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Kinematics of Fluids in Motion

2.1. Methods of describing fluid motion.

There are two methods for studying fluid motion mathematically. These are Lagrangian and Eulerian (flux) methods and refer to 'individual time-rate of change' and 'local time rate of change' respectively.

(I) Lagrangian method. **[Garhwal 2005; Meerut 2009, 10, 12]**

In this method we study the history of each fluid particle, *i.e.* any fluid particle is selected and is pursued on its onward course observing the changes in velocity, pressure and density at each point and at each instant. Let (x_0, y_0, z_0) be the coordinates of the chosen particle at a given time $t = t_0$. At a later time, $t = t$, let the coordinates of the same particle be (x, y, z). Since the chosen particle is any particle in the fluid, the coordinates (x, y, z) will be functions of t and also of their initial values (x_0, y_0, z_0), so that

$$x = f_1(x_0, y_0, z_0, t), \qquad y = f_2(x_0, y_0, z_0, t), \qquad z = f_3(x_0, y_0, z_0, t). \qquad ...(1)$$

Let u, v, w and a_x, a_y, a_z be the components of velocity and acceleration respectively. Then, we have

$$u = \partial x / \partial t, \qquad v = \partial y / \partial t \qquad w = \partial z / \partial t \qquad ...(2)$$

and

$$a_x = \partial^2 x / \partial t^2, \qquad a_y = \partial^2 y / \partial t^2, \qquad a_z = \partial^2 z / \partial t^2 \qquad ...(3)$$

Remark 1. The fundamental equations of motion in Lagrangian form are non-linear and hence it leads to many difficulties while solving a problem. In fact, the present method is employed with an advantage only in some one-dimensional (involving one space coordinate) problems. Hence we need to think about another method of describing fluid motion.

Remark 2. This method resembles that of dynamics of a particle in so far as (x, y, z) are dependent on t. However, in Lagrangian method of fluid dynamics (x, y, z) are dependent on four independent variables x_0, y_0, z_0, t.

(II) Eulerian method. **[Ranchi 2010, Agra 2005; Garhwal 2005; Meerut 2009, 2010, 12]**

In this method we select any point fixed in space occupied by the fluid and study the changes which take place in velocity, pressure and density as the fluid passes through this point. Let u, v, w be the components of velocity at the point (x, y, z) at time t. Then, we have

$$u = F_1(x, y, z, t), \qquad v = F_2(x, y, z, t), \qquad w = F_3(x, y, z, t). \qquad ...(4)$$

For a particular value of t, (4) exhibits the motion at all points in the fluid at that time. Again for a particular point (x, y, z), u, v, w are functions of t, which define the mode of variations of velocity at that point.

Remark 1. The point under consideration being fixed, x, y, z and t are independent variables and hence dx/dt d^2x/dt^2 etc. have no meaning in this method.

Remark 2. In Lagrangian method a particular fluid particle is identified and changes in velocity etc. are studied as that fluid particle moves onwards. On the other hand, in Eulerian method the individual fluid particles are not identified. Instead, a point in fluid is chosen and changes in velocity etc. are studied as the fluid passes through the chosen fixed point.

Relationship between the Lagrangian and Eulerian methods.

[Garhwal 2001, 05; Meerut 2005]

In order to establish relationship between the two methods, we investigate a relation between the particle parameters and space parameters.

(*i*) **Lagrangian to Eulerian.** Suppose $\phi(x_0, y_0, z_0, t)$ be some physical quantity involving Lagrangian description

$$\phi = \phi(x_0, y_0, z_0, t) \qquad ...(5)$$

Since Lagrangian description is given, (1) holds. Solving (1) for x_0, y_0, z_0 we have

$$x_0 = g_1(x, y, z, t), \qquad y_0 = g_2(x, y, z, t), \qquad z_0 = g_3(x, y, z, t) \qquad ...(6)$$

Using (6), (5) reduces to

$$\phi = \phi\,[g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t), t], \qquad ...(7)$$

which expresses ϕ in terms of Eulerian description.

(*ii*) **Eulerian to Lagrangian.** Suppose $\psi(x, y, z, t)$ be some physical quantity involving Eulerian description

$$\psi = \psi(x, y, z, t). \qquad ...(8)$$

Since Eulerian description is given, (4) holds. Again, (2) holds for the proposed Lagrangian description. Hence (2) and (4) yield

$$dx/dt = F_1(x, y, z, t), \qquad dy/dt = F_2(x, y, z, t), \qquad dz/dt = F_3(x, y, z, t) \qquad ...(9)$$

The integration of (9) involves three constants of integration which may be taken as initial coordinates x_0, y_0, z_0 of the fluid particle. Thus the integration of (9) leads to the well known equations of Lagrange (1). Using (1), (8) reduces to

$$\psi = \psi\,[f_1(x_0, y_0, z_0, t), f_2(x_0, y_0, z_0, t)\, f_3(x_0, y_0, z_0, t), t], \qquad ...(10)$$

which expresses ψ in terms of Lagrangian description.

2.2. Illustrative solved examples.

Ex. 1. *The velocity components for a two-dimensional fluid system can be given in the Eulerian system by* $u = 2x + 2y + 3t$, $v = x + y + t/2$.

Find the displacement of a fluid particle in the Lagrangian system.

[kanpur 2000, 05, Rajasthan 2003, Rohalkhand 2005]

Sol. Given $\qquad u = 2x + 2y + 3t, \qquad v = x + y + t/2 \qquad ...(1)$

In terms of the displacement x and y, the velocity components u and v may be represented by

$$u = dx/dt, \qquad v = dy/dt \qquad ...(2)$$

From (1) and (2), we have

$$dx/dt = 2x + 2y + 3t, \qquad dy/dt = x + y + t/2 \qquad ...(3)$$

Let $D \equiv d/dt$. Then equations (3) become

$$(D-2)\,x - 2y = 3t \qquad ...(4)$$

$$-x + (D-1)\,y = t/2 \qquad ...(5)$$

Operating (5) by $(D-2)$, we have

$$-(D-2)\,x + (D-2)(D-1)\,y = (1/2) \times (D-2)\,t$$

or

$$-(D-2)\,x + (D^2 - 3D + 2)\,y = (1/2) - t \qquad ...(6)$$

Adding (4) and (6), we have

$$(D^2 - 3D)\, y = (1/2) + 2t \qquad ...(7)$$

Auxiliary equation of (7) is $D^2 - 3D = 0$. Solving for D, it gives D = 0, 3. Hence complementary function (C.F.) is given by

$$\text{C.F.} = c_1 + c_2 e^{3t}$$

Next, the particular integral (P.I.) is given by

$$P.I. = \frac{1}{D^2 - 3D}\left(\frac{1}{2} + 2t\right)$$

$$= \frac{1}{-3D(1 - D/3)}\left(\frac{1}{2} + 2t\right) = -\frac{1}{3D}\left(1 - \frac{1}{3}D\right)^{-1}\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left(1 + \frac{1}{3}D + ...\right)\left(\frac{1}{2} + 2t\right) = \frac{1}{3D}\left(\frac{1}{2} + 2t + \frac{1}{3} \times 2\right)$$

$$= -\frac{1}{3} \cdot \frac{1}{D}\left(2t + \frac{7}{6}\right) = -\frac{1}{3}\left(2 \times \frac{t^2}{2} + \frac{7}{6} \times t\right) = -\frac{t^2}{3} - \frac{7t}{18}$$

Hence the general solution of (7) is

$$y = c_1 + c_2 e^{3t} - (t^2/3) - (7t/18) \qquad ...(8)$$

From (8), $\qquad dy/dt = 3c_2 e^{3t} - (2t/3) - (7/18) \qquad ...(9)$

Re-writing the second equation of (3), we get

$$x = dy/dt - y - (t/2) \qquad ...(10)$$

Putting the values of y and dy/dt given by (8) and (9) in (10), we get

$$x = 3c_2 e^{3t} - \frac{2}{3}t - \frac{7}{18} - c_1 - c_2 e^{3t} + \frac{1}{3}t^2 + \frac{7}{18}t - \frac{1}{2}t$$

or $\qquad x = -c_1 + 2c_2 e^{3t} + (t^2/3) - (7t/9) - (7/18) \qquad ...(11)$

We now use the following initial conditions :

$$x = x_0, \qquad y = y_0 \qquad \text{when} \qquad t = t_0 = 0 \qquad ...(12)$$

Using (12), (8) and (11) reduce to

$$y_0 = c_1 + c_2 \qquad \text{and} \qquad x_0 = -c_1 + 2c_2 - (7/18) \qquad ...(13)$$

Solving (13) for c_1 and c_2, we have

$$c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54} \qquad \text{and} \qquad c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54} \qquad ...(14)$$

Using (14), (11) and (8) give

$$x = \frac{1}{3}x_0 - \frac{2}{3}y_0 + \frac{1}{3}\left(2x_0 + 2y_0 + \frac{7}{9}\right)e^{3t} - \frac{7}{9}t + \frac{1}{3}t^2 - \frac{7}{27} \qquad ....(15)$$

and $\qquad y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 + \frac{1}{3}\left(x_0 + y_0 + \frac{7}{18}\right)e^{3t} - \frac{7}{18}t + \frac{1}{3}t^2 - \frac{7}{54} \qquad ....(16)$

(15) and 16 give the desired displacements x and y in the Langrangian system involving the initial positions x_0 and y_0 and the time, t.

Ex. 2. *For a two-dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$. Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.* **[I.A.S. 1999]**

Sol. Given $\quad u = x + y + 2t \quad$ and $\quad v = 2y + t. \quad$ ...(*i*)

In terms of the displacements x and y, we have

$$u = dx/dt \quad \text{and} \quad v = dy/dt. \quad \text{...(2)}$$

From (1) and (2), $\quad dx/dt = x + y + 2t \quad$ ...(3)

and $\quad dy/dt = 2y + t \quad$ or $\quad dy/dt - 2y = t. \quad$ ...(4)

Integrating factor (I.F.) of (4) = $e^{\int(-2)dt} = e^{-2t}$ and solution of (4) is

$$ye^{-2t} = c_1 + \int t(e^{-2t})dt, \quad c_1 \text{ being an arbitrary constant}$$

or $$ye^{-2t} = c_1 + t\left(-\frac{1}{2}e^{-2t}\right) - \int 1\cdot\left(-\frac{1}{2}e^{-2t}\right)dt = c_1 - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} = c_1 - \frac{1}{4}(2t+1)e^{-2t}$$

or $$y = c_1e^{2t} - (2t + 1)/4. \quad \text{...(5)}$$

Substituting the above value of y in (3), we have

$$\frac{dx}{dt} = x + c_1e^{2t} - \frac{1}{4}(2t+1) + 2t \quad \text{or} \quad \frac{dx}{dt} - x = c_1e^{2t} + \frac{1}{4}(6t-1) \quad \text{...(6)}$$

I.F. of (6) = $e^{\int(-1)dt} = e^{-t}$ and solution of (6) is

$$xe^{-t} = c_2 + \int e^{-t}\left[c_1e^{2t} + \frac{1}{4}(6t-1)\right]dt = c_2 + c_1e^t + \int \frac{(6t-1)}{4}e^{-t}dt$$

or $$xe^{-t} = c_2 + c_1e^t + \frac{6t-1}{4}(-e^{-t}) - \int\left(\frac{6}{4}\right)(-e^{-t})\,dt$$

or $$xe^{-t} = c_2 + c_1e^t - \frac{1}{4}(6t-1)\,e^{-t} - \frac{6}{4}e^{-t} = c_2 + c_1e^t - \frac{1}{4}e^{-t}(6t+5)$$

or $$x = c_2e^t + c_1e^{2t} - (6t+5)/4. \quad \text{...(7)}$$

We now use the following initial conditions :

$$x = x_0, \quad y = y_0 \quad \text{when} \quad t = t_0 = 0. \quad \text{...(8)}$$

Using (8), (5) and (7) reduce to

$$y_0 = c_1 - (1/4) \quad \text{and} \quad x_0 = c_2 + c_1 - (5/4). \quad \text{...(9)}$$

Solving (9) for c_1 and c_2, $\quad c_1 = y_0 + (1/4), \quad c_2 = x_0 - y_0 + 1. \quad$ ...(10)

Using (10), (7) and (5) reduce to

$$x = (x_0 - y_0 + 1)e^t + (y_0 + 1/4)e^{2t} - (6t + 5)/4 \quad \text{...(11)}$$

and $$y = (y_0 + 1/4)e^{2t} - (2t + 1)/4. \quad \text{...(12)}$$

(11) and (12) give the desired displacements x and y in the Lagrangian system involving the initial positions x_0, y_0 and the time t.

Ex. 3. *The velocity distribution of a certain two-dimensional flow is given by $u = Ay + B$ and $v = Ct$, where A, B, C are constants. Obtain the equation of the motion of fluid particles in Lagrangian method.*

Sol. Let **r** (x, y) be the position of the given particle at any time t. Then the path lines for the fluid particle are given by

$$\mathbf{q} = \frac{d\mathbf{r}}{dt} \quad \Rightarrow \quad u\mathbf{i} + v\mathbf{j} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j}).$$

$$\Rightarrow \qquad u = dx/dt = Ay + B \qquad \ldots(1)$$

and $$v = dy/dt = Ct. \qquad \ldots(2)$$

Integrating (2), $y = (1/2) \times Ct^2 + c_1$ where c_1 is a constant of integration ...(3)

Initially, let $y = y_0$, $t = 0$. Then (3) gives $c_1 = y_0$

So (3) gives $$y = (1/2) \times Ct^2 + y_0. \qquad \ldots(4)$$

Substituting the above value of y in (1), we get

$$\frac{dx}{dt} = A\left(\frac{1}{2}Ct^2 + y_0\right) + B \qquad \text{so that} \qquad x = A\left(\frac{1}{6}Ct^3 + y_0 t\right) + Bt + c_2, \qquad \ldots(5)$$

where c_2 is a constant of integration.

Initially, let $x = x_0$, $t = 0$. Then (5) gives $c_2 = x_0$.

So (5) gives $$x = A\left(\frac{1}{6}Ct^3 + y_0 t\right) + Bt + x_0. \qquad \ldots(6)$$

The required equation of motion is given by (4) and (6).

Ex. 4. (*a*) *The velocities at a point in a fluid in the Eulerian system are given by*

$$u = x + y + z + t, \qquad v = 2(x + y + z) + t, \qquad w = 3(x + y + z) + t.$$

Obtain the displacements of a fluid particle in the Lagrangian system. **[Garhwal 2000]**

(*b*) *The velocity field at a point in fluid is given by*

$$q = [x + y + z + t, 2(x + y + z) + t, 3(x + y + z) + t].$$

Obtain the velocity of a fluid partifcle which is at (x_0, y_0, z_0) *initially.*

Sol. (*a*) In terms of the displacements x, y and z, the velocity components u, v and w, may also be represented by

$$u = dx/dt, \qquad v = dy/dt \qquad \text{and} \qquad w = dz/dt \qquad \ldots(1)$$

Using (1) and the given values of u, v, and w, we have

$$dx/dt = x + y + z + t \qquad \ldots(2)$$

$$dy/dt = 2(x + y + z) + t \qquad \ldots(3)$$

$$dz/dt = 3(x + y + z) + t \qquad \ldots(4)$$

Let $D \equiv d/dt$. Then (2) and (3) yield

$$(D - 1)x - y = z + t \qquad \ldots(5)$$

$$-2x + (D - 2)y = 2z + t \qquad \ldots(6)$$

Operating (5) by $(D - 2)$ and then adding the resulting equation to (6), we have

$$(D - 2)(D - 1)x - 2x = (D - 2)(z + t) + 2z + t$$

or $$(D^2 - 3D)x = Dz + 1 - t \qquad \ldots(7)$$

Next, multiplying both sides of (5) by 2, operating (6) by $(D - 1)$ and adding the resulting equations, we have

$$-2y + (D - 1)(D - 2)y = 2(z + t)(D - 1)(2z + t)$$

or $$(D^2 - 3D)y = 2Dz + 1 + t \qquad \ldots(8)$$

Re-writing (4), we have $$(D - 3)z = 3x + 3y + t$$

or $$(D^2 - 3D)(D - 3)z = 3(D^2 - 3D)x + 3(D^2 - 3D)y + (D^2 - 3D)t$$

or $$(D^3 - 6D^2 + 9D)z = 3Dz + 3 - 3t + 6Dz + 3 + 3t - 3, \text{ using (7) and (8)}$$

or $$(D^3 - 6D^2)z = 3 \qquad \ldots(9)$$

Auxiliary equation of (9) is $D^3 - 6D^2 = 0$. Solving for D, it gives $D = 0, 0, 6$.

Hence, C.F. $= c_1 + c_2 t + c_3 e^{6t}$, c_1, c_2 and c_3 being arbitrary constants.

Next, $$\text{P.I.} = \frac{1}{D^3 - 6D^2} 3 = 3 \times \frac{1}{-6D^2(1 - D/6)} 1 = -\frac{1}{2D^2}\left(1 - \frac{D}{6}\right)^{-1} 1$$

$$= -\frac{1}{2D^2}\left(1 + \frac{D}{6} + \dots\right) 1 = -\frac{1}{2D^2} 1 = -\frac{1}{2} \times \frac{t^2}{2}$$

Hence the general solution of (9) is

$z = c_1 + c_2 t + c_3 e^{6t} - (t^2/4)$, c_1, c_2 and c_3 being arbitrary constants ...(10)

Re-writing (3) and (4), we have

$$(D - 2)\, y - 2z = 2x + t \quad \text{...(11)}$$

and $$-3y + (D - 3)\, z = 3x + t \quad \text{...(12)}$$

As before, (11) and (12) give

$$(D^2 - 5D)y = 2Dx + 1 - t \quad \text{...(13)}$$

and $$(D^2 - 5D)\, z = 3Dx + 1 + t \quad \text{...(14)}$$

But from (2), $(D - 1)x = y + z + t$

$\Rightarrow$ $(D^2 - 5D)(D - 1)x = (D^2 - 5D)y + (D^2 - 5D)\, z + (D^2 - 5D)\, t$

or $(D^3 - 6D^2 + 5D)x = 2Dx + 1 - t + 3Dx + 1 + t - 5$, using (13) and (14)

or $$(D^3 - 6D^2)x = -3 \quad \text{...(15)}$$

As before, the general solution is $x = a_1 + a_2 t + a_3 e^{6t} + (1/4) \times t^2$ ...(16)

Re-writing (4) and (2), we have

$$(D - 3)\, z - 3x = 3y + t \quad \text{...(17)}$$

$$-z + (D - 1)\, x = y + t \quad \text{...(18)}$$

As before, (17) and (18) give

$$(D^2 - 4D)\, z = 3Dy + 1 + 2t \quad \text{...(19)}$$

and $$(D^2 - 4D)x = Dy + 1 - 2t \quad \text{...(20)}$$

But from (3), $(D - 2)\, y = 2x + 2z + t$

$\Rightarrow$ $(D^2 - 4D)(D - 2)y = 2(D^2 - 4D)x + 2\, (D^2 - 4D)\, z + (D^2 - 4D)t$

or $(D^3 - 6D^2 + 8D)y = 2Dy + 2 - 4t + 6Dy + 2 + 4t - 4$, using (19) and (20)

or $$(D^3 - 6D^2)y = 0 \quad \text{...(21)}$$

As before, the general solution is $y = b_1 + b_2 t + b_3 e^{6t}$ ...(22)

Also suppose $x = x_0$, $y = y_0$ $z = z_0$ when $t = t_0 = 0$ ...(23)

Using (23), (16), (22) and (10) give

$x_0 = a_1 + a_3$, $y_0 = b_1 + b_3$, $z_0 = c_1 + c_3$

so that $a_1 = x_0 - a_3$, $b_1 = y_0 - b_3$, $c_1 = z_0 - c_3$ ...(24)

Using (24), (16), (22) and (10), we have

$$x = x_0 - a_3 + a_2\, t + a_3 e^{6t} + (1/4) \times t^2 \quad \text{...(25)}$$

$$y = y_0 - b_3 + b_2 t + b_3 e^{6t} \quad \text{...(26)}$$

$$z = z_0 - c_3 + c_2 t + c_3 e^{6t} - (1/4) \times t^2 \quad \text{...(27)}$$

Substituting these values of x, y and z into (2), (3) and (4), we have

$$a_2 + 6a_3 e^{6t} + t/2 = x_0 + y_0 + z_0 - (a_3 + b_3 + c_3) + (a_2 + b_2 + c_2)t + (a_3 + b_3 + c_3)\, e^{6t} + t \quad \text{...(28)}$$

$$b_2 + 6b_3 e^{6t} = 2(x_0 + y_0 + z_0) - 2(a_3 + b_3 + c_3) + 2(a_2 + b_2 + c_2)t + 2(a_3 + b_3 + c_3)\, e^{6t} + t \quad \text{...(29)}$$

$$c_2 + 6c_3 e^{6t} - (t/2) = 3(x_0 + y_0 + z_0) - 3(a_3 + b_3 + c_3) + 3(a_2 + b_2 + c_2)t$$
$$+ 3(a_3 + b_3 + c_3)e^{6t} + t \quad \text{...(30)}$$

(28), (29) and (30) are identities. So equating coefficients of t, e^{6t} and absolute terms, these identities give

$$x_0 + y_0 + z_0 - (a_3 + b_3 + c_3) = a_2 \qquad ...(31a)$$

$$a_3 + b_3 + c_3 = 6a_3 \qquad ...(31b)$$

$$a_2 + b_2 + c_2 + 1 = 1/2 \qquad ...(31c)$$

$$2(x_0 + y_0 + z_0) - 2(a_3 + b_3 + c_3) = b_2 \qquad ...(32a)$$

$$2(a_3 + b_3 + c_3) = 6b_3 \qquad ...(32b)$$

$$2(a_2 + b_2 + c_2) + 1 = 0 \qquad ...(32c)$$

$$3(x_0 + y_0 + z_0) - 3(a_3 + b_3 + c_3) = c_2 \qquad ...(33a)$$

$$3(a_3 + b_3 + c_3) = 6c_3 \qquad ...(33b)$$

$$3(a_2 + b_2 + c_2) + 1 = -(1/2) \qquad ...(33c)$$

From (31c) or (32c) or (33c), we have

$$a_2 + b_2 + c_2 = -(1/2) \qquad ...(34)$$

Adding (31a), (32a) and (33a), we get

$$6[(x_0 + y_0 + z_0) - (a_3 + b_3 + c_3)] = a_2 + b_2 + c_2$$

or $$6[(x_0 + y_0 + z_0) - (a_3 + b_3 + c_3)] = -(1/2), \text{ by } (34)$$

or $$a_3 + b_3 + c_3 = x_0 + y_0 + z_0 + (1/12) \qquad ...(35)$$

Using (35), (31b), (32b) and (33b) give

$$a_3 = (1/6) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(36)$$

$$b_3 = (1/3) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(37)$$

$$c_3 = (1/2) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(38)$$

Again, using (35), (31a), (32a) and (33a) give

$$a_2 = -1/12, \qquad b_2 = -1/6, \qquad \text{and} \qquad c_2 = -1/4 \qquad ...(39)$$

Substitutiting the above values of a_2, b_2, c_2, a_3, b_3 and c_3 into (25), (26) and (27) and simplifying, we have

$$x = (5/6) \times x_0 - (1/6) \times y_0 - (1/6) \times z_0 + (1/6) \times (x_0 + y_0 + z_0 + 1/12)e^{6t} - t/12 + t^2/4 - (1/72) \quad ...(40)$$

$$y = -(1/3) \times x_0 + (2/3) \times y_0 - (1/3) \times z_0 + (1/3) \times (x_0 + y_0 + z_0 + 1/12)\, e^{6t} - t/6 - (1/36) \quad ...(41)$$

$$z = -(1/2) \times x_0 - (1/2) \times y_0 + (1/2) \times z_0 + (1/2) \times (x_0 + y_0 + z_0 + 1/12)\, e^{6t} - t/4 - t^2/4 - (1/24) \quad ...(42)$$

which give the desired displacements.

Part. (b) Let u', v', w' be the components of the velocity in Lagrangian system. Then using (40), (41) and (42), we have

$$u' = \partial x / \partial t = \left(x_0 + y_0 + z_0 + 1/12\right) e^{6t} - (1/12) + (t/2) \qquad ...(43)$$

$$v' = \partial y / \partial t = 2\left(x_0 + y_0 + z_0 + 1/12\right) e^{6t} - (1/6) \qquad ...(44)$$

$$w' = \partial z / \partial t = 3\left(x_0 + y_0 + z_0 + 1/12\right) e^{6t} - (1/4) - (t/2) \qquad ...(45)$$

The required velocity is $u'i + v'j + w'k$, where u', v' and w' are given by (43), (44) and (45) respectively.

Velocity of a fluid particle.

Let the fluid particle be at P at any time t and let it be at Q at time $t + \delta t$ such that

$$\overrightarrow{OP} = \mathbf{r} \qquad \text{and} \qquad \overrightarrow{OQ} = \mathbf{r} + \delta\mathbf{r}.$$

Then in the interval δt the movement of the particle is $\overrightarrow{PQ} = \delta\mathbf{r}$ and hence the velocity of the fluid particle $\mathbf{q}$ at P is given by

$$\mathbf{q} = \lim_{\delta t \to 0} (\delta\mathbf{r} / \delta t) = d\mathbf{r} / dt,$$

assuming such a limit to exist uniquely. Taking the fluid as continuous, the above assumption is justified. Clearly $\mathbf{q}$ is a function of $\mathbf{r}$ and t and hence it can be expressed as $\mathbf{q} = f(\mathbf{r}, t)$. If u, v, w are the components of $\mathbf{q}$ along the axes, we have $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

2.4. Material, local and convective derivatives. **(Meerut 2009, 2011)**

Suppose a fluid particle moves from $P(x, y, z)$ at time t to $Q(x + \delta x, y + \delta y, z + \delta)$ at time $t + \delta t$. Further suppose $f(x, y, z, t)$ be a scalar function associated with some property of the fluid (*e.g.* the pressure or density etc.). Let the total change of f due to movement of the fluid particle from P to Q be δf. Then, we have

$$\delta f = (\partial f / \partial x)\delta x + (\partial f / \partial y)\delta y + (\partial f / \partial z)\delta z + (\partial f / \partial t)\delta t$$

or $$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t} \qquad ...(1)$$

Let $$\left.\begin{array}{ll} \lim_{\delta t \to 0} \frac{\delta f}{\delta f} = \frac{Df}{Dt} \text{ or } \frac{df}{dt}, & \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = u, \\ \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt} = v \quad \text{and} & \lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \frac{dz}{dt} = w \end{array}\right\} \qquad ...(2)$$

where $\boldsymbol{q} = (u, v, w)$ is the velocity of the fluid particle at P. Making $\delta t \to 0$ and using (2), (1) reduces to

$$\frac{Df}{Dt} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \qquad ...(3)$$

But $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad ...(4)$$

and $$\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k} \qquad ...(5)$$

From (4) and (5), $$\mathbf{q}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad ...(6)$$

Using (6) and (3) reduces to $$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{q}\cdot\nabla)f \qquad ...(7)$$

Again, suppose $\mathbf{g}$ (x, y, z, t) be a vector function associated with some property of the fluid (*e.g.* velocity etc.). Then proceeding as above, we have

$$\frac{D\mathbf{g}}{Dt} = \frac{\partial \mathbf{g}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{g} \qquad ...(8)$$

From (7) and (8), we have for both scalar and vector functions

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla \qquad ...(9)$$

D/Dt is called the material (*or particle or substantial*) *derivative*. It is also spoken of as *differentiation following the motion of the fluid*. The first term on R.H.S. of (9), namely $\partial/\partial t$, is called the *local derivative* and it is associated with time variation at a fixed position. The second term on R.H.S. of (9), namely $\mathbf{q}\cdot\nabla$, is called the *convective derivative* and it is associated with the change of a physical quantity f or $\mathbf{g}$ due to motion of the fluid particle.

Note. The operator D/Dt signifies that we are calculating the rate of change of a physical quantity f or $\mathbf{g}$ associated with the same fluid particle as it moves about. The symbol d/dt is also used for the material derivative D/Dt.

2.5A. Acceleration of a fluid particle. [Kanpur 2004]

Suppose a fluid particle moves from P (x, y, z) at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Let

$$\mathbf{q} = (u, v, w) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad ...(1)$$

be the velocity of the fluid particle at P and let $\boldsymbol{q}+\delta\boldsymbol{q}$ be the velocity of the same fluid particle at Q. Then, we have

$$\delta \boldsymbol{q} = \frac{\partial \boldsymbol{q}}{\partial x}\delta x + \frac{\partial \boldsymbol{q}}{\partial y}\delta y + \frac{\partial \boldsymbol{q}}{\partial z}\delta z + \frac{\partial \boldsymbol{q}}{\partial t}\delta t \quad \text{or} \quad \frac{\delta \boldsymbol{q}}{\delta t} = \frac{\partial \boldsymbol{q}}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial t} \quad \text{...(2)}$$

Let
$$\lim_{\delta t \to 0}\frac{\delta \mathbf{q}}{\delta t} = \frac{D\mathbf{q}}{Dt} \text{ or } \frac{d\mathbf{q}}{dt}, \qquad \left.\begin{aligned} \lim_{\delta t \to 0}\frac{\delta x}{\delta t} &= \frac{dx}{dt} = u, \\ \lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt} = v \quad \text{and} \quad \lim_{\delta t \to 0}\frac{\delta z}{\delta t} &= \frac{dz}{dt} = w \end{aligned}\right\} \quad \text{...(3)}$$

Making $\delta t \to 0$ and using (3), (2) reduces to

$$\boldsymbol{a} = \frac{D\mathbf{q}}{Dt} = u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} + \frac{\partial \mathbf{q}}{\partial t} \quad \text{...(4)}$$

Let
$$\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k} \quad \text{...(5)}$$

From (1) and (5),
$$\mathbf{q}\cdot\nabla = u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z) \quad \text{...(6)}$$

Using (6), (4) may be re-written as

$$\boldsymbol{a} = \frac{D\mathbf{q}}{Dt} = (\mathbf{q}\cdot\nabla)\mathbf{q} + \frac{\partial \mathbf{q}}{\partial t}, \quad \text{...(7)}$$

which shows that the acceleration $\boldsymbol{a}$ of a fluid particle of fixed identity can be expressed as the material derivative of the velocity vector $\mathbf{q}$.

(*i*) **Components of acceleration in cartesian coordinates** (x, y, z). **(Meerut 2010)**

Let $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Then (4) yields

$$a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = u\frac{\partial}{\partial x}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + v\frac{\partial}{\partial y}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + w\frac{\partial}{\partial z}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + \frac{\partial}{\partial t}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$\therefore \quad a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

$$a_y = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$

and
$$a_z = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}.$$

(*ii*) **Components of acceleration** $(a_r, a_\theta \cdot a_z)$ **in cylindrical coordinates (r, θ, z) with velocity components** $(v_r, v_\theta \cdot v_z)$.

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

(*iii*) **Components of acceleration $(a_r, a_\theta \cdot a_\phi)$ in spherical polar coordinates (r, θ, ϕ) with velocity components $(v_r, v_\theta \cdot v_\phi)$.**

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\phi v_\theta \cot\theta}{r}$$

2.5B. Acceleration in cartesian coordinates (an alternative proof).

Let $P(x, y, z)$ be any point within the fluid. Let u, v, w be components of velocity of the element of the fluid at P.

Let $$u = f(x, y, z, t) \qquad \ldots(1)$$

Let particle which is at $P(x, y, z)$ at time t move to $Q(x + u\delta t, y + v\delta t, z + w\delta t)$ after a short interval δt. If $u + \delta u$ be *x-component* of velocity at Q, then

$$u + \delta u = f(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t)$$

$$= f(x, y, z) + \left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}\right)\delta t$$

+ terms containing higher power of δt, by Taylor's theorem

$$\therefore \quad u + \delta u = u + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)\delta t + \ldots, \text{ using (1)} \qquad \ldots(2)$$

Let a_x, a_y, a_z be the components of acceleration of the element of the fluid at P. Then,

$$a_x = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta t \to 0} \frac{(u + \delta u) - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t})\delta t - u}{\delta t}, \quad \text{using (2)}$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \; = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = \frac{Du}{Dt},$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

which in known as *material or substantial derivative.*

$$\therefore \quad a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad \ldots(3)$$

Similarly, we have

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial y}{\partial z}, \qquad ...(4)$$

and
$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}, \qquad ...(5)$$

2.6. Illustrative solved examples.

Ex. 1. *If the velocity distribution is* $\mathbf{q} = \mathbf{i}\, Ax^2y + \mathbf{j}\, By^2zt + \mathbf{k}Czt^2$, *where A, B, C, are constants, then find the the acceleration and velocity components.*

[Agra 2005; Garhwal 2001; Kanpur 2001; Meerut 2009, 2010, 2011]

Sol. The acceleration $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is given by

$$\boldsymbol{a} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} \qquad ...(1)$$

Also $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \mathbf{i}Ax^2y + \mathbf{j}By^2zt + \mathbf{k}Czt^2 \qquad ...(2)$

Hence, $\quad u = Ax^2y, \qquad v = By^2zt, \qquad w = Czt^2 \qquad ...(3)$

Using (2) and (3), (1) reduces to

$$\boldsymbol{a} = By^2z\mathbf{j} + 2Czt\mathbf{k} + Ax^2y \times (2Axy\mathbf{i}) + By^2zt(Ax^2\mathbf{i} + 2Byzt\mathbf{j}) + Czt^2(By^2t\mathbf{j} + Ct^2\mathbf{k})$$
$$= A(2Ax^3y^2 + Bx^2y^2zt)\,\mathbf{i} + B\,(y^2z + 2By^3\,z^2t^2 + Cy^2zt^3)\mathbf{j} + C(2zt + Czt^4)\,\mathbf{k}$$

The components of the acceleration (a_x, a_y, a_z) are given by

$a_x = A(2Ax^3y^2 + Bx^2y^2zt), \qquad a_y = B\,(y^2z + 2By^3z^2t^2 + Cy^2zt^3), \qquad a_z = C(2zt + Czt^4)$

Ex. 2. *The velocity components of a flow in cylindrical polar coordinates are* $(r^2 z\cos\theta, rz\sin\theta, z^2t)$. *Determine the components of the acceleration of a fluid particle.*

Sol. Let v_r, v_θ, v_z be the components of velocity in cylindrical polar coordinates (r, θ, z). Then, we have

$$v_r = r^2 z\cos\theta, \qquad v_\theta = rz\sin\theta, \qquad v_z = z^2t \qquad ...(1)$$

Let a_r, a_θ and a_z be the components of acceleration. Then using (1), we have

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$= 0 + (r^2z\cos\theta)\,(2rz\cos\theta) + \{(rz\sin\theta)/r\}(-2rz\sin\theta) + (z^2t)(r^2\cos\theta) - (rz\sin\theta)^2/r$$

$$= rz^2(2r^2\cos^2\theta - 3\sin^2\theta + rt\cos\theta)$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$= (r^2z\cos\theta)(z\sin\theta) + \{(rz\sin\theta)/r\}(rz\cos\theta) + (z^2t)(r\sin\theta) + (1/r)\,(r^2z\cos\theta)(rz\sin\theta)$$

$$= z^2r\sin\theta\,(3r\cos\theta + t)$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

$$= z^2 + (r^2z\cos\theta)\times 0 + \{(rz\sin\theta)/r\}\times 0 + (z^2t)\,(2zt) = z^2(1 + 2t^2z).$$

2.7. Significance of the equation of continuity, (or conservation of mass.)

[Kurukshetra 1999; Meerut 2010; Himachal 2002, 09, 10; Garhwal 2005; Kanpur 2003]

The law of conservation of mass states that fluid mass can be neither created nor destroyed. The equation of continuity aims at expressing the law of conservation of mass in a mathematical form. Thus, in continuous motion, the equation of continuity expresses the fact that the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

2.8. The equation of continuity (or equation of conservation of mass) by Euler's method.

[Kurukshatra 1999; Himachal 2010; Kanpur 2003, 05, 08; Meerut 2003, 10 Purvanchal 2004, 05]

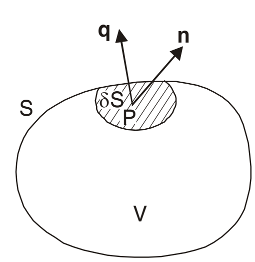
Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let $P(x, y, z)$ be any point of S and let $\rho(x, y, z, t)$ be the fluid density at P at any time t. Let δS denote element of the surface S enclosing P. Let $\mathbf{n}$ be the unit outward-drawn normal at δS and let $\mathbf{q}$ be the fluid velocity at P. Then the normal component of $\mathbf{q}$ measured outwards from V is $\mathbf{n}\cdot\mathbf{q}$. Thus,

Rate of mass flow accross $\delta S = \rho(\mathbf{n}\cdot\mathbf{q})\,\delta S$

$\therefore$ Total rate of mass flow across S

$$= \int_S \rho\,(\mathbf{n}\cdot\mathbf{q})\,dS = \int_V \nabla\cdot(\rho\,\mathbf{q})\,dV$$

(By Gauss divergence theorem)



$\therefore$ Total rate of mass flow into $V = -\int_V \nabla\cdot(\rho\,\mathbf{q})\,dV$ ...(1)

Again, the mass of the fluid within S at time $t = -\int_V \rho\,dV$

$\therefore$ Total rate of mass increase within $S = \dfrac{\partial}{\partial t}\int_V \rho\, d\, V = \int_V \dfrac{\partial \rho}{\partial t}\, d\, V$ ...(2)

Suppose that the region V of the fluid contains neither sources nor sinks (*i.e.* there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from (1) and (2), we have

$$\int_V \frac{\partial \rho}{\partial t}\, d\, V = -\int_V \nabla\cdot(\rho\mathbf{q})\, d\, V \qquad \text{or} \qquad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{q})\right] d\, V = 0$$

which holds for arbitrary small volumes V, if $\dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0.$ ...(3)

Equation (3) is called the *equation of continuity*, or the *conservation of mass* and it holds at all points of fluid free from sources and sinks.

