt{\mu}}$ the particle reaches O with velocity $a\sqrt{\mu}$. At O, the attraction ceases but due to velcoity $a\sqrt{\mu}$ in the negative direction, the particle crosses O and moves towards the point A'. As soon as the particle crosses O it is attracted towards O and due to this, its velocity decreases and becomes zero at A'. At A' although the velocity of particle becomes zero, but due to attraction it again moves towards O and reaches O with velocity $a\sqrt{\mu}$. Due to this velocity the particle moves from O towards A and again stops at A where its velocity becomes zero. Thus the motion is repeated again and again. Thus the motion is repeated again and again. Thus the motion is **oscillatory** *i.e.*, from A to A' and back to A and so on.

Period. The time from A to A' is known as the **period** of the motion and this is given by

$$T=2.\frac{\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\mu}}$$

The frequency is the number of complete oscillations in one second, so that if n be the frequency and T the periodic time then,

$$n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}$$

The distance $a\ (=OA)$, *i.e.*, the distance of the centre from one of the position of rest is called the **amplitude.**

From (i), we have

$$\frac{d^2x}{dt^2}+\mu x=0.$$

The most general solution of this equation is

$$x=a\cos\{\sqrt{(\mu)}\,t+\xi\}.$$

In this relation the quantity ξ is called the **epoch** and the angle $(\sqrt{(\mu)}\, t + \xi)$ is called **argument.** The particle is at its maximum distance at time t_0, where

$$\sqrt{(\mu)}\; t_0 + \xi = 0$$

i.e.,
$$t_0 = -\frac{\xi}{\sqrt{\mu}}.$$

The **phase** at a point is defined as the time that has elapsed since the particle was at its maximum distance in the positive direction, *i.e.,*

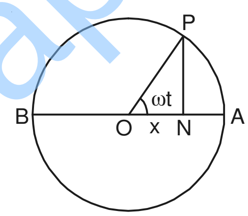
$$t - t_0 = t + \frac{\xi}{\sqrt{\mu}} = \frac{\sqrt{(\mu)}\; t + \xi}{\sqrt{\mu}},$$

this is phase at time t.

3.1.1 Geometrical Representaion of S.H.M.

If a particle describes a circle with constant angular velocity, the foot of the perpendicualr from the particle on a diameter moves with S.H.M.

Let a point P moves along the circumference of circle with centre O, with uniform speed. Let P describes equal arcs in equal times. Let the rate of description of $\angle POA$ is ω.



Let N be the foot of the perpendicular drawn from P on the diameter AOA'.

Let the particle starts from A and reaches P after time t. Then

$$\angle AOP = \omega t.$$

If $ON = x$ and $OP = a$, then from ΔOPN, we have

$$x = a \cos \omega t \qquad ...(i)$$

Hence,
$$\frac{dx}{dt} = -a\omega \sin \omega t \qquad ...(ii)$$

and
$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t = -\omega^2 x \qquad ...(iii)$$

Equations (ii) and (iii) represent the velocity and acceleration of N. Also, as P moves along the circumference of the circle, N oscillates from A to A' and back to A. Thus the motion of N is periodic. The periodic time will be the time taken by P in moving once along the whole circumference of the circle, *i.e.,* the time taken by P to turn through an angle 2π with the uniform rate of ω.

$\therefore$ Periodic time of $N = \dfrac{2\pi}{\omega}$

EXAMPLES

1. *Show that the particle executing S.H.M. requires* $\frac{1}{6}$ *th of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.*

Sol. The equation of motion of the particle executing S.H.M. is

$$\frac{d^2x}{dt^2} = -\mu x \qquad ...(i)$$

then the period $= \dfrac{2\pi}{\sqrt{\mu}} = T$ (say) ...(ii)

From (i), we get

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2),$$

where a is amplitude.

or $$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{[(a^2 - x^2)]}$$

negative sign indicates that particle is moving towards the centre.

or $$dt = -\frac{dx}{\sqrt{\mu}\sqrt{[(a^2 - x^2)]}}$$

Hence, the time T, taken by the particle in moving from the position of maximum displacement, *i.e.*, $x = a$ to position when the displacement is half the amplitude, *i.e.*, $x = \frac{1}{2}a$ will be

$$T_1 = \frac{1}{\sqrt{\mu}}\int_a^{a/2} \frac{dx}{\sqrt{[(a^2 - x^2)]}} = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\left(\frac{x}{a}\right)\right]_a^{a/2}$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(1)\right] = \frac{1}{\sqrt{\mu}}\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi}{3\sqrt{\mu}} = \frac{1}{6}T, \qquad \text{from (ii)}$$

2. *Show that if the displacement of a particle moving in a straight line is expressed by the equation* $x = a\cos nt + b\sin nt$, *a S.H.M. whose amplitude is* $\sqrt{(a^2 - b^2)}$ *and period is* $2\pi/n$.

Sol. We have

$$x = a\cos nt + b\sin nt \qquad \text{...(i)}$$

$$\frac{dx}{dt} = -an\sin nt + bn\cos nt$$

and $$\frac{d^2x}{dt^2} = -an^2\cos nt - bn^2\sin nt$$

$$= -n^2(a\cos nt + b\sin nt)$$

or $$\frac{d^2x}{dt^2} = -n^2x$$

This equation is of the form $\frac{d^2x}{dt^2} = -\mu x,$

hence represents S.H.M., where $\mu = n^2$

$\therefore$ Time period $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{n^2}} = \frac{2\pi}{n}.$

Also, amplitude of the motion is the value of x when velocity (dx/dt) is zero.

$\therefore$ by $\dfrac{dx}{dt} = 0$, we get $-an \sin t + bn \cos nt = 0$

or $\tan nt = b/a$. This gives $\sin nt = \dfrac{b}{\sqrt{a^2 + b^2}}$

and $$\cos nt = \frac{a}{\sqrt{a^2 + b^2}}$$

$\therefore$ Amplitude, from (i)

$$= a \cdot \frac{a}{\sqrt{a^2 + b^2}} + b \cdot \frac{b}{\sqrt{a^2 + b^2}} = \frac{a^2 + b^2}{\sqrt{(a^2 + b^2)}}$$

$$= \sqrt{(a^2 + b^2)}.$$

3. *At the ends of three successive seconds the distances of a point moving with S.H.M. from its mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is* $\dfrac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$.

Sol. If x be the distance of the particle from its mean position at time t, then for S.H.M.

$$x = a \cos \sqrt{\mu}\, t \quad \text{...(i)}$$

where μ is the intensity and a the amplitude of the S.H.M.
According to the given problem

$$1 = a \cos \{\sqrt{\mu}\,(T - 1)\} \quad \text{...(ii)}$$

$$5 = a \cos (\sqrt{\mu}\, T) \quad \text{...(iii)}$$

and $$5 = a \cos \{\sqrt{\mu}\,(T + 1)\} \quad \text{...(iv)}$$

considering three successive seconds as $T - 1, T$ and $T + 1$.
Adding (ii) and (iv), we get

$$1 + 5 = a [\cos \{\sqrt{\mu}\,(T - 1)\} + \cos \{\sqrt{\mu}\,(T + 1)\}]$$

or $6 = a [2 \cos (\sqrt{\mu}\, T) \cos \sqrt{\mu}\,]$ or $3 = 5 \cos (\sqrt{\mu})$, from (iii)

or $\cos \sqrt{\mu} = \dfrac{3}{5}$ or $\sqrt{\mu} = \cos^{-1}\left(\dfrac{3}{5}\right)$

$\therefore$ The period of a complete oscillation

$$= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$$

4. *A point executes S.H.M. such that in two of its positions the velocities are u, v and the corresponding accelerations* α, β. *Show that the distance between the positions is* $\dfrac{v^2 - u^2}{\alpha - \beta}$ *and* *amplitude of the motion is* $\dfrac{\{(v^2-u^2)(\alpha^2 v^2-\beta^2 u^2)\}^{1/2}}{(\beta^2-\alpha^2)}$.

Sol. Let at distances x_1 and x_2 from the centre the velocities be u and v and accelerations be α and β respectively.

Let μ be the intensity and a the amplitude of S.H.M., then we have

$$\frac{d^2x}{dt^2}=-\mu x \qquad \text{...(i)}$$

and
$$\left(\frac{dx}{dt}\right)^2=\mu(a^2-x^2) \qquad \text{...(ii)}$$

From (i), we have

$$\alpha=-\mu x_1 \qquad \text{...(iii)}$$

and
$$\beta=-\mu x_2 \qquad \text{...(iv)}$$

From (ii), we get

$$u^2=\mu(a^2-x_1^2) \qquad \text{...(v)}$$

and
$$v^2=\mu(a^2-x_2^2) \qquad \text{...(vi)}$$

Adding (iii) and (iv), we get $\alpha+\beta=-\mu(x_1+x_2)$

and from (v) and (vi), $v^2-u^2=\mu(x_1^2-x_2^2)$

Dividing, we get
$$\frac{v^2-u^2}{\alpha+\beta}=\frac{\mu(x_1^2-x_2^2)}{-\mu(x_1+x_2)}$$
$$=x_2-x_1$$

i.e., the required distance between the positions $=x_2-x_1=\dfrac{v^2-u^2}{\alpha+\beta}$

Again, from (iii) and (iv), we get

$$\beta^2-\alpha^2=\mu^2(x_2^2-x_1^2)$$

$$\therefore \quad \frac{v^2-u^2}{\beta^2-\alpha^2}=\frac{\mu(x_1^2-x_2^2)}{\mu^2(x_2^2-x_1^2)}=-\frac{1}{\mu}$$

From (v)
$$a^2=\frac{u^2}{\mu}+x_1^2=(u^2/\mu)+\left(\frac{\alpha^2}{\mu^2}\right), \text{ From (iii)}$$
$$=-\frac{u^2(v^2-u^2)}{(\beta^2-\alpha^2)}+\frac{\alpha^2(v^2-u^2)^2}{(\beta^2-\alpha^2)}$$
$$=\frac{(v^2-u^2)}{(\beta^2-\alpha^2)^2}[-u^2(\beta^2-\alpha^2)+\alpha^2(v^2-u^2)]$$
$$=(v^2-u^2)(v^2\alpha^2-u^2\beta^2)/(\beta^2-\alpha^2)^2$$

or
$$\text{amplitude}=a=\frac{\{(v^2-u^2)(v^2\alpha^2-u^2\beta^2)\}^{1/2}}{(\beta^2-\alpha^2)}.$$

5. *A particle is performing a S.H.M. of period T about a centre O and it passes through a point P where $OP=b$ with velocity v in the direction OP prove that time which elapses before it returns to P is*

$$\left(\frac{T}{\pi}\right)\tan^{1}\left(\frac{vT}{2\pi b}\right).$$

Sol. Given that $OP = b$. Let us take a point Q on the line OP, such that $OQ = x$. Then the velocity at Q is given by

$$\frac{dx}{dt} = \sqrt{\mu}\,\sqrt{[(a^2 - x^2)]} \qquad \text{...(i)}$$

where $OA = a$ is the amplitude of S.H.M.

Now, it is given that velocity at $P = v$, *i.e.*, at $x = b$. Hence from (i) we get

$$v = \sqrt{\mu}\,\sqrt{[(a^2 - b^2)]} \qquad \text{..(ii)}$$

Also, from (i), we get

$$\frac{dx}{\sqrt{[(a^2 - x^2)]}} = \sqrt{(\mu)}\, dt$$

Integrating, we get

$$\sin^{-1}(x/a) = \sqrt{(\mu)}\, t + c,$$

where c is an arbitrary constant.

Let at $t = 0, x = 0$ then $c = 0$.

$$\therefore \quad \sin^{-1}(x/a) = \sqrt{(\mu)}\, t$$

or $$x = a \sin\{\sqrt{(\mu)}\, t\} \qquad \text{...(iii)}$$

Let t' be the time taken in moving from O to P, then from (iii),

$$b = a \sin(\sqrt{\mu}\, t')$$

or $$t' = \left(\frac{1}{\sqrt{\mu}}\right) \sin^{-1}\left(\frac{b}{a}\right) \qquad \text{...(iv)}$$

Also, time taken from O to A

$$= \frac{1}{4}(\text{period}) = \frac{1}{4}\left(\frac{2\pi}{\sqrt{\mu}}\right) = \frac{\pi}{2\sqrt{\mu}}.$$

$\therefore$ Time taken in moving from P to A

$$= \frac{\pi}{2\sqrt{\mu}} - t'$$

$\therefore$ Required time $= 2\times$ time taken in moving from P to A

$$= 2\left(\frac{\pi}{2\sqrt{\mu}} - t'\right) = 2\left(\frac{\pi}{2\sqrt{\mu}} - \frac{1}{\sqrt{\mu}} \sin^{-1}\frac{b}{a}\right) = \frac{2}{\sqrt{\mu}}\left[\frac{\pi}{2} - \sin^{-1}\frac{b}{a}\right]$$

Also, time period $T = \dfrac{2\pi}{\sqrt{\mu}}$

$$\therefore \quad \frac{1}{\sqrt{\mu}} = \frac{T}{2\pi} \qquad \text{...(v)}$$

Also, let $\dfrac{\pi}{2} - \sin^{-1}\dfrac{b}{a} = \theta$ ...(vi)

$$\therefore \quad \frac{b}{a} = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\therefore \quad \tan\theta = \frac{\sqrt{[(a^2 - b^2)]}}{b} = \frac{v}{b\sqrt{\mu}} \qquad \text{from (ii)}$$

$$= \frac{vT}{2\pi b}, \qquad \text{from (vi)}$$

or $$\theta = \tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

Substituting the values of $\frac{1}{\mu}$ and θ.

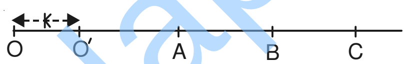
i.e., $\left(\frac{\pi}{2} - \sin^{-1}\frac{b}{a}\right)$ in (v), we get

the required time $= \frac{2T}{2\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right) = \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$.

6. *If in a S.H.M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of the force; show that the period T is given by the equation*

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

Sol. Let O be the centre of force and O' the fixed point from which the distances of the points A, B and C are a, b and c respectively, velocities at A, B, C are u, v and w. Let $OO' = k$. Then the distances of A, B and C from O will be $a+k, b+k$ and $c+k$ respectively.



We know that velocity V at a distance x from the centre of force is given by

$$V^2 = \mu(A^2 - x^2),$$

where μ is the intensity and A the amplitude of S.H.M.

$\therefore$ at A we get

$$u^2 = \mu\{A^2 - (a+k)^2\}$$

or $$\frac{u^2}{\mu} = A^2 - a^2 - k^2 - 2ak$$

or $$\left(\frac{u^2}{\mu} + a^2\right) + 2ak + (k^2 - A^2) = 0 \quad \text{...(i)}$$

Similarly, at B and C, we get

$$\left(\frac{v^2}{\mu} + b^2\right) + 2bk + (k^2 - A^2) = 0 \quad \text{...(ii)}$$

and $$\left(\frac{w^2}{\mu} + c^2\right) + 2ck + (k^2 - A^2) = 0 \quad \text{...(iii)}$$

Eliminating $2k$ and $(k^2 - A^2)$ from (i), (ii) and (iii), we get

$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

or $$\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} = -\mu \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

or $$\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} = \mu (b-c)(c-a)(a-b)$$

Also, periodic time $T = \dfrac{2\pi}{\sqrt{\mu}}$ or $\mu = \dfrac{4\pi^2}{T^2}$

$\therefore$ $$\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \frac{4\pi^2}{T^2}(b-c)(c-a)(a-b)$$

7. *A particle oscillates with S.H.M. of amplitude a and periodic time T. Find the expression of the velocity v in terms of (i), a, T and x; (ii) a, T and t and also prove that*

$$\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}.$$

Sol. If μ be the intenstiy of S.H.M. we have

$$v^2 = \mu (a^2 - x^2) \quad \text{...(i)}$$

$$x = a \cos \sqrt{(\mu)}\, t \quad \text{...(ii)}$$

and $$T = 2\pi / \sqrt{\pi} \quad \text{...(iii)}$$

(i) From (iii), $\mu = 4\pi^2 / T^2$

$\therefore$ From (i) $v^2 = \dfrac{4\pi^2}{T^2}(a^2 - x^2)$.

(ii) From (i) and (ii)

$$v^2 = \mu [a^2 - a^2 \cos^2 \sqrt{(\mu)}\, t]$$

$$= \mu a^2 \sin^2 \sqrt{(\mu)}\, t$$

$$= \frac{4\mu^2}{T^2} a^2 \sin^2 \left(\frac{2\pi t}{T}\right)$$

(iii) $\int_0^T v^2 dt$

$$= \int_0^T \frac{4\pi^2}{T^2} a^2 \sin^2 \left(\frac{2\pi t}{T}\right) dt$$

$$= \frac{2a^2\pi^2}{T^2} \int_0^T 2 \sin^2 \left(\frac{2\pi t}{T}\right) dt$$

$$= \frac{2a^2\pi^2}{T^2} \int_0^T \left(1 - \cos \frac{4\pi t}{T}\right) dt$$

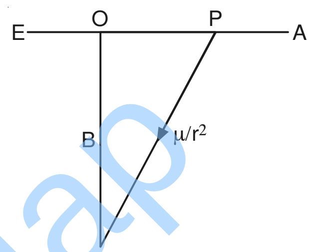
$$= \frac{2a^2\pi^2}{T^2} \left(t - \frac{T}{4\pi} \sin \frac{4\pi t}{T}\right)_0^T$$

$$= \frac{2a^2\pi^2}{T^2}\left\{\left(T - \frac{T}{4\pi}\sin 4\pi\right) - 0\right\}$$

$$= \frac{2a^2\pi^2}{T}.$$

8. *A small bead P can slide on a smooth wire AB, being acted upon by a force per unit mass equal μ / CP^2, where C is outside AB. Show that time of a small oscillation about its position of equilibrium is $(2\pi/\sqrt{\mu})\, b^{3/2}$, where b is the perpendicular distance of C from AB.*

Sol. Let O be the foot of the perpendicular from C on AB and let O be the origin. Let P be the position of bead at time t and $OP = x$, $CP = r$.



Here, the force along $PC = \frac{\mu}{r^2}$, hence

Component of force along PO

$$= \frac{\mu}{r^2}\cos\theta = \frac{\mu x}{r^3}$$

$$= \frac{\mu x}{(b^2 + x^2)^{3/2}} = \frac{\mu x}{b^3}\left(1 + \frac{x^2}{b^2}\right)^{-3/2}$$

$$= \frac{\mu x}{b^3}\left[1 - \frac{3}{2}\frac{x^2}{b^2} + \ldots\right]$$

$$= \frac{\mu x}{b^3}, \text{ neglecting higher powers of } \frac{x}{b}.$$

Hence the equation of motion of bead will be

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{b^3}.$$

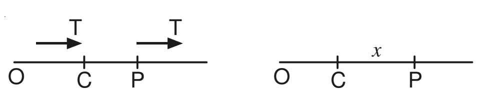
This is the standard form of S.H.M. and its period

$$= \frac{2\pi}{\{\sqrt{(\mu/b^3)}\}} = \frac{2\pi}{\sqrt{\mu}}\, b^{3/2}.$$

9. *A particle P moves in a straight line OCP being attracted by a force $m\mu$. PC, always directed towards C whilst C moves along OC with a constant acceleration f. If initially C was at rest at the origin O, then P was at a distance c from O moving with velocity V, prove that the distance of P from O at any time t is*

$$\left(\frac{f}{\mu} + c\right)\cos\sqrt{\mu}\,t + \frac{V}{\sqrt{\mu}}\sin\sqrt{\mu}\,t - \frac{f}{\mu} + \frac{1}{2}ft^2.$$

Sol. Intially, let O and C are coincident. Let after time t, $CP = x$. Since C is itself moving with constant acceleration f, we consider the motion of P relative to C. For this purpose we impose the constant acceleration f upon the whole system in the opposite direction. Let at any instant $CP = x$,



then the particle P will be attracted towards C with a force μx per unit of mass.

Eqation of motion of P relative to C is

$$\frac{d^2x}{dt^2} = -\mu x - f = -\mu\left(x - \frac{f}{\mu}\right) \qquad \text{...(i)}$$

Solution to this differential equation will be

$$x - \frac{f}{\mu} = A \cos \sqrt{\mu} t + B \sin \sqrt{\mu}\, t \qquad \text{...(ii)}$$

Differentiating, we get

$$\frac{dx}{dt} = -A\sqrt{\mu} \sin \sqrt{\mu}\, t + B\sqrt{\mu} \cos \sqrt{\mu}\, t \qquad \text{...(iii)}$$

Given conditions are, at $t = 0,\ x = c,\ \frac{dx}{dt} = V$. Hence we have from (ii) and (iii),

$$A = c - \frac{f}{\mu},\ B = \frac{V}{\sqrt{\mu}}$$

Therefore,

$$x = \frac{f}{\mu} + \left(c - \frac{f}{\mu}\right) \cos \sqrt{\mu}\, t + \frac{V}{\mu} \sin \sqrt{\mu}\, t \qquad \text{...(iv)}$$

Equation (iv) gives the distance of P from the point C after time t, but in time t the point C itself will move a distance $\frac{1}{2} ft^2$. Hence the distance of P from O at any instant is equal to

$$\frac{f}{\mu} + \frac{1}{2} ft^2 + \left(c - \frac{f}{\mu}\right) \cos \sqrt{\mu}\, t + \frac{V}{\sqrt{\mu}} \sin \sqrt{\mu}\, t.$$

EXERCISE

1. In a S.H.M., at what distance from the centre will the velocity be half of the maximum.

[**Ans.** $\pm \frac{1}{2} a\sqrt{3}$]

2. The speed v of a point P which moves in a straight line is given by the relation $v^2 = a - bx^2$, where x is the distance of the point P from a fixed point on the path, a and b being constants. Show that motion of P is simple harmonic and determine its amplitude and period.

[**Ans.** $\sqrt{(a/b)}, 2\pi b$]

3. A particle moves in a straight line and its velocity at a distance x from the origin is $k\sqrt{[(a^2 - b^2)]}$, where k and a are constants. Prove that the motion is simple harmonic and find the amplitude and the periodic time of the motion.

[**Ans.** $a, \frac{2\pi}{k}$]

4. A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distance from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a complete oscillation is $\frac{2\pi}{\theta}$, where $\theta = \cos^{-1}\left\{\frac{x_1 + x_3}{2x_2}\right\}$.

5. A body is attached to one end inextensible string and the other end moves in a vertical line with S.H.M. of amplitude a, taking n complete oscillations per second, show that the string will not remain tight during the motion unless $n^2 < (g/4\pi^2 a)$.

6. Show that in a S.H.M. of amplitude a and period T, the velocity v at a distance, x from the centre is given by the relation $v^2T^2 = 4\pi^2(a^2 - x^2)$.

7. The speed v of a particle moving along the axis OX is given by the relation $v^2 = n^2(8ax - x^2 - 12a^2)$. Prove that the motion is simple harmonic, with amplitude $2a$ and that the time taken from $x = 4a$ to $x = 6a$ is $\frac{\pi}{2n}$. What is the periodic time. $\left[\textbf{Ans. } \frac{\pi}{2n}\right]$

8. The speed v of the point P which moves in a line is given by the relation $v^2 = a + 2bx - cx^2$, where x is the distance of the point P from a fixed point on the path and a, b, c are constants. Show that the motion is simple harmonic if c is positive and determine the period.

$\left[\textbf{Ans. } \frac{2\pi}{\sqrt{c}}\right]$

9. If time t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $f^3 \frac{d^2t}{dv^2}$, f being the acceleration.

10. In a S.H.M., if the velocities at distance b and c from the centre of force be respectively u and v, then prove that the frequency n of oscillation is given by

$$4\pi^2 n^2 (b^2 - c^2) = v^2 - u^2.$$

11. A particle moves with S.H.M. in a straight line. In the first second after starting from rest, it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is $2a^2/(3a - b)$.

12. A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b - a)/v$.

13. A particle of mass m is attached towards a fixed point O with a force (m/μ) times the distance from O. If initially it is projected towards O with a velocity v from a point distant c from O, find the amplitude of its oscillations.

$\left[\textbf{Ans. } \left(\frac{v^2}{\mu} + c^2\right)^{1/2}\right]$

14. In a S.H.M. of period $\frac{2\pi}{\omega}$ if the initial displacement be x_0 and the initial velocity u_0, prove that

(i) Amplitude $= \sqrt{\{x_0^2 + (u_0^2/\omega^2)\}}$

(ii) Position at time $t = \sqrt{\{x_0^2 + (u_0^2 + \omega^2)\}}\ \cos\left[t - \left(\frac{1}{\omega}\right)\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right]$

(iii) Time to the position at rest $= \left(\frac{1}{\omega}\right)\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$

15. A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distances of the particle from the two ends, prove that the period of small transverse oscillation of mass m is

$$\frac{2\pi}{\sqrt{\{T(a + b)/mab\}}}$$

16. A particle starts from rest under an acceleration $k^2 x$ directed towards a fixed point and after time t another particle starts from the same position under the same acceleration. Show that the particle will collide at time $\left[(\pi / k) + \frac{1}{2} t\right]$ after the start of the first particle provided $t < 2\pi / k$.

17. Assuming that the gravity inside the earth varies as the distance from its centre, show that a train, starting from rest and moving under gravity only, would take the same time to travors smooth straight airless tunnel between any two points of the earth's surface. Find the time.

[**Ans.** $42\frac{1}{2}$ minutes nearly]

3.2 ELASTIC STRINGS AND SPRINGS (HOOKE'S LAW)

According to Hooke's Law,

$$\text{stress} \propto \text{strain} \qquad \text{or} \qquad \text{stress} = \lambda \text{ (strain)}$$

where λ is called modulus of elasticity.

In case of elastic strings or springs the stress is the tension and strain is the increase in length per unit legnth or the increase in natural length divided by the natural length.

Thus if a is natural length and l is the extended length then tension T of the string is given by

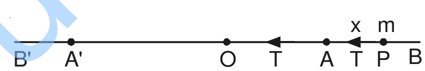
$$T = \lambda \frac{l - a}{a}.$$

3.2.1 A Horizontal Elastic String

One end of an elastic string whose modulus of elasticity is λ and whose natural length is a, is tied to a fixed point on a smooth horizontal table, and the other end is tied to a mass m lying on the table. The particle is pulled to a distance where the extension of the string becomes b, and then let go. Discuss the motion and find the period of one complete oscillation.

Let O be the fixed point and OA the natural length *i.e.*, $OA = a$. The particle is pulled out to a point B and then released.

Let P $(AP = x)$ be the position of the particle at time t, then by Hooke's law, the tension T in the string will be



$$T = \lambda \frac{x}{a} \qquad \text{...(i)}$$

The equation of motion of m will be

$$m \frac{d^2 x}{dt^2} = -T = -\lambda \frac{x}{a}$$

or

$$\frac{d^2 x}{dt^2} = -\frac{\lambda}{am} x \qquad \text{...(ii)}$$

This show that the motion is simple harmonic so long as there is extension in the string, *i.e.*, x is not equal to zero. The period of the S.H.M. is

$$\frac{2\pi}{\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}} = 2\pi\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}.$$

Hence time of describing BA will be $\frac{1}{4}\cdot 2\pi\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]} = \frac{\pi}{2}\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}$

From (ii), we have $\quad v\frac{dv}{dx} = -\frac{\lambda}{am}x$

Integrating, $\quad \frac{1}{2}v^2 = -\frac{\lambda}{am}\frac{x^2}{2} + C$

where C is constant of integration.

At B, $x = b, v = 0$, hence $\quad C = \frac{\lambda}{am}\cdot\frac{1}{2}b^2$

Therefore $\quad v^2 = \frac{\lambda}{am}(b^2 - x^2) \quad$ ...(iii)

Since the particle is moving from B towards O, *i.e.*, x decreases as t increases, we get

$$\frac{dx}{dt} = -\sqrt{\left[\left\{\frac{\lambda}{am}(b^2 - x^2)\right\}\right]} \quad \text{...(iv)}$$

when the particle reaches the point A, $x = 0$ and hence from (i) and (iv) we have $T = 0$ and $v = -\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}b$. Hence the S.H.M. ceases and the particle moves towards O with constant velocity $\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}b$ till it reaches the point A' on the other side of O where $OA' = OA = a$.

Thus the particle moves from A to A' with constant velocity $\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}b$ and time taken to describe this path

$$= \frac{2a}{\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}b} = \frac{2a}{b}\sqrt{\left[\left(\frac{am}{b}\right)\right]}$$

As soon as the particle goes beyond A', the string becomes again extended so that tension comes into play and the motion is again simple harmonic till the particle reaches the point B', where $AB = A'B' = b$. Here the velocity becomes zero and particle stops. The particle will then retrace its path under S.H.M. till it will reach to A' and then again under constant velocity $\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}b$ till the point A and then again under S.H.M. till the point B where particle will stop.

This motion will again be repeated.

Obviously, the motion from B to A, A' to B', B' to A' and A to B is simple harmonic and the total time for this motion is equal to $2\pi\sqrt{\left[\left(\frac{\lambda}{am}\right)\right]}$.

The motion from A to A' and A' to A in under constant velocity $\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}\,b$ and the total time for this motion is $2\,.\,\frac{2a}{b}\sqrt{\left[\left(\frac{am}{b}\right)\right]}$.

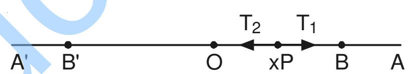
Thus the complete motion is oscillatory and the time for one complete oscillation is

$$2\pi\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}+\frac{4a}{b}\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}=2\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}\left(\pi+\frac{2a}{b}\right).$$

EXAMPLES

1. *A light elastic string of modulus λ is stretched to double its natural length and is tied to two fixed points distant 2a apart. A particle of mass m, tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed point and released. Prove that the time of a small oscillation is $\pi\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}$ and the maximum velocity is $\sqrt{\left[\left(\frac{\lambda a}{m}\right)\right]}$ whose λ is modulus of elasticity.*

Sol. Let the string be tied to two fixed points A and A'. Let O, the middle point of AA' be the origin. Let the particle tied at O is displaced to B, where $OB = a/2$. Let P be the position of the particle m at time t and then tensions in the string be T_1 and T_2 along PA and PA'. Let $OP = x$.



Then the equation of motion of P will be

$$m\frac{d^2x}{dt^2}=T_1-T_2$$

Now, $$T_1=\lambda\,.\,\frac{a-x-a/2}{a/2}=\lambda\left(1+\frac{2x}{a}\right)$$

since $a-x$ is extended length and $a/2$ natural length of this portion. Similarly

$$T_2=\lambda\frac{a+x-a/2}{a/2}=\lambda\left(1+\frac{2x}{a}\right)$$

Hence from (i) $$m\frac{d^2x}{dt^2}=\lambda\left[\left(1-\frac{2x}{a}\right)-\left(1+\frac{2x}{a}\right)\right]$$

or $$\frac{d^2x}{dt^2}=-\frac{4\lambda}{am}x. \qquad \text{...(ii)}$$

It is clear from equation (ii) that the motion of the particle is simple harmonic and particle oscillates between B and B' and its periodic time is

$$2\pi\sqrt{\left[\left(\frac{am}{4\lambda}\right)\right]}=\pi\sqrt{\left[\left(\frac{am}{\lambda}\right)\right]}.$$

Now, from (ii) $$v\frac{dv}{dx}=-\frac{4\lambda}{am}x$$

Integrating, $$\frac{1}{2}v^2=-\frac{2\lambda}{am}x^2+C,$$

where C is the constant of integration. At B, $x=a/2$, $v=0$; therefore $C=\frac{4a}{m}$. Hence

$$v^2=\frac{4\lambda}{am}\left(\frac{a^2}{4}-x^2\right) \quad ...(iii)$$

From (iii) it is clear that the velocity is maximum at $x=0$ and then its maximum value will be

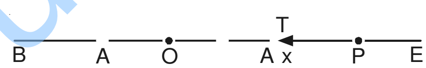
$$v_{\max}=\sqrt{\left[\left(\frac{\lambda a}{m}\right)\right]}.$$

2. *One end of elastic string whose modulus of elasticity is λ and whose natural length is a, is tied to a fixed point on a smooth horizontal table, and the other end is tied to a mass m lying on the table. The particle is pulled to a distance where the extension of the string becomes b and then let go. Describe the character of motion and show that the period of complete oscillation is*

$$2\left\{\pi+\frac{2a}{b}\right\}\sqrt{\left[\left(\frac{ma}{\lambda}\right)\right]}.$$

Sol. Let O be the fixed point on the table, to which one end of the string is tied. $OA=a$ is the natural length of the string. A particle of mass m is tied to the string at A. Let the particle is now pulled to B where $AB=b$ and then released.

Let P is the position of the particle at any time t, where $AP=x$. When the particle is at P, the extended length of the string $=OP=a+x$.



$\therefore$ Tension in the string

$$AO=\lambda\frac{(a+x)-a}{a}=\frac{\lambda x}{a}.$$

The only force acting on the particle at P is the tension. Hence the equation of motion will be

$$m\frac{d^2x}{dt^2}=-\frac{\lambda x}{a}$$

Negative sign indicates that the tension is in the direction of x decreasing.

Hence $$\frac{d^2x}{dt^2}=-\frac{\lambda}{am}x \quad ...(i)$$

This represent S.H.M. whose time period

$$= 2\pi \sqrt{\left[\left(\frac{ma}{\lambda}\right)\right]} \quad \text{...(ii)}$$

Again from (i), we get

$$v\frac{dv}{dx} = -\frac{\lambda}{am}x, \text{ as } \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

Integrating, we get

$$v^2 = -\frac{\lambda}{ma}x^2 + C,$$

where C is the constant of integration. At B, $x = b, v = 0$

$$\therefore \quad 0 = -\frac{\lambda}{ma}b^2 + C \quad \text{or} \quad C = \frac{\lambda}{ma}b^2$$

Substituting the value of C, we get

$$v^2 = \frac{\lambda}{ma}(b^2 - x^2) \quad \text{...(iii)}$$

When the particle reaches A, then $x = 0$. Then its velocity,

$$v = \sqrt{\left[\left(\frac{\lambda}{ma}\right)\right]}\, b$$

At A the tension on the particle will be zero, which means that the particle is not influenced by any force. So it moves with velocity $\sqrt{\left[\left(\frac{\lambda}{ma}\right)\right]}\, b$ and will pass through O; and go to the left of O to the point A'. When it reaches A' the string again becomes stretched and the tension comes into play and the motion once again becomes S.H.M. to B. The particle comes to instantaneous rest at B', moves back to A' under S.H.M. From A to A' the motion is under uniform velocity while from A to B it is simple harmonic.

Hence, time of one complete oscillation

$$= 4\,[\text{time form } B \text{ to } O] = 4\,[\text{time from } B \text{ to } A + \text{time from } A \text{ to } O]$$

$$= 4\left[\frac{1}{4} \times \text{periodic time} + \frac{\text{distance } AO}{\text{uniform velocity}}\right]$$

$$= 4 \times \left[\frac{1}{4} \times 2\pi\sqrt{\left[\left(\frac{ma}{\lambda}\right)\right]} + \frac{a}{\sqrt{\left[\left(\frac{\lambda}{ma}\right)\right]}\, b}\right]$$

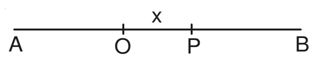
$$= 4\left[\frac{\pi}{2}\sqrt{\left(\frac{ma}{\lambda}\right)} + \frac{a}{b}\sqrt{\left(\frac{ma}{\lambda}\right)}\right]$$

$$= 2\left[\left(\pi + \frac{2a}{b}\right)\sqrt{\left(\frac{ma}{\lambda}\right)}\right].$$

3. *A particle is performing S.H.M. in the line joining two points A and B on a smooth plane and is connected with these points by elastic strings of natural lengths a and a' the moduli of elasticity being λ and λ' respectively. Show that the periodic time is*

$$2\pi\sqrt{\left[\left(\frac{m}{(\lambda/a)+(\lambda'/a')}\right)\right]}$$

Sol. Let OA and OB be the elastic strings whose natural lengths are a and a' respectively. Let O be the position of equilibrium of mass m, λ and λ' being the moduli of elasticity of the strings OA and OB respectively.



Then in the position of equilibrium tension in the string $OA =$ tension in the string OB

i.e., $$\frac{\lambda}{a}l = \frac{\lambda'}{a'}l' \qquad \text{...(i)}$$

where l and l' are the extensions in OA and OB.

Let at time t, P be the displaced position of the particle of mass m, such that $OP = x$. Then tension in the string $PB = \left(\frac{\lambda'}{a'}\right)(l'-x)$ acting in the direction PB and tension in the string $PA = (\lambda/a)(l+x)$, acting in the direction PA. Hence the equation of motion of the particle will be

$$m\frac{d^2x}{dt^2} = \left(\frac{\lambda'}{a'}\right)(l'-x) - \left(\frac{\lambda}{a}\right)(l+x) = \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'} + \frac{\lambda}{a}\right)x$$

or $$\frac{d^2x}{dt^2} = -\left\{\frac{\left(\frac{\lambda}{a}+\frac{\lambda'}{a'}\right)}{m}\right\}x, \qquad \text{from (i)}$$

This is the standard form of S.H.M. Hence the required time period

$$= \frac{2\pi}{\sqrt{\left[\frac{\left(\frac{\lambda}{a}+\frac{\lambda'}{a'}\right)}{m}\right]}} = 2\pi\sqrt{\left\{\frac{m}{\left(\frac{\lambda}{a}+\frac{\lambda'}{a'}\right)}\right\}}$$

EXERCISES

1. A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. Show that the time of an oscillation is $2\pi\sqrt{\left[\left(\frac{mll'}{T(l+l')}\right)\right]}$, where l, l' are the extensions of the string beyond their natural lengths.

2. An elastic string of natural length $(a+b)$ where $a>b$ modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to and fro through a distance $b(\sqrt{a}+\sqrt{b})/\sqrt{a}$ in a periodic time $\pi(\sqrt{a}+\sqrt{b})\sqrt{[(m/\lambda)]}$.

3. Two light elastic strings are fastened to a particle of mass m and their other ends to fixed points so that the strings are taut. The modulus of each is λ, the tension T and lengths a and b. Show that the period of one oscillation along the line of the string is

$$2\pi\sqrt{\left[\left(\frac{mab}{(T+b)(a+b)}\right)\right]}$$

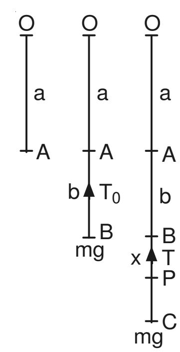
4. Prove that the work done against the tension in stretching a light elastic string is equal to the product of its extension and the mean of the initial and final tension.

5. Two particles of masses m_1 and m_2 are tied to the end of an elastic string of natural length a and modulus λ. They are placed on a smooth table so that the string is just taut and m_2 is projected with any velocity directly away from m_1. Prove that the string will become slack after the lapse of time

$$\pi\sqrt{\left(\frac{am_1m_2}{\lambda(m_1+m_2)}\right)}$$

6. Two masses m_1 and m_2 are connected by a spring of such a strength that when m_1 is held fixed m_2 performs n complete vibrations per second. Show that if m_2 be held fixed, m_1 will make $n\sqrt{\left(\frac{m_2}{m_1}\right)}$, and if both be free, they will make $n\sqrt{\{(m_1+m_2)/m_1\}}$ vibrations per second, the vibrations in each case being in the line of spring.

3.3 VERTICAL ELASTIC STRING



A light elastic string of natural length a and modulus of elasticity λ is suspended by one end, to the other end is tied a particle of mass m, the particle is slightly pulled down and released. Discuss the motion.

Let O be the fixed point at which upper end of the string is tied. Let $OA = a$ be the natural length of the string. When weight mg is tied at the lower end A then let b the extension in the string in equilibrium, where $AB = b$. Let T_0 be the tension in this case. Then

$$T_0 = mg = \lambda\frac{b}{a} \qquad \text{...(i)}$$

Now, the particle is pulled vertically down wards to a point $C\,(BC = c)$ and then released. Since the tension in the string when particle is at C is greater than the weight of the particle; hence it will move upwards.

Let $P\,(BP = x)$ be the position of the particle after time t and then let T be the tension in the string, then by Hooke's Law,

$$T = \lambda\,.\frac{b+x}{a} = mg + \lambda\frac{x}{a} \qquad \text{...(ii)}$$

[from (i)]

and equation of motion will be

$$m\frac{d^2x}{dt^2} = mg - T$$

$$= mg - mg - \lambda\frac{x}{a} \qquad \text{[from (ii)]}$$

or
$$\frac{d^2x}{dt^2} = -\frac{\lambda}{am}x$$

or
$$= -\frac{g}{b}x, \qquad \text{[from (i)]} \qquad \text{...(iii)}$$

$$v\frac{dv}{dx} = -\frac{g}{b}x$$

Integrating, we get

$$\frac{v^2}{2} = -\frac{g}{2}\frac{x^2}{2} + A,$$

where A is constant of integration.
At C, $x = c, v = 0$, hence

$$A = \frac{g}{b}\frac{c^2}{2}$$

$$\therefore \qquad v^2 = \frac{g}{b}(c^2 - x^2) \qquad \text{...(iv)}$$

The motion given by (iii) is simple harmonic, having B, the position of equilibrium as the centre of oscillation. Now, let us consider the following two cases :

Case I. If $c \le b$, the motion given by (iii) is purely simple harmonic, centre of oscillation B, with amplitude c and period

$$2\pi\sqrt{[(b/g)]}.$$

Case II. If $c > b$ but less than $\sqrt{[(b^2 + 4ab)]}$, the particle in its upwards motion goes above A, but at A the string becomes slack and tension becomes zero. Hence S.H.M. ceases and the particle rises against gravity till the velocity becomes zero.

From (iv) the velocity at A is $u = \sqrt{\left[\left\{\frac{g}{b}(c^2 - b^2)\right\}\right]}$ in the upward direction and the particle rises up to the height $\frac{u^2}{2g} = \frac{c^2 - b^2}{2b}$ above O.

This will be true if the height risen above O is not greater than a, the natural length of the string, otherwise the motion again becomes S.H.M. Condition that motion above O may not be S.H.M. is

$$\frac{c^2 - b^2}{2b} < 2a, \text{ i.e., } c < \sqrt{[(b^2 + 4ab)]}$$

Remarks : (i) In the case of a spring the law of compression is same as law of extension. Thus the tension operates even when particle rises above A. Hence in this case equation (ii) holds good through out the motion. The period of motion will be $2\pi\sqrt{[(b/g)]}$.

(ii) While solving problems on vertical elastic strings the position of equilibrium must be obtained first.

EXAMPLES

1. *An elastic string without weight, of which the unstretched length is l and modulus of elasticity is the weight of n oz, is suspended by one end and a mass of m oz, is attached to the other end. Show that the time of a small vertical oscillation is*

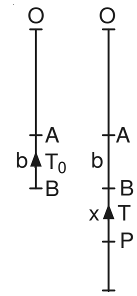
$$2\pi\sqrt{\left[\left(\frac{ml}{ng}\right)\right]}.$$

Sol. Let O be the fixed point, l natural length and b and extension when m *oz* mass is attached to the string. The string is then pulled down to a small distance and then released. Let when weight is at P at time t, then extension in the string is $b + x$. Then we have

$$T_0 = mg = \lambda\frac{b}{l}$$

or $$mg = ng\frac{b}{l} \quad \text{(since } \lambda = ng) \qquad \text{...(i)}$$

Equation of motion will be



$$m\frac{d^2x}{dt^2} = mg - T \qquad \text{...(ii)}$$

But $$T = ng\,.\frac{b+x}{l} = mg + \frac{ng}{l}x, \quad \text{[from (i)]}$$

Hence from (ii)

$$m\frac{d^2x}{dt^2} = mg - mg - \frac{ng}{l}x$$

or $$\frac{d^2x}{dt^2} = -\frac{ng}{ml}x.$$

Hence the motion is S.H.M. and the period of motion

$$= \frac{2\pi}{\sqrt{[(ng/ml)]}} = 2\pi\sqrt{\left[\left(\frac{ml}{ng}\right)\right]}.$$

2. *A light elastic string of natural length l has one end fixed at a point A; and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1. Show that if the stone be dropped from rest at A, it will come to an instantaneous rest at a depth $\sqrt{[(l_1^2 - l^2)]}$ below the equilibrium position.*

Sol. $AB = l$ is the natural length of the string. Let m be the mass of the stone and C its position of equilibrium. According to given condition

$$AC = l_1 \quad \text{or} \quad BC = (l_1 - l)$$

Then at C, $$mg = \left(\frac{\lambda}{l}\right)(l_1 - l) \qquad \text{...(i)}$$

where λ is the modulus of elasticity.