Cor. 1. Since $\nabla\cdot(\rho\mathbf{q}) = \rho\nabla\cdot\mathbf{q} + \nabla\rho\cdot\mathbf{q}$, other forms of (3) are

$$\partial\rho/\partial t + \rho\nabla\cdot\mathbf{q} + \nabla\rho\cdot\mathbf{q} = 0, \quad ...(4)$$

$$D\rho/Dt + \rho\nabla\cdot\mathbf{q} = 0, \quad ...(5)$$

and $$D(\log\rho)/Dt + \nabla\cdot\mathbf{q} = 0. \quad ...(6)$$

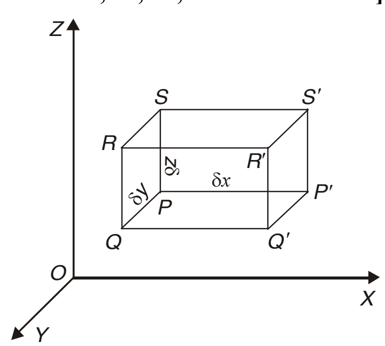
Cor. 2. For an incompressible and heterogeneous fluid the density of any fluid particle is invariable with time so that $D\rho/Dt = 0$. Then (5) gives

$$\nabla\cdot\mathbf{q} = 0 \text{ i.e. div } \mathbf{q} = 0 \quad \text{or} \quad \partial u/\partial x + \partial v/\partial z + \partial w/\partial z = 0 \quad \text{if} \quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}.$$

Cor. 3. For an incompressible and homogeneous fluid, ρ is constant and hence $\partial\rho/\partial t = 0$. Then (3) gives $\nabla\cdot(\rho\mathbf{q}) = 0$ *i.e.* $\nabla\cdot\mathbf{q} = 0$ or $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, as ρ is constant.

2.9. The equation of continuity in cartesian coordinates.

[Garhwal 2005; I.A.S. 1999; Kanpur 2011; Meerut 2002; Agra 1997; Bombay 1998' G.N.D.U. Amritsar 2000, 03, 05; Rohilkhand 2005]



Let there be a fluid particle at P (x, y, z). Let $\rho\,(x, y, z, t)$ be the density of the fluid at P at any time t and let u, v, w be the velocity components at P parallel to the rectangular coordinate axes. Construct a small parallelepiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes, having P at one of the angular points as shown in figure. Then, we have

Mass of the fluid that passes in through the face $PQRS$

$= (\rho\delta y\delta z)\, u$ per unit time $= f\,(x, y, z)$ say ...(1)

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S'$

$= f(x+\delta x, y, z)$ per unit time $= f(x,y,z) + \delta x \dfrac{\partial}{\partial x} f(x,y,z) + \ldots$ ...(2)

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the element (rectangular parallelepiped) due to flow through the faces PQRS and $P'Q'R'S'$ by using (1) and (2)

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$$= f(x,y,z) - \left[f(x,y,z) + \delta x \cdot \frac{\partial}{\partial x} f(x,y,z) + \dots \right]$$

$$= -\delta x \cdot \frac{\partial}{\partial x} f(x,y,z), \text{ to the first order of approximation } = -\delta x \cdot \frac{\partial}{\partial x}(\rho u\, \delta y\, \delta z), \text{ by (1)}$$

$$= -\delta x\, \delta y\, \delta z \frac{\partial(\rho u)}{\partial x} \qquad \text{...(3)}$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces $PP'S'S$ and $QQ'RR'$ $\quad = -\delta x\, \delta y\, \delta z \dfrac{\partial(\rho v)}{\partial y}$ ...(4)

and the net gain in mass per unit time within the element due to flow through the faces $PP'Q'Q$ and $SS'R'R$ $\quad = -\delta x\, \delta y\, \delta z \dfrac{\partial(\rho w)}{\partial z}$ ...(5)

$\therefore$ Total rate of mass flow into the elementary parallelepiped

$$= -\delta x\, \delta y\, \delta z \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \qquad \text{...(6)}$$

Again, the mass of the fluid within the chosen element at time t $= \rho \delta x\, \delta y\, \delta z$

$\therefore$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho \delta x\, \delta y\, \delta z) = \delta x\, \delta y\, \delta z \frac{\partial \rho}{\partial t} \qquad \text{...(7)}$$

Suppose that the chosen region (bounded by the elementary parallelepiped) of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$\delta x\, \delta y\, \delta z \frac{\partial \rho}{\partial t} = -\delta x\, \delta y\, \delta z \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right]$$

or
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \text{...(8)}$$

or
$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z} = 0$$

or
$$\left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right] \rho + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or $$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \qquad ...(9)$$

which is the desired equation of continuity in cartesian coordinates and it holds at all point of the fluid free from sources and sinks.

Remark. If the fluid is homogeneous and incompressible, ρ is constant and (9) reduces to

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(10)$$

Further, if the fluid is heterogeneous and incompressible, ρ is a function of x, y, z and t such that $D\rho/Dt = 0$. Hence the corresponding equation of continuity is again given by (10).

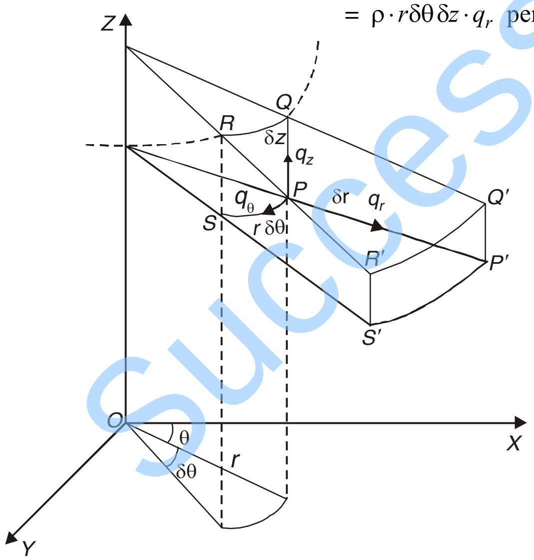
2.10. The equation of continuity in cylindrical coordinates. (Kanpur 2009)

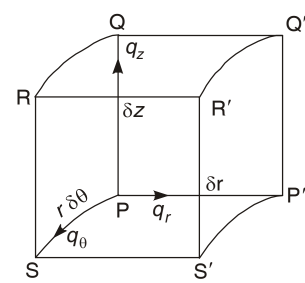
[Agra 2005, Himachal 1998, Meerut 2000, 01, Garhwal 2000, Rajasthan 1998]

Let there be a fluid particle at P whose cylindrical coordinates are (r, θ, z), where $r \geq 0, 0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$. Let $\rho(r, \theta, z, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelepiped ($PQRS$, $P'Q'R'S'$) with its edges $SS' = \delta r$, arc $SP = r\delta\theta$ and $PQ = \delta z$. Let q_r, q_θ and q_z be the velocity components in the direction of the elements SS', arc SP and PQ respectively. Then, we have

Mass of the fluid that passes in through the face $PSRQ$

$$= \rho \cdot r\delta\theta\,\delta z \cdot q_r \text{ per unit time } = f(r, \theta, z), \text{ say} \qquad ...(1)$$





$\therefore$ Mass of the fluid that passes out through the opposite face $P'S'R'Q'$

$$= f(r + \delta r, \theta, z) \text{ per unit time } = f(r, \theta, z) + \delta r\frac{\partial}{\partial r}f(r, \theta, z) + ... \qquad ...(2)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the chosen elementary parallelepiped $(PQRS, P'Q'R'S')$ due to flow through the faces $PSRQ$ and $P'S'R'Q'$ by using (1) and (2)

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$$= f(r, \theta, z) - \left[f(r, \theta, z) + \delta r \cdot \frac{\partial}{\partial r}f(r, \theta, z) + ...\right]$$

$= -\delta r \cdot \dfrac{\partial}{\partial r} f(r, \theta, z)$, tothe first order of approximation $= -\delta r \cdot \dfrac{\partial}{\partial r}(\rho\, r\, \delta\theta\, \delta z\, q_r)$, by (1)

$$= -\delta r\, \delta\theta\, \delta z \frac{\partial(\rho r q_r)}{\partial r} \qquad ...(3)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces

$SRR'S'$ and $QPP'Q'$ $\quad = -\delta r\, \delta\theta\, \delta z \dfrac{\partial}{\partial \theta}(\rho\, q_\theta)$ ...(4)

and the net gain in mass per unit time within the element due to flow through the faces $PSS'P'$ and

$QRR'Q'$ $\quad = -\delta r\, \delta\theta\, \delta z \dfrac{\partial}{\partial z}(\rho r\, q_z) = -r \delta r\, \delta\theta\, \delta z \dfrac{\partial(\rho q_z)}{\partial z}$ ...(5)

$\therefore$ Total rate of mass flow into the chosen element

$$= -\delta r\, \delta\theta\, \delta z \left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r \frac{\partial}{\partial z}(\rho q_z)\right] \qquad ...(6)$$

Again, the mass of the fluid within the element at time t $= \rho r\, \delta r\, \delta\theta\, \delta z$

$\therefore$ Total rate of mass increase within the element $= \dfrac{\partial}{\partial t}(\rho r\, \delta r\, \delta\theta\, \delta z) = r \delta r\, \delta\theta\, \delta z \dfrac{\partial \rho}{\partial t}$ ...(7)

Suppose that the chosen region of the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$r\, \delta r\, \delta\theta\, \delta z \frac{\partial \rho}{\partial t} = -\delta r\, \delta\theta\, \delta z \left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r \frac{\partial}{\partial z}(\rho q_z)\right]$$

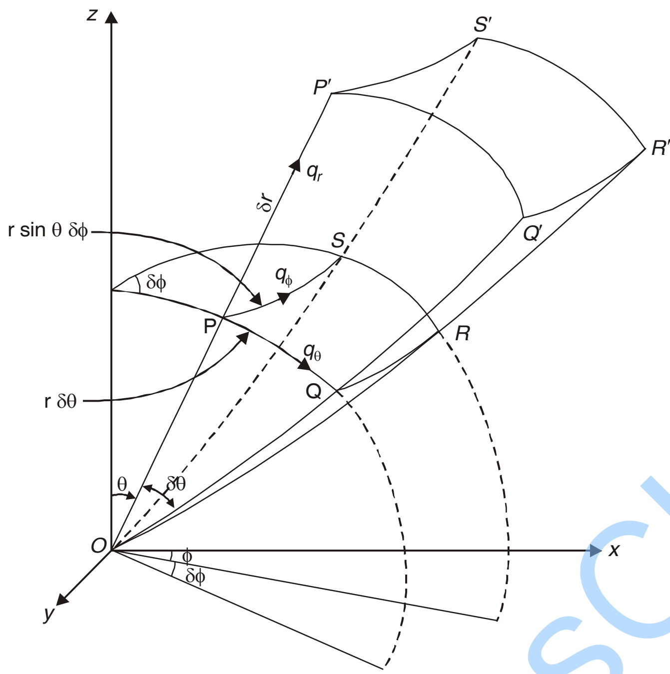
or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{1}{\partial z}(\rho q_z) = 0, \qquad ...(8)$$

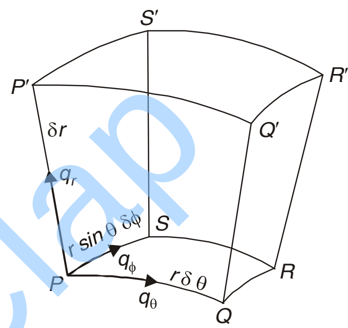
which is the desired equation of continuity in cylindrical coordinates and it holds at all points of the fluid free from sources and sinks.

2.11. The equation of continuity in spherical polar coordinates.

[Meerut 2008; Garhwal 1995, 96; Rajasthan 1997; Rohilkhand 2000]

Let there be a fluid particle at P whose spherical polar coordinates are (r, θ, ϕ), where $r \geq 0, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi$. Let $\rho(r, \theta, \phi, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelopiped $(PQRS, P'Q'R'S')$ with its edges $PP' = \delta r$, arc $PQ = r\delta\theta$, arc $PS = r \sin\theta \delta\phi$. Let q_r, q_θ and q_ϕ be the velocity components in the direction of the elements PP', arc PQ and arc PS respectively. Then, we have





Mass of the fluid that passes in through the face $PQRS$

$$= \rho \cdot r\delta\theta\,\delta r \sin\theta\,\delta\phi \cdot q_r \text{ per unit time } = f(r,\theta,\phi), \text{ say} \qquad ...(1)$$

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S'$

$$= f(r+\delta r,\theta,\phi) = f(r,\theta,\phi) + \delta r \frac{\partial}{\partial r}(r,\theta,\phi) + ... \qquad ...(2)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the chosen elementary parallelopiped $(PQRS, P'Q'R'S')$ due to flow through the faces $PQRS$ and $P'Q'R'S'$ by using (1) and (2)

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$$= f(r,\theta,\phi) - \left[f(r,\theta,\phi) + \delta r \cdot \frac{\partial}{\partial r}(r,\theta,\phi) + ... \right]$$

$$= -\delta r \cdot \frac{\partial}{\partial r} f(r,\theta,\phi), \text{ to the first order of approximation}$$

$$= -\delta r \cdot \frac{\partial}{\partial r}(\rho r^2 \sin\theta\, q_r \delta\theta\, \delta\phi), \text{ by (1)} \qquad ...(3)$$

Similarly the net gain in mass per unit time within the element due to flow through the faces $PSS'P'$ and $QRR'Q'$

$$= -r\delta\theta \frac{\partial}{r\delta\theta}(\rho \cdot \delta r \cdot r \sin\theta\, \delta\phi \cdot q_\theta) \qquad ...(4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PQQ'P'$ and $SRR'S'$

$$= -r\sin\theta\,\delta\phi \frac{1}{r\sin\theta} \frac{\partial}{\partial\phi}(\rho \cdot \delta r\, r\,\delta\theta \cdot q_\theta) \qquad ...(5)$$

∴ Total rate of mass flow into the elementary parallelepiped

$$= -\delta r\,\delta\theta\,\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] \quad ...(6)$$

Again, the mass of the fluid within the chosen element at time $t = -\rho\delta r\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\phi$

∴ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi) = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t} \quad ...(7)$$

Suppose that the chosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$-\delta r\,\delta\theta\,\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t}$$

or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0, \quad ...(8)$$

which is the desired equation of continuity in spherical polar coordinates and it holds at all points of the fluid free from sources and sinks.

2.11A. Generalised orthogonal curvilinear coordinates

Let the rectangular cartesian coordinates (x, y, z) of any point P in space be expressed in terms of three independent, single-valued and continuously differentiable scalar point functions u_1, u_2, u_3 as follows :

$$\left.\begin{aligned} x &= x(u_1, u_2, u_3)\\ y &= y(u_1, u_2, u_3)\\ z &= z(u_1, u_2, u_3)\end{aligned}\right\} \quad ...(1)$$

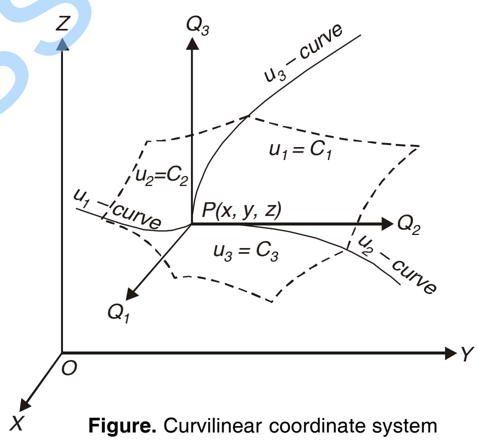


Figure. Curvilinear coordinate system

Suppose that the Jacobian of x, y, z with respect to u_1, u_2, u_3 does not vanish, that is,

$\dfrac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} \neq 0$. Then the transformation (1) can be inverted, *i.e.*, u_1, u_2, u_3 can be expressed in terms of x, y, z giving

$$u_1 = u_1(x, y, z), \qquad u_2 = u_2(x, y, z), \qquad u_3 = u_3(x, y, z). \quad ...(2)$$

Thus to each point $P(x, y, z)$ we can assign a unique set of new coordinates (u_1, u_2, u_3) called the *curvilinear coordinates* of P. In this sense the equations (1) or (2) may be interpreted as defining a *transformation of coordinates*.

The surfaces $u_1(x, y, z) = C_1$, $u_2(x, y, z) = C_2$, $u_3(x, y, z) = C_3$, where C_1, C_2, C_3 are constants, are called *coordinate surfaces* and each pair of these surfaces intersect in curves called *coordinate curves or lines*. The surfaces $u_2 = C_2$ and $u_3 = C_3$ intersect in a curve along which the coordinate 'u_1' alone varies and hence it is called u_1-curve or line. Similarly, we have u_2- line and u_3- line. The coordinate axes are determined by the tangents PQ_1, PQ_2 and PQ_3 to the coordinate curves $u_1 = C_1$, $u_2 = C_2$, $u_3 = C_3$. Note carefully that the directions of these coordinate axes depend on the chosen point P of space and consequently the unit vectors associated with them are not necessarily constant.

If at every point $P(x, y, z)$, the coordinate axes are mutually perpendicular, then u_1, u_2, u_3 are called *orthogonal curvilinear coordinates* of P.

The line element ds in cartesian coordinates in given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \qquad ...(3)$$

Now, from (1), we have

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 + \frac{\partial x}{\partial u_3} du_3, \qquad dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 + \frac{\partial y}{\partial u_3} du_3$$

and

$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 + \frac{\partial z}{\partial u_3} du_3 \cdot$$

Substituting these values of dx, dy and dz in (3) and using the fact that by orthogonal property coefficents of $du_1 du_2$, $du_2\, du_3$ and $du_3 du_1$ must vanish in the result so obtained, we have

$$(ds)^2 = h_1^{\,2}\,(du_1)^2 + h_2^{\,2}\,(du_2)^2 + h_3^{\,2}\,(du_3)^2, \qquad ...(4)$$

where

$$\left.\begin{aligned} h_1 &= (\partial x/\partial u_1)^2 + (\partial y/\partial u_1)^2 + (\partial z/\partial u_1)^2 \\ h_2 &= (\partial x/\partial u_2)^2 + (\partial y/\partial u_2)^2 + (\partial z/\partial u_2)^2 \\ h_3 &= (\partial x/\partial u_3)^2 + (\partial y/\partial u_3)^2 + (\partial z/\partial u_3)^2, \end{aligned}\right\} \qquad ...(5)$$

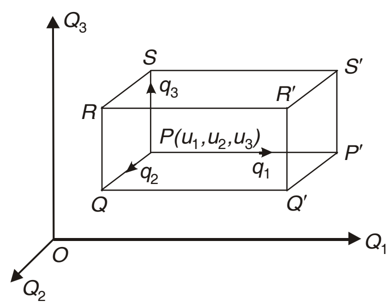
and

h_1, h_2, h_3 being known as *scale factors*.

2.11B. Equation of continuity in generalised orthogonal curvilinear coordinates

(Kanpur 2007, 10)

Let there be a fluid particle at P whose orthogonal curvilinear coordinates are (u_1, u_2, u_3). Let $\rho\,(u_1, u_2, u_3, t)$ be the density of fluid at P at any time t and let q_1, q_2, q_3 be the velocity components at P along PP', PQ and PS respectively. Consider an infinitesimal parallelepiped $PQRS$, $P'Q'R'S'$ with one vertex at P as shown in the figure. Then we know that the lengths of edges of parallelopiped are $PP' = h_1 \delta u_1$, $PQ = h_2\, \delta u_2$ and $PS = h_3\, \delta u_3$. Areas of the faces are $h_2 h_3\, \delta u_2 \delta u_3$, $h_3 h_1 \delta u_3\, \delta u_1$ and $h_1\, h_2\, \delta u_1 \delta u_2$ and volume of the parallelepiped is $h_1\, h_2\, h_3\, \delta u_1\, \delta u_2\, \delta u_3$.



Then, mass of the fluid that passes in through the face $PQRS$

$$= \rho(h_2 \delta u_2\, h_3 \delta u_3) q_1 \text{ per unit time} = f(u_1, u_2, u_3), \text{ say.} \qquad ...(1)$$

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S'$

$$= f(u_1 + \delta u_1, u_2, u_3) \text{ per unit time} = f(u_1, u_2, u_3) + \delta u_1 \frac{\partial}{\partial u_1} f(u_1, u_2, u_3) + ... \qquad ...(2)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the elementary parallelepiped due to flow through the faces $PQRS$ and $P'Q'R'S'$

= Mass that enters in through the face $PQRS$ − Mass that leaves through the face $P'Q'R'S'$

$= f(u_1, u_2, u_3) - [f(u_1, u_2, u_3) + \delta u_1 \dfrac{\partial}{\partial u_1} f(u_1, u_2, u_3) + ...]$, by (1) and (2)

$= -\delta u_1 \dfrac{\partial}{\partial u_1} f(u_1, u_2, u_3)$, to the first order of approximation

$= -\delta u_1 \dfrac{\partial}{\partial u_1} (pq_1 h_2 h_3 \delta u_2 \delta u_3)$, using (1)

$$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_1} (\rho q_1 h_2 h_3). \qquad ...(3)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces

$PP'S'S$ and $QQ'R'R$ $$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) \qquad ...(4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PP'Q'Q$

and $SS'R'R$ $$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2). \qquad ..(5)$$

From (3), (4) and (5), total rate of mass flow into the elementary parallelopiped

$$= -\delta u_1 \delta u_2 \delta u_3 \left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right]. \qquad ...(6)$$

Again the mass of the fluid within the chosen element at time $t = \rho h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3$.

$\therefore$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3) = h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3 \frac{\partial \rho}{\partial t}. \qquad ...(7)$$

Now, by the law of conservation of fluid mass, the rate of increase of mass of the fluid within the element must be equal to the rate of flowing into the element. Hence, from (6) and (7), we have

$$h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3 \frac{\partial \rho}{\partial t} = -\delta u_1 \delta u_2 \delta u_3 \left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right]$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right] = 0 \qquad ...(8)$$

This is the required equation of continuity in orthogonal curvilinear coordinates (u_1, u_2, u_3).

Deductions. (*i*) Rectangular Cartesian Coordinates (*x, y, z*)

Then, $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$

$\Rightarrow$ $h_1 = h_2 = h_3 = 1$, $u_1 = x$, $u_2 = y$ and $u_3 = z$.

Also, here $q_1 = u$, $q_2 = v$ and $q_3 = w$

In this case the equation of continuity (8) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0.$$

Deduction (*ii*). Cylinderical coordinates (*r*, θ, *z*)

Cylindrical coordinates (r, θ, z) are defined by means of equations

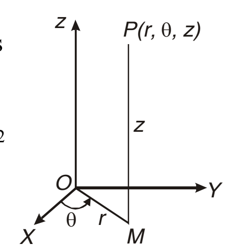
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

where $r \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$.

Here $(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$

$\Rightarrow$ $h_1 = 1$, $h_2 = r$, $h_3 = 1$, $u_1 = r$, $u_2 = \theta$, $u_3 = z$.

Also, here $q_1 = q_r$, $q_2 = q_\theta$ and $q_3 = q_z$



Cylindrical Coordinates

In this case the equation of continuity (8) becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_r r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z r)\right] = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_r r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0.$$

Deduction (*iii*). Spherical coordinates (*r*, θ, φ).

Spherical coordinates (r, θ, ϕ) are defined by means of equations

$$x = r \sin\theta \cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

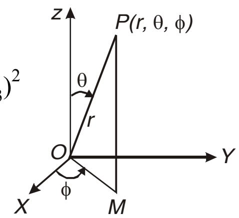
where $\quad r \ge 0, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi.$

Here $(ds)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2 = (h_1du_1)^2 + (h_2du_2)^2 + (h_3\, du_3)^2$

$\Rightarrow h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \quad u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi.$

Also, here $\quad q_1 = q_r, \qquad q_2 = q_\theta \quad$ and $\quad q_3 = q_\phi$

In this case the equation of continuity (8) becomes



Spherical Coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(\rho r^2\sin\theta\, q_r) + \frac{\partial}{\partial\theta}(\rho r\sin\theta\, q_\theta) + \frac{\partial}{\partial\phi}(\rho r q_\phi)\right] = 0$$

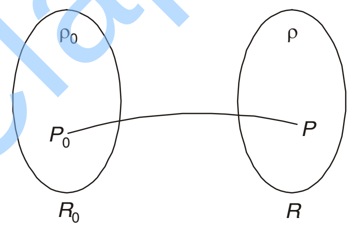
or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0\cdot$$

2.12A. The Equation of continuity by the Lagrangian method.

[G.N.D.U - Amritsar 2000, Meerut 2005, 07, Rohilkhand 2005, Kanpur 2003, 04]

Let R_0 be the region occupied by portion of a fluid at the time $t = 0$, and R the region occupied by the same fluid at any time t.

Let (a, b, c) be the initial co-ordinates of a fluid particle P_0 enclosed in this element and ρ_o be its density.



Then mass of the fluid element at $t = 0$ is $\rho_o\, \delta a\, \delta b\, \delta c$.

Let P be the subsequent position of P_0 at time t and let ρ be the density of the fluid there.

Then mass of the fluid element at $t = t$ is $\rho\,\delta x\,\delta y\,\delta z$.

From the law of conservation of mass, the mass contained inside a given volume of fluid remains unchanged throughout the motion. Thus, the total mass inside R_0 must be equal to the total mass inside R.

$$\therefore \qquad \iiint_{R_0} \rho_0\, \delta a\, \delta b\, \delta c = \iiint_R \rho\, \delta x\, \delta y\, \delta z \qquad \text{...(1)}$$

From the advanced calculus, we have

$$\delta x\ \delta y\ \delta z = \mathbf{J}\ \delta a\ \delta b\ \delta c \qquad \text{...(2)}$$

where
$$\text{Jacobian } J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} \qquad \text{...(3)}$$

Using (2), (1) may be re-written as

$$\iiint_{R_0} \rho_0\, \delta a\, \delta b\, \delta c = \iiint_{R_o} \rho\, \mathbf{J}\, \delta a\, \delta b\, \delta c \quad \text{or} \quad \iiint_{R_0} (\rho_0 - \rho\,\mathbf{J})\,\delta a\,\delta b\,\delta c = 0 \qquad \text{...(4)}$$

which holds for all regions R_0 if
$$\rho_0 - \rho\,\mathbf{J} = 0, \qquad \text{...(5)}$$
which is the equation of continuity in Lagrangian form.

2.12B. Equivalence between Eulerian and Lagrangian forms of equations of continuity.

[Meerut 2007, Kurukshetra 1997]

Refer figure of Art. 2.12A. Let R_0 be the region occupied by portion of a fluid at the time $t = 0$, and R the region occupied by the same fluid at any time t. Let (a, b, c) be the initial coordinates of a fluid particle P_0 enclosed in this element and ρ_0 be its density. Then mass of the fluid at $t = 0$ is $\rho_0\, \delta a\, \delta b\, \delta c$. Let P be the subsequent position of P_0 at time t and let ρ be the density of the fluid there. Then mass of the fluid element at $t = t$ is $\rho\,\delta x\,\delta y\,\delta z$.

The velocity components in the two systems are connected by the equations

$$u = dx/dt, \qquad v = dy/dt, \qquad w = dz/dt \qquad ...(1)$$

Also, $x = x\,(a, b, c, t), \qquad y = y\,(a, b, c, t), \qquad z = z\,(a, b, c, t)$ ...(2)

$$\therefore \quad \frac{\partial u}{\partial a} = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) \quad \text{so that} \quad \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial u}{\partial a} \quad \text{etc.} \qquad ...(3)$$

The equation of continuity in the Lagrangian form is

$$\rho \mathbf{J} = \rho_0, \qquad ...(4)$$

where
$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} \qquad ...(5)$$

Also, the equation of continuity in the Eulerian form is

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad ...(6)$$

Differentiating both sides of (5) *w.r.t.* 't' and using (3), we get

$$\frac{dJ}{dt} = \begin{vmatrix} \partial u/\partial a & \partial u/\partial b & \partial u/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} + \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial v/\partial a & \partial v/\partial b & \partial v/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} + \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial w/\partial a & \partial w/\partial b & \partial w/\partial c \end{vmatrix}$$

or
$$dJ/dt = J_1 + J_2 + J_3, \qquad ...(7)$$

Since $\dfrac{\partial u}{\partial a} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial a}$ etc, J_1 can be re-written as (after interchanging its rows and columns) Thus, we have

$$J_1 = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix},$$

[the last two determinants vanish because they possess two identical columns]

$\therefore$ $$J_1 = \frac{\partial u}{\partial x} J, \text{ using (5)}$$

Similarly, we have $$J_2 = \frac{\partial v}{\partial y} J \quad \text{and} \quad J_3 = \frac{\partial w}{\partial z} J$$

$\therefore$ (7) becomes $$\frac{dJ}{dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad ...(8)$$

Derivation of Eulerian form from Lagragin form :

From (4), $$\frac{d\rho}{dt}(\rho J) = \frac{d}{dt}(\rho_0) = 0 \quad \text{or} \quad \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$$

$\therefore$ $$\frac{d\rho}{dt} J + \rho J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \text{ using (8)}$$

or $$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0,$$

which is (6) *i.e.* Eulerian form of equation of continuity.

Derivation of Lagrangian form from Eulerian form :

From (6), $$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$\therefore$ $$\frac{d\rho}{dt} + \rho\left(\frac{1}{\mathbf{J}} \frac{d\mathbf{J}}{dt}\right) = 0 \text{ using (8)}$$

or $$J\frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(\rho J) = 0 \qquad ...(9)$$

Integrating (9), $$\rho J = \rho_0,$$

which is (4) *i.e.* Lagrangian equation of continuity.

2.13. Some symmetrical forms of the equation of continuity.

The equation of continuity takes a simplified form in cases when the motion of the fluid possesses certain symmetrical properties as shown below :

(*i*) **Cylindrical Symmetry.** Let there be a fluid particle at P whose cylindrical coordinates are (r, θ, z). Due to cylindrical symmetry, let $q_r\,(r,\, t)$ be the velocity at P perpendicular to the axis OZ and let $\rho(r, t)$ be the density of the fluid at P. Consider an element of the fluid consisting of two cylinders of radii r and $r + \delta r$ with OZ as axis, bounded by planes at unit distance apart. Then, we have

Rate of flow across the inner surface $= \rho q_r (2\pi r) = f(r, t)$, say ...(1)

Rate of flow across the outer surface $= f(r + \delta r, t)$ ...(2)

Rate of change of mass within the element $= \dfrac{\partial}{\partial t}(\rho \cdot 2\pi r \cdot \delta r)$ ...(3)

Suppose the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass within the element must be equal to the rate of mass flowing into the element. Hence from (1), (2) and (3), we have

$$\frac{\partial}{\partial t}(\rho \cdot 2\pi r \cdot \delta t) = f(r,t) - f(r+\delta r, t)$$

or $$2\pi r\delta r\frac{\partial \rho}{\partial t} = f(r,t) - \left[f(r,t) + \delta r\frac{\partial}{\partial r}f(r,t) + \dots\right], \text{ expanding by Taylor's theorem}$$

or $$2\pi r\delta r\frac{\partial \rho}{\partial t} = -\delta r\frac{\partial}{\partial r}f(r,t), \text{ to first order of approximation}$$

or $$2\pi r\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r}(2\pi r\rho q_r), \text{ by (1)}$$

or $$2\pi r\frac{\partial \rho}{\partial t} = -2\pi\frac{\partial}{\partial r}(r\rho q_r) \qquad \text{or} \qquad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho q_r) = 0 \qquad \text{...(4)}$$

If ρ is constant, $\partial\rho/\partial t = 0$ and (4) reduces to

$$\frac{\partial}{\partial r}(r\rho q_r) = 0 \qquad \text{...(5)}$$

Integrating (5) *w.r.t.* '*r*', we have

$$r\rho q_r = \rho g(t) \qquad \text{or} \qquad r\, q_r = g(t) \qquad \text{...(6)}$$

If the flow is steady, $g(t)$ reduces to an absolute constant. Thus, for a steady flow

$$r\, q_r = C, \text{ where } C \text{ is a constant.} \qquad \text{..(7)}$$

Note. The relation (4) may be also be derived as a special case of equation(8) of Art. 2.10 by using the cylindrical symmetry (*i.e.* $\partial/\partial\theta = 0$ and $\partial/\partial z = 0$)

(*ii*) **Spherical Symmetry.** Let there be a fluid particle at P whose spherical polar coordinates are (r, θ, ϕ). Due to spherical symmetry, let $q_r(r, t)$ be the velocity at P in the direction of OP and let $\rho(r, t)$ be the density of the fluid at P. Consider an element of the fluid consisting of two concentric spheres of radii r and $r + \delta r$ with O as centre. Then, we have

Rate of flow across the inner surface $= \rho q_r \cdot 4\pi r^2 = f(r,t)$. say ...(1)

Rate of flow across the outer surface $= f(r+\delta r, t)$ ...(2)

Rate of change of mass within the element $= \dfrac{\partial}{\partial t}(\rho \cdot 4\pi r^2 \cdot \delta r)$ ...(3)

Then as in part (*i*) above, we have

$$\frac{\partial}{\partial t}(4\pi r^2 \rho\, \delta r) = f(r,t) - f(r+\delta r, t)$$

or $$4\pi r^2 \delta r\frac{\partial \rho}{\partial t} = f(r,t) - \left[f(r,t) + \delta r\frac{\partial}{\partial r}f(r,t) + \dots\right], \text{ expanding by Taylor's theorem}$$

or $$4\pi r^2 \delta r\frac{\partial \rho}{\partial t} = -\delta r\frac{\partial}{\partial r}f(r,t), \text{ to first order of approximation}$$

or $$4\pi r^2 \delta r \frac{\partial \rho}{\partial t} = -\delta r \frac{\partial}{\partial r}(\rho q_r \cdot 4\pi r^2), \text{ by (1)}$$

or $$4\pi r^2 \frac{\partial \rho}{\partial t} = -4\pi \frac{\partial}{\partial r}(r^2 \rho q_r) \qquad \text{or} \qquad \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho q_r) = 0 \qquad \text{...(4)}$$

If ρ is constant, $\partial \rho / \partial t = 0$ and (4) reduces to

$$\frac{\partial}{\partial t}(r^2 \rho q_r) = 0 \qquad \text{...(5)}$$

Integrating (5) *w.r.t.* 'r', we have

$$r^2 \rho q_r = \rho g(t) \qquad \text{or} \qquad r^2 q_r = g(t) \qquad \text{...(6)}$$

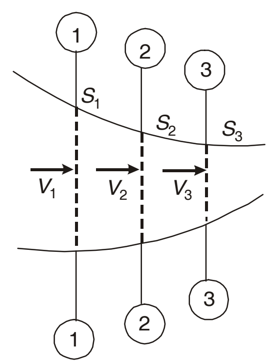
If the flow is steady, $g(t)$ reduces to an absolute constant. Thus, for a steady flow

$$r^2 q_r = C, \text{ where } C \text{ is a constant.} \qquad \text{...(7)}$$

Note. The relation (4) may be derived as a special case of equation (8) of Art. 2.11 by using the spherical symmetry $(i.e., \partial/\partial\theta = 0, \partial/\partial\phi = 0)$.