Let the stone be dropped from rest at A, the motion from A to B will be due to gravity only as there will be no tension. If v be the velocity of the particle when it reaches B, then

$$v = \sqrt{[(2gl)]} \qquad \text{...(ii)}$$

At B, the string becomes taut and for the motion below B, the tension of the string comes into play. Let P be any displaced position of the stone. Let $CP = x.$ Then at P the forces acting on the stone being (i) its weight mg acting vertically downwards and (ii) tension $\left(\frac{\lambda}{l}\right)(l_1 - l + x)$ in the string acting vertically upwards. Hence the equation of motion will be

$$m\left(\frac{d^2x}{dt^2}\right) = mg - \frac{\lambda}{l}(l_1 - l + x)$$

$$= mg - mg - \left(\frac{\lambda}{l}\right)x, \qquad \text{from (i)}$$

or $$\frac{d^2x}{dt^2} = -\frac{\lambda}{ml}.x = -\frac{gx}{(l_1 - x)},$$ from (i)

or $$v\frac{dv}{dx} = -\frac{gx}{(l_1 - l)}$$

Integrating,

$$\frac{v^2}{2} = -\frac{g}{(l_1 - l)}\frac{x^2}{2} + A.$$

At $B, v = \sqrt{[(2gl)]}$ and $x = -(l_1 - l)$

$\therefore$ $$2gl = -\frac{g}{(l_1 - l)}(l_1 - l)^2 + A$$

or $$A = 2gl + g(l_1 - l)$$

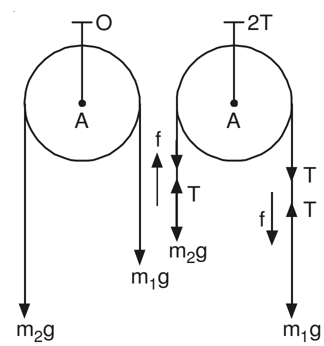
$\therefore$ $$v^2 = 2gl + g(l_1 - l) - \left(\frac{g}{l_2 - l}\right)x^2$$ ...(iii)

If the particle comes to instantaneous rest at D, such that $CD = d$, then from (iii), we get

$$0 = 2gl + g(l_1 - l) - \frac{gd^2}{(l_1 - l)}$$

or $$d^2 = (l_1^2 - l^2) \quad \text{or} \quad d = \sqrt{[(l_1^2 - l^2)]}.$$

3. *A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity ng. If masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point of suspension is $l(1 + 2M/n)$, where M is the harmonic mean between m_1 and m_2.*



Sol. Let O be the fixed point which the pulley is suspended. Let λ be the modulus of elasticity, then

$$\lambda = ng$$ ...(i)

Let us consider the motion of masses m_1 and m_2. Let f be the acceleration of the system and T the tension in the string round the pulley. Then the equation of motion will be

$$m_1 g - T = m_1 f$$

and $$T - m_2 g = m_2 f$$

which give

$$T = \frac{2m_1m_2}{(m_1 + m_2)}g = Mg, \text{ where}$$

$$M = \frac{2m_1m_2}{m_1 + m_2}.$$

Hence pressure on the pulley due to the masses m_1 and $m_2 = 2T = 2Mg$.

Now, the problem reduces to the motion of a mass $2M$ hanging from one end of an elastic spring of natural length l whose other end is fixe at O.

Let B be its position of equilibrium such that $AB = d$ (say),

Then at B,

weight of the mass $2M$ = Tension in the spring

i.e., $$2Mg = (\lambda / l)\, d = (ng / l)\, d \text{ from (i)}$$

or $$d = \frac{2lM}{n}$$

In case of vertical spring whose one end is fixed we know that the motion is simple harmonic about the posiion of equilibrium of the mass attached to the other end.

Hence required depth $= OB = OA + AB$

$$= l + d = l + \frac{2lM}{n} = l\left[1 + \left(\frac{2M}{n}\right)\right].$$

EXERCISES

1. A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that

$$g\,(t_1^2 - t_2^2) = 4\pi^2\,(c_1 - c_2).$$

2. A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\left(\frac{a}{g}\right)}\left(\frac{4}{3}\pi + 2\sqrt{3}\right)$ where a is the natural length of the string.

3. A heavy particle attached to a fixed point by an elastic string hangs freely stretching the string by a quantity e. It is drawn by an additional distance f and then let go. Determine the height to which it will rise if $f^2 - e^2 = 4ae$; e being unstretched length of the string.

[**Ans.** $f + e + 2a$]

4. A mass m hangs from a fixed point by a light spring and is given small vertical displacement. Show that the motion is simple harmonic. If l is length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is $l - (g/4\pi^2 n^2)$.

5. A light elastic string whose natural length is a has one end fixed to a point O and to the other end is attached a wieght which in equilibrium would produce an extension e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae + e^2)}$ below the position of equilibrium.

6. A heavy particle of mass m is attached to one end of an elastic string of natural length l whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic, and that if the greatest depth of the particle below O is $l\cot^2(\theta/2)$, the modulus of elasticity of the string is $\frac{1}{2}mg\tan^2\theta$ and that the particle attains this depth in time $\sqrt{(2l/g)}\,[1 + (\pi - \theta)\cot\theta]$, where θ is a positive acute angle.

7. One and of a light elastic string of natural length a and modulus $2mg$ is attached to a point O and the other end to a particle of mass m. The particle initially held at rest at O, is allowed to fall. Find the greatest extension of the string and show that the particle will reach O again after a time t equal to

$$(\pi + 2 - \tan^{-1} 2)\sqrt{\left(\frac{2a}{g}\right)}$$

8. A heavy particle of mass m is attached to one end of an elastic string of natural length l feet, whose modulus of elasticity is equal to the weight of the particle and the other end is fixed at O, the particle is let fall from rest at O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2 + \sqrt{3})\,l$ feet. show that this depth is attained in time

$$\sqrt{(l/g)}\,\{\sqrt{2}+\pi-\cos^{-1}(1/\sqrt{3})\}\text{ seconds}.$$

9. A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity 2 mg, whose other end is fixed at O. The particle is let fall from A, when A is vertically above O and $OA = a$. Show that its velocity will be zero at B, where $OB = 3a$; calculate also time from A to B.

$$\left[\textbf{Ans. } \frac{1}{2}\sqrt{\left(\frac{a}{2g}\right)}\left\{4\sqrt{2}+\pi+2\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

10. A heavy particle is attached to an inextensible string to a fixed point from which the particle is allowed to fall freely. When the particle is in its lowest position the string is of twice its natural length. Prove that the modulus is four times the weight of the particle and find the time during which the string is extended beyond its natural length.

$$\left[\textbf{Ans. } \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\left\{\pi-2\cos^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

11. The bodies of mases M and M' are attached to the lower end of an elastic string whose upper end is fiexd and hang at rest, M' falls off; show that distance of M from the upper end of string at time t is $a+b+c\cos\{\sqrt{(g/b)}\,t\}$, where a is the unstretched length of the string, b and c the distances by which it would be stretched when supporting M and M' respectively.

12. A heavy particle is attached to one end of a fine elastic string, the other end of which is fixed. The unstretched length of the string is a and its modulus of elasticity is n times the weight of the particle, is pulled vertically downwards till the length of the string is a' and is then let go from rest. Show that the time it returns to this position is

$$2(\pi-\theta+\theta'+\tan\theta-\tan\theta')\sqrt{\left(\frac{a}{ng}\right)}$$

where θ and θ' are positive acute angles given by

$$\sec\theta=\frac{na'}{a}-n-1,\ \sec^2\theta'=\sec^2\theta-4n.$$

13. A heavy particle is attached to one point of a uniform light elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is $2\pi\sqrt{(mh/2\lambda)}$, where λ is the coefficient of elasticity of the string and h the harmonic mean of the unstretched lengths of the two parts of the string.

14. Two particles of masses M and $2M$ are connected by an inextensible string passing over a smooth peg. From the particle M another equal particle hangs by an elastic string of natrual length a and modulus Mg. The system is initially supported with the string vertical, the first being taut and the second at its natural length and then released. Show that the motion is S.H.M. with period $\pi\sqrt{(3a/g)}$ and the extension of the second string at time t is

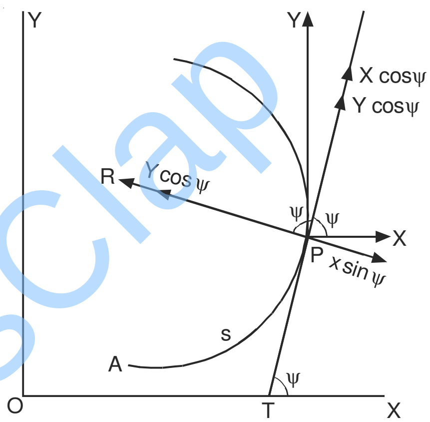
$$a\,[1-\cos\{2t\sqrt{(g/3a)}\}]$$

❑❑❑

Motion on Smooth and Rough Plane

4.1 MOTION ON A SMOOTH PLANE CURVE

A particle is compelled to move on a smooth plane curve under the action of given forces in the plane, to find the motion.



Let P be the position of a particle of mass m at time t and let the arc AP be s, where A is some fixed point on the curve. Let the components of the force acting at P be X and Y parallel to the axes OX and OY respectively. Let R be the reaction at P. Let the tangent at P makes an angle ψ with the x-axis.

In the problems of this type one should use tangential and normal accelerations. The impressed forces must be resolved along the tangent and normal and equated to the effective forces in those directions. Hence, the effective forces are $m\frac{d^2s}{dt^2}$ and $m\frac{v^2}{\rho}$. The impressed forces are X, Y and normal reaction R.

The equations of motion of the particle along the tangent and normal to the curve at P are

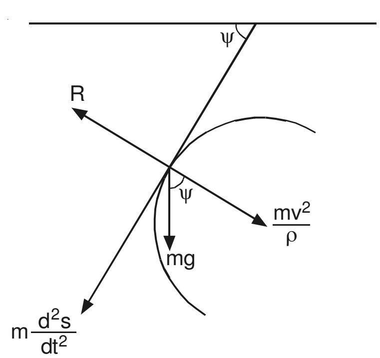
$$m\frac{d^2s}{dt^2} = X\cos\psi + Y\sin\psi \qquad ...(i)$$

and
$$m\frac{v^2}{\rho} = R - X\sin\psi - Y\cos\psi \qquad ...(ii)$$

where ρ is the radius of curvature of the curve at P. These two equations determine the motion of the particle.

4.2 MOTION ON A SMOOTH PLANE CURVE UNDER GRAVITY

To determine the motion of a particle on a smooth vertical plane curve under gravity.



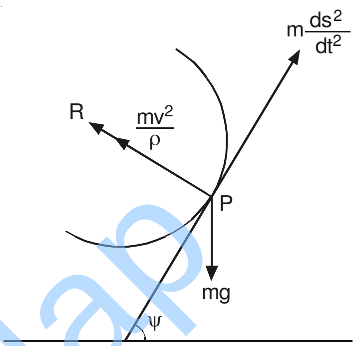
(i) When the particle is moving outside the restraining curve, so that the normal reaction offered by the curve is away from the centre of curvature of the curve. Let ψ be the angle which the tangent to the curve at any point makes with the horizontal line. The impressed forces are the weight of the particle and the reaction of the curve and they must be equal to the effective forces $m\frac{d^2s}{dt^2}$ along the tangent and $m\frac{v^2}{\rho}$ along the normal, if s is measured from the highest point of the curve. Hence, we have, the tangential effective force

$$m\frac{d^2s}{dt^2} = mg\sin\psi \quad \text{...(i)}$$

The normal effective force

$$\frac{mv^2}{\rho} = mg\cos\psi - R \quad \text{...(ii)}$$

(ii) When the particle is moving inside the curve, so that R is in the same direction $\frac{mv^2}{\rho}$. If s is measured from the lowest point on curve in this case, then we have



$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{...(iii)}$$

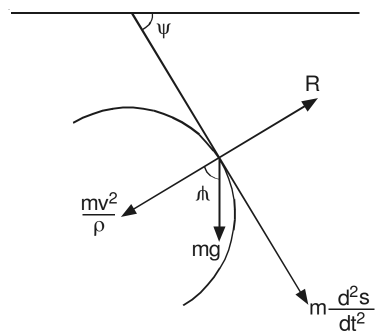
and $$\frac{mv^2}{\rho} = R - mg\cos\psi \quad \text{...(iv)}$$

Equations (i) and (ii) or (iii) and (iv) are quite sufficient to determine the motion, as the case may be.

EXAMPLES

1. *A particle slides on the curve* $x = 2\sqrt{\{a(y-a)\}}$ *with a velocity due to a fall from the horizontal x-axis, the y-axis being vertically downwards. Find the pressure on the curve at any time and time of sliding from* $y = b$ *to* $y = c$.

Solution. The equations of motion will be



$$m\frac{d^2s}{dt^2} = mg\sin\psi \quad \text{...(i)}$$

and $$\frac{mv^2}{\rho} = mg\cos\psi - R \quad \text{...(ii)}$$

Since the velocity of the particle is due to the fall from the horizontal x-axis, we have

$$v = \sqrt{(2gy)}$$

To find the pressure R from (ii) let us find out ρ and ψ. Now from the equation of the curve; so have

$$x^2 = 4a(y-a)$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{2a} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{2a}$$

Now, $$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2} = \frac{\left(1+\frac{x^2}{4a^2}\right)^{3/2}}{1/2a} = \frac{(4ay)^{3/2}}{4a^2} = \frac{2y^{3/2}}{a^{1/2}}$$

Also, $$\frac{dy}{dx} = \tan\psi = \frac{x}{2a}$$

$$\therefore \quad \cos\psi = \sqrt{\left(\frac{a}{y}\right)}$$

Thus, from (ii), we have

$$R = mg\cos\psi - \frac{mv^2}{\rho} = mg\sqrt{\left(\frac{a}{y}\right)} - \frac{m.2gy.a^{1/2}}{2y^{3/2}}$$

$$= mg\sqrt{\left(\frac{a}{y}\right)} - mg\sqrt{\left(\frac{a}{y}\right)} = 0$$

Hence, the pressure at any point is zero.
To find the time of sliding from $y = b$ to $y = c$, let us find a relation between y and t.

Now,
$$\frac{dy}{dt} = \frac{dy}{ds}\cdot\frac{ds}{dt} = \sqrt{(2gy)}\,\frac{dy}{ds}$$

$$= \sqrt{(2gy)}\sin\psi = \sqrt{(2gy)}\sqrt{(1-\cos^2\psi)}$$

$$= \sqrt{(2gy)}\sqrt{\left(1-\frac{a}{y}\right)} = \sqrt{\{2g(y-a)\}}$$

or
$$\sqrt{(2g)}\,dt = \frac{dy}{\sqrt{(y-a)}}$$

To get the required time, let us integrate between $y = b$ to $y = c$. This gives

$$\int_0^T \sqrt{(2g)}\,dt = \int_b^c \frac{dy}{\sqrt{(y-a)}}$$

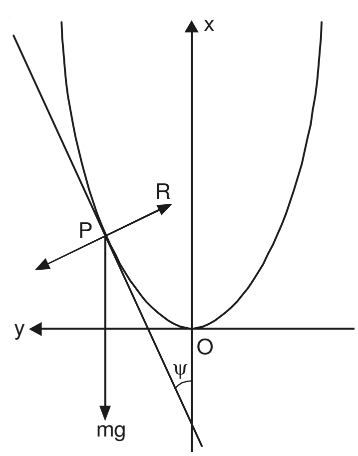
or
$$\sqrt{(2g)}\,T = 2\left[\sqrt{y-a}\right]_b^c \quad \text{or} \quad T = \sqrt{\left(\frac{2}{g}\right)}[\sqrt{(c-a)} - \sqrt{(b-c)}]$$

2. *A wire, in the form of the parabola* $y^2 = 4ax$, *is fixed with its axis vertical and vertex downwards. If a small smooth bead of mass m can slides on the wire and is released from rest at one end of the latus rectum, find its acceleration along the tangent when it is at a point x above the vertex and show that the pressure on the wire is* $2mg\left(\frac{a}{a+x}\right)^{3/2}$.

Solution. The equation of the parabola is

$$y^2 = 4ax \qquad \text{...(i)}$$

$$\therefore \quad \tan\psi = \frac{dy}{dx} = \frac{2a}{y} = \sqrt{\left(\frac{a}{x}\right)}$$



Hence,
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

or
$$\rho = \frac{(4ay+4a^2)^{3/2}}{4a^2}$$

Applying the equation of energy, the velocity at $P(x, y)$ is given by

$$\frac{1}{2}mv^2 = mg(a-x)$$

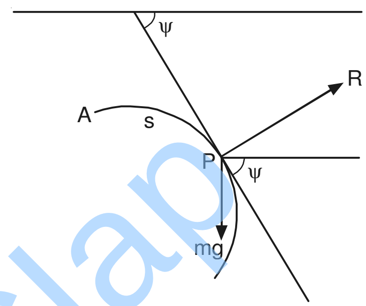
Whence $v^2 = 2g(a-x)$, since $m \neq 0$, we also have

$$\frac{mv^2}{\rho} = R - mg\sin\psi$$

Hence, $R = \frac{mv^2}{\rho} + mg\sin\psi = mg\left\{\frac{\sqrt{a}\,(a-x)}{(x+a)^{3/2}} + \frac{\sqrt{a}}{(x+a)^{3/2}}\right\} = 2mg\left(\frac{a}{a+x}\right)^{3/2}$

Also, $\frac{d^2s}{dt^2} = -g\sin\psi = -g\left(\frac{a}{a+x}\right)^{1/2}$

3. *A particle slides down the smooth curve $y = a\sinh\frac{x}{a}$, the axis of x being horizontal and aixs of y downwards, starting from rest at the point where the tangent is inclined at an angle α to the horizontal. Show that it will leave the curve when it has fallen through a vertical distance $a\sec\alpha$.*



Solution. The equation of motion are

$$mg\frac{dv}{ds} = mg\sin\psi \quad \text{...(i)}$$

and $$\frac{mv^2}{\rho} = mg\cos\psi - R \quad \text{...(ii)}$$

From (i), $v\frac{dv}{ds} = g\cdot\frac{dy}{ds}$ $\left(\because \frac{dy}{ds} = \sin\psi\right)$

$\therefore$ $v\,dv = g\,dy$

Integrating, $v^2 = 2gy + A$,

where A is the constant of integration.

Initially, when $y = y_0$, say $v = 0$,

$$\therefore \quad A = -2gy_0$$

$$\therefore \quad v^2 = 2g(y - y_0) \quad \text{...(iii)}$$

Putting for v^2 in equation (ii), we get

$$\frac{2mg(y-y_0)}{\rho} = mg\cos\psi - R$$

when particle leaves the curve, then $R = 0$.

$$\therefore \quad m.2g(y-y_0) = mg\cos\psi\,\rho \quad \text{...(iv)}$$

Now, $$y = a\sin h\frac{x}{a}$$

$$\therefore \quad \frac{dy}{dx} = \cos h\frac{x}{a} = \tan\psi$$

or $$\tan\psi = \sqrt{\left(1 + \sinh^2\frac{x}{a}\right)} = \sqrt{\left(1 + \frac{y^2}{x^2}\right)}$$

Initially, when $\psi = \alpha, y = y_0$

$$\therefore \quad \tan\alpha = \sqrt{\left(1 + \frac{y_0^2}{a^2}\right)} \quad \text{...(v)}$$

Now,
$$\rho \cos \psi = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \cdot \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$
$$= \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}} = a^2 \frac{\left(1+\cos h^2 \frac{x}{a}\right)}{\sin h \frac{x}{a}}$$

or
$$\rho \cos \psi = a \frac{1+1+\sin h^2 \frac{x}{a}}{\sin h \frac{x}{a}} = a^2 \frac{\left(2+\frac{y^2}{a^2}\right)}{y} = \frac{2a^2+y^2}{y}$$

Hence, from (iv), we get
$$2\,(y-y_0) = \frac{2a^2+y^2}{y} \quad \text{or} \quad 2y^2 - 2yy_0 = 2a^2 + y^2$$

or
$$y^2 - 2yy_0 + y_0^2 = 2a^2 + y_0^2$$

or
$$(y-y_0)^2 = 2a^2 + a^2(\tan^2\alpha - 1) = a^2(1+\tan^2\alpha) \qquad \text{by (v)}$$

or
$$(y-y_0)^2 = a^2 \sec^2 \alpha$$

∴
$$y - y_0 = a \sec \alpha$$

Hence, the particle wil leave when it has fallen through a vettical distance $a \sec \alpha$.

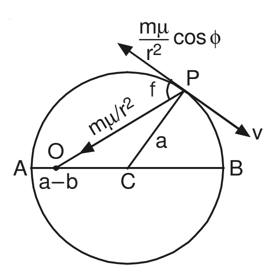
4. *A small bead, of mass m, moves on a smooth circular wire, being acted upon by a central attraction* $\frac{m\mu}{(distance)^2}$ *to a point within the circle situated at a distance b from the centre. Show that in order that the bead may more completely round the circle, its velocity of projection at the point of the wire nearest the centre of force must not be less than* $\sqrt{\left\{\frac{4\mu b}{(a^2-b^2)}\right\}}$.

Solution. Let O be the centre of attraction. Then $OA = a - b$, $OB = a + b$. Let the veloicity at A be V and that at P be v.

The Central attraction is $\frac{m\mu}{r^2}$.

The equation of motion will be
$$mv\frac{dv}{ds} = -\frac{m\mu}{r^2}\cos\phi = -\frac{m\mu}{r^2}\cdot\frac{dr}{ds} \qquad \left[\because \cos\phi = \frac{dr}{ds}\right]$$

or
$$vdv = -\frac{\mu}{r^2}\,dr$$



Integrating, we get
$$\int v\,dv = \int -\frac{\mu}{r^2}\,dr$$

or
$$\frac{1}{2}v^2 = \frac{\mu}{r} + A \qquad \text{...(i)}$$

when the particle is at A, where $r = a - b$, the velocity is V.

∴
$$\frac{1}{2}V^2 = \frac{\mu}{a-b} + A \qquad \text{...(ii)}$$

Subtracting (i) and (ii), we get

$$\frac{1}{2}v^2 - \frac{1}{2}V^2 = \frac{\mu}{r} - \frac{\mu}{a-b} \qquad \text{...(iii)}$$

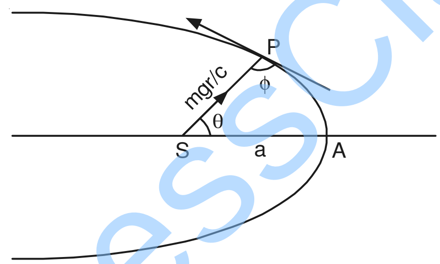
In order that the particle may move completely round the circle its velocity v should not vanish till the particle has reached the other end of the diameter, *i.e.*, B where $r = a + b$. Hence, the least velocity of projection will be obtained by putting $v = 0$ and $r = a + b$ in (iii).

$$\therefore \quad 0 - \frac{1}{2}V^2 = \frac{\mu}{a+b} - \frac{\mu}{a-b} = -\frac{2\mu b}{a^2 - b^2}$$

$$\therefore \quad V^2 = \frac{4\mu b}{a^2 - b^2} \quad \text{or} \quad V = \sqrt{\left(\frac{4\mu b}{a^2 - b^2}\right)}$$

5. *A particle is projected from the vertex of a smooth parabolic tube of latus rectum 4a along the tube and is acted upon by a repulsive force* $\frac{mgr}{c}$ *from the focus. If the velocity of projection is that which would be acquired in moving from focus to the vertex, prove that the time of describing angle* θ *about the focus is* $2\sqrt{\left(\frac{c}{g}\right)} \log \tan\left(\frac{\pi + \theta}{4}\right)$.

Solution. The polar equation of a parabola of latus rectum $4a$ referred to focus as pole is



$$\frac{2a}{r} = 1 + \cos\theta$$

or
$$r = \frac{2a}{2\cos^2\frac{\theta}{2}} = a\sec^2\frac{\theta}{2}$$

Let V be the velocity from focus to vertex, *i.e.*, from $r = 0$ to $r = a$ under the given force. Then

$$\left[\frac{v^2}{2}\right]_0^V = \frac{g}{c}\left[\frac{r^2}{2}\right]_0^r$$

or
$$V^2 = \frac{g}{c}a^2 \qquad \text{...(i)}$$

This will be the velocity of projection.

Tangential equation of motion will be

$$mv\frac{dv}{ds} = mg\frac{r}{c}\cos\phi = mg\frac{r}{c}\frac{dr}{ds} \quad \text{or} \quad v\,dv = \frac{g}{c}r\,dr$$

Integrating, we get

$$\left[\frac{v^2}{2}\right]_V^v = \frac{g}{c}\left[\frac{r^2}{2}\right]_0^r$$

$\therefore \qquad \dfrac{v^2}{2} - \dfrac{V^2}{2} = \dfrac{g}{c}\left(\dfrac{r^2}{2} - \dfrac{a^2}{2}\right) = \dfrac{gr^2}{2c} - \dfrac{V^2}{2}$ by (i)

$\therefore \qquad v^2 = \dfrac{g}{c} r^2 \quad \text{or} \quad v = \dfrac{ds}{dt} = \sqrt{\left(\dfrac{g}{c}\right)^2}$

or $\qquad \dfrac{ds}{d\theta} \cdot \dfrac{d\theta}{dt} = \sqrt{\left(\dfrac{g}{c}\right)} a \sec^2 \dfrac{\theta}{2}$ ...(ii)

Now $r = a\sec^2 \dfrac{\theta}{2}$; $\therefore \quad \dfrac{dr}{d\theta} = 2a \sec\dfrac{\theta}{2} \sec\dfrac{\theta}{2} \tan\dfrac{\theta}{2}\dfrac{1}{2}$

$\therefore \qquad \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = a\sec^2\dfrac{\theta}{2}\sqrt{\left(1+\tan^2\dfrac{\theta}{2}\right)}$

$= a \sec^3 \dfrac{\theta}{2}$

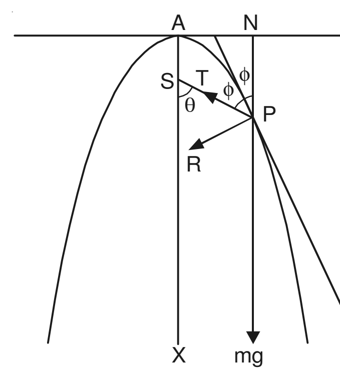
Hence, from (ii), we get

$a\sec^3\dfrac{\theta}{2} \cdot \dfrac{d\theta}{dt} = \sqrt{\left(\dfrac{g}{c}\right)} a \sec^2\dfrac{\theta}{2} \quad \text{or} \quad \sqrt{\left(\dfrac{c}{g}\right)} \int_0^\theta \sec\dfrac{\theta}{2}\, d\theta = \int_0^r dt$

or $\qquad \sqrt{\left(\dfrac{c}{g}\right)}\left[2\log\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{4}\right)\right]_0^\theta = T \quad \text{or} \quad 2\sqrt{\left(\dfrac{c}{g}\right)}\log\tan\dfrac{\pi+\theta}{4} = T$

6. *A heavy ring of mass m is free to move on a smooth fixed parabolic wire of latus rectum 4a whose axis is vertical and vertex upwards and is attached to one end of an elastic string of natural length a and modulus of elasticity 2 mg whose other end is fixed at the focus. The ring is projected from the vertex of the parabola with velocity* $2\sqrt{(ga)}$. *Prove that that when the focal distances is r, its velocity is* $\left\{2g\dfrac{r}{a}(3a-r)\right\}^{1/2}$ *and the pressure on the wire is*

$$\frac{1}{\sqrt{(ar)}}(3r-4a)\, mg$$



Solution. Taking SX as initial line and S on pole, the equation of the parabola will be

$$\frac{2a}{r} = 1 + \cos(\pi - \theta)$$

$$= 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

At A where $\theta = \pi$,

$$\frac{2a}{r} = 2\sin^2\frac{\pi}{2} = 2$$

$\therefore \quad r = a$

In a parabola tangent at any point bisects the angle between the focal radius and perpendicular from the point on the directrix.

Also, $\qquad \angle PSX = \angle SPN$ (alternate angles)

$\therefore \qquad \theta = \phi + \phi = 2\phi \quad \text{or} \quad \phi = \dfrac{\theta}{2}$

or $\qquad \sin\phi = \sin\dfrac{\theta}{2} = \sqrt{\left(\dfrac{a}{r}\right)}$

where ϕ is the angle between tangent and radius vector.
Also, pedal equation of parabola is

$$p^2 = ar \quad i.e., \quad p = \sqrt{a}\,.\sqrt{r}$$

$$\therefore \quad \frac{dp}{dr} = \frac{\sqrt{a}}{2\sqrt{r}}$$

$$\therefore \quad \rho = r\frac{dr}{dr} = r\,.\frac{2\sqrt{r}}{\sqrt{a}} = 2r\sqrt{\left(\frac{r}{a}\right)}$$

Also tension in an elastic string

$$= \frac{\lambda\,(\text{Extension})}{\text{Natural length}} = 2mg\,\frac{(r-a)}{a}$$

The equation of motion will be

$$mv\frac{dv}{ds} = mg\cos\phi - T\cos\phi$$

$$= \left\{mg - 2mg\,\frac{(r-a)}{a}\right\}\frac{dr}{ds} \qquad \text{...(i)}$$

or $$v\,dv = \frac{g}{a}(3a - 2r)\,dr$$

$$\therefore \quad v^2 = \frac{2g}{a}(3ar - r^2) + A$$

At A, $r = a$, then $v^2 = 4ag$, given

$$4ag = \frac{2gt}{a}\,.\,2a^2 + A, \quad \therefore \; A = 0$$

Hence, $$v^2 = \frac{2g}{a}(3ar - r^2) \qquad \text{...(ii)}$$

Again the normal equation of motion will be

$$m\frac{v^2}{\rho} = R - mg\sin\phi + T\sin\phi$$

Putting the values of $v^2\rho$, T and $\sin\phi$, we get

$$R = m\,.\frac{2gr}{a}(3a - r)\,.\frac{1}{2r}\sqrt{\left(\frac{a}{r}\right)} - \left\{mg + 3mg\,\frac{(r-a)}{a}\right\}\sqrt{\left(\frac{a}{r}\right)}$$

$$= \frac{mg}{a}\sqrt{\left(\frac{a}{r}\right)}\,[3a - r - a - 2r + 2a] = \frac{mg}{\sqrt{ar}}(4a - 3r)$$

EXERCISES

1. A particle slides down a catenary, whose plane is vertical and vertex upwards, the velocity at any point being due to the fall from the directrix. Prove that the pressure at any point varies inversely as the distance of that point from the directrix.

2. A small bead is projected with any velocity along a smooth circular wire under the action of a force varying inversely as the fifth power of the distance from a centre of force situated on the circumference. Prove that the pressure on the wire is constant.

3. A particle of mass m moves in a smooth circular tube of radius a under the action of a force equal to $m\mu \times$ distance to a point inside the tube at a distance from its centre. If the particle be placed very nearly at its greatest distance from the centre of force, show that it will describe the quadrant ending at its least distance in time

$$\left(\frac{a}{\mu c}\right)\log(\sqrt{2} + 1)$$

4. A particle is projected horizontally from the lowest point of smooth elliptic arc whose major axis $2a$ is vertical and moves under gravity along the concave side. Prove that it will leave the curve at same point if the velocity of projection lies between $\sqrt{(2ga)}$ and $\sqrt{[ga(5-e^2)]}$; and if the velocity have the later value, prove that the particle will continue to move round the ellipse in time

$$2\sqrt{\left(\frac{a}{g}\right)}\int_0^{n}\left\{\frac{1-e^2\cos^2\phi}{3-e^2+2\cos\phi}\right\}^{1/2} d\phi$$

5. From the lowest point of a smooth hollow cylinder whose crosss-section in one half of the lemniscate $r^2 = a^2\cos 2\theta$ with axis vertical and node downwards, a particle is projected with velocity V, along the inner surface in the plane of the cross-section. Show that it will make a complete revolution if $3V^2 > 7ag$.

6. A smooth parabolic tube is placed, vertex downwards, in a vertical plane. A particle slides down the tube from rest under influence of gravity. Prove that in any position the reaction of the tube is $2w \cdot \frac{h+a}{\rho}$, where w is the weight of the particle, ρ the radius of curvature, $4a$ the latus rectum.

7. A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attached to pole of the spiral by a force $m\mu$ (distance)$^{-2}$ and starts from rest at a distance b from the pole. Show that if the equation of the spiral be $r = ae^{\theta\cos\alpha}$, the time of arriving at the pole is $\frac{\pi}{2}\sqrt{\left(\frac{b^2}{2\mu}\right)}\sec\alpha$.

8. From the lowest point of a smooth hollow cylinder whose cross-section is an ellipse of major axis $2a$ and minor axis $2b$ and whose minor axis is vertical, a particle is projected from the lowest point in a vertical plane perpendicular to the axis of cylinder. Show that it will leave the cylinder if the velocity of projection lies between $\sqrt{(2gb)}$ and $\sqrt{\left(g\cdot\frac{a^2+4b^2}{b}\right)}$.

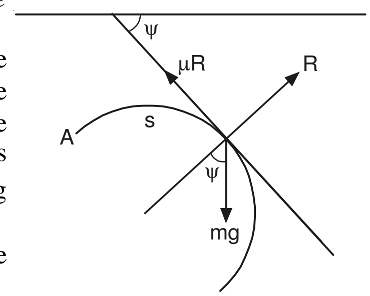
9. A small bead moves on a thin ellipse wire under a force to the focus equal to $\frac{\mu}{r^2}+\frac{\lambda}{r^3}$ and is projected from a point on the wire distant R from the focus with the velocity which would cause it to describe the ellipse freely under a force $\frac{\mu}{r^2}$. Show that the reaction of the wire is $\frac{\lambda}{\rho}\left(\frac{1}{r^2}-\frac{1}{ar}+\frac{1}{R^2}\right)$, where ρ is the radius of curvature.

10. An elastic string of modulus λ is attached at one end to a focus of a smooth wire in the shape of an ellipse of latus rectum $2l$ and major axis $2a$. The other end of the string is attached to a small ring of unit mass which can slide on the wire which is fixed with its plane horizontal. If the ring be slightly displaced from its position of unstable equilibrium at the end of major axis of the ellipse, show that its angular velocity about the focus when the string becomes slack is

$$\sqrt{\left\{\frac{\lambda l}{a^3}(a-l)\right\}}.$$

4.3 MOTION ON A ROUGH CURVE UNDER GRAVITY

A particle slides down a rough curve in a vertical plane under gravity, to discuss the motion.



Let P be the position of the particle at time t. At P, let the tangent is making an angle ψ with any fixed horizontal line and the arcural distance of P measured from a fixed point A be s. Let R be the normal reaction. Since the particle slides downwards, hence the force of friction μR acts upwards along the tangent at P.

Resolving forces along the tangent and normal at P, we get the equations of motions are

$$mv\left(\frac{dv}{ds}\right) = mg \sin \psi - \mu R$$

or $$\frac{1}{2} m\left(\frac{dv^2}{ds}\right) = mg \sin \psi - \mu R$$

and $$m\left(\frac{v^2}{\rho}\right) - mg \cos \psi - R$$

Eliminating R from (i) and (ii) by multiplying (ii) by μ and subtracting from (i), we get

$$\frac{1}{2} m \frac{dv^2}{ds} - \mu m \frac{v^2}{\rho} = mg \sin \psi - \mu\, mg \cos \psi$$

or $$\frac{dv^2}{ds}\rho - 2\mu v^2 = 2g\rho(\sin \psi - \mu \cos \psi)$$

or $$\frac{dv^2}{ds} \cdot \frac{ds}{d\psi} - 2\mu v^2 = 2g\rho\,(\sin \psi - \mu \cos \psi) \qquad \left(\because\ \rho = \frac{ds}{d\psi}\right)$$

or $$\frac{dv^2}{d\psi} - 2\mu v^2 = 2g\rho\,(\sin \psi - \mu \cos \psi)$$

This is a linear differential equation in v^2 whose integrating factor is $e^{-2\mu\psi}$. Hence, its solution will be

$$v^2 . e^{-2\mu\psi} = 2g \int \rho \,.\, 2^{-2\mu\psi} (\sin \psi - \mu \cos \psi)\, d\psi + c,$$

where c is a constant of integration.

When the equation of the curve is given, ρ can be determined in terms of ψ. Hence by substituting the value of ρ in right hand side of (iii) it can be easily integrated. The value of c can be determine by initial conditions.

Thus, from (iii) we can find the value of v^2 in any position and then by substituting for v^2 in (ii) the value of R can be determined.

EXAMPLES

1. *A particle is projected horizontally with velocity V along the inside of a rough vertical circle from the lowest point. Prove that if it completes the circle, it will return to the lowest point with a velocity v given by*

$$v^2 = V^2 e^{-4\pi\mu} + \frac{2ga}{1+4\mu^2}(1 - 2\mu^2)(1 - e^{-4\pi\mu})$$

Solution. Equations of motions will be

$$mv\frac{dv}{ds} = -\mu R - mg \sin \theta \qquad \text{...(i)}$$

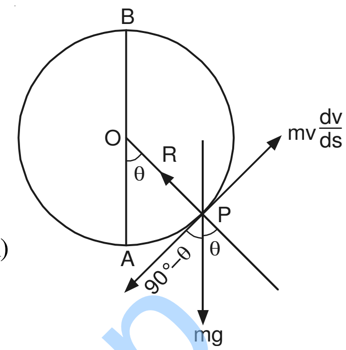
and $$m\frac{v^2}{\rho} = R - mg\cos\theta \qquad \text{...(ii)}$$

For circle, we have $\rho = a$. Putting the values of R from (ii) into (i), we get

$$\frac{m}{2}\cdot\frac{dv^2}{ds} = -\mu m\frac{v^2}{a} - \mu\, mg\cos\theta - mg\sin\theta$$

or $$\frac{m}{2}\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds} = -\mu m\cdot\frac{v^2}{a} - mg(\mu\cos\theta + \sin\theta)$$

Since $s = a\theta$ $\therefore$ $$\frac{d\theta}{ds} = \frac{1}{a}$$



$\therefore$ $$\frac{dv^2}{d\theta} + 2\mu v^2 = -2ga(\mu\cos\theta + \sin\theta) \qquad \text{...(iii)}$$

This is a linear equation. Hence, its I.F. $= e^{\int 2\mu d\theta} = e^{2\mu\theta}$

Hence, the solution of (iii) will be

$$v^2 \cdot e^{2\mu\theta} = -2ga\int(\mu e^{2\mu\theta}\cos\theta + e^{2\mu\theta}\sin\theta)\,d\theta$$

or $$v^2\cdot e^{2\mu\theta} = -2ga\left[\mu\cdot\frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\cos\theta+\sin\theta) + \frac{e^{2\mu\theta}}{1+4\mu^2}(2\mu\sin\theta - \cos\theta)\right] + A,$$

where A is the constant of integration.

or $$v^2\cdot e^{2\mu\theta} = -2ga\cdot\frac{e^{2\mu\theta}}{(1+4\mu^2)}[3\mu\sin\theta - (1-2\mu^2)\cos\theta] + A \qquad \text{...(iv)}$$

Initially, when $\theta = 0, v = V$ (given)

$\therefore$ $$V^2.1 = -\frac{2ga}{1+4\mu^2}[0-(1-2\mu^2)] + A$$

$\therefore$ $$A = V^2 - \frac{2ga}{1+4\mu^2}(1-2\mu^2)$$

By putting the value of A in (iv), we get

$$v^2 e^{2\mu\theta} = -2ga\cdot\frac{e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta - (1-2\mu^2)\cos\theta] + V^2 - \frac{2ga}{(1+4\mu^2)}(1-2\mu^2) \qquad \text{...(v)}$$

When the particle returns to the lowest point, it would have described an angle $\theta = 2\pi$ and let the velocity be v_1. Then from (v), we get

$$v_1^2 e^{4\pi\mu} = \frac{-2ga}{1+4\mu^2}e^{4\pi\mu}[0-(1-2\mu^2).1] + V^2 - \frac{ga}{(1+4\mu^2)}(1-2\mu^2)$$

Dividign by $e^{4\pi\mu}$, we get

$$v_1^2 = V^2 e^{-4\pi\mu} + \frac{2ga}{1+4\mu^2}(1-2\mu^2)(1-e^{-4\pi\mu})$$

2. *A particle under no forces is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole. Show that it will arrive at the pole in time*

$$\frac{a}{V}\frac{1}{\cos\alpha - \mu\sin\alpha}$$

α *being the angle of the spiral and* $(\mu < \cot\alpha)$ *the coefficient of friction.*

Solution. The equation of the equiangular spiral is

$$r = ae^{\theta\cot\alpha} \qquad \text{...(i)}$$

Also, $\phi = \alpha$ and $\psi = \theta + \phi = \theta + \alpha$ and its pedal equation is $p = r \sin \alpha$;

$$\therefore \quad \frac{dp}{dr} = \sin \alpha \quad \text{and} \quad \rho = r\frac{dr}{dp} = r \operatorname{cosec} \alpha.$$

The particle is moving under no forces and hence we have the following equations of motion.

$$mv\frac{dv}{ds} = \mu R \qquad \text{...(ii)}$$

and
$$m\frac{v^2}{\rho} = R \qquad \text{...(iii)}$$

or
$$mv\frac{dv}{ds} = \mu \,.\, m\frac{v^2}{\rho}$$

or
$$\frac{1}{2}\frac{dv^2}{d\theta}\,.\,\frac{d\theta}{ds} = \frac{\mu v^2}{r \operatorname{cosec} \alpha}$$

or
$$\frac{1}{2}\frac{dv^2}{d\theta}\,.\,r\frac{d\theta}{ds} = \frac{\mu v^2}{\operatorname{cosec} \alpha}$$

But $\tan \phi = r\frac{d\theta}{dr}$; $\therefore$ $\sin \phi = r\,.\,\frac{d\theta}{ds} = \sin \alpha$, because $\phi = \alpha$.

$$\therefore \quad \frac{1}{2}\frac{dv^2}{d\theta}\sin \alpha - \mu v^2 \sin \alpha = 0$$

or
$$\frac{dv^2}{d\theta} - 2\mu v^2 = 0$$

This is a linear differential equation whose integrating factor $= e^{-2\mu\theta}$.
Hence, the solution of the equation will be

$$v^2 . e^{-2\mu\theta} = c^2 \quad \text{or} \quad v = ce^{\mu\theta},$$

where c^2 is the constant of integration.

From (i), we have
$$\left(\frac{r}{a}\right)^{1/\cot \alpha} = e^{\theta}$$

or
$$\left(\frac{r}{a}\right)^{\mu \tan \alpha} = e^{\mu\theta}$$

$$\therefore \quad v = c\left(\frac{r}{a}\right)^{\mu \tan \alpha}$$

Initially, when $r = a, v = -V$ (towards the pole) $\therefore$ $c = -V$

$$\therefore \quad v = \left(\frac{r}{a}\right)^{\mu \tan \alpha}(-V) \quad \text{or} \quad v = -V\left(\frac{r}{a}\right)^{\mu \tan \alpha}$$

or
$$\frac{ds}{dt} = -V\,.\,\frac{r^{\mu \tan \alpha}}{a^{\mu \tan \alpha}} \quad \text{or} \quad \frac{ds}{dr}\,.\,\frac{dr}{dt} = -V\,.\,\frac{r^{\mu \tan \alpha}}{a^{\mu \tan \alpha}}$$

But $\cos \phi = \frac{dr}{ds}$ or $\cos \alpha = \frac{dr}{ds}$, $\because \phi = \alpha$

$$\therefore \quad \frac{1}{\cos \alpha}\,.\,\frac{dr}{dt} = -\frac{V}{a^{\mu \tan \alpha}} r^{\mu \tan \alpha} \quad \text{or} \quad -\frac{a^{\mu \tan \alpha}}{V \cos \alpha}\int_a^0 r^{-\mu \tan \alpha}\, dr = \int_0^t dt$$

or
$$-\frac{a^{\mu \tan \alpha}}{V \cos \alpha}\left[\frac{r^{-\mu \tan \alpha + 1}}{-\mu \tan \alpha + 1}\right]_\alpha^0 = 1$$

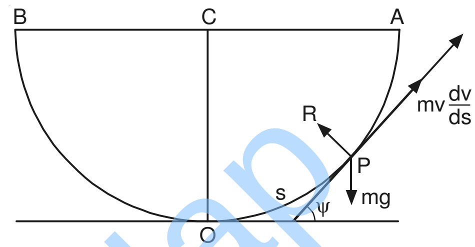
$$\therefore \qquad t = -\frac{a^{\mu \tan\alpha}}{V\cos\alpha}\cdot\frac{0 - a^{-\mu\tan\alpha+1}}{-\mu\tan\alpha+1} = \frac{a^{\mu\tan\alpha - \mu\tan\alpha+1}}{V\cos\alpha}\cdot\frac{\cos\alpha}{\cos\alpha - \mu\sin\alpha}$$

$$= \frac{a}{V}\cdot\frac{1}{\cos\alpha - \mu\sin\alpha}.$$

3. *A particle slides in a vertical plane down a rough cycloidal arc where axis is vertical and vertex downwards starting from a point where the tangent makes an angle* θ *with the horizon and coming to rest at the vertex; show that* $\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta$.