2.14. Equation of continuity of a liquid flow through a channel or a pipe.

Let an incompressible liquid continuously flow through a channel or a pipe whose cross-sectional area may or may not be fixed. Then the quantity of liquid passing per second is the same at all sections.



Suppose some liquid is flowing through a tapering pipe as shown in figure. Let S_1, S_2, S_3 be areas of the pipe at sections 1-1, 2-2, 3-3 respectively. Further, let V_1, V_2 and V_3 be velocities of the liquid at sections 1-1, 2-2, 3-3 respectively. Let Q_1, Q_2, Q_3 be the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 respectively. Then

$$Q_1 = S_1 V_1, \qquad Q_2 = S_2 V_2, \qquad Q_3 = S_3 V_3 \qquad \text{...(1)}$$

From the law of conservation of mass, the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 must be the same. Hence

$$Q_1 = Q_2 = Q_3 = \dots \text{ and so on.}$$

Thus, $S_1 V_1 = S_2 V_2 = S_3 V_3 = \dots$ is the equation continuity.

2.15. Working rule of writing the equation of continuity.

Let P be any fluid particle and let ρ be density at P. With P as one corner construct a parallelepiped whose edges are $\lambda\delta\alpha, \mu\delta\beta, \nu\delta\gamma$, in the chosen coordinate system. Let

Lengths of elements	:	$\lambda\delta\alpha$,	$\mu\delta\beta$,	$\nu\delta\gamma$,
Components of velocity	:	u	v	w

Now calculate the rate of the excess of the flow-in over flow-out along the first length by taking the negative derivative with respect to the first length of the product (density × velocity in the first direction × product of remaining lengths) and finally multiplying this by first length itself. We thus obtain

$$-\lambda\delta\alpha \frac{\partial}{\lambda\partial\alpha}(\rho u \mu\delta\beta\, \nu\delta\gamma).$$

Similarly calculate the rates of the excess of the flow in over the flow-out along the remaining two lengths and obtain

$$-\mu\delta\beta \frac{\partial}{\mu\partial\beta}(\rho v \lambda\delta\alpha\, \nu\delta\gamma) \qquad \text{and} \qquad -\nu\delta\gamma \frac{\partial}{\nu\partial\gamma}(\rho w \lambda\delta\alpha\, \mu\delta\beta).$$

Now, the total mass of fluid in the element

= density × product of the three edges of the element = $\rho\lambda\,\delta\alpha\,\mu\delta\beta\,\nu\,\delta\gamma$.

Hence the rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho\,\lambda\delta\alpha\,\mu\delta\beta\;\nu\delta\gamma)$

For the equation of continuity, we have

Rate of increase in mass of the element

= Total rate of the excess of the flow-in over the flow out along the three lengths of the element

$$i.e.\quad \frac{\partial}{\partial t}(\rho\,\lambda\delta\alpha\,\mu\delta\beta\;\nu\delta\gamma) = -\lambda\delta\alpha\frac{\partial}{\lambda\partial\alpha}(\rho u\,\mu\delta\beta\,\nu\delta\gamma) - \mu\delta\beta\frac{\partial}{\mu\partial\beta}(\rho v\,\lambda\delta\alpha\;\nu\delta\gamma) - \nu\delta\gamma\frac{\partial}{\nu\partial\gamma}(\rho w\lambda\,\delta\alpha\,\mu\delta\beta),$$

which on simplification yields the desired equation of the continuity.

2.16. Illustrative solved examples.

Ex. 1. *The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0, \quad \text{where } u \text{ is the velocity at distance } r.$$

[Meerut 2011; Rohilkhand 2005; Himachal 2003; Kanpur 2004]

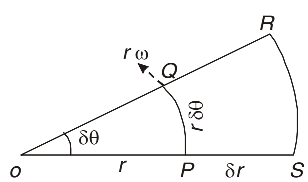
Sol. Here we have spherical symmetry. Proceed as in case (*ii*) Art. 2.13 upto equation (4) and obtain (noting that $q_r = u$ in the present problem).

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left\{r^2 u\frac{\partial\rho}{\partial r} + \rho\frac{\partial}{\partial r}(r^2\rho u)\right\} = 0$$

or
$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0.$$

Ex. 2. *A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is* $\partial\rho/\partial t + \partial(\rho\omega)/\partial\theta = 0,$ *where* ω is the angular velocity of a particle whose azimuthal angle is θ at time t

(Meerut 2009; Ranchi 2010)



Sol. Here the motion is confined in a plane. Consider a fluid particle P, whose polar coordinates are (r, θ). Let P describe a circle of radius r. With P as one corner, consider an element $PQRS$ such that $PS = \delta r$ and arc $PQ = r\delta\theta$. Here there is no motion of the fluid along PS. The rate of the excess of the flow-in over the flow-out along PQ

$$= -r\delta\theta\frac{\partial}{r\partial\theta}(\rho\cdot r\omega\cdot\delta r).$$

Again, the total mass of the fluid within the element = $\rho\cdot\delta r\cdot r\delta\theta$.

The rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho r\;\delta r\;\delta\theta)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial t}(\rho r\;\delta r\;\delta\theta) = -r\delta\theta\frac{\partial}{r\partial\theta}(\rho r\omega\,\delta r) \qquad \text{or} \qquad r\,\delta r\,\delta\theta\frac{\partial\rho}{\partial r} = -r\delta r\,\delta\theta\frac{\partial}{\partial\theta}(\rho\omega)$$

or
$$\partial\rho/\partial t + \partial(\rho\omega)/\partial\theta = 0,$$

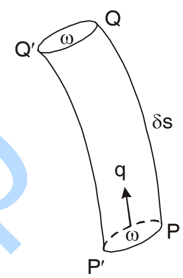
Ex. 3. *If* ω *is the area of cross-section of a stream filament prove that the equation of continuity is*

$$\frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial s}(\rho\omega q)=0,$$

where δs *in an element of arc of the filament in the direction of flow and q is the speed.*

[Garhwal 1993, 95]

Sol. Let $PP'Q'Q$ be stream filament whose area of cross-section is ω and arc $PQ=\delta s$. The rate of the excess of the flow-in over the flow-out along PQ



$$=-\delta s\frac{\partial}{\partial s}(\rho\omega q)$$

Again, the total mass of the fluid within the stream filament is $\rho\omega\delta s$.

$\therefore$ the rate of increase in mass of the stream filament $=\dfrac{\partial}{\partial t}(\rho\omega\delta s)$.

Hence the equation of continuity in given by

$$\frac{\partial}{\partial t}(\rho\omega\,\delta s)=-\delta s\frac{\partial}{\partial s}(\rho q\omega) \qquad \text{or} \qquad \frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial s}(\rho q\omega)=0.$$

Ex. 4. (*i*) *A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is* u_0 *to become* $\rho_0\phi(vt-x)$. *Prove that the velocity u (at time t and distance x from the origin) is given by*

$$v+\frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}$$

(*ii*) *A gas is moving in a uniform straight tube. Prove that if the density be f(at –x) at a point where t is the time and x is the distance of the point from an end of the tube, its velocity is*

$$\frac{af(at-x)+(v-a)f(at)}{f(at-x)},$$

where v is the velocity at that end of the tube and a is a constant.

Sol. (*i*) Let ρ be the density and u the velocity at a distance x. Then we are given that

$$\rho=\rho_0\phi(vt-x) \qquad \text{...(1)}$$

Again the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}=0 \qquad \text{...(2)}$$

From (1), $\dfrac{\partial\rho}{\partial t}=\rho_0 v\,\phi'(vt-x)$, and $\dfrac{\partial\rho}{\partial x}=-\rho_0\phi'(vt-x)$ ...(3)

Using (1) and (3), (2) reduces to

$$\rho_0 v\,\phi'(vt-x)+\rho_0\phi(vt-x)\frac{\partial u}{\partial x}-u\rho_0\phi'(vt-x)=0 \quad \text{or} \quad (v-u)\phi'(vt-x)+\phi(vt-x)\frac{\partial u}{\partial x}=0$$

or

$$\frac{du}{v-u}+\frac{\phi'(vt-x)}{\phi(vt-x)}dx=0$$

Integrating, $\quad -\log (v-x) - \log \phi (vt-x) = -\log C$, C being an arbitrary constant

or $$(v-u)\,\phi\,(vt-x) = C \qquad ...(4)$$

Given, $u = u_0$ when x = 0 so that $(v-u_0)\,\phi\,(vt) = C$. With this value of C, (4) reduces to

$$(v-u)\,\phi\,(vt-x) = (v-u_0)\,\phi\,(vt) \qquad \text{or} \qquad u = v + \frac{(u_0 - v)\,\phi\,(vt)}{\phi\,(vt-x)}$$

(*ii*) Do just like (*i*) yourself.

Ex. 5. *A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0,$$

where u,v are the velocity perpendicular and parallel to z.

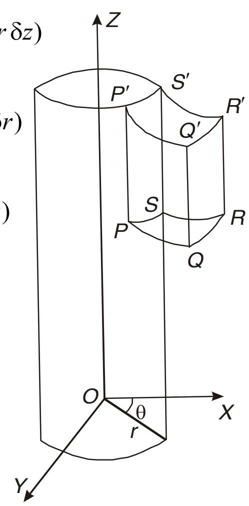
[Agra 2003; Rohilkhand 2002, Kanpur 2000, 08; Meerut 1999, 2002, 2012]

Sol. Consider a fluid particle P, whose cylindrical coordinates are (r, θ, z). With P as one corner construct an element (curvilinear parallelepiped $PQRS$, $P'Q'R'S'$) with edges $PQ = \delta r$ $PS = r\delta\theta$ and $PP' = \delta z$.

Let ρ be the density of the fluid at P.

Since the lines of motion lie on the surface of co-axial cylinder, there is no motion along PQ. Hence the rate of the excess of the flow-in over flow-out along PQ vanishes. Again, we have

Rate of excess of flow-in over flow-out along $PS = -r\delta\theta \dfrac{\partial}{r\partial\theta}(\rho u\,\delta r\,\delta z)$



Rate of excess of flow-in over flow-out along $PP' = -\delta z \dfrac{\partial}{\partial z}(\rho v r\,\delta\theta\,\delta r)$

Again, the rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho r\,\delta\theta\,\delta r\,\delta z)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial r}(\rho r\,\delta\theta\,\delta r\,\delta z) = -\delta\theta \frac{\partial}{\partial \theta}(\rho u\,\delta r\,\delta z) - \delta z \frac{\partial}{\partial z}(\rho v r\,\delta\theta\,\delta r)$$

or $$r\delta\theta\,\delta r\,\delta z \frac{\partial \rho}{\partial t} + \delta r\,\delta\theta\,\delta z \frac{\partial}{\partial \theta}(\rho u) + r\delta r\,\delta\theta\,\delta z \frac{\partial}{\partial z}(\rho v) = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0.$$

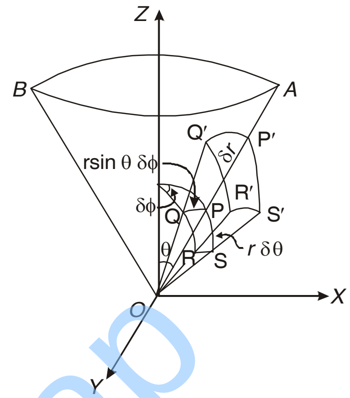
Ex. 6. *If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common surface, prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial r}(\rho u) + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{\mathrm{r}}\frac{\partial}{\partial \theta}(\rho w) = 0,$$

where u and w are the velocity components in the directions in which r and ϕ *increase.*

[Agra 2001; Garhwal 2000; Meerut 2001, 02, 03, 04, G.N.D.U. Amritsar 1998; Rohilkhand 2004]

Sol. Let O, the vertex of cones, be the origin and let OZ, their common axis, be the axis of z. Let OAB be a cone of semi-vertical angle θ. Consider a fluid particle P whose spherical polar coordinates are (r, θ, ϕ).. With P as one corner construct an element (curvilinear parallelepiped $PQRS$, $P'Q'R'S'$) with edges $PP' = \delta r$, $PS = r\delta\theta$ and $PQ = r\sin\theta\,\delta\phi$.



Since the lines of motion are curves on the surfaces of cones, there would be no motion perpendicular to the surface of the cone *i.e.*, the velocity in the θ-direction (in the direction of PS) is zero. Further, given that u and w are velocities along PP' and PQ respectively. Since velocity along PS is zero, the excess of flow-in over flow-out along PS vanishes. Again, we have

Rate of excess of flow-in over flow-out along PP'

$$= -\delta r \frac{\partial}{\partial r}(\rho u \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) = -\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial r}(r^2 u\rho)$$

Rate of excess of flow-in over flow-out along PQ

$$= -r\sin\delta\phi \frac{\partial}{r\sin\theta\partial\phi}(\rho w \cdot \delta r \cdot r\delta\theta) = -r\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\phi}(\rho w)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho\delta r \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t} = -\delta r\,\delta\theta\,\delta\phi\left[\sin\theta \frac{\partial}{\partial r}(r^2 u\rho) + r\frac{\partial}{\partial\phi}(\rho w)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho w) = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left[r^2\frac{\partial(\rho u)}{\partial r} + 2r\rho u\right] + \frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\phi} = 0$$

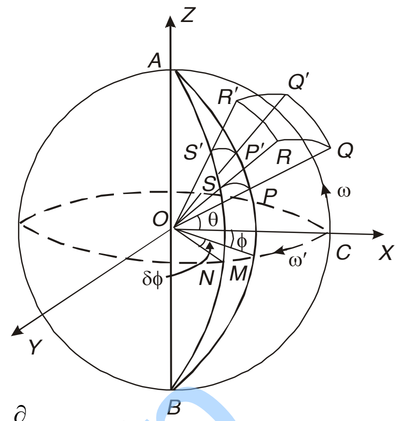
or $$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial r} + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{r}\frac{\partial(\rho w)}{\partial\phi} = 0.$$

Ex. 7. *If every particle moves on the surface of a sphere prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + \cos\theta + \frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta) = 0,$$

ρ *being the density,* θ, ϕ *the latitude and longitude of any element,* ω *and* ω' *the angular velocities of the element in latitude and longitude respectively.* **(I.A.S. 1991)**

Sol. Consider a fluid particle P on the semi-circle APB making an angle ϕ with semi-circle ACB. Suppose that OP makes an angle θ with OC. With P as one corner construct and elementary parallelepiped $PQRS$, $P'Q'R'S$ on the surface of edges $PQ = \delta r$, $PP' = r\delta\theta$ an $PS = r\cos\theta\delta\phi$.



Let ρ be the density of the fluid at P.

Since every particle moves on the surface of the sphere, there will be no velocity normal to the surface of the sphere (*i.e.* radial direction or along PQ). Again the velocities along PP' and PS are $r\omega$ and $r\cos\theta\omega'$ because ω and ω' are the angular velocities in latitude and longitude respectively.

Since velocity along PQ is zero, the rate of excess of flow-in over flow-out along PQ vanishes. Further, we have

Rate of excess of flow-in over flow-out along PP'

$$= -r\delta\theta \frac{\partial}{r\partial\theta}(\rho \cdot r\omega \cdot \delta r \cdot r\cos\theta\,\delta\phi) = -r^2 \delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\theta}(\rho\omega\cos\theta)$$

Rate of excess of flow-in over flow-out along PS

$$= -r\cos\theta\,\delta\phi \frac{\partial}{r\cos\theta\,\partial\phi}(\rho \cdot r\cos\theta\,\omega' \cdot \delta r \cdot r\delta\theta) = -r^2 \delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)$$

Again, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho \cdot \delta r \cdot r\delta\theta \cdot r\cos\theta\,\delta\phi) = r^2 \cos\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2 \cos\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t} = -r^2 \delta r\,\delta\theta\,\delta\phi \left[\frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)\right]$$

or
$$\frac{\partial\rho}{\partial t}\cos\theta + \frac{\partial}{\partial\theta}(\rho\,\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta) = 0.$$

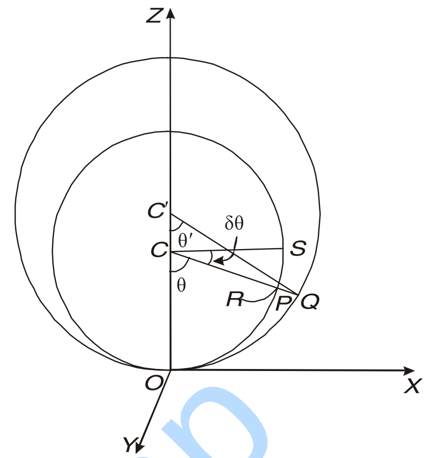
Ex. 8. *If the lines of motion are curves on the surfaces of spheres, all touching the plane of xy at the origin O, the equation of continuity is*

$$r\sin\theta \frac{\partial\rho}{\delta t} + \frac{\partial}{\partial\phi}(\rho v) + \sin\theta \frac{\partial}{\partial\theta}(\rho u) + \rho u(1 + 2\cos\theta) = 0,$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity and ϕ *the inclination of the plane PCO to a fixed plane through the axis of z.*

[Agra 1995; Garhwal 2001; I.A.S. 1999, Rohilkhand 2000, U.P.P.C.S. 2002; Rajasthan 1998]

Sol. Let C and C' be the centres of the two consecutive spheres of radii r and $r + \delta r$ respectively touching the plane of xy at the origin O as shown in the figure. Clearly, $CC' = OC' - OC = (r + \delta r) - r = \delta r$. Consider a fluid particle P on the inner sphere. Produce CP so as to meet the outer sphere at Q. Let S be a consecutive point on the circle in the given plane PCO so that $\angle SCO = \theta + \delta\theta$. Hence $\angle SCP = \angle SCO - \angle PCO = (\theta + \delta\theta) - \theta = \delta\theta$. Let PR be an elementary arc in a plane perpendicular to the plane PCO.



In $\Delta CC'Q$, we have form Trigonometry,

$$C'Q^2 = CC'^2 + CQ^2 - 2CC' \cdot CQ \cos QCC'$$

or $$(r+\delta r)^2 = \delta r^2 + (r+PQ)^2 - 2\delta r\,(r+PQ)\cos(\pi-\theta)$$

or $$2r\delta r = 2r\,PQ + PQ^2 + 2r\,\delta r\cos\theta + 2\delta r\,PQ\cos\theta$$

Since PQ and δr are small quantities, to first order of approximation we have

$$2r\delta r\,(1-\cos\theta) = 2r\,PQ$$

or $$PQ = (1-\cos\theta)\delta r.$$

With P as one corner consider an elementary parallelepiped with edges $PQ = (1-\cos\theta)\,\delta r$, $PS = r\delta\theta$ and $PR = r\sin\theta\,\delta\phi$, where ϕ is the angle that the plane PCO makes with a fixed plane through the z axis (say, the plane XOZ). The value of PR can be calculated by rotating the plane PCO about Z-axis through an angle $\delta\phi$.

Since the lines of motion are curves on the surfaces of spheres touching the plane of xy, there would be no motion along PQ, *i.e.,* the velocity along PQ is zero. Further, given that u and v are the velocity components along the edges PS and PR in the direction of θ and ϕ increasing.

Since velocity along PQ is zero, the rate of excess of flow in over flow-out along PQ vanishes. Further, we have

Rate of excess of flow-in over flow-out along PS

$$= -r\delta\theta \frac{\partial}{r\partial\theta}\{\rho u \cdot (1-\cos\theta)\,\delta r \cdot r\sin\theta\,\delta\phi\}$$

$$= -r\,\delta r\,\delta\theta\,\delta\phi\left[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u \cdot \left\{\cos\theta(1-\cos\theta) + \sin^2\theta\right\}\right]$$

$$= -r\,\delta r\,\delta\theta\,\delta\phi[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u\,\{\cos\theta(1-\cos\theta) + (1-\cos^2\theta)\}]$$

$$= -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi[\sin\theta\frac{\partial}{\partial\theta}(\rho u) + \rho u\,(1+2\cos\theta)]$$

Next, rate of excess of flow-in over flow-out along PR

$$= -r\sin\theta\,\delta\phi\frac{\partial}{r\sin\theta\,\partial\phi}\{\rho v\,(1-\cos\theta)\,\delta r \cdot r\delta\theta\} = -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho v)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}\{\rho \cdot (1-\cos\theta)\,\delta r \cdot r\,\delta\theta \cdot r\sin\theta\,\delta\phi\} = r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\delta t} = -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi[\sin\theta\frac{\partial}{\partial\theta}(\rho u) + \rho u\,(1+2\cos\theta)]$$

$$-r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho v)$$

or $$r\sin\theta\frac{\partial\rho}{\partial t}+\sin\theta\frac{\partial}{\partial\theta}(\rho u)+\frac{\partial}{\partial\phi}(\rho v)+\rho u(1+2\cos\theta)=0.$$

Ex. 9. *Show that in a two-dimensional incompressible steady flow field the equation of continuity is satisfied with the velocity components in* rectangular coordinates given by

$$u(x,y)=\frac{k(x^2-y^2)}{(x^2+y^2)^2}, \qquad v(x,y)=\frac{2kxy}{(x^2+y^2)^2},$$

where k is an arbitrary constant. **[Meerut 1994; Rolhilkhand 2001, 03, 04]**

Sol. The equation of continuity for incompressible steady flow in cartesian coordinates is

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0 \qquad \text{...(1)}$$

For a two dimensional flow in xy-plane, $w = 0$ so that (1) reduce to

$$\partial u/\partial x+\partial v/\partial y=0 \qquad \text{...(2)}$$

Differentiating the given values of u and v partially *w.r.t* 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x}=k(x^2-y^2)\frac{(-2)\times(2x)}{(x^2+y^2)^3}+\frac{k\times 2x}{(x^2+y^2)^2}=-4kx\frac{(x^2-y^2)}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2} \qquad \text{...(3)}$$

$$\frac{\partial v}{\partial y}=2kxy\frac{(-2)\times(2y)}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2}=-\frac{8kxy^2}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2} \qquad \text{...(4)}$$

From (3) and (4), we have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{-4kx^3+4kxy^2-8kxy^2}{(x^2+y^2)^3}+\frac{4kx}{(x^2+y^2)^2}=\frac{-4kx^3+4kxy^2-8kxy^2+4kx(x^2+y^2)}{(x^2+y^2)^3}=0$$

Hence, the equation of continuity (2) is satisfied.

Ex. 10. *Consider a two dimensional incompressible steady flow field with velocity components in spherical coordinates* (r,θ,ϕ) *given by*

$$v_r=c_1\left(1-\frac{3}{2}\frac{r_0}{r}+\frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta, \qquad v_\phi=0, \qquad v_\theta=-c_1\left(1-\frac{3}{4}\frac{r_0}{r}-\frac{1}{4}\frac{r_0^3}{r^3}\right)\sin\theta, \qquad r\ge r_0>0$$

where c_1 and r_0 are arbitrary constants. Is the equation of continuity satisfied.

Sol. The equation of continuity in spherical polar coordinates is given by (using Art. 2.11 with notations : $q_r=v_r$, $q_\theta=v_\theta$, $q_\phi=v_\phi$)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r)+\frac{1}{r\sin\theta}\frac{1}{\partial\theta}(\rho\sin\theta\, v_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\phi)=0$$

For a two-dimensional incompressible steady flow with $v_\phi=0$, we have ρ = constant and $\partial\rho/\partial t=0$. Hence for the present flow, the equation of continuity is given by

$$\frac{\rho}{r^2}\frac{\partial(r^2v_r)}{\partial r}+\frac{\rho}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta)=0 \quad\text{or}\quad \frac{1}{r^2}\left[r^2\frac{\partial v_r}{\partial r}+2rv_r\right]+\frac{1}{r\sin\theta}\left[\sin\theta\frac{\partial v_\theta}{\partial\theta}+\cos\theta\cdot v_\theta\right]=0$$

or $$\frac{\partial v_r}{\partial r}+\frac{2v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\theta}{r}\cot\theta=0 \qquad \text{...(1)}$$

From the given values of v_r and v_θ, we have

$$\frac{\partial v_r}{\partial r} = c_1\left(0 + \frac{3}{2}\frac{r_0}{r^2} - \frac{3r_0^3}{2r^4}\right)\cos\theta \qquad \text{...(2)}$$

and $$\frac{\partial v_\theta}{\partial \theta} = -c_1\left(1 - \frac{3r_0}{4r} - \frac{r_0^3}{4r^3}\right)\cos\theta \qquad \text{...(3)}$$

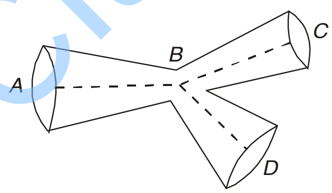
Using (2) and (3), we have

$$\text{L.H.S. of (1)} = c_1\left(\frac{3}{2}\frac{r_0}{r^2} - \frac{3r_0^3}{2r^4}\right)\cos\theta + \frac{2c_1}{r}\left(1 - \frac{3}{2}\frac{r_0}{r} + \frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta - \frac{c_1}{r}\left(1 - \frac{3r_0}{4r} - \frac{r_0^3}{4r^3}\right)\cos\theta$$

$$-\frac{c_1}{r}\left(1 - \frac{3}{4}\frac{r_0}{r} - \frac{r_0^3}{4r^3}\right)\cot\theta\sin\theta = 0, \text{ on simplification}$$

Hence, the equation of continuity (1) is satisfied.

Ex. 11. *A pipe branches into two pipes C and D as shown in the adjoining figure. The pipe has diameter of 45 cm at A, 30 cm at B, 20 cm at C and 15 cm at D. Determine the discharge at A, if the velocity at A is 2m/sec. Also determine the velocities at B and D, if the velocity at C is 4 m/sec.*



Sol. Let S_A, S_B, S_C, and S_D be areas of cross sections and let V_A, V_B, V_C, V_D, be velocities at A, B, C and D respectively. Then, we have

$$S_A = \pi\left(\frac{0.45}{2}\right)^2 = 0.159 \text{ square meters}, \qquad S_B = \pi\left(\frac{0.3}{2}\right)^2 = 0.0706 \text{ square meters}$$

$$S_C = \pi\left(\frac{0.2}{2}\right)^2 = 0.0314 \text{ square meters}, \qquad S_D = \pi\left(\frac{0.15}{2}\right)^2 = 0.01767 \text{ square meters}$$

Also, given that $V_A = 2$ m/sec and $V_C = 4$ m/sec

Let Q_A, Q_B, Q_C, and Q_D, be discharges at A, B, C and D respectively. Remembering that Discharge = area of cross-section × velocity, we have

$$Q_A = S_A V_A = 0.159 \times 2 = 0.318 \text{ m}^3/\text{sec}$$

From the equation of continuity (refer Art. 2.14), we have

$$S_A V_A = S_B V_B \qquad \text{so that} \qquad V_B = \frac{S_A V_A}{S_B} = \frac{Q_A}{S_B} = \frac{0.318}{0.0706} = 4.5 \text{ m/sec.}$$

Again, from the geometry of flow, we have

$$Q_A = Q_C + Q_D \qquad \text{or} \qquad 0.138 = S_C V_C + S_D V_D$$

or $0.138 = 0.0314 \times 4 + 0.01767 \times V_D$ so that $V_D = 10.6$ m/sec

Ex. 12. *The diameters of a pipe at the sections A and B are 200 mm and 300 mm respectively. If the velocity of water flowing through the pipe at section A is 4m/s, find*

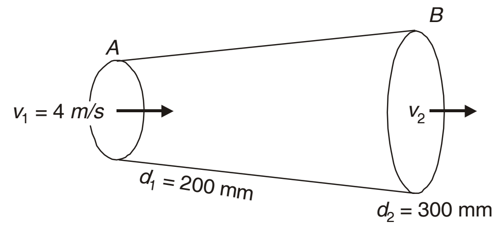
(i) Discharge through the pipe (ii) velocity of water at section B.

Sol. Radii r_1 and r_2 at the section A and B are given by

$r_1 = d_1/2 = 100$ mm $= 0.1$ m,
$r_2 = d_2/2 = 150$ mm $= 0.15$ m

S_1 = area of section $A = \pi r_1^2$

$= \pi \times (0.1)^2 = 0.0314 m^2$



v_1 = velocity at section A = 4m/s (given)

S_2 = area of section $B = \pi r_2^2 = \pi \times (0.15)^2 = 0.0707 m^2$

(*i*) **To determine discharge Q through the pipe.** We have

$$Q = S_1 v_1 = 0.0314 \times 4 = 0.1256 \text{ m}^3/\text{s}.$$

(*ii*) **To determine velocity v_2 of water at section B :** Here the continuity equation is

$$S_1 v_1 = S_2 v_2 \quad \Rightarrow \quad v_2 = \frac{S_1 v_1}{S_2} = \frac{0.0314 \times 4}{0.0707} = 1.77 \, m/s$$

Ex.13. *A pipe A 450 mm in diameter branches into two pipe B and C of diameters 300 mm and 200 mm respectively. If the average velocity in 450 mm diameter pipe is 3 m/s, find (i) Discharge through 450 mm diameter pipe (ii) Velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s.*

Sol. S_1 = area of section $A = \pi (d_1/2)^2 = (\pi/4) \times (0.45)^2 = 0.159 \text{ m}^2$

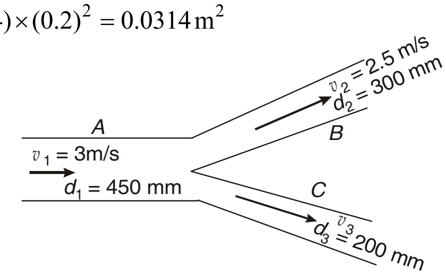
S_2 = area of section $B = \pi (d_2/2)^2 = (\pi/4) \times (0.3)^2 = 0.0707 \text{ m}^2$

S_3 = area of section $C = \pi (d_3/2)^2 = (\pi/4) \times (0.2)^2 = 0.0314 \text{ m}^2$

(*i*) **To find dischange Q_1 through A :**

$$Q_1 = S_1 v_1 = 0.159 \times 3 = 0.477 \text{ m}^3/\text{s}.$$

(*ii*) **To find velocity v_3 in pipe C.**



By continuity equation,we have

$S_1 v_1 = S_2 v_2 + S_3 v_3$ so that

$v_3 = (S_1 v_1 - S_2 v_2)/S_3$. ...(1)

But by part (*i*), $S_1 v_1 = 0.477$ m^3/s.

Also $S_2 v_2 = 0.0707 \times 2.5 = 0.1767$ m^3/s.

Hence (1) reduces to $v_3 = \dfrac{0.4770 - 0.1767}{0.0314} = 9.55 \, m/s.$

Ex. 14. *In a three dimensional incomperessible flow, the velocity components in y and z directions are $v = ax^3 - by^2 + cz^2, w = bx^3 - cy^2 + az^2 x$. Determine the missing component of velocity distribution such that continuity equation is satisfied.*

Sol. Given $v = ax^3 - by^2 + cz^2$ and $w = bx^3 - cy^2 + az^2 x$. ...(1)

The continuity equation for an incompressible fluid flow is

$$(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$$

or $\partial u/\partial x - 2by + 2azx = 0$ or $\partial u/\partial x = 2by - 2azx.$

Integrating (2) *w.r.t.* 'x', $u = 2byx - 2az \times (x^2/2) + f(y, z),$ ...(2)

where $f(y, z)$ is an arbitrary function which is independent of x.