Solution. Intrinsic equation of a cycloid is

$$s = 4a\sin\psi \qquad \text{...(i)}$$



$\rho = \dfrac{ds}{d\psi} = 4a\cos\psi$. Its length $= 8a$

Also $\psi = \dfrac{\pi}{2}$ for cusp and $\psi = 0$ at the vertex.

Since the particle is sliding down the arc, the force of friction acts in upwards direction. Hence, equations of motion will be

$$mv\frac{dv}{ds} = \mu R - mg\sin\psi \qquad \text{...(ii)}$$

and
$$m\frac{v^2}{\rho} = R - mg\cos\psi \qquad \text{...(iii)}$$

Eliminating R between (ii) and (iii), we get

$$mv\frac{dv}{ds} = \mu\left(m\frac{v^2}{\rho} + mg\cos\psi\right) - mg\sin\psi$$

or
$$\frac{1}{2}\frac{dv^2}{ds} = \mu\frac{v^2}{\rho} + g\,(\mu\cos\psi - \sin\psi)$$

Multiplying both sides by 2ρ, we get

$$\frac{dv^2}{ds}\cdot\frac{ds}{d\psi} - 2\mu v^2 = 2g\,4a\cos\psi\,(\mu\cos\psi - \sin\psi)$$

$$\because \qquad \rho = \frac{ds}{d\psi} = 4a\cos\psi$$

or
$$\frac{dv^2}{d\psi} - 2\mu v^2 = 8ag\cos\psi\,(\mu\cos\psi - \sin\psi) \qquad \text{...(iv)}$$

This is linear differential equation and its integrating factor is $e^{-2\mu\psi}$.
Hence, solution of this equation will be

$$v^2.e^{-2\mu v} = 8ag\int e^{-2\mu v}\cos\psi\,(\mu\cos\psi - \sin\psi)\,d\psi + A \qquad \text{...(v)}$$

where A is constant of integration.

Put $e^{-\mu\psi}(\mu\cos\psi - \sin\psi) = z$.

$$\therefore \qquad [-\mu e^{-\mu\psi}(\mu\cos\psi - \sin\psi) + e^{-\mu\psi}(-\mu\sin\psi - \cos\psi)]\,d\psi = dz$$

or
$$-e^{-\mu\psi}[1+\mu^2]\cos\psi\,d\psi = dz$$

$$\therefore \qquad e^{-\mu\psi}\cos\psi\,d\psi = -\frac{dz}{1+\mu^2} \qquad \text{...(vi)}$$

Hence, from (v) and (vi), we get

$$v^2 e^{-2\mu\psi} = 8ag \int e^{-\mu\psi} . \cos\psi\, e^{-\mu\psi} (\mu \cos\psi - \sin\psi)\, d\psi + A$$

$$= 8ag \int z . \frac{-dz}{1+\mu^2} + A$$

$$= -\frac{8agz^2}{2(1+\mu^2)} + A$$

or $$v^2 e^{-2\mu\psi} = -\frac{8ag}{1+\mu^2} e^{-2\mu\psi} (\mu\cos\psi - \sin\psi)^2 + A \qquad \text{...(vii)}$$

We are given that $v = 0$ when $\psi = 0$.

$\therefore$ $A = \dfrac{4ag}{(1+\mu^2)} e^{-2\mu\theta} (\mu\cos\theta - \sin\theta)^2$ or $v^2 = \dfrac{4ag}{(1+\mu^2)} e^{-2\mu\theta} (\mu\cos\theta - \sin\theta)^2$

$$- \frac{4ag}{1+\mu^2} e^{-2\mu\psi} (\mu\cos\psi - \sin\psi)^2 \qquad \text{...(viii)}$$

Again, we are given that it comes to rest at the vertex where $\psi = 0$.

Hence, putting $v = 0$ and $\psi = 0$ in (viii), we get

$$\frac{4ag}{1+\mu^2} e^{-2\mu\theta} (\mu\cos\theta - \sin\theta)^2 = \frac{4ag}{1+\mu^2} (\mu)^2$$

or $$e^{-2\mu\theta} (\mu\cos\theta - \sin\theta)^2 = \mu^2$$

or $$\mu = \pm e^{-\mu\theta} (\mu\cos\theta - \sin\theta) \quad \text{or} \quad \mu e^{\mu\theta} = \pm (\mu\cos\theta - \sin\theta)$$

Taking negative sign, we get

$$\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta$$

4. *A small bead is threaded on a rough rigid wire in the form of an equiangular spiral of angle* α. *The bead is projected away from the pole with any velocity. Prove that the intervals of time between the successive instants at which the bead is moving in the same direction as at first form a G.P. of common ratio* $e^{2\pi(\mu + \cot\alpha)}$.

Solution. The equations of motions are

$$mv \frac{dv}{ds} = -\mu R \qquad \text{...(i)}$$

and $$\frac{mv^2}{\rho} = R \qquad \text{...(ii)}$$

Eliminating R, we get

$$mv \frac{dv}{ds} = -\mu . m \frac{v^2}{\rho} \quad \text{or} \quad \frac{dv}{ds} . \frac{ds}{d\psi} = -\mu v$$

or $$\frac{dv}{d\psi} = -\mu v \quad \text{or} \quad \frac{dv}{v} = -\mu\, d\psi.$$

Integrating, $\log v = -\mu\psi + \log k$,
where k is a constant.

or $$\log \frac{v}{k} = -\mu\psi \quad \text{or} \quad v = ke^{-\mu\psi} = \frac{ds}{dt}$$

or $$\frac{ds}{d\psi} . \frac{d\psi}{dt} = ke^{-\mu\psi} \qquad \text{...(iii)}$$

Now, the equation of equiangular spiral of angle α is $r = ae^{\theta\cot\alpha}$ and its pedal equation is $p = r\sin\alpha$ and $\phi = \alpha$. Hence,

$$\frac{ds}{d\psi} = \rho = r\frac{dr}{dp} = \frac{r}{\sin\alpha} = r \operatorname{cosec} \alpha. \qquad \text{Also } \psi = \theta + \phi = \theta + \alpha.$$

Hence, from (iii), we get

$$\frac{r}{\sin\alpha} \cdot \frac{d\psi}{dt} = ke^{-\mu\psi} \quad \text{or} \quad \frac{d\psi}{dt} = \frac{k \sin\alpha e^{-\mu\psi}}{r} = \frac{k \sin\alpha\, e^{-\mu\psi}}{ae^{\theta \cot\alpha}}$$

or $$\frac{d\psi}{dt} = \frac{k \sin\alpha}{a} e^{-\mu\psi} e^{-(\psi-\alpha)\cot\alpha} \qquad \because \ \psi = \theta + \alpha$$

or $$\frac{d\psi}{dt} = \frac{k \sin\alpha}{a} e^{\alpha \cot\alpha} \cdot e^{-(\mu + \cot\alpha)\psi}$$

or $$e^{(\mu+\cot\alpha)\psi}\, d\psi = \frac{k \sin\alpha}{a} e^{\alpha\cot\alpha}\, dt$$

Integrating, $$\frac{e^{(\mu+\cot\alpha)\psi}}{\mu + \cot\alpha} = \frac{k\sin\alpha}{a} e^{\alpha\cot\alpha}\, t + A,$$

where A is constant of integration. Since α, k, a and μ are constants, above equation can be put in the form

$$e^{(\mu+\cot\alpha)\psi} = Bt + c \qquad \text{...(iv)}$$

Let initially, when $t = 0, \psi = \psi_0$

$\therefore$ $$c = e^{(\mu+\cot\alpha)\psi_0} \qquad \text{...(v)}$$

Since we are to find the intervals, when the particles is moving in the same direction as at first, hence we give to ψ the values $\psi_0 + 2\pi, \psi_0 + 4\pi, \psi_0 + 6\pi$ and so on and let the corresponding values of t be t_1, t_2, t_3 and so on. We are to find the values of $t_2 - t_1, t_3 - t_2$ and so on.

Putting these values in (iv), we get

$$e^{(\mu+\cot\alpha)(\psi_0+2\pi)} = Bt_1 + c$$

or $$a^{(\mu+\cot\alpha)\psi_0} e^{(\mu+\cot\alpha)2\pi} - c = Bt_1$$

or $$c\,\{e^{(\mu+\cot\alpha)2\pi} - 1\} = Bt_1 \qquad \text{by (v)}$$

Similarly, $$c\,\{e^{(\mu+\cot\alpha)4\pi} - 1\} = Bt_2$$

and $$c\,\{e^{(\mu+\cot\alpha)6\pi} - 1\} = Bt_3 \text{ and so on.}$$

$\therefore$ $$t_2 - t_1 = \frac{c}{B} e^{(\mu+\cot\alpha)2\pi} [e^{(\mu+\cot\alpha)2\pi} - 1]$$

$$t_3 - t_2 = \frac{c}{B} e^{(\mu+\cot\alpha)4\pi} [e^{(\mu+\cot\alpha)2\pi} - 1]$$

Obviously, above intervals of time $t_2 - t_1, t_3 - t_2, \ldots$ form a G.P. whose common ratio is $e^{(\mu+\cot\alpha)2\pi}$.

5. *A particle starts from rest from the cusp of a rough cycloid whose axis is vertical and vertex downwards. Show that its velocity at the vertex is to its velocity at the same point when the cycloid is smooth as*

$$(e^{-\mu\pi} - \mu^2)^{1/2} : \sqrt{(1+\mu^2)}$$

where μ *is the coefficient of friction. Further show that the particle will certainly come to rest before reaching the vertex if the coefficient of friction be* 0.5, *having given that* $\log 2 = .69315$.

Solution. Proceeding exactly as in Example 3, we arrive at result (vii) as

$$v^2 e^{-2\mu\psi} = -\frac{4ag}{1+\mu^2} e^{-2\mu v} (\mu\cos\psi - \sin\psi)^2 + A$$

Since the particle starts from rest at the cusp when $\psi = \frac{\pi}{2}$, we have

$$0 = -\frac{4ag}{1+\mu^2} e^{-\mu\pi} (0-1)^2 + A$$

$\therefore$ $$A = \frac{4ag}{1+\mu^2} e^{-\mu\pi}$$

$\therefore$ $$e^{-2\mu\psi} = \frac{4ag}{1+\mu^2} e^{-\mu\pi} - \frac{4ag}{1+\mu^2} e^{-2\mu\psi} (\mu \cos\psi - \sin\psi)^2$$

Let v_1 be the velocity at the vertex where $\psi = 0$. Then we have from above by putting $v = v_1$ and $\psi = 0$.

$\therefore$ $$v_1^2 \,.\, 1 = \frac{4ag}{1+\mu^2} e^{-\mu\pi} - \frac{4ag}{1+\mu^2} 1 \,.\, (\mu \,.\, 1 - 0)^2$$

or $$v_1^2 = \frac{4ag}{1+\mu^2} (e^{-\mu\pi} - \mu^2) \qquad \text{...(i)}$$

Let v_2 be the velocity at the vertex when the cycloid is smooth, then putting $\mu = 0$ in (i), we get

$$v_2^2 = \frac{4ag}{1} (1-0) = 4ag \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$\frac{v_1^2}{v_2^2} = \frac{e^{-\mu\pi} - \mu^2}{1+\mu^2}$$

or $$v_1 : v_2 = (e^{-\mu\pi} - \mu^2)^{1/2} : \sqrt{(1+\mu^2)}$$

Now, let the paricle comes to rest at the lowest point, *i.e.*, vertex.

Then $v_1 = 0$ and hence from (i),

$e^{-\mu x} - \mu^2 = 0$ or $\mu^2 e^{\mu\pi} = 1.$ Hence, it will coem to rest before reaching the vertex if

$$\mu^2 e^{\mu\pi} > 1 \quad \text{or} \quad \mu e^{\mu\pi/2} > 1 \quad \text{or if} \quad \log\mu + \frac{\mu\pi}{2} \log e > \log 1$$

or $$\log\mu + \frac{\mu\pi}{2} \log e > 0 \quad \text{or} \quad \log\frac{1}{2} + \frac{1}{4} \,.\, \frac{22}{7} \,.\, 1 > 0$$

since $\mu = .5 = \frac{1}{2}$ and $\pi = \frac{22}{7}$

or $$\text{if } \frac{22}{28} - \log 2 > 0 \quad \text{or} \quad \frac{22}{28} - .69315 > 0$$

which is true as $\frac{22}{28} = .785.$

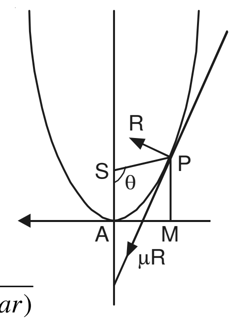
Hence, if $\mu = .5,$ the particle will come to rest before reaching the vertex.

6. *A rough parabolic wire with latus rectum 4a is placed with its axis vertical and vertex downwards and a bead is projected along it from the lowest point with velocity u. Show that the bead will come to rest at a distance* $\frac{a}{n^2}$ *from the focus where*

$$\mu \cos^{-1} n = \log \left\{ n \sqrt{\left(1 + \frac{u^2}{2ag}\right)} \right\}.$$

Solution. Taking S as pole and SX as initial line, the polar equation of the parabola will be

$$\frac{2a}{r} = 1 + \cos\theta \quad \text{or} \quad r = a \sec^2 \frac{\theta}{2}$$



∴ $\frac{dr}{d\theta} = 2a \sec\frac{\theta}{2} \sec\frac{\theta}{2} \tan\frac{\theta}{2} \cdot \frac{1}{2}$

or $\frac{1}{r}\frac{dr}{d\theta} = \tan\frac{\theta}{2}$ or $\cot\phi = \tan\frac{\theta}{2}$

∴ $\phi = 90° - \frac{\theta}{2}$

and $\psi = \theta + \phi = 90° + \frac{\theta}{2}$ ...(ii)

Also, $p = r\sin\phi = r\sin\left(90° + \frac{\theta}{2}\right) = r\cos\frac{\theta}{2} = r\sqrt{\left(\frac{a}{r}\right)} = \sqrt{(ar)}$

∴ $\frac{dp}{dr} = \frac{\sqrt{a}}{2\sqrt{r}}$. Hence, $\rho = r\frac{dr}{dp}$

$= 2r\sqrt{\left(\frac{r}{a}\right)} = 2a\sec^3\frac{\theta}{2}$ ...(iii)

Also, when $r = \frac{a}{n^2}$, then $a\sec^2\frac{\theta}{2} = \frac{a}{n^2}$

or $\cos\frac{\theta}{2} = n$

∴ $\frac{\theta}{2} = \cos^{-1} n$ ...(iv)

The equations of motion will be

$$mv\frac{dv}{ds} = -\mu R - mg\cos\phi = -\mu R - mg\sin\frac{\theta}{2} \quad \text{by (ii)} \qquad ...(v)$$

and $m\frac{v^2}{\rho} = R - mg\sin\phi = R - mg\cos\frac{\theta}{2}$ ...(vi)

Eliminating R between (v) & (vi), we get

$$mv\frac{dv}{ds} = -\mu\left(\frac{mv^2}{\rho} + mg\cos\frac{\theta}{2}\right) - mg\sin\frac{\theta}{2}$$

or` $\frac{1}{2}\frac{dv^2}{ds} + \frac{\mu v^2}{\rho} = -g\left(\mu\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)$

Multiplying both sides by 2ρ and putting

$\rho = 2a\sec^3\frac{\theta}{2}$ by (iii),

we get $\frac{dv^2}{ds} \cdot \frac{ds}{d\psi} + 2\mu v^2 = -2g \cdot 2a\sec^3\frac{\theta}{2}\left(\mu\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)$

$\frac{dv^2}{d\theta} \cdot \frac{d\theta}{d\psi} + 2\mu v^2 = -4ag\left(\mu\sec^2\frac{\theta}{2} + \tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\right)$

or $\frac{dv^2}{d\theta} \cdot 2 + 2\mu v^2 = -4ag\left(\mu\sec^2\frac{\theta}{2} + \tan\frac{\theta}{2}\sec^2\theta\right)$ $\left[\because \psi = 90° + \frac{\theta}{2}, \therefore \frac{d\theta}{d\psi} = 2\right]$

or $\frac{dv^2}{d\theta} + \mu v^2 = -2ag\left(\mu\sec^2\frac{\theta}{2} + \tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\right)$

This is linear equation in v^2 and its I.F. $= e^{\mu\theta}$. Hence, the solution of this equation will be

$$v^2 . e^{\mu\theta} = -2ag\int e^{\mu\theta}\left(\tan\frac{\theta}{2}\sec\frac{\theta}{2}\sec\frac{\theta}{2} + \mu\sec^2\frac{\theta}{2}\right)d\theta + A$$

$$= -2ag\left[e^{\mu\theta}\sec^2\frac{\theta}{2} - \int e^{\mu\theta}.\mu\sec^2\frac{\theta}{2}\,d\theta + \int e^{\mu\theta}\mu\sec^2\frac{\theta}{2}\,d\theta\right] + A$$

or $$v^2 e^{\mu\theta} = -2ag\, e^{\mu\theta}\sec^2\frac{\theta}{2} + A$$

Initially, for the vertex $\theta = 0$ and $v = u$

$\therefore$ $$u^2 = -2ag + A \quad \text{or} \quad A = u^2 + 2ag$$

$\therefore$ $$v^2 e^{\mu\theta} = -2ag e^{\mu\theta}\sec^2\frac{\theta}{2} + u^2 + 2ag$$

The bead will come to rest when $v = 0$

$\therefore$ $$2ag\, e^{\mu\theta}\sec^2\frac{\theta}{2} = u^2 + 2ag$$

or $$e^{\mu\theta}\sec^2\frac{\theta}{2} = 1 + \frac{\mu^2}{2ag}$$

or $$e^{\mu\theta/2} = \cos\frac{\theta}{2}\sqrt{\left(1 + \frac{u^2}{2ag}\right)}$$

$$= n\sqrt{\left(1 + \frac{u^2}{2ag}\right)}, \quad \text{by (iv)}$$

Taking log of both sides, we get

$$\frac{\mu\theta}{2} = \log\left[n\sqrt{\left(1 + \frac{u^2}{2ag}\right)}\right]$$

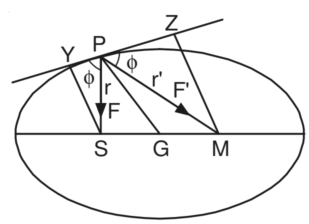
$$\mu\cos^{-1} n = \log\left[n\sqrt{\left(1 + \frac{u^2}{2ag}\right)}\right] \qquad \text{by (iv).}$$

7. *A particle P of unit mass describes on ellipse under an attraction f to focus S and attraction f' to the other focus H. If $SP = r$ and $HP = r'$, prove that*

$$\frac{1}{r^2}\frac{d}{dr}(r^2 f) = \frac{1}{r'^2}\frac{d}{dr'}(r'^2 f')$$

Hence, show that if one force obeys the Newtonian law, so also must obey the other. If the forces are equal, then each varies inversely as the product of focal distance of P.

Solution. There is a force f at P towards focus S, where $SP = r$ and a force f' at P towards focus H where $HP = r'$.



We know that tangent and normal at any point on an ellipse are equally inclined to the focal radii SP and HP.

$\therefore$ $$\angle SPY = \angle HPZ = \phi$$

$$\sin\phi = \frac{SY}{SP} = \frac{SY}{r}$$

and $$\sin\phi = \frac{HZ}{HP} = \frac{HZ}{r'}$$

$$\sin^2\phi = \frac{SY.HZ}{rr'} = \frac{b^2}{rr'} \qquad \because SY.HZ = b^2$$

Hence, $$\sin\phi = \frac{b}{\sqrt{(rr')}}$$

Again we know that in an ellipse,

$$x = a \cot t, \quad y = b \sin t$$

$$\therefore \quad \rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left[\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right]} = \frac{(a^2 \sin^2 t - b^2 \cos^2 t)^{3/2}}{ab}$$

$$= \frac{[a^2(1-\cos^2 t) + b^2 \cos^2 t]^{3/2}}{ab} = \frac{[a^2 - (a^2 - b^2)\cos^2 t]^{3/2}}{ab}$$

$$= \frac{(a^2 - a^2e^2 \cos^2 t)^{3/2}}{ab} = \frac{(a^2 - e^2x^2)^{3/2}}{ab} = \frac{[(a-ex)(a+ex)]^{3/2}}{ab} = \frac{(SP.HP)^{3/2}}{ab}$$

$$= \frac{(rr')^{3/2}}{ab} \qquad \text{...(ii)}$$

Also, $\quad SP + HP = 2a \quad$ or $\quad r + r' = 2a$

Resolving the forces along the tangent, we have

$$v\frac{dv}{ds} = f \cos\phi - f' \cos\phi = f\frac{dr}{ds} - f'\frac{dr'}{ds}, \qquad \because \cos\phi = \frac{dr}{ds}.$$

$$\therefore \quad \frac{1}{2}dv^2 = -(f\,dr + f'dr') \qquad \text{...(iii)}$$

Resolving along the normal, we get

$$\frac{v^2}{\rho} = f \sin\phi + f' \sin\phi \qquad [\because m = 1]$$

or $$v^2 = (f + f')\rho \sin\phi \qquad \text{...(iv)}$$

Putting for v^2 from (iv) in (iii), we get

$$d[(f+f')\rho \sin\phi] = -2f\,dr\,2f'\,dr'$$

or $$d\left\{(f+f')\frac{(rr')^{3/2}}{ab}\cdot\frac{b}{\sqrt{(rr')}}\right\} = -2f\,dr - 2f'\,dr' \qquad \text{by (i) and (ii)}$$

or $$\frac{1}{a}d[(f+f')\,rr'] = -2f\,dr - 2f'\,dr'$$

or $$d(frr') + d(f'\,rr') = -2a(fdr + f'\,dr')$$

or $$\frac{d}{dr}(frr')\,dr + \frac{d}{dr'}(f'\,rr')\,dr' = -2a(f\,dr + f'\,dr')$$

Now, we know that in an ellipse sum of the focal distances is constant and equal to $2a$.

$$\therefore \quad r + r' = 2a, \qquad \therefore \quad dr = -dr'$$

$$\therefore \quad \frac{d}{dr}(frr')\,dr - \frac{d}{dr'}(f'\,rr')\,dr = -2a(f\,dr - f'\,dr)$$

or $$\frac{d}{dr}(frr') - \frac{d}{dr'}(f'rr') = -(r+r')(f - f')$$

$$\left(rr'\frac{df}{dr} + fr' + fr\frac{dr'}{dr}\right) - \left(rr'\frac{df'}{dr'} + f'r + f'r'\frac{dr}{dr'}\right) = -f(r+r') + f'(r+r')$$

Putting $\frac{dr}{dr'} = \frac{dr'}{dr} = -1$, we get

$$\left(rr'\frac{dr}{dr} + fr' - fr\right) - \left(rr'\frac{df'}{dr'} + f'r + f'r'\right) = -f(r+r') + f'(r+r')$$

or $$rr' \frac{df}{dr} + 2fr' = rr' \frac{df'}{dr'} + 2f'r.$$

Dividing throughout by rr', we get

$$\frac{df}{dr} + \frac{2f}{r} = \frac{df'}{dr'} + \frac{2f'}{r'}$$

or $$\frac{1}{r^2}\left(r^2 \frac{df}{dr} + 2r \, . \, f\right) = \frac{1}{r^2}\left(r'^2 \frac{df'}{dr'} + 2r'r'\right)$$

or $$\frac{1}{r^2}\frac{d}{dr}(r^2 f) = \frac{1}{r'^2}\frac{d}{dr'}(r'^2 f') \qquad ...(ii)$$

Now if f obeys Newtonian Law, then $f = \frac{\mu}{r^2}$ or $r^2 f = \mu$, *i.e.*, constant.

Hence, from (iv), we get

$$\frac{1}{r}\frac{d}{dr}(\mu) = \frac{1}{r'^2}\frac{d}{dr'}(r'^2 f') \quad \text{or} \quad 0 = \frac{1}{r'^2}\frac{d}{dr'}(r'^2 f')$$

$$\therefore \quad r'^2 f' = \text{cons tant} = \mu' \quad \therefore \quad f' = \frac{\mu'}{r'^2}$$

Hence, clearly f' also obeys Newtonian Law.
From above

$$f = \frac{\mu}{r^2} \text{ and also } f' = \frac{\mu'}{r'^2} \quad \therefore \quad ff' = \frac{\mu\mu'}{r^2 r'^2}$$

But if he forces be equal, *i.e.*, $f' = f$.

then $$f^2 = \frac{\mu\mu'}{r^2 r'^2} \quad \text{or} \quad f = \frac{\sqrt{(\mu\mu')}}{rr'} = \frac{k}{rr'}$$

i.e., force varies inversely as the product of the focal distances of the point.

EXERCISES

1. A particle falls from a position of limiting equilibrium near the top of a nearly smooth glass sphere. Show that it will leave he sphere at the point whose radius is inclined to the vertical at an angle $\alpha + \mu\left[2 - \frac{4\alpha}{3 \sin \alpha}\right]$ where $\cos \alpha = \frac{2}{3}$ and μ is the small coefficient of friction.

2. A particle is projected along the inner surface of a rough sphere and is acted on by no forces. Show that it will return to the point of projection at the end of time $\frac{a}{\mu V}(e^{2\mu\pi} - 1)$, where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.

3. A rough cycloid has its plane vertical and the line joining its cusp horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45° to the vertical. Show that the coefficient of friction satisfies the equation

$$3\mu\pi + 4 \log (1 + \mu) = 2 \log 2$$

4. A bead moves along a rough curve wire which is such that it changes its direction of motion with constant angular velocity. Show that a possible form of wire is an equiangular spiral.

5. A smooth wire in the form of a parabola of latus rectum $4a$ is fixed in a horizontal place. A small ring of mass m can slide on it and is attached to the focus by a light elastic string of natural length a and modulus λ. If the ring is held at the end of the latus rectum and released, show that its distance from the axis of the parabola at time t is

$$2a \cos\left[t\sqrt{\left(\frac{\lambda}{4am}\right)}\right]$$

❒❒❒

Motion in a Resisting Medium

5.1 INTRODUCTION

We are familiar with several motions in vacuum, for example the motion of a projectile in vacuum. In vacuum the moving body does not experience any resistance to its motion but when a body moves in a medium like air, it feels a resistance to its motion and hence it is called the *motion in a resisting medium.* It has been observed that resistance increases as the velocity of the body increases and hence the resistance may be assumed to be some function of the velocity. *The resisting force always acts in the opposite direction of motion and is non-conservative and hence the principle of conservation of energy can not be applied.*

All the laws of resistance, in fact, are more or less empiric.

5.2 TERMINAL VELOCITY

Let a particle of mass m be falling under gravity in air and let the resistance be proportional to v^n, where v is the velocity at a distance x. Then equation of motion will be

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = g - kv^n \quad \text{..(i)}$$

where g is acceleration due to gravity. The negative sign to kv^n shows that there is retardation due to air resistance. Equation (i) shows that with the increase in velocity the acceleration $\frac{d^2x}{dt^2}$ goes on decreasing till the particle acquires a velocity which makes the right hand side of (i) vanish. Let this velocity be V, then

$$0 = g - kV^n \quad \text{or} \quad V = \left(\frac{g}{k}\right)^{1/n} \quad \text{...(ii)}$$

After v has reached the value V, $\frac{dv}{dt}$ vanishes. Hence, it ceases to increase any more and thus v can never exceed the value V.

If we consider the case in which the particle is projected downwards with a velocity greater than V, then the right hand side of (i) will be negative and hence $\frac{dv}{dt}$ will be negative. It means that from the beginning of motion there is retardation. Due to this retardation, the velocity of particle will go on decreasing till it reaches the velocity V. When this velocity is reached, $\frac{dv}{dt} = 0$ and then velocity ceases to decrease.

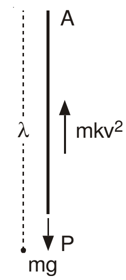
Obviously, in either case the ultimate velocity is V. This velocity V is called the **terminal velocity.**

For example, a rain drop falling on the earth can not give an idea of the height from which it is coming as it will attain terminal velocity much earlier than it has fallen on the ground, while for a feather falling from the top of a tower, it is easy to see that the feather will have practically the same constant velocity in the last four or five metres.

5.3 MOTION IN A VERTICAL LINE DOWNWARDS

A particle falls under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to find the motion of the particle if it starts from rest.

Let a particle falls, from rest, vertically downwards from A. Let P be its position when it has described a distance x in time t. The forces acting at P will be :



(i) The weight mg of the particle acting vertically downwards, and

(ii) The force mkv^2 due to resistance acting vertically upwards.

Hence, the equation of motion will be

$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = mg - mkv^2$$

or $$v\frac{dv}{dx} = g\left(1 - \frac{k}{g}v^2\right) \qquad \text{...(i)}$$

Let V be the terminal velocity, *i.e.*, the velocity when the downward acceleration is zero. Then from (i)

$$0 = g\left(1 - \frac{k}{g}V^2\right) \quad \text{or} \quad 1 - \frac{k}{g}V^2 = 0$$

or $$V^2 = \frac{g}{k} \qquad \text{...(ii)}$$

From (ii) and (i), we get

$$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{V^2}\right) \qquad \text{...(iii)}$$

or $$\frac{v\,dv}{V^2 - v^2} = \frac{g}{V^2}dx$$

Integrating, we get $$\log(V^2 - v^2) = -\frac{2gx}{V^2} + C,$$

where C is constant of integration.

Initially at A, $x = 0, v = 0$. $\therefore$ $C = \log V^2$

Hence, $$\log(V^2 - v^2) = -\frac{2gx}{V^2} + \log V^2$$

or $$\log\left(\frac{V^2 - v^2}{V^2}\right) = -\frac{2gx}{V^2} \quad \text{or} \quad \frac{V^2 - v^2}{V^2} = e^{(-2gx/V^2)}$$

$$v^2 = V^2\left(1 - e^{(-2gx/V^2)}\right) \qquad \text{...(iv)}$$

Equation (iv) gives the velocity in any position.

Now from equation (iii), we get

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{V^2}\right) = g\frac{(V^2 - v^2)}{V^2} \quad \text{or} \quad \frac{dv}{(V^2 - v^2)} = \frac{g}{V^2}dt$$

Integrating, we get

$$\frac{1}{V}\tanh^{-1}\left(\frac{v}{V}\right) = \frac{gt}{V} + A,$$

where A is constant of integration.

Initially, at A, $v = 0$ when $t = 0$, $\therefore A = 0.$

Hence, $$\frac{1}{V}\tanh^{-1}\frac{v}{V} = \frac{gt}{V^2}$$

or $$v = V\tanh\left(\frac{gt}{V}\right) \qquad \text{...(v)}$$

Equation (v) gives velocity at any time.
Equating the values of v from (iv) and (v), we get

$$V^2 \tanh^2\left(\frac{gt}{V}\right) = V^2 (1 - e^{-2gx/v^2})$$

or $$e^{-2gx/V^2} = 1 - \tanh^2\left(\frac{gt}{V}\right) = \text{sech}^2\left(\frac{gt}{V}\right)$$

or $$e^{2gx/V^2} = \cosh^2\left(\frac{gt}{V}\right)$$

or $$\frac{2gx}{V^2} = 2\log\cosh\left(\frac{gt}{V}\right)$$

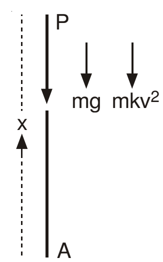
or $$x = \left(\frac{V^2}{g}\right)\log\cosh\left(\frac{gt}{V}\right) \quad \text{...(vi)}$$

This equation gives relation between x and t.

5.3.1 Motion in a Vertical Line in Upward Direction

A particle is projected upwards under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion.

Let a particle of mass m be projected from A with initial velocity u, vertically under gravity. Let P be its position when it has described a distance x in time t. The forces acting on the particle at P will be



(i) Weight of the particle mg, acting vertically downwards, and

(ii) The force mkv^2 due to resistance acting vertically downwards.

Hence, the equation of motion will be

$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = -mg - mkv^2$$

or $$v\frac{dv}{dx} = -g\left(1 + \frac{k}{g}v^2\right) \quad \text{...(i)}$$

Let V be the terminal velocity, *i.e.*,the velocity when the downwards acceleration is zero, then from (i) of 5.3.

$$0 = g\left(1 - \frac{k}{g}V^2\right) \quad \text{or} \quad V^2 = \frac{g}{k} \quad \text{...(ii)}$$

Substituting from (ii) in (i), we get

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^2}{V^2}\right) \quad \text{or} \quad \frac{v\,dv}{V^2 + v^2} = -\frac{g}{V^2}dx$$

Integrating, we get

$$\log(V^2 + v^2) = -\frac{2gx}{V^2} + C,$$

where C is constant of integration.

Initially, at A, $v = u$, $x = 0$;

$$\therefore \quad C = \log(V^2 + u^2)$$

$$\therefore \quad \log(V^2 + v^2) = -\frac{2gx}{V^2} + \log(V^2 + u^2)$$

or $$\log\left(\frac{V^2 + u^2}{V^2 + v^2}\right) = \frac{2gx}{V^2} \quad \text{...(iv)}$$

This gives velocity in any position, Equation (ii) may be written as

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2) \qquad \left[\because \; v\frac{dv}{dx} = \frac{dv}{dt}\right]$$

or
$$\frac{dv}{V^2 + v^2} = -\frac{g}{V^2}\,dt$$

Integrating, we get

$$\frac{1}{V}\tan^{-1}\frac{v}{V} = -\frac{gt}{V^2} + A,$$

where A is constant of integration.

At A, $v = u$ where $t = 0$, $\therefore \quad A = \frac{1}{V}\tan^{-1}\frac{u}{V}$

$\therefore$
$$\frac{1}{V}\tan^{-1}\frac{u}{V} = -\frac{gt}{V^2} + \frac{1}{V}\tan^{-1}\frac{u}{V}$$

or
$$t = \frac{V}{g}\left[\tan^{-1}\left(\frac{u}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right] \qquad \text{...(v)}$$

This is the relation between velocity and time.

5.4 RESISTANCE PROPORTIONAL TO THE VELOCITY

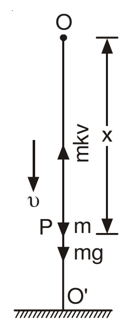
(a) A particle falls under gravity (supposed constant) from rest in a medium whose resistance varies as the velocity; to discuss the motion.

Let at time t, the particle has fallen through a distance x when v is its velocity. The forces acting at the particle at P will be

(i) The weight mg of the particle acting vertically downwards, and

(ii) The force mkv due to resistance acting vertically upwards.

Then the equation of motion will be



$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = mg - mkv$$

or
$$v\frac{dv}{dx} = \left(g - \frac{k}{g}v\right) \qquad \text{...(i)}$$

Let V be the terminal velocity, then

$$0 = g - \frac{k}{g}V \quad \text{or} \quad V = \frac{g^2}{k}.$$

$\therefore$
$$v\frac{dv}{dx} = g - \frac{g}{V}v$$

or
$$\frac{dv}{dt} + \frac{g}{V}v = g \qquad \text{...(ii)} \quad \left[\because \; v\frac{dv}{dx} = \frac{dv}{dt}\right]$$

This is a linear differential equation whose

$$\text{I.F.} = e^{\int (g/V)\,dt} = e^{(g/V)t}$$

Hence solution of (ii) will be

$$ve^{(g/v)t} = g\int e^{(g/V)t}\,dt + A \quad \text{or} \quad ve^{(g/V)t} = Ve^{(g/V)t} + A,$$

where A is constant of integration.

Initially, $t = 0, v = 0$, $\quad \therefore A = -V$

$\therefore$
$$ve^{(g/V)t} = Ve^{(g/V)t} - V$$

or
$$v = V - Ve^{-(g/V)t} \qquad \text{...(iii)}$$

or
$$\frac{dx}{dt} = V - Ve^{-(g/V)t}$$

Inetgrating, we get $\quad x = vt + \frac{V^2}{g} e^{-\frac{g}{V}t} + B,$

where B is constant of integration.

Initially, $t = 0, x = 0, \quad \therefore \quad B = -\frac{V^2}{g}$

Hence, $$x = Vt - \frac{V^2}{g} + \frac{V^2}{g} e^{-\frac{g}{V}t} \quad \text{...(iv)}$$

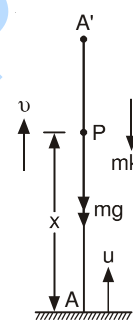
This is the relation between x and t.

(b) A particle is projected upwards with a velocity u in a medium whose resistance varies as the velocity to discuss the motion.

Let the particle be projected from A with initial velocity u, vertically up-wards, under gravity. Let P be its position when it has described a distance x time t. The force acting on the particle at P will be

(i) The weight mg of the particle acting vertically downwards, and

(ii) The force mkv due to resistance acting vertically downwards.



Hence, the equation of motion will be

$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = -mg - mkv$$

or $$v\frac{dv}{dx} = -g\left(1 + \frac{k}{g}v\right) \quad \text{...(i)}$$

Let V be the terminal velocity, then $V = \frac{g}{k}$.

$\therefore$ $$v\frac{dv}{dx} = -g - \frac{g}{V}v$$

or $$v\frac{dv}{dx} = -\frac{g}{V}(V + v) \quad \text{...(ii)}$$

or $$\frac{v\,dv}{V + v} = -\frac{g}{V}dx$$

or $$\left[1 - \frac{V}{V + v}\right]dv = -\frac{g}{V}dx$$

Integrating,

$$v - V\log(V + v) = -\frac{g}{V}x + A,$$

where A is constant of integration.

Initially, $x = 0, v = u, \ \therefore \ A = u - V\log(V + u)$

$\therefore$ $$v - u - V\log\frac{V + v}{u + V} = -\frac{g}{V}x$$

or $$x = \frac{V^2}{g}\log\frac{V + u}{V + u} + \frac{V}{g}(u - v) \quad \text{...(iii)}$$

Let H be the maximum height then at $x = H, v = 0$. Hence,

$$H = \frac{V^2}{g}\log\frac{V}{V + v} + \frac{Vu}{g}$$

Equation (iii) gives relation between x and v.

Again equation (ii) may be written as

$$\frac{dv}{dt} + \frac{g}{V} = -g \qquad \left[\because v\frac{dv}{dx} = \frac{dv}{dt}\right]$$

This is a linear equation whose I.F. $= e^{(g/V)t}$. Hence, its solution will be

$$ve^{(g/V)t} = -g \int e^{(g/V)t}\, dt + B$$

$$= -Ve^{(g/V)t} + B,$$

where B is constant of integration. Initially, $t = 0$, $v = u$,

$\therefore$ $$B = u + V$$

$\therefore$ $$v = (u + V)\, e^{-(g/V)t} - V \qquad \text{...(v)}$$

This is the relation between v and t. Again, (v) can be written as

$$\frac{dx}{dt} = (u + V)\, e^{-(g/V)t} - V \quad \text{or} \quad dx = [(u + V)\, e^{-(g/V)t} - V]\, dt$$

Integrating, we get

$$x = -(u + V)\frac{V}{g}\, e^{-(g/V)t} - Vt + C,$$

where C is constant of integration.

Initially, $t = 0$, $x = 0$ $\therefore$ $C = (u + V)\dfrac{V}{g}$.

$\therefore$ $$x = \frac{V}{g}(u + V)\,[1 - e^{-(g/V)t}] - Vt \qquad \text{...(vi)}$$

This is the relation between x and t.

From relation (v) it is clear that v decreases as t increases till at the highest point, where $v = 0$, we have

$$e^{-(g/V)t} = \frac{V}{u + V}$$

or $$t = \frac{V}{g}\log\left(\frac{u + V}{V}\right) \qquad \text{...(vii)}$$

From (vi) and (vii) the greatest height H will be

$$H = -\frac{V^2}{g} - \frac{V^2}{g}\log\frac{u + V}{V} + \frac{V}{g}(u + V)$$

$$= \frac{uV}{g} - \frac{V^2}{g}\log\left(\frac{u + V}{V}\right) \qquad \text{...(viii)}$$

5.5 RESISTANCE VARYING AS *n*TH POWER OF THE VELOCITY

A particle falls under gravity (supposed constant), in a resisting medium whose resistance varies as nth power of the velocity; to find the motion of the particle if it starts from rest.

In this case, the equation of motion will be

$$m\frac{d^2x}{dt^2} = mv\frac{dv}{dx} = mg - mkv^n$$

or $$v\frac{dv}{dx} = g\left(1 - \frac{k}{g}v^n\right) \qquad \text{...(i)}$$

Let V be the terminal velocity, *i.e.*, the velocity when the downwards acceleration is zero, then from (i)

$$0 = g\left(1 - \frac{k}{g}V^n\right) \quad \text{or} \quad V^n = \frac{g}{k} \qquad \text{...(ii)}$$

Hence, from (i); we have

$$v\frac{dv}{dx} = g\left(1 - \frac{v^n}{V^n}\right)$$

or
$$\frac{v\,dv}{\left(1-\frac{v^n}{V^n}\right)} = g\ dx$$

or
$$\left[1+\frac{v^n}{V^n}+\frac{v^{2n}}{V^{2n}}+\frac{v^{3n}}{V^{3n}}+...\right] v\ dv = g\ dx$$

Integrating,
$$\left[\frac{v^2}{2}+\frac{v^{n+2}}{(n+2)\,V^n}+\frac{v^{2n+2}}{(2n+2)\,V^{2n}}+...\right] = gx + A$$

where A is constant of integration.
Initially, $v = 0,\ x = 0;\ \therefore\ A = 0.$
Hence,
$$x = \frac{v^2}{g}\left[\frac{1}{2}+\frac{1}{(n+2)}\left(\frac{v}{V}\right)^n+\frac{1}{(2n+2)}\left(\frac{v}{V}\right)^{2n}+\frac{1}{(3n+2)}\left(\frac{v}{V}\right)^{3n}+...\right] \qquad ...(iv)$$

This is the relation between v and x.
Again, equation (iii) can be written as
$$\frac{dv}{dt} = g\left(1-\frac{v^n}{V^n}\right) \qquad \left[\because\ v\frac{dv}{dx}=\frac{dv}{dt}\right]$$

$$\therefore \quad gdt = \frac{dv}{\left(1-\frac{v^n}{V^n}\right)}$$

Since $\frac{v}{V}<1,$ hence expanding by Binomial theorem, we get
$$gdt = \left(1+\frac{v^n}{V^n}+\frac{v^{2n}}{V^{2n}}+\frac{v^{3n}}{V^{3n}}+...\right) dv$$

Integrating, we get
$$gt = \left[v+\frac{v^{n+1}}{(n+1)\,V^n}+\frac{v^{2n+1}}{(2n+1)\,V^{2n}}+\frac{v^{3n+1}}{(3n+1)\,V^{3n}}+...\right] + B,$$

where B is constant of integration.
Initially, $v = 0$ and $t = 0,\ \therefore\ B = 0$. Hence,
$$t = \frac{v}{g}\left[1+\frac{1}{(n+1)}\left(\frac{v}{V}\right)^n+\frac{1}{(2n+1)}\left(\frac{v}{V}\right)^{2n}+\frac{1}{(3n+1)}\left(\frac{v}{V}\right)^{3n}+...\right] \qquad ...(v)$$

This is the relation between v and t.

EXAMPLES

1. *Show that a particle projected upwards with a velocity U in a medium whose resistance varies as the square of the velocity will return to the point of projection wtih velocity $v_1 = \frac{UV}{\sqrt{U^2+V^2}}$ after a time $\frac{V}{g}\left(\tan^{-1}\frac{U}{V}+\tanh^{-1}\frac{v_1}{V}\right)$ where V is the terminal velocity.*

Solution. We know that the equation of motion when the particle is moving upwards is given by
$$mv\frac{dv}{dx} = -\,mg - mkv^2 \qquad ...(i)$$

Now, the particle moving upwards reaches the highest point B and then return back moving in the downwards path. We know that the equation of motion when the particle is descending is given by

$$mv\frac{dv}{dy} = mg - mkv^2 \quad \text{...(ii)}$$

If V be the terminal velocity, then putting $v = V$ and $v\frac{dv}{dy} = 0$ in (ii), we get

$$0 = mg - mkV^2 \quad \text{or} \quad V^2 = \frac{g}{k} \quad \text{...(iii)}$$

Hence, from (i)

$$v\frac{dv}{dx} = -g\left(1 + \frac{v^2}{V^2}\right) \quad \text{[using (iii)]} \quad \text{...(iv)}$$

or
$$\frac{2v\,dv}{V^2 + v^2} = -\frac{2g}{V^2}dx$$

Integrating, we get

$$\log(V^2 + v^2) = -\frac{2g}{V^2} + A$$

where A is constant of integration.
Initially, when $x = 0, V = U$.

$\therefore$
$$A = \log(V^2 + U^2)$$

$\therefore$
$$\log(V^2 + v^2) = -\frac{2gx}{V^2} + \log(V^2 + U^2)$$

or
$$x = \frac{V^2}{2g}\log\left(\frac{V^2 + U^2}{V^2 + v^2}\right) \quad \text{...(v)}$$

Let h be the greatest height attained by the particle, the putting $x = h, v = 0$ in (v), we get

$$h = \frac{V^2}{2g}\log\left(\frac{V^2 + U^2}{V^2}\right) \quad \text{...(vi)}$$

Again, we may write equation (iv) as

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2) \quad \left[\because \frac{dv}{dt} = v\frac{dv}{dx}\right]$$

or
$$\frac{dv}{V^2 + v^2} = \frac{g}{V^2}dt \quad \text{...(vii)}$$

Let t_1 be the time taken by the particle in reaching the highest point where is velocity become zero. Then by integrating (vii) between the limits $t = 0$ to $t = t_1$ and $v = U$ to $v = 0$, we get

$$\left[\frac{1}{V}\tan^{-1}\frac{v}{V}\right]_U^0 = -\frac{g}{V^2}[t]_0^{t_1}$$

or
$$-\frac{1}{V}\tan^{-1}\frac{U}{V} = -\frac{gt_1}{V^2} \quad \text{or} \quad t_1 = \frac{V}{g}\tan^{-1}\frac{U}{V} \quad \text{...(viii)}$$

Now, after reaching the highest point, the particle stars moving downwards from rest. Therefore, its equation of motion will be (ii), and it may written as

$$v\frac{dv}{dy} = g\left(1 - \frac{k}{g}v^2\right) \quad \text{or} \quad v\frac{dv}{dy} = g\left(1 - \frac{v^2}{V^2}\right)$$

or
$$v\frac{dv}{dy} = \frac{g}{V^2}(V^2 - v^2) \quad \text{...(ix)}$$

or
$$\frac{v\,dv}{(V^2 - v^2)} = -\frac{g}{V^2}dy$$

Integrating, we get

$$\log(V^2 - v^2) = -\frac{2gy}{V^2} + B,$$

where B is constant of integration.
Initially, at highest point $y = 0, v = 0$

$\therefore$ $$B = \log V^2$$

Hence, $$\log(V^2 - v^2) = -\frac{2gy}{V^2} + \log V^2$$

or $$\frac{2gy}{V^2} = \log\frac{V^2}{V^2 - v^2}$$

or $$y = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v^2}\right)$$

Let v_1 be the velocity of particle when it return back the point of projection, then $y = h, v = v_1$, hence from (x).

$$h = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v_1^2}\right)$$

or $$\frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2} = \frac{V^2}{2g}\log\frac{V^2}{V^2 - v_1^2}$$ [from (vi)]

or $$\log\frac{V^2 + U^2}{V^2} = \log\frac{V^2}{V^2 - v_1^2}$$

or $$\frac{V^2 + U^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$$

or $$V^4 = V^4 - V^2 v_1^2 + U^2 V^2 - U^2 v_1^2$$

or $$v_1^2 = \frac{U^2 V^2}{U^2 + V^2} \text{ or } v_1 = \frac{UV}{\sqrt{(U^2 + V^2)}} \quad \text{...(xi)}$$

Equation (ix) may be written as

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2)$$

or $$dt = \frac{V^2}{g}\frac{dv}{V^2 - v^2} \quad \text{...(xii)}$$

Let t_2 be the time taken by the particle in reaching the point of projection, then integrating (xii) between the limits $t = 0$ to $t = t_1$ and $v = 0$ to $v = v_1$, we get

$$[t]_0^{t_2} = \frac{V^2}{g}\int_g^{v_1}\frac{dv}{V^2 - v^2}$$

or $$t_2 = \frac{V^2}{g}\cdot\frac{1}{V}\left[\tanh^{-1}\frac{v}{V}\right]_0^{v_1} = \frac{V}{g}\left[\tanh^{-1}\frac{v_1}{V} - \tanh^{-1} 0\right]$$

or $$t_2 = \frac{V}{g}\tanh^{-1}\frac{v_1}{V} \quad \text{...(xiii)}$$

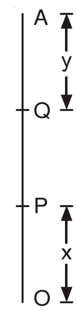
Adding (viii) and (xiii), the required time will be

$$t_1 + t_2 = \frac{V}{g}\left[\tan^{-1}\frac{u}{V} + \tanh^{-1}\frac{v_1}{V}\right] \quad \text{...(xiv)}$$

2. *A heavy particle is projected upwards with velocity u in a medium the resistance of which is* $gu^{-2}\tan^2\alpha$ *times the square of the velocity,* α *being a constant. Show that the particle will return to the point of projection with velocity* $u\cos\alpha$ *after a time*

$$ug^{-1}\cot\alpha\left[\alpha+\log\frac{\cos\alpha}{1-\sin\alpha}\right]$$

Solution. Let the particle be projected upwards from a point O with velocity u. Let v be the velocity at a point P distance x from O. Then



Resistance $= gu^{-2}\tan^2\alpha\,.\,v^2$

Now, equation of motion when the particle is rising will be

$$mv\frac{dv}{dx}=-mg-mgu^{-2}\tan^2\alpha\,.\,v^2$$

or $$v\frac{dv}{dx}=-\left(\frac{g}{u^2}\right)(u^2+v^2\tan^2\alpha) \quad ...(i)$$

or $$\frac{2v\tan^2\alpha\,dv}{u^2+v^2\tan^2\alpha}=-2\tan^2\alpha\,.\,\frac{g}{u^2}\,dx$$

Integrating, we get

$$\log(u^2+v^2\tan^2\alpha)=-\tan^2\alpha\,.\,\frac{gx}{u^2}+A$$

where A is constant of integration.

Initially, $x=0,\ v=u$

$\therefore$ $$A=\log(u^2+u^2\tan^2\alpha)=\log u^2\sec^2\alpha$$

Hence, $$\log(u^2+v^2\tan^2\alpha)=-\left(\frac{2gx}{u^2}\right)\tan^2\alpha+\log u^2\sec^2\alpha$$

or $$\left(\frac{2gx}{u^2}\right)\tan^2\alpha=\log(u^2\sec^2\alpha)-\log(u^2+v^2\tan^2\alpha) \quad ...(ii)$$

Let h be the maximum height attained by the particle. Then $v=0, x=h$.
Hence, from (ii), we get

$$\left(\frac{2gh}{u^2}\right)\tan^2\alpha=\log(u^2\sec^2\alpha)-\log u^2$$

or $$h=\left(\frac{u^2}{2g}\right)\cot^2\alpha\,.\,\log(\sec^2\alpha) \quad ...(iii)$$

Equation (i) may be written as

$$\frac{dv}{dt}=-\frac{g}{u^2}(u^2+v^2\tan^2\alpha)$$

or $$dt=-\frac{u^2}{g}\,.\,\frac{dv}{(u^2+v^2\tan^2\alpha)} \quad ...(iv)$$

Let t_1 be the time taken by the particle to reach the highest point B.

Integrating (iv) between the limits $t=0$ to $t=t_1$ and $v=u$ to $v=0$, we get

$$[t]_0^{t_1}=-\frac{u^2}{g}\left[\frac{1}{u\tan\alpha}\,.\,\tan^{-1}\frac{v\tan\alpha}{u}\right]_0^u$$

or $$t_1=\frac{u}{g\tan\alpha}\tan^{-1}(\tan\alpha)=\frac{u\alpha}{g\tan\alpha} \quad ...(v)$$

From the highest point, particle starts moving downwards from rest. Let v be the velocity at a point Q distant y from B. Equation of motion in downwards direction will be

$$mv\frac{dv}{dy} = mg - mgu^{-2}\tan^2\alpha\, v^2$$

or $$v\frac{dv}{dy} = \frac{g}{u^2}(u^2 - v^2\tan^2\alpha) \quad \text{...(vi)}$$

or $$-\frac{2v\,dv}{u^2 - v^2\tan^2\alpha} = \frac{2g}{u^2}dy$$

Integrating, $$\cot^2\alpha \log(u^2 - v^2\tan^2\alpha) = -\frac{2gy}{u^2} + B,$$

where B is constant of integration.
Initially, when $y = 0$, $v = 0$.

$\therefore$ $$B = \cot^2\alpha \log u^2$$

$\therefore$ $$\cot^2\alpha\log(u^2 - v^2\tan^2\alpha) = -\frac{2gy}{u^2} + \cot^2\alpha\,.\log u^2$$

or $$\frac{2gy}{u^2} = \cot^2\alpha\log\frac{u^2}{u^2 - v^2\tan^2\alpha} \quad \text{...(vii)}$$

Let v_1 be the velocity of return to the point of projection A, then putting $y = h, v = v_1$ in (vii), we get

$$\frac{2hg}{u^2} = \cot^2\alpha\log\frac{u^2}{u^2 - v_1^2\tan^2\alpha}$$

or $$\log\sec^2\alpha = \log\frac{u^2}{u^2 - v_1^2\tan^2\alpha} \quad \text{[putting for } h \text{ from (iii)]}$$

or $$\sec^2\alpha = \frac{u^2}{u^2 - v_1^2\tan^2\alpha} \quad \text{or} \quad u^2\sec^2\alpha - v_1^2\tan^2\alpha\sec^2\alpha = u^2$$

or $$u^2 - v_1^2\tan^2\alpha = \cos^2\alpha\,.\,u^2 \quad \text{or} \quad v_1^2\tan^2\alpha = u^2\sin^2\alpha \quad \text{or} \quad v_1^2 = u^2\cos^2\alpha$$

or $$v_1 = u\cos\alpha \quad \text{...(viii)}$$

This is the velocity when particle comes back to the point of projection.
Equation (vi) may be written as

$$\frac{dv}{dt} = \frac{g}{u^2}(u^2 - v^2\tan^2\alpha)$$

or $$dt = \frac{u^2}{g}\cdot\frac{dv}{u^2 - v^2\tan^2\alpha} \quad \text{...(ix)}$$

Let t_2 be the time taken to reach the point A. Then integrating (ix) between the limits $t = 0$ to $t = t_2$ and $v = 0$ to $v = u\cos\alpha$, we get

$$[t]_0^{t_2} = \frac{u^2}{g}\int_0^{u\cos\alpha}\frac{dv}{u^2 - u^2\tan^2\alpha}$$

or $$t_2 = \frac{u^2}{g}\cdot\frac{1}{2u\tan\alpha}\left[\log\frac{u + v\tan\alpha}{u - v\tan\alpha}\right]_0^{u\cos\alpha}$$

$$= \frac{u}{2g\tan\alpha}\log\frac{u + u\sin\alpha}{u - u\sin\alpha} = \frac{u}{2g\tan\alpha}\log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

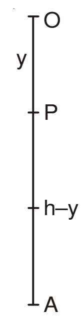
$$= \frac{u}{2g \tan \alpha} \log \frac{1 - \sin^2 \alpha}{(1 - \sin \alpha)^2} = \frac{u}{2g \tan \alpha} \log \frac{\cos^2 \alpha}{(1 - \sin \alpha)^2}$$

$$= \frac{u}{g \tan \alpha} \log \left(\frac{\cos \alpha}{1 - \sin \alpha} \right) \qquad \text{...(x)}$$

$\therefore$ Required time $= t_1 + t_2$

$$= \frac{u}{g \tan \alpha} \left[\alpha + \log \frac{\cos \alpha}{1 - \sin \alpha} \right] \qquad \text{[from (v) and (x)]}$$

3. *Two particles move in a medium whose resistance varies as square of the velocity. One is let fall from a height h and the other projected upwards at the same instant with initial velocity sufficient to carry it to a height h. Show that the particles meet at a depth y below the highest point given by* $\cosh \beta \cos (\alpha - \beta) = 1$, *where* $gy = V^2 \log \cosh \beta$ *and* $gh = V^2 \sec \alpha, V$ *being the terminal velocity.*



Solution. Let the particles meet at P after time t at a depth y below O.

Then from (vi) of 5.3, we get

$$y = \frac{V^2}{g} \log \cosh \frac{g}{V} t$$

$$\therefore \qquad gy = V^2 \log \cosh \beta \qquad \text{...(i)}$$

where $\beta = \frac{g}{V} t$.

Again, if a particle be projected upwards with velocity u, then from (iv) of 5.3.1, we have

$$\frac{2g}{V^2} x = \log \frac{V^2 + u^2}{V^2 + v^2} \qquad \text{...(ii)}$$

If h be the greatest height, then $v = 0$ and $x = h$.

$$\therefore \qquad \frac{2g}{V^2} h = \left(\frac{V^2 + u^2}{V^2} \right) = \log \left(1 + \frac{u^2}{V^2} \right) \qquad \text{...(iii)}$$

Let $\frac{u}{V} = \tan \alpha$, then

$$\frac{2g}{V^2} h = \log (1 + \tan^2 \alpha) = \log \sec^2 \alpha = 2 \log \sec \alpha$$

$$\therefore \qquad gh = V^2 \log \sec \alpha \qquad \text{...(iv)}$$

Let t be the time when the velocity is v at P where $x = OP = h - y$, then, since this t is same as in (i) as the particle is projected at the same time when the other falls from (v) of 5.3.1, we have

$$t = \frac{V}{g} \left(\tan^{-1} \frac{u}{V} - \tan^{-1} \frac{v}{V} \right) \qquad \text{...(v)}$$

and
$$\frac{2g}{V^2} (h - y) = \log \frac{V^2 + u^2}{V^2 + v^2} \qquad \text{by (ii)}$$

$$\therefore \qquad \frac{2gh}{V} - \frac{2g}{V^2} y = \log \left(1 + \frac{u^2}{V^2} \right) - \log \left(1 + \frac{v^2}{V^2} \right)$$

or
$$\frac{2g}{V^2} y = \log \left(1 + \frac{v^2}{V^2} \right), \qquad \text{by (iii)}$$

or $$2 \log \cosh \beta = \log \left(1 + \frac{v^2}{V^2}\right), \quad \text{by (i)}$$

or $$\cosh^2 \beta = 1 + \frac{v^2}{V^2} \quad \text{or} \quad \cosh^2 \beta - 1 = \frac{v^2}{V^2}$$

or $$\sinh \beta = \frac{v}{V} \quad \text{...(vi)}$$

Now from (v), putting the values of (i) and (vi), we get

$$\beta = \tan^{-1} \tan \alpha - \tan^{-1} \sinh \beta$$

or $$\tan^{-1} (\sinh \beta) = (\alpha - \beta)$$

$\therefore$ $$\sinh \beta = \tan (\alpha - \beta)$$

$\therefore$ $$\cos (\alpha - \beta) = \frac{1}{\sqrt{[1 + \tan^2 (\alpha - \beta)]}} = \frac{1}{\sqrt{(1 + \sinh^2 \beta)}} = \frac{1}{\cosh \beta}$$

$\therefore$ $$\cosh \beta \cos (\alpha - \beta) = 1 \quad \text{...(vii)}$$

Results (i) and (vii) prove the results.

4. *A particle is projected wtih velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is* μ *times the cube of the velocity. Show that the distance it has described in time t is* $\frac{1}{\mu V}[\sqrt{(1 + 2\mu t V^2)} - 1]$ *and that its velocity is* $\frac{V}{\sqrt{(1 + 2\mu t V^2)}}$.

Solution. In this case, since the particle is moving in a horizontal plane, hence its weight mg will not act. Hence, the only force acting on the particle will be that due to resistance and equal to $-m\mu v^2$.

The equation of motion of the particle will be

$$m \frac{dv}{dt} = -m\mu v^2 \quad \text{or} \quad -\left(\frac{dv}{v^2}\right) = \mu \, dt$$

Integrating, we get

$$\frac{1}{2v^2} = \mu t + A, \quad \text{where } A \text{ is constant of integration.}$$

Initially, when $t = 0$, $v = V$, $\therefore$ $A = \frac{1}{2V^2}$

$\therefore$ $$\frac{1}{2v^2} = \mu t + \frac{1}{2V^2} \quad \text{or} \quad \frac{1}{v^2} = \frac{2\mu t V^2 + 1}{V^2}$$

or $$v = \frac{V}{\sqrt{(1 + 2\mu t V^2)}} \quad \text{...(i)}$$

Let x be the distance traversed by the particle in time t. Then from equation (i), we have

$$\frac{dx}{dt} = \frac{V}{\sqrt{(1 + 2\mu t V^2)}} \quad \text{or} \quad dx = \frac{V dt}{\sqrt{(1 + 2\mu t V^2)}}$$

integrating :

$$x = \frac{V (1 + 2\mu t V^2)^{1/2}}{2\mu V^2 \times \frac{1}{2}} + B,$$

where B is constant of integration

or $$x = \frac{1}{\mu V}\sqrt{(1+2\mu t V^2)} + B \qquad \text{...(ii)}$$

Initially, when $t = 0,\ x = 0\ \therefore\ B = -\frac{1}{\mu V}$

Hence, equation (ii) becomes

$$x = \frac{1}{\mu V}\sqrt{(1+2\mu t V^2)} - \frac{1}{\mu V}$$

or $$x = \frac{1}{\mu V}[\sqrt{(1+2\mu t V^2)} - 1] \qquad \text{...(iii)}$$

5. *A particle of mass m is projected vertically under gravity; the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is*

$$\left(\frac{V^2}{g}\right)[\lambda - \log(1-\lambda)]$$

where V is the terminal velocity of the particle and λV *is the initial velocity.*

Solution. In this case the particle is projected under gravity and hence the equation of motion will be

$$mv\frac{dv}{dx} = -mg - mkv$$

or $$v\frac{dy}{dx} = -\left(1+\frac{k}{g}v\right) \qquad \text{...(i)}$$

To find the terminal velocity V, the equation of motion in the downwards direction will be

$$mv\left(\frac{dv}{dx}\right) = mg - mkv \qquad \text{...(ii)}$$

We know that the terminal velocity of the particle is that velocity for which its downwards acceleration is zero. Hence, if V be the terminal velocity, then from (ii), we have

$$mg - mkV = 0 \quad \text{or} \quad V = \frac{g}{k} \qquad \text{...(iii)}$$

Hence, (i) becomes

$$v\frac{dv}{dx} = -g\left(1+\frac{v}{V}\right) = -\frac{g}{V}(V+v)$$

or $$\frac{vdV}{V+v} = -\frac{g}{V}dx \quad \text{or} \quad \frac{(v+V)-V}{v+V}dv = -\frac{g}{V}dx$$

or $$\left\{1 - \frac{V}{v+V}\right\}dv = -\frac{g}{V}dx$$

Integrating, we get

$$v - V\log(V+v) = -\frac{gx}{V} + A \qquad \text{...(iv)}$$

Initially, when $x = 0,\ v = \lambda V$

$\therefore$ $$A = \lambda V - V\log(V+\lambda V)$$

Substituting this value of A in (iv), we get

$$v - V\log(V+v) = -\frac{gx}{V} + \lambda V - V\log(V+\lambda V)$$

or $$\frac{gx}{V} = \lambda V - V\log\{V(1+\lambda)\} - v + V\log(V+v)$$

$$= \lambda V - V\log V - V\log(1+\lambda) - v + V\log(V+v) \qquad \text{...(v)}$$

Let h be the greatest height attained by the particle. Then at $x = h,\ v = 0$.

Hence, from equation (v), we get

$$\frac{gh}{V} = \lambda V - V \log V - V \log (1+\lambda) - 0 + V \log V$$

or
$$\frac{gh}{V} = \lambda V - V \log (1+\lambda)$$

or
$$h = \left(\frac{V^2}{g}\right)[\lambda - \log (1+\lambda)].$$

EXERCISES

1. A particle is projected vertically upwards with velocity V and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity V' with which the particle returns to the point of projection is given by

$$\frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}$$

2. A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is

$$\left(\frac{1}{k}\right) \log \cosh \{t \sqrt{(gk)}\}$$

If the particle were ascending, show that at any instant the distance from the highest point of its path is

$$\left(\frac{1}{k}\right) \log \sec \{t \sqrt{(gk)}\},$$

where t now denotes the time it will take to reach its highest point.

3. A heavy particle is projected upwards in a medium the resistance of which varies as the square of the velocity. It has a kinetic energy K in its upwards path at a given point, when it passes through the same point on the way down, show that its loss of energy is $\dfrac{K^2}{K+K'}$, where K' is the limit to which energy approaches in its downwards course.

4. A particle moves from rest at a distance a from a point O under the action of a force to O equal to μ times the distance per unit of mass; if the resisatnce of the medium in which it moves be k times the square of the velocity per unit mass, show that the square of the velocity, when it is at a distance x from O, is

$$\frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2}[1 - e^{2k(x-a)}]$$

Show also that when it first comes to rest it will be at a distance b given by

$$(1-2bk)\, e^{2bk} = (1+2ak)\, e^{-2ak}$$

5. An attracting force varying as the distance acts on a particle initially at rest at a distance a. Show that if V be the velocity when the particle is at a distance x and v' the velocity when the resistance of air is taken into account, then $v' = V\left[1 - \dfrac{k(2a+x)(a-x)}{3(a+x)}\right]$ nearly, the resistance being k times the square of the velocity, k being very small.

6. A particle moving in a straight line is subjected to a resistance kv^3, where v is the velocity. Show that if v is the velocity at time t, when the distance is s,

$$v = \frac{u}{1+uks}; \quad t = \left(\frac{s}{u}\right) + \frac{1}{2} ks^2$$

7. A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{gm^2}{\mu^2}\left[e^{-(\mu/m)t}+\frac{\mu t}{m}-1\right]$$

8. If the resistance vary as the fourth power of the velocity, the energy of m lbs. at a depth x below the highest point when moving in a vertical line under gravity will be $E\tan\left(\frac{(mgx)}{E}\right)$ when rising and $E\tan h\left(\frac{mgx}{E}\right)$ when falling, where E is the terminal energy in the medium.

9. A heavy particle is projected vertically upwards with a velocity U in a medium, the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.

$$\left[\textbf{Ans. } h=\frac{V^2}{6g}\log\frac{U^2+V^2-UV}{(U+V)^2}+\frac{\pi V^2\sqrt{3}}{18g}+\frac{V^2\sqrt{3}}{3g}\tan^{-1}\left(\frac{3U-V}{V\sqrt{3}}\right)\right]$$

10. A heavy particle is projected in a resisting medium, the resistance varying as velocity. If v_1 and v_2 are its velocities at any point in its upward and downwards path and t the interval between its passage through this point; prove that

$$v_1+v_2=gt,\ \ V-v_2=(V+v_1)\,.\,e^{gt/V}$$

where V is the terminal velocity.

11. A particle falls from rest under gravity through a distance x in a medium whose resistance varies as square of the velocity. If v be the velocity actually acquired by it. v_0 the velocity it would have acquired, had there been no resisting medium and V the terminal velocity, show that

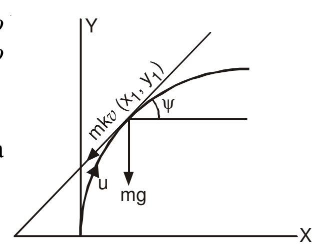
$$\frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots$$

12. A particle of mass m falls from rest at a distance a from the centre of the earth, the motion meeting with a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity, show that the kinetic energy at a distance x from the centre is $mgr^2\left[\frac{1}{x}-\frac{1}{a}+2\mu\left(1-\frac{x}{a}\right)-2\mu\log\left(\frac{a}{x}\right)\right]$ the square of μ being neglected and r the radius of the earth.

13. A particle is projected in a resisting medium whose resistance varies as $(\text{velocity})^n$ and it comes to rest after describing a distance h in time t, find the values of h and t_1 and show that h is finite to $n<2$, and infinite if $n=$ or >2, whilst t is finite if $n<1$, but infinite if $n=$ or >1.

5.6 MOTION OF PROJECTILES IN A RESISTING MEDIUM

A particle is projected under gravity and a resistance equal to mk (velocity) with a velocity u at an angle α to the horizon; to discuss the motion.



Let the particle be projected from O with velocity u at an angle α to the horizon. Let O be the origin. Let $P(x, y)$ be its position after a time t. The forces acting on the particle at P will be :

(i) weight of the particle mg, acting vertically downwards, and

(ii) the resistance force mkv along the tangent in the direction PT, We know that acceleration along x and y axes are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ respectively. Hence, equations of motions along coordinate axes will be

$$m\frac{d^2x}{dt^2} = -mkv\cos\psi$$

or $$\frac{d^2x}{dt^2} = -k\frac{ds}{dt}\cdot\frac{dx}{ds} \qquad \left[\because \cos\psi = \frac{dx}{ds}\right]$$

or $$\frac{d^2x}{dt^2} = -k\frac{dx}{dt}$$

or $$\frac{d^2x/dt^2}{dx/dt} = -k \qquad \text{...(i)}$$

and $$m\frac{d^2y}{dt^2} = mkv\sin\psi - mg$$

or $$\frac{d^2y}{dt^2} = -k\cdot\frac{ds}{dt}\cdot\frac{dy}{ds} - g, \qquad [\because \sin\psi = dy/ds]$$

or $$\frac{d^2y}{dt^2} = -k\frac{dy}{dt} - g \qquad \text{...(ii)}$$

Integrating (i), we get

$$\log\left(\frac{dx}{dt}\right) = -kt + A,$$

where A is constant of integration.

Initially, when $t = 0$; $\frac{dx}{dt} =$ initial horizontal component of velocity $= u\cos\alpha$

$\therefore$ $$A = \log(u\cos\alpha)$$

$\therefore$ $$\log\left(\frac{dx}{dt}\right) = -kt + \log(u\cos\alpha)$$

or $$\log\left(\frac{dx/dt}{u\cos\alpha}\right) = -kt$$

or $$\frac{dx}{dt} = u\cos\alpha\,.\,e^{-kt} \qquad \text{...(iii)}$$

we can write equation (ii) as

$$\left[\frac{k\,d^2y/dt^2}{k\,dy/dt + g}\right] = -k$$

Integrating, we get

$$\log\left(k\frac{dv}{dt} + g\right) = -kt + B,$$

where B is constant of integration.

Initialy, when $t = 0$, $\frac{dy}{dt} = u\sin\alpha$

$\therefore$ $$B = \log(ku\sin\alpha + g)$$

$\therefore$ $$\log\left(\frac{kdy/dt + g}{ku\sin\alpha + g}\right) = -kt$$

or $$k\frac{dy}{dt} + g = (ku\sin\alpha + g)\,e^{-kt} \qquad \text{...(iv)}$$

Equations (iii) and (iv) give the **horizontal and vertical components** of velocity of the particle at any time t.

Now, we can write equation (iii) as

$$dx = u\cos\alpha \,.\, e^{-kt}\,dt$$

Integrating, $$x = -\frac{u}{k}\cos\alpha\, e^{-kt} + C,$$

where C is constant of integration.

Initially, at O, $x = 0, t = 0,$

$$\therefore \quad C = \frac{u}{k}\cos\alpha$$

$$\therefore \quad x = \frac{u}{k}\cos\alpha\,(1 - e^{-kt}) \qquad \text{...(v)}$$

Equation (iv) may be written as

$$kdy + gdt = (ku\sin\alpha + g)\,e^{-kt}\,dt$$

Integrating, we get

$$ky + gt = -\frac{1}{k}(ku\sin\alpha + g)e^{-kt} + D,$$

where D is constant of integration,

Initially, at 0, $y = 0, t = 0$

$$\therefore \quad D = \frac{1}{k}(ku\sin\alpha + g)$$

$$\therefore \quad ky + gt = \frac{g + ku\sin\alpha}{k}(1 - e^{-kt}) \qquad \text{...(vi)}$$

Equations (v) and (vi) represent the horizontal and vertical distances travelled by the particle in time t. These equations are called the **parametric equations of the trajectory.**

To obtain Cartesian equation, we will eliminate t between these equations. From (v)

$$1 - e^{-kt} = \frac{kx}{u\cos\alpha}, \quad \therefore \quad e^{-kt} = 1 - \frac{kx}{u\cos\alpha}$$

or $$t = -\frac{1}{k}\log\left(1 - \frac{kx}{u\cos\alpha}\right)$$

Substituting these values of t and e^{-kt} in (vi), we get

$$ky - \frac{g}{k}\log\left(1 - \frac{kx}{u\cos\alpha}\right) = \frac{g + ku\sin\alpha}{k}\cdot\frac{kx}{u\cos\alpha}$$

or $$y = \frac{g}{k^2}\log\left(1 - \frac{kx}{u\cos\alpha}\right) + \frac{x}{ku\cos\alpha}(g + ku\sin\alpha) \qquad \text{...(vii)}$$

This is the **Cartesian equation of path of the particle.**

Deductions

(a) Range on the horizontal Plane

Let the required range be R. The coordinates of the point where the particle strikes the horizontal plane are $(R, 0)$. This point $(R, 0)$ must satisfy the equation (vii) of trajectory. Thus,

$$0 = \frac{g}{k^2}\log\left(1 - \frac{kR}{u\cos\alpha}\right) + \frac{R}{ku\cos\alpha}(g + ku\sin\alpha) \qquad \text{...(viii)}$$

Approximate value of R, if k is small.

From equation (viii), we get

$$0 = \frac{g}{k^2}\left[-\frac{kR}{u\cos\alpha} - \frac{k^2R^2}{2u^2\cos^2\alpha} - \frac{k^3R^3}{3u^3\cos^3\alpha} - \ldots\right] + \frac{R}{ku\cos\alpha}(g + ku\sin\alpha)$$

Neglecting $K^3, \ldots$ etc., we have

$$-g - \frac{kRg}{2u\cos\alpha} - \frac{k^2R^2g}{3u^2\cos^2\alpha} + g + ku\sin\alpha = 0$$

or $$-\frac{kRg}{2u\cos^2\alpha} - \frac{k^2R^2g}{3u^2\cos^2\alpha} + ku\sin\alpha = 0$$

or $$R = \frac{2u^2\sin\alpha\cos\alpha}{g} - \frac{2kR^2}{3u\cos\alpha} \qquad \ldots(ix)$$

$\therefore$ $$R = \frac{2u^2\sin\alpha\cos\alpha}{g} \text{ for first approximation}$$

Substituting this value of R in right hand side of (ix), we get

$$R = \frac{2u^2\sin\alpha\cos\alpha}{g} - \frac{2k}{3u\cos\alpha}\left\{\frac{2u^2\sin\alpha\cos\alpha}{g}\right\}^2$$

$$= \frac{2u^2\sin\alpha\cos\alpha}{g} - \frac{8}{3}\frac{ku^3\cos\alpha\sin^2\alpha}{g^2} \qquad \ldots(x)$$

This is value of R up to second apporoximation.

(b) Time of Fight

Let T be the time of flight, *i.e.,* the time required to strike the horizontal plane through the point of projection. Thus, during the time T the particle will describe zero vertical distance. Hence, putting $y = 0, t = T$ in equation (vi), we get $y = 0, t = T$

$$gT = \frac{g + ku\sin\alpha}{k}(1 - e^{-kT})$$

or $$gT = \frac{1}{k}(g + ku\sin\alpha)\left[1 - \left\{1 - kT + \frac{k^2T^2}{2!} - \frac{k^3T^3}{3!} + \ldots\right\}\right]$$

or $$gT = (g + ku\sin\alpha)\left[T - \frac{kT^2}{2} + \frac{k^2T^3}{6} + \ldots\right]$$

or $$gT = gT + kTu\sin\alpha - \frac{1}{2}gkT^2 - \frac{1}{2}k^2T^2 u\sin\alpha + \frac{1}{6}gk^2T^3 + \ldots$$

or $$0 = \frac{1}{2}kt\,[2u\sin\alpha - gT - kT\,u\sin\alpha + \frac{1}{3}gkT^2 + \ldots]$$

or $$T = \frac{2u\sin\alpha}{g} + \frac{k}{g}\left(\frac{gT^2}{3} - Tu\sin\alpha\right) \qquad \ldots(xi)$$

[neglecting k^2, k^3 etc.]

$\therefore$ $$T = \frac{2u\sin\alpha}{g}, \text{ upto first approximation.}$$

Substituting this value of T in right hand side of equation (xi), we have the value of T up to second approximation as

$$T = \frac{2u\sin\alpha}{g} + \frac{k}{g}\left[\frac{g}{3}\cdot\frac{4u^2\sin^2\alpha}{g^2} - \frac{2u\sin\alpha}{g}\cdot u\sin\alpha\right]$$

$$= \frac{2u\sin\alpha}{g} - \frac{2}{3}\cdot\frac{k^2\sin^2\alpha}{g^2}. \qquad \ldots(xii)$$

(c) Greatest height and time to reach this height

Let h be greatest height attained by the particle and t_1 the time for the same. At the highest point velocity will be horizontal and hence the vertical component of velocity, *i.e.*, $\frac{dy}{dt}$ will be zero. Hence, by putting $\frac{dy}{dt} = 0$ and $t = t_1$ in equation (iv), we get

$$g = (ku \sin\alpha + g)\, e^{-kt_1}$$

or $$e^{kt_1} = \frac{1}{g}(ku \sin\alpha + g)$$

$\therefore$ $$t_1 = \frac{1}{k}\log\left(\frac{k}{g}u\sin\alpha\right) \quad \text{...(xiii)}$$

Now, let us put $y = h, t = t_1$ in (vi), we get

$$kh + gt_1 = \frac{g + ku\sin\alpha}{k}(1 - e^{-kt})$$

or $$kh + \frac{g}{k}\log\left(1 + \frac{k}{g}u\sin\alpha\right) = \frac{g + ku\sin\alpha}{k} - \frac{1}{k}g \quad \text{[by (xiii)]}$$

or $$h = \frac{1}{k}u\sin\alpha - \frac{g}{k^2}\log\left(k + \frac{k}{g}u\sin\alpha\right) \quad \text{...(xiv)}$$

(d) Time to greatest height is less than half the time of flight.

Now, we want to prove that $t_1 < \frac{1}{2}T$, *i.e.*, $2t_1 - T < 0$.

Now, $$2t_1 - T = \frac{2}{k}\log\left(1 + \frac{k}{g}u\sin\alpha\right) - \left(\frac{2u\sin\alpha}{g} - \frac{2}{3}\frac{ku^2\sin^2\alpha}{g^2}\right)$$

[from (xii) and (xiii)]

$$= \frac{2}{k}\left[\frac{k}{g}u\sin\alpha - \frac{1}{2}\frac{h^2}{g^2}u^2\sin^2\alpha\right] - \left(\frac{2u\sin\alpha}{g} - \frac{2}{3}\frac{ku^2\sin^2\alpha}{g^2}\right)$$

$$= -\frac{ku^2\sin^2\alpha}{g^2} - \frac{2ku^2\sin^2\alpha}{3g^2} < 0.$$

5.6 MOTION OF A PROJECTILE IN A RESISTING MEDIUM, RESISTANCE VARYING AS SQUARE OF VELOCITY

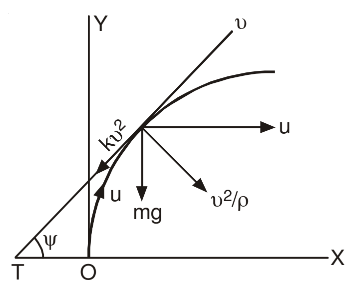
Let P be the position of particle at time t and u b the horizontal component of velocity, *i.e.*,

$$u = v\cos\psi$$

Equations of motion in normal and horizontal directions are

$$\frac{v^2}{\rho} = g\cos\psi \quad \text{...(i)}$$

$$\frac{du}{dt} = -kv^2\cos\psi \quad \text{...(ii)}$$



But $\rho = -\frac{ds}{d\psi}$,

negative sign is because ψ is decreasing when s increasing

$\therefore$ $$g\cos\psi = -v^2\frac{d\psi}{ds}$$

$$= -v\cdot\frac{ds}{dt}\cdot\frac{d\psi}{ds} = -v\frac{d\psi}{dt} \quad \text{...(iii)}$$

From (ii), $$\frac{du}{d\psi}\cdot\frac{d\psi}{dt} = -kv^2 \cos\psi$$

or $$\frac{du}{u^3} = \frac{k}{g}\sec^3\psi\, d\psi \qquad [\because\ u = v\cos\psi]$$

Integrating, we get

$$-\frac{1}{2u^2} = \frac{k}{2g}[\sec\psi\tan\psi + \log(\sec\psi + \tan\psi)] + A \qquad \text{...(iv)}$$

Initially, $u = u_0$ and $\psi = \alpha$

$$\therefore \quad A = -\frac{1}{2u_0^2} = -\frac{k}{2g}[\sec\alpha\tan\alpha + \log(\sec\alpha + \tan\alpha)]$$

$$\therefore \quad \frac{1}{u^2} = \frac{1}{u_0^2} + \frac{k}{2g}[\sec\alpha\tan\alpha - \sec\psi\tan\psi] + \frac{k}{2g}\log\left(\frac{\sec\alpha + \tan\alpha}{\sec\psi + \tan\psi}\right) \qquad \text{...(v)}$$

Also, $$\frac{du}{dt} = -kv^2\cos\psi = -kvu = -ku\frac{ds}{dt}$$

$$\therefore \quad \frac{du}{u} = -k\, ds$$

Integrating,

$$u = u_0 e^{-ks} \qquad [\because\ u = u_0,\ s = 0] \qquad \text{...(vi)}$$

Putting this value in (v), we have

$$e^{2ks} = 1 + \frac{u_0^2 k}{2g}[\sec\alpha\tan\alpha - \sec\psi\tan\psi] + \frac{u_0^2 k}{2g}\log\left(\frac{\sec\alpha + \tan\alpha}{\sec\psi + \tan\psi}\right) \qquad \text{...(vii)}$$

This is kinetic equation of path of projectile.

Remarks : (i) From (v) it is clear that as $s \to \infty, u \to 0$ *i.e.*, the particle moves vertically ultimately.

The terminal velocity $V = \sqrt{(g/k)}$

(ii) From equation (iv) we have

$$\frac{\sec^3\psi}{\rho} = -k[\tan\psi\sec\psi + \log(\sec\psi + \tan\psi)] + \frac{1}{\rho_0} \qquad \text{...(viii)}$$

$$\left[\because \rho = \frac{v^2}{g}\sec\psi = \frac{u^2\sec^2\psi}{g}\right]$$

If $\rho = \rho_0, \psi = 0$ at the highest point.

5.7 TRAJECTORY IN A RESISTING MEDIUM WHEN RESISTANCE VARIES AS (VELOCITY)^n

If a particle describes a trajectory under gravity in a resisting medium whose resistance is equal to mk (velcoity)^n, to find the motion.

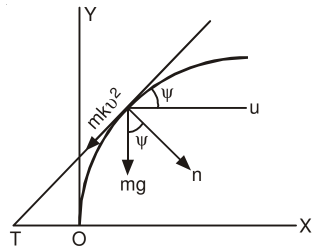
Let the particle be projected from O. Let P be the position of particle at any time t. Let at P, v be its velocity and u the horizontal component of velocity, so that

$$u = v\cos\psi \qquad \text{...(i)}$$

Equations of motion along the normal and parallel to x-axis are

$$\frac{mv^2}{\rho} = mg\cos\psi$$

or $$\frac{v^2}{\rho} = g\cos\psi \qquad \text{...(ii)}$$



and $$m\frac{du}{dt} = -mkv^n \cos\psi$$

or $$\frac{du}{dt} = -kv^n \cos\psi \quad ...(iii)$$

We have $\rho = -\frac{ds}{d\psi}$ because ψ decreases as s increases.

Putting this value of ρ in (ii), we get

$$v^2\left(-\frac{d\psi}{ds}\right) = g\cos\psi \quad \text{or} \quad -v\frac{ds}{dt}\cdot\frac{d\psi}{ds} = g\cos\psi$$

or $$v\frac{d\psi}{dt} = -g\cos\psi \quad ...(iv)$$

From (iii), $$\frac{du}{d\psi}\cdot\frac{d\psi}{dt} = -kv^n\cos\psi$$

or $$\frac{du}{d\psi}\cdot\left(-\frac{g\cos\psi}{v}\right) = -kv^n\cos\psi \quad \text{[from (vi)]}$$

or $$\frac{du}{d\psi}\left(-\frac{g\cos\psi}{v}\right) = \frac{k}{g}(u\sec\psi)^{n+1} \quad \text{[from (i)]}$$

or $$-n\frac{du}{u^{n-1}} = \frac{nk}{g}\sec^{n+1}\psi\, d\psi$$

Integrating we get $\frac{1}{u^n} = -\frac{nk}{g}\int\sec^{n+1}\psi\, d\psi + A$

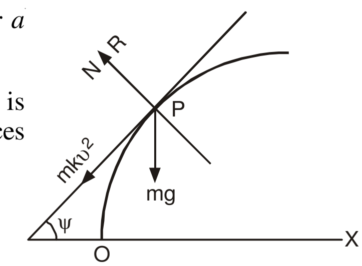
where A is the constant of integration

or $$\frac{1}{(v\cos\psi)} = -\frac{nk}{g}\int\sec^{n+1}\psi\, d\psi + A$$

This gives velocity v at any position, constant A being determined by initial conditions.

5.8. MOTION ON A SMOOTH CURVE

A bead moves on a smooth wire in a vertical plane under a resistance equal to k (velocity)^2; to find the motion.



Let P the position of the bead at any time t. Since bead is moving on the wire, hence normal reaction will also act. The forces acting on the bead at P will be

(i) weight mg of the bead acting vertically downwards.

(ii) The resistance force mkv^2 along the tangent at P, and

(iii) Normal reaction R of the wrie along PN.

The equations of motion along the tangent and normal at P are

$$mv\frac{dv}{ds} = mg\sin\psi - mkv^2 \quad \text{or} \quad v\frac{dv}{ds} + kv^2 = g\sin\psi$$

or $$\frac{1}{2}\frac{dv^2}{ds} + dv^2 = g\sin\psi \quad \text{or} \quad \frac{dv^2}{d\psi}\frac{d\psi}{ds} + 2kv^2 = 2g\sin\psi$$

or $$\frac{dv^2}{d\psi} + 2k\rho v^2 = 2g\rho\sin\psi \quad \left[\because \rho = \frac{ds}{d\psi}\right] \quad ...(1)$$

And other equation (along normal) is :

$$\frac{v^2}{\rho} = g\cos\psi - R \quad ...(2)$$

If the equation to the curve is given, ρ can be determined. The equation can be integrated after substituting for ρ.

Particle Case : If the curve is a circle of radius a, then $\rho = a$. Equation (i) becomes

$$\frac{dv^2}{d\psi} + 2kav^2 = 2ga \sin\psi \qquad ...(3)$$

This is a linear differential equation. Its integrating factor

$$= e^{\int 2ak\, d\psi} = e^{2ak\psi}$$

$\therefore$ Solution of (3) is

$$v^2 e^{2ak\psi} = 2ga \int e^{2ak\psi} \sin\psi\, d\psi + C = 2ag \frac{e^{2ak\psi}}{1+4a^2k^2}[2ak \sin\psi - \cos\psi] + C$$

or
$$v^2 = \frac{2ag}{1+4a^2k^2}[2ak\sin\psi - \cos\psi] + Ce^{ak\psi} \qquad ...(4)$$

Equation (4) gives velocity of the bead at any point ψ.

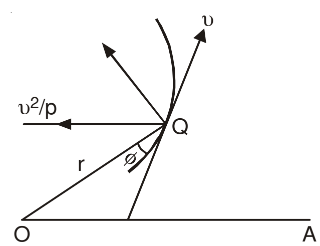
5.9. ORBITS IN A RESISTING MEDIUM (CENTRAL FORCE)

(a) Resistance varying as velocity.