Ex. 15. *Water flows through a pipe of length l which tapers from the entrance radius r_1 to the exist radius r_2 If the entrance velocity is V_1 and the relation between r_1 and r_2 is given by $r_2 = r_1 \pm ml$, where m is the slope, prove that the exist velocity V_2 is*

$$V_2 = V_1\left[1 - \frac{\pm 2m(l/r_1) + m^2(l/r_1)^2}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2}\right].$$

Sol. If S_1 and S_2 be the areas of cross-sections of the pipe at the entrance and exist, then $S_1 = \pi r_1^2$ and $S_2 = \pi r_2^2$. From the equation of continuity, we have

$$S_1 V_1 = S_2\, V_2 \qquad \text{or} \qquad \pi r_1^2 V_1 = \pi r_2^2 V_2$$

Thus, $V_2 = \dfrac{r_1^2 V_1}{r_2^2} = \dfrac{r_1^2 V_2}{(r_1 \pm ml)^2}$, as given $r_2 = r_1 \pm$ ml

$$= \frac{V_1}{\{1 \pm m(l/r_1)\}^2} = \frac{V_1}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2} = V_1\left[1 - \frac{\pm 2m(l/r_1) + m^2(l/r_1)^2}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2}\right]$$

Ex. 16. *Determine the constants* l, m *and* n *in order that the velocity* $\mathbf{q} = \{(x + lr)\mathbf{i} + (y + mr)\mathbf{j} + (z + nr)\mathbf{k}\}/\{r(x + r)\}$, *where* $r = (x^2 + y^2 + z^2)^{1/2}$ *may satisfy the equation of continuity* for a liquid. **[Bhopal 2000; Meerut 1996]**

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then we have

$$u = \frac{x + lr}{r(x+r)}, \qquad v = \frac{y + mr}{r(x+r)}, \qquad w = \frac{z + nr}{r(x+r)}, \qquad \text{...(1)}$$

Also, given $r = (x^2 + y^2 + z^2)^{1/2}$ so that $r^2 = x^2 + y^2 + z^2$. ...(2)

From (2), differentiating partially *w.r.t.* 'x', we have

$$2r(\partial r/\partial x) = 2x \qquad \text{so that} \qquad \partial r/\partial x = x/r. \qquad \text{...(3)}$$

Similarly, from (2), $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$. ...(4)

From (1), $$\frac{\partial u}{\partial x} = \frac{1}{r(x+r)}\frac{\partial}{\partial x}(x + lr) + (x + lr)\frac{\partial}{\partial x}\left\{\frac{1}{r(x+r)}\right\}$$

$$= \frac{1}{r(x+r)}\left(1 + l\frac{\partial r}{\partial x}\right) + (x + lr)\left[-\frac{1}{r^2}\frac{\partial r}{\partial x} - \frac{1}{r(x+r)^2}\left(1 + \frac{\partial r}{\partial x}\right)\right]$$

$$= \frac{1}{r(x+r)}\left(1 + l\frac{x}{r}\right) + (x + lr)\left[-\frac{1}{r^2}\cdot\frac{x}{r} - \frac{1}{r(x+r)^2}\left(1 + \frac{x}{r}\right)\right], \text{ by (3)}$$

$$= \frac{r + xl}{r^2(x+r)} - \frac{(x + lr)}{r^2}\left(\frac{x}{r} + \frac{1}{x+r}\right) \qquad \text{...(5)}$$

Also, $$\frac{\partial v}{\partial y} = \frac{1}{r(x+r)}\frac{\partial}{\partial y}(y + mr) + (y + mr)\frac{\partial}{\partial y}\left\{\frac{1}{r(x+r)}\right\}$$

$$= \frac{1}{r(x+r)}\left(1 + m\frac{\partial r}{\partial y}\right) + (y + mr)\left\{-\frac{1}{r^2}\frac{\partial r}{\partial y} - \frac{1}{r(x+r)^2}\left(0 + \frac{\partial r}{\partial y}\right)\right\}$$

$$= \frac{1}{r(x+r)}\left(1 + m\frac{y}{r}\right) + (y + mr)\left\{-\frac{1}{r^2}\frac{y}{r} - \frac{1}{r(x+r)^2}\frac{y}{r}\right\}, \text{ by (4)}$$

$$= \frac{r + my}{r^2(x+r)} - \frac{(y + mr)}{r^2}\left[\frac{y}{r} + \frac{y}{(x+r)^2}\right]. \qquad \text{...(6)}$$

Similarly, $$\frac{\partial w}{\partial z}=\frac{r+nz}{r^2(x+r)}-\left(\frac{z+nr}{r^2}\right)\left[\frac{z}{r}+\frac{z}{(x+r)^2}\right]. \qquad ...(7)$$

For the given velocity to satisfy the equation of continuity, we must have

$$(\partial u/\partial x)+(\partial v/\partial y)+(\partial w/\partial z)=0 \qquad ...(8)$$

or $$\frac{r+xl}{r^2(x+r)}-\frac{(x+lr)}{r^2}\left(\frac{x}{r}+\frac{1}{x+r}\right)+\frac{r+my}{r^2(x+r)}-\frac{(y+mr)}{r^2}\left[\frac{y}{r}+\frac{y}{(x+r)^2}\right]$$

$$+\frac{r+nz}{r^2(x+r)}-\left(\frac{z+nr}{r^2}\right)\left[\frac{z}{r}+\frac{z}{(x+r)^2}\right]=0\text{, by (5), (6) and (7)}$$

Multiplying both sides by $r^3\ (x+r)^2$, we have

$r\,(r+xl)\,(x+r)-(x+lr)\,(x+r)\,[x(x+r)+r]+r(r+my)\,(x+r)$

$-(y+mr)\,[y(x+r)^2+yr]+r(r+nz)\,(x+r)-(z+nr)\,[z(x+r)^2+zr]=0$

or $r^2\{r(1-l)+x(1-l)-my-nz\}=0$, on simplification.

This is satisfied by all values of x, y, z, if and only if $l=1$, $m=0$ and $n=0$.

Ex.17. *From the law of conservation of mass, show that whether the flow field represented by* $u=-3x+y^2-1/x$ *and* $v=x^2+3y+y\log x$ *is a possible velocity field for two-dimenation incomprenible fluid flow.*

Sol. Here $$\partial u/\partial x+\partial v/\partial y=-3+2/x^2+3+\log x\neq 0,$$

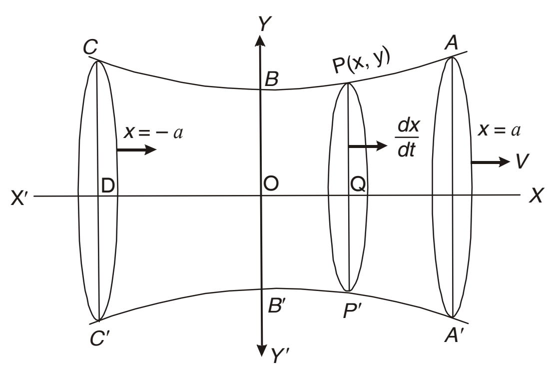
showing that the equation of continuity is not satsfied Hence the given flow does not represent a possible two-dimenation fluid flow

Ex. 18. *Liquid flows through a pipe whose surface is the surface of revolution of the curve* $y=a+(kx^2/a)$ *about the x - axis* $(-a\le x\le a)$. *If the liquid enters at the end* $x=-a$ *of the pipe with velicity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from* $x=-a$ *to* $x=a$ *is* $\{2a/V(1+k)^2\}\ \{1+(2k/3)+(k^2/5)\}$. *Assume that k is so small that flow remains apprecially one dimensional thoughtout.* **[I.A.S. 1999]**

Sol. Re-writing the given curve, we have

$$y-a=kx^2/a \quad \text{or} \quad (x-0)^2=(a/k)\,(y-a), \qquad ...(1)$$

which is a parabola ABC whose vertex is B (0, a). When the given curve (1) *revolved about x-axis,* we get surface of revolution. Figure shows a portion of the above mentioned surface bounded by circular ends $CC'(x=-a)$ and $AA'(x=a)$.



Let $P(x, y)$ be any point on (1). Then from (1), we have

$$PQ = y = a + kx^2/a. \qquad ...(2)$$

Also $C(-a, CD)$ lies on (1), Hence, we have

$$CD - a = k(-a)^2/a \quad \text{so that} \quad CD = a(1 + k) \qquad ...(3)$$

Velocity at section CC' is given to be V. Again, velocity of arbitrary section PQ is dx/dt. If S_1 and S_2 be areas of sections at C and P respectively, then

$$S_1 = \pi\, CD^2 = \pi a^2(1+k^2) \quad \text{and} \quad S_2 = \pi y^2 = \pi(a + kx^2/a)^2.$$

Since the motion is regarded as one-dimensional, by equation of continuity (expressing equal rates of volumetric flow across the cross-sections at CC' and PP'), we have

$$\pi[a(1+k)\}^2 V = \pi(a + kx^2/a)^2(dx/dt)$$

or
$$dt = \frac{1}{a^2V(1+k)^2}\left(a + \frac{kx^2}{a}\right)^2 dx. \qquad ...(4)$$

Let the required time of travelling from $x = -a$ to $x = a$ be T. Then integrating w.r.t 't' between $t = 0$ to $t = T$ and integrating w.r.t 'x' between corresponding limits $x = -a$ and $x = a$, (4) gives

$$\int_0^T dt = \frac{1}{a^2V(1+k)^2}\int_{-a}^{a}\left(a + \frac{kx^2}{a}\right)^2 dx.$$

or
$$T = \frac{1}{V(1+k)^2}\int_{-a}^{a}\left(1 + \frac{kx^2}{a^2}\right)^2 dx = \frac{2}{V(1+k)^2}\int_0^a\left(1 + \frac{kx^2}{a^2}\right)^2 dx$$

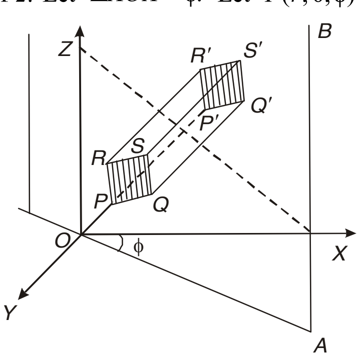
[Since the integrand in an even function]

$$= \frac{2}{V(1+k)^2}\int_0^a\left(1 + \frac{2kx^2}{a^2} + \frac{k^2x^4}{a^4}\right)^2 dx = \frac{2}{V(1+k)^2}\left[x + \frac{2kx^3}{3a^2} + \frac{k^2x^5}{5a^4}\right]_0^a$$

$$= \{2a/V(1+k)^2\}\{1 + (2k/3) + (k^2/5)\}$$

Ex. 19. *Each particle of a mass of liquid moves in a plane through the axis of z; find the equation of continuity.*

Sol. Let $ZOAB$ be a plane passing of through the axis of z. Let $\angle XOA = \phi$. Let $P(r, \theta, \phi)$ be the position of fluid particle of a mass of fluid moving on the plane $ZOAB$. We contruct a parallelopiped with edges PQ, PR and PP' such that $PQ = \delta r$, $PR = r\,\delta\theta$ and $PP' = r\sin\theta\delta\phi$. Clearly, the edges PQ and PR lie on the plane $ZOAB$ while PP' is perpendicular to the plane. Since the fluid particle move only on the plane $ZOAB$, there would be no motion along PP'.



Let u and v be velocity components of the fluid along PQ and PR respectively, We now use working rule of Art 2.15 for writting the equation continuity

The rate of the excess of flow-in over the flow-out along PQ

$$= -\delta r\frac{\partial}{\partial r}(\rho ur\,\delta\theta\, r\sin\theta\,\delta\phi) = -\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial r}(\rho u r^2)$$

Again, the rate of the excess of flow-in over the flow-out along PR

$$= -r\delta\theta \frac{\partial}{r\,\partial\theta}(\rho v \delta r\, r\sin\theta\,\delta\phi) = -r\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\theta}(\rho\, v \sin\theta)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho\,\delta r \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial \rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t} = -\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial r}(\rho\, u\, r^2) - r\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\theta}(\rho\, v \sin\theta)$$

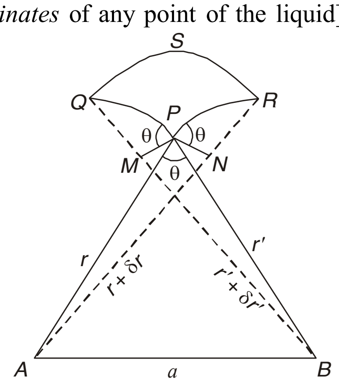
or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho\, u\, r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\, v\sin\theta) = 0$$

Ex. 20. *In the motion of a homogeneous liquid in two dimensions the velocity at any point is given by* v, v' *along the directions which pass through the fixed points distant 'a' from one another. Show that the equation of continuity is*

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \frac{r^2 + r'^2 - a^2}{2r\,r'}\left(\frac{\partial v}{\partial r'} + \frac{\partial v'}{\partial r}\right) + \frac{v}{r} + \frac{v'}{r'} = 0,$$

where r *and* r' *are distances of any point of the liquid from the fixed points.* **[Osmania 2005]**

[In this example (r, r') are known as the *di-polar co-ordinates* of any point of the liquid]



Sol. Let A and B be two given fixed points such that $AB = a$. Let r and r' be the distances of any point P of the liquid from A and B respectively. With A as centre draw two circular arcs PQ and RS with r and $r + \delta r$ as radii. Similarly, with B as centre draw two circular arcs PR and QS with r' and $r' + \delta r'$ as radii. Then, we have $AR = r + \delta r$ and $BQ = r' + \delta r'$.

Since arcs PQ and PR are very small hence we can assume that arcs PQ and PR are approximately equal to straight lines PQ and PR respectively.

Let $\angle APB = \theta$. Draw PN and PM perpendicular to AR and BQ. Then,

$AN = r$, $BM = r'$ $NR = \delta r$, $MQ = \delta r'$, $\angle NPR = \theta$ and $\angle MPQ = \theta$

From right-angled ΔPRN, $\sin\theta = NR/PR$ $\Rightarrow$ $PR = (\delta r)/\sin\theta$

Similarly, from ΔPMQ, $\sin\theta = QM/PQ$ $\Rightarrow$ $PQ = (\delta r')/\sin\theta$

Since v and v' are velocities along AP and BP respectively so velocity along normal to PR is $v' + v\cos\theta$ and velocity along normal to PQ is $v + v'\cos\theta$.

Since the liquid is homogeneous, so ρ = constant.

The rate of mass of the liquid flowing through PQ.

$$= \rho \times PQ \times (\text{velocity perpendicular to } PQ) = \rho(\delta r'/\sin\theta)\,(v + v'\cos\theta)$$

$\therefore$ the rate of the excess of the flow-in over the flow-out along PQ

$$= -\delta r \frac{\partial}{\partial r}\left\{\rho \frac{\delta r'}{\sin\theta}(v + v'\cos\theta)\right\} = -\rho \delta r\, \delta r' \frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right)$$

Similarly, the rate of the excess of the flow-in over the flow-out along PR

$$= -\delta r' \frac{\partial}{\partial r}\left\{\rho \frac{\delta r}{\sin\theta}(v' + v\cos\theta)\right\} = -\rho\, \delta r\, \delta r' \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right)$$

Now, area $PQSR = PQ \cdot PR \sin QPR = \dfrac{\delta r'}{\sin\theta} \times \dfrac{\delta r}{\sin\theta} \times \sin(\pi - \theta)$

$\therefore$ the rate of increase of mass of the liquid in $PQSR$

$$= \frac{\partial}{\partial t}\left(\rho \frac{\delta r'}{\sin\theta} \times \frac{\delta r}{\sin\theta} \times \sin\theta\right) = \rho \frac{\delta r\, \delta r'}{\sin\theta}\frac{\partial \rho}{\partial t} = 0, \text{ as } \rho = \text{constant}.$$

Hence the equation of continuity is given by

$$0 = -\rho\, \delta r\, \delta r' \frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right) - \rho\, \delta r\, \delta r' \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right)$$

or
$$\frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right) + \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right) = 0$$

or
$$\frac{1}{\sin\theta}\left(\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r}\cos\theta + v'\frac{\partial \cos\theta}{\partial r}\right) - \frac{v + v'\cos\theta}{\sin^2\theta}\cos\theta \frac{\partial \theta}{\partial r}$$
$$+ \frac{1}{\sin\theta}\left(\frac{\partial v'}{\partial r'} + \frac{\partial v}{\partial r'}\cos\theta + v\frac{\partial \cos\theta}{\partial r'}\right) - \frac{v' + v\cos\theta}{\sin^2\theta}\cos\theta \frac{\partial \theta}{\partial r'} = 0$$

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r}\cos\theta + v'\frac{\partial \cos\theta}{\partial r} - \frac{v + v'\cos\theta}{\sin^2\theta}\cos\theta\left(-\frac{\partial \cos\theta}{\partial r}\right)$$
$$+ \frac{\partial v'}{\partial r'} + \frac{\partial v}{\partial r'}\cos\theta + v\frac{\partial \cos\theta}{\partial r'} - \frac{v' + v\cos\theta}{\sin^2\theta}\cos\theta\left(-\frac{\partial \cos\theta}{\partial r'}\right) = 0$$

or
$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \left(\frac{\partial v'}{\partial r} + \frac{\partial v}{\partial r'}\right)\cos\theta + v'\frac{\partial \cos\theta}{\partial r} + v\frac{\partial \cos\theta}{\partial r'}$$
$$+ \frac{\cos\theta}{\sin^2\theta}\left\{(v + v'\cos\theta)\frac{\partial \cos\theta}{\partial r} + (v' + v\cos\theta)\frac{\partial \cos\theta}{\partial r'}\right\} = 0 \quad \text{...(1)}$$

Using cosine formula of trigonometry in ΔABP, we have

$$\cos\theta = \frac{r^2 + r'^2 - a^2}{2rr'} = \frac{r}{2r'} + \frac{r'}{2r} - \frac{a^2}{2rr'} \quad \text{...(2)}$$

$$\therefore \quad \frac{\partial(\cos\theta)}{\partial r} = \frac{1}{2r'} - \frac{r'}{2r^2} + \frac{a^2}{2r^2 r'} = \frac{1}{r'} - \frac{1}{r}\left(\frac{r}{2r'} + \frac{r'}{2r} - \frac{a^2}{2rr'}\right) = \frac{1}{r'} - \frac{\cos\theta}{r}$$

Similarly, from (2), we have $\quad \dfrac{\partial \cos\theta}{\partial r'} = \dfrac{1}{r} - \dfrac{\cos\theta}{r'}$

Substituting the above values of $\partial(\cos\theta)/\partial r$ and $\partial(\cos\theta)/\partial r'$ in (1), we have

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \left(\frac{\partial v'}{\partial r} + \frac{\partial v}{\partial r'}\right)\cos\theta + v'\left(\frac{1}{r'} - \frac{\cos\theta}{r}\right) + v\left(\frac{1}{r} - \frac{\cos\theta}{r'}\right)$$

$$+\frac{\cos\theta}{\sin^2\theta}\left\{(v+v'\cos\theta)\left(\frac{1}{r'}-\frac{\cos\theta}{r}\right)+(v'+v\cos\theta)\left(\frac{1}{r}-\frac{\cos\theta}{r'}\right)\right\}=0$$

or $$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+\frac{v'}{r'}+\frac{v}{r}-\left(\frac{v'}{r}+\frac{v}{r'}\right)\cos\theta$$

$$+\frac{\cos\theta}{\sin^2\theta}\left\{\frac{v}{r'}(1-\cos^2\theta)+\frac{v'}{r}(1-\cos^2\theta)\right\}=0$$

or $$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+\frac{v'}{r'}+\frac{v}{r}-\left(\frac{v'}{r}+\frac{v}{r'}\right)\cos\theta+\left(\frac{v}{r'}+\frac{v'}{r}\right)\cos\theta=0$$

or $$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\frac{r^2+r'^2-a^2}{2rr'}\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)+\frac{v}{r}+\frac{v'}{r'}=0, \text{ using (2)}$$

EXERCISE 2(C)

1. Determine the equation of continuity by vector apprach for incompressible fluid. Interpret it physically. **[Meerut 2003]**

2. A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is

$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_r)+\frac{\partial}{\partial z}(\rho q_z)=0,$ where q_r and q_z are velocities perpendicular and parallel to z-axis.

[**Hint.** Do as in Ex. 5 of Art 2.16 by taking $u = q_r,\ v = q_\theta$] **[Meerut 1999, 2001, 02]**

3. Water is flowing through a pipe 10 cm diameter with an average velocity of 10 m/sec. What is the rate of discharge of the water? Also determine the velocity at the other end of the pipe, if the diameter of the pipe is gradually changes to 20 cm.

[**Ans.** Discharge = 0.7854 m^3/sec; velocity = 2.5 m/sec.]

4. Homogeneous liquid moves so that the path the any particle P lies in the plane POX, where OX is fixed axis. Prove that if $OP = r$ and the $\angle XOP=\theta$, the equation of continuity may be written as
$$\frac{\partial}{\partial r}(u\,r^2)-\frac{\partial}{\partial\mu}(v\,r\,sin\,\theta)=0,$$
where u, v are the component velocities along and perpendicular to OP in the plane POX and $\mu=\cos\theta$.

[**Hint:** $\mu=\cos\theta$ so that $d\mu=\sin\theta\,d\theta$. Also ρ = constant. Proceed as Ex. 19 of Art. 2.16 by taking OX in place of OZ]

5. Does the three-dimensional incompressible flow given by

$$u(x,y,z)=\frac{kx}{(x^2+y^2+z^2)^{3/2}},\quad v(x,y,z)=\frac{ky}{(x^2+y^2+z^2)^{3/2}},\quad w(x,y,z)=\frac{kz}{(x^2+y^2+z^2)^{3/2}}$$

satisfy the equation of continuity? K is an arbitrary constant. Thus show the above motion is kinematically possible for an incompressible fluid. **[Purvanchal 2005]**

6. Does the two-dimensional incompressible flow given by

$$v_r(r,\theta)=c_1\left(\frac{1}{r^2}-1\right)cos\,\theta,\qquad v_\theta(r,\theta)=c_1\left(\frac{1}{r^2}+1\right)\sin\theta\qquad (r>0)$$

where c_1 is an arbitrary non-zero constant, satisfy the equation of continuity?

Boundary conditions (kinematical).

When fluid is in contact with a rigid solid surface (or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact:

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.

Let $\mathbf{n}$ denote a normal unit vector drawn at the point P of the surface of contact and let $\mathbf{q}$ denote the fluid velocity at P. When the rigid surface of contact is at rest, we must have $\mathbf{q.n} = 0$ at each point of the surface. This expresses the condition that the normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point as shown in Fig. (*i*).

Next, let the rigid surface be in motion and let $\mathbf{u}$ be its velocity at P (refer Fig (*ii*)]. Then we must have

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{or} \qquad (\mathbf{q} - \mathbf{u}) \cdot \mathbf{n} = 0,$$

which expresses the fact that there must be no normal velocity at P between boundary and fluid, that is, the velocity of the fluid relative to the boundary is tangential to the boundary at its each point.

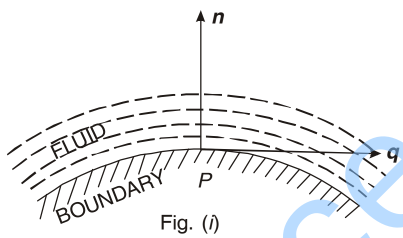


Fig. (*i*)

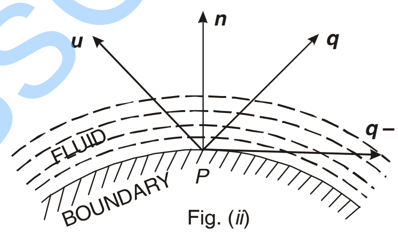


Fig. (*ii*)

Remark. For inviscid fluid the above condition must be satisfied at the boundary. However, for viscous fluid (in which there is no slip), the fluid and the surface with which contact is maintained must also have the same tangential velocity at P.

Boundary conditions (physical). The above mentioned kinematical boundary conditions must hold independently of any particular physical hypothesis. In the case of a non-viscous fluid in contact with rigid boundaries (fixed or moving), the following additional condition must be satisfied:

The pressure of the fluid must act normal to the boundary.

Again, let S denote the surface of separation of two fluids (which do not mix). Then the following additional condition must be satisfied :

The pressure must be continuous at the boundary as we pass from one side of S to the other.

2.18. Conditions at a boundary surface.

[Garhwal 1996, Kanpur 2002, 03, Meerut 1997, Rajasthan 2000, Rohilkhand 2001, 04, Purvanchal 2004]

We propose to derive the differential equation satisfied by a boundary surface of a fluid. Thus, we discuss the following problem :

To find the condition that the surface $F(\mathbf{r}, t) = 0$ *or* $F(x, y, z, t) = 0$ *may be a boundary surface*. For figure, refer figure (*ii*) of Art. 2.17.

Let P be a point on the moving boundary surface $\qquad F(\mathbf{r}, t) = 0. \qquad$ ...(1)

where the fluid velocity is $\mathbf{q}$ and the velocity of the surface is $\mathbf{u}$.

Now in order to preserve contact, the fluid and the surface with which contact is to be maintained must have the same velocity normal to the surface. Thus, we have

$$\mathbf{q}\cdot\mathbf{n}=\mathbf{u}\cdot\mathbf{n} \qquad \text{or} \qquad (\mathbf{q}-\mathbf{u})\cdot\mathbf{n}=0, \qquad ...(2)$$

where **n** in the unit normal vector drawn at P on the boundary surface (1). We know that the direction ratios of **n** are $\partial F/\partial x,\ \partial F/\partial y,\ \partial F/\partial z$. Again,

$$\nabla F=(\partial F/\partial x)\,\mathbf{i}+(\partial F/\partial y)\,\mathbf{j}+(\partial F/\partial z)\mathbf{k}, \qquad ...(3)$$

which shows that **n** and ∇F are parallel vectors and hence we may write $\mathbf{n}=k\nabla F$. With this value of **n**, (2) reduces to

$$(\mathbf{q}-\mathbf{u})\cdot k\nabla F=0 \qquad \text{so that} \qquad \mathbf{q}\cdot\nabla F=\mathbf{u}\cdot\nabla F \qquad ...(4)$$

Let P (**r**, t) move to a point $Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t)$ in time δt. Then Q must satisfy the equation of the boundary surface (1), at time $t+\delta t$, namely

$$F(\mathbf{r}+\delta\mathbf{r}, t+\delta t)=0$$

Expanding by Taylor's theorem, the above equation gives

$$F(\mathbf{r},t)+\delta\mathbf{r}\cdot\nabla\mathbf{F}+\delta t\left(\frac{\partial F}{\partial t}\right)=0 \qquad \text{or} \qquad \frac{\partial F}{\partial t}+\frac{\delta\mathbf{r}}{\delta t}\cdot\nabla\mathbf{F}=0, \text{ using}(1) \qquad ...(5)$$

Proceeding to the limits as $\delta\mathbf{r}\to 0,\ \delta t\to 0$ and noting that

$$\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt}=\mathbf{u}, \qquad (5) \text{ gives} \qquad \partial F/\partial t+\mathbf{u}\cdot\nabla\mathbf{F}=0 \qquad ...(6)$$

or
$$\partial F/\partial t+\mathbf{q}\cdot\nabla\mathbf{F}=0, \text{ using } (4) \qquad ...(7)$$

which is the required condition for $F(r, t)$ to be a boundary surface.

Remark 1. Let **q** = u**i** + v**j** + w**k**. Then (7) may be re-written as

$$\frac{\partial F}{\partial t}+(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\cdot\left(\frac{\partial F}{\partial x}\mathbf{i}+\frac{\partial F}{\partial y}\mathbf{j}+\frac{\partial F}{\partial z}\mathbf{k}\right)=0$$

or
$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \qquad \text{or} \qquad \frac{DF}{Dt}=0, \qquad ...(8)$$

where
$$D=\partial/\partial t+u(\partial/\partial x)+v(\partial/\partial y)+w(\partial/\partial z)$$

(8) presents the required condition in cartesian coordinates for F (x, y, z, t) = 0 to be a boundary surface. **[Agra 2006, Meerut 1997]**

Remark 2. The normal velocity of the boundary

$$=\mathbf{u}\cdot\mathbf{n}=\mathbf{u}\cdot\frac{\nabla\mathbf{F}}{|\nabla F|}=\frac{-(\partial F/\partial t)}{|(\partial F/\partial x)\mathbf{i}+(\partial F/\partial y)\mathbf{j}+(\partial F/\partial z)\mathbf{k}|}, \text{ by (3) and (6)}$$

$$=\frac{-(\partial F/\partial t)}{\sqrt{\{(\partial F/\partial x)^2+(\partial F/\partial y)^2+(\partial F/\partial z)^2\}}} \qquad ...(9)$$

$$=\frac{u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)}{\sqrt{\{(\partial F/\partial x)^2+(\partial F/\partial y)^2+(\partial F/\partial z)^2\}}}, \text{ using (8)} \qquad ...(10)$$

Remark 3. When the boundary surface is at rest, then $\partial F/\partial t = 0$ and hence the condition (8) reduces to

$$u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(11)}$$

2.19. Illustrative solved examples.

Ex. 1. *Show that the surface* $\dfrac{x^2}{a^2k^2t^4} + kt^2\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right) = 1$ *is a possible form of boundary surface of a liquid at time t.* **[I.A.S. 1992; Punjab 2002; Rohilkhand 2001]**

Sol. The given surface

$$F(x, y, z, t) = \frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(2)}$$

and the same values of u, v , w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(3)}$$

From (1), $\dfrac{\partial F}{\partial t} = -\dfrac{4x^2}{a^2k^2t^5} + 2kt\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)$, $\dfrac{\partial F}{\partial x} = \dfrac{2x}{a^2k^2t^4}$, $\dfrac{\partial F}{\partial y} = \dfrac{2kt^2y}{b^2}$, $\dfrac{\partial F}{\partial z} = \dfrac{2kt^2z}{c^2}$

With these values, (2) reduces to

$$-\frac{4x^2}{a^2k^2t^5} + 2kt\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \frac{2xu}{a^2k^2t^4} + \frac{2kt^2yv}{b^2} + \frac{2kt^2zw}{c^2} = 0,$$

or

$$\frac{2x}{a^2k^2t^4}\left(u - \frac{2x}{t}\right) + \frac{2kyt}{b^2}(y + vt) + \frac{2ktz}{c^2}(z + wt) = 0,$$

which is identically satisfied if we take

$$u = 2x/t, \qquad v = -y/t, \qquad w = -z/t \qquad \text{...(4)}$$

From (4), $\dfrac{\partial u}{\partial x} = \dfrac{2}{t}$, $\quad \dfrac{\partial v}{\partial y} = -\dfrac{1}{t}$, $\quad \dfrac{\partial w}{\partial z} = -\dfrac{1}{t}$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4).

Ex. 2. (*i*) *Determine the restrictions on* f_1, f_2, f_3 *if* $(x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) = 1$ *is a possible boundary surface of a liquid.*

[Agra 2005; I.A.S.1995; Kanpur 2011; Meerut 2000]

(*ii*) Show that $(x^2/a^2)f(t) + (y^2/b^2)\,\phi(t) + (z^2/c^2)\,\psi(t) = 1$ *is a possible from of the boundary surface if* $f(t)\,\phi(t)\,\psi(t) = 1$.

Sol. (*i*) The given surface

$$F(x, y, z, t) = (x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$(\partial F/\partial t) + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{....(2)}$$

and the same values of u, v w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(3)}$$

Using dashes for differentiation with respect to t, (1) gives

$$\frac{\partial F}{\partial t}=\frac{x^2}{a^2}f_1'(t)+\frac{y^2}{b^2}f_2'(t)+\frac{z^2}{c^2}f_3'(t),\qquad \frac{\partial F}{\partial x}=\frac{2x}{a^2}f_1(t),\qquad \frac{\partial F}{\partial y}=\frac{2y}{b^2}f_2(t),\qquad \frac{\partial F}{\partial z}=\frac{2z}{c^2}f_3(t)$$

With these values, (2) reduces to

$$\frac{x^2f_1'}{a^2}+\frac{y^2f_2'}{b^2}+\frac{z^2f_3'}{c^2}+\frac{2xf_1u}{a^2}+\frac{2yf_2v}{b^2}+\frac{2zf_3w}{c^2}=0$$

or $$\frac{2xf_1}{a^2}\left(u+\frac{xf_1'}{2f_1}\right)+\frac{2yf_2}{b^2}\left(v+\frac{yf_2'}{2f_2}\right)+\frac{2zf_3}{c^2}\left(w+\frac{zf_3'}{2f_3}\right)=0$$

which is identically satisfied if we take

$$u=-\frac{xf_1'}{2f_1},\qquad v=-\frac{yf_2'}{2f_2},\qquad w=-\frac{zf_3'}{2f_3} \qquad ...(4)$$

From (4), $$\frac{\partial u}{\partial x}=-\frac{f_1'}{2f_1};\qquad \frac{\partial v}{\partial y}=-\frac{f_2'}{2f_2};\qquad \frac{\partial w}{\partial z}=-\frac{f_3'}{2f_3} \qquad ...(5)$$

Now the required restriction will be obtained if the above velocity components satisfy (3). Hence, we get

$$-\frac{f_1'}{2f_1}-\frac{f_2'}{2f_2}-\frac{f_3'}{2f_3}=0 \qquad \text{or} \qquad \frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}=0$$

Integrating, $\log f_1 + \log f_2 + \log f_3 = \log c$

or $\log (f_1 f_2 f_3) = \log c$ or $f_1 f_2 f_3 = c$, where c is an arbitrary constant.

(*ii*) Proceed as in the above example. There is no loss of generality if c is taken as unity.

Ex. 3. *Show that* $(x^2/a^2)\tan^2 t+(y^2/b^2)\cot^2 t=1$ *is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.*

[Garhwal 2005; I.A.S. 1997; Kanpur 1999, 2004, 08; Rajasthan 2004; Meerut 2003, 05; Rohilkhand 2002; 05; Purvanchel 2004]

Sol. For the present two dimensional motion $(\partial F/\partial z=0$ and $\partial w/\partial z=0)$, the surface

$$F(x, y, t)=(x^2/a^2)\tan^2 t+(y^2/b^2)\cot^2 t-1=0 \qquad ...(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t+u(\partial F/\partial x)+v(\partial F/\partial y)=0 \qquad ...(2)$$

and the same values of u and v satisfy the equation of continuity

$$\partial u/\partial x+\partial v/\partial y=0 \qquad ...(3)$$

From (1), $$\frac{\partial F}{\partial t}=\frac{x^2}{a^2}\cdot 2\tan t\sec^2 t-\frac{y^2}{b^2}\cdot 2\cot t\,\mathrm{cosec}^2 t,\quad \frac{\partial F}{\partial x}=\frac{2x}{a^2}\tan^2 t,\quad \frac{\partial F}{\partial y}=\frac{2y}{b^2}\cot^2 t$$

With these values, (2) reduces to

$$\frac{x\tan t}{a^2}(x\sec^2 t+u\tan t)+\frac{y\cot t}{b^2}(-y\,\mathrm{cosec}^2 t+v\cot t)=0,$$

which is identically satisfied if we take

$x\sec^2 t+u\tan t=0$ and $-y\,\mathrm{cosec}^2 t+v\cot t=0$

i.e. $u = -\dfrac{x}{\sin t \cos t}$ and $v = \dfrac{y}{\sin t \cos t}$ ...(4)

From (4), $\dfrac{\partial u}{\partial x} = -\dfrac{1}{\sin t \cos t}$ and $\dfrac{\partial v}{\partial y} = \dfrac{1}{\sin t \cos t}$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v. Hence (1) is a possible bounding surface with velocity components given by (4).

Using remark 2 of Art. 2.18 (with $\partial F/\partial z = 0$ here), the normal velocity

$$= \frac{u(\partial F/\partial x) + v(\partial F/\partial y)}{\sqrt{\{(\partial F/\partial x)^2 + (\partial F/\partial y)^2\}}} = \frac{-\dfrac{x}{\sin t \cos t}\cdot\dfrac{2x\tan^2 t}{a^2} + \dfrac{y}{\sin t\cos t}\cdot\dfrac{2y\cot^2 t}{b^2}}{\left\{\left(\dfrac{2x\tan^2 t}{a^2}\right)^2 + \left(\dfrac{2y\cot^2 t}{b^2}\right)^2\right\}^{1/2}}$$

$$= \frac{a^2y^2\cot t\,\mathrm{cosec}^2 t - b^2x^2\tan t\sec^2 t}{\sqrt{(x^2b^4\tan^4 t + y^2a^4\cot^4 t)}}$$

Ex. 4. (*a*) *Show that the ellipsoid* $x^2/(a^2k^2t^{2n}) + kt^n\{(y/b)^2 + (z/c)^2\} = 1$ *is a possible form of the boundary surface of a liquid. Derive also velocity components.*

(Kanpur 2009; 2010; Meerut 2007)

(*b*) Show that the variable ellipsoid $x^2/(a^2k^2t^4) + kt^2\{(y/b)^2 + (z/c)^2\} = 1$ is a possible form for the boundary surface at any time *t*. **(Kanpur 2007)**

Sol. (a) The given surface

$$F(x,y,z,t) = x^2/(a^2k^2t^{2n}) + kt^n\{(y/b)^2 + (z/c)^2\} - 1 = 0 \qquad \text{... (1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condtion

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(2)}$$

and the same values of u, v w stisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \qquad \text{...(3)}$$

From (1), $$\frac{\partial F}{\partial t} = -\frac{x^2}{a^2k^2}\cdot\frac{2n}{t^{2n+1}} + nkt^{n-1}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right),$$

$$\frac{\partial F}{\partial t} = \frac{2x}{a^2k^2t^{2n}}, \qquad \frac{\partial F}{\partial y} = \frac{2kt^n y}{b^2} \qquad \text{and} \qquad \frac{\partial F}{\partial z} = \frac{2kt^n z}{c^2}.$$

With these values, (2) reducces to

$$-\frac{x^2}{a^2k^2}\frac{2n}{t^{2n+1}} + nkt^{n-1}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \frac{2xu}{a^2k^2t^{2n}} + \frac{2kt^n vy}{b^2} + \frac{2kt^n zw}{c^2} = 0$$

or $$\left(u - \frac{nx}{t}\right)\frac{2x}{a^2k^2t^{2n}} + \left(v + \frac{ny}{2t}\right)\frac{2kyt^n}{b^2} + \left(w + \frac{nz}{2t}\right)\frac{2kzt^n}{c^2} = 0,$$

which is identically satisfied if we take

$u - (nx/t) = 0$, $v + (ny/2t) = 0$ and $w + (nz/2t) = 0$

or $u = nx/t$, $v = -\,ny/2t$ and $w = -\,nz/2t$. ...(4)

From (4), $\partial u/\partial x = n/t$, $\partial v/\partial y = -n/2t$ and $\partial w/\partial z = -n/2t$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4)

(b) Proceed as in part (a) by taking $n = 2$

Ex. 5. *Show that the ellipsoid*

$$\frac{x^2}{a^2 e^{-t}\cos(t+\pi/4)}+\frac{y^2}{b^2 e^{t}\sin(t+\pi/4)}+\frac{z^2}{c^2\sec 2t}=1$$

is a possible form of boundary surface of a liquid for any time t and determine the velocity **q** *of any particle on this boundary. Also prove that the equation of continuity is satisfied.*

Sol. The given surface

$$F(x, y, z, t) = (x^2/a^2)e^t \sec(t+\pi/4) + (y^2/b^2)e^{-t}\operatorname{cosec}(t+\pi/4) + (z^2/c^2)\cos 2t - 1 = 0 \quad ...(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \quad ...(2)$$

and the same values of u, v, w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad ...(3)$$

From (1), $$\frac{\partial F}{\partial t}=\frac{x^2}{a^2}e^t\sec\left(t+\frac{\pi}{4}\right)+\frac{x^2}{a^2}e^t\sec\left(t+\frac{\pi}{4}\right)\tan\left(t+\frac{\pi}{4}\right)-\frac{y^2}{b^2}e^{-1}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)$$
$$-\frac{y^2}{b^2}e^{-t}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)\cot\left(t+\frac{\pi}{4}\right)-\frac{2z^2}{c^2}\sin 2t$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}e^t\sec\left(t+\frac{\pi}{4}\right),\qquad \frac{\partial F}{\partial y}=\frac{2y}{b^2}e^{-t}\operatorname{cosec}\left(t+\frac{\pi}{4}\right),\qquad \frac{\partial F}{\partial z}=\frac{2z}{c^2}\cos 2t$$

With these values, (2) reduces to

$$\frac{xe^t}{a^2}\sec\left(t+\frac{\pi}{4}\right)\left[2u+x\left\{1+\tan\left(t+\frac{\pi}{4}\right)\right\}\right]+\frac{ye^{-t}}{b^2}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)\left[2v-y\left\{1+\cot\left(t+\frac{\pi}{4}\right)\right\}\right]$$
$$+(2z/c^2)\times(w\cos 2t - z\sin 2t)=0,$$

which is identically satisfied if we take

$$2u + x\{1+\tan(t+\pi/4)\}=0,\qquad 2v - y\{1+\cot(t+\pi/4)\}=0,\qquad w\cos 2t - z\sin 2t = 0$$

or $$u=-(x/2)\times\{1+\tan(t+\pi/4)\},\qquad v=-(y/2)\times\{1+\cot(t+\pi/4)\},\qquad w=z\tan 2t$$

Using these values of u, v, w on the boundary for all t, we have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{2}\left\{1+\tan\left(t+\frac{\pi}{4}\right)\right\}-\frac{1}{2}\left\{1+\cot\left(t+\frac{\pi}{4}\right)\right\}+\tan 2t$$

$$=\frac{1-\tan^2(t+\pi/4)}{2\tan(t+\pi/4)}+\tan 2t=\cot\left(2t+\frac{\pi}{2}\right)+\tan 2t=-\tan 2t+\tan 2t=0,$$

showing that the equation of continuity is satisfied.

EXERCISE 2 (D)

1. Show that $(x^2/a^2)f(t)+(y^2/b^2)\phi(t)=1$, where $f(t)\phi(t)=1$ is a possible form of the boundary surface of a liquid. **[Kanpur 2006]**

2. Show that $(x^2/a^2)f(t)+(y^2/b^2)f(t)=1$ is a possible from of the boundary surface of a liquid.

3. Show that $(x^2/a^2)\cos^2 t + (y^2/b^2)\sec^2 t = 1$ is a possible form for the boundary surface. **[I.A.S. 2007]**

4. Show that $(x^2/a^2) f(t) + y^2/b^2 + (z^2/c^2) f(t) = 1$ is a possible form of the boundary surface of a liquid.

5. A sphere of radius r moves with a steady velocity components (U, V, W) through an initially stationary fluid. If t be measured from the instant the sphere was at the origin, prove that

$$(u - U)(x - Ut) + (v - V)(y - Vt) + (w - W)(z - Wt) = 0$$

where (u, v, w) are components of velocity on the sphere at any point.

6. The parabolic profile $y = kx^{1/2}$ moves in the negative x-direction with a velocity U through a fluid which was initially stationary. If u and v are the instantaneous velocity components of a fluid particle on boundary, show that $v/(u + U) = k^2/2y$.

2.20. Streamline or line of flow.

[I.A.S. 1995; Kurkshetra 1998; U. P. P. C. S. 2000, Agra 2004, 2009 Kanpur 2000, 04, Meerut 2001, 02, 05, 12; G. N. D. U. Amritsar 1999]

A streamline is a curve drawn in the fluid so that its tangent at each point is the direction of motion (*i.e.* fluid velocity) at that point.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a straight line and let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ be the fluid velocity at P. Then $\mathbf{q}$ is parallel to $d\mathbf{r}$ at P on the steamline. Thus, the equation of streamlines is given by

$$\mathbf{q} \times d\mathbf{r} = \mathbf{0} \qquad \text{...(1)}$$

i.e., $$(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$$

or $$(v\,dz - w\,dy)\,\mathbf{i} + (w\,dx - u\,dz)\,\mathbf{j} + (u\,dy - v\,dx)\,\mathbf{k} = \mathbf{0}$$

whence $$v\,dz - w\,dy = 0, \qquad w\,dx - u\,dz = 0, \qquad u\,dy - v\,dx = 0$$

so that $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad \text{...(2)}$$

The equations (2) have a double infinite set of solutions. Through each point of the flow field where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ do not all vanish, there passes one and only one streamline at a given instant. This fact can be verified by employing the well known existence theorem for the system of equations (2). If the velocity vanishes at a given point, various singularities occur there. Such a point is known as a *critical point* or *stagnation point.*

2.21. Path line or path of a particle.

[Meerut 2012; Kanpur 2000, 02]

A path line is the curve or trajectory along which a particular fluid particle travels during its motion.

The differential equation of a path line is $$d\mathbf{r}/dt = \mathbf{q} \qquad \text{...(1)}$$

so that $$dx/dt = u, \qquad dy/dt = v \qquad \text{and} \qquad dz/dt = w \qquad \text{...(2)}$$

where $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \text{and} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + w\mathbf{k}.$$

Remark. Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then the path line is determined from equations

$$\left.\begin{aligned} dx/dt &= u(x, y, z, t) \\ dy/dt &= v(x, y, z, t) \\ dz/dt &= w(x, y, z, t) \end{aligned}\right\} \qquad \text{...(3)}$$

with initial conditions $$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0 \qquad \text{...(4)}$$

Streamline or line of flow. **[I.A.S. 1995; Kurkshetra 1998; U. P. P. C. S. 2000, Agra 2004, 2009 Kanpur 2000, 04, Meerut 2001, 02, 05, 12; G. N. D. U. Amritsar 1999]**

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Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a straight line and let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ be the fluid velocity at P. Then $\mathbf{q}$ is parallel to $d\mathbf{r}$ at P on the steamline. Thus, the equation of streamlines is given by

$$\mathbf{q} \times d\mathbf{r} = \mathbf{0} \qquad ...(1)$$

i.e., $$(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$$

or $$(vdz - wdy)\,\mathbf{i} + (wdx - udz)\,\mathbf{j} + (udy - vdx)\,\mathbf{k} = \mathbf{0}$$

whence $$vdz - wdy = 0, \qquad wdx - udz = 0, \qquad udy - vdx = 0$$

so that $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad ...(2)$$

The equations (2) have a double infinite set of solutions. Through each point of the flow field where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ do not all vanish, there passes one and only one streamline at a given instant. This fact can be verified by employing the well known existence theorem for the system of equations (2). If the velocity vanishes at a given point, various singularities occur there. Such a point is known as a *critical point* or *stagnation point.*

2.21. Path line or path of a particle. **[Meerut 2012; Kanpur 2000, 02]**

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The differential equation of a path line is $$d\mathbf{r}/dt = \mathbf{q} \qquad ...(1)$$

so that $$dx/dt = u, \qquad dy/dt = v \qquad \text{and} \qquad dz/dt = w \qquad ...(2)$$

where $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \text{and} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + w\mathbf{k}.$$

Remark. Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then the path line is determined from equations

$$\left.\begin{aligned} dx/dt &= u(x, y, z, t) \\ dy/dt &= v(x, y, z, t) \\ dz/dt &= w(x, y, z, t) \end{aligned}\right\} \qquad ...(3)$$

with initial conditions $$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0 \qquad ...(4)$$

2.22. Streak lines or filament lines. [Kanpur 2000; Meerut 2005, 12]

A streak line is a line on which lie all those fluid particles that at some earlier instant passed through a certain point of space. Thus, a streak line presents the instantaneous pictures of the position of all fluid particles, which have passed through a given point at some previous time. When a dye is injected into a moving fluid at some fixed point, the visible lines produced in the fluid are streak lines which have passed through the injected point.

The equation of the streak line at time t can be derived by Lagrangian method, Suppose that a fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) in the course of time. Then by using the Lagrangian method of description, we have

$$f_1(x_0, y_0, z_0, t) = x_1, \qquad f_2(x_0, y_0, z_0, t) = y_1, \qquad f_3(x_0, y_0, z_0, t) = z_1 \qquad \text{...(1)}$$

Solving (1) for x_0, y_0, z_0, we have

$$x_0 = g_1(x_1, y_1, z_1, t), \qquad y_0 = g_2(x_1, y_1, z_1, t), \qquad z_0 = g_3(x_1, y_1, z_1, t) \qquad \text{...(2)}$$

Now a streak line is the locus of the positions (x, y, z) of the particles which have passed through the fixed point (x_1, y_1, z_1). Hence the equation of the streak line at time t is given by

$$x = h_1(x_0, y_0, z_0, t), \qquad y = h_2(x_0, y_0, z_0, t), \qquad z = h_3(x_0, y_0, z_0, t) \qquad \text{...(3)}$$

Substituting the values of x_0, y_0, z_0 in (3), the desired equation of streak line passing through (x_1, y_1, z_1) at time t is given by

$$x = h_1(g_1, g_2, g_3, t), \qquad y = h_2(g_1, g_2, g_3, t), \qquad z = h_3(g_1, g_2, g_3, t) \qquad \text{...(4)}$$

2.23. Difference between the streamlines and path lines. [Agra 2005]

It is important to note that streamlines are not, in general, the same as the path lines. Streamlines show how each particle is moving at a given instant of time while the path lines present the motion of the particles at each instant. Except in the case of steady motion, u, v, w are always functions of the time and hence the streamlines go on changing with the time, and the actual path of any fluid particle will not in general coincide with a streamline. To understand this, take three consecutive points P, Q, R on a streamline at time t. Then a particle moving through P at this instant will move along PQ but as soon as it arrives at Q at time $t + \delta t$, QR is no longer the direction of the velocity at Q and the particle will therefore cease to move along QR and move instead in the direction of the new velocity at Q. However, in the case of steady motion the streamlines remain unchanged as time progresses and hence they are also the path lines.

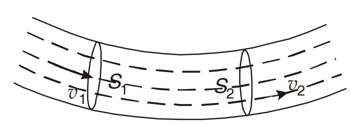
2.24. Stream tube (or tube of flow) and stream filament.

If we draw the streamlines from each point of a closed curve in the fluid, we obtain a tube called the *stream tube*.

A stream tube of infinitesimal cross-section is known as a *stream filament*.

Remark 1. Since there is no movement of fluid across a streamline, no fluid can enter or leave the steam tube except at the ends. So in the case of the steady motion, a stream tube behaves like an actual solid tube through which the fluid is flowing. Due to steady flow, the walls of the tube are fixed in space and hence the motion through the stream tube would remain unchanged on replacing the walls of the tube by a rigid boundary.

Remark 2. Consider a steam filament of liquid in steady motion. Let the cross-sectional area of the filament be so small that the velocity is the same at each point of this area, which may be taken perpendicular to the direction of the velocity. Let v_1, v_2 be the speeds of the flow at places where the cross-sectional areas are S_1, S_2. Let the liquid be incompressible. From the law of conservation of mass, the total quantity of liquid flowing across each section of the filament must be the same. Thus, we have



$$v_1 S_1 = v_2 S_2,$$

from which we arrive at the following theorem :

Theorem : *The product of the speed and cross sectional area is constant along a stream filament of a liquid in steady motion.*

It follows from the above theorem that a stream filament is widest at places where the speed is least and is narrowest at places where the speed is greatest. Furthermore, the stream filament cannot terminate at a point within the liquid unless the velocity is infinite there, which is never possible. Leaving this exceptional case, it follows that, in general, stream filaments are either closed or terminate at the boundary of a liquid. The same results are true for stream lines, because the cross-section of the filament may be considered as small as we please.

2.25. Illustrative solved examples.

Ex. 1. *Obtain the streamlines of a flow* $u = x,\ v = -y$.

OR *If the velocity* **q** *is given* $\mathbf{q} = x\mathbf{i} - y\mathbf{j}$, ***determine the equations of the streamlines.*** **[Meerut 2012]**

Sol. For two-dimensional flow ($w = 0$), we have

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = x\mathbf{i} - y\mathbf{j}$$

so that $u = x,\qquad v = -y,\qquad w = 0$

Streamlines are given by $(dx)/u = (dy)/v = (dz)/w$

i.e. $(dx)/x = (dy)/(-y) = (dz)/0$ so that $(dx)/x + (dy)/y = 0$ and $dz = 0$

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1$ and $z = c_2$

The required straight lines are given by the curves of intersection of $xy = c_1$ and $z = c_2$, c_1 and c_2 being arbitrary constants.

Ex. 2. *The velocity components in a three-dimensional flow field for an incompressible fluid are* $(2x, -y, -z)$. *Is it a possible field? Determine the equations of the streamline passing through the point* (1, 1, 1). Sketch the streamlines.

Sol. Here $u = 2x,\qquad v = -y,\qquad w = -z$

Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e. \qquad \frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z} \qquad ...(1)$$

Taking the first two members of (1), we have

$$\frac{dx}{2x} = \frac{dy}{-y} \qquad \text{or} \qquad \frac{dx}{x} + 2\cdot\frac{dy}{y} = 0$$

Integrating, $\log x + 2\log y = \log c_1$ or $xy^2 = c_1$. ...(2)

Again, taking the first and third members of (1) and proceeding as above, we get

$$xz^2 = c_2. \qquad ...(3)$$

Here c_1 and c_2 are arbitrary constants. The streamlines are given by the curves of intersection of (2) and (3). The required streamline passes through (1, 1, 1) so that $c_1 = 1$ and $c_2 = 1$. Thus, the desired stream line is given by the intersection of $xy^2 = 1$ and $xz^2 = 1$.

We also have $\partial u/\partial x = 2,\qquad \partial v/\partial y = -1,\qquad \partial w/\partial z = -1$

so that $$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$$

showing that the equation of continuity is satisfied for the given flow field for an incompressible fluid. Hence the given velocity components correspond to a possible field.

Ex. 3. *The velocity field at a point in fluid is given as* $\mathbf{q} = (x/t, y, 0)$. *Obtain path lines and streak lines.* **[Agra 2008; Meerut 2002; 04]**

Sol. Here $\mathbf{q} = (u, v, w) = (x/t, y, 0)$

so that $u = x/t,\qquad v = y,\qquad w = 0$ ...(1)

The equations of path lines are

$$dx/dt = u, \qquad dy/dt = v, \qquad dz/dt = w$$

i.e. $$dx/dt = x/t, \qquad dy/dt = y, \qquad dz/dt = 0 \qquad ...(2)$$

Suppose that (x_0, y_0, z_0) are coordinates of the chosen fluid particle at time $t = t_0$. Then

$$x = x_0, \qquad y = y_0, \qquad z = z_0 \qquad \text{when } t = t_0 \qquad ...(3)$$

From (2), $(1/x)dx = (1/t)dt$ giving $\log x = \log t + \log c_1$

i.e. $x = tc_1$, c_1 being an arbitrary constant ...(4)

Using initial conditions (3), (4) gives

$$x_0 = t_0 c_1 \qquad \text{so that} \qquad c_1 = x_0/t_0$$

$\therefore$ From (4), $x = (x_0\, t)/t_0$ ...(5)

Similarly, integrating $dy/dt = y$ *i.e.* $(1/y)\, dy = dt$, we get

$$\log y - \log c_2 = t \qquad i.e. \qquad y = c_2 e^t \qquad ...(6)$$

Using (3), (6) gives $y_0 = c_2\, e^{t_0}$ *i.e.* $c_2 = y_0\, e^{-t_0}$

$\therefore$ From (6), $y = y_0\, e^{-t_0}$ ...(7)

Finally, integrating $dz/dt = 0$, we get $z = c_3$ ...(8)

Using (3), $z_0 = c_3$ so that $z = z_0$ ...(9)

Hence the required path lines are given by

$$x = (x_0 t)/t_0 \qquad y = y_0 e^{t-t_0}, \qquad z = z_0. \qquad ...(10)$$

Let the fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) at time $t = s$ where $t_0 \le s \le t$. Then (10) gives

$$x_1 = (x_0 s)/t_0, \qquad y_1 = y_0 e^{s-t_0}, \qquad z_1 = z_0$$

so that $$x_0 = (x_1 t_0)/s, \qquad y_0 = y_1 e^{t_0 - s}, \qquad z_0 = z_1 \qquad ...(11)$$

wherein s is the parameter. Substituting equations (11) into (10), we obtain the equation of streak line passing through (x_1, y_1, z_1) at times t as

$$x = (x_1 t)/s, \qquad y = y_1 e^{t-s}, \qquad z = z_1. \qquad ...(12)$$

Remark. It is easily seen from the above example that for a steady flow, streak lines are identical to path lines, and hence they coincide with streamlines.

Ex. 4. *Find the streamlines and paths of the particles when*

$$u = x/(1+t), \qquad v = y/(1+t), \qquad w = z/(1+t). \qquad \textbf{[I.A.S. 1994]}$$

Sol. Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e. \qquad \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

i.e. $$(dx)/x = (dy)/y = (dz)/z \qquad ...(1)$$

Taking the first two members of (1), we get $x/y = c_1$ ...(2)

Taking the last two members of (1), we get $y/z = c_2$ ...(3)

The desired streamlines are given by the intersection of (2) and (3).

The paths of the particle are given by

$$dx/dt = u, \qquad dy/dt = v, \qquad dz/dt = w$$

i.e. $$dx/dt = x/(1+t), \qquad dy/dt = y/(1+t), \qquad dz/dt = z/(1+t)$$

giving $\dfrac{dx}{x} = \dfrac{dt}{1+t}$, $\dfrac{dy}{y} = \dfrac{dt}{1+t}$, $\dfrac{dz}{z} = \dfrac{dt}{1+t}$

Integrating, $x = c_3\,(1 + t)$, $y = c_4\,(1 + t)$, $z = c_5\,(1 + t)$

which give the desired paths of the particles, c_3, c_4 and c_5 being arbitrary constants.

Ex. 5. *Consider the velocity field given by* $\mathbf{q} = (1 + At)\mathbf{i} + x\mathbf{j}$. *Find the equation of the streamline at* $t = t_0$ *passing through the point* (x_0, y_0). *Also obtain the equation of the path line of a fluid element which comes to* (x_0, y_0) *at* $t = t_0$. *Show that, if* $A = 0$ *(i.e. steady flow), the streamline and path line coincide.*

Sol. Since $\mathbf{q} = u\mathbf{i} + v\mathbf{j}$, $u = 1 + At$ and $v = x$. Hence the streamline at $t = t_0$ is given by

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{1+At_0} = \frac{dy}{x} \quad \text{or} \quad x\,dx = (1 + At_0)\,dy$$

Integrating, $(1 + At_0)y = x^2/2 + c$, c being an arbitrary constant. ...(*i*)

But $y = y_0$, when $x = x_0$, so we get $(1 + At_0)y_0 = x_0^2/2 + c$ ...(*ii*)

Subtracting (*ii*) from (*i*) to eliminate c, we get $(1 + At_0)\,(y - y_0) = (x^2 - x_0^2)/2$, ...(1)

which is the required streamline.

We now determine required path line. Consider a fluid element passing through (x_0, y_0) at $t = t_0$.Then its coordinates (x, y) at any instant t (which define the path line) may be written as

$$x = x\,(x_0, y_0\,,\, t), \qquad y = y\,(x_0, y_0, t) \qquad \text{...(2)}$$

Now the path line is given by

$$dx/dt = u = 1 + At \qquad \text{...(3)}$$

and

$$dy/dt = v = x \qquad \text{...(4)}$$

Integrating (3) and using the condition $x = x_0$ at $t = t_0$, we get

$$x - x_0 = (t - t_0) + A\,(t^2 - t_0^2)/2 \qquad \text{...(5)}$$

Using (5), (4) may be re-written as

$$dy/dt = x_0 + (t - t_0) + A\,(t^2 - t_0^2)/2 \qquad \text{...(6)}$$

Integrating (6) and using the condition $y = y_0$ at $t = t_0$, we get

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0)\right] + \frac{1}{6}A(t^3 - t_0^3) - \frac{1}{2}At_0^2(t - t_0) \qquad \text{...(7)}$$

Equations (5) and (7) together give the equation of the path line in parametric form with t as parameter. On elimination of t between (5) and (7), we will get equation of path-line in cartesian coordinates x, y. The resulting equation so obtained will be different from the equation (1) of the streamline.

When $A = 0$, the equation of the streamline (1) gives

$$y - y_0 = (x^2 - x_0^2)/2 \qquad \text{...(8)}$$

and parametric equation of path line given by (5) and (7) reduce to

$$x - x_0 = t - t_0, \qquad y - y_0 = (t - t_0)\left[x_0 + (1/2) \times (t - t_0)\right] \qquad \text{...(9)}$$

Eliminating t from (9), the equation of path line is

$$y - y_0 = (x^2 - x_0^2)/2.$$

Thus, the streamline coindes with the path line.

Ex. 6. *Prove that if the speed is everywhere the same, the streamlines are straight lines.*

Sol. Let u, v, w be the constant speed components of the speed of the fluid particle. Then the equation of the streamlines are given by $(dx)/u = (dy)/v = (dz)/w$ ...(1)

Taking first and second and then first and third fractions in (1), we get

$$vdx - udy = 0 \quad \text{and} \quad wdx - udz = 0.$$

Integrating, $vx - uy = c_1$ and $wx - uz = c_2$, ...(2)

where c_1 and c_2 are arbitray constants of integration.

The required streamlines are given by the straight lines of intersection of two planes given by (2).

Ex. 7. *Find the equation of the streamlines for the flow* $\mathbf{q} = -\mathbf{i}\,(3y^2) - \mathbf{j}\,(6x)$ *at the point* (1, 1).

Sol. Here $\mathbf{q} = u\mathbf{i} + v\mathbf{j} = -\mathbf{i}\,(3y^2) - j\,(6x) \quad \Rightarrow \quad u = -3y^2, \quad v = -6x,$ ...(1)

The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{-3y^2} = \frac{dy}{-6x} \quad \text{or} \quad 6xdx - 3y^2\,dy = 0.$$

Integrating, $3x^2 - y^3 = c$, c being an arbitrary constant. ...(2)

At the point (1, 1), (2) gives $3 - 1 = c$ or $c = 2$.

Hence, from (2), the required equation of the streamline is $3x^2 - y^3 = 2$.

Ex. 8. *The velocity components in a two-dimensonal flow field for an incompressible fluid are given by* $u = e^x \cosh y$ *and* $v = -e^x \sinh y$. *Determine the equation of the streamlines for this flow.* **[Agra 2003]**

$k \neq 0$

Sol. The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{du}{v} \quad \text{or} \quad \frac{dx}{e^x \cosh y} = \frac{dy}{-e^x \sinh y} \quad \text{or} \quad \coth y\, dy = -dx.$$

Integrating, $\log \sinh y - \log c = -x$ or $\sinh y = ce^{-x}$,

where c is a constant of integration.

Ex. 9. *For an incompressible homogeneous fluid at the point (x, y, z) the velocity distribution is given by* $u = -(c^2 y/r^2)$, $v = c^2 x/r^2$, $w = 0$, *where r denotes the distance from the z-axis. Show that it is a possible motion and determine the surface which is orthogonal to streamlines.*

Sol. Since r is distance of point (x, y, z) from the z-axis, we have $r = (x^2 + y^2)^{1/2}$. Hence given velocity distribution becomes

$$u = -\{c^2 y/(x^2 + y^2)\}, \qquad v = (c^2 x)/(x^2 + y^2), \qquad w = 0 \qquad \text{...(1)}$$

From (1), $\partial u/\partial x = -\{2c^2 yx/(x^2 + y^2)^2\}$, $\quad \partial v/\partial y = (2c^2 xy)/(x^2 + y^2)^2$, $\quad \partial w/\partial z = 0$

$\therefore \; \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, showing that the equation of continutiy is satisfied and so the motion specified by (1) is possble.

The surfaces which are orhogonal to streamlines $(dx)/u = (dy)/v = (dz)/w$ are given by*

$$udx + vdy + wdz = 0 \quad \text{or} \quad -\{c^2 y/(x^2 + y^2)\}\,dx + \{c^2 x/(x^2 + y^2)\}dy = 0$$

or $\quad -y\,dx + x\,dy = 0 \quad$ or $\quad (1/y)dy = (1/x)\,dx$,

Integrating, $\log y = \log k + \log x$ or $y = kx$, k being an arbitrary constant.

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* Refer chapter 3 in part II of author's Ordinary and Partial Differential Equations published by S. Chand & Co., New Delhi

Ex.10. *Determine the streamlines and the path lines of the particle when the components of the velocity field are given by* $u = x/(1+t)$, $v = y/(2+t)$ *and* $w = z/(3+t)$. *Also state the condition for which the steamlines are identical with path lines.* **[I.A.S. 2000]**

Sol. Streamlines are given by $dx/u = dy/v = dz/w$

or $$(1+t)(1/x)dx = (2+t)(1/y)dy = (3+t)(1/z)dz. \qquad ...(1)$$

Taking the first two members of (1), we have

$$(1/x)dx + (t/x)dx = (2/y)dy + (t/y)dy$$

or $$(1/x)dx - (2/y)dy = t\{(1/y)dy - (1/x)dx\}.$$

Integrating, $\log x - 2\log y = t(\log y - \log x) + \log c_1$, c_1 being an arbitrary constant

or $\log(x/y^2) = \log\{c(y/x)^t\}$ so that $(y/x)^t = x/c_1y^2$. ...(2)

Similarly, taking the last two members of (1), we have

or $\log(y^2/z^3) = \log\{c_2(y/z)^t\}$ or $(y/z)^t = y^2/c_2z^3$. ...(3)

The desired stremlines at a given instant $t = t_0$ are given by the intersection of the surfaces (2) and (3) by substituting t_0 for t.

Again, the path lines are given by

$$dx/dt = u, \qquad dy/dt = v \qquad dz/dt = w$$

or $$dx/dt = x/(1+t), \qquad dy/dt = y/(2+t), \qquad dz/dt = z/(3+t),$$

giving $$dx/x = dt/(1+t), \qquad dy/y = dt/(2+t), \qquad dz/z = dt/(3+t).$$

Integraing, $x = c_3(1+t)$, $y = c_4(2+t)$, $z = c_5(3+t)$, c_3, c_4, c_5 being arbitrary constants which gives the desired paths of the given particle in terms of the parameter t.

Condition under which the stremlines and path linear are indentical.

In the case of steady motion the steamlines remain unchanged as time progresses and hence they are identical with the path lines.

Ex. 11. *In the steady motion of homogenous liquid if the surfaces* $f_1 = a_1$, $f_2 = a_2$, *define the streamlines prove that the most general values of the velocity components u, v, w are*

$$F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x, y)}. \qquad \textbf{(Meerut 2008)}$$

Sol. The motion being steady, the streamlines will be independent of time. It follows that the functions f_1 and f_2 will be functions of x, y, z. We have

$$f_1 = a_1 \quad \Rightarrow \quad df_1 = 0 \quad \Rightarrow \quad \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0 \qquad ...(1)$$

$$f_1 = a_2 \quad \Rightarrow \quad df_2 = 0 \quad \Rightarrow \quad \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz = 0. \qquad ...(2)$$

From (1) and (2), by cross multiplication, we have

$$\frac{dx}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{dy}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}} = \frac{dz}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}$$

or $$\frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_3}, \qquad ...(2)$$

where $$J_1 = \frac{\partial(f_1, f_2)}{\partial(y, z)}, \qquad J_2 = \frac{\partial(f_1, f_2)}{\partial(z, x)} \qquad \text{and} \qquad J_3 = \frac{\partial(f_1, f_2)}{\partial(x, y)}. \qquad ...(3)$$

We know that the equation of streamlines are given by

$$(dx)/u = (dy)/v = (dz)/w.$$

Comparing (2) and (3), $u = FJ_1, \quad v = FJ_2, \quad w = FJ_3,$ ....(5)

where F is an arbitrary function. We now proceed to find F.

For the given liquid motion to be possible, the equation of continuity must be satisfied, *i.e*

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

or
$$\frac{\partial}{\partial x}(FJ_1) + \frac{\partial}{\partial y}(FJ_2) + \frac{\partial}{\partial z}(FJ_3) = 0 \qquad ...(6)$$

or
$$F\left(\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}\right) + \left(J_1\frac{\partial F}{\partial x} + J_2\frac{\partial F}{\partial y} + J_3\frac{\partial F}{\partial z}\right) = 0. \qquad ...(7)$$

By the property of Jocabains, $\partial J_1/\partial x + \partial J_2/\partial y + \partial J_3/\partial z = 0.$ ...(8)

Using (8), (7) becomes $J_1(\partial F/\partial x) + J_2(\partial F/\partial y) + J_3(\partial F/\partial z) = 0$

or
$$\frac{\partial F}{\partial x}\frac{\partial(f_1, f_2)}{\partial(y, z)} + \frac{\partial F}{\partial y}\frac{\partial(f_1, f_2)}{\partial(z, x)} + \frac{\partial F}{\partial z}\frac{\partial(f_1, f_2)}{\partial(x, y)} = 0, \text{ using (3)}$$

or
$$\frac{\partial F}{\partial x}\left(\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}\right) + \frac{\partial F}{\partial z}\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right) = 0$$

or
$$\begin{vmatrix} \partial F/\partial x & \partial F/\partial y & \partial F/\partial z \\ \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial x \end{vmatrix} = 0 \quad \text{or} \quad \frac{\partial(F_1, f_1, f_2)}{\partial(x, y, z)} = 0,$$

showing that F, f_1 and f_2 are not independent and hence F is a function of f_1, f_2 only. Therefore, $F = F(f_1, f_2)$.

Hence, from (2) and (5), the values of the velocity components u, v, w respectively are given by FJ_1, FJ_2, FJ_3, that is,

$$F_1(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)}, \qquad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)}, \qquad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x, y)}.$$

EXERCISE 2 (E)

1. Determine the streamlines and streaklines for the flow whose velocity field is given by
$u = -x + t + 2, \qquad v = y - t + 2.$ **[Meerut 2005]**

2. Find the streamlines and path lines of the two-dimensional velocity field $u = x/(1 + t)$, $v = y$, $w = 0$. **[Agra 2002, 2004]**

[**Ans.** $z = c_1$, $y = c_2 x^{1+t}$; $x = a_1(1 + t)$, $y = a_2 e^t$, $z = a_3$]

3. Distinguish between path lines and streamlines.

4. Find the streamline and path of the paticle when u = $(2xt)/(1 + t^2)$, $v = (2yt)/(1+t^2)$, $w = (2zt)/(1 + t^2)$. **[Purvanchal 2007]**

The velocity potential or velocity function.

[Meerut 2005, 09; Rohilkhand 2004, 05]

Suppose that the fluid velocity at time t is $\mathbf{q} = (u, v, w)$. Further suppose that at the considered instant t, there exists a scalar function ϕ (x, y, z, t), uniform throughout the entire field of flow and such that

$$-d\phi = u\,dx + v\,dy + w\,dz \qquad \text{...(1)}$$

i.e.

$$-\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz \qquad \text{...(2)}$$

Then the expression on the R.H.S. of (1) is an exact differential and we have

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(3)}$$

$$\therefore \qquad \mathbf{q} = -\nabla\phi = -\text{grad}\,\phi. \qquad \text{...(4)}$$

ϕ is called the *velocity potential*. The negative sign in (4) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \mathbf{q} = 0, \qquad i.e. \qquad \text{curl } \mathbf{q} = 0 \qquad \text{...(5)}$$

or

$$\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathbf{0} \qquad \text{...(6)}$$

Remark 1. The surfaces $\phi\,(x, y, z, t) = \text{const.}$ ...(7)

are called the *equipotenials*. The streamlines

$$dx/u = dy/v = dz/w \qquad \text{...(8)}$$

are cut at right angles by the surfaces given by the differential equation

$$udx + vdy + wdz = 0 \qquad \text{...(9)}$$

and the condition for the existence of such orthogonal surfaces is the condition that (9) may posses a solution of the form (7) at the considered instant t, the analytical condition being

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad \text{...(10)}$$

When the velocity potential exists, (3) holds. Then

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y} = 0, \qquad i.e., \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \text{...(11)}$$

Similarly, $\partial u/\partial z = \partial w/\partial x$ and $\partial v/\partial x = \partial u/\partial y$ ...(12)

Using (11) and (12), we find that the condition (10) is satisfied. Hence surfaces exist which cut the streamlines orthogonallly. We also conclude that at all points of field of flow the equipotentials are cut orthogonally by the streamlines.

Remark 2. When (5) holds, the flow is known as the *potential kind*. It is also known as *irrototional*. For such flow the field of **q** is *conservative*.