Let P be the central force and kv the resistance, then equations of motion along normal and transverse directions will be

$$\frac{v^2}{\rho} = P\sin\psi \qquad ...(i)$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -kv\sin\phi \qquad ...(ii)$$



Putting $vp = h = r^2\frac{d\theta}{dt}$, which is not constant here, we get

$$\frac{dh}{dt} = -kvr\sin\phi = -kvp = -kh.$$

Integrating, we get

$$h = h_0 e^{-kt}, \qquad \text{where } h = h_0, \text{ when } t = 0$$

Hence,
$$P = \frac{v^2}{\rho\sin\phi} = \frac{v^2}{r\frac{dr}{dp}\cdot\frac{p}{r}} = \frac{h^2}{p^3}\frac{dp}{dr} = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= h_0^2u^2e^{-kt}\left(u + \frac{d^2u}{d\theta^2}\right) \qquad ...(iii)$$

(b) Resistance varying as square of velocity

The equations of motion will be

$$\frac{v^2}{\rho} = P\sin\phi,$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -kv^2\sin\phi$$

Putting $up = h = r^2\frac{d\theta}{dt}$, we get

$$\frac{dh}{dt} = -kv^2 r \sin\phi = -kv^2 p = -kvh = -kh\frac{ds}{dt}$$

$$\therefore \quad \frac{dh}{h} = -k\,ds$$

Integrating, $h = h_0 e^{-ks}$, where $h = h_0$ where $s = 0$.

Hence,
$$P = h^2 u^2 \left(u + \frac{d^2u}{d\theta^2}\right)$$
$$= h_0^2 u^2 e^{-2ks}\left(u + \frac{d^2u}{d\theta^2}\right) \quad \text{...(iv)}$$

(c) If the resistance be *R*.

The equations of motion wil be

$$\frac{v^2}{\rho} = P\sin\phi, \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -R\sin\phi$$

$$\therefore \quad v^2 = P\rho\sin\phi = P\,.\,r\frac{dr}{dp}\,.\,\frac{p}{r} = P\rho\frac{dr}{dp}.$$

$$\therefore \quad h^2 = v^2 p^2 = Pp^3\frac{dr}{dp} \quad \text{...(v)}$$

Hence,
$$R = \frac{1}{r\sin\phi}\frac{dh}{dt} = -\frac{1}{p}\frac{dh}{dt}$$
$$= -\frac{1}{p}\,.\,\frac{dh}{ds}\,.\,\frac{ds}{dt} = -\frac{v}{p}\frac{dh}{ds} = \frac{h}{p^2}\frac{dh}{ds}$$
$$= -\frac{1}{2p^2}\frac{dh^2}{ds} = \frac{1}{2p^2}\,.\,\frac{d}{ds}\left(Pp^3\frac{dr}{dp}\right) \quad \text{...(vi)}$$

EXAMPLES

1. *If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is maximum, show that the angle* α *which the direction of projection makes with the vertical is given by*

$$\frac{\mu(1+\mu\cos\alpha)}{\mu+\cos\alpha} = \log(1+\mu\sec\alpha),$$

where μ *is the ratio of the velocity of projection to the terminal velocity.*

Solution. In the present case resistance = kv.

Let the particle be falling under gravity. Hence, downward acceleration $= g - kv$. When this acceleration $= g - kv$. When this acceleration will be zero, the velocity will be the terminal velocity, *i.e.,*

$$g - kV = 0 \quad \text{or} \quad V = \frac{g}{k} \quad \text{...(i)}$$

Equations of motion in horizontal and vertical directions will be

$$\frac{d^2x}{dt^2} = -kv\cos\psi = -k\frac{ds}{dt}\,.\,\frac{dx}{ds} \qquad \left(\because \cos\psi = \frac{dx}{ds}\right)$$
$$= -k\frac{dx}{dt} \quad \text{..(ii)}$$

and $\dfrac{d^2y}{dt^2} = -kv\sin\psi - g = -k\,.\,\dfrac{ds}{dt}\,.\,\dfrac{dy}{ds} - g$

$$= -\left(k\frac{dy}{dt} + g\right) \qquad \left[\because \sin\psi = \frac{dy}{ds}\right] \qquad \text{...(iii)}$$

In this case the direction of projection makes an angle α with the vertical, *i.e.*, $\frac{\pi}{2} - \alpha$ with the horizon.

Then as in 5.7, we get

$$x = \frac{u\cos\left(\frac{\pi}{2} - \alpha\right)}{k}(1 - e^{-kt})$$

$$= \frac{u\sin\alpha}{k}(1 - e^{-kt}) \qquad \text{...(iv)}$$

and
$$ky + gt = \frac{g + ku\sin\left(\frac{\pi}{2} - \alpha\right)}{k}(1 - e^{-kt})$$

$$= \frac{g + ku\cos\alpha}{k}(1 - e^{-kt}) \qquad \text{...(v)}$$

Let R be the range on the horizontal plane. The value of x will be the range of R if $y = 0$. Hence, from (iv) and (v), R will be

$$R = \frac{u\sin\alpha}{k}(1 - e^{-kt}) \qquad \text{...(vi)}$$

where t is given by
$$gt = \frac{ku\cos\alpha + g}{k}(1 - e^{-kt}) \qquad \text{...(vii)}$$

For R to the maximum $\frac{dR}{d\alpha} = 0$. Hence, from (vi), we get

$$\frac{dR}{d\alpha} = \frac{u\cos\alpha}{k}(1 - e^{-kt}) + \frac{u\sin\alpha}{k}e^{-kt}\,.\,k\,.\,\frac{dt}{d\alpha}$$

or
$$0 = \cos\alpha\,(1 - e^{-kt}) + k\sin\alpha\, e^{-kt}\,.\,\frac{dt}{d\alpha} \qquad \text{...(viii)}$$

Differentiating (vii) w.r.t. α, we get

$$g\frac{dt}{d\alpha} = (ku\cos\alpha + g)\,e^{-kt}\frac{dt}{d\alpha} - u\sin\alpha\,(1 - e^{-kt})$$

or
$$\{(ku\cos\alpha + g)\,e^{-kt} - g\}\frac{dt}{d\alpha} = u\sin\,(1 - e^{-kt}) \qquad \text{...(ix)}$$

From equation (viii), we get

$$k\sin\alpha\, e^{-kt}\,.\,\frac{dt}{dx} = -\cos\alpha\,(1 - e^{-kt}) \qquad \text{...(x)}$$

Dividing (ix) by (x), we get

$$\frac{(ku\cos\alpha + g)\,e^{-kt} - g}{k\sin\alpha\,.\,e^{-kt}} = -\frac{u\sin\alpha}{\cos\alpha}$$

or
$$(ku\cos^2\alpha + g\cos\alpha)\,e^{-kt} - g\cos\alpha = -ku\sin^2\alpha\, e^{-kt}$$

or
$$e^{-kt} = \frac{g\cos\alpha}{ku + g\cos\alpha}$$

$\therefore$
$$e^{kt} = \left(\frac{ku + g\cos\alpha}{g\cos\alpha}\right) = \left(\frac{ku}{g\cos\alpha} + 1\right)$$

$\therefore$
$$t = \frac{1}{k}\log\left(1 + \frac{ku}{g\cos\alpha}\right) \qquad \text{...(xii)}$$

Substituting the values of t and e^{-kt} in (vii), we get

$$\frac{g}{k}\log\left(1+\frac{ku}{g\cos\alpha}\right)=\left(u\cos\alpha+\frac{g}{k}\right)\left\{1-\frac{g\cos\alpha}{ku+g\cos\alpha}\right\}$$

or $$\frac{g}{k}\log\left(1+\frac{ku}{g\cos\alpha}\right)=\left(u\cos\alpha+\frac{g}{k}\right)\left(\frac{ku}{g\cos\alpha+ku}\right)$$

or $$V\log\left(1+\frac{u}{V\cos\alpha}\right)=(u\cos\alpha+V)\left\{\frac{u}{\frac{g}{k}\cos\alpha+u}\right\}$$ [from (i)]

or $$V\log(1+\mu\sec\alpha)=V\left(\frac{u}{V}\cos\alpha+1\right)\left[\frac{1}{\frac{V}{u}\cos\alpha\, u}\right]$$ $\left[\because \frac{u}{V}=\mu\right]$

or $$\log(1+\mu\sec\alpha)=(\mu\cos\alpha+1)\left[\frac{1}{\frac{1}{\mu}\cos\alpha+1}\right]=\frac{\mu(1+\mu\cos\alpha)}{\mu+\cos\alpha}$$

2. *If a particle is projected at an angle* α *with a velocity u in a medium whose resistance varies as square of velocity, then prove that*

$$\rho\rho'\cos^3\psi\cos^2\psi'=\rho_0^2$$

where ρ *and* ρ' *are the radii of curvature at two points at equal arcural distances from the vertex and* ψ,ψ' *the incinations to the horizon of the tangents at the these points and* ρ_0 *is the radius of curvature at the vertex.*

Solution. From 5.8, equation (v), we have

$$u=u_0e^{-ks}\quad\text{and}\quad\frac{v^2}{\rho}=g\cos\psi$$ $(v=u\sec\psi)$

$\therefore$ $$\rho=\frac{v^2}{g}\sec\psi=\frac{u^2}{g}\sec^2\psi=\frac{u_0^2}{g}e^{-2ks}\sec^3\psi$$

$\therefore$ $$\rho\cos^3\psi=\frac{u_0^2}{g}e^{-2ks}$$

and $$\rho'\cos^3\psi'=\frac{u_0^2}{g}e^{-2k(-s)}=\frac{u_0^2}{g}e^{2ks}$$

$\therefore$ $$\rho\rho'\cos^3\psi\cos^3\psi'=\frac{u_0^4}{g^2}=\frac{1}{\rho_0^2}$$ $\left[\text{Since }\rho_0=\frac{u_0^2}{g},\text{ if } s=0\right]$

3. *If the resistance of the air to a particle's motion is n times the weight and the particle be projected horizontaly with velocity V, show that its velocity, when moving at an angle* ψ *to the horizontal is*

$$V(1-\sin\psi)^{(n-1)/2}(1+\sin\psi)^{-(n+1)/2}$$

Solution. Equations of motion are

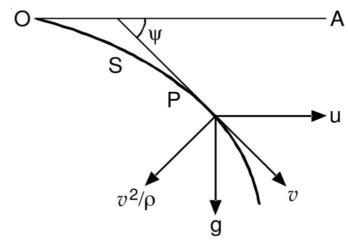
$$\frac{du}{dt}=-ng\cos\phi \qquad \text{...(i)}$$

and $$\frac{v^2}{\rho}=g\cos\psi \qquad \text{...(ii)}$$

and $$\rho = \frac{ds}{d\psi} \quad \text{...(iii)}$$

as s increases when ψ increase.

Also, $u = v \cos\psi$



$\therefore$ $$\frac{du}{d\psi} \cdot \frac{d\psi}{dt} = -n\frac{v^2}{\rho} = -nv \cdot \frac{ds}{dt} \cdot \frac{d\psi}{ds}$$

or $$\frac{du}{d\psi} = \frac{-v}{\cos\psi} = -nu\sec\psi$$

Integrating,

$$\log u = -n\log(\sec\psi + \tan\psi) + A,$$

where A is constant of integration.

Initially, $u = V, \psi = 0, \therefore\ C = \log V$

$\therefore$ $$\log u = -n\log(\sec\psi + \tan\psi) + \log V$$

$\therefore$ $$u = \frac{V}{(\sec\psi + \tan\psi)^n}$$

$\therefore$ $$v = u\sec\psi = \frac{V\cos^{-1}\psi}{(1+\sin\psi)^n} = \frac{V(1-\sin^2\psi)^{n-1/2}}{(1+\sin\psi)}$$

$$= u(1-\sin\psi)^{n-1/2}(1+\sin\psi)^{-(n+1)/2}$$

4. *A shot is fired in atmosphere in which the resistance varies as the cube of the velocity. if f be the retardation when the shot is ascending at an inclination α to the horizon, f_0 when it is moving horizontally and f' when it is descending at an inclination α to the horizon, prove that*

$$\frac{1}{f} + \frac{1}{f'} = \frac{2\cos^2\alpha}{f_0} \text{ and } \frac{1}{f'} - \frac{1}{f} = \frac{2\sin\alpha}{g}(3 - 2\sin^2\alpha)$$

Solution. Let u be the horizontal component of velocity v at the point where tangent to this point makes an angle ψ with the horizontal. Thus,

$$u = v\cos\psi \quad \text{...(i)}$$

Now, when the shot is ascending at an inclination α, then resistance is

$$kv^3 = f$$

Hence, from (i), we get

$$u^2 = \left(\frac{f}{k}\right)\cos^3\psi \quad \text{...(ii)}$$

Equations of motion along the normal and parallel to x-axis will be

$$\frac{v^2}{\rho} = g\cos\psi \quad \text{...(iii)}$$

and $$\frac{du}{dt} = -kv^3\cos\psi \quad \text{...(iv)}$$

Also, $$\rho = -\frac{ds}{d\psi},$$

since in this case ψ decreases as s increases.

From (v) and (iii), we get

or $$v^2\left(-\frac{d\psi}{ds}\right) = -g\cos\psi \quad \text{or} \quad v \cdot \frac{ds}{dt} \cdot \frac{d\psi}{ds} = -g\cos\psi$$

or $$v\frac{d\psi}{dt} = -g\cos\psi \quad \text{...(vi)}$$

From equation (iv), we get

$$\frac{du}{d\psi}\cdot\frac{d\psi}{dt} = -kv^3\cos\psi \quad \text{or} \quad \frac{du}{d\psi} = \frac{k}{g}v^4, \qquad \text{from (vi)}$$

or
$$\frac{du}{d\psi} = \frac{k}{g}u^4\sec^4\psi$$

or
$$\frac{-3du}{u^4} = -\frac{3k}{g}\sec^4\psi$$

Integrating, we get

$$\frac{1}{u^3} = \int -\frac{3k}{g}\sec^2\psi\,(1+\tan^2\psi)\,d\psi + C$$

$$= -\frac{3k}{g}\left(\tan\psi + \frac{1}{3}\tan^3\psi\right) + C,$$

where C is constant of integration.

when $\psi = 0$, *i.e.*, particle is moving horizontally, let $u = u_0$

$\therefore$
$$C = \frac{1}{u_0^3}$$

$\therefore$
$$\frac{1}{u^3} - \frac{1}{u_0^3} = -\frac{3k}{g}\left(\tan\psi + \frac{1}{3}\tan^3\psi\right) \qquad \text{...(vii)}$$

Also, when particle is moving horizontally,

i.e., $\psi = 0,\ f = f_0,\ u = u_0$

Hence from (ii), we get

$$u_0^3 = \frac{f_0}{k} \qquad \text{...(viii)}$$

Substituting values of u^3 and u_0^3 from (ii) and (viii) in (vii), we get

$$\frac{k}{f\cos^3\psi} - \frac{k}{f_0} = -\frac{3k}{g}\left(\tan\psi + \frac{1}{3}\tan^3\psi\right)$$

or
$$\frac{k}{f\cos^3\psi} = \frac{1}{f_0} - \frac{1}{g}\cdot\frac{\sec\psi\,(3-2\sin^2\psi)}{\cos^3\psi} \qquad \text{...(ix)}$$

Now, when $\psi = \alpha$, $f = f$ (given), and when $\psi = -\alpha$, $f = f'$ (given). Thus, equation (ix) provides

$$\frac{1}{f\cos^3\alpha} = \frac{1}{f_0} - \frac{1}{g}\cdot\frac{\sin\alpha\,(3-2\sin^2\alpha)}{\cos^3\alpha} \qquad \text{...(x)}$$

and
$$\frac{1}{f'\cos^2\alpha} = \frac{1}{f'} + \frac{1}{g}\frac{\sin\alpha\,(3-2\sin^2\alpha)}{\cos^3\alpha} \qquad \text{...(xi)}$$

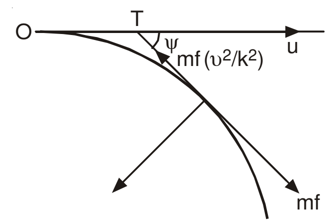
Adding (x) and (xi), we get

$$\frac{1}{f} + \frac{1}{f'} = \frac{2\cos^3\alpha}{f_0}$$

Subtracting (x) from (xi), we get

$$\frac{1}{f'} - \frac{1}{f} = \frac{2\sin\alpha\,(3-2\sin^2\alpha)}{g}$$

5. *If a particle of mass m be acted upon by equal constant forces mf tangentially and normally to the path and if the resistance be mf* $\frac{v^2}{k^2}$, *prove that intrinsic equation of the path is* $k^2\,(e^{2fs/k^2}-1) = u^2\,(e^{2\psi}-1)$, *where u is the velocity of projection.*



Solution. Since the particle is not projected in a vertical plane hence weight of the particle will not be considered.

The equations of motion of the particle along the tangent and normal are

$$mv\frac{dv}{ds} = mf - mf\frac{v^2}{k^2}$$

and $$\frac{mv^2}{\rho} = mf$$

or $$\frac{1}{2}\frac{dv^2}{ds} + f\frac{v^2}{k^2} = f \quad \text{...(i)}$$

and $$\frac{v^2}{f} = \rho = \frac{ds}{d\psi} \quad \text{...(ii)}$$

As ψ increases when s increases. Hence $\rho = +\frac{ds}{d\psi}$.

From (i), $$\frac{dv^2}{ds} + \frac{2f}{k^2}v^2 = 2f.$$

This is linear equation whose integrating factor $= e^{2fs/k^2}$. Hence, its solution will be $v^2.e^{2fs/k^2} = \int 2f.e^{2fs/k^2}.ds + A.$

or $$v^2.e^{2fs/k^2} = k^2.e^{2fs/k^2} + A,$$

where A is the constant of integration.

Initially, $s = 0, v = u, \therefore A = u^2 - k^2.$

$\therefore$ $$v^2 = k - (u^2 - k^2)\,e^{-2fs/k^2}.$$

Hence, by putting the value of v^2 in equation (ii), we get

$$\frac{k^2}{f} + \frac{u^2 - k^2}{f}e^{-2fs/k^2} = \frac{ds}{d\psi}$$

or $$e^{2fs/k^2}.\frac{ds}{d\psi} - \frac{k^2}{f}e^{2fs/k^2} = \frac{u^2 - k^2}{f}$$

Put $$\frac{k^2}{2f}e^{2fs/k^2} = z, \quad \therefore \quad e^{2fs/k^2}.\frac{ds}{d\psi} = \frac{dz}{d\psi}$$

$\therefore$ $$\frac{dz}{d\psi} - 2z = \frac{u^2 - k^2}{f}$$

This is a linear equation whose I.F. $= e^{-2\psi}$. Hence, its solution will be

$$z.e^{-2\psi} = \int \frac{u^2 - k^2}{f}e^{-2\psi}\,d\psi + B$$

or $$\frac{k^2}{2f}.2fs/k^2.e^{-2\psi} = \frac{u^2 - k^2}{-2f}e^{-2\psi} + B,$$

where B is constant of integration.

Initially, when $s = 0, \psi = 0.$

$\therefore$ $$\frac{k^2}{2f} = -\frac{u^2}{2f} + \frac{k^2}{2f} + B, \quad \therefore B = \frac{u^2}{2f}$$

$\therefore$ $$\frac{k^2}{2f}.e^{2fs/k^2}.e^{-2\psi} = \frac{u^2 - k^2}{-2f}e^{-2\psi} + \frac{u^2}{2f}$$

or $$k^2 e^{2fs/k^2} = -u^2 + k^2 + u^2 e^{2\psi}$$

or $$k^2 (e^{2fs/k^2} - 1) = u^2 (e^{2\psi} - 1)$$

This is the required intrinsic equation.

EXERCISES

1. A particle is projected with a velocity whose horizontal and vertical components are U and V from a point in a medium whose resistance per unit of mass is k times the speed. Obtain the equation of the depth, and prove that if k is small, the horizontal range is approximately $\frac{2UY}{g} - \frac{8UV^2 k}{3g^2}$.

2. A particle acted on by gravity is projected in a medium, the resistance of which varies as the velocity. Show that its acceleration retains a fixed direction and diminishes without limit to zero.

3. In the case of a flat trajectory with initial velocity u and resistance equal to μ (velocity)^2, show that the path of projectile approximately is

$$y = x \tan \alpha - \frac{gx^2}{2u^2} - \frac{\mu g}{3u^3} x^3 - \ldots$$

4. A heavy particle is projected in a resisting medium. If v be the velocity at any time, ϕ be inclination to the vertical of the direction of motion and f the retardation prove that

$$\frac{1}{v}\frac{dv}{d\phi} = \cos t\ \phi + \frac{f}{g \sin \phi}$$

5. If the resistance per unit mass is $g\left(\frac{v}{V}\right)^2$ prove that

$$\frac{du}{ds} = -\left(\frac{g}{V^2}\right) u^2, \quad \frac{d\psi}{ds} = \left(\frac{g}{u^2}\right) \cos^3 \psi$$

where u is the horizontal component of velocity.

6. A heavy bead of mass m slides on a smooth wire in the shape of a cycloid, whose axis is vertical and vertex upwards in a medium whose resistance is $mv^2/2c$ and the distances of starting point from the vertex is c, show that the time of descent to the cusp is $\sqrt{\left[\frac{8a(4a-c)}{gc}\right]}$, where $2a$ is the length of the axis of cycloid.

7. A particle moving in resisting medium acted upon by a central force (μ/r^n), if the path be an equiangular spiral of angle α, whose pole is at the centre of force, show that the resistance is

$$\frac{n-3}{3} \cdot \frac{\mu \cos \alpha}{r^n}$$

8. A bead moves on a smooth wire in the form of a circle in a vertical plane under a resistance $\{= k\ (\text{velocity})^2\}$. Find the velocity of the bead at any point.

9. Prove that by a proper choice of axes the equation of the path of the projectile in a resisting medium can be put in the form $y + ax = b \log x$.

10. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time $\left(\frac{1}{k}\right) \log \left\{1 + \left(\frac{2ku}{g}\right) \sin \alpha\right\}$.

11. A heavy particle describes a path given by

$$\cos\psi = f(\rho\cos\psi);$$

show that the law of resistance is given by

$$Rv\frac{df}{dv} = -g\sqrt{(1-f^2)}\,\frac{d}{dv}(vf),$$

when $f = f\left(\frac{v^2}{g}\right)$.

12. Prove that in the motion of a projectile in a resisting medium the equation $\frac{d^2y}{dx^2} = -\frac{g}{u^2}$ is satisfied whatever be the law of resistance, u being the horizontal component of velocity the axes of x and y being horizontal and vertically upwards. If the resistance is constant and equal to kg, show that the velocity at any point is given by

$$v(1-\sin\psi)^k = u_0(\cos\psi)^{k-1},$$

ψ being the slope and u_0 the velocity at the highest point.

❑ ❑ ❑

Motion of Particles of Varying Mass

6.1 MOTION WHEN MASS VARIES

We come across a number of problems in mechanics in which moving mass changes with time. Obviously, the motion of a body will be affected by the material added or removed. For example, a rocket ejecting burned fuel in the form of hot gases looses its mass while a rain drop acquires additional mass by condensations it falls through a cloud. Such type of motion will be discussed ni this section.

We know that the equation $P = mf$ holds only if the moving mass m remains constant throughout the motion.

Let the moving mass does not remain constant. Let m be its mass and v the velocity at time t. Then by Newton's second law

$$P = \left(\frac{d}{dt}\right)(mv) \qquad \text{...(i)}$$

Let there be an increment δm of mass in time δt and let this increment δm be moving with velocity V.

The increment in the momentum of the particle in time δt

$$= m\,(v + \delta v - v) + \delta m\,(v + \delta v - V) \qquad \text{...(ii)}$$

$$= m\,\delta v + \delta m\,(v + \delta v - V)$$

The impulse of force in time $\delta t = P\,.\,\delta t$ ...(iii)

Hence equating (ii) and (iii) and taking limits at $\delta t \to 0,$ we get

$$P = m\left(\frac{dv}{dt}\right) + v\left(\frac{dm}{dt}\right) - V\,(dmdt) \qquad \text{[neglecting } \delta m\,\delta v \text{]}$$

or $$m\left(\frac{dv}{dt}\right) + v\left(\frac{dm}{dt}\right) = P + V\left(\frac{dm}{dt}\right)$$

or $$\left(\frac{d}{dt}\right)(vm) = P + V\left(\frac{dm}{dt}\right) \qquad \text{...(iv)}$$

If $V = 0,$ the equation (iv) reduces to (i).

Equation (iv) is the equation of motion when mass varies.

EXAMPLES

1. *A spherical raindrop, falling freely, receives in each instant an increase of volume equal to λ times its surface at that instant; find the velocity at the end of time t, and the distance fallen through in that time.*

Solution. Let m be the mass and r the radius of the drop when it has fallen through a distance x in time t. Also, let v be the velocity at that instant, so that

$$\frac{dx}{dt} = v \qquad \text{...(i)}$$

Since drop is falling freely under gravity, hence

$$P = mg$$

Now, m = mass of drop at time $t = \left(\frac{4}{3}\right)\pi r^3 \rho$

$$\therefore \quad \frac{dm}{dt} = 4\pi r^2 \rho \left(\frac{dr}{dt}\right)$$

But, given rate of increase of mass

$$\frac{dm}{dt} = \lambda\,(4\pi r^2)\,\rho$$

$\therefore$ Equating two values of $\frac{dm}{dt}$, we get

$$4\pi r^2 \rho \left(\frac{dr}{dt}\right) = \lambda\rho\,(4\pi r^2) \quad \text{or} \quad \left(\frac{dr}{dt}\right) = \lambda$$

Integrating, $\quad r = \lambda t + A$

where A is constant of integration.

Initially, at $t = 0, r = a$ (say) $\therefore \; A = a \therefore r = \lambda t + a$

$$\therefore \quad m = \frac{4}{3}\pi\rho\,(a + \lambda t)^3 \qquad \text{[from (iii) and (iv)]}$$

Since the mass is picked up from rest, hence $V = 0$.

Hence, from the equation of variable mass

$$\left(\frac{d}{dt}\right)(mv) = P + V\left(\frac{dm}{dt}\right), \quad \text{we get}$$

$$\left(\frac{d}{dt}\right)[v\,.\,4/3\,\pi\rho\,(a + \lambda t)^3] = \frac{4}{3}\pi\rho\,(a + \lambda t)^3\, g \qquad \text{[from (ii) and (iv)]}$$

$$\left(\frac{d}{dt}\right)[v(a + \lambda t)^3] = g\,(a + \lambda t)^3$$

Integrating, we get

$$v\,(a + \lambda t)^3 = g\,(a + \lambda t)^4/4 + B,$$

where B is constant of integration.

Initially, $v = 0, t = 0$; $\quad \therefore \quad B = -\left(\frac{ga^4}{4\lambda}\right)$

$$\therefore \quad v\,(a + \lambda t)^3 = \left(\frac{g}{4\lambda}\right)[(a + \lambda t)^4 - a^4]$$

$$\text{or} \quad v = \left(\frac{g}{4\lambda}\right)\left[(a + \lambda t) - a^4/(a + \lambda t)^3\right] \qquad \text{...(vi)}$$

This is expression for velocity at any time t.

Now, from (i) and (vi), we get

$$\frac{dx}{dt} = \frac{g}{4\lambda}\left[(a + \lambda t) - \frac{a^4}{(a + \lambda t)^3}\right]$$

Integrating, $\quad x = \frac{g}{4\lambda}\left[(a + \lambda t)^2/2\lambda + a^4/2\lambda\,(a + \lambda t)^2\right] + C$

where C is constant of integrating. Initially, $x = 0$ at $t = 0$

$$\therefore \quad C = -\left(\frac{g}{8x^2}\right)(a^2 + a^2)$$

Hence, $x = \left(\frac{g}{8\lambda^2}\right)[(a+\lambda t)^2 + a^4/(a+\lambda t)^2 - 2a^2]$

$= \left(\frac{g}{8\lambda^2}\right)[\{(a+\lambda t) - a^2\}/(a+\lambda t)]^2$

$= \left(\frac{g}{8\lambda^2}\right)[\{(a+\lambda t)^2 - a^2\}/(a+\lambda t)]^2$

$= [\{gt^2\,(2a+\lambda t^2)\}/\{8\,(a+\lambda t)^2\}]$ ...(vii)

This is the required expression for distance.

2. *A spherical drop of liquid falling freely in a vapour acquires mass by condensation at a constant rate c. Show that the velocity after falling from rst in time t is*

$$\left(\frac{1}{2}\right)gt\left[1+\left\{\frac{M}{(M+ct)}\right\}\right]$$

where M is the initial mass of the drop.

Solution. Since the drop is falling freely under gravity, hence

$$P = mg$$

where m is initial mass of the drop at any time t.

Let M be the initial mass of the drop, then

$$m = M + ct \qquad \text{...(i)}$$

Also $v = 0$, since the mass is picked up from rest.

Hence, equation of motion when mass varies will be

$$\left(\frac{d}{dt}\right)(vm) = mg + V\left(\frac{dm}{dt}\right) \qquad [\because V = 0]$$

In this case, it will becomes $\left(\frac{d}{dt}\right)(vm) = mg$

or $\left(\frac{d}{dt}\right)\{v\,(M+ct)\} = (M+ct)\,g,$ [from (i)]

Integrating, $v\,(M+ct) = [\{g\,(M+ct)^2\}/2c] + A,$

where A is constant of integration.

Initially at $t=0, v=0$; $\therefore\ A = -\left(\frac{gM^2}{2c}\right)$

Hence, $v\,(M+ct) = \left(\frac{g}{2c}\right)(M+ct)^2 - \left(\frac{gM^2}{2c}\right)$

or $v = \left(\frac{g}{2c}\right)(M+ct) - \left[\left(\frac{gM^2}{2c}\right)(M+ct)\right]$

$$= \left(\frac{g}{2c}\right)\left[\frac{\{(M+ct)^2 - M^2\}}{(M+ct)}\right]$$

$$= \left(\frac{g}{2c}\right)\left[\frac{(2mct + c^2t^2)}{(M+ct)}\right] = \left(\frac{gt}{2}\right)\left[\frac{(2M+ct)}{(M+ct)}\right]$$

$$= \left(\frac{gt}{2}\right)\left[\frac{\{(M+ct)+M\}}{(M+ct)}\right] = \left(\frac{gt}{2}\right)\left[1+\left\{\frac{M}{(M+ct)}\right\}\right]$$

3. *Snow slides off a roof clearing away a part of uniform breadth; show that if it all slide at once, the time in which the roof will be cleared is* $\sqrt{\{6\pi a / g \sin \alpha\}} \dfrac{(2/3)!}{(1/6)!}$ *but that, if the top move first and gradually set the rest in motion, the acceleration is (1/3) g sin* α *and the time will be* $\sqrt{(6a / g \sin \alpha)}$, *where* α *is the inclination of the roof and a the original length of the snow.*

Solution. Let y be the length of the snow on the roof at time t and b the constant breadth

$$\therefore \qquad m = yb\rho\,;\ \ v = -\frac{dy}{dt},\ f = g \sin \alpha$$

Hence, equation of motion $\left(\dfrac{d}{dt}\right)(mv) = \rho$, becomes

$$\left(\frac{d}{dt}\right)[-yb\rho\,(dy/dt)] = yb\rho g \sin \alpha \qquad [\because \rho = mb]$$

or
$$\left(\frac{d}{dt}\right)\left[y\left(\frac{dy}{dt}\right)\right] = -gy \sin \alpha$$

or
$$\left(\frac{d}{dy}\right)\left[y\left(\frac{dy}{dt}\right)\right]\left(\frac{dy}{dt}\right) = -gy \sin \alpha$$

or
$$\left[y\left(\frac{dy}{dt}\right)\right]\left(\frac{d}{dy}\right)\left[y\left(\frac{dy}{dt}\right)\right] = -gy^2 \sin \alpha$$

Integrating,
$$\left(\frac{1}{2}\right)\left[y\left(\frac{dy}{dt}\right)\right]^2 = A - \frac{1}{3} gy^3 \sin \alpha, \qquad \text{...(i)}$$

where A is constant of integration.

Initially, at $y = a, \dfrac{dy}{dt} = 0, \therefore\ A = \dfrac{1}{3} ga^3 \sin \alpha$

Putting the value of A in (i), we get

$$\left(\frac{1}{2}\right)y^2\left(\frac{dy}{dt}\right)^2 = \frac{1}{3}(a^3 - y^3)\, g \sin \alpha$$

or
$$\frac{dy}{dt} = -\sqrt{\left(\frac{2}{3} g \sin \alpha\right)}\left\{\sqrt{a^3 - y^3} / y\right\}$$

or
$$dt = -\sqrt{\left(\frac{3}{2} g \sin \alpha\right)}\left\{\frac{y}{\sqrt{(a^3 - y^3)}}\right\} dy \qquad \text{...(ii)}$$

Integrating (ii) from $y = a$ to $y = b$, the time t in which the roof will be cleared off is given by

$$t = -\sqrt{\left(\frac{3}{2} g \sin \alpha\right)} \int_a^0 \left\{\frac{y}{\sqrt{(a^3 - y^3)}}\right\} dy$$

Putting $y^{3/2} = a^{3/2} \sin \theta,\ \therefore\ \dfrac{3}{2} y^{1/2}\, dy = a^{3/2} \cos \theta\, d\theta$

or
$$t = \sqrt{\left(\frac{3}{2} g \sin \alpha\right)} \int_0^{\pi/2} (a^{1/2} \sin^{1/3} \theta \,.\, \frac{2}{3} a^{3/2} \cos \theta\, d\theta) / (a^{3/2} \cos \theta)$$

$$= \sqrt{\left(\frac{6a}{g \sin \alpha}\right)} \int_0^{\pi/2} \frac{1}{3} \sin^{1/3} \theta \, d\theta$$

$$= \sqrt{\left(\frac{6a}{g \sin \alpha}\right)} \left[\frac{1}{3} . \left(\Gamma \frac{2}{3} . \Gamma \frac{1}{3}\right) / \left(2 . \Gamma \frac{7}{6}\right)\right]$$

$$= \sqrt{\left(\frac{6\pi a}{g \sin \alpha}\right)} \left(\Gamma \frac{2}{3} / \Gamma \frac{1}{6}!\right)$$

Let z be the portion of the roof cleared off in time t, then equation of motion will be

$$\left(\frac{d}{dt}\right)\left(z \frac{dz}{dt}\right) = gz \sin \alpha.$$

or
$$z \left(\frac{dz}{dt}\right) . \left(\frac{d}{dz}\right)\left(z \frac{dz}{dt}\right) = gz^2 \sin \alpha$$

Integrating,
$$\frac{1}{2}\left(z \frac{dz}{dt}\right)^2 = \frac{1}{3} gz^3 \sin \alpha + c, \quad \text{...(iii)}$$

where c is constant of integration.

Initially, $z = 0, \dfrac{dz}{dt} = 0, \quad \therefore \quad c = 0$

Hence, from (iii), we get
$$\left(\frac{dz}{dt}\right)^2 = \frac{2}{3} gz \sin \alpha \quad \text{...(iv)}$$

Differentiating (iv) with respect to t, we get

$$2\left(\frac{dz}{dt}\right)\left(\frac{d^2 z}{dt^2}\right) = \frac{2}{3} g \left(\frac{dz}{dt}\right) \sin \alpha \quad \text{or} \quad \frac{d^2 z}{dt^2} = \frac{1}{3} g \sin \alpha$$

Hence, the acceleration is $\dfrac{1}{3} g \sin \alpha.$

Again, from (iv), we get
$$dt = \sqrt{\left(\frac{3}{2} g \sin \alpha\right)} \frac{dz}{\sqrt{z}}.$$

Integrating from $z = 0$ to $z = a,$ the required time t is given by

$$t = \sqrt{\left(\frac{3}{2} g \sin \alpha\right)} \left(\frac{a^{1/2}}{1/2}\right) = \sqrt{\left(\frac{6a}{g \sin \alpha}\right)}$$

4. *A uniform chain of length 1 and mass ml is coiled on the floor and a mass mc is attached to one end and projected vertically upwards wth velocity* $\sqrt{(2gh)}$. *Show that according as the chain does or does not completely leave the floor, the velocity of the mass finally reaching the floor again, is the velocity due to a fall through a height* $\dfrac{1}{3}[2l - c + a^2 / (l + c)^2]$ *or* $a - c$, *where* $a^3 = c^2 (c + 3h)$.

Solution. Let the length of the portion of the chain uncoiled in time t be x. Then the equation of motion will be

$$\left(\frac{d}{dt}\right)\left\{\frac{(mx + mc)}{(dx / dt)}\right\} = - g (mx + mc)$$

or $$(x+c)\left(\frac{dx}{dt}\right)\left(\frac{d}{dx}\right)\left\{(x+c)\left(\frac{dx}{dt}\right)\right\}=-g\,(x+c)^2$$

Integrating, we get $$\frac{1}{2}\left\{(x+c)\left(\frac{dx}{dt}\right)\right\}^2=-\frac{1}{3}g(x+c)^3+A, \quad \text{...(i)}$$

where A is constant of integration.

Initially, $x=0,\ dx/dt=\sqrt{(2gh)},\ \therefore\ A=\frac{1}{3}gc^2\,(3h+c)$

Hence, (i) becomes

$$(x+c)^2\left(\frac{dx}{dt}\right)^2=\frac{2}{3}g\,[c^2\,(3h+c)-(x+c)^3]$$

$$=\frac{2}{3}g\,\{a^3-(x+c)^3\} \quad \text{...(ii)}$$

where $a^3=c^2\,(c+3h)$.

Now, we have

Case I. Let the coil does not completely leave the floor when the mass comes to rest at a height x above the floor. Putting $\frac{dx}{dt}=0$ in (ii), we get

$$0=a^3-(x+c)^3 \quad \text{or} \quad x=a-c$$

Hence, the velocity of the mass on reaching the floor again is that due to fall through a height $a-c$.

Case II. Let the initial velocity be sufficient enough so that the coil leaves the floor and mass does not come to rest. Let V be the velocity of the mass just at the time coil leaves the floor, then putting $x=l,\ \frac{dx}{dt}=V$ in (ii), we get $(l+c)^2\,V^2=\frac{2}{3}g\,\{g^3-(l+c)^3\}$.

Hence, total height attained by the mass

$$=l+V^2/2g=l+\frac{1}{3}\{a^3/(l+c)^2-(l+c)\}$$

$$=\frac{1}{3}[2l-c+a^3/(l+c)^2] \quad \text{...(iii)}$$

Hence, the velocity of the mass on reaching the floor again is that due to a fall through a height given by (iii).

5. *A rocket whose total initial mass (fuel + shell) is m_0, ejects fuel at a constant rate cm_0 and at a velocity V relative to the case. Show that lowest rate of fuel consumption that will permit the rocket to rise at once is $c=g/V$. Assuming this design condition is wet, find the greatest speed and height reached by the rocket.*

Solution. We know that the equation of motion of variable mass is

$$\left(\frac{d}{dt}\right)(vm)=P+V_1\left(\frac{dm}{dt}\right) \quad \text{...(i)}$$

where m is mass at time t.

Here $P=-mg$ (since mass is moving in upward direction)

$$\frac{dm}{dt}=-cm_0 \text{ as mass is ejected.} \quad \text{...(ii)}$$

$V_1=$ velocity of ejecting mass $=v-V$

Integrating (ii), we have $m = -cm_0 t + A$

Initially, $m = m_0, t = 0 \therefore A = m_0$

$$m = m_0(1 - ct) \qquad \text{...(iii)}$$

Substituting the values of P and V_1 in equation (i), we get

$$\left(\frac{dv}{dt}\right)m + v\left(\frac{dm}{dt}\right) = -mg + (v - V)\left(\frac{dm}{dt}\right)$$

or $$m\left(\frac{dv}{dt}\right) = -mg - V\left(\frac{dm}{dt}\right) \text{ and } \frac{dv}{dt} = -g + (Vcm_0)/m, \qquad \text{from (ii)} \quad \text{...(iv)}$$

or $$\frac{dv}{dt} = -[-g + \{Vc/(1-ct)\}], \qquad \text{from (iii)} \quad \text{...(v)}$$

This equation holds for all values of t.

Initially, at $t = 0$, we have $\left(\frac{dv}{dt}\right)_{t=0} = -g + Vc$

In order, the rocket rise at once dv/dt must be positive.

i.e., $$Vc - g > 0 \quad \text{or} \quad c > \frac{g}{V}$$

$\therefore$ minimum value of $c = \frac{g}{V}$.

Now, let us assume that masses are so arranged that the rocket lifts at time zero $(c > g/V)$, integrating (v), we get

$$v = \int_0^t [-g + Vc/(1-ct)]\,dt = -gt - V\log(1-ct) \qquad \text{...(vi)}$$

$$\therefore \quad \frac{dx}{dt} = -gt - V\log(1-ct)$$

Integrating, we get $$x = \int_0^t [-gt - V\log(1-ct)\,dt]$$

or $$x = -\left(\frac{1}{2}\right)gt^2 - v\,[t\log(1-ct) + c\int_0^t t\{dt/(1-ct)\}]_0^t$$

$$= -\left(\frac{1}{2}\right)gt^2 - Vt\log(1-ct) + V\int_0^t [(1-ct-1)/(1-ct)]\,dt$$

$$= -\left(\frac{1}{2}\right)gt^2 - Vt\log(1-ct) + V\int_0^t [1 - 1/(1-ct)]\,dt$$

$$= -\left(\frac{1}{2}\right)gt^2 - Vt\log(1-ct) + V[t + (1/c)\log(1-c)]$$

$$= -\left(\frac{1}{2}\right)gt^2 + (V/c)(1-ct)[\log(1-ct) - 1] + (V/c) \qquad \text{...(vii)}$$

Now, let M be the mass of the shell (or case). The rocket will be lightest when all fuel is burnt, *i.e.,* when $m = -M$. Putting this value of m in (iii), we get

$$M = m_0(1-ct) \quad \text{or} \quad ct = 1 - M/m_0 \qquad \text{...(viii)}$$

The speed v will be maximum when all fuel is burnt, *i.e.,* when (viii) holds.

Using this value of t in (vi), we get

$$V_{\max} = -\left(\frac{g}{ct}\right)\left[1 - \left(\frac{M}{m_0}\right)\right] - V\log\left(\frac{M}{m_0}\right) \qquad \text{...(ix)}$$

And at this instant, the height x will be

$$x = -\left(\frac{g}{2c^2}\right)[1 - (M/m_0)]^2 + (V/c)[(M/m_0)\log(M/m_0) + 1(1 - M/m_0)] \qquad \text{...(x)}$$

When the shell (rocket) reaches this height, it moves freely as projectile under gravity with initial vertical velocity v_{maximum}.

Hence, additional height reached $= (v_{\max})^2 / (2g)$

$$= \left(\frac{1}{2}g\right)[g^2c^2 (1 - M/m_0)^2 + 1/2 (\log M/m_0)^2$$

$$+ (2gV/c)(1 - M/m_0)^2 \times \log M/m_0] \quad \text{...(xi)}$$

Adding (ix) and (x), we get

$$x_{\max} = \left(\frac{V}{c}\right)\left(1 - \frac{M}{m_0} + \log\frac{M}{m_0}\right) + \left(\frac{V^2}{2g}\right)\left(\log\frac{M}{m_0}\right)^2$$

EXERCISES

1. A spherical raindrop of radius a cms falls from rest through a vertical height h, receiving throughout the motion an accumulation of condensed vapour at the rate of k grammes per square cm. per second, no vertical force but gravity acting. Show that when it reaches the ground its radius will be

$$\sqrt{(2hg)}\,[1 + \sqrt{(ga^2/2hk^2)}]$$

2. A mass in the form of a solid cylinder, the area of whose cross-section is A, moves parallel to its axis, being acted on by a constant force F, through a uniform cloud of fine dust of volume density ρ which is moving in a direction opposite to that of cylinder with constant velocity V. If all the dust that meets the cylinder clings to it, the cylinder starts from rest and its initial mass was m;show that the velocity after time t is $(mV + Ft/k) - V$ and the distance described in that time is

$$\frac{k}{A\rho} - Vt - \frac{m}{A\rho} \quad \text{where } k = \sqrt{(m^2 + 2mA\rho vt + AF\rho t^2)}.$$

3. A particle of mass M is at rest and beings to move under the action of a constant force F in a fixed direction. It encounters the resistance of a stream of fine dust moving in the opposite direction with velocity V, which deposits matter on it at a constant rate c. Show that its mass will be m when it has travelled a distance

$$\left(\frac{k}{c^2}\right)[m - M(1 + \log m/M)], \qquad \text{where } k = F - cV.$$

4. A chain of length l is coiled at the edge of a table. One end is fastened to a particle whose mass is equal to that of the whole chain and the other end is put over the edge. Show that immediately after leaving the table, the particle is moving with velocity $\frac{1}{2}\sqrt{(5/6)\,gl}$.