Remark 3. The equation of continuity of an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(13)}$$

Suppose that the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(14)}$$

Using (14), (13) reduces to

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0, \qquad ...(15)$$

showing that ϕ is a harmonic function satisfying the Laplace euation $\nabla^2\phi = 0$, where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad ...(16)$$

2.27. The Vorticity Vector. **[Kanpur 2004, Garhwal 2005]**

Let $\mathbf{q} = ui + vi + wk$ be the fluid velocity such that curl $\mathbf{q} \neq \mathbf{0}$. Then the vector

$$\boldsymbol{\Omega} = \text{curl}\,\mathbf{q} \qquad ...(1)$$

is called the *vorticity vector*.

Let $\boldsymbol{\Omega}_x, \boldsymbol{\Omega}_y, \boldsymbol{\Omega}_z$ be the components of $\boldsymbol{\Omega}$ in cartesian coordinates. Then (1) reduces to

$$\boldsymbol{\Omega}_x\mathbf{i} + \boldsymbol{\Omega}_y\mathbf{j} + \boldsymbol{\Omega}_z\mathbf{k} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so that $$\boldsymbol{\Omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \boldsymbol{\Omega}_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \boldsymbol{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Note. Some authors use $\xi, \eta, \zeta,$ for $\boldsymbol{\Omega}_x, \boldsymbol{\Omega}_y, \boldsymbol{\Omega}_z$ and define $\boldsymbol{\Omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = (1/2) \times \text{curl}\,\mathbf{q}$. Thus, we have

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Remark 1. In the two-dimensional cartesian coordinates, the vorticity is given by

$$\boldsymbol{\Omega}_z = \partial v/\partial x - (\partial u/\partial y)$$

Remark 2. In the two-dimensional polar coordinates the vorticity is given by

$$\Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 3. The vorticity components in cylindrical polar coordinates (r, θ, z) are given by

$$\boldsymbol{\Omega}_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \qquad \boldsymbol{\Omega}_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \qquad \boldsymbol{\Omega}_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 4. The vortcity components in spherical polar coordinates (r, θ, ϕ) are given by

$$\boldsymbol{\Omega}_r = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi}{r}\cot\theta, \quad \boldsymbol{\Omega}_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}, \quad \boldsymbol{\Omega}_\phi = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

2.28. Vortex line **[Agra 2004, 2009; Garhwal 2005]**

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\boldsymbol{\Omega}$.

Let $\boldsymbol{\Omega} = \boldsymbol{\Omega}_x\mathbf{i} + \boldsymbol{\Omega}_y\mathbf{j} + \boldsymbol{\Omega}_z\mathbf{k}$ and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a vortex line. Then $\boldsymbol{\Omega}$ is parallel to $d\mathbf{r}$ at P on the vortex line. Hence the equation of vortex lines is given by

$$\boldsymbol{\Omega} \times d\mathbf{r} = \mathbf{0},$$

i.e.
$$(\Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$$

or $$(\Omega_y dz - \Omega_z dy)\mathbf{i} + (\Omega_z dx - \Omega_x dz)\mathbf{j} + (\Omega_x dy - \Omega_y dx)\mathbf{k} = \mathbf{0}$$

whence $$\Omega_y dz - \Omega_z dy = 0, \qquad \Omega_z dx - \Omega_x dz = 0, \qquad \Omega_x dy - \Omega_y dx = 0$$

so that $$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \qquad \text{...(1)}$$

(1) gives the disired equations of vortex lines.

2.29. Vortex tube and vortex filament.

It we draw the vortex lines from each point of a closed curve in the fluid. we obtain a tube callled the *vortex tube*.

A vortex tube of infinitesimal cross-section is known as vortex *filament* or simply a *vortex*.

Remark. It will be shown that vortex lines and tubes cannot originate or terminate at internal points in a fluid. They can only form closed curves or terminate on boundaries. [For proof, refer Art. 11.2 of chapter 11].

2.30. Rotational and irrotationl motion.

[Agra 2011; Garhwal 2005; I.A.S. 2000; G.N.D.U. Amritser 2003; Meerut 2002, 09, 10]

The motion of a fluid is said to be *irrotational* when the vorticity vector $\boldsymbol{\Omega}$ of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be *rotational*.

Since $\boldsymbol{\Omega} = \text{curl}\,\mathbf{q}$ and $$\boldsymbol{\Omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k},$$

we conclude that the motion is irrotation if $\text{curl}\,\mathbf{q} = \mathbf{0}$

or $$\partial w/\partial y = \partial v/\partial z, \qquad \partial u/\partial z = \partial w/\partial x, \qquad \partial v/\partial x = \partial u/\partial y,$$

When the motion is irrotational i.e. when $\text{curl}\,\mathbf{q} = \mathbf{0}$, then $\mathbf{q}$ must be of the form (– grad ϕ) for some scalar point function ϕ (say) because curl grad ϕ = 0. Thus velocity potential exists whenever the fluid motion is irrotational. Again notice that when velocity potential exists, the motion is irrotational because $\mathbf{q} = -\,\text{grad}\,\phi \Rightarrow \text{curl}\,\mathbf{q} = -\,\text{curl grad}\,\phi = \mathbf{0}$.

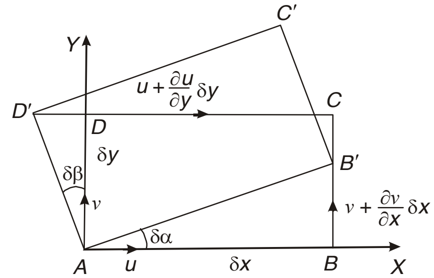
Thus, the fluid motion is irrotational if and only if the velocity potential exists. **(Meerut 2009, 10)**

Rotational motion is also said to be *vortex motion*. Again by definition it follows that there are no vortex lines in an irrotational fluid motion.

2.31. The angular velocity vector. **[Kanpur 2003]**

Consider a rectangular element in two-dimensional flow such that $AB = \delta x$ and $AD = \delta y$ as shown in the figure. Upon rotating about A during a small interval δt, let the element assume the shape indicated by $A'B'C'D'$ in figure, B' and D' approximately lying on BC and CD produced.

Let u, v be the components of velocity at A. Then the components of velocity along BC and DC are respectively $v + (\partial v/\partial x)\delta x$ and $u + (\partial u/\partial y)\delta y$.



$\therefore$ velocity of B relative to A along $BC = \dfrac{\partial v}{\partial x}\delta x$

and velocity of D relative to A along $DC = \dfrac{\partial u}{\partial y}\delta y$.

$$\therefore \quad BB' = \frac{\partial v}{\partial x}\delta x \delta t \qquad \text{and} \qquad DD' = -\frac{\partial u}{\partial y}\delta y \delta t$$

Hence, the angular velocity of AB about z-axis *i.e.* perendicular to the plane through A

$$= \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{\tan \delta\alpha}{\delta t} \qquad [\because\ \delta\alpha \text{ is small } \Rightarrow\ \delta\alpha = \tan\delta\alpha]$$

$$= \lim_{\delta t \to 0} \frac{BB'/\delta x}{\delta t} = \lim_{\delta t \to 0} \frac{\frac{\partial v}{\partial x}\delta x\,\delta t}{\delta x\,\delta t} = \frac{\partial v}{\partial x}.$$

Again, the angular velocity of AD about z-axis

$$= \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{\tan \delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{DD'/\delta y}{\delta t} = -\lim_{\delta t \to 0} \frac{\frac{\partial u}{\partial y}\delta y\,\delta t}{\delta y\,\delta t} = -\frac{\partial u}{\partial y}.$$

Let $\overline{\omega}_z$ denote the averge of the angular velocities of AB and AD. Then, we have

$$\overline{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \qquad \text{...(1)}$$

The average angular velocity components $\overline{\omega}_x$, $\overline{\omega}_y$ and $\overline{\omega}_z$ in the case of three-dimensional flows may be obetained in a similar manner as follows:

$$\overline{\omega}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \overline{\omega}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \overline{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{...(2)}$$

Hence the angular velocity vector $\boldsymbol{\omega}$ of a fluid element is given by

$$\boldsymbol{\omega} = \mathbf{i}\,\overline{\omega}_x + \mathbf{j}\,\overline{\omega}_y + \mathbf{k}\,\overline{\omega}_z$$

or

$$\boldsymbol{\omega} = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

Thus, $\boldsymbol{\omega} = (1/2) \times \text{curl } \mathbf{q}$ or $2\boldsymbol{\omega} = \text{curl } \mathbf{q}$ ...(3)

But the vorticity vector Ω is given by $\boldsymbol{\Omega} = \text{curl } \mathbf{q}$ ...(4)

From (3) and (4), we have $\boldsymbol{\Omega} = 2\boldsymbol{\omega}$

Remark 1. $\boldsymbol{\omega}$ is also called the *rotation*. The condition for the two dimensional flow to be irrotational is that the rotation w_z is everywhere zero *i.e.* $\partial v/\partial x = \partial u/\partial y$.

Again, the condition for irrotationality in three-dimendional flow is that,

$$\Omega_x = \Omega_y = \Omega_z = 0 \text{ everwhere in the flow, } i.e.$$

$$\partial w/\partial y = \partial v/\partial z \qquad \partial u/\partial z = \partial w/\partial x \qquad \partial v/\partial x = \partial u/\partial y$$

Remark 2. A flow, in which the fluid particle also rotate (*i.e.* possess some angular velocity) about their own axes,while flowing, is said to be a *rotational flow*. Again a flow, in which the fluid particles do not rotate about their own axes, and retain their original orientations, is said to be an *irrotational flow*.

2.32. Illustrative solved examples.

Ex. 1. *Give examples of irrotational and rotational flows.* **[Agra 2011, Garhwal 2005]**

Sol. Consider parallel flow with uniform velocity. For example, let there be a fluid motion with the following velocity components:

$$u = kx, \quad k \neq 0 \qquad v = 0, \qquad w = 0 \qquad \text{...(1)}$$

Then, $\partial w/\partial y = \partial v/\partial z$ $\quad \partial u/\partial z = \partial w/\partial x$ $\quad \partial v/\partial x = \partial u/\partial y$

Hence the flow is irrotational.

Next, consider a two-dimensional shear flow with the following velocity components;

$$u = ky, \quad v = 0, \quad w = 0, \quad (k \neq 0) \qquad ...(2)$$

Then $\quad \Omega_z = \partial v/\partial x - \partial u/\partial y = -k \neq 0$

Hence the rotation Ω_z is non-zero and so the flow is rotational.

Ex. 2. *Determine the vorticity components when velocity distribution is given by* $\mathbf{q} = \mathbf{i}\,(Ax^2yt) + \mathbf{j}\,(By^2zt) + \mathbf{k}\,(Czt^2)$ *where A, B, and C are constants.*

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Hence, here we have

$$u = Ax^2yt, \qquad v = By^2zt, \qquad w = Czt^2 \qquad ...(1)$$

The vorticity components $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ are given by

$$\mathbf{\Omega}_x = \partial w/\partial y - (\partial v/\partial z) = 0 - By^2t = -By^2t, \qquad \mathbf{\Omega}_y = \partial u/\partial z - (\partial w/\partial x) = 0 - 0 = 0$$

and $\quad \mathbf{\Omega}_z = \partial v/\partial x - (\partial u/\partial y) = 0 - Ax^2t = -Ax^2t.$

Ex. 3. (*a*) *Test whether the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2}$ $(k = \text{const})$, *is a possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.*

[Kanpur 2006; I.A.S. 1996, Rohilkhand 2003, 04]

(*b*) *Determine the velocity potential for the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2}$, $(k = \text{const})$.

[Agra 2007]

Sol. (*a*) Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then here

$$u = -\frac{k^2 y}{x^2 + y^2}, \qquad v = \frac{k^2 x}{x^2 + y^2}, \qquad w = 0 \qquad ...(1)$$

The equation of continuity for an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(2)$$

Form (1), $\quad \dfrac{\partial u}{\partial x} = \dfrac{2k^2xy}{(x^2 + y^2)^2}, \qquad \dfrac{\partial v}{\partial y} = -\dfrac{2k^2xy}{(x^2 + y^2)^2}, \qquad \dfrac{\partial w}{\partial z} = 0$

Hence (2) is satisfied and so the motion specified by given $\mathbf{q}$ is possible.

The equation of the streamlines are $\quad dx/u = dy/v = dz/w$

i.e.
$$\frac{dx}{-k^2y/(x^2 + y^2)} = \frac{dy}{k^2x/(x^2 + y^2)} = \frac{dz}{0} \qquad ...(3)$$

Taking the last fraction, $\quad dz = 0 \quad$ so that $\quad z = c_1. \qquad ...(4)$

Taking the first two fractions in (3) and simplifying, we get

$$dx/(-y) = dy/x \qquad \text{or} \qquad 2xdx + 2ydy = 0$$

Integrating, $\quad x^2 + y^2 = c_2$, c_2 being an arbitrary constant $\qquad ...(5)$

(4) and (5) together give the streamlines. Clearly, the streamlines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

Again $\quad \text{curl}\,\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{k^2 y}{x^2+y^2} & \dfrac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} = k^2\left\{\dfrac{y^2-x^2}{(x^2+y^2)^2} + \dfrac{x^2-y^2}{(x^2+y^2)^2}\right\}\mathbf{k} = \mathbf{0}.$

Hence the flow is of the potential kind and we can find velocity potential $\phi\ (x, y, z)$ such that $\mathbf{q} = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \qquad ...(6)$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2} \qquad ...(7)$$

$$\partial\phi/\partial z = -w = 0 \qquad ...(8)$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x, y)$.

Integrating (6), $\phi\ (x, y) = k^2 \tan^{-1}(x/y) + f(y)$, where $f(y)$ is an arbitrary function, ...(9)

From (9), $$\frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2} \qquad ...(10)$$

Comparing (7) and (10), we have $f'(y) = 0$ so that

$f(y)$ = constant.

Since the constant can be omitted while writting velocity potential, the required velocity potential can be taken as [refer equation (9)]

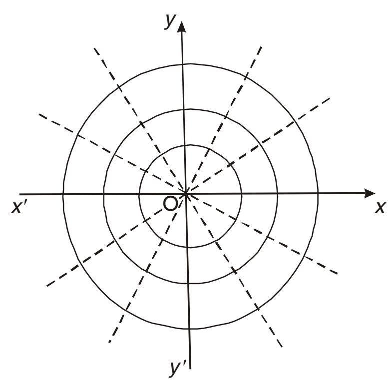
$$\phi(x, y) = k^2 \tan^{-1}(x/y) \qquad ...(11)$$

The equipotentials are given by

$$k^2 \tan^{-1}(x/y) = \text{constant} = k^2 \tan^{-1} c$$

or $\quad x = cy$, c being a constant

which are planes through the z-axis. They are intersected by the streamlines as shown in the figure. Dotted lines represent equipotentials and ordinary lines represent streamlines.



(*b*) Proceed as in part (*a*) upto equation (11). Then the required velocity potential is given by (11).

Ex. 4. *The velocity in the flow field is given by*

$$\mathbf{q} = \mathbf{i}\,(Az - By) + \mathbf{j}\,(Bx - Cz) + \mathbf{k}\,(Cy - Ax)$$

where A, B, C are non-zero constants. Determine the equation of the votex lines.

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, Then we have

$$u = Az - By, \qquad v = Bx - Cz, \qquad w = Cy - Ax \qquad ...(1)$$

Let $\Omega_x, \Omega_y, \Omega_z$ be vorticity components. Then

$$\boldsymbol{\Omega}_x = \partial w/\partial y - (\partial v/\partial z) = C + C = 2C, \qquad \boldsymbol{\Omega}_y = \partial u/\partial z - (\partial w/\partial x) = A + A = 2A,$$

and $$\boldsymbol{\Omega}_z = \partial v/\partial x - (\partial u/\partial y) = B + B = 2B.$$

The equation of the vortex lines are $dx/\Omega_x = dy/\Omega_y = dz/\Omega_z$

i.e. $dx/(2C) = dy/(2A) = dz/(2B)$ ...(2)

Taking the first two members in (2) and integrating, we get

$Ax - Cy = C_1$, C_1 being an arbitrary constant ...(3)

Next, taking the last two membes in (2) and integrating, we get

$By - Az = C_2$, C_2 being an arbitrary constants ...(4)

The required vortex lines are the straight lines of the intersection of (3) and (4).

Ex. 5. *At a point in an incomressible fluid having spherical polar co-ordinates* (r, θ, ϕ), *the velocity components are* $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, *where M is a constant. Show the velocity is of the potential kind. Find the velocity potential and the equations of the stream lines.*

Sol. Here $q_r = 2Mr^{-3}\cos\theta$, $q_\theta = Mr^{-3}\sin\theta$, $q_\phi = 0$.

Then, we have $\mathbf{q} = 2Mr^{-3}\cos\theta\,\mathbf{e_r} + Mr^{-3}\sin\theta\,\mathbf{e_\theta} + 0\,\mathbf{e_\phi}$ and hence

$$\text{curl}\,\mathbf{q} = \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e_\theta} & r\sin\theta\,\mathbf{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ q_r & q_\theta & q_\phi \end{vmatrix} = \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e_\theta} & r\sin\theta\;\mathbf{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 2\mathbf{M}r^{-3}\cos\theta & \mathbf{M}r^{-3}\sin\theta & 0 \end{vmatrix}$$

$= 0$, on simplification

Hence the flow is of the potential kind.

Let $F(r, \theta, \phi)$ be the required velocity potential. We have used F for velocity potential to avoid confusion. Then by definition

$$-\frac{\partial F}{\partial r} = q_r = 2Mr^{-3}\cos\theta, \quad -\frac{1}{r}\frac{\partial F}{\partial\theta} = q_\theta = Mr^{-3}\sin\theta, \quad \text{and} \quad \frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi} = q_\phi = 0$$

$$\therefore \qquad dF = \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial\theta}d\theta + \frac{\partial F}{\partial\phi}d\phi.$$

or $$dF = -(2Mr^{-3}\cos\theta)\,dr - (Mr^{-2}\sin\theta)\,d\theta + 0\cdot d\phi = d(Mr^{-2}\cos\theta)$$

Integrating, $F = Mr^{-2}\cos\theta$. omiting constant of integration, for it has no significance in F)

Finally, the streamlines are given by

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\phi}{q_\phi} \qquad i.e., \qquad \frac{dr}{2Mr^{-3}\cos\theta} = \frac{rd\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta d\phi}{0}$$

given $d\phi = 0$ and $2\cot\theta\,d\theta = (1/r)\,dr$.

Integrating, the equation of the streamlines are given by

$\phi = C_1$ and $r = C_2\sin^2\theta$, C_1 and C_2 being arbitrary constants.

The equation ϕ = constant shows that the required streamlines lie in a plane which pass through the axis of symmetry $\theta = 0$.

Ex. 6. (*a*) *Show that* $u = -\dfrac{2xyz}{(x^2+y^2)^2}$, $v = \dfrac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w = \dfrac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrotational.

[Garhwal 2004; Agra 2004; Kerala 2001; I.A.S. 2000, 2002 Meerut 2002, 04]

(*b*) *Show that a fluid of constant density can have a velocity* **q** *given by*

$$\mathbf{q} = \left[-\frac{2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{x^2+y^2} \right]$$

Find the vorticity vector. **[Kanpur 2007; I.A.S. 1988, 98, 2000]**

Sol. Part (a). Here, we have

$$\frac{\partial u}{\partial x} = -2yz\frac{1\cdot(x^2+y^2)^2 - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = -2yz\frac{x^2+y^2-4x^2}{(x^2+y^2)^3} = -2yz\frac{y^2-3x^2}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = z\frac{-2y(x^2+y^2)^2 - 2(x^2+y^2)\cdot 2y(x^2-y^2)}{(x^2+y^2)^4} = -2yz\frac{x^2+y^2+2(x^2-y^2)}{(x^2+y^2)^3} = -2yz\frac{3x^2-y^2}{(x^2+y^2)^3}$$

and $$\partial w/\partial z = 0$$

Hence the equation of continuity $$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

is satisfied and so the liquid motion is possible.

Furthermore, we have

$$\mathbf{\Omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\mathbf{\Omega}_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

and $$\mathbf{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} = 0$$

$$\therefore \quad \partial w/\partial y = \partial v/\partial z, \qquad \partial u/\partial z = \partial w/\partial x, \qquad \partial v/\partial x = \partial u/\partial y$$

and hence the motion is irrotational.

Part (b). Let **q** = (u, v, w). Then we have the same values of u, v, w as in part (a). By definition, the vorticity vector $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k} = \mathbf{0}, \text{ using part (a)}$$

Ex. 7. *Show that* $\phi = (x-t)(y-t)$ *represents the velocity potential of an incompressible two dimensional fluid. Show that the streamlines at time 't' are the curves*

$$(x-t)^2 - (y-t)^2 = \text{constant},$$

and that the paths of the fluid particles have the equations

$$\log(x-y) = (1/2)\times\{(x+y) - a(x-y)^{-1}\} + b, \text{ where } a, b \text{ are constants.}$$

Sol. Given $$\phi = (x-t)(y-t) \qquad \ldots (1)$$

From (1), we have

$$u = -\partial\phi/\partial x = -(y-t), \qquad v = -\partial\phi/\partial y = -(x-t) \text{ and so } \qquad \partial u/\partial x = 0 = \partial v/\partial y$$

Thus the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied. Hence ϕ given by (1) represents the velocity potential of an incompressible two-dimensional flow.

Again, the equaiton of streamlines are given by

$$\frac{dx}{u} = \frac{dv}{v} \qquad \text{or} \qquad \frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$$

or $$(x-t)\,dx-(y-t)\,dy=0$$

Integrating, $$(x-t)^2-(y-t)^2=\text{constant}$$

Finally, the paths of particles are given by

$$u=dx/dt=-(y-t) \quad \text{and} \quad v=dy/dt=-(x-t)$$

$\therefore$ $$dx/dt=t-y \qquad ...(2)$$

and $$dy/dt=t-x \qquad ...(3)$$

From (2) and (3), $$dx/dt+dy/dt=2t-(x+y) \qquad ...(4)$$

Let $x+y=z$ so that $dx/dt+dy/dt=dz/dt$ ...(5)

Then (4) gives $dz/dt=2t-z$ or $dz/dt+z=2t$ ...(6)

which is a linear differential equation.

Its integrating factor = $e^{\int dt}=e^t$. Here solution of (6) is

$$ze^t=\int 2te^t\,dt+c_1=2t\cdot e^t-\int(2)e^t\,dt+c_1=2te^t-2e^t+c_1$$

$\therefore$ $z=2t-2+c_1e^{-t}$ or $x+y=2t-2+c_1e^{-t}$, by (5) ...(7)

Again from (2) and (3), $dx/dt-(dy/dt)=x-y$

or $$\frac{dx-dy}{dt}=x-y \quad \text{or} \quad \frac{dx-dy}{x-y}=dt$$

Integrating, $\log(x-y)-\log c_2=t$ or $x-y=c_2\,e^t$. ...(8)

Using (7) and (8), we have

$\therefore$ $$x+y-a(x-y)^{-1}=2t-2+c_1e^{-t}-\frac{a}{c_2}e^{-t}=2t-2, \quad \text{taking} \quad c_1=\frac{a}{c_2}$$

$\therefore$ $$(1/2)\times\{(x+y)-a\,(x-y)^{-1}\}=t-1 \qquad ...(9)$$

But from (8), $e^t=(x-y)/c_2$ so that $t=\log(x-y)-\log c_2$

$\therefore$ $$t-1=\log(x-y)-(\log c_2+1)$$

$\therefore$ $t-1=\log(x-y)-b$, taking $b=-(\log c_2+1)$ ...(10)

Using (10), (9) reduces to the required equations

$$(1/2)\times\{(x+y)-a\,(x-y)^{-1}\}=\log(x-y)+b.$$

Ex. 8(*a*). *If the velocity of an incompressible fluid at the point* (x, y, z) *is given by*

$$\left(\frac{3xz}{r^5},\ \frac{3yz}{r^5},\ \frac{3z^2-r^2}{r^5}\right)$$

prove that the liquid motion is possible and that the velcity potential is $(\cos\theta)/r^2$. *Also determine the streamlines.*

Sol. Here $$u=\frac{3xz}{r^5}, \qquad v=\frac{3yz}{r^5}, \qquad w=\frac{3z^2-r^2}{r^5}=\frac{3z^2}{r^5}-\frac{1}{r^3} \qquad ...(1)$$

where $$r^2=x^2+y^2+z^2 \qquad ...(2)$$

From (2), $\partial r/\partial x=x/r$, $\partial r/\partial y=y/r$, $\partial r/\partial z=z/r$ ...(3)

From (1), (2) and (3), we have

$$\frac{\partial u}{\partial x}=3z\left[\frac{1}{r^5}+(-5x)r^{-6}\frac{\partial r}{\partial x}\right]=\frac{3z}{r^5}-\frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y} = 3z\left[\frac{1}{r^5} + (-5y)r^{-6}\frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2 z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - 15z^2\, r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{9z}{r^5} - \frac{15z^3}{r^7}$$

$\therefore$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = \frac{15z}{r^5} - \frac{15z}{r^7}\times r^2 = 0.$$

Since the equation of continuity is satisfied by the given values of u, v and w, the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(udx + vdy + wdz), \text{ by definition of } \phi$$

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2 - r^2}{r^5}dz\right] = \frac{r^2dz - 3z\,(xdx + ydy + zdz)}{r^5}$$

Thus,
$$d\phi = \frac{r^3dz - 3r^2 zdr}{(r^3)^2} = d\left(\frac{z}{r^3}\right), \text{ using (2)}$$

Intergrating ,
$$\phi = z/r^3$$

[Omitting constant of integration, for it has no significance in ϕ]

In spherical polar coordinates (r, θ, ϕ), we know that $z = r\cos\theta$. Hence the required potential is given by
$$\phi = (r\cos\theta)/r^3 = (\cos\theta)/r^2$$

We now obtain the streamlines. The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \textit{i.e.,} \qquad \frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2 - r^2)/r^5}$$

or
$$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - r^2} \qquad \text{...(4)}$$

Taking the first two members of (4) and simplifying, we get

$dx/x = dy/y$ or $dx/x - dy/y = 0$

Integrating, $\log x - \log y = \log c_1$ *i.e.* $x/y = c_1$, c_1 being a constant ...(5)

Now, each member in (4) $= \dfrac{xdx + ydy + zdz}{3x^2z + 3y^2z + 3z^3 - r^2z} = \dfrac{xdx + ydy + zdz}{3z(x^2+y^2+z^2) - r^2 z}$

$$= \frac{xdx + ydy + zdz}{3z(x^2+y^2+z^2) - z(x^2+y^2+z^2)} = \frac{xdx + ydy + zdz}{2z(x^2+y^2+z^2)}, \text{ by (2)} \qquad \text{...(6)}$$

Taking the first member in (4) and (6), we get

$$\frac{dx}{3xz} = \frac{xdx + ydy + zdz}{2z(x^2+y^2+z^2)} \qquad \text{or} \qquad \frac{2}{3}\frac{dx}{x} = \frac{1}{2}\frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}$$

Integrating, $(2/3) \times \log x = (1/2) \times \log (x^2 + y^2 + z^2) + \log c_2$

or $x^{2/3} = c_2\,(x^2 + y^2 + z^2)^{1/2}$, c_2 being an arbitrary constant ...(7)

The required streamlines are the curves of intersction of (5) and (7).

Ex. 8(*b*). If velocity distributon of an incompressible fluid at point (x, y, z) is gives by $\{3xz/r^5,\ 3yz/r^5\ (kz^2 - r^2)/r^5\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential. **[I.A.S. 2001]**

Sol. Here $$u = \frac{3xz}{r^5}, \qquad v = \frac{3yz}{r^5}, \qquad w = \frac{kz^2 - r^2}{r^5} = \frac{kz^2}{r^5} - \frac{1}{r^3}, \qquad ...(1)$$

where $$r^2 = x^2 + y^2 + z^2 \qquad ...(2)$$

From (2), $$\partial r/\partial x = x/r, \qquad \partial r/\partial y = y/r \qquad \text{and} \qquad \partial r/\partial z = z/r \qquad ...(3)$$

Now proceed as in solved Ex. 8(*a*) and obtain

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2 z}{r^7}, \qquad \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2 z}{r^7} \qquad ...(4)$$

and $$\frac{\partial w}{\partial z} = \frac{2kz}{r^5} - 5kz^2 r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7} \qquad ...(5)$$

Since (1) gives a possible liquid motion, the equation of continuity must be satisfied and so

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

or $$\frac{(2k+9)z}{r^5} - \frac{15z}{r^7}(x^2 + y^2 + z^2) = 0 \quad \text{or} \quad \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}\cdot r^2 = 0, \text{ using (2),(4) and (5)}$$

or $(2k - 6)\, z/r^5 = 0$ so that $2k - 6 = 0$ giving $k = 3$.

Substituting the above value of k in (1), we have

$$u = (3xz)/r^5, \qquad v = (3yz)/r^5, \qquad w = (3z^2 - r^2)/r^5. \qquad ...(6)$$

Using (6) and proceeding as in Ex. 8(*a*), the required velocity potential ϕ is given by $\phi = z/r^3$.

Ex. 9. (*a*) *Show that if the velocity potential of an irrotatioanl fluid motion is equal to*

$$A\,(x^2 + y^2 + z^2)^{-3/2}\, z \tan^{-1}(y/x)$$

the lines of flow will be on the series of the surfaces $x^2 + y^2 + z^2 = c^{2/3}\,(x^2 + y^2)^{2/3}$.

[Agra 2004, 06; Kanpur 2002, 11; Meerut 2004]

(*b*) *If the velocity potential of a fluid is* $\phi = (z/r^3)\tan^{-1}(y/x)$ *where* $r^2 = x^2 + y^2 + z^2$, *then show that the streamlines lie on the surfaces* $x^2 + y^2 + z^2 = c\,(x^2 + y^2)^{2/3}$, *c being an arbitrary constant.* **[I.A.S. 2008]**

Sol. (*a*) The velocity potential ϕ is given by

$$\phi(x, y, z) = A(x^2 + y^2 + z^2)^{-3/2} z \tan^{-1}(y/x) = Ar^{-3} z \tan^{-1}(y/x) \qquad ...(1)$$

where $$r^2 = x^2 + y^2 + z^2 \qquad ...(2)$$

so that $$\partial r/\partial x = x/r, \qquad \partial r/\partial y = y/r, \qquad \partial z/\partial r = z/r \qquad ...(3)$$

$\therefore$ $$u = -\frac{\partial \phi}{\partial x} = 3Azxr^{-5}\tan^{-1}\frac{y}{x} + \frac{Azyr^{-3}}{x^2 + y^2}$$

$$v = -\frac{\partial \phi}{\partial y} = 3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{x^2 + y^2}$$

$$w = -\frac{\partial \phi}{\partial z} = 3Az^2 r^{-5}\tan^{-1}\frac{y}{x} - Ar^{-3}\tan^{-1}\frac{y}{x}$$

The equation of lines of flow are given by $dx/u = dy/v = dz/w$

i.e. $$\frac{dx}{3Azxr^{-5}\tan^{-1}\dfrac{y}{x} + \dfrac{Azyr^{-3}}{x^2 + y^2}} = \frac{dy}{3Azyr^{-5}\tan^{-1}\dfrac{y}{x} - \dfrac{Azxr^{-3}}{x^2 + y^2}} = \frac{dz}{A(3z^2 r^{-5} - r^{-3})\tan^{-1}\dfrac{y}{x}} \qquad ...(4)$$

Each member of (4) is $= \dfrac{xdx + ydy + zdz}{(3x^2 + 3y^2 + 3z^2) r^{-2} - 1} = \dfrac{xdx + ydy}{(3x^2 + 3y^2)/r^2}$ (on simplification)

or $$\frac{xdx + ydy + zdz}{2} = \frac{r^2 (xdx + ydy)}{3(x^2 + y^2)}$$

or $$\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{2}{3} \cdot \frac{2xdx + 2ydy}{x^2 + y^2} \qquad ...(5)$$

Integrating (5), $\log (x^2 + y^2 + z^2) = (2/3) \times \log (x^2 + y^2) + (2/3) \times \log c$

or $x^2 + y^2 + z^2 = c^{2/3} (x^2 + y^2)^{2/3}$, c being an arbitrary constant ...(6)

(6) gives the required series of the surfaces on which the desired lines of flow will lie.

(*b*). Proceed like part (*a*) by taking $A = 1$. Thus obtain (5). Integrating (5), $\log (x^2 + y^2 + z^2) = (2/3) \times \log (x^2 + y^2) + \log c$ giving $x^2 + y^2 + z^2 = c (x^2 + y^2)^{2/3}$, c being an arbitrary constant.

Ex. 10. *Given $u = -Wy$, $v = Wx$, $w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of flow.*

Sol. Given $u = -Wy, \quad v = Wx, \quad w = 0$ ...(1)

$\therefore \quad \partial u/\partial x = 0, \quad \partial v/\partial y = 0, \quad \partial w/\partial z = 0$

so that $\partial u/\partial x + \partial y/\partial y + \partial w/\partial z = 0.$ ...(2)

(2) shows that the equation of continuity is satisfied and so the motion specified by (1) is possible. The equations of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w} \qquad i.e., \qquad \frac{dx}{-Wy} = \frac{dy}{Wx} = \frac{dz}{0}$$

giving $xdx + ydy = 0$ and $dz = 0$

Integrating, $x^2 + y^2 = c_1$ and $z = c_2$, c_1 and c_2 being arbitrary constants ...(3)

Hence the streamlines are circles given by the intersection of surfaces (3).

The surfaces which cut the stream lines orthogonally are

$$udx + vdy + wdz = 0$$

i.e. $-Wydx + Wxdy = 0$ or $dx/x - dy/y = 0$

Integrating, $x/y = c$ or $x = cy$, c being an arbitrary constant, ...(4)

which repressents a plane through z-axis and cuts the stream lines (3) orthogonally

Now $udx + vdy + wdz = -Wydx + Wxdy$ ...(5)

Here $\dfrac{\partial}{\partial y}(-Wy) = -W$ and $\dfrac{\partial}{\partial x}(Wx) = W.$ ..(6)

Hence $udx + vdy + wdz$ is not a perfect differential and so the velocity potential does not exist. Again, we have

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -Wy & Wx & 0 \end{vmatrix} = 2W\mathbf{k}.$$

Since curl $\mathbf{q} \neq \mathbf{0}$, the motion is rotational. Notice that a rigid body rotating about z-axis with constant vector angular velocity $2W\mathbf{k}$ will produce the above type of motion.

Ex. 11. *Prove that the liquid motion is possible when velocity at (x, y, z) is given by*

$$u = (3x^2 - r^2)/r^5, \qquad v = 3xy/r^5, \qquad w = 3xz/r^5,$$

where $r^2 = x^2 + y^2 + z^2$*, and the streamlines lines are the intersection of the surfaces* $(x^2 + y^2 + z^2)^3 = c\,(y^2 + z^2)^2$ *by the planes passing through OX. State if the motion is irrotational giving reasons for your answer.* **[Kanpur 2011; Agra 2008]**

Sol. Given $\quad u = (3x^2 - r^2)/r^5, \quad v = 3xy/r^5, \quad w = 3xz/r^5$ ...(1)

For the motion to be possible, we must show that the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(2)}$$

must be satisfied.

From (1), $$\frac{\partial u}{\partial x} = \frac{[6x - 2r(\partial r/\partial x)]r^5 - 5r^4(\partial r/\partial x)(3x^2 - r^2)}{r^{10}} \qquad \text{...(3)}$$

But $$r^2 = x^2 + y^2 + z^2 \qquad \text{...(4)}$$

From (4), $\quad \partial r/\partial x = x/r, \quad \partial r/\partial y = y/r \quad$ and $\quad \partial r/\partial z = z/r$ ...(5)

Using (5), (3) gives

$$\frac{\partial u}{\partial x} = \frac{(6x - 2x)r^5 - 5r^3 x\,(3x^2 - r^2)}{r^{10}} = \frac{3x(3r^2 - 5x^2)}{r^7}$$

Similarly, $$\frac{\partial v}{\partial y} = \frac{3x(r^2 - 5y^2)}{r^7} \quad \text{and} \quad \frac{\partial w}{\partial z} = \frac{3x(r^2 - 5z^2)}{r^7}.$$

$\therefore$ L.H.S. of (2) $= \dfrac{3x[5r^2 - 5(x^2 + y^2 + z^2)]}{r^7} = 0$, using (4)

(2) is satisfied. So the liquid motion is possible. The equation of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{or} \quad \frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} \qquad \text{...(6)}$$

Taking the last two members (6), we get

$dy/y = dz/z \quad$ giving $\quad y = az$, a being an arbitrary constant ...(7)

which is a plane passsing through OX.

Now each member of (6) $= \dfrac{xdx + ydy + zdz}{x(3r^2 - r^2)} = \dfrac{ydy + zdz}{3x(y^2 + z^2)}$

Thus, $$\frac{3(2xdx + 2ydy + 2zdz)}{x^2 + y^2 + z^2} = \frac{2(2ydy + 2zdz)}{y^2 + z^2}$$

Integrating, $\quad 3 \log (x^2 + y^2 + z^2) = 2 \log (y^2 + z^2) + \log c$

or $\quad (x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, c being an arbitrary constant ...(8)

The required streamlines are given by the intersection of surfaces (8) by the planes (7) passing through OX.

Finally, to show that the motion is irrotational, we should verify the conditions:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{...(9)}$$

From (11), we have

$$\frac{\partial u}{\partial y} = -\frac{3y(5x^2 - r^2)}{r^7}, \qquad \frac{\partial u}{\partial z} = -\frac{3z(5x^2 - r^2)}{r^7}, \qquad \frac{\partial v}{\partial x} = \frac{3y(r^2 - 5x^2)}{r^7},$$

$$\frac{\partial v}{\partial z} = -\frac{15xyz}{r^7}, \qquad \frac{\partial w}{\partial x} = \frac{3z(r^2 - 5x^2)}{r^7}, \qquad \frac{\partial w}{\partial y} = -\frac{15xyz}{r^7}.$$

With these values, conditions (9) are all satisfied. Hence the motion is irrotational.

Ex. 12. *Show that in the motion of a fluid in two dimensions if the coordinates (x, y) of an element at any time be expressed in terms of the initial coordinates (a, b) and the time, the motion is irrotational, if* $\dfrac{\partial(\dot{x}, x)}{\partial(a,b)} = \dfrac{\partial(\dot{y}, y)}{\partial(a,b)} = 0.$ [Here $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$].

[Agra 2010; G.N.D.U. Amritser 2003, 05; Kanpur 1999, 2007; Meerut 2003]

Sol. Let u and v be the velocity components parallel to x-and y-axes respectively so that $\dot{x} = dx/dt = u$, $\dot{y} = dy/dt = v$. Now, we have

$$\left.\begin{aligned} \frac{\partial u}{\partial a} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}, \qquad \frac{\partial u}{\partial b} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} \\ \frac{\partial v}{\partial a} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}, \qquad \frac{\partial v}{\partial b} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b} \end{aligned}\right\} \qquad \text{...(1)}$$

$$\therefore \quad \frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = \frac{\partial(u, x)}{\partial(a,b)} + \frac{\partial(v, y)}{\partial(a,b)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial a} & \dfrac{\partial u}{\partial b} \\ \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial v}{\partial a} & \dfrac{\partial v}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \end{vmatrix} = \frac{\partial u}{\partial a}\frac{\partial x}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial v}{\partial b}\frac{\partial y}{\partial a}$$

$$= \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right) + \frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial y}{\partial a}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b}\right),$$

[Using (1)]

$$= \frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right) + \frac{\partial v}{\partial a}\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right) = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\begin{vmatrix} \partial x/\partial a & \partial x/\partial b \\ \partial y/\partial a & \partial y/\partial b \end{vmatrix}$$

$$\therefore \qquad \frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial(x, y)}{\partial(a,b)} \qquad \text{...(2)}$$

Lagrangian equation of continuity in two dimensional case is given by

$$\rho_0 = \rho J \qquad \text{or} \qquad \rho\frac{\partial(x, y)}{\partial(a,b)} = \rho_0 \qquad \text{...(3)}$$

From (3), we find that $\partial(x, y)/\partial(a,b) \neq 0,$ so (2) shows that

$$\frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = 0 \quad \text{if and only if} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$$ *i.e.* the motion is irrotational.

Ex. 13. *Show that all necessary conditions can be satisfied by a velocity potential of the form* $\phi = \alpha x^2 + \beta y^2 + \gamma z^2$, *and a bounding surface of the form* $F = ax^4 + by^4 + cz^4 - \chi(t) = 0$, *where* $\chi(t)$ *is a given function of the time and* $\alpha, \beta, \gamma, a, b, c$ *are suitable functions of the time.*

[Kanpur 2003; Himachel 1994; Gerhwal 1998; I.A.S. 1998; Kurkshetra 2000]

Sol. The given expressions for velocity potential and bounding surface are respectively

$$\phi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 \qquad \text{...(1)}$$

and
$$F(x, y, z, t) = ax^4 + by^4 + cz^4 - \chi(t) \qquad \text{...(2)}$$

The following conditions must be satisfied :

(*i*) ϕ satisfies the Laplace's equation, namely,

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0 \qquad \text{...(3)}$$

(*ii*) F satisfies the condition for boundary surface, namely,

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(4)}$$

From (1), $\quad \partial^2\phi/\partial x^2 = 2\alpha, \quad \partial^2\phi/\partial y^2 = 2\beta,$ and $\quad \partial^2\phi/\partial z^2 = 2\gamma$

Hence (2) will be satisfied if

$$2\alpha + 2\beta + 2\gamma = 0 \quad \text{or} \quad \alpha + \beta + \gamma = 0 \qquad \text{...(5)}$$

for which α, β, γ must be some suitable functions of time.

Now from (2), we have (by using dots for differentiation with respect to time)

$$\partial F/\partial t = x^4\dot{a} + y^4\dot{b} + z^4\dot{c} - \dot{\chi}, \quad \partial F/\partial x = 4ax^3, \quad \partial F/\partial y = 4by^3, \quad \partial F/\partial z = 4cz^3 \qquad \text{...(6)}$$

Remember that if ϕ is the velocity potential function, then u, v, w are given by

$$u = -\partial\phi/\partial x, \quad v = -\partial\phi/\partial y, \quad \text{and} \quad w = -\partial\phi/\partial z \qquad \text{...(7)}$$

Using (1) and (7), we have

$$u = -2\alpha x, \quad v = -2\beta y, \quad \text{and} \quad w = -2\gamma z \qquad \text{...(8)}$$

Using (6) and (8), (4) reduces to

$$x^4(\dot{a} - 8a\alpha) + y^4(\dot{b} - 8b\beta) + z^4(\dot{c} - 8c\gamma) - \dot{\chi} = 0 \qquad \text{...(9)}$$

Since all the points on the surface (2) must also simultaneously satisfy (9), we have

$$\frac{\dot{a} - 8a\alpha}{a} = \frac{\dot{b} - 8b\beta}{b} = \frac{\dot{c} - 8c\gamma}{c} = \frac{\dot{\chi}}{\chi} \qquad \text{...(10)}$$

Taking first and the fourth members of (10), we get

$$\dot{a}/a = 8\alpha + \dot{\chi}/\chi$$

Integrating, $$\log a = 8\int \alpha\, dt + \log \chi \qquad \text{...(11)}$$

Simlarly, $$\log b = 8\int \beta\, dt + \log \chi \qquad \text{...(12)}$$

and $$\log c = 8\int \gamma\, dt + \log \chi \qquad \text{...(13)}$$

In view of (5), α, β and γ are known. Hence equations (11) to (13) determine a, b and c as functions of t.

Thus, velocity potential ϕ given by (1) and the bounding surface $F = 0$ given by (2) satisfy the necessary conditions if a, b, c, α, β and γ are some suitable functions of time.

Ex. 14. *Show that the velocity poremtial* $\phi = (a/2)\times(x^2 + y^2 - 2z^2)$ *satisfies the Laplace equation. Also determine the streamlines.* **[Nagpur 2003, I.A.S. 2002]**

Sol. We know that the velocity **q** of the fluid is given by

$$\mathbf{q} = -\nabla\phi = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{a}{2}(x^2 + y^2 - 2z^2)\right\}$$

or
$$\mathbf{q} = -(a/2) \times (2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}). \qquad ...(1)$$

But
$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}. \qquad ...(2)$$

Comparing (1) and (2), $\quad u = -ax, \quad v = -ay, \quad w = 2az.$

The equaitons of streamlines are given by $\quad dx/u = dy/v = dz/w$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \quad \text{or} \quad \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z}. \qquad ...(3)$$

Taking the first two fractions of (3), $\quad (1/x)dx = (1/y)\,dy.$

Integrating, $\quad \log x = \log y + \log c_1 \quad$ or $\quad x = c_1 y. \qquad ...(4)$

Taking the last two fractions of (3), $\quad (2/y)dy + (1/z)dz = 0$

Integrating, $\quad 2\log y + \log z = \log c_2 \quad$ or $\quad y^2 z = c_2. \qquad ...(5)$

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that
$$\phi = (a/2)\times(x^2 + y^2 - 2z^2). \qquad ...(6)$$

From (6), $\quad \partial\phi/\partial x = ax, \quad \partial\phi/\partial y = ay \quad$ and $\quad \partial\phi/\partial z = -2az$

$\Rightarrow \quad \partial^2\phi/\partial x^2 = a, \quad \partial^2\phi/\partial y^2 = a \quad$ and $\quad \partial^2\phi/\partial z^2 = -2a$

$\therefore \quad \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = a + a - 2a \quad$ or $\quad \nabla^2\phi = 0,$

showing that ϕ satisfies the Laplace equation.

Ex. 15. *Show that* $\phi = x\,f(r)$ *is a possible form for the velocity potential of an incompressible liquid motion. Given that the liquid speed* $q \to 0$ *as* $r \to \infty$, *deduce that the surfaces of constant speed are* $(r^2 + 3x^2)r^{-8}$ = *constant.*

Sol. Given
$$\phi = xf(r). \qquad ...(1)$$

$\therefore$
$$\mathbf{q} = -\nabla\phi = -\nabla[xf(r)] = -[f(r)\nabla x + x\nabla f(r)]. \qquad ...(2)$$

Now, $\quad r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad 2r(\partial r/\partial x) = 2x \quad \Rightarrow \quad \partial r/\partial x = x/r. \qquad ...(3)$

Similarly, $\quad \partial r/\partial y = y/r \quad$ and $\quad \partial r/\partial z = z/r. \qquad ...(4)$

Also,
$$\nabla x = [\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)]x = \mathbf{i}$$

and
$$\nabla f(r) = [\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)]f(r)$$

$$= \mathbf{i}\, f'(r)(\partial r/\partial x) + \mathbf{j}\, f'(r)(\partial r/\partial y) + \mathbf{k}\, f'(r)(\partial r/\partial z)$$

$$= \mathbf{i}\, f'(r)(x/r) + \mathbf{j}\, f'(r)(y/r) + \mathbf{k}\, f'(r)(z/r), \text{ by (3) and (4)}$$

$$= (1/r) f'(r)(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = (1/r) f'(r)\mathbf{r}.$$

$\therefore$ (2) $\Rightarrow$ $$\mathbf{q} = -f(r)\mathbf{i} - (x/r) f'(r)\mathbf{r}. \qquad ...(5)$$

For a possible motion of an incompressible fluid, we have

$$\nabla \cdot \mathbf{q} = 0 \quad \text{or} \quad \nabla \cdot (-\nabla\phi) = 0 \quad \text{or} \quad \nabla^2 \phi = 0$$

or $$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial/\partial z^2)[x\, f(r)] = 0, \text{ using (1)} \qquad ...(6)$$

Now, $$\frac{\partial^2}{\partial x^2}[x\, f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x\, f(r)\}\right] = \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$$

Also $$\frac{\partial^2}{\partial y^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \frac{\partial^2}{\partial z^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ (6) becomes $$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0. \qquad ...(7)$$

Now, $$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'\frac{x}{r}, \text{ using (3).} \qquad ...(8)$$

and $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$$

$$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x} = \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}.$$

$\therefore$ $$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \qquad ...(9)$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f' \qquad ...(10)$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f' \qquad ...(11)$$

Adding (9), (10) and (11), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{x^2+y^2+z^2}{r^2}f'' - \frac{x^2+y^2+z^2}{r^3}f'$$

$$= \frac{3f'}{r} + f'' - \frac{f'}{r}, \text{ as } x^2+y^2+z^2 = r^2.$$

$\therefore$ $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 2f'/r + f'' \qquad ...(12)$$

Using (8) and (12), (7) reduces to

$$\frac{2f'x}{r} + x\left(\frac{2f'}{r} + f''\right) = 0 \qquad \text{or} \qquad f'' + \frac{4f'}{r} = 0$$

or
$$f''/f' + 4/r = 0.$$

Integrating $\quad \log f' + 4\log r = \log c_1 \quad$ so that $\quad f' = c_1 r^{-4}, \quad$ ...(13)

Integrating (13), $\quad f = -(c_1/3) \times r^{-3} + c_2$, c_2 being an arbitrary constant ...(14)

Substituting the values of f' and f from (13) and (14) in (5), we get

$$\mathbf{q} = -\{(c_1/3r^2) - c_2\}\mathbf{i} - (c_1 x/r^5)\,\mathbf{r} \qquad \text{...(15)}$$

Given that $\mathbf{q} \to 0$ as $r \to \infty$, hence (15) shows that $c_2 = 0$.

$\therefore$ from (15),
$$\mathbf{q} = \frac{c_1}{3r^3}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) \qquad \text{...(16)}$$

Now,
$$q^2 = \mathbf{q}\cdot\mathbf{q} = \frac{c_1^2}{9r^6}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)\cdot\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) = \frac{c_1^2}{9r^6}\left[\mathbf{i}\cdot\mathbf{i} - \frac{6x}{r^2}\mathbf{r}\cdot\mathbf{i} + \frac{9x^2}{r^4}\mathbf{r}\cdot\mathbf{r}\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2 r^2}{r^4}\right), \quad \text{as} \quad \mathbf{r}\cdot\mathbf{r} = r^2 \quad \text{and} \quad \mathbf{r}\cdot\mathbf{i} = x$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right) = \frac{c_1^2}{9r^8}(r^2 + 3x^2).$$

Hence the required surfaces of constant speed are

$q^2 =$ constant $\quad$ or $\quad (c_1^2/9r^8)(r^2 + 3x^2) =$ constant $\quad$ or $\quad (r^2 + 3x^2)r^{-8} =$ constant.

Ex. 16. *What is the irrotional velocity field associated with the velocity potential* $\phi = 3x^2 - 3x + 3y^2 + 16t^2 + 12zt$. *Does the flow field satisfy the incompressible continuty equation?*

Sol. The velocity field is given by

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial}{\partial x}(3x^2 - 3x + 3y^2 + 16t^2 + 12zt) = -6x + 3 \qquad \text{...(1)}$$

and
$$v = -\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}(3x^2 - 3x + 3y^2 + 16t^2 + 12zt) = -6y. \qquad \text{...(2)}$$

Here $\quad \partial u/\partial x = -6 \quad$ and $\quad \partial v/\partial x = -6. \quad$ ...(3)

The continuity equation for an incompressible fluid is

$$(\partial u/\partial x) + (\partial v/\partial y) = 0. \qquad \text{...(4)}$$

Using (3) in (4) we find – 6 – 6 = 0, which is absurd. Hence the velocity field given by (1) and (2) does not satisfy the continuity equation (4).

Ex. 17. *The velocity potential function* ϕ *is given by* $\phi = -(xy^3/3) - x^2 + (x^3y/3) + y^2$. *Determine the velocity components in x and y directions and show that* ϕ *represents a possible case of flow.*

Sol. Here $u = -\partial\phi/\partial x = (y^3/3) + 2x - x^2 y,$ $\qquad v = -\partial\phi/\partial y = xy^2 - (x^3/3) - 2y.$

$\therefore \qquad \partial u/\partial x = 2 - 2xy \qquad$ and $\qquad \partial v/\partial y = 2xy - 2.$

Hence $\partial u/\partial x + \partial v/\partial y = 0,$ showing that the continuity equation is satisfied so ϕ represents a possible case of flow.

Ex. 18. *Prove that the velocity potentials* $\phi_1 = x^2 - y^2$ and $\phi_2 = r^{1/2}\cos(\theta/2)$ *are solutions of the Laplace equation and the velocity potential* $\phi_3 = (x^2 - y^2) + r^{1/2}\cos(\theta/2)$ *satisfies* $\nabla^2\phi_3 = 0.$

Sol. The Laplace's equation in cartesian and cylindrical polar coordinates are given by

$$\nabla^2\phi_1 = \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \qquad \text{and} \qquad \nabla^2\phi_2 = \frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r} = 0,$$

Here $\partial^2\phi_1/\partial x^2 = 2$ and $\partial^2\phi_1/\partial y^2 = -2.$ So $\nabla^2\phi_1 = 2 - 2 = 0.$ ...(1)

Next, $\dfrac{\partial\phi_2}{\partial r} = \dfrac{1}{2}r^{-1/2}\cos\dfrac{\theta}{2},\quad \dfrac{\partial^2\phi_2}{\partial r^2} = -\dfrac{1}{4}r^{-3/2}\cos\dfrac{\theta}{2},\quad \dfrac{\partial^2\phi_2}{\partial\theta^2} = -\dfrac{r^{1/2}}{4}\cos\dfrac{\theta}{2}$

$$\therefore \qquad \nabla^2\phi_2 = -\frac{1}{4r^{3/2}}\cos\frac{\theta}{2} - \frac{1}{4r^{3/2}}\cos\frac{\theta}{2} + \frac{1}{2r^{3/2}}\cos\frac{\theta}{2} = 0. \qquad \text{...(2)}$$

(1) and (2) show that ϕ_1 and ϕ_2 satisfy Laplace's equation.

Now, $\qquad \phi_3 = (x^2 - y^2) + r^{1/2}\cos(\theta/2) = \phi_1 + \phi_2$

$\Rightarrow \qquad \nabla^2\phi_3 = \nabla^2(\phi_1 + \phi_2) = \nabla^2\phi_1 + \nabla^2\phi_2 = 0 + 0 = 0,$ by (1) and (2).

Hence ϕ_3 satisfies $\nabla^2\phi_3 = 0.$

Ex. 19. *Find the vorticity of the fluid motion for the given velocity components : (i) $u = A(x + y)$, $v = -A(x + y)$, (ii) $u = 2Axz$, $v = A(c^2 + x^2 - z^2)$, (iii) $u = Ay^2 + By + c$, $v = 0$, Here A, B, C as constants.*

Sol. The vorticity vector $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = (\partial w/\partial y - \partial v/\partial z)\mathbf{i} + (\partial u/\partial z - \partial w/\partial x)\mathbf{j} + (\partial v/\partial x - \partial u/\partial y)\mathbf{k} \qquad \text{...(1)}$$

(*i*) Using (1), $\qquad \mathbf{\Omega} = (0)\mathbf{i} + (0)\mathbf{j} + (-A - A)\mathbf{k} = -2A\mathbf{k}.$

(*ii*) Using (1), $\qquad \mathbf{\Omega} = (0 - 2Az)\mathbf{i} + (2Ax - 0)\mathbf{j} + (2Ax - 0)\mathbf{k} = 2A(-z\mathbf{i} + x\mathbf{j} + x\mathbf{k}).$

(*iii*) Using (1), $\qquad \mathbf{\Omega} = (0)\mathbf{i} + (0)\mathbf{j} + [0 - (2Ay + B)]\mathbf{k} = -(2Ay + B)\mathbf{k}.$

Ex. 20. *Find the vorticity in the spherical coordinates for the velocity components* $v_r = (1 - A/r^3)\cos\theta,$ $v_\theta = -(1 + A/2r^3)\sin\theta,$ $v_\phi = 0.$ *Here A is a constant. Find the nature of the fluid motion.*

Sol. Refer remark 4 of Art. 2.27. Let $\mathbf{\Omega}(\Omega_r, \Omega_\theta, \Omega_\phi)$ be the vorticity vector in the spherical polar coordinates (r, θ, ϕ). Then, we have

$$\Omega_r = \frac{1}{r}\frac{\partial v_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_\phi}{r}\cot\theta = 0, \qquad \Omega_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} = 0,$$

and $\Omega_\phi = \dfrac{\partial v_\theta}{\partial r} + \dfrac{v_\phi}{r} - \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta}$

$$= -\frac{\partial}{\partial r}\left(1+\frac{A}{2r^3}\right)\sin\theta - \frac{1}{r}\left(1+\frac{A}{2r^3}\right)\sin\theta - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(1-\frac{A}{r^3}\right)\cos\theta\right]$$

$$= \frac{3A\sin\theta}{2r^4} - \frac{1}{r}\left(1+\frac{A}{2r^3}\right)\sin\theta + \frac{1}{r}\left(1-\frac{A}{r^3}\right)\sin\theta = \left(\frac{3A}{2r^4} - \frac{1}{r} - \frac{A}{2r^4} + \frac{1}{r} - \frac{A}{r^4}\right)\sin\theta = 0.$$

Since $\Omega_r = \Omega_\theta = \Omega_\phi = 0$, the motion is irrotational.

Ex. 21. *If the fluid be in motion with a velocity potential* $\phi = z\log r$, *and if the density at a point fixed in space be independent of the time, show that the surfaces of equal density are of the forms* $r^2\{\log r - (1/2)\} - z^2 = f(\theta, \rho)$, *where* ρ *is the density at* (z, r, θ)

Sol. The surfaces of equal density are given by $\rho(z, r, \theta) = \text{constant}$

or $\dfrac{D\rho}{Dt} = 0$ or $\dfrac{\partial\rho}{\partial t} + (\mathbf{q}\cdot\nabla)\rho = 0$

or $\dfrac{\partial\phi}{\partial r}\dfrac{\partial\rho}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial\phi}{\partial\theta}\dfrac{\partial\rho}{\partial\theta} + \dfrac{\partial\phi}{\partial z}\dfrac{\partial\rho}{\partial z} = 0$ and $\mathbf{q} = -\nabla\phi$. ...(1)

Also, given $\phi = z\log r$. ...(2)

Then, using (2), (1) reduces to $z(\partial\rho/\partial r) + r\log r\,(\partial\rho/\partial z) = 0$. ...(3)

(3) is of the form of *Lagrange's equation $Pp + Qq = R$ and so here Lagrange's subsidiary equations are

$$\frac{dr}{r} = \frac{dz}{r\log r} = \frac{d\rho}{0}. \qquad ...(4)$$

Third fraction of (4) gives $d\rho = 0$ so that $\rho = c_1$, ...(5)

where c_1 is an arbitary constant.

Taking the first and the second fractions in (4), we have $2r\log r\,dr - 2z\,dz = 0$.

Integrating, $\int 2r\log r\,dr - z^2 = c_2$ or $(\log r)\,r^2 - \int \frac{1}{r}\cdot r^2\,dr - z^2 = c_2$, integrating by parts

or $r^2\log r - (r^2/2) - z^2 = c_2$, c_2 being an arbitrary constant. ...(6)

From (5) and (6), the solution of (3) is given by

$$r^2\{\log r - (1/2)\} - z^2 = f(\theta, \rho),\ f \text{ being an arbitrary function}$$

which are the surfaces of equal density.

EXERCISE 2(F)

1. Show that the following velocity field is a possible case of irrotational flow of an incompressible flow $u = yzt$, $v = zxt$, $w = xyt$.

2. Show that the equation of an incompressible fluid moving irrotationally is given by $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0$, where ϕ is the velocity potential.

3. λ denoting a variable parameter, and f a given function, find the condition that

* Refer chapter 2 of part III in author's "Ordinary and partial differential equations" published by S. Chand & Co., New Delhi.

$f(x, y, \lambda) = 0$ should be a possible system of streamlines for steady irrotational motion in two dimensions.

4. Find the vorticity in polar coordinates for the following velocity components:

(*i*) $v_r = r\sin\theta, v_\theta = 2r\cos\theta$ (*ii*) $v_r = (A/r)\cos\theta, v_\theta = 0$ (*iii*) $v_r = A/r,, v_\theta = 0$

(*iv*) $v_r = \left(1 - A/r^2\right)\cos\theta,$ $v_\theta = -\left(1 + A/r^2\right)\sin\theta - (B/r).$

5. If $u = \dfrac{ax - by}{x^2 + y^2}$, $v = \dfrac{ay + bx}{x^2 + y^2}$, $w = 0$, investigate the nature of the motion of the liquid.

[**Ans.** Irrotational]

6. Establish the relation $\boldsymbol{\Omega} = 2\,\boldsymbol{\omega}$ connecting the angular velocity $\boldsymbol{\omega}$ and the vorticity vector $\boldsymbol{\Omega}$.

(Meerut 2000, 2010)

7. Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and irrotational.

[**Hint.** Since the motion is irrotational, there exists velocity potential ϕ such that $\mathbf{q} = -\nabla\phi$. Further $\partial\rho/\partial t = 0$ as the liquid is incompressible. Hence the equation of continuity $\partial\rho/dt + \rho\nabla\cdot\mathbf{q} = 0$ reduces to $0 + \rho\nabla\cdot(-\nabla\phi) = 0$ or $\nabla^2\phi = 0$

i.e., $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0,$ which is Laplace's equation as required]

8. Prove that a surface of the form $ax^4 + by^4 + cz^4 - \chi(t) = 0$ is a possible form of a boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid being

$$\phi = (\beta - \gamma)x^2 + (\gamma - \alpha)y^2 + (\alpha - \beta)z^2$$

where $\chi, \alpha, \beta, \gamma$ are given functions of time and a, b, c are suitable functions of time.

OBJECTIVE QUESTIONS ON CHAPTER 2

Multiple choice questions

Choose the correct alternative from the following questions

1. If the motion is irrotational, we have

(*i*) $\mathbf{w} = (1/2) \times curl\ \mathbf{q} = \mathbf{0}$ (*ii*) $\mathbf{w} = curl\ \mathbf{q} = \mathbf{0}$

(*iii*) $\mathbf{w} = div\ \mathbf{q} = \mathbf{0}$ (*iv*) None of these. **[Agra 2012; Kanpur 2003]**

2. The condition that the surface $F(x, y, z, t) = 0$ may be bounding surface is

(*i*) $DF/Dt = 1$ (*ii*) $DF/Dt = 0$

(*iii*) $DF/Dt = 2$ (*iv*) None of these **[Kanpur 2002, 2003]**

3. With usual notations

(*i*) $\mathbf{q} = -\nabla\phi$ (*ii*) $\mathbf{q} = \nabla\phi$

(*iii*) $|\mathbf{q}| = \nabla^2\phi$ (*iv*) None of these **[Kanpur 2003]**

4. Differential equations of the path lines are

(*i*) $dx/u = dy/v = dz/w$ (*ii*) $dx/dt = u, dy/dt = v, dz/dt = w$

(*iii*) $dx/\xi = dy/\eta = dz/\zeta$ (*iv*) None of these **[Kanpur 2002]**

5. Velocity potential ϕ satisfies the following equation

(*i*) Bernoulli (*ii*) Cauchy (*iii*) Laplace (*iv*) None of these

6. In usual notations, $\rho\dfrac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_0$, is the equation of continuity in

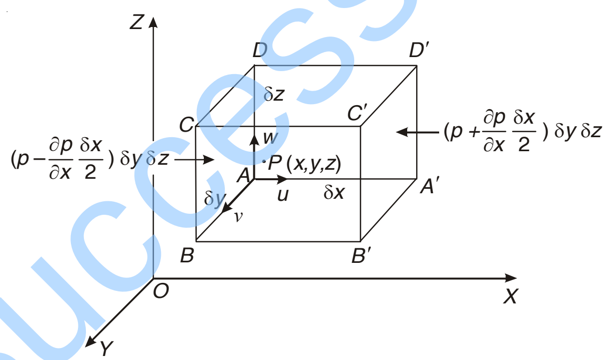
(*i*) Cartesian coordinates (*ii*) Euler's form

Equations of Motion of Inviscid Fluids

Euler's equations of motion.

[Meerut 2012; Kanpur 1999; 02, 04; Agra 2005; Garhwal 2001, 05; G.N.D.U. Amritsar 1999; Rohilkhand 2001; Rajasthan 1997, 98; U.P.P.C.S. 1998, Purvanchel 2004; Kurukshetra 1997]

Let p be the pressure and ρ be density at a point P (x, y, z) in an inviscid (perfect) fluid. Consider an elementary parallelepiped with edges of lengths $\delta x, \delta y, \delta z$ parallel to their respective coordinate axes having P at its centre as shown in figure. Let (u, v, w) be the components of velocity and (X, Y, Z) be the components of external force per unit mass at time t at P. Then if $p = f(x, y, z)$, we have



Force on the plane through P parallel to $ABCD = p\,\delta y\,\delta z$.

$\therefore$ Force on the face $ABCD = f\left(x - \dfrac{1}{2}\delta x, y, z\right)\delta y\,\delta z$

$$= \left\{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \dots\right\}\delta y\,\delta z, \text{ expanding by Taylor's theorem}$$

and force on the face $A'B'C'D' = f\left(x + \dfrac{1}{2}\delta x, y, z\right)\delta y\,\delta z = \left\{f + \dfrac{1}{2}\delta x \dfrac{\partial f}{\partial x} + \dots\right\}\delta y\,\delta z$

$\therefore$ The net force in x-direction due to forces on $ABCD$ and $A'B'C'D'$

$$= \left\{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \dots\right\}\delta y\,\delta z - \left\{f + \frac{1}{2}\delta x \frac{\partial f}{\partial z} + \dots\right\}\delta y\,\delta z$$

$$= -(\partial f/\partial x)\,\delta x\,\delta y\,\delta z, \text{ to first order of approximation}$$

$$= -(\partial p/\partial x)\,\delta x\,\delta y\,\delta z, \qquad \text{as} \qquad p = f(x, y, z)$$

The mass of the element is $\rho\,\delta x\,\delta y\,\delta z$. Hence the external force on the element in x-direction is $X\rho\,\delta x\,\delta y\,\delta z$. Also we know that Du/Dt is the total acceleration of the element in x-direction.

By Newton's second law of motion, the equation of motion in x-direction is

Mass × (acceleration in x-direction) = Sum of the components of external forces in x-direction.

i.e.
$$\rho\,\delta x\,\delta y\,\delta z\,\frac{Du}{Dt} = X\rho\,\delta x\,\delta y\,\delta z - \frac{\partial p}{\partial x}\delta x\,\delta y\,\delta z$$

or
$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}. \qquad \ldots(1)$$

Similarly, the equations of motion in y and z-directions are, respectively

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \ldots(2)$$

and
$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots(3)$$

Re-writing (1), (2) and (3), the so called *Euler's dynamical equations* of motion in cartesian coordinates are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \ldots(4)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \ldots(5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots(6)$$

Alternativ form (Vector method). **[Delhi 1997, Punjab 2003, Kurukshetra 2000]**

Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $F = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$. Then, since

$$\mathbf{i}(\partial p/\partial x) + \mathbf{j}(\partial p/\partial y) + \mathbf{k}(\partial p/\partial z) = \nabla p,$$

(1), (2) and (3) may be combined to yield
$$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(7)$$

which is called the *Euler's equation of motion*.

But
$$\frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{q} \qquad \ldots(8)$$

Using (8), (7) may be re-written as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots(9)$$

Again,
$$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2[\mathbf{q}\times\operatorname{curl}\mathbf{q} + (\mathbf{q}\cdot\nabla)\,\mathbf{q}]$$

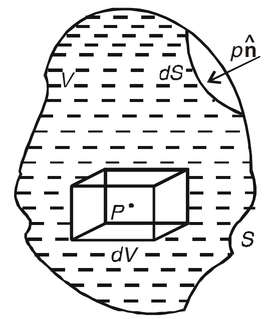
so that
$$(\mathbf{q}\cdot\nabla)\,\mathbf{q} = (1/2)\times\nabla\mathbf{q}^2 - \mathbf{q}\times\operatorname{curl}\mathbf{q} \qquad \ldots(10)$$

Using (10), (9) takes the form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times\operatorname{curl}\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

or
$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q}\times\operatorname{curl}\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla q^2. \qquad \ldots(11)$$

3.1A. The equation of motion of an inviscid fluid (Vector method)



Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid and moving with it, so that it contains the same fluid particles at every instant.

By Newton's second law of motion,
the total force acting on this mass of fluid
= the rate of change in linear momentum ...(1)

The mass of fluid under consideration is subjected to the following two forces : (*i*) The normal pressure thrusts on the boundary.
(*ii*) The external force **F** (say) per unit mass.

Let ρ be the density of the fluid particle P within the closed surface and let dV be the volume enclosing P. The mass of element ρdV will always remain constant. Let **q** be the velocity of fluid particle P, then the momentum **M** of the volume V is given by

$$\mathbf{M} = \int_V \mathbf{q}\,\rho\, dV, \qquad \text{...(2)}$$

where the integral has been taken over the entire volume V.

The time rate of change of linear momentum is given by differentiating (2) w.r.t. 't' as

$$\frac{D\mathbf{M}}{Dt} = \frac{D}{Dt}\int_V \mathbf{q}\,\rho\, dV = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV + \int_V \mathbf{q}\,\frac{D}{Dt}(\rho\, dV)$$

or

$$\frac{D\mathbf{M}}{Dt} = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV, \qquad \text{...(3)}$$

noting that the second integral vanishes because the mass ρdV remains constant for all time.

Here D/Dt is the well known material derivative (or differentiation following the motion of the fluid) and is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla. \qquad \text{... (4)}$$

If **F** be the external force per unit mass acting on fluid particle P, then the total force on the volume V is given by

$$\int_V \mathbf{F}\,\rho\, dV. \qquad \text{...(5)}$$

Finally, if p be the normal pressure thrust at a point of the surface element dS, the total force on the surface S

$= \int_S p\,(-\hat{\mathbf{n}})\, dS$, (negative sign is taken because surface force acts inwards and $\hat{\mathbf{n}}$ is unit vector along the outward normal)

$= -\int_V \nabla p\, dV$, by Gauss theorem

$\therefore$ The total force acting on the volume $V = \int_V \mathbf{F}\,\rho\, dV - \int_V \nabla p\, dV = \int_V (\mathbf{F}\rho - \nabla p)\, dV$. ...(6)

By Newton's second law as stated in (1), we have

$$\int_V (\mathbf{F}\rho - \nabla p)\, dV = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV \quad \text{or} \quad \int_V \left(\rho\frac{D\mathbf{q}}{Dt} - \rho\mathbf{F} + \nabla p\right) dV = 0. \qquad \text{...(7)}$$

Since the volume V enclosed by surface S is arbitrary, (7) holds if

$$\rho \frac{D\mathbf{q}}{Dt} - \rho \mathbf{F} + \nabla p = \mathbf{0} \qquad \text{or} \qquad \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(8)$$

which is known as *Euler's equation of motion*. It is also known as the *equation of motion by flux method*.