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Central Orbits

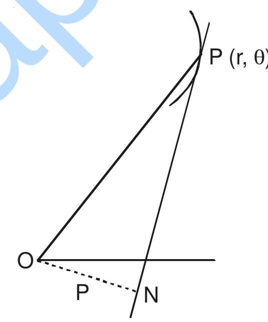
7.1 DEFINITIONS

(i) Central force : A force which is directed towards a fixed point O is called *central force*. The fixed point O is called the *centre of force*.

(ii) Central Orbit : The path described by the particle moving along a plane curve under a central force is called the *central orbit*.

7.2 DIFFERENTIAL EQUATION OF CENTRAL ORBIT (POLAR FORM)

A particle is moving in a plane with an acceleration F which is always directed towards a fixed point O in the plane, to find the differential equation of the path.



Let a particle moves in a plane curve under an acceleration F which is always directed towards a fixed point O in the plane. Let OX be a fixed line in the plane of orbit. Let O be the pole and OX as the initial line. Let the position of the particle at time t be $P(r, \theta)$.

Since the acceleration F is always directed towards O, hence the particle has only radial acceleration towards O of magnitude F and no transverse acceleration.

Hence, the equation of motion in the radial and transverse directions will be

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F \qquad \text{...(i)}$$

and
$$\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \qquad \text{...(ii)}$$

Integrating (ii), we get

$$r^2 \frac{d\theta}{dt} = \text{constant} = h \text{ (say)} \qquad \text{...(iii)}$$

To integrate these equations, let

$$r = \frac{1}{u}$$

$\therefore$
$$\frac{dr}{dt} = -\frac{1}{u^2}\cdot\frac{du}{dt} = -\frac{1}{u^2}\cdot\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

or
$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\left\{\frac{h}{r^2}\right\}, \qquad \therefore \text{ from (iii) } \frac{d\theta}{dt} = \frac{h}{r^2}.$$

or
$$\frac{dr}{dt} = -h\frac{du}{d\theta} \qquad \text{...(iv)}$$

and
$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left\{-h\frac{du}{d\theta}\right\} = -h\frac{d}{d\theta}\left\{\frac{du}{d\theta}\right\}\cdot\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}\cdot\frac{h}{r^2}, \qquad \text{from (iii)}$$

or $$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting the values of $\frac{d\theta}{dt}$ and $\frac{d^2r}{dt^2}$ from (iii) and (v) in (i), we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\left\{\frac{h}{r^2}\right\}^2 = -F$$

or $$h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = F \qquad ...(v)$$

or $$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \qquad ...(vi)$$

This is the required differential equation of the central orbit in polar form as it is satisfied by the coordinates (u, θ) of a point on the orbit.

7.2.1 Differential equation of a central orbit (pedal form)

To obtain differential equation to the path in `p' and `r' for a particle moving in a plane with acceleration F, directed towards a fixed point O.

We know that if p be the length of the perpendicular drawn from the pole to the tangent to a curve at any point $p(r, \theta)$, then

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left\{\frac{dr}{d\theta}\right\}^2$$

Putting $u = \frac{1}{r}$, we have $\frac{du}{d\theta} = -\frac{1}{r^2}\left\{\frac{dr}{d\theta}\right\}$

$\therefore$ Above equation reduces to

$$\frac{1}{p^2} = u^2 + \left\{\frac{du}{d\theta}\right\}^2 \qquad ...(i)$$

Differentiating (i) with respect to θ, we get

$$-\frac{2}{p^3}\cdot\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

$$-\frac{1}{p^3}\frac{dp}{d\theta} = \left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{d\theta}$$

$$= \left(\frac{F}{h^2u^2}\right)\cdot\left(-\frac{1}{r^2}\cdot\frac{dr}{d\theta}\right), \text{ from (vi) of 7.2 above}$$

and $$\frac{du}{d\theta} = -\left(\frac{1}{r^2}\right)\frac{dr}{d\theta} \quad \text{or} \quad \frac{1}{p^3}\frac{dp}{d\theta} = \frac{F}{h^2}\frac{dr}{d\theta}$$

or $$\frac{h^2}{p^3}\cdot\frac{dp}{dr} = F \qquad ...(ii)$$

This is the required differential equation of the central orbit in the pedal form.

EXAMPLES

1. *Find the law of force to the pole if the path of the particle is cardioid* $r = a(1 + \cos\theta)$.

Solution. It is given that

$$u = \frac{1}{r} = \frac{1}{a(1+\cos\theta)}$$

or $$\log u = -\log a - \log(1 + \cos\theta)$$

Differentiating with respect to θ,

$$\frac{1}{u}\cdot\frac{du}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

$\therefore$ $$\frac{du}{d\theta} = u\tan\frac{\theta}{2}$$

or $$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2} = u\tan^2\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2}$$

$\therefore$ $$u + \frac{d^2u}{d\theta^2} = u + u\tan^2\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2} = \frac{3u}{2}\sec^2\frac{\theta}{2} = \frac{3u}{(1+\cos\theta)}$$

$$= 3au^2$$

But $$u + \frac{d^2u}{d\theta^2} = \frac{F}{h^2u^2}$$

$\therefore$ $$F = 3ah^2u^4 = \frac{3ah^2}{r^4}$$

i.e., $$F \propto \frac{1}{r^4}.$$

2. *A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.*

Solution. The equation of the circle with pole on its circumference is

$$r = a\cos\theta$$

or $$\frac{1}{u} = a\cos\theta \quad \text{...(i)}$$

or $$1 = au\cos\theta$$

Differentiating both sides with respect to θ, we get

$$0 = a\frac{du}{d\theta}\cos\theta + au(-\sin\theta)$$

or $$\frac{du}{d\theta} = u\tan\theta$$

or $$\frac{d^2u}{d\theta^2} = u\sec^2\theta + \frac{du}{d\theta}\tan\theta$$

$$= \frac{1}{a}\sec^2\theta + \frac{1}{a}\sec\theta\tan^2\theta$$

$$= \frac{1}{a}\sec^3\theta(\sec^2\theta + \tan^2\theta)$$

$$= \frac{1}{a}\sec\theta(2\sec^2\theta - 1) \quad \text{...(iii)}$$

Now, the differential equation of the path will be

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2},$$

or $$\frac{1}{a}\sec\theta(2\sec^2\theta - 1) + u = \frac{P}{h^2u^2}, \quad \text{from (iii)}$$

or $$\frac{1}{a}\sec\theta(2\sec^2\theta - 1) + \frac{1}{a}\sec\theta = \frac{P}{h^2u^2}, \quad \text{from (i)}$$

or $$\frac{2}{a}\sec^2\theta = \frac{P}{h^2u^2} \quad \text{or} \quad P = \frac{2}{a}(au)^3 \cdot h^2u^2, \quad \text{from (i)}$$

or $$P = 2a^2h^2u^5 = \frac{2a^2h^2}{r^5} \quad \text{or} \quad P \propto \frac{1}{r^5}$$

3. *A particle describes the curve* $r^n = a^n \cos n\theta$ *under a force F to the pole. Find the law of force.*

Solution. The equation of given curve is

$$r^n = a^n \cos n\theta$$

or $$1 = a^n u^n \cos n\theta, \qquad \because \ u = \frac{1}{r}$$

Differentiating both sides w.r.t. θ, we get

$$0 = a^n \left[u^n (-n \sin n\theta) + nu^{n-1} \frac{du}{d\theta} \cos n\theta \right]$$

or $$\frac{du}{d\theta} = u \tan n\theta$$

Again differentiating w.r.t. θ, we get

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \frac{du}{d\theta} \tan n\theta$$

$$= un \sec^2 n\theta + u \tan^2 n\theta$$

The differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

or $$(un \sec^2 \theta + u \tan^2 n\theta) + u = \frac{F}{h^2u^2}$$

or $$u(n+1)\sec^2 n\theta = \frac{F}{h^2u^2}$$

or $$F = h^2u^3 (n+1) \sec^2 n\theta$$

$$= \frac{h^2}{r^3}(n+1)\left(\frac{a^n}{r^n}\right)^2, \qquad \because \ \cos n\theta = \frac{r^n}{a^n}$$

or $$F = \frac{h^2 (n+1) a^{2n}}{r^{2n+3}} \quad \text{or} \quad F \propto \frac{1}{r^{2n+3}}$$

4. *A particle describes the curve* $au = \tan h(\theta/\sqrt{2})$ *under a force F to the pole. Find the law of force.*

Solution. The equation of given curve is

$$au = \tan h\left(\frac{\theta}{\sqrt{2}}\right) \qquad \text{...(i)}$$

Differentiating with respect to θ, we get

$$a\frac{du}{d\theta} = \frac{1}{\sqrt{2}} \sec h^2 \left(\frac{\theta}{\sqrt{2}}\right)$$

or $$a\frac{d^2u}{d\theta^2} = -\sec h^2\left(\frac{\theta}{\sqrt{2}}\right) \tan h\left(\frac{\theta}{\sqrt{2}}\right) \qquad \text{...(ii)}$$

The differentiating equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

or $$-\frac{1}{a}\sec h^2\frac{\theta}{\sqrt{2}}\tan h\frac{\theta}{\sqrt{2}}+\frac{1}{a}\tan h\frac{\theta}{\sqrt{2}}=\frac{F}{h^2u^2},\quad \text{from (i), (ii)}$$

or $$\frac{1}{a}\tan h\frac{\theta}{\sqrt{2}}\left[1-\sec h^2\frac{\theta}{\sqrt{2}}\right]=\frac{F}{h^2u^2}$$

or $$u\left[\tan h^2\left(\frac{\theta}{\sqrt{2}}\right)\right]=\frac{F}{h^2u^2},\quad \text{from (i)}$$

or $$F=h^2a^2u^5=\frac{h^2a^2}{r^5}\quad\text{or}\quad F\propto\frac{1}{r^5}.$$

5. *A particle describes the curve* $p^2 = ar$ *under a force F to the pole, find the law of force.*
Solution. The equation of given curve is

$$p^2 = ar \quad ...(i)$$

Differentiating with respect to r,

$$2p\,.\frac{dp}{dt}=a$$

or $$\frac{dp}{dr}=\frac{a}{2p}=\frac{a}{2\sqrt{(ar)}}\quad\text{from (i)}$$

$$=\frac{1}{2}\sqrt{\left(\frac{a}{r}\right)}\quad ....(ii)$$

We have the pedal equation of central orbit as

$$F=\frac{h^2}{p^3}\cdot\frac{dp}{dr}=\frac{h^2}{(ar)^{3/2}}\cdot\frac{1}{2}\sqrt{\left(\frac{a}{r}\right)}=\frac{h^2}{2a}\cdot\frac{1}{r^2}$$

$\therefore$ $$F\propto\frac{1}{r^2}.$$

EXERCISES

1. A particle describes the equiangular spiral $r = ae^{\theta\cot\alpha}$ under a force F to the pole. Fnid the law of force. $\left[\textbf{Ans. } F\propto\frac{1}{r^3}\right]$
2. Find the law of force, if the path of the particle is $r^n = A\cos n\theta + B\sin n\theta$. $\left[\textbf{Ans. } F\propto\frac{1}{r^{2n+3}}\right]$
3. Find the law of force, if the path of the particle is $r^2 = a^2\cos 2\theta$. $\left[\textbf{Ans. } F\propto\frac{1}{r^7}\right]$
4. A particle is describing an ellipse under a force to a pole, find the law of force. $\left[\textbf{Ans. } F\propto\frac{1}{r^2}\right]$
5. Find the force to the pole when an particle describes the curve $r = a\sin\theta$. $\left[\textbf{Ans. } F\propto\left\{\frac{2n^2a^2}{r^5}-\frac{(n^2-1)}{r^3}\right\}\right]$

6. A particle describes the curve $au = \dfrac{\cos h\theta - 2}{\cos h\theta + 1}$ under a force F to the pole. Find the law of force. $\left[\textbf{Ans. } F \propto \dfrac{1}{r^4}\right]$

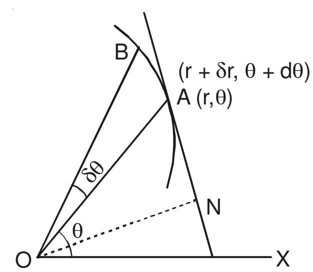
7. A particle describes the curve $r^n \cos n\theta = a^n$ under a force F to the pole. Find the law of force. [**Ans.** $F \propto r^{2n-3}$]

8. A particle describes the curve $r \cos hn\theta = a$ under a force F to the pole. Find the law of force. $\left[\textbf{Ans. } F \propto \dfrac{1}{r^2}\right]$

9. A particle describes a parabola $2p^2 = lr$ under a force to its pole, find the law of force. $\left[\textbf{Ans. } F \propto \dfrac{1}{r^2}\right]$

10. A particle describes the curve $\dfrac{b^2}{p^2} = \dfrac{2a}{r} + 1$ under a force F to the pole, find the law of force. $\left[\textbf{Ans. } F \propto \dfrac{1}{r^2}\right]$

7.3 AREAL VELOCITY



Let A and B be the two neighbouring positions of the particle at times t and $t + \delta t$ respectively, moving along the curve. With O as pole, let coordinates of A and B be (r, θ) and $(r + \delta r, \theta + \delta\theta)$ respectively.

Then in time δt, the sectorial area OAB swept out by OA

$$= \frac{1}{2} OA \,.\, OB \sin \delta\theta$$

$$= \frac{1}{2} \,.\, r\,(r + \delta r) \sin \delta\theta$$

$$= \frac{1}{2} r^2 d\theta, \text{ to the first approximation.}$$

Hence rate of description of the sectorial area OAB as radius vector passes through OA

$$= \lim_{\delta t \to 0} \left[\frac{\frac{1}{2} r^2 \delta\theta}{\delta t}\right] = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} h, \text{ from equation (iii), of 7.2}$$

Hence, the rate of description of the sectorial area is constant or in other words *the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time.*

This rate of description of sectorial area is defined as the *areal velcoity* of particle at A about the fixed point O.

Again, sectorial area OAB

$$= \frac{1}{2} (\text{base } AB) \,.\, \text{perpendicular from } O \text{ on } AB.$$

$\therefore$ Rate of description of sectorial area OAB

$$= \lim_{\delta t \to 0} \left[\frac{\frac{1}{2} . AB . \text{perpendicular form } O \text{ on } AB}{\delta t} \right]$$

$$= \frac{1}{2} \lim_{\delta t \to 0} \left[\frac{AB}{\delta s} . \frac{\delta s}{\delta t} . (\text{perp. form } O \text{ on } AB) \right]$$

Now, as $\delta t \to 0, B \to A$ and secant $AB \to$ tangent at A to the curve. Hence, perpendicular distance from O on $AB \to p$ and $\frac{AB}{\delta s} \to 1.$

Hence, from (ii), we get
rate of description of sectorial area OAB

$$= \frac{1}{2}\left[1 . \frac{ds}{dt} . p\right] = \frac{1}{2} vp \quad \text{...(iii)}$$

Hence, from (i) and (iii), we get

$$\frac{1}{2} h = \frac{1}{2} vp$$

or

$$h = vp \quad \text{or} \quad v = \frac{h}{p} \quad \text{...(iv)}$$

Hence, *the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.*

Again from (iv)

$$v = \frac{h}{p} \quad \text{or} \quad v^2 = \frac{h^2}{p^2} \quad \text{or} \quad \frac{v^2}{h^2} = \frac{1}{p^2}$$

We have from differential calculus

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left\{\frac{dr}{d\theta}\right\}^2 = u^2 + \left\{\frac{du}{d\theta}\right\}^2, \quad \text{where } u = \frac{1}{r}$$

$\therefore$

$$\frac{v^2}{h^2} = u^2 + \left\{\frac{du}{d\theta}\right\}^2$$

or

$$v^2 = h^2\left[u^2 + \left\{\frac{du}{d\theta}\right\}^2\right] \quad \text{...(v)}$$

7.4 AREAS AND APSIDAL DISTANCE

An apse is a point on a central orbit at which the radius vector drawn from the centre of the force is a maximum or minimum. The length of the radius vector at such a point is known as **apsidal distance** *and the angle between two apsidal distances is called the* **apsidal angle.**

As $u = \frac{1}{r}$, hence r will be maximum or minimum according as u will be minimum or maximum and from differential calculus, we know that $\frac{du}{d\theta} = 0.$

Also, we know that

$$\tan \phi = r\frac{d\theta}{dr} = r\frac{d\theta}{du} . \frac{du}{dr}$$

or

$$\tan \phi = r\frac{d\theta}{du}\left\{-\frac{1}{r^2}\right\} = -\frac{1}{r}\frac{d\theta}{du} = -u\frac{d\theta}{du}$$

$\therefore \qquad \frac{du}{d\theta} = -u \cot \phi = 0.$

Hence, $\qquad \cot \phi = 0 \quad \text{or} \quad \phi = \frac{\pi}{2}.$

But ϕ is the angle between the tangent and radius vector. Hence, *at an apse tangent is perpendicular to radius vector.*

7.4.1 Properties of The apse Line

If the central force is a single valved function of the distance, every apse line divides the orbit into two equal and similar portions and thus there can only be two apse distances.

Let the central force be F, which is single valued depending upon the distances only say $F(r)$.

$\therefore \qquad F = F(r) \qquad$ ...(i)

Equation of motion gives

$$u + \frac{d^2u}{d\theta^2} = \frac{F}{h^2u^2}$$

Multiplying by $2\frac{du}{d\theta}$ and integrating with respect to θ, we get

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2}{h^2}\int \frac{F}{u^2}\,du \quad \text{or} \quad u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2}{h^2}\int F(r)\,.\,r^2\,.\,d\left(\frac{1}{r}\right)$$

$$= -\frac{2}{h^2}\int F(r)\,dr$$

But $\frac{1}{p^2} = u^2 + \left\{\frac{du}{d\theta}\right\}^2$

$\therefore \qquad \frac{1}{p^2} = -\frac{2}{h^2}\int f(r)\,dr.$

It is clear that p is a function of r and hence ϕ is also a function of r only.
Above relation also gives,

$$\frac{h^2}{p^2} = -2\int F(r)\,dr$$

$\therefore \qquad v^2 = -2\int F(r)\,dr \qquad$ ...(ii)

This gives that velocity and therefore, acceleration depends upon radius vector only, *i.e.,* for the same r, velocity remains the same.

Equation (ii) shows that F is a single valued function of the distance r, so the velocity is same at the same distance r and does not depend on the direction of the motion. Also, acceleration is same at the same distance because P is single valued function of r. Thus both velocity and acceleration are the same distance r from the centre.

Hence if at an apse, direction of velocity is reversed it will describe a symmetrical curve on both sides of the apsidal distance. When the particle arrives at the second apse, the path is again symmetrical about this second apse. But this is possible only if the next, *i.e.,* third apsidal distance is equal to the one before it, *i.e.,* first. Hence, there are only two different apsidal distances.

7.5 VELOCITY IN A CIRCLE

If v be the velocity with which a particle is projected at $r = a$ from the centre of force at right angles to the radius vector, then the particle describes a circle of radius a and the velocity v is called *velocity in a circle.*

Thus, along the radius vector,

$$\frac{v^2}{a} = F = [F(r)]_{at\ r=a}$$

or $$v^2 = aF(a)$$

Velocity from infinity to $r = a$.

It is the velocity acquired by the particle in falling from infinity to a point $r = a$ under the attraction $F = R(r)$ towards the centre of force.

Hence, $$\frac{v^2}{2} = -\int_\infty^a F(r)\,dr$$

Velocity of fall to the point of projection :

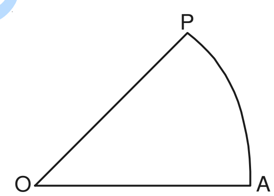
If a particle falls from the centre of repulsion under a force F to a point at a distance r. Then

$$\frac{1}{2}v^2 = \int_0^r F\,dr.$$

7.6 Time in an Orbit

The time of passing through one point A to another point P of a central orbit is given from the equation

$$h = r^2\frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}$$



where x, y are the coordinates of P with centre of force as origin.
Also, time for any are AP is given by

$$t = \frac{\text{area } AOP}{\frac{1}{2}h}.$$

EXAMPLES

1. *A particle moves with a central acceleration* $\mu\left\{r + \frac{a^4}{r^3}\right\}$ *being projected from an apse at a distance a with a velocity* $2\sqrt{\mu}a$. *Prove that it describes the curve* $r^2\,(2 + \cos\sqrt{3}\theta) = 3a^2$.

Solution. We have the differential equation of path as

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \qquad \text{...(i)}$$

Given that $$F = \mu\left(r + \frac{a^4}{r^3}\right) = \mu\left(\frac{1}{u} + a^4u^3\right)$$

Hence, from (i), we get

$$h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \mu\left(\frac{1}{u^3} + a^4u\right)$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left(-\frac{1}{2}u^{-2} + \frac{1}{2}a^4u^2\right) + C, \qquad \text{...(ii)}$$

where C is the constant of integration.

Initially, $r = a$, *i.e.*, $u = \frac{1}{a}$,

$$v = 2\sqrt{\mu}a \quad \text{and} \quad \frac{du}{d\theta} = 0 \quad \text{(at an apse).}$$

Hence, from (ii) we get

$$4\mu a^2 = h^2 . \frac{1}{a^2} = 2\mu\left[-\frac{1}{2}a^2 + \frac{1}{2}a^2\right] + C$$

$\therefore$ $$h^2 = 4\mu a^2 \quad \text{and} \quad C = 4\mu a^2$$

Hence, from (ii), we get

$$4\mu a^2\left[\left\{\frac{du}{d\theta}\right\}^2 + u^2\right] = \mu\left[-\frac{1}{u^2} + a^4u^2\right] + 4\mu a^2$$

or $$4a^2\left[\frac{du}{d\theta}\right]^2 = -\frac{1}{u^2} + a^4u^2 + 4a^2 - 4a^2u^2 = \frac{4a^2u^2 - 3a^4u^4 - 1}{u^2}$$

$$= \frac{-\left[a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}}\right]^2 + \frac{4}{3} - 1}{u^2} = \frac{\frac{1}{3} - \left\{a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}}\right\}^2}{u^2}$$

or $$2a^2\frac{du}{d\theta} = -\frac{1}{u}\sqrt{\left[\frac{1}{3} - \left(a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}}\right)^2\right]}$$

or $$\frac{-2\sqrt{3}\,a^2u\,du}{\sqrt{\left[\frac{1}{3} - \left(a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}}\right)^2\right]}} = \sqrt{3}d\theta$$

Let $$a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}} = t$$

$\therefore$ $$2\sqrt{3}a^2u\,du = dt$$

or $$\frac{-dt}{\sqrt{\left(\frac{1}{3} - t^2\right)}} = \sqrt{3}d\theta$$

Integrating, we get

$$\cos^{-1}(t\sqrt{3}) = \sqrt{3}\theta + A \qquad \text{...(iii)}$$

where A constant of integration.

Initially, $u = \frac{1}{a}$, *i.e.*, $t = \left(\sqrt{3} - \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$ and let $\theta = 0$, then $A = 0$.

Hence, from (iii), we get

$$\cos^{-1}(t\sqrt{3}) = \theta\sqrt{3} \quad \text{or} \quad t\sqrt{3} = \cos(\theta\sqrt{3})$$

or $$\sqrt{3}\left(a^2u^2\sqrt{3} - \frac{2}{\sqrt{3}}\right) = \cos(\theta\sqrt{3}) \quad \text{or} \quad 3a^2u^2 - 2 = \cos(\theta\sqrt{3})$$

or $$3a^2u^2 = 2 + \cos(\sqrt{3}\theta) \quad \text{or} \quad \frac{3a^2}{r^2} = 2\cos(\sqrt{3}\theta)$$

or $$r^2(2 + \cos\sqrt{3}\theta) = 3a^2$$

This is the required equation of the path.

2. *A particle moves under a force* $m\mu\{3au^4 - 2(a^2 - b^2)u^2\}$ $a > b$ *and is projected from an apse at a distance (a + b) with velocity* $\frac{\sqrt{\mu}}{a+b}$. *Show that the equation of its path is* $r = a + b\cos\theta$.

Solution. We know that the differential equation of the path is

$$\frac{d^2u}{d\theta} + u = \frac{F}{h^2u^2}$$

We have $F = \mu \{3au^4 - 2(a^2 - b^2)u^5\}$

Hence, the path becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\{3au^2 - 2(a^2 - b^2)u^3\}$$

or $$h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \mu\{3au^2 - 2(a^2 - b^2)u^3\}$$

multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\{2au^3 - (a^2 - b^2)u^4\} + c \qquad \text{...(i)}$$

where c is constant of integration.

Initially, $r = a + b$, *i.e.*, $u = \frac{1}{(a+b)}$, $v = \frac{\sqrt{\mu}}{(a+b)}$, $\frac{du}{d\theta} = 0$

$\therefore$ From (i),

$$\frac{\mu}{(a+b)^2} = h^2\left[0 + \frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2 - b^2)}{(a+b)^4}\right] + c$$

or $$\frac{\mu}{(a+b)^2} = \frac{h^2}{(a+b)^2} = \mu\left[\frac{2a - (a-b)}{(a+b)^3}\right] + c$$

or $$\frac{\mu}{(a+b)^2} = \frac{h}{(a+b)^2} = \frac{\mu}{(a+b)^2} + c$$

$\therefore$ $$h = \sqrt{\mu},\ c = 0$$

Then, from (i)

$$\mu\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu[2au^3 - (a^2 - b^2)u^4]$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2 - b^2)u^4 - u^2$$

Putting $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$ we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} - \frac{(a^2 - b^2)}{r^4} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = 2ar - (a^2 - b^2) - r^2 = b^2 - (r-a)^2$$

or $$\frac{dr}{d\theta} = \pm\sqrt{\{b^2 - (r-a)^2\}}$$

or $$-\frac{dr}{\sqrt{\{b^2 - (r-a)^2\}}} = d\theta$$

Integrating, $\cos^{-1}\left\{\frac{(r-a)}{b}\right\} = \theta + c_1$

Initially, $r = a + b, \theta = 0,$ so we get $c_1 = 0$

$$\therefore \quad \cos^{-1}\left\{\frac{r-a}{b}\right\} = \theta \quad \Rightarrow \quad r - a = b\cos\theta$$

or $\quad r = a + b\cos\theta.$

This is the required equation of the path.

3. *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is* $\sqrt{2}$ *times the velocity for a circle of radius a. Show that the equation of its path is* $r\cos(\theta/\sqrt{2}) = a.$

Solution. Let v_1 be the velocity for a circle of radius a with a central acceleration $\propto (\text{distance})^{-3}$. Then

$$\frac{v_1^2}{a} = \frac{\mu}{a^3} \quad \text{or} \quad v_1^2 = \frac{\mu}{a^2} \quad \text{or} \quad v_1 = \frac{\sqrt{\mu}}{a}$$

Hence, if v_2 be the velocity of projection, then

$$v_2 = \sqrt{2}.\ v_1 = \sqrt{2}\,.\frac{\sqrt{\mu}}{a} = \frac{\sqrt{(2\mu)}}{a} \qquad \text{...(i)}$$

Now, the differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2},$$

it is given that $F = \left(\frac{\mu}{r^3}\right) = \mu u^3,$ where $u = \frac{1}{r}.$

$$\text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2u^2} \quad \text{or} \quad h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \mu u$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left(\frac{u^2}{2}\right) + C \qquad \text{...(ii)}$$

Initially, $r = a, i.e., u = \frac{1}{a}, \left(\frac{du}{d\theta}\right) = 0$ and $v = \sqrt{(2\mu)}/a.$

Hence, from (ii), we have

$$\frac{2\mu}{a^2} = \frac{h^2}{a^2} = \frac{\mu}{a^2} + C \quad \text{or} \quad h^2 = 2\mu \quad \text{and} \quad C = \frac{\mu}{a^2}$$

Hence, from (ii) we get

$$2\mu\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\left(u^2 + \frac{1}{a^2}\right)$$

$$\text{or} \quad \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{2}\left(u^2 + \frac{1}{a^2}\right) \quad \text{or} \quad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{2}\left(\frac{1}{a^2} - u^2\right) = \frac{1 - a^2u^2}{2a^2}$$

$$\text{or} \quad \frac{a\,du}{\sqrt{[(1 - a^2u^2)]}} = \frac{1}{\sqrt{2}}\,d\theta$$

Integrating, we have $$\sin^{-1}(au) = \left(\frac{\theta}{\sqrt{2}}\right) + A, \quad \text{...(iii)}$$

where A is the constant of integration.

Initially, $u = \frac{1}{a}$ and $\theta = 0$, $\quad \therefore \quad A = \frac{\pi}{2}$

Hence, from (iii) we get

$$\sin^{-1}(au) = \frac{1}{2}\pi + \left(\frac{\theta}{\sqrt{2}}\right)$$

or $$au = \sin\left(\frac{\pi}{2} + \frac{\theta}{\sqrt{2}}\right) = \cos\left(\frac{\theta}{\sqrt{2}}\right)$$

or $$a = r\cos\left(\frac{\theta}{\sqrt{2}}\right).$$

This is the equation of the path.

4. *A particle moves with a central acceleration* $[\mu / (\text{distance})^5]$ *and projected from the apse at a distance a with a velocty equal to n times that which would be acquired in falling from infinity, show that the other apsidal distance is* $a / \sqrt{(n^2 - 1)}$.

If n = 1 and particle be projected in any direction, show that the path is a circle passing through the centre of force.

Solution. If v be the velocity at a distance x from the centre, then $v\frac{dv}{dx} = -\frac{\mu}{x^5}$, negative sign indicates that the particle is moving towards the centre.

Hence, if V be the velocity from infinity to a distance a from the centre then

$$\int_0^V v\,du = -\mu\int_\infty^a \frac{1}{x^5}\,dx \quad \text{or} \quad V^2 = 2\mu\left[\frac{1}{4x^4}\right]_\infty^a = \frac{\mu}{2a^4}$$

or $$V = \sqrt{\left(\frac{\mu}{2a^4}\right)}$$

Hence, the velocity of projection of the particle

$$= nV = n\sqrt{\left(\frac{\mu}{2a^4}\right)} \quad \text{...(i)}$$

We know that the differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \quad \text{...(ii)}$$

or $$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^5}{h^2u^2} \text{ or } h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \mu u^3$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \mu\frac{u^4}{2} + C, \quad \text{...(iii)}$$

where C is constant of infegration.

Initially, $v = n\sqrt{\left(\frac{\mu}{2a^4}\right)}$, $\frac{du}{d\theta} = 0$ and $u = \frac{1}{a}$

$\therefore$ From (iii), we get

$$\frac{n^2\mu}{2a^4} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{2a^4} + C \quad \text{or} \quad h^2 = \frac{n^2\mu}{(2a^2)} \text{ and } C = \frac{\mu(n^2-1)}{2a^4}$$

Hence, from (iii), we get

$$\frac{n^2\mu}{2a^2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{\mu u^4}{2} + \frac{\mu(n^2-1)}{2a^4}$$

or
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{a^2n^2}[a^4u^4 + (n^2-1)]$$

or
$$\left(\frac{du}{d\theta}\right)^2 = \left[\frac{1}{a^2n^2}\right][a^4u^4 - a^2n^2u^2 + (n^2-1)] \qquad \text{...(iv)}$$

Now, at any apse $\frac{du}{d\theta} = 0$, so form (iv) the apsidal distances are given by

$$a^4u^4 - a^2n^2u^2 + (n^2-1) = 0 \quad \text{or} \quad (n^2-1)r^4 - a^2n^2r^2 + a^4 = 0$$

Let r_1^2 and r_2^2 be the roots of this quadratic equation in r^2. Then

$$r_1^2 \cdot r_2^2 = \frac{a^4}{(n^2-1)} \quad \text{or} \quad r_1r_2 = \frac{a^2}{\sqrt{(n^2-1)}}$$

But one of the apsidal distance is a. Let $r_1 = a$.

$\therefore$
$$ar_2 = \frac{a^2}{\sqrt{[(n^2-1)]}} \quad \text{or} \quad r_2 = \frac{a}{\sqrt{[(n^2-1)]}}$$

If $n = 1$, from (iv), we have

$$\left[\frac{du}{d\theta}\right]^2 = \frac{1}{a^2}(a^4u^4 - a^2u^2) = u^2(a^2u^2 - 1)$$

or
$$\frac{du}{u\sqrt{[(a^2u^2-1)]}} = d\theta$$

Integrating, we get

$$\sec^{-1}(au) = \theta + A$$

Initially, $u = \frac{1}{a}$ and $\theta = 0$, $\therefore$ $A = 0$. Hence,

$$\sec^{-1}(au) = \theta$$

or
$$au = \sec\theta \quad \text{or} \quad \frac{a}{r} = \sec\theta \quad \text{or} \quad r = a\cos\theta$$

This is the polar equation of a circle passing through the pole, *i.e.,* centre of force.

5. *A particle subject to a force producing an acceleration* $\mu(r+2a)/r^5$ *towards the origin is projected from the point (a, 0) with a velocity equal to the velocity from infinity at an angle* $\cot^{-1} 2$ *with the initial line. Show that the equation to the path is*

$$r = a(1 + 2\sin\theta)$$

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

Here,
$$F = \frac{\mu(r+2a)}{r^5} = \frac{\mu\left(\frac{1}{u} + 2a\right)}{1/u^5} = \mu(u^4 + 2au^5)$$

$\therefore$ $$\frac{d^2u}{d\theta^2} + u = \frac{\mu (u^4 + 2au^5)}{h^2u^2}$$

or $$h^2 \left(\frac{d^2u}{d\theta^2} + u \right) = \mu (u^2 + 2au^3)$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we have

$$v^2 = h^2 \left\{ \left(\frac{du}{d\theta} \right)^2 + u^2 \right\}$$

$$= 2\mu \left\{ \frac{1}{3}u^3 + \frac{1}{2}au^4 \right\} + c \quad \text{...(i)}$$

Now, the velocity of projection of the particle is equal to the velocity acquired by a particle in falling from infinity to the point of projection under given acceleration. Let v_1 be the velocity thus acquired. We have,

$$v\frac{dv}{dx} = -\mu \left\{ \frac{x + 2a}{x^5} \right\}$$

or $$v\,dv = -\mu \left\{ \frac{1}{x^4} + \frac{2a}{x^5} \right\} dx$$

$\therefore$ $$\int_{v=0}^{v_1} v\,d\theta = -\mu \int_{x=0}^{a} \left(\frac{1}{x^4} + \frac{2a}{x^5} \right) dx$$

or $$\frac{v_1^2}{2} = -\mu \left[-\frac{1}{3x^3} - \frac{2a}{4x^4} \right]^a$$

$$= \mu \left[\frac{1}{3a^3} + \frac{1}{2a^3} \right]$$

or $$v_1^2 = \frac{5\mu}{(3a^3)}$$

Also, from $p = r \sin \phi$, we have initially, $p_0 = a \sin \phi_0$.

Here, $$\phi_0 = \cot^{-1} 2 \quad \text{or} \quad \cot \phi_0 = 2$$

or $$\sin \phi_0 = \frac{1}{\sqrt{5}}$$

$\therefore$ $$p_0 = a \sin \phi_0 = \frac{a}{\sqrt{5}}$$

Also, we have $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$

$\therefore$ Initially, $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{p_0^2} = \frac{5}{a^2}$

$\therefore$ From (i), initially, we have

$$\frac{5\mu}{3a^3} = h^2 \frac{5}{a^2} = 2\mu \left[\frac{1}{3a^3} + \frac{1}{2a^3} \right] + c$$

or $$\frac{5\mu}{3a^3} = \frac{5h^2}{a^2} = \frac{5\mu}{3a^3} + c$$

$\therefore$ $$h^2 = \frac{\mu}{3a} \text{ and } c = 0.$$

$\therefore$ From (i), we get

$$\frac{\mu}{3a}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[\frac{u^3}{3} + \frac{au^4}{2}\right]$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2$$

Putting $u = \frac{1}{r}, \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = 2ar + 3a^2 - r^2 = 4a^2 - (r-a)^2$$

$$\frac{dr}{\sqrt{\{(2a)^2 - (r-a)^2\}}} = d\theta$$

Integrating, we get

$$\sin^{-1}\left\{\frac{(r-a)}{2a}\right\} = \theta + c_1$$

Initially, $\theta = 0, r = a \therefore c_1 = 0$

Hence, $$\sin^{-1}\left\{\frac{r-a}{2a}\right\} = \theta$$

or $$r - a = 2a \sin\theta$$

or $$r = a(1 + 2\sin\theta)$$

This is the required equation of the path.

6. *A particle of mass m moves under a central attractive force* $m\mu\left(\frac{5}{r^3} + \frac{8c^2}{r^5}\right)$. *and is projected from an apse at a distance c with velocity* $\frac{3\sqrt{\mu}}{c}$, *prove that the orbit is* $r = c\cos\frac{2}{3}\theta$, *and that it will arrive at the origin after a time* $\frac{\pi c^2}{8\sqrt{\mu}}$.

Solution. Here, the central acceleration is

$$F = \mu\left(\frac{5}{r^3} + \frac{8c^2}{r^5}\right) = \mu(5u^3 + 8c^2u^5)$$

The differential equation of the central orbit is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

or $$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu}{u^2}(5u^3 + 8c^2u^5) = \mu(5u + 8c^2u^3)$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\,(5u^2 + 4c^2u^4) + c \qquad \text{...(i)}$$

Initially, at an apse $r = c, u = \frac{1}{c}, \frac{du}{d\theta} = 0, v = 3\sqrt{\mu}/c,$

$\therefore$ From (i), we get

$$\frac{9\mu}{c^2} = \frac{h^2}{c^2} = \mu\left(\frac{5}{c^2} + \frac{4}{c^2}\right) + C$$

or
$$\frac{9\mu}{c^2} = \frac{h^2}{c^2} = \frac{9\mu}{c^2} + C$$

$\therefore$
$$h^2 = 9\mu, \quad c = 0$$

Substituting these values in (i), we get

$$9\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\,[5u^2 + 4c^2u^4]$$

or
$$9\left(\frac{du}{d\theta}\right)^2 = 4c^2u^4 - 4u^2$$

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we get

or
$$9\,.\,\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = 4\left(\frac{c^2}{r^4} - \frac{1}{r^2}\right)$$

$$9\left(\frac{dr}{d\theta}\right)^2 = 4\,(c^2 - r^2)$$

or
$$\frac{dr}{d\theta} = -\frac{2}{3}\sqrt{(c^2 - r^2)}$$

or
$$\frac{2}{3}\,d\theta = \frac{-dr}{\sqrt{(c^2 - r^2)}}$$

Integrating, $\frac{2}{3}\theta + A = \cos^{-1}\left(\frac{r}{c}\right)$, where A is constant of integration.

Initially, when $r = c$, let $\theta = 0$. Then, $A = \cos^{-1} 1 = 0$

$\therefore$
$$\frac{2}{3}\theta = \cos^{-1}\left(\frac{r}{c}\right)$$

or
$$\frac{r}{c} = \cos\frac{2}{3}\theta$$

or
$$r = c\cos\frac{2}{3}\theta$$

This is the required equation of the path.
Again, we have

$$h = r^2\left(\frac{d\theta}{dt}\right)$$

But $h = 3\sqrt{\mu}$ and $r = c\cos\frac{2}{3}\theta$

$\therefore$
$$dt = \frac{r^2}{h}d\theta = \frac{c^2\cos^2\frac{2}{3}\theta}{3\sqrt{\mu}}\,d\theta \qquad \text{...(ii)}$$

At the point of projection, we have taken $\theta = 0$. Also, at the point O, $r = 0$.
Putting $r = 0$ in the equation of path, we get

$$0 = c\cos\frac{2}{3}\theta$$

i.e.,
$$\frac{2}{3}\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3}{4}\pi$$

So, at O, $\theta = \frac{3}{4}\pi$. Let t_1 be the time from the point of projection to the point.
Then, integration (ii), we get

$$t_1 = \int_0^{3\pi/4} \frac{c^2}{3\sqrt{\mu}}\cos^2\left(\frac{2}{3}\theta\right)d\theta$$

Put $\frac{2}{3}\theta = \phi$, so that $\frac{2}{3}d\theta = d\phi$. When $\theta = 0, \phi = 0$ and when $\theta = \frac{3}{4}\pi, \phi = \frac{\pi}{2}$.

$$\therefore \quad t_1 = \int_0^{\pi/2} \frac{c^2}{3\sqrt{\mu}}(\cos^2\phi)\frac{2}{3}d\phi$$

$$= \frac{c^2}{2\sqrt{\mu}}\int_0^{\pi/2}\cos^2\phi\, d\phi$$

$$= \frac{c^2}{2\sqrt{\mu}}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi = \frac{\pi c^2}{8\sqrt{\mu}}.$$

7. *A particle is moving with central acceleration* $\mu(r^5 - c^4 r)$ *being projected from an apse at a distance c with velocity* $\sqrt{\frac{2\mu}{3}}\,c^3$, *show that its path is the curve* $x^4 + y^4 = c^4$.

Solution. Here $P = \mu(r^5 - c^4 r)$

$$= \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right) \qquad \left[\because r = \frac{1}{u}\right]$$

$\therefore$ Differential equation of central orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \quad \text{becomes}$$

$$\frac{d^2u}{d\theta^2} + u = \mu\left[\frac{\frac{1}{u^5} - \frac{c^4}{u}}{h^2u^2}\right]$$

or
$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\left[\frac{1}{u^7} - \frac{c^4}{u^3}\right]$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating above, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$= 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right] + C_1 \qquad ..(1)$$

From initial conditions, *i.e.*, at an apse;

$$u = \frac{1}{c}, \frac{du}{d\theta} = 0$$

and $$v = \sqrt{\frac{2\mu}{3}}\, c^3.$$

$\therefore$ $$\frac{2\mu}{3} c^6 = \frac{h^2}{c^2} = 2\mu\left(-\frac{c^6}{6} + \frac{c^6}{2}\right) + C_1$$

$\therefore$ $$h^2 = \frac{2\mu}{3} c^8 \quad \text{and} \quad C_1 = 0$$

Substituting values of h^2 and C_1 in (1), we get

$$\frac{2\mu}{3} c^8 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right]$$

or $$\frac{c^8}{3}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right]$$

or $$c^8\left(\frac{du}{d\theta}\right)^2 = 3\left[-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right] - c^8u^2$$

$$= \frac{1}{2u^6}[-1 + 3c^4u^4 - 2c^8u^8]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} + \frac{3}{2}c^4u^4 - c^8u^8\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

or $$c^4\frac{du}{d\theta} = \pm\frac{1}{4u^3}\sqrt{1 - (4c^4u^4 - 3)^2}$$

$\therefore$ $$-\frac{4.4c^4u^3 du}{\sqrt{1 - (4c^4u^3 - 3)^2}} = 4d\theta$$

Putting $4c^4u^4 - 3 = V$, so that $4.4c^4u^3\, du = dV$ we get

$$-\frac{dV}{\sqrt{1 - V^2}} = 4d\theta$$

Now integrating above, we get

$$\cos^{-1}(V) = 4\theta + C_2$$

or $$\cos^{-1}(4c^4u^4 - 3) = 4\theta + C_2 \qquad ..(2)$$

Intially when $u = \frac{1}{c}$, then $\theta = 0$; $C_2 = 0$

$\therefore$ (2) becomes

$$\cos^{-1}(4c^4u^4 - 3) = 4\theta$$

or $$4c^4u^4 = 3 + \cos 4\theta$$

or $$4c^4 = r^4[3 + \cos 4\theta]$$

$$= r^4[2 + (2\cos^2\theta - 1)^2]$$

$$= r^4[4 + 8\cos^4\theta - 8\cos^2\theta]$$

$$= 4r^4 [1 + 2\cos^4\theta - 2\cos^2\theta]$$
$$= 4r^4 [\cos^4\theta + (1 - \cos^2\theta)]^2$$

or $$c^4 = r^4 [\cos^4\theta + \sin^4\theta]$$

or $$c^4 = x^4 + y^4 \qquad [\because\ x = r\cos\theta,\ y = r\sin\theta]$$

EXERCISES

1. A particle describes an orbit with a central acceleration $\mu u^3 - \lambda u^5$, being projected from an apse at distance a with a velocity equal to that from infinity, show that the path is $r = a\cos h\dfrac{\theta}{n}$, where $n^2 + 1 = \dfrac{2\mu a^2}{\lambda}$.

2. A particle moves is a curve under a centarl acceleration so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that law of force is inverse cube and path is an equiangular spiral.

3. A particle moving with a central acceleration $\mu/(\text{distance})^3$ is projected from an apse at a distance a with velocity *V*, show that the path is $r\cosh\left[\dfrac{\sqrt{(\mu - a^2V^2)}}{aV}\theta\right] = a$, or $r\cos\left(\dfrac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta\right) = a$ according as $V \gtrless$ the velocity from infinity.

4. In a central orbit the force is $\mu u^3 (3 + 2a^2u^2)$; if the particle be projected at a distance a with a velocity $\sqrt{(5\mu/a^2)}$ in a direction making an angle $\tan^{-1}\left(\dfrac{1}{2}\right)$ with the radius, show that the equation of the path is $r = a\tan\theta$.

5. A particle is acted on by a central repulsive force which varies as the *n*th power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point, show that the equation of the path is of the form $r^{(n+3)/2}\cos\dfrac{1}{2}(n+3)\,\theta = \text{constant}$.

6. A particle of mass *m* moves under a central force $m\mu/(\text{distance})^3$ and is projected at a distance a from the centre of force with the velocity which at angle α to the radius would be acquired by a fall from rest at infinity to the point of projection, prove that the orbit is an equiangular spiral.

7. A particle is projected from an apse at a distance a with the velocity from infinity, the acceleration being μu^7, show that the equation to its path is $r^2 = a^2\cos 2\theta$.

8. A particle moves with central acceleration $(\mu u^2 + \lambda u^3)$ and the velocity of projection at disatnce *R* is *V*. Show that the particle will ultimately go off to infinity if

$$V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}.$$

9. A particle of mass *m* is attached to a fixed point by an elastic string of natural length *a*, the coefficient of elasticity *nmg*. It is projected from an apse at a distance a with velocity $\sqrt{(2pgh)}$; show that the other apsidal distance is given by the equation

$$nr^2(r - a) - 2pha(r + a) = 0$$

10. A particle is projected with velocity $\sqrt{\left(\frac{2\mu}{3c^3}\right)}$ from a point P in a field of attractive force $\frac{\mu}{r^4}$ to a point O distant C from P, where r denotes the distance from O.
If the direction of projection makes an angle 45° with PO, prove that the orbit is cardioid and the particle will arrive at O after a time $\left(\frac{3\pi}{4} - 2\right)\sqrt{\left(\frac{3c^5}{4}\right)}$.

11. A particle subject to a central force per unit of mass equal to $\mu\{2(a^2+b^2)u^5 - 3a^2b^2u^7\}$ is projected at the distance a with velocity $\frac{\sqrt{(\mu)}}{a}$ in a direction at right angles to the initial distance. Show that the path in the curve $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

12. A particle moves under a repulsive force $= \left[\frac{m\mu}{(\text{distance})^2}\right]$ and is projected from an apse at a distance a with velocity V. Show that the equation to the path is $r\cos p\theta = a$, and the angle θ, described in time t is $\frac{1}{p}\tan^{-1}\left(\frac{pVt}{a}\right)$, where $p^2 = \frac{(\mu + a^2V^2)}{a^2V^2}$.

13. A particle subject to a central attractive acceleration $\sqrt{\left(\frac{\mu}{r^3}\right)} + f$ is projected from an apse at a distance a with a velocity $\frac{\sqrt{\mu}}{a}$. Prove that the at any subsequent time t, $r = a - \frac{1}{2}ft^2$.

14. A particle moves with a central acceleration $\lambda^2(8au^2 + a^4u^5)$, it is projected with velocity λ from an apse at a distance $a/3$ from the origin, show that equation to its path is

$$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)} = \cot\frac{\theta}{\sqrt{6}}$$

15. A particle moves under a central force $m\lambda[3a^3u^4 + 8au^2]$, it is projected from an apse at a distance a from the centre of force with velocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half the first, and that the equation to path is $2r = a\left[1 + \sec h\frac{\theta}{\sqrt{5}}\right]$.

16. If the law of force is $\mu\left(u^4 - \frac{10}{9}au^5\right)$ and the particle be projected from an apse at a distance $5a$ with a velocity equal to $\sqrt{\left(\frac{5}{7}\right)}$ of that in a circle at the same distance, show that the orbit is the limacon $r = a(3 + 2\cos\theta)$.

17. Show the only law for a central attraction for which velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.

18. A particle is acted on by a repulsive central force $\frac{\mu r}{(r^2 - 9c^2)^2}$ is projected from an apse at a distance c with velocity $\sqrt{\left(\frac{\mu}{8c^2}\right)}$. Find the equation of its path and show that the time to the cusp is $\frac{4}{3}\pi c^2\sqrt{\left(\frac{2}{\mu}\right)}$.

[**Ans.** $8p^2 = 9c^2 - r^2$]

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Planetary Motion

8.1 NEWTONIAN LAW OF ATTRACTION

According to this law the mutual attraction between two particles of masses m_1 and m_2 and placed at a distance r apart is $\gamma \cdot \frac{m_1 m_2}{r^2}$, where γ is universal constant. This law is found to hold good in the case of the motion of all planets in the solar system. Therefore motion of earth about the Sun, that of planets about the earth or of moon about the Sun is governed by this law. Here in this chapter we will discuss the case of central orbits when the force is an attraction varying inversely as the square of the distance from the centre of force.

8.2

A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to $\frac{\mu}{(\text{distance})^2}$*; Show that the path is a conic section, and distinguish between the three cases that arise.*

This is a case of central orbit as force is always directed towards a fixed point.

Here $$P = \frac{\mu}{r^2}.$$

Also pedal form of differential equation is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P = \frac{\mu}{r^2} \quad \text{or} \quad \frac{h^2}{p^3}\,dp = \frac{\mu}{r^2}\,dr.$$

Integrating above, we get

$$-\frac{h^2}{2p^2} = -\frac{\mu}{r} + A \quad \text{or} \quad \frac{h^2}{p^2} = \frac{2\mu}{pr} + B.$$

Also we know that $h = pv$; hence the equation reduces to

$$v^2 = \frac{2\mu}{r} + B. \qquad \therefore \qquad v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + B.$$

Also we know that pedal equation of ellipse, parabola and hyperbola (that branch which is nearer to centre of force) all referred to focus as pole are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1,\; p^2 = ar \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1,$$

respectively, where $2a$ and $2b$ are the lengths of major and minor or transverse and conjugate axes.

Now comparing equation (1) with these pedal equations of ellipse, parabola and hyperbola, we get

Case 1. With ellipse

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{-1}; \qquad \therefore \qquad h^2 = \frac{\mu b^2}{a} \text{ and } B = -\frac{\mu}{a}.$$

Therefore (1) reduces to

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left[\frac{2}{r} - \frac{1}{a}\right].$$

From $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, we see that $\boldsymbol{v^2 < \frac{2\mu}{r}}$.

Case 2. With parabola.

On comparing (1) with pedal equation of parabola, we get $\mathbf{B = 0}$ and $\mathbf{h^2} = 2a\mu$.

$\therefore$ (1) reduces to $\mathbf{v^2} = \frac{\mathbf{2\mu}}{\mathbf{r}}$.

Case 3. With hyperbola.

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{1}; \mathbf{h^2} = \mu\frac{\mathbf{b^2}}{\mathbf{a}} \text{ and } \mathbf{B} = \frac{\mu}{\mathbf{a}}.$$

$\therefore$ (1) reduces to $v^2 = \frac{2\mu}{r} + \frac{\mu}{a} = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$.

From $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$, we see that $\mathbf{v^2} > \frac{\mathbf{2\mu}}{\mathbf{r}}$

Thus from above three cases, we conclude that $v^2 = \frac{2\mu}{r} + B$ always represents a conic section whose focus is at the centre of force and it represents an ellipse, parabola or hyperbola according as B is – ve, 0 or + ve.

Also we have found that

if $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, it is an ellipse;

if $v^2 = \frac{2\mu}{r}$, it is a parabola;

if $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$, it is a hyperbola;

or from above relations, we conclude that

if $v^2 < \frac{2\mu}{r}$, it is an ellipse;

if $v^2 = \frac{2\mu}{r}$, it is a parabola;

if $v^2 > \frac{2\mu}{r}$, it is a hyperbola;

Also we have found that

$h^2 = \mu\frac{b^2}{a} = \mu l$ in case of ellipse;

$h^2 = 2a\mu = \mu l$ in case of parabola;

$h^2 = \mu\frac{b^2}{a} = \mu l$ in case of hyperbola.

Therefore in all the three cases.

$h^2 = \mu l$

or $h = \{\sqrt{(\mu l)}\}$ [where l is semi-latus rectum].

We observe that velocity at any point in independent of the direction of the velocity.

Cor 1. *If a particle be projected from a point at a distance R from the focus with velocity V in any direction, the path will be an ellipse, parabola or hyperbola according as*

$$V^2 <=> \frac{2\mu}{R}$$

Here we observe that $\frac{2\mu}{R}$ is the square of the velocity the paraticle will acquire in falling from infinity to the distance R from the centre as

$$v\frac{dv}{dr} = -\frac{\mu}{r^2}; \qquad \int_0^{V_1} v\,dv = -\mu\int_\infty^R \frac{1}{r^2}\,dr$$

or $$\frac{V_1^2}{2} = \left[\frac{\mu}{r}\right]_\infty^R = \frac{\mu}{R}; \qquad V_1^2 = \frac{2\mu}{R}$$

Therefore the path will be an ellipse, parabola or hyperbola according as the velocity at any points is $<=>$ the velocity the particle will acquire in falling from infinity to that point.

Cor 2. The velocity V_2 for the description of a circle of R is given by

$$\frac{V_2^2}{R} = \frac{\mu}{R} \text{ (normal acceleration)}$$

$$\therefore \qquad V_2^2 = \frac{\mu}{R} = \frac{1}{2}\cdot\frac{2\mu}{R} \text{ or } V^2 = \frac{1}{\sqrt{2}}\sqrt{\left\{\frac{2\mu}{R}\right\}}$$

As we have already proved above that $\sqrt{\left\{\frac{2\mu}{R}\right\}}$ is velocity the particle would acquire in falling from infinity to the point distant R from the centre, hence

$$V_2 = \frac{1}{\sqrt{2}} \text{ (velocity from infinity)}$$

or velocity for the description of a circle of radius $R = \frac{1}{\sqrt{2}}$ (velocity from infinity) to the distance R from the centre.

Note. *In the above article, when the central acceleration was towards the focus, the branch of hyperbola described in the one nearest to the focus.*

Now if we consider the case when the central acceleration is away from the focus, then the further branch of hyperbola is described.

Now in this case equation of motion (orbit) is

$$\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{\mu}{r^2} \qquad \text{or} \qquad \frac{h^2}{p^3}\,dp = -\frac{\mu}{r^2}\,dr.$$

Integrating above, we get

$$v^2 = \frac{h^2}{p^2} = -\frac{2\mu}{r} + A. \qquad \text{...(1)}$$

Also the pedal equation of further branch of hyperbola is

$$\frac{b^2}{p^2} = 1 - \frac{2a}{r} \qquad \text{...(2)}$$

Now comparing (1) and (2), we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{A}{1}. \qquad \therefore \quad h^2 = \mu\frac{b^2}{a} \text{ and } A = \frac{\mu}{a}.$$

Therefore (1) reduces to

$$v^2 = \mu\left[\frac{1}{a} - \frac{2}{r}\right].$$

8.3 KEPLER'S LAWS

There are three laws discovered by the astronomer Kepler connecting the motion of various planets about the sun :

(i) *Each planet describes an ellipse with the Sun as one of its foci.*

(ii) *The areas described by the radii drawn from the planet to the Sun are, in the same orbit, proportioanl to the times of describing them.*

(iii) *The squares of the periodic times of the planets are proportional to the cubes of the major axes of their orbits.*

Deductions

(1) We have proved in chapter of central forces that if the orbit is an ellipse under an acceleration towards one of its foci, then the law of the acceleration is that of the inverse square of the distance from the centre (focus). Therefore we conclude that *the acceleration of each planet towards the Sun varies inversely as the square of its distance from the Sun.*

(2) From the second law we find that the rate of description of sectorial area is constant and this is true only when *the acceleration of the planet and therefore force on it is directed towards the sun.*

(3) We know that periodic time of a central orbit under inverse law is given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \quad \text{or} \quad T^2 = \frac{4\pi^2 a^3}{\mu},$$

i.e., the square of periodic time varies as the cube of the major axes.

Hence from the third law we conclude that μ is same for all the planets, *i.e., absolute acceleration* μ *(acceleration at unit distance from the sun) is the same for all planets.*

8.4 ACCURATE VALUE OF μ

Kepler's third law is true only on the supposition that the Sun is fixed and mass of the Sun is so great in comparison to that of the planet that the effect on its motion on account of its attraction is negligible.

A more accurate form is obtained below :

Let S be the mass of the Sun and P that of any planet and γ the constant of gravitation.

Now mutual attraction between them at any instant, when distance between them is r, is given by $\gamma \frac{SP}{r^2}$.

Therefore planet's acceleration $\alpha = \gamma \frac{SP}{r^2 . P} = \gamma \frac{S}{r^2}$

and sun's acceleration $\beta = \gamma \frac{S.P}{r^2 S} = \gamma \frac{P}{r^2}$.

and the relative acceleration of the planet with respect to the sun is $\alpha + \beta$.

$$\therefore \qquad \alpha + \beta = \frac{\gamma (S + P)}{r^2} = \frac{\mu}{r^2}.$$

Therefore here $\qquad \mu = \gamma (S + P)$.

Also periodic time is given by $T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$;

therefore exact periodic time $T = \frac{2\pi}{\sqrt{[\gamma (S + P)]}} a^{3/2}$. ...(1)

Now let T_1 be the periodic time of another planet of mass P_1 and a_1 be semi-major axes of its orbit; then

$$T_1 = \frac{2\pi}{\sqrt{[\gamma(S+P_1)]}} a_1^{3/2}. \quad \text{...(2)}$$

Therefore from (1) and (2), we get

$$\frac{T^2}{T_1^2} = \frac{S+P_1}{S+P} \cdot \frac{a^3}{a_1^3} \quad \text{or} \quad \frac{S+P}{S+P_1} \cdot \frac{T^2}{T_1^2} = \frac{a^3}{a_1^3}.$$

If D be the mean distance (distance between centres) of the planet from the Sun, then we can also say

$$T = \frac{2\pi}{\sqrt{[\gamma(S+P)]}} D^{3/2}. \quad \text{...(3)}$$

Similarly if p be the mass of satellite of planet P and the mean distance between their centres be a, then the periodic time t of satellite is given by

$$t = \frac{2\pi}{\sqrt{[\gamma(P+p)]}} d^{3/2}. \quad \text{...(4)}$$

Therefore from (3) and (4), we get

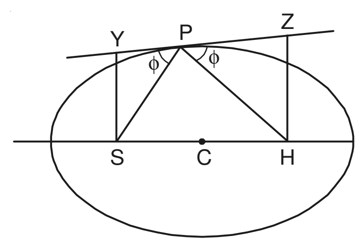
$$\frac{T^2}{t^2} = \frac{P+p}{S+P} \cdot \frac{D^3}{d^3} \quad \text{or} \quad \frac{S+P}{P+p} \cdot \frac{T^2}{t^2} = \frac{D^3}{d^3}.$$

8.5.

In solving questions in this chapter we will require help of properties of ellipse. So some of the important properties are given below :

(1) *The product of perpendiculars drawn from the foci on tangent at any point on ellipse is constant and is equal to square of the semi-minor axis of the ellipse, i.e.,*

$$SY.HZ = b^2$$



(2) *The sum of the focal distances of a point on an ellipse is 2a, where 2a is length of major axis of ellipse i.e.,*

$$SP + HP = 2a.$$

(3) *The length of latus rectum is* $2\frac{b^2}{a}$, *where*

$$b^2 = a^2(1-e^2),$$

e being the eccentricity of the ellipse.

(4) *The tangent and normal at any point are each equally inclined to focal radii of that point.*

EXAMPLES

1. *A particle moves with a central acceleration* $\frac{\mu}{(\text{distance})^2}$; *it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is*

$$\sin^{-1}\frac{\mu}{VR\left\{V^2 - \frac{2\mu}{R}\right\}^{1/2}}.$$

Sol. We know that in the case of hyperbola

$$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right].$$

Therefore in this case

$$V^2 = \mu\left[\frac{2}{R} + \frac{1}{a}\right] \qquad \text{as } [v = V, r = R \text{ given}]$$

or $$V^2 - \frac{2\mu}{R} = \frac{\mu}{a} \qquad ...(1)$$

Also we know that $h = pv$.

Let $p = p_0$ initially.

$$h = p_0 V = R \sin \alpha V \qquad (\because\ p_0 = R \sin \alpha) \qquad ...(2)$$

Also we know that $h = \sqrt{(l\mu)} = \sqrt{\left\{\mu \frac{b^2}{a}\right\}}$.

Therefore (2) reduces to

$$\sqrt{(\mu a)} = R \sin \alpha . V \quad \text{or} \quad \mu a = R^2 \sin^2 \alpha . V^2$$

or $$\frac{\mu \times \mu}{\left(V^2 - \frac{2\mu}{R}\right)} = R^2 V^2 \sin^2 \alpha \qquad \left\{\because \text{ from (1), } a = \frac{\mu}{V^2 - \frac{2\mu}{R}}\right\}$$

or $$\sin^2 \alpha = \frac{\mu^2}{R^2 V^2 \left\{V^2 - \frac{2\mu}{R}\right\}} \quad \text{or} \quad \sin \alpha = \frac{\mu}{RV\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}$$

or $$\alpha = \sin^{-1} \frac{\mu}{V\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}.$$

2. *A particle describes an ellipse under a force* $\frac{\mu}{[distance]^2}$ *towards the focus. If it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is*

$$\frac{2\pi}{\sqrt{\mu}}\left\{\frac{2}{r} - \frac{V^2}{\mu}\right\}^{-3/2}$$

Sol. Since orbit described is an ellipse, therefore

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \text{or} \quad \frac{V^2}{\mu} = \frac{2}{r} - \frac{1}{a}$$

or $$\frac{1}{a} = \left(\frac{2}{r} - \frac{V^2}{\mu}\right). \qquad ...(1)$$

Now periodic time is given by $T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$. ...(2)

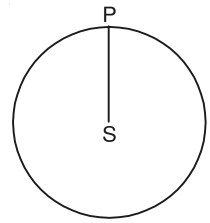
Therefore from (1) and (2), we get

$$T = \frac{2\pi}{\sqrt{\mu}}\left(\frac{2}{r} - \frac{V^2}{\mu}\right)^{-3/2}$$

3. *If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the Sun in a time which is* $\frac{\sqrt{2}}{8}$ *times the period of the planet's revolution.*

Sol. Let a be the radius circular path and the particle be suddenly stopped at point P, *i.e.*, its velocity suddenly reduces to zero; then the planet will begin to move in the straight line PS towards S under the acceleration

$$\frac{\mu}{(\text{distance})^2}$$



Now equation of motion of the particle while falling from P to S is

$$v\frac{dv}{dr} = -\frac{\mu}{r^2} \text{ (} r \text{ decreasing)} \quad \text{or} \quad v\,dv = -\frac{\mu}{r^2}\,dr.$$

On integrating, we get $\quad v^2 = \frac{2\mu}{r} + A.$

Now when $r = a, v = 0;\quad \therefore\quad A = -\frac{2\mu}{a}$

Therefore $\quad v^2 = 2\mu\left[\frac{1}{r} - \frac{1}{a}\right] = \frac{2\mu(a-r)}{ar}.$

$$\therefore \quad v = \frac{dr}{dt} = -\sqrt{(2\mu)}\sqrt{\left(\frac{a-r}{ar}\right)} \text{ (} r \text{ decreasing).}$$

$$\therefore \quad \int_a^0 \sqrt{\left[\frac{r}{a-r}\right]}\,dr = -\sqrt{\left[\frac{2\mu}{a}\right]}\int_0^t dt,$$

where r is time of reaching the planet to the Sun.

$$\therefore \quad -\sqrt{\left[\frac{2\mu}{a}\right]}\,t = \int_a^0 \sqrt{\left[\frac{r}{a-r}\right]}\,dr.$$

Put $r = a\cos^2\theta,\ dr = -2a\cos\theta\sin\theta\,d\theta.$

$$\therefore \quad -\sqrt{\left[\frac{2\mu}{a}\right]}\,t = -\int_0^{\pi/2} 2a\cos^2\theta\,d\theta.$$

$$\text{or} \quad \sqrt{\left[\frac{2\mu}{a}\right]}\,t = 2a\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$\text{or} \quad \sqrt{\left(\frac{2\mu}{a}\right)}\,t = 2a\,\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{or} \quad t = \frac{\pi a^{3/2}}{2\sqrt{(2\mu)}}$$

If T be the periodic time of planet's revolution, then.

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \qquad \therefore \quad \frac{t}{T} = \frac{2\sqrt{(2\mu)}}{2\pi a^{3/2}}$$

$$\therefore \quad \frac{t}{T} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8} \quad \text{or} \quad t = \frac{\sqrt{2}}{8}T.$$

4. *If the velocity of a body in an elliptic orbit, major axis 2a, is the same at a certain point P, whether the orbit is being described in a periodic time T about one focus S or in periodic time T' about the other focus S', prove that*

$$SP = \frac{2aT}{T+T'} \quad \textit{and} \quad S'P = \frac{2aT}{T+T'}.$$

Sol. As we know that sum of focal distances of any point on the ellipse is equal to $2a$, so if $SP = r,$ then

$$S'P = 2a - r.$$

Now when S is the focus, then

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right] \quad \text{and} \quad T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}. \qquad \text{...(1)}$$

Now when S' is the focus, then

$$v^2 = \mu'\left[\frac{2}{2a-r} - \frac{1}{a}\right] \text{ and } T' = \frac{2\pi a^{3/2}}{\sqrt{\mu'}}. \quad \text{...(2)}$$

$\therefore$ From (1) and (2), we get

$$\mu\left[\frac{2}{r} - \frac{1}{a}\right] = \mu'\left[\frac{2}{2a-r} - \frac{1}{a}\right]. \quad \text{...(3)}$$

But $\mu = \dfrac{4\pi^2 a^3}{T^2}$ from (1) and $\mu' = \dfrac{4\pi^2 a^3}{T'^2}$ from (2)

Therefore (3) reduces to

$$\frac{4\pi^2 a^3}{T^2}\left[\frac{2}{r} - \frac{1}{a}\right] = \frac{4\pi^2 a^3}{T'^2}\left[\frac{2}{2a-r} - \frac{1}{a}\right]$$

or $$\frac{1}{T^2} \cdot \frac{2a-r}{ar} = \frac{1}{T'^2} \cdot \frac{r}{a(2a-r)} \quad \text{or} \quad T^2 r^2 = T'^2 (2a-r)^2$$

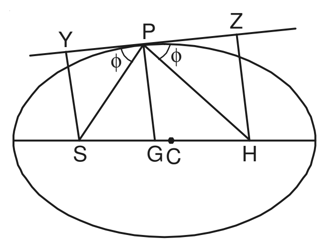
or $$Tr = T'(2a-r) \quad \text{or} \quad (T+T')\, r = 2aT'$$

or $$r = \frac{2aT'}{T+T'} \quad i.e., \quad SP = \frac{2aT'}{T+T'}.$$

Therefore $$S'P = 2a - \frac{2aT'}{T+T'} \qquad [\text{as } S'P = 2a - SP]$$

or $$S'P = \frac{2aT}{T+T'}.$$

5. *A particle describes an ellipse about a centre of force at the focus; show that any point of its path, the angular velocity about the other focus varies inversely as the square of the normal at that point.*



Sol. $r^2 \dfrac{d\theta}{dt} = h = pv$ in a central orbit where $\dfrac{d\theta}{dt}$ is angular velcoity about the focus S. Therefore the angular velocity (say ω) about the other focus H by above formula is given by

$$HP^2 . \omega = HZ . v, \qquad [\because \ p = HZ]$$

or $$HP^2 \omega = HZ . \frac{h}{SY},$$

$$\left[\because \ h = pv, \ \therefore \ v = \frac{h}{p} \text{ or } v = \frac{h}{SY}\right].$$

$\therefore$ $$\omega = \frac{h . HZ}{SY . HP^2} \quad \text{or} \quad \omega = \frac{h . HZ}{SY . HP . HP}. \quad \text{...(1)}$$

Now since the Δs HZP and SYP are similar,

$$\therefore \quad \frac{HZ}{HP} = \frac{SY}{SP}.$$

Therefore from (1),

$$\omega = \frac{h}{SY . HP} \cdot \frac{HZ}{HP} = \frac{h}{SY . HP} \cdot \frac{SY}{SP} = \frac{h}{SP . HP}. \quad \text{...(2)}$$

But $SP = a + ex' = a + ae \cos\phi$ and $HP = (a - ae') = a - ae\cos\phi$.

$\therefore$ from (2),

$$\omega = \frac{h}{(a + ae\cos\phi)(a - ae\cos\phi)} = \frac{h}{a^2 - a^2 e^2 \cos^2\phi}$$

or $$\omega = \frac{h}{a^2 - (a^2 - b^2)\cos^2\phi} = \frac{h}{b^2\cos^2\phi + a^2\sin^2\phi}. \quad ...(3)$$

Normal at P, whose co-ordinates are $(a\cos\phi, b\sin\phi)$, is given by

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2$$

or $$ax\sec\phi - by\,\text{cosec}\,\phi = a^2 - b^2.$$

Now above normal meets the major axis at G, where $y = 0$.

Hence co-ordiantes of G are $\left[\frac{a^2 - b^2}{a}\cos\phi, 0\right]$.

Therefore $$PG^2 = \left[a\cos\phi - \frac{a^2 - b^2}{a}\cos\phi\right]^2 + b^2\sin^2\phi$$

$$= \frac{b^2}{a^2}\cos^2\phi + b^2\sin^2\phi$$

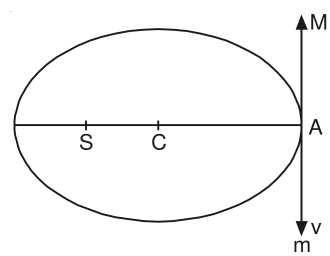
or $$PG^2 = \frac{b^2}{a^2}(b^2\cos^2\phi + a^2\sin^2\phi)$$

or $$(b^2\cos^2\phi + a^2\sin^2\phi) = \frac{b^2}{a^2}\cdot\frac{1}{PG^2}$$

$\therefore$ from (3), $$\omega = \frac{h}{b^2\cos^2\phi + a^2\sin^2\phi} = h\,\frac{b^2}{a^2}\,\frac{1}{PG^2}.$$

6. *A planet of mass M and periodic time T, when at its greatest distance from the Sun, comes into collision with a meteor of mass m, moving in the same orbit in the opposite direction with velocity v; if m/M be small, show that* major axis of the planet's path is reduced by

$$\frac{4m}{M}\frac{vT}{\pi}\sqrt{\left(\frac{1-e}{1+e}\right)}$$



Sol. Let v be the velocity of the planet and meteor at A, before collision. Since orbit is same for planet and meteor, hence their velocities are equal at A.

$$\therefore \quad v^2 = \mu\left[\frac{2}{a + ae} - \frac{1}{a}\right]$$

$$= \frac{\mu}{a}\cdot\frac{1-e}{1+e}. \quad ...(1)$$

[since $SA = SC + CA = ae - a$]

Let V be the velocity of combined body after the collision.

Since momentum is unaltered by collision, by the principle of conservation of momentum, we get

$$(M + m)V = Mv - mv$$

or $$V = \frac{M - m}{M + m}v = \left(1 - \frac{m}{M}\right)\left(1 + \frac{m}{M}\right)^{-1} v$$

$$= \left\{1 - \frac{m}{M}\right\}\left\{1 - \frac{m}{M}\right\} v \text{ nearly} \quad \left\{\text{as } \frac{m}{M} \text{ is small}\right\}$$

$$= \left\{1 - \frac{m}{M} - \frac{m}{M}\right\} v \qquad \left\{\text{neglecting } \left(\frac{m}{M}\right)^2\right\}$$

or $$V = \left\{1 - \frac{2m}{M}\right\} v \qquad \text{...(2)}$$

Hence from (2), we observe that

$$V^2 < v^2 \text{ or } V^2 < \frac{2\mu}{a + ae} \quad \text{as} \quad v^2 = \mu\left\{\frac{2}{a+ae} - \frac{1}{a}\right\} \quad \text{or} \quad V^2 < \frac{2\mu}{SA},$$

Therefore the subsequent path of the combined body is ellipse. Let its major axis be $2a'$.
Now for the new path *i.e.,* ellipse at A,

$$V^2 = \mu\left[\frac{2}{a+ae} - \frac{1}{a'}\right] \qquad \text{...(3)}$$

or $$\left(1 - \frac{2m}{M}\right)^2 v^2 = \mu\left(\frac{2}{a+ae} - \frac{1}{a'}\right) \qquad \text{[from (2)]}$$

or $$\left(1 - \frac{4m}{M}\right) v^2 = \mu\left(\frac{2}{a+ae} - \frac{1}{a'}\right) \qquad \left\{\text{neglecting } \left(\frac{m}{M}\right)^2 \text{ and higher powers}\right\}$$

or $$\left(1 - \frac{4m}{M}\right) . \frac{1-e}{1+e} = \mu\left(\frac{2}{a+ae} - \frac{1}{a'}\right) \qquad \text{[from (1)]}$$

or $$\left\{1 - \frac{4m}{M}\right\} \frac{1}{a} . \frac{1-e}{1+e} = \frac{2}{a+ae} - \frac{1}{a'}$$

or $$\frac{1}{a'} = \frac{1}{a}\left(\frac{2}{1+e} - \frac{1-e}{1+e} + \frac{4m}{M} . \frac{1-e}{1+e}\right) = \frac{1}{a}\left(1 + \frac{4m}{M} . \frac{1-e}{1+e}\right)$$

or $$a' = a\left(1 + \frac{4m}{M}\frac{1-e}{1+e}\right)^{-1} = a\left(1 - \frac{4m}{M} . \frac{1-e}{1+e}\right) \text{ nearly.}$$

$\therefore$ $$2a' - 2a = -\frac{4m}{M}\frac{1-e}{1+e} . 2a.$$

or major axis of subsequent path is reduced by

$$\left(\frac{4m}{M} . \frac{1-e}{1+e} . 2a\right).$$

Also $$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}; \ \therefore vT = \sqrt{\left(\frac{\mu}{a} . \frac{1-e}{1+e}\right)} \frac{2\pi a^{3/2}}{\sqrt{\mu}} \qquad \left(\text{as } v^2 = \frac{\mu}{a} . \frac{1-e}{1+e}\right)$$

or $$vT = 2a\pi\sqrt{\left(\frac{1-e}{1+e}\right)}; \quad \therefore \quad 2a = \frac{vT}{\pi}\sqrt{\left(\frac{1+e}{1-e}\right)}$$

$\therefore$ $$\frac{4m}{M} . \frac{1-e}{1+e} 2a = \frac{4m}{M} . \frac{1-e}{1+e} . \frac{vT}{\pi}\sqrt{\left(\frac{1+e}{1-e}\right)}.$$

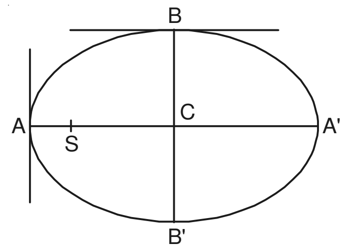
Hence major axis of subsequent path is reduced by

$$\frac{4m}{M}\frac{vT}{\pi}\sqrt{\left(\frac{1-e}{1+e}\right)}.$$

7. *A particle is moving in ellipse of eccentricity e, under the acceleration* $\frac{\mu}{r^2}$ *to a focus; when the particle is nearest to a focus, the acceleration is suddenly replaced by an acceleration* μr *towards the centre of the ellipse. If the particle continues to move in the same ellipse, prove that*

$$\mu = \mu'(1-e^2)\,a^2.$$

Sol. Let v be the velocity of particle at point A which is nearest to the focus S; then



$$v^2 = \mu\left(\frac{2}{AS} - \frac{1}{a}\right)$$

$$= \mu\left(\frac{2}{a-ae} - \frac{1}{a}\right)$$

or $$v^2 = \frac{\mu}{a}\frac{1+e}{1-e}. \qquad ...(1)$$

Now the centre of force is shifted to centre *i.e.,* at point C and new force is $\mu' r$.
Now distance of point A from C is a and velocity of particle at A is v.

Also $\frac{du}{d\theta} = 0$ at point A, being an apse at A

The differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{\mu'}{h^2u^2} \qquad \left[\text{as } P = \mu' r = \frac{\mu'}{\mu}\right].$$

Integrating above, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = -\frac{2\mu'}{2u^2} + A, \qquad ...(2)$$

Now when $u = \frac{1}{a}, \frac{du}{d\theta} = 0$ and $v^2 = \frac{\mu}{a}\frac{1+e}{1-e}$ from (1),

$$\therefore \qquad \frac{\mu}{a}\cdot\frac{1+e}{1-e} = \frac{h^2}{a^2} = -\mu' a^2 + A.$$

Therefore $au\cdot\frac{1+e}{1-e}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = -\frac{\mu'}{u^2} + \frac{\mu}{a}\cdot\frac{1+e}{1-e} + \mu' a^2.$

Therefore equation (2) reduces to

$$au\cdot\frac{1+e}{1-e}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = -\frac{\mu'}{u^2} + \frac{\mu}{a}\cdot\frac{1+e}{1-e} + \mu' a^2. \qquad ...(3)$$

Now as the particle continues to move in the same ellipse, the next apse will be at point B *i.e.,* next apse be $r = b$.

Therefore when $r = b$ or $u = \frac{1}{b}, \frac{du}{d\theta} = 0.$

Therefore in (3), putting $u = \frac{1}{b}$ and $\frac{du}{d\theta} = 0$, we get

$$a\mu\frac{1+e}{1-e}\cdot\frac{1}{b^2} = -\mu' b^2 + \frac{\mu}{b}\cdot\frac{1+e}{1-e} + \mu' a^2$$

or $$\frac{\mu}{a}\cdot\frac{1+e}{1-e}\cdot\left(\frac{a^2}{b^2} - 1\right) = \mu'(a^2 - b^2)$$

or $$\frac{\mu}{a} \cdot \frac{1+e}{1-e} \cdot \frac{(a^2-b^2)}{a^2(1-e^2)} = \mu'(a^2-b^2)$$

or $$\frac{\mu}{a} \cdot \frac{1+e}{1-e} \cdot \frac{1}{a^2(1-e)(1+e)} = \mu' \quad \text{or} \quad \mu = \mu' a^3 (1-e)^2.$$

EXERCISES

1. Show that the velocity of a particle moving in an ellipse about a centre in the focus is compounded of two constant velocities, $\frac{\mu}{h}$ perpendicular to the radius and $\frac{\mu e}{h}$ perpendicular to the major axis. Hence deduce that the radial velocity is given by the equation

$$r^2\left(\frac{dr}{dt}\right)^2 = \frac{\mu}{a}\{a(1+e)-r\}\{r-a(1-e)\}$$

2. A planet is describing an ellipse about the Sun as focus. Show that its velocity away from the Sun is greatest when the radius vector to the planet is at right angles to the major axis of path and that is then $\frac{2\pi ae}{T\sqrt{(1-e^2)}}$, where $2a$ is the major axis, e the eccentricity and T the periodic time.

3. Prove that the time taken by the earth to travel over half its orbit, remote from the Sun, separated by the minor axis is 2 days more than half the year. The eccentricity of the orbit is $\frac{1}{60}$.

4. A body is moving in an ellipse about a centre of force in the focus. When it arrives at P, the direction of motion is turned through a right angle, the speed being unaltered. Show that the body will describe an ellipse whose eccentricity varies as the distance of P from the centre.

5. A comet is moving in a parabola about the Sun as focus. When at the end of its latus rectum its velocity suddenly becomes altered in the ratio of $n:1$, where $n<1$. Show that the comet will describe an ellipse whose eccentricity is $\sqrt{(1-2n^2+2n^4)}$ and whose major axis is $\frac{l}{1-n^2}$ where $2l$ was the latus rectum of the parabolic path.

6. A particle is describing a parabola under a force to the focus. It meets and coalesces with another particle of n times its mass which was at rest before the impact. Show that the composite body will describe an ellipse whose eccentricity is given by

$$1-e^2 = \frac{4n(n+2)}{(n+1)^4}\cos^2\frac{\theta}{2},$$

where θ is measured from the apse in the parabola.

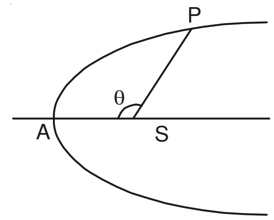
7. A particle of mass m when at any point of the ellipse of semi-major axis a is split by an explosion into two particles of masses m_1 and m_2. If the particle m_1 describes a parabola, prove the semi-major axis of the orbit described by m_2 is $\frac{\mu m_2 a}{\mu m - aE}$, where $\frac{E}{2}$ is the energy generated by the explosion.

8. A body describes an ellipse under a force to the focus and when at the extremity of the minor axis, moving towards the nearer apse, it receives a blow in the direction of the other focus which cause it to move towards the centre of the ellipse. Show that the eccentricity of the new orbit is $(3e^2-3+e^{-2})^{1/2}$ and that the major axis is turned through angle whose tangent is

$$\frac{ab}{a(2e^2-1)}.$$

8.6

(a) Time to Describe a given arc of Parabolic Orbit Starting from the Vertex



We know that polar equation of parabola is

$$\frac{1}{r} = 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

or $$r = \frac{l}{2}\sec^2\frac{\theta}{2} = a\sec^2\frac{\theta}{2}$$

[as $l = 2a$ in parabola].

Also we know that $r^2\dfrac{d\theta}{dt} = h$

or $$h\,dt = r^2\,d\theta \quad \text{or} \quad h\int_0^t dt = \int_0^\theta r^2\,d\theta$$

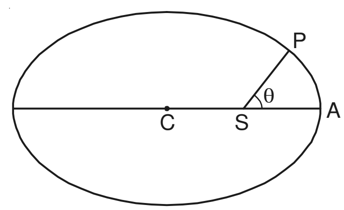
or $$ht = \int_0^\theta a^2\sec^2\frac{\theta}{2}\,d\theta \qquad \left\{\text{as } r = a\sec^2\frac{\theta}{2}\right\}$$

or $$ht\,a^2\int_0^\theta \sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta = 2a^2\left(\tan\frac{\theta}{3} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

or $$\sqrt{(\mu\,2a)}\;t = 2a^2\left(\tan\frac{\theta}{2} \pm \frac{1}{3}\tan^3\frac{\theta}{2}\right) \qquad [\text{as } g = \sqrt{(\mu l)},\ l = 2a]$$

or $$t = \sqrt{\left(\frac{2a^2}{\mu}\right)}\left\{\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right\}.$$

(b) To Find the Time of Description of given arc of an Elliptic Orbit Starting from the Nearer End of the Major Axis.



Polar equation of an ellipse with S as focus and SA as initial line is

$$\frac{1}{r} = 1 + e\cos\theta, \quad \text{where } e < 1.$$

Also $$r^2\frac{d\theta}{dt} = h \quad \text{or} \quad h\,dt = r^2\,d\theta$$

or $$h\int_0^t dt = \int_0^\theta r^2\,d\theta \quad \text{or} \quad ht = l^2\int_0^\theta \frac{1}{(1+e\cos\theta)^2}\,d\theta. \qquad \text{...(1)}$$

Now $$\frac{d}{d\theta}\left\{\frac{\sin\theta}{1+e\cos\theta}\right\} = \frac{(1+e\cos\theta)\cos\theta + e\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{e+\cos\theta}{(1+e\cos\theta)^2} = \frac{e^2 + e\cos\theta}{e\,(1+e\cos\theta)^2}$$

$$= \frac{(1+e\cos\theta)\,(1-e^2)}{e\,(1+e\cos\theta)^2} \quad [\text{as } e < 1]$$

or $$\frac{d}{d\theta}\left\{\frac{\sin\theta}{1+e\cos\theta}\right\} = \frac{1}{e\,(1+e\cos\theta)} - \frac{1-e^2}{e}\,\frac{1}{(1+e\cos\theta)^2}$$

or $$\frac{1-e^2}{e\,(1+e\cos\theta)^2} = \frac{1}{e\,(1+e\cos\theta)} - \frac{d}{d\theta}\left\{\frac{\sin\theta}{1-e\cos\theta}\right\}$$

On integrating above, we get

$$\frac{1-e^2}{e}\int\frac{1}{(1+e\cos\theta)^2}\,d\theta=\frac{1}{e}\int\frac{1}{1+e\cos\theta}\,d\theta-\frac{\sin\theta}{1+e\cos\theta}$$

or $$\int\frac{1}{(1+e\cos\theta)^2}\,d\theta=\frac{1}{(1-e^2)}\int\frac{1}{(1+e\cos\theta)}\,d\theta-\frac{e}{1-e^2}\cdot\frac{\sin\theta}{1+e\cos\theta}$$

Also we know that

$$\int\frac{d\theta}{1+e\cos\theta}=\frac{2}{\sqrt{(1-e)}}\left\{\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}\text{ if } e<1.$$

$\therefore$ $$\int\frac{1}{(1+e\cos\theta)^2}\,d\theta=\frac{2}{(1-e^2)^{3/2}}\left\{\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-\frac{e}{1-e^2}\frac{\sin\theta}{(1+e\cos\theta)}$$

Therefore from (1),

$$ht=l^2\int_0^{\theta}\frac{1}{(1+e\cos\theta)^2}\,d\theta.$$

or $$ht=l^2\left[\left\{\frac{2}{(1-e^2)^{3/2}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}-\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\right\}\right]$$

Now as $h=\sqrt{(\mu l)}=\sqrt{\{\mu . a\,(1-e^2)\}}$,

and $$l=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a\,(1-e^2).$$

$\therefore$ $$\sqrt{\{\mu a\,(1-e^2)\}}\,t=\frac{2a^2(1-e^2)^2}{(1-e^2)^{3/2}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}-\frac{ea^2(1-e^2)^2}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}$$

or $$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right].$$

(c) To find the time of description of a given arc of a hyperbolic orbit.

Polar equation of hyperbola is

$$\frac{l}{r}=1+e\cos\theta,\quad\text{where } e>1.$$

Also we know that

$$r^2\frac{d\theta}{dt}=h\quad\text{or}\quad dt=r^2\,d\theta$$

or $$\int_0^t h\,dt=\int_0^{\theta}r^2\,d\theta$$

or $$\int_0^t h\,dt=\int_0^{\theta}\frac{l^2}{(1+e\cos\theta)^2}\,d\theta. \qquad \text{...(1)}$$

Now $$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)=\frac{\cos\theta\,(1+e\cos\theta)+e\sin^2\theta}{(1+e\cos\theta)^2}$$

$$=\frac{\cos\theta+e\cos^2\theta+e\sin^2\theta}{(1+e\cos\theta)^2}=\frac{e+\cos\theta}{(1+e\cos\theta)^2}$$

$$=\frac{e\cos\theta+e^2}{e\,(1+e\cos\theta)^2}=\frac{1+e\cos\theta+e^2-1}{e\,(1+e\cos\theta)^2}$$

or $$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)=\frac{1}{e\,(1+e\cos\theta)}+\frac{e^4-1}{e\,(1+e\cos\theta)^2}\qquad\text{(because } e>1)$$

or $$\frac{1}{(1+e\cos\theta)^2} = \frac{e}{e^2-1}\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right) - \frac{1}{e^2-1}\frac{1}{(1+e\cos\theta)}$$

Integrating both sides of above, we get

$$\int \frac{1}{(1+e\cos\theta)^2}\,d\theta = \frac{e}{e^2-1}\frac{\sin\theta}{1+e\cos\theta} - \frac{1}{e^2-1}\int \frac{1}{(1+e\cos\theta)}\,d\theta.$$

Therefore (1) reduces to

$$ht = \frac{l^2 e\sin\theta}{(e^2-1)(1+e\cos\theta)} - \frac{l^2}{e^2-1}\int_0^{\theta} \frac{1}{(1+e\cos\theta)}\,d\theta$$

or $$ht = \frac{l^2 e\sin\theta}{(e^2-1)(1+e\cos\theta)} - \frac{l^2}{(e^2-1)}\cdot\frac{1}{\sqrt{(e^2-1)}}\log\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}$$

as $e>1$

or $$t = \frac{l^2}{h}\left\{\frac{e\sin\theta}{(e^2-1)(1+e\cos\theta)} - \frac{1}{(e^2-1)^{3/2}}\log\frac{\sqrt{(e-1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}.$$

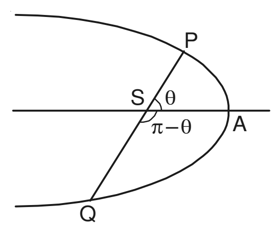
Now $$\frac{l^2}{h} = \frac{l^2}{\sqrt{(\mu l)}} = \frac{l^{3/2}}{\sqrt{\mu}} = \frac{a^{3/2}(e^2-1)^{3/2}}{\mu}$$

$$\therefore \quad t = \frac{a^{2/3}}{\sqrt{\mu}}\left\{e\sqrt{(e-1)}\,\frac{\sin\theta}{1+e\cos\theta} - \log\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}$$

EXAMPLES

1. *Find the time T of describing an arc C of a parbolic orbit Newtonian law; if C be bounded by a focal chord, prove that*

$$T \propto \int (focal\ chord)^{3/2}.$$



Sol. We have already found the time of description of given arc of parabolic orbit as

or $$\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{1}{2}\theta + \frac{1}{2}\tan^3\frac{1}{2}\theta\right].$$

Let there be a focal chord PQ inclined at an angle θ to the initial line, so that the vectorial angle of Q will be $(\pi-\theta)$ below the initial line.

Therefore the time of describing the arc bounded by the focal chord PQ is

$$T = \sqrt{\left(\frac{2a^3}{\mu}\right)}\left\{\tan\frac{1}{2}\theta + \frac{1}{2}\tan^3\frac{1}{2}\theta\right\}_{-(\pi-\theta)}^{\theta}$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta + \cot\frac{1}{2}\theta + \frac{1}{3}\cot^3\frac{1}{2}\theta\right]$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta + \frac{1}{3}\left(\tan^3\frac{\theta}{2} + \cot^3\frac{\theta}{2}\right)\right]$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\frac{\sin^2\frac{1}{2}\theta + \cos^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} + \frac{1}{3}\,\frac{\sin^6\frac{1}{2}\theta\cos^2\frac{1}{2}\theta}{\sin^3\frac{1}{2}\theta\cos^3\frac{1}{2}\theta}\right]$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\frac{2}{\sin\theta} + \frac{8}{3}\,\frac{\left(\sin^2\frac{1}{2}\theta + \cos^2\frac{1}{2}\theta\right)\left(\sin^4\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta + \cos^4\frac{1}{2}\theta\right)}{\sin^3\theta}\right]$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\frac{2}{\sin\theta} + \frac{8}{3\sin^3\theta}\left\{\left(\sin^2\frac{1}{2}\theta + \cos^2\frac{1}{2}\theta\right)^2 - 3\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta\right\}\right]$$

$$= \sqrt{\left(\frac{2a^3}{\mu}\right)}\left\{\frac{2}{\sin\theta} + \frac{8}{3\sin^3\theta}\left(1 - 3\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta\right)\right\}$$

$$= \sqrt{2\left(\frac{2a^3}{\mu}\right)}\left\{\frac{3\sin\theta + 4\left(1 - \frac{3}{4}\sin^2\right)}{3\sin^3\theta}\right\} = \sqrt{2\left(\frac{2a}{\mu}\right)}\left(\frac{3\sin^2\theta + 4 - 3\sin^2 6}{3\sin^3\theta}\right)$$

or $\quad T = 2\sqrt{\left(\frac{2a^3}{\mu}\right)}\,\frac{4}{3}\operatorname{cosec}\theta = \frac{1}{3}\sqrt{\left(\frac{2}{\mu}\right)}\,[4a\operatorname{cosec}^2\theta]^{3/2}$

$$= \frac{1}{3} = \sqrt{\left(\frac{2}{\mu}\right)}\,[\text{focal chord}]^{3/2}.$$

Since $\quad PQ = SP + SQ = \dfrac{l}{1+\cos} + \dfrac{l}{1+\cos(\pi+\theta)}$

[as vectorial angle of $Q = \pi + \theta$ measured anti-clockwise]

$$= \frac{l}{1+\cos\theta} + \frac{l}{1-\cos\theta} = 2l\operatorname{cosec}^2\theta$$

or $\quad PQ = 4a\operatorname{cosec}^2\theta \qquad$ (as $l = 2a$).

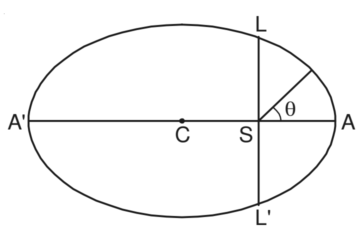
$\therefore \quad T = \dfrac{1}{3}\sqrt{\left(\dfrac{2}{\mu}\right)}\,[\text{focal chord}]^{3/2}$

or $\quad T \propto (\text{focal chord})^{3/2}$

2. *Prove that time taken to describe two portions into which an ellipse is divided by the latus rectum through the centre of the force are in a ratio*

$$\{\cos^{-1}e - e\sqrt{(1-e^2)}\} : \{\pi - \cos^{-1}e + e\sqrt{(1-e^2)}\}.$$

Sol. We have already found the time t of describing an arc of elliptic orbit given by



$$t = \frac{a^{3/2}}{\sqrt{\mu}} \left[2 \tan^{-1} \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \frac{\theta}{2} - e\sqrt{(1-e^2)} \frac{\sin\theta}{1+e\cos\theta} \right]$$

Now in the above formula if we put $\theta = \pi/2$ it will give us time of description of arc AL : let it be t_1.

$$\therefore \qquad t_1 = \frac{a^{3/2}}{\sqrt{\mu}} \left[2 \tan^{-1} \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \frac{\pi}{4} - e\sqrt{(1-e^2)} \frac{\sin(\pi/2)}{1+e\cos(\pi/2)} \right]$$

$$\text{or} \qquad t_1 = \frac{a^{3/2}}{\sqrt{\mu}} \left[2 \tan^{-1} \sqrt{\left(\frac{1-e}{1+e}\right)} - e\sqrt{(1-e)} \right].$$

Now $\quad 2 \tan^{-1} \sqrt{\left(\frac{1-e}{1+e}\right)} = 2 \tan^{-1} \tan \frac{z}{2} = \tan^{-1} \sqrt{\left(\frac{1-\cos z}{1+\cos z}\right)}$, by putting $e = \cos z$

or $\quad 2 \tan^{-1} \sqrt{\left(\frac{1-e}{1+e}\right)} = 2, \frac{z}{2} = z = \cos^{-1} e.$

Therefore (1) reduces to

$$t_1 = \frac{a^{3/2}}{\sqrt{\mu}} [\cos^{-1} e - \sqrt{(1-e^2)}]$$

The time of describing the arc $LA'L' = 2t_1$

or time of describing the arc LAL'

$$= \frac{2a^{3/2}}{\sqrt{\mu}} [\cos^{-1} e - e\sqrt{(1-e^2)}].$$

Now since the period to describe an ellipse is $\frac{2\pi a^{3/2}}{\sqrt{\mu}}$

Therefore the time to describe the arc $LA'L'$ is

$$\frac{2a^{3/2}}{\sqrt{\mu}} - \frac{2a^{3/2}}{\sqrt{\mu}} [\cos^{-1} e - e\sqrt{(1-e^2)}]$$

$$\text{or} \qquad \frac{2a^{3/2}}{\sqrt{\mu}} [\pi - \cos^{-1} e + e\sqrt{(1-e^2)}].$$

Hence the ratio of times of describing the arcs $LA'L$ and $LA'L'$ is

$$[\cos^{-1} e - e\sqrt{(1-e^2)}] : [\pi - \cos^{-1} e + e\sqrt{(1-e^2)}].$$

EXERCISES

1. Prove that in a parabolic orbit the time taken to move from the vertex to a point distant r from the focus is

$$\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{(2r-l)},$$

where $2l$ is latus rectum.

2. If the period of a planet be 365 days and the eccentricity $e=\frac{1}{60}$, show that times of describing the two halves of the orbit bounded by the latus rectum through the centre of force are

$$\frac{365}{2}\left[1\pm\frac{1}{15\pi}\right]\text{nearly}.$$

3. The perihelion distance of a planet describing a parabolic orbit is $\frac{1}{n}$ of the radius of the earth's path supposed circular, show that the time that the comet remain with the earth's orbit is

$$\frac{2}{3\pi}\,\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)}\text{ of a year}.$$

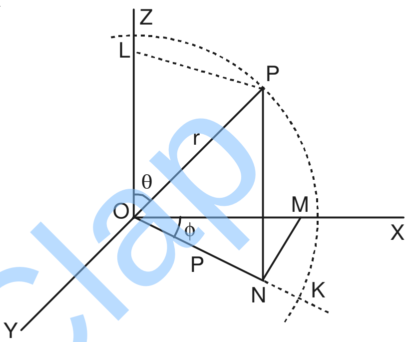
Also prove that the longest time that the comet will remain within the earth's orbit is $\frac{2}{3\pi}$ of an year.

❑❑❑

Motion in Three Dimensions

9.1 ACCELERATION OF A PARTICLE IN POLAR CO-ORDINATES

To find the acceleration of particle in terms of polar coordinates.



Let at any instant particle be at P whose coordinates be r, θ and ϕ, where r is the distance of a P from a fixed origin O, θ is the angle that OP makes with z-axis and ϕ is the angle that the plane ZOP makes with a fixed plane ZOX (Z-X plane).

Draw PN perpendicular to the plane XOY and let $ON = \rho$.

Then the acceleration of P are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$, where x, y and z are the coordinates of P.

The polar coordinates of N, which in always is the plane XOY are ρ and ϕ. This point N will have radial and transverse accelerations whose magnitudes are

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 \text{ along } ON,$$

and
$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) \text{ perpendicular to } ON$$

Also the acceleration of P relative to N is $\frac{d^2z}{dt^2}$ along NP.

Hence, the acceleration of P are

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 \text{ along } LP,$$

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) \text{ perpendicular to the plane } ZPK$$

and
$$\frac{d^2z}{dt^2} \text{ parallel to } OZ.$$

Since $z = r\cos\theta$ and $\rho = r\sin\theta$, the acceleration $\frac{d^2z}{dt^2}$ and $\frac{d^2\rho}{dt^2}$, along and perpendicular to OZ is the plane ZPK, are equivalent to $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ along OP (radial acceleration) and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ perpendicular to OP (transverse acceleration) in the plane ZPK.

Also the acceleation $-\rho\left(\frac{d\phi}{dt}\right)^2$ along LP is equivalent to $-\rho\sin\theta\left(\frac{d\phi}{dt}\right)^2$ along OP and $-\rho\cos\theta\left(\frac{d\phi}{dt}\right)^2$ perpendicular to OP.

Hence, if α, β, γ be the accelerations of P respectively along OP, perpendicular to OP in the plane ZPK in the direction of θ increasing, and perpendicular to the plane ZPK in the direction of ϕ increasing, we have

$$\alpha = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 - \rho\sin\theta\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2$$

$$\beta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \rho\cos\theta\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 \quad ...(2)$$

and $$\gamma = \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right) = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right) \quad ...(3)$$

Cor. Acceleration in Cylindrical Coordinates : Sometimes we consider the motion of P with reference to the coordinates z, ρ and ϕ, called cylindrical coordinates.

As in 9.1, the accelerations are then

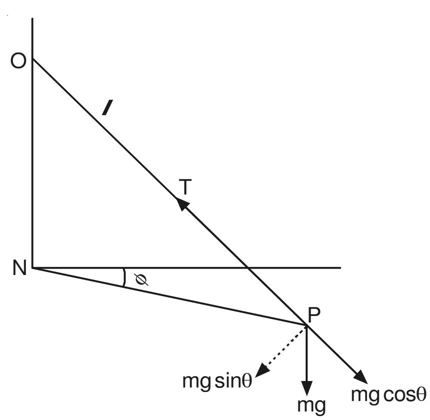
$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 \text{ along } LP,$$

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) \text{ perpendicular to the plane } ZPK$$

and $$\frac{d^2z}{dt^2} \text{ parallel to } OZ.$$

9.2

A particle is attached to one end of a string of length l, the other end of which is tied to a fixed point O. When the string in inclined at an acute angle α to the downward-drawn vertical the particle is projected horizontally and perpendicular to the string with a velocity V; to find the resulting motion.



Taking O as origin, the components of weight mg of the particle are $mg\cos\theta$ (along r), $-mg\sin\theta$ (perpendicular to OP) and 0 for P, we have $r = l$. Hence, the equations of motion will be $\left[\text{as } \frac{dl}{dt} = \frac{d^2 l}{dt^2} = 0\right]$

$$-l\left(\frac{d\theta}{dt}\right)^2 - l\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 = \frac{T}{m} + g\cos\theta \quad \text{...(1)}$$

$$l\frac{d^2\theta}{dt^2} - l\cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2 = -g\sin\theta \quad \text{...(2)}$$

and

$$\frac{1}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right) = 0 \quad \text{...(3)}$$

Equation (3) gives

$$\sin^2\theta\frac{d\phi}{dt} = \text{constant} = \sin^2\alpha\left[\frac{d\phi}{dt}\right]_{\text{at } 0}$$

$$= \frac{V\sin\alpha}{l} \quad \text{...(4)}$$

$\therefore$

$$\frac{d\phi}{dt} = \frac{V\sin\alpha}{l\sin^2\theta}$$

Substituting the value of $\frac{d\phi}{dt}$ in (2), we get

$$\frac{d^2\theta}{dt^2} - \frac{V^2\sin^2\alpha}{l^2}\frac{\cos\theta}{\sin^3\theta} = -\frac{g}{l}\sin\theta \quad \text{...(5)}$$

Multiplying by $2\frac{d\theta}{dt}$, and integrating, we get

$$\int 2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}d\theta - \int\frac{2V^2\sin^2\alpha}{l^2}\frac{\cos\theta}{\sin^3\theta}d\theta = -\int\frac{2g}{l}\sin\theta\, d\theta$$

i.e.,

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2\sin^2\alpha}{l^2}\cdot\frac{1}{\sin\theta} = \frac{2g}{l}\cos\theta + A,$$

at O, we have $\theta = \alpha, \frac{d\theta}{dt} = 0,$ we get

$$0 + \frac{V^2\sin^2\alpha}{l^2}\cdot\frac{1}{\sin^2\alpha} = \frac{2g}{l}\cos\alpha + A \quad \text{...(6)}$$

Subtracting (6) from (5), we get

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{V^2\sin^2\alpha}{l^2}\left(\frac{1}{\sin^2\alpha} - \frac{1}{\sin^2\theta}\right) - \frac{2g}{l}(\cos\alpha - \cos\theta)$$

$$= \frac{V^2\sin^2\alpha}{l^2}\left(\frac{\sin^2\theta - \sin^2\alpha}{\sin^2\alpha\sin^2\alpha}\right) - \frac{2g}{l}(\cos\alpha - \cos\theta)$$

$$= \frac{V^2}{l^2}\left(\frac{\cos^2\alpha - \cos^2\theta}{\sin^2\theta}\right) - \frac{2g}{l}(\cos\alpha - \cos\theta)$$

$$= \frac{2g}{l}(\cos\alpha - \cos\theta)\left[\frac{V^2}{2gl}\left(\frac{\cos\alpha - \cos\theta}{\sin^2\theta}\right) - 1\right]$$

$$= \frac{2g}{l}(\cos\alpha - \cos\theta)\left(2n^2 \frac{\cos\alpha + \cos\theta}{\sin^2\theta} - 1\right) \text{ where } V^2 = 4lgn^2.$$

Hence, $\frac{d\theta}{dt}$ is again zero-when

$$2n^2(\cos\alpha + \cos\theta) = \sin^2\theta$$

i.e., when $\quad 2n^2\cos\alpha + 2n^2\cos\theta = 1 - \cos^2\theta$

i.e., when $\quad \cos^2\theta + 2n^2\cos\theta + 2n^2\cos\alpha - 1 = 0$

i.e., when $\quad \cos\theta = -n^2 \pm \sqrt{(1 - 2n^2\cos\alpha + n^4)}$

The negative sign gives inadmissisable value for θ. The only inclination at which $\frac{d\theta}{dt}$ again vanishes is when $\theta = \theta_1$, when

$$\cos\theta_1 = -n^2 + \sqrt{(1 - 2n^2\cos\alpha + n^4)}$$

The motion is therefore confined between value α and θ_1 of the θ.

The motion of the particle is along above or below the starting point, according as $\theta_1 \geq \alpha$,

i.e., according as $\cos\theta_1 \leq \cos\alpha$,

i.e., according as $\sqrt{(1 - 2n^2\cos\alpha + n^4)} \geq n^2 + \cos\alpha$,

i.e., according as $1 - 2n^2\cos\alpha \geq \cos\alpha + 2n^2\cos\alpha$

i.e., according as $n^2 \geq \frac{\sin^2\alpha}{4\cos\alpha}$

i.e., according as $\frac{V^2}{4lg} \geq \frac{\sin^2\alpha}{4\cos\alpha}$

i.e., according as $V^2 \leq lg\sin\alpha\tan\alpha$.

The tension of the string at any instant is now given by equation (1). Let T does not vanish during the motion.

The square of the velocity at any instant

$$= \left(l\frac{d\theta}{dt}\right)^2 + \left(l\sin\theta\frac{d\phi}{dt}\right)^2$$

$$= l^2\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\right]$$

Hence, the principle of conservation of energy gives

$$\frac{1}{2}ml^2\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\right] = \frac{1}{2}mV^2 - mgl(\cos\alpha - \cos\theta)$$

Hence, from (1), we get

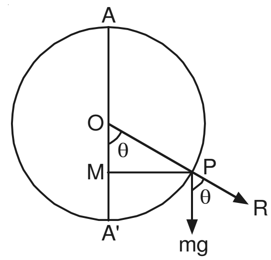
$$\frac{T}{m} = g\cos\theta + \frac{(\text{velocity})^2}{l} = g\cos\theta + \frac{V^2 - 2gl(\cos\alpha - \cos\theta)}{l}$$

$$= \frac{V^2}{l} + g(3\cos\theta - 2\cos\alpha).$$

EXAMPLES

1. *A heavy particle moves in a smooth sphere. Show that, if the velocity be that due to the level of the centre, the reaction of the surface will vary as the depth below the centre.*

Solution. At time t, let P be the position of the particle of mass m, such that OP makes an angle θ with the vertical through O, the centre of the sphere. Let PM be the perpendicular from P on the vertical line AA'. Then velocity of the particle at P



$$= (2g \, . \, OM) = \sqrt{(2ga \cos \theta)}$$

where a is the radius of the the sphere.

Let R be the reaction of the surface at P, then $OP = a$, the radius of the sphere being constant. Hence, the equations of motion are

$$-a\left(\frac{d\theta}{dt}\right)^2 - a \sin^2 \theta \left(\frac{d\theta}{dt}\right)^2 = -\frac{R}{m} + g \cos \theta \qquad \text{...(1)}$$

$$a \frac{d^2\theta}{dt^2} - a \cos \theta \sin \theta \left(\frac{d\theta}{dt}\right)^2 = -g \sin \theta \qquad \text{...(2)}$$

and
$$\frac{1}{\sin \theta} \cdot \frac{d}{dt}\left(\sin^2 \theta \, . \frac{d\phi}{dt}\right) = 0 \qquad \text{...(3)}$$

Integration of equation (3), gives

$$\sin^2 \theta \, . \frac{d\phi}{dt} = A \ \text{(constant)}$$

Let when $\theta = \alpha$, $\frac{d\theta}{dt} = 0$, $\frac{dr}{dt} = 0$.

$\therefore$
$$\left[a \sin \theta \frac{d\phi}{dt}\right]_{\theta = \alpha} = V, \quad i.e., \quad \frac{d\phi}{dt} = \frac{V}{a \sin \alpha}$$

so that
$$A = \sin^2 \alpha \, . \frac{V}{a \sin \alpha} = \frac{V \sin \alpha}{a}$$

and hence
$$\sin^2 \theta \frac{d\phi}{dt} = \frac{V \sin \alpha}{a} \qquad \text{...(4)}$$

Substituting the value of $\frac{d\phi}{dt}$ from (4) to (2), we get

$$\frac{d^2\theta}{dt^2} - \cos \theta \sin \theta \, . \frac{v^2 \sin^2 \alpha}{a^2 \sin^4 \theta} = -\frac{g}{a} \sin \theta$$

or
$$\frac{d^2\theta}{dt^2} - \frac{V^2}{a^2} \sin^2 \alpha \, . \frac{\cos \theta}{\sin^3 \theta} = -\frac{g}{a} \sin \theta$$

Multiplying the equation by $2 \frac{d\theta}{dt}$ and integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \sin^2 \alpha}{a^2} \cdot \frac{1}{\sin^2 \theta} = \frac{2g}{a} \cos \theta + B$$

when $\theta = \alpha, \frac{d\theta}{dt} = 0$, then

$$\frac{V^2}{a^2} = \frac{2g}{a} \cos \alpha + B$$

i.e., $$B = \frac{V^2}{a^2} - \frac{2g}{a}\cos\alpha$$

Thus, $$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2\sin^2\alpha}{a^2}\cdot\frac{1}{\sin^2\theta} = \frac{2g}{a}\cos\theta + \frac{V^2}{a^2} - \frac{2g}{a}\cos\alpha$$

or $$\left(\frac{d\theta}{dt}\right)^2 = -\frac{V^2}{a^2}\left(\frac{\sin^2\alpha}{\sin^2\theta} - 1\right) - \frac{2g}{a}(\cos\alpha - \cos\theta)$$

$$= -\frac{V^2}{a^2\sin^2\theta}(\sin^2\alpha - \sin^2\theta) - \frac{2g}{a}(\cos\alpha - \cos\theta)$$

$$= \frac{V^2}{a^2\sin^2\theta}(\sin^2\theta - \sin^2\alpha) - \frac{2g}{a}(\cos\alpha - \cos\theta)$$

Substituting the values of $\left(\frac{d\theta}{dt}\right)^2$ and $\left(\frac{d\phi}{dt}\right)^2$ in (1), we get

$$\frac{R}{m} = \frac{V^2}{a\sin^2\theta}(\sin^2\theta - \sin^2\alpha) - 2g(\cos\alpha - \cos\theta) + \frac{aV^2\sin^2\alpha\sin^2\theta}{a^2\sin^4\theta} + g\cos\theta$$

$$= \frac{V^2}{a\sin^2\theta}(\sin^2\theta - \sin^2\alpha) - 2g(\cos\alpha - \cos\theta) + \frac{V^2\sin^2\alpha}{a\sin^2\theta} + g\cos\theta$$

$$= \frac{V^2}{a} - 2g\cos\alpha + 3g\cos\theta$$

But when $\theta = \alpha$, $V = \sqrt{(2ga\cos\alpha)}$

$\therefore$ $$\frac{R}{m} = 3g\cos\theta \text{ or } R = \frac{3g}{am}(a\cos\theta) = \frac{3g}{am}\cdot OM$$

i.e., $R \propto OM$.

Hence, the reaction of the surface varies as the depth below the centre.

2. *A particle moves on a smooth sphere under no force except the pressure of the surface. Show that the path is given by the equation* $\cot\theta = \cot\beta\cos\phi$, *where* θ *and* ϕ *are its angular coordinates.*

Solution. Since there are no forces, except the pressure, the equations of motion are

$$a\left(\frac{d\theta}{dt}\right)^2 + a\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 = \frac{R}{m}, \quad ...(1)$$

$$a\frac{d^2\theta}{dt^2} - a\cos\theta\sin\theta\left(\frac{d\phi}{dt}\right)^2 = 0$$

i.e., $$\frac{d^2\theta}{dt^2} = \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 \quad ...(2)$$

and $$\frac{1}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right) = 0 \quad ...(3)$$

Here a is the radius of the sphere.

Integration of (3) gives

$$\sin^2\theta \frac{d\phi}{dt} = A \text{ (constant)}$$

i.e., $$\frac{d\phi}{dt} = \frac{A}{\sin^2\theta} \quad \text{...(6)}$$

By putting the value of $\frac{d\phi}{dt}$ from (4) in (2), we get

$$\frac{d^2\theta}{dt^2} = \sin\theta\cos\theta \, . \, \frac{A^2}{\sin^4\theta} = \frac{A^2\cos\theta}{\sin^3\theta}$$

Multiplying both sides by $2\frac{d\theta}{dt}dt$ and integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 = -\frac{A^2}{\sin^2\theta} + B.$$

Initially, when $\theta = \beta, \frac{d\theta}{dt} = 0.$

$\therefore$ $$B = \frac{A^2}{\sin^2\beta},$$

Thus, $$\left(\frac{d\theta}{dt}\right)^2 = A^2\left(\frac{1}{\sin^2\beta} - \frac{1}{\sin^2\theta}\right)$$

Dividing (4) by (5), we have

$$\frac{d\phi}{dt} = \frac{\sin\beta}{\sin\theta\sqrt{(\sin^2\theta - \sin^2\beta)}} = \frac{\operatorname{cosec}^2\theta}{\sqrt{\cot^2\beta - \cot^2\theta}}$$

Integrating, $$\phi = \cos^{-1}\frac{\cot\theta}{\cos\beta} + C$$

If $\phi = 0$ when $\theta = \beta, \therefore C = 0.$

Hence, $$\phi = \cos^{-1}\left(\frac{\cot\theta}{\cot\beta}\right)$$

or $$\cos\phi = \frac{\cot\theta}{\cot\beta} \quad \text{or} \quad \cot\theta = \cot\beta\cos\phi$$

3. *A smooth circular cone of angle 2α, has its axis vertical and its vertex, which is pierced with a small hole, downwards. A mass M hangs at rest by a string which passes through the vertex, and a mass m attached to the upper end describes a horizontal circle on the inner surface of the cone. Find the time T of a complete revolution and show that small oscillations about the steady motion take place in the time*

$$T \operatorname{cosec}\alpha\sqrt{\left(\frac{M+m}{3m}\right)}.$$

Solution. Let T_1 be the tension is the string and l the total length of the string. Let P be the position of mass m in the initial state such that the length $OP = r$, O being the vertex of the cone. Therefore, the length of the hanging portion of the string below O is $(l - r)$.

In this case, $\theta = \alpha$, hence, $\frac{d\theta}{dt} = 0, \frac{d^2\theta}{dt^2} = 0.$ Hence, equations of motion will be

$$m\left[\frac{d^2r}{dt^2} - r\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2\right] = -T_1 - mg\cos\alpha \qquad ...(1)$$

$$\frac{1}{r\sin\alpha}\cdot\frac{d}{dt}\left(r^2\sin\alpha\frac{d\phi}{dt}\right) = 0 \qquad ...(2)$$

and
$$M\frac{d^2}{dt^2}(l - r) = Mg - T_1 \qquad ...(3)$$

The equation (3) gives
$$-M\frac{d^2r}{dt^2} = Mg - T_1 \qquad ...(4)$$

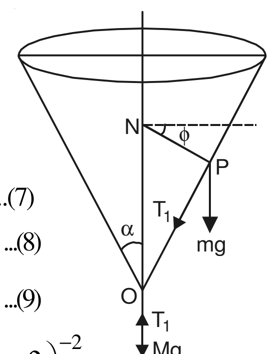
subtracting (4) from (1), we get

$$(M+m)\frac{d^2r}{dt^2} - mr\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2 = -mg\cos\alpha - Mg \qquad ...(5)$$

The equation (2) gives

$$r^2\frac{d\phi}{dt} = A \qquad ...(6)$$

For the steady motion,



$$\frac{d^2r}{dt^2} = 0, \quad r = d$$

and therefore, (5) and (6), give on putting $\frac{d\phi}{dt} = \omega,$

$$m\omega^2 d\sin^2\alpha = g(M + m\cos\alpha), \qquad ...(7)$$

and
$$A = d^2\omega \qquad ...(8)$$

Also the time period T is given by $T = \frac{2\pi}{\omega}$. ...(9)

If we put $r = d + \rho$, $\frac{d^2r}{dt^2} = \frac{d^2\rho}{dt^2}$ and $\frac{d\phi}{dt} = \frac{A}{r^2} = \frac{d^2\omega}{(d+\rho)^2} = \frac{d^2\omega}{d^2}\left(1 + \frac{\rho}{d}\right)^{-2}$

$$= \omega\left[1 - \frac{2\rho}{d}\right]$$ neglecting squares of ρ as ρ is small

Now we have from (5)

$$(m+M)\frac{d^2\rho}{dt^2} = m(d+\rho)\sin^2\alpha\,\omega^2\left(1 + \frac{2\rho}{d}\right)^2 - g(M + m\cos\alpha)$$

$$= m\omega^2(d+\rho)\left(1 - \frac{4\rho}{d}\right)\sin^2\alpha - m\omega^2 d\sin^2\alpha$$

from (7) and also neglecting the squares of ρ

$$= m\omega^2\sin^2\alpha\,(d - 3\rho) - m\omega^2 d\sin\alpha,$$ neglecting squares of ρ

$$= -3m\omega^2\sin^2\alpha\,.\,\rho.$$

This represents a S.H.M. of time period

$$= 2\pi\sqrt{\frac{M+m}{2m\omega^2\sin^2\alpha}} = \frac{2\pi}{\omega}\operatorname{cosec}\alpha\sqrt{\left(\frac{M+m}{3m}\right)}$$

$$= T\operatorname{cosec}\alpha\sqrt{\left(\frac{M+m}{3m}\right)} \qquad \text{[from (9)]}$$

EXERCISES

1. A heavy particle is projected horizontally along the inner surface of a smooth spherical shell of radius $\frac{a}{\sqrt{2}}$ with velocity $\sqrt{\frac{2ag}{3}}$ at a depth $\frac{2a}{3}$ below the centre. Show that it will rise to a height $\frac{a}{3}$ above the centre, and that the pressure on the sphere just vanishes at the height point of the path.

2. A heavy particle is projected with velocity V from the end of a horizontal diameter of a sphere of radius a along the inner surface, the direction of projection making an angle β with the equator. It the particle never leaves the surface prove that $3\sin^2\beta < 2 + \left(\frac{V^2}{3ga}\right)^2$.

3. A particle constrained to move on a smooth spherical surface is projected horizontally from a point at the level of the centre so that its angular velocity relative to the centre is ω. If $\omega^2\alpha$ be very great compared with g, show that its depth c below the level of the centre at time t is $\frac{2g}{\omega^2}\sin^2\frac{\omega t}{2}$ approximately.

4. A Smooth hollow right circular cone is placed with its vertex downwards and axis vertical, and at a point on its interior surface at a height h above the vertex a particle is projected horizontally along the surface with a velocity $\sqrt{\frac{2gh}{n^2+n}}$. Show that the lowest point of path will be at a height $\frac{4}{n}$ above the vertex of the cone.

5. A particle is projected horizontally along the interior surface of a smooth hemisphere whose axis is vertical and whose vertex is downwards. The point of projection being at the angular distance β from the lowest point. Show that the particle may just ascend to the rim of the hemisphere is $\sqrt{(2ag\sec\beta)}$.

6. A particle moves on the inner surface of a smooth cone, of vertical angle 2α, being acted on by a force towards the vertex of the cone, and its direction of motion always cuts the generators at a constant angle β. Find the motion and the law of force.

[**Ans.** $F \propto \frac{1}{r^3}$, $r = r_0 e^{\sin\alpha\cot\beta\,.\,\phi}$]

9.3 ACCELERATIONS ALONG THE TANGENT, THE PRINCIPAL NORMAL AND THE BINORMAL

A particle is moving along any curve in three dimensions; to find its accelerations along (i) the tangent to the curve, (ii) the principal normal, and (iii) the binormal.

Let (x, y, z) be the coordinates of the point at time t. The accelerations parallel to the axes of coordinates are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$.

Now,
$$\frac{dx}{dt} = \frac{dx}{ds}\cdot\frac{ds}{dt}$$

$\therefore$
$$\frac{d^2x}{dt^2} = \frac{dx}{ds}\cdot\frac{d^2s}{dt^2} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 \quad ...(1)$$

So, $$\frac{d^2y}{dt^2} = \frac{dy}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 \qquad ...(2)$$

and $$\frac{d^2z}{dt^2} = \frac{dz}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2z}{ds^2}\left(\frac{ds}{dt}\right)^2 \qquad ...(3)$$

(i) The direction cosines of the tangent are $\frac{dx}{ds}$, $\frac{dy}{ds}$ and $\frac{dz}{ds}$,

Hence, the acceleration along it

$$= \frac{dx}{ds} \cdot \frac{d^2x}{dt^2} + \frac{dy}{ds} \cdot \frac{d^2y}{dt^2} + \frac{dz}{ds} \cdot \frac{d^2z}{dt^2}$$

$$= \frac{dx}{ds}\left[\frac{dx}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2\right] + \frac{dy}{ds}\left[\frac{dy}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2\right]$$
$$+ \frac{dz}{ds}\left[\frac{dz}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2z}{ds^2}\left(\frac{dz}{ds}\right)^2\right]$$

$$= \frac{d^2s}{dt^2}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right]$$
$$+ \left(\frac{ds}{dt}\right)^2\left[\frac{dx}{ds} \cdot \frac{d^2x}{ds^2} + \frac{dy}{ds} \cdot \frac{d^2y}{ds^2} + \frac{dz}{ds}\,\frac{d^2z}{ds^2}\right] \qquad ...(4)$$

We have $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$,

differentiating w.r.t. s, we get

$$\frac{dx}{ds}\,\frac{d^2x}{ds^2} + \frac{dy}{ds} \cdot \frac{d^2y}{ds^2} + \frac{dz}{ds} \cdot \frac{d^2z}{ds^2} = 0 \qquad ...(5)$$

By putting these values in equation (4), we get

$$\text{Acceleration along the tangent } = \frac{d^2s}{dt^2} \qquad ...(6)$$

(ii) The direction cosines of the principal normal are $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$ and $\rho\frac{d^2z}{ds^2}$, where ρ is the radius of curvature.

Hence, the acceleration along principal normal

$$= \rho\frac{d^2x}{ds^2} \cdot \frac{d^2x}{dt^2} + \rho\frac{d^2y}{ds^2} \cdot \frac{d^2y}{dt^2} + \rho\frac{d^2z}{ds^2} \cdot \frac{d^2z}{dt^2}$$

$$= \rho\frac{d^2x}{ds^2}\left[\frac{dx}{ds}\,\frac{d^2s}{dt^2} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2\right] + \rho\frac{d^2y}{ds^2}\left[\frac{dy}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2\right]$$
$$+ \rho\frac{d^2z}{ds^2}\left[\frac{dz}{ds} \cdot \frac{d^2s}{dt^2} + \frac{d^2z}{ds^2}\left(\frac{ds}{dt}\right)^2\right]$$

[from (1), (2) and (3)]

$$= \rho \frac{d^2 s}{dt^2}\left[\frac{dx}{ds}.\frac{d^2 x}{ds^2}+\frac{dy}{ds}.\frac{d^2 y}{ds^2}+\frac{dz}{ds}.\frac{d^2 z}{ds^2}\right]$$

$$+\rho\left(\frac{ds}{dt}\right)^2\left[\left(\frac{d^2 x}{ds^2}\right)^2+\left(\frac{d^2 y}{ds^2}\right)^2+\left(\frac{d^2 z}{dx^2}\right)^2\right] \quad ...(7)$$

We have $\left[\left(\frac{d^2 x}{dt^2}\right)^2+\left(\frac{d^2 y}{dt^2}\right)^2\left(\frac{d^2 z}{dt^2}\right)^2\right]=\frac{1}{\rho^2}$.

and putting the value from (5), we get

Acceleration along the principal normal

$$= \rho\left(\frac{ds}{dt}\right)^2 \times \frac{1}{\rho^2}=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \quad ...(8)$$

(iii) The direction cosines of the binormal are proportional to

$$\frac{dy}{ds}\frac{d^2 z}{ds^2}-\frac{dz}{ds}.\frac{d^2 y}{ds^2}, \frac{dz}{ds}.\frac{d^2 z}{ds^2}-\frac{dx}{ds}.\frac{d^2 z}{ds^2} \text{ and } \frac{dx}{ds}.\frac{d^2 y}{ds^2}-\frac{dy}{ds}.\frac{d^2 x}{ds^2}.$$

On multiplying (1), (2) and (3) in succession by these and adding, the result is zero, *i.e.,* the acceleration in the direction of the binormal vanishes

Remark : Let $(l_1, m_1, n_1), (l_2, m_2, n_2)$ and (l_3, m_3, n_3) are the direction cosines of the tangent, the principal normal, and the binormal, then equations (1), (2), (3) can be written as

$$\frac{d^2 s}{dt^2}=l_1\frac{d^2 s}{dt^2}+l_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\}$$

$$\frac{d^2 y}{dt^2}=m_1\frac{d^2 s}{dt^2}+m_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\}$$

and

$$\frac{d^2 z}{dt^2}=n_1\frac{d^2 s}{dt^2}+n_2\left\{\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2\right\}$$

These equations show that the accelerations along the axes are the components of an acceleration $\frac{d^2 s}{dt^2}$ along the tangent, an acceleration $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$ along the principal normal, and nothing is the direction of the binomal.

Hence, for a particle describing a $v\frac{dv}{dt}$, along the tangent and $\frac{v^2}{\rho}$ along the principal normal, which lies in the osculating plane of the curve.

9.4

A particle moves in a curve, there being no friction, under force such as occur in nature. Show that the change in its kinetic energy as it passes from one position to the other is independent of the path pursued and depends on its initial and final positions.

Proof : Let *X, Y, Z* be the components of the forces. Resolving the forces along the tangent to the path, we have

$$m\frac{d^2 s}{dt^2}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}$$

Multiplying by $2\frac{ds}{dt}$ and integrating, we get

$$m\left(\frac{ds}{dt}\right)^2 = 2\int(X\,dx + Y\,dy + Z\,dz)$$

$$\therefore \qquad \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \int(X\,dx + Y\,dy + Z\,dz)$$

Since the forces are such as occur in nature, hence the components are one-valued functions of distances from fixed points. So, the quantity $X\,dx + Y\,dy + Z\,dz$ is the differential of some function $\phi(x, y, z)$, so that

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \phi(x, y, z) + C$$

where $\frac{1}{2}mv_0^2 = \phi(x_0, y_0, z_0) + C.$

Here (x_0, y_0, z_0) is the starting point an v_0 the initial velocity.

Hence $$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \phi(x, y, z) - \phi(x_0, y_0, z_0)$$

The right-hand member of this equation depends only on the position of the initial point and on that of the point of the path under consideration, and is quite independent of the path pursued.

9.5 MOTION ON A SMOOTH SURFACE

If the particle moves on a surface whose equation in $f(x, y, z) = 0$, let the direction-cosines of the normal point *(x, y, z)* of its path be (l_1, m_1, n_1) so that

$$\frac{l_1}{df/dx} = \frac{m_1}{df/dy} = \frac{n_1}{df/dz}$$

$$= \frac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{[(df/dx)^2 + (df/dy)^2 + (df/dz)^2]}}$$

$$= \frac{1}{\sqrt{[(df/dx)^2 + (df/dy)^2 + (df/dz)^2]}} \qquad ...(1)$$

Now, if R be the normal reaction, we have

$$m\frac{d^2x}{dt^2} = X + Rl_1,\quad m\frac{d^2y}{dt^2} = Y + Rm_1 \text{ and } m\frac{d^2z}{dt^2} = Z + Rn_1,$$

where *X, Y, Z* are the components of the impressed forces.

Multiplying these equations by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ and adding, we have

$$\frac{1}{2}m\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}$$

for the coefficient of R

$$= l_1\frac{dx}{dt} + m_1\frac{dy}{dt} + n_1\frac{dz}{dt}$$

$$= \left(l_1\frac{dx}{ds} + m_1\frac{dy}{ds} + n_1\frac{dz}{ds}\right)\frac{ds}{dt}$$

$= \frac{ds}{dt}$ (the cosines of the angle between a tangent line to the surface and the normal)

$= 0.$

Hence, on integration,

$$\frac{1}{2}mv^2 = \int (X\,dx + Y\,dy + Z\,dz)$$

Also, on eliminating R, the path on the surface is given by

$$\frac{m\dfrac{d^2x}{dt^2} - X}{l_1} = \frac{m\dfrac{d^2y}{dt^2} - Y}{m_1} = \frac{n\dfrac{d^2z}{dt^2} - Z}{n_1}$$

EXAMPLES

1. *A smooth helix is placed with its axis vertical and a small bead slides down it under gravity. Show that it makes its first revolution from rest in time* $2\sqrt{\left(\dfrac{\pi a}{g \sin\alpha \cos\alpha}\right)}$ *where* α *is angle of the helix.*

Solution. Equations of a helix are

$$x = a\cos\theta,\quad y = a\sin\theta,\quad z = a\,\theta\tan\alpha,$$

so that $\dfrac{dx}{dt} = -a\sin\theta\,\dfrac{d\theta}{dt}$, $\dfrac{dy}{dt} = a\cos\theta\,\dfrac{d\theta}{dt}$, $\dfrac{dz}{dt} = \tan\alpha\,.\,\dfrac{d\theta}{dt}$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= [a^2\sin^2\theta + a^2\cos^2\theta + a^2\tan^2\alpha]\left(\frac{d\theta}{dt}\right)^2$$

$$= a^2(1 + \tan^2\alpha)\left(\frac{d\theta}{dt}\right)^2$$

$$= a^2\sec^2\alpha\left(\frac{d\theta}{dt}\right)^2.$$

Here a is the radius and α the angle of the helix.

Now, if z_0 be initial value of z, then by the principle of energy

$$2(z - z_0)\,g = \left(\frac{ds}{dt}\right)^2 = a^2\sec^2\alpha\left(\frac{d\theta}{dt}\right)^2$$

or
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a^2\sec^2\alpha}(z - z_0)$$

we have $z = a\,\theta\tan\alpha$ and $z_0 = a\,\theta_0\tan\alpha$

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{2ga\tan\alpha}{a^2\sec^2\alpha}(\theta - \theta_0) = \frac{2g\sin\alpha\cos\alpha}{a}(\theta - \theta_0)$$

$$\therefore \quad \sqrt{\left(\frac{2g\sin\alpha\cos\alpha}{a}\right)}\,dt = \frac{d\theta}{\sqrt{(\theta - \theta_0)}}$$

Integrating,

$$t\sqrt{\left(\frac{2g\sin\alpha\cos\alpha}{a}\right)} = \int_{\theta_0}^{\theta_0 + 2\alpha}\frac{d\theta}{\sqrt{(\theta - \theta_0)}} = 2\,[\sqrt{(\theta - \theta_0)}\,]_{\theta_0}^{\theta + 2\pi} = 2\sqrt{(2\pi)}$$

Hence,
$$t = 2\sqrt{\left(\frac{\pi a}{g\sin\alpha\cos\alpha}\right)}.$$

2. *A particle moving on a paraboloid of revolution under a force parallel to the axis crosses the meridians at a constant angle. Show that the force varies inversely as the fourth power of the distance from the axis.*

Solution. Let the equation to the paraboloid in cylindrical coordinates be

$$\rho^2 = 4az \qquad ...(1)$$

The equation of motion is

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2 \frac{d\phi}{dt}\right) = 0$$

i.e.,

$$\rho^2 \frac{d\phi}{dt} = \text{constant} = A \text{ (say)} \qquad ...(2)$$

Let the force F parallel to axis cross the meridians at a constant angle β then

$$\frac{r \sin\theta \dfrac{d\phi}{dt}}{r \dfrac{d\theta}{dt}} = \tan\beta,$$

i.e.,

$$r\frac{d\theta}{dt} = r \sin \frac{d\phi}{dt} \cot\beta,$$

But we know that

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = r\sin\theta\cot\beta\frac{d\theta}{dt}$$

$$= \rho\cos\beta \,.\, \frac{d\phi}{dt} \qquad [\because \rho = r\sin\theta] \qquad ...(3)$$

If z_0 be the initial value of z, then by the principle of energy

$$\int_{z_0}^{z} F\,dz = \frac{1}{2}\rho^2\left(\frac{d\phi}{dt}\right)^2[1+\cot^2\beta] = \frac{1}{2}\rho^2 \operatorname{cosec}^2\beta\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{A^2}{2\sin^2 . \rho^2} \qquad \text{from (2)}$$

$$= \frac{A^2}{8a\sin^2\beta . z}, \qquad \text{from (1)}$$

Integrating,

$$F = -\frac{A^2}{8a\sin^2\beta . z^2} = -\frac{2aA^2}{\sin^2\beta}.\frac{2}{\rho^4}, \qquad \text{from (1)}$$

$\therefore\ F \propto \dfrac{1}{\rho^2}$, *i.e.,* the force varies inversely as the fourth power of the distance from the axis.

EXERCISES

1. A particle, without weight, slides on a smooth helix of angle α and radius a under a force to a fixed point on the axis equal to $m\mu$ (distance). Show that the reaction of the curve can not vanish unless the greatest velocity of the particle is $a\sqrt{\mu}\sec\alpha$.
2. A smooth paraboloid is placed with its axis vertical and vertex downwards, the latus-rectum of the generating parabola being $4a$. A heavy particle is projected horizontally with velocity V at a height h above the lowest point. Show that the particle is again moving horizontally when its height is $\dfrac{V^2}{2g}$. Show also that the reaction of the paraboloid at any point is inversely proportional to the corresponding radius of curvature of the generating parabola.

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