Deduction of Lamb's hydrodynamical equations.

Using (4), (8) may be rewritten as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\, \mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p. \qquad \ldots(9)$$

But $\nabla\,(\mathbf{q} \cdot \mathbf{q}) = 2\,[\mathbf{q} \times \text{curl}\,\mathbf{q} + (\mathbf{q} \cdot \nabla)\,\mathbf{q}]$

so that $(\mathbf{q} \cdot \nabla)\,\mathbf{q} = (1/2) \times \nabla\,(\mathbf{q} \cdot \mathbf{q}) - \mathbf{q} \times \text{curl}\,\mathbf{q}$

or $$(\mathbf{q} \cdot \nabla)\,\mathbf{q} = \nabla\,(\mathbf{q}^2/2) + \text{curl}\,\mathbf{q} \times \mathbf{q}. \qquad \ldots(10)$$

Using (10), (9) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\,(\mathbf{q}^2/2) + (\text{curl}\,\mathbf{q}) \times \mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p. \qquad \ldots(11)$$

Now, the vorticity vector $\boldsymbol{\Omega}$ is given by $\boldsymbol{\Omega} = \text{curl}\,\mathbf{q}$. ...(12)

Using (12), (11) may be rewritten as

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\,(\mathbf{q}^2/2) + \boldsymbol{\Omega} \times \mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(13)$$

which is known as *Lamb's hydrodynamical equation*. The main advantage of it lies in the fact that it is invariant under a change of co-ordinate system.

3.1B. Conservative field of force.

In a conservative field of force, the work done by the force $\boldsymbol{F}$ of the field in taking a unit mass from one point to the other is independent of the path of motion.

Thus, if $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, then a scalar point function $V(x, y, z)$ exists such that

$$Xdx + Ydy + Zdz = -\,dV \qquad \text{or} \qquad \mathbf{F} = -\nabla V$$

so that $X = -\partial V/\partial x, \quad Y = -\partial V/\partial y, \quad Z = -\partial V/\partial z.$

V is said to be **force potential** and it measures the potential energy of the field.

3.2A. Euler's equations of motion in cylindrical coordinates.

Euler's equation of motion is $$\frac{D\boldsymbol{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots(1)$$

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the components of external force in r, θ, z directions. Then we know that

$$\frac{D\boldsymbol{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}, \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r}, \frac{Dq_z}{Dt}\right), \qquad \mathbf{F} = (F_r, F_\theta, F_z), \qquad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right).$$

Substituting in (1), and equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we obtain Euler's equations of motion in cylindrical coordinates as:

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\ \frac{Dq_z}{Dt} &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\}, \qquad \ldots(2)$$

where $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial t} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z \frac{\partial}{\partial z}. \qquad \ldots(3)$$

3.2B. Euler's equations of motion in spherical coordinates. [Garhwal 2005]

Euler's equation of motion is $$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots(1)$$

Let (q_r, q_θ, q_ϕ) be the velocity components and (F_r, F_θ, F_ϕ) be the components of external force in r, θ, ϕ directions. Then we know that

$$\frac{D\mathbf{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r}, \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi \cot\theta}{r} \right)$$

$\mathbf{F} = (\mathrm{F}_r, \mathrm{F}_\theta, \mathrm{F}_\phi)$, and $\nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} \right).$

Substituting in (1) and equating the coefficients of **i**, **j**, **k** we obtain Euler's equations of motion in spherical polar coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\ \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi \cot\theta}{r} &= F_\theta - \frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial \phi} \end{aligned}\right\}, \qquad \ldots(2)$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \qquad \ldots(3)$$

3.2C. An important theorem.

If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the streamlines and vortex lines.

Proof. Euler's equation in vector form is given by (Refer equation (11), Art. 3.1)

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl } \mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla q^2 \qquad \ldots(i)$$

For steady flow, $\partial \mathbf{q} / \partial t = \mathbf{0}.$

Since the external forces are conservative, there exists force potential V such that $\mathbf{F} = -\nabla V$. Further, density being a function of pressure p only, there must be a function P such that $\nabla P = (1/\rho)\nabla p$. Using these facts, (i) reduces to

$$\nabla (V + P + q^2/2) = \mathbf{q} \times \text{curl}\, \mathbf{q} \qquad \ldots(ii)$$

Let $\quad \mathbf{\Omega} = \text{curl}\, \mathbf{q} =$ vorticity vector.

Then $$\nabla (V + P + q^2/2) = \mathbf{q} \times \mathbf{\Omega} \qquad \ldots(iii)$$

Let $$\mathbf{n} = \nabla (V + P + q^2/2) \qquad \ldots(iv)$$

Then (*iii*) reduces to $$\mathbf{n} = \mathbf{q} \times \mathbf{\Omega} \qquad \ldots(v)$$

From (*v*), we get $$\mathbf{n} \cdot \mathbf{q} = (\mathbf{q} \times \mathbf{\Omega}) \cdot \mathbf{q} = (\mathbf{q} \times \mathbf{q}) \cdot \mathbf{\Omega} = 0$$

and $$\mathbf{n} \cdot \mathbf{\Omega} = (\mathbf{q} \times \mathbf{\Omega}) \cdot \mathbf{\Omega} = \mathbf{q} \cdot (\mathbf{\Omega} \times \mathbf{\Omega}) = 0.$$

These results show that $\mathbf{n}$ is perpendicular to both $\mathbf{q}$ and $\mathbf{\Omega}$.

Since ∇f is perpendicular everywhere to the surface f = constant, (*iv*) shows that $\mathbf{n}$ is perpendicular to the family of surfaces

$$V + P + q^2/2 = C. \qquad \ldots(vi)$$

Thus $\mathbf{q}$ and $\mathbf{\Omega}$ are both tangential to the surfaces (*vi*). Hence (*vi*) contains the streamlines and vortex lines.

Another Form : *Prove that for steady motion of an inviscid isotropic fluid* $p = f(\rho), \int \frac{dp}{\rho} + \frac{1}{2} q^2 + \Omega = const.$ *over a surface containing the streamlines and vortex lines. Comment on the nature of this constant.*

3.3. Working rule for solving problems.

(*i*) Read and remember all equations of motion given in Art. 3.1, 3.1A, 3.2A and 3.2B. Use an appropriate one in the given problem.

(*ii*) Read and remember all equations of continuity given in Art. 2.8 to 2.14 of chapter 2. Use an appropriate one in the given problem.

(*iii*) Physical relations connecting p and ρ may be used. If the given fluid is at constant temperature, then use $p = k\rho$, where k is a constant. When the change is adiabatic, the relation $p = k\rho^\gamma$ is used.

(*iv*) Given initial and boundary conditions are used.

3.4. Illustrative solved examples.

Ex. 1. *A sphere of radius R, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density* ρ, *which is at rest at infinity. If the pressure at infinity is* Π, *show that the pressure at the surface of the sphere at time t is*

$$\Pi + \frac{1}{2}\rho \left\{ \frac{d^2 R^2}{dt^2} + \left(\frac{dR}{dt} \right)^2 \right\}.$$ **[Kanpur 2008; Meerut 2007; Bombay 2000; I.A.S. 1996]**

If $R = a\,(2 + \cos nt)$, show that, to prevent cavitation in the fluid, Π must not be less than $3\rho a^2 n^2$.

Sol. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin), and time t only. Let p be pressure at a distance r'. Let P be the pressure on the surface of the sphere of radius R and V be the velocity there. Then the equation of continuity is

$$r'^2 v' = R^2 V = F(t) \qquad \ldots(1)$$

From (1), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

Again equation of motion is $$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating with respect to r', (3) reduces to

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \; C \text{ being an arbitrary constant}$$

When $r' = \infty$, then $v' = 0$ and $p = \Pi$ so that $C = \Pi/\rho$. Then, we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \text{or} \qquad p = \Pi + \frac{1}{2}\rho\left[2\frac{F'(t)}{r'} - v'^2\right]. \qquad \ldots(4)$$

But $p = P$ and $v' = V$ when $r' = \mathrm{R}$. Hence (4) gives

$$\mathrm{P} = \Pi + \frac{1}{2}\rho\left[\frac{2}{\mathrm{R}}\{F'(t)\}_{r'=R} - \mathrm{V}^2\right] \qquad \ldots(5)$$

Also $V = dR/dt$. Hence using (1), we have

$$\{F'(t)\}_{r'=R} = \frac{d}{dt}(R^2 V) = \frac{d}{dt}\left(R^2 \frac{d\mathrm{R}}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}\cdot\frac{dR^2}{dt}\right)$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + \frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt} = \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V and $\{F'(t)\}_{r'=R}$, (5) reduces to

$$P = \Pi + \frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

or $$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]. \qquad \ldots(6)$$

Second Part: From $r'^2 v' =$ constant, we conclude that v' is maximum when r' is minimum *i.e.* $r' = R$. So pressure is minimum on $r' = R$ by using Bernoulli's theorem [Refer Chapter 4].

Given $$R = a\,(2 + \cos nt) \qquad \ldots(7)$$

$\therefore$ $$dR/dt = -\,an \sin nt$$

and $$dR^2/dt = 2a^2\,(2 + \cos nt)\,(-\,n \sin nt)$$

$\therefore$ $$d^2R^2/dt^2 = -\,2a^2n^2\,(2 + \cos nt)\cos nt + 2a^2n^2 \sin^2 nt$$

With the above values, (6) reduces to

$$P = \Pi + (3/2)\times\rho a^2 n^2 \sin^2 nt - a^2 n^2 \rho\,(2\cos nt + \cos^2 nt) \qquad \ldots(8)$$

From (7), R varies from $3a$ to a. Thus the sphere has the greatest radius $3a$ when $nt = 0$ or $2m\pi$. Clearly as the sphere shrinks from $R = 3a$, there is a possibility of a cavitation there because pressure would be minimum there. Hence the minimum value of pressure P' (say) on the surface of the sphere is given by replacing $t = 0$ or $nt = 2m\pi$ in (8). We thus obtain

$$P' = \Pi - 3\rho a^2 n^2. \qquad \ldots(9)$$

To prevent cavitation in the fluid. P' given by (9) must be positive *i.e.* Π must not be less than $3\rho a^2 n^2$.

Ex. 2. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure* Π *and contains a spherical cavity of radius a, filled with a gas at pressure* $m\Pi$; *prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na, where n is determined by the equation* $1 + 3m \log n - n^3 = 0$. **(Kanpur 2010)**

If m be nearly equal to 1, *the time of an oscillation will be* $2\pi\sqrt{(a^2\rho/3\pi)}$, ρ *being the density of the fluid.* **[Kanpur 2008; Agra 1998; I.A.S. 1994; Meerut 1999]**

Sol. As in Ex. 1, let at any time t, v' be the velocity at a distance r' and p' the pressure there. Also let v be the velocity at a distance r and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \qquad \ldots(1)$$

From (1),

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

i.e.

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating with respect to r', (3) gives

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{p'}{\rho}, \text{ } C \text{ being an arbitrary constant}$$

When $r' = \infty$, then $v' = 0$, $p' = \Pi$ so that $C = \Pi/\rho$. Hence, the above equation yields

$$\therefore \qquad -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p'}{\rho} \qquad \ldots(4)$$

Since gas inside cavity obeys Boyle's law, we get

$$(4/3)\times \pi a^3 m\Pi = (4/3)\times \pi r^3 p \qquad \text{so that} \qquad p = (a^3 m\Pi)/r^3$$

When $r' = r$ then $v' = v$, $p' = p = (a^3 m\Pi)/r^3$. So (4) gives

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3} \qquad \ldots(5)$$

From (1),

$$F'(t) = 2r\frac{dr}{dt}\cdot v + r^2\cdot\frac{dv}{dt} = 2rv^2 + r^2\frac{dv}{dr}\frac{dr}{dt}, \qquad \text{as} \qquad v = \frac{dr}{dt}$$

or

$$F'(t) = 2rv^2 + r^2 v\,(dv/dr)$$

Hence (5) reduces to

$$-\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3}$$

or $$rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{\Pi}{\rho} + \frac{a^3 m\Pi}{\rho r^3} \qquad \ldots(6)$$

Multiplying both sides of (6) by $2r^2 dr$, we get

$$2r^3 v dv + 3r^2 v^2 dr = \left(-\frac{2\Pi r^2}{\rho} + \frac{2a^3 m\Pi}{\rho r}\right)dr \quad \text{or} \quad d(r^3 v^2) = \left(-\frac{2\Pi r^2}{\rho} + \frac{2a^3 m\Pi}{\rho r}\right)dr$$

Integrating, $r^3 v^2 = -\dfrac{2\Pi}{3\rho}r^3 + \dfrac{2a^3 m\Pi}{\rho}\log r + C'$, C' being an arbitrary constant ...(7)

Initially, when $r = a$, then $v = 0$. Hence (7) $\Rightarrow$ $C' = \dfrac{2\Pi a^3}{3\rho} - \dfrac{2a^3 m\Pi}{\rho}\log a$

$\therefore$ From (7), $$r^3 v^3 = \frac{2\Pi}{3\rho}(a^3 - r^3) + \frac{2a^3 m\Pi}{\rho}\log\left(\frac{r}{a}\right) \qquad \ldots(8)$$

Since the radius of the sphere oscillates between a and na, we have $v = 0$, when $r = a$ and $r = na$. Putting $v = 0$ and $r = na$ in (8), we have

$$0 = \frac{2\Pi}{3\rho}\left\{a^3 - n^3 a^3 + 3ma^3 \log\left(\frac{na}{a}\right)\right\}$$

so that $1 + 3m \log n - n^3 = 0$, as $a \neq 0$

Second Part. Let m be nearly equal to 1. Then, we take $r = a + x$ where x is small. Again, $v = dr/dt = dx/dt = \dot{x}$. Hence, taking $m = 1$, (8) reduces to

$$(a + x)^3 \dot{x}^2 = \frac{2\Pi}{3\rho}\left\{a^3 - (a + x)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(\frac{a + x}{a}\right)$$

or $$a^3\left(1 + \frac{x}{a}\right)^3 \dot{x}^2 = \frac{2\Pi a^3}{3\rho}\left\{1 - \left(1 + \frac{x}{a}\right)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(1 + \frac{x}{a}\right)$$

or $$\left(1 + \frac{x}{a}\right)^3 \dot{x}^2 = \frac{2\Pi}{3\rho}\left\{1 - \left(1 + \frac{3x}{a} + \frac{3x^2}{a^2} + \ldots\right)\right\} + \frac{2a^3\Pi}{\rho}\left\{\frac{x}{a} - \frac{1}{2}\frac{x^2}{a^2} + \ldots\right\}$$

or $$\dot{x}^2 = \frac{2\Pi}{3\rho}\left(1 + \frac{x}{a}\right)^{-3}\left[-\frac{9}{2}\frac{x^2}{a^2} + \ldots\right] = \frac{2\Pi}{3\rho}\left(1 - \frac{3x}{a} + \frac{6x^2}{a^2} - \ldots\right)\left[-\frac{9}{2}\frac{x^2}{a^2} + \ldots\right]$$

or $$\dot{x}^2 = -[3\Pi x^2/\rho a^2], \text{ neglecting higher powers of } x$$

Differentiating the above relation with respect to t, we get

$$2\dot{x}\ddot{x} = -\frac{3\Pi}{\rho a}\cdot 2x\dot{x} \quad \text{or} \quad \ddot{x} = -\frac{3\Pi}{\rho a^2}x,$$

which represents the standard equation of simple harmonic motion and hence the required time of oscillation (*i.e.* periodic time) is given by

$$2\pi/\sqrt{(3\Pi/\rho a^2)} \quad i.e. \quad 2\Pi\sqrt{(\rho a^2/3\Pi)}.$$

Ex. 3. *A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if the shell suddenly disappears, the initial pressure at any point of the fluid at distance r from the centre is*

$$\frac{2}{3}\pi\gamma\rho^2(b-a)(r-a)\left(\frac{a+b}{r}+1\right).$$ **[Bombay 1998]**

Sol. As in Ex. 1, let at time t, v' be the velocity at a distance r' from the centre and let the radius of the inner spherical cavity be r. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad \ldots(1)$$

From (1),

$$\frac{dv'}{dt} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

Attraction at distance $r' = \dfrac{(4/3)\times\pi\gamma\rho(r'^3 - r^3)}{r'^2}$,

where γ is the usual constant of gravitation.

Hence the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

i.e.

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating with respect to r', we obtain

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Initially, when $t = 0$, then $v' = 0$, $r = a$ and $p = P$ (say). Hence (3) yields

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + C \qquad \ldots(4)$$

But, $P = 0$ when $r' = a$ and $r' = b$. So (4) gives

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{a^2}{2} + a^2\right) + C \qquad \ldots(5)$$

and

$$-\frac{F'(0)}{b} = -\frac{4}{3}\pi\gamma\rho\left(\frac{b^2}{2} + \frac{a^3}{b}\right) + C \qquad \ldots(6)$$

Subtracting (6) from (5), we have

$$F'(0)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{4}{3}\pi\gamma\rho\left\{\frac{b^2 - a^2}{2} + a^2\left(\frac{a}{b} - 1\right)\right\}$$

$$\therefore \qquad F'(0)\frac{a-b}{ab} = \frac{4}{3}\pi\gamma\rho\left\{\frac{(b-a)(b+a)}{2} + \frac{a^2(a-b)}{b}\right\}$$

or $$F'(0) = -(2/3) \times \pi \gamma \rho ab(a+b) + (4/3) \times \pi \gamma \rho a^3 \qquad \ldots(7)$$

Multiplying (5) by a and (6) by b and then subtracting, we get

$$0 = \frac{4}{3}\pi\gamma\rho\left(\frac{b^3}{2} - \frac{a^3}{2}\right) + C(a-b)$$

or $$C(b-a) = (2/3)\times\pi\gamma\rho(b-a)(b^2 + a^2 + ba)$$

or $$C = (2/3)\times\pi\gamma\rho(a^2 + b^2 + ab) \qquad \ldots(8)$$

Putting the values of $F'(0)$ and C in (4), we get

$$-\frac{1}{r'}\left\{-\frac{2}{3}\pi\gamma\rho ab(a+b) + \frac{4}{3}\pi\gamma\rho a^3\right\} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + \frac{2}{3}\pi\gamma\rho(a^2 + b^2 + ab)$$

$$\therefore \qquad P = \frac{2}{3}\pi\gamma\rho^2\left\{a^2 + b^2 + ab - \frac{ab(a+b)}{r'} - r'^2\right\}$$

or $$P = \frac{2}{3}\pi\gamma\rho^2(r'-a)(b-r')\left(\frac{a+b}{r'} + 1\right). \qquad \ldots(9)$$

For the required result, replace r' by r in (9).

Ex. 4. *Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated ; prove that if* Π *be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is*

$$\Pi\frac{\log r - \log b}{\log a - \log b}.$$

[Kanpur 2000; Meerut 2000; Agra 1995; I.A.S. 2006]

Sol. Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder $|z| = b$. Hence the free surface would be cylindrical. Thus the liquid velocity v' will be radial and v' will be function of r' (the radial distance from the centre of the cylinder $|z| = b$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t' = F'(t)/r' \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Initially when $t = 0$, $v' = 0$, $p = P$. So (3) $\Rightarrow$ $F'(0)\log r' = -(P/\rho) + C \qquad \ldots(4)$

Again, $P = \Pi$ when $r' = a$ and $P = 0$ when $r' = b$. So (3) yields

$\therefore \qquad F'(0)\log a = -(\Pi/\rho) + C \qquad$ and $\qquad F'(0)\log b = C \qquad \ldots(5)$

Solving (5) for $F'(0)$ and C, we have

$$C = -\log b \frac{\Pi}{\rho \log(a/b)}, \qquad F'(0) = -\frac{\Pi}{\rho \log(a/b)}.$$

Putting these values in (4), we get

$$\frac{P}{\rho} = \frac{\Pi}{\rho \log(a/b)} \log r' - \frac{\Pi}{\rho \log(a/b)} \log b$$

or
$$P = \Pi \frac{\log r' - \log b}{\log(a/b)} = \Pi \frac{\log r' - \log b}{\log a - \log b}. \qquad \ldots(6)$$

For the required result, replace r' by r in (6).

Ex. 5. *A mass of liquid of density* ρ *whose external surface is a long circular cylinder of radius a which is subject to a constant pressure* Π, *surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed; show that if* v *is the velocity at the internal surface, when the radius is r, then*

$$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2 \log\{(r^2 + a^2 - b^2)/r^2\}}.$$ **[Garhwal 2000, Meerut 2006, Kanpur 2011]**

Sol. When the inner cylinder is suddenly destroyed, the motion of the liquid will take place along the radii of the normal sections of the cylinder. Hence the velocity will be function of r' (the radial distance from the centre of the cylinder $|z| = a$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad \ldots(1)$$

From (1), $\qquad \partial v'/\partial t = F'(t)/r' \qquad \ldots(2)$

The equation of motion is $\qquad \dfrac{\partial v'}{\partial t} + v' \dfrac{\partial v'}{\partial r'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or $\qquad \dfrac{F'(t)}{r} + \dfrac{\partial}{\partial r'}\left(\dfrac{1}{2}v'^2\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$, using (2)

Integrating, $\qquad F'(t)\log r' + \dfrac{1}{2}v'^2 = -\dfrac{p}{\rho} + C$, being an arbitrary constant $\qquad \ldots(3)$

Let r and R be the radii of the internal and external surfaces of the cylinder and let v and V be the velocities there at any time t. Hence, we have

When $\quad r' = r, \quad v' = v, \quad p = 0 \qquad \ldots(4)$

and when $\quad r' = R, \quad v' = V, \quad p = \Pi \qquad \ldots(5)$

Using (4) and (5), (2) reduces to

$$F'(t)\log r + v^2/2 = C \qquad \ldots(6)$$

and
$$F'(t)\log R + V^2/2 = -\Pi/\rho + C \qquad \ldots(7)$$

Subtracting (7) from (6), we have

$$F'(t)(\log r - \log R) + (v^2 - V^2)/2 = \Pi/\rho \qquad \ldots(8)$$

From (1), $$rv = RV = F(t) \qquad \ldots(9)$$

But $v = dr/dt$ and $V = dR/dt$. So (9) becomes

$$2rdr = 2RdR = 2F(t)dt \qquad \ldots(10)$$

Also $$R^2 - r^2 = a^2 - b^2 \qquad \ldots(11)$$

From (9), $$F'(t) = \frac{d}{dt}(rv) = \frac{d}{dr}(rv)\cdot\frac{dr}{dt} = v\frac{d}{dr}(rv), \text{ as } v = \frac{dr}{dt}$$

Putting the values of $F'(t)$ and V, (8) gives

$$v\frac{d}{dr}(rv)\cdot\log\frac{r}{R} + \frac{1}{2}\left(v^2 - \frac{r^2v^2}{R^2}\right) = \frac{\Pi}{\rho} \quad \text{or} \quad rv\frac{d}{dr}(rv)\cdot\log\frac{r}{R} + \frac{1}{2}rv^2\left(1 - \frac{r^2}{R^2}\right) = \frac{\Pi r}{\rho}$$

or $$\frac{1}{2}\frac{d}{dr}\left\{(rv)^2\right\}\cdot\log\frac{r}{R} + \frac{1}{2}r^2v^2\left(\frac{1}{r} - \frac{r}{R^2}\right) = \frac{\Pi r}{\rho} \quad \text{or} \quad \frac{d}{dr}\left(\frac{1}{2}r^2v^2\log\frac{r}{R}\right) = \frac{\Pi r}{\rho}, \qquad \ldots(12)$$

where we have used (10) *i.e.* $RdR = rdr$.

Integrating (12), $$\frac{1}{2}r^2v^2\log\frac{r}{R} = \frac{\Pi r^2}{2\rho} + C', \quad C' \text{ being an arbitrary contant} \qquad \ldots(13)$$

But $v = 0$ when $r = b$. So $C' = -\pi b^2/2\rho$.

$\therefore$ From (13), $$r^2v^2\log\frac{r}{R} = \frac{\Pi}{\rho}(r^2 - b^2)$$

or $$r^2v^2\log\left(\frac{r}{R}\right)^2 = \frac{2\Pi}{\rho}(r^2 - b^2)$$

or $$v^2 = \frac{2\Pi(r^2 - b^2)}{\rho r^2\log(r^2/R^2)} = \frac{2\Pi(r^2 - b^2)}{\rho r^2\log(R^2/r^2)^{-1}} = -\frac{2\Pi(r^2 - b^2)}{\rho r^2\log(R^2/r^2)} = \frac{2\Pi(b^2 - r^2)}{\rho r^2\log(R^2/r^2)}$$

Thus, $$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2\log\left\{(r^2 + a^2 - b^2)/r^2\right\}}, \text{ using (11)}$$

Ex. 6. *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at an infinite distance is* Π *and is such that the work done by this pressure on a unit area through a unit of length is one-half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time filling up the cavity will be* $\pi a(\rho/\pi)^{1/2}\left\{2 - (3/2)^{3/2}\right\}$, *a being the initial radius of the cavity, and* ρ *the density of the fluid.*

[Meerut 1999]

Sol. At any time t, let v' be the velocity at a distance r' and p be the pressure there. Let r be the radius of the cavity at that time and v be the velocity there. Then equation of continuity is

$$r'^2v' = \text{F}(t) = r^2v \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{p}{\rho} + C$, C beings an arbitrary constant

But $v' = 0$ and $p = \Pi$ when $r' = \infty$. So $C = \Pi/\rho$. Hence the above equation yields

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} + \frac{\Pi - p}{\rho} \qquad \ldots(3)$$

Also $v' = v$ and $p = 0$ when $r' = r$. So from (3), we get

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\mu}{r} + \frac{\Pi}{\rho} \qquad \ldots(4)$$

From (1), $F'(t) = \frac{d}{dt}(r^2 v) = 2r\frac{dr}{dt}\cdot v + r^2\frac{dv}{dt} = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt}$

$$= 2rv^2 + r^2 v\frac{dv}{dr}, \qquad \text{as} \qquad v = \frac{dr}{dt}$$

Using the above value of $F'(t)$, (4) gives

$$-\frac{1}{r}\left\{2rv^2 + r^2 v\frac{dv}{dr}\right\} + \frac{1}{2}v^2 = \frac{\mu}{r} + \frac{\Pi}{\rho} \qquad \text{or} \qquad 2rvdv + 3v^2 dr = -2\left(\frac{\mu}{r} + \frac{\Pi}{\rho}\right)dr$$

or $\quad 2r^3 vdv + 3v^2 r^2 dr = -2r^2\left(\frac{\mu}{r} + \frac{\Pi}{\rho}\right)dr \qquad$ or $\qquad d(r^3 v^2) = -2\left(\mu r + \frac{\Pi}{\rho}r^2\right)dr$

Integrating, $r^3 v^2 = -\left(\mu r^2 + \frac{2\Pi}{3\rho}r^3\right) + C'$, C' being an arbitrary constant $\qquad \ldots(5)$

Initially, when $r = a$, $v = 0$. So $C' = \mu a^2 + (2\Pi/3\rho)a^3$.

$\therefore$ From (5), $\qquad r^3 v^2 = \mu(a^2 - r^2) + \frac{2\Pi}{3\rho}(a^3 - r^3). \qquad \ldots(6)$

Since the work done by Π is half the work done by the attractive force, we have

$$\Pi \times 1 \times 1 = \frac{1}{2}\int_{\infty}^{a}\left(-\frac{\mu}{r^2}\right)\rho\, dr \qquad \text{so that} \qquad \mu = \frac{2\Pi a}{\rho}.$$

Putting this value of μ in (6), we get

$$r^3 v^3 = \frac{2\Pi a}{\rho}(a^2 - \rho^2) + \frac{2\Pi}{3\rho}(a^3 - r^3)$$

or $\quad r^3 v^2 = \frac{2\Pi}{3\rho}\{3a(a^2 - r^2) + a^3 - r^3\} \qquad$ or $\qquad v^2 = \frac{2\Pi}{3\rho}\frac{\{3a(a^2 - r^2) + a^3 - r^3\}}{r^3}$

or $$\frac{dr}{dt} = -\left(\frac{2\Pi}{3\rho}\right)^{1/2}\frac{\{3a(a^2 - r^2) + a^3 - r^3\}^{1/2}}{r^{3/2}} \qquad \ldots(7)$$

wherein negative sign is taken as r decreases when t increases.

Let T be the time of filling the cavity. Then we have, $r = a$ when $t = 0$ and $r = 0$ when $t = T$. Hence (7) gives on integration

$$\int_0^T dt = -\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_a^0 \frac{r^{3/2}dr}{\{3a(a^2-r^2)+a^3-r^3\}^{1/2}}$$

$$\therefore \qquad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^a \frac{r^{3/2}dr}{(r+2a)\sqrt{(a-r)}} \qquad \ldots(8)$$

Put $r = a\sin^2\theta$ so that $dr = 2a\sin\theta\cos\theta$. Then (8) reduces to

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta\cdot 2a\sin\theta\cos\theta\, d\theta}{a(2+\sin^2\theta)\cdot a^{1/2}\cos\theta} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{\sin^4\theta\, d\theta}{2+\sin^2\theta}$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \left(\sin^2\theta - 2 + \frac{4}{2+\sin^2\theta}\right) d\theta$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[\frac{\pi}{4} - \pi + 4\int_0^{\pi/2} \frac{d\theta}{2+\sin^2\theta}\right] = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{2\sec^2\theta + \tan^2\theta}\right]$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{2+3\tan^2\theta}\right] = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\int_0^{\infty} \frac{dt}{(2/3)+t^2}\right]$$

[Putting $\tan\theta = t$ and $\sec^2\theta\, d\theta = dt$]

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left\{-\frac{3\Pi}{4} + \frac{4}{3}\cdot\sqrt{\frac{3}{2}}\left[\tan^{-1}\left(t\sqrt{\frac{3}{2}}\right)\right]_0^{\infty}\right\} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\left(\frac{3}{2}\right)^{1/2}\cdot\frac{1}{2}\pi\right]$$

$$= a\pi\left(\frac{\rho}{\Pi}\right)^{1/2} \left\{-\frac{3}{2}\times\left(\frac{3}{2}\right)^{1/2} + \frac{4}{3}\times\frac{3}{2}\right\} = \pi a\left(\frac{\rho}{\Pi}\right)^{1/2} \left[2 - \left(\frac{3}{2}\right)^{3/2}\right].$$

Ex. 7. *A spherical hollow of radius a initially exists in an infinite fluid, subject to constant pressure at infinity. Show that the pressure at distance* r' *from the centre when the radius of the cavity is r is to the pressure at infinity as* $3r^2r'^4 + (a^3-4r^3)r'^3 - (a^3-r^3)r^3 : 3r^2r'^4$

[Garhwal 2000]

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there. Again, let v be the velocity of the inner surface of radius r. Then the equation of continuity is

$$r'^2v' = F(t) = r^2v \qquad \ldots(1)$$

From (1), $\qquad \partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$

The equation of motion is $\qquad \dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or $\qquad \dfrac{F'(t)}{r'^2} + \dfrac{\partial}{\partial r'}\left(\dfrac{1}{2}v'^2\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$, using (2)

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Let Π be the pressue at infinity. Thus $v' = 0$ and $p = \Pi$ when $r' = \infty$. So (3) gives $C = \Pi/\rho$. Then (3) reduces to

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \ldots(4)$$

But $p = 0$ and $v' = v$ when $r' = r$. Then (4) gives

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \qquad \ldots(5)$$

From (1), $$F'(t) = \frac{d}{dt}(r^2 v) = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt} = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt}$$

$$= 2rv^2 + r^2 v\frac{dv}{dr} \qquad \left[\because \ v = \frac{dr}{dt}\right]$$

Using the above value of $F'(t)$, (5) gives

$$-\frac{1}{r}\left\{2rv^2 + r^2 v\frac{dv}{dr}\right\} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \quad \text{or} \quad -rv\frac{dv}{dr} - \frac{3}{2}v^2 = \frac{\Pi}{\rho} \qquad \ldots(6)$$

Multiplying both sides by $(-2r^2\,dr)$, (6) gives

$$2r^3 v\,dv + 3r^2 v^2 dr = -\frac{2\Pi}{\rho}r^2 dr \quad \text{or} \quad d(r^3 v^2) = -\frac{2\Pi}{\rho}r^2 dr$$

Integrating, $$r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C', \quad C' \text{ being an arbitrary constant.} \qquad \ldots(7)$$

But when $r = a$, $v = 0$. Hence $C' = (2\Pi a^3)/(3\rho)$

$\therefore$ From (7), $$r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) \qquad \ldots(8)$$

Putting the value of v from (8) in (5), we get

$$F'(t) = r\left(\frac{1}{2}v^2 - \frac{\Pi}{\rho}\right) = r\left[\frac{\Pi}{3\rho}\frac{a^3 - r^3}{r^3} - \frac{\Pi}{\rho}\right]$$

or $$F'(t) = \frac{\Pi}{3\rho}\frac{a^3 - 4r^3}{r^2}. \qquad \ldots(9)$$

From (1), $$v' = (r^2 v)/r'^2 \qquad \ldots(10)$$

Using (9) and (10), (4) reduces to

$$\frac{\Pi - p}{\rho} = -\frac{1}{r'}\cdot\frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2} + \frac{1}{2}\frac{v^2 r^4}{r'^4} = -\frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2 r'} + \frac{\Pi}{3\rho}\cdot\frac{r(a^3 - r^3)}{r'^4}, \quad \text{using (8)}$$

$$\therefore \quad \frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2 r'} - \frac{\Pi}{3\rho}\cdot\frac{r(a^3 - r^3)}{r'^4}$$

or $$\frac{p}{\Pi} = \frac{3r^2 r'^4 + (a^3 - 4r^3) r'^3 - (a^3 - r^3) r^3}{3r^2 r'^4},$$

which gives the required ratio of two pressures under consideration

Ex. 8. *A solid sphere of radius a is surrounded by a mass of liquid whose volume is* $(4\pi c^3)/3$ *and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface, when its radius is x, is given by*

$$\dot{x}^2 x^3 [(x^3 + c^3)^{1/3} - x] = \left(\frac{2\Pi}{3\rho} + \frac{2\mu c^3}{9}\right)(a^3 - x^3)(c^3 + x^3)^{1/3},$$

where ρ *is the density,* Π *the external pressure,* μ *the absolute force and* $\dot{x} = dx/dt$.

[Agra 2000; Himanchal 1999]

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there. Let r and R be the radii and v and V the velocities of the inner and outer surfaces at time t. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v = R^2 V \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$$

The equation of motion is

$$\frac{\partial v'}{dt} + v'\frac{\partial v'}{\partial r'} = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r'},$$ where here $\mu r'^2$ is the attractive force

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{\mu r'^3}{3} - \frac{p}{\rho} + C,$$ C being an arbitrary constant ...(3)

Now, when $r' = r$, $v' = v$ and $p = 0$

and when $r' = R$, $v' = \text{V}$ and $p = \Pi$

$\therefore$ (3) yields $$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{\mu r^3}{3} + \text{C} \qquad \ldots(4)$$

and $$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = -\frac{\mu R^3}{3} - \frac{\Pi}{\rho} + C \qquad \ldots(5)$$

Subtracting (4) from (5), we have

$$F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{2}(v^2 - V^2) = \frac{\mu}{3}(r^3 - R^3) - \frac{\Pi}{\rho}$$

But $(4/3)\times\pi \text{R}^3 - (4/3)\times\pi r^3 = (4/3)\times\pi c^3$ so that $r^3 - R^3 = -c^3$.

$$\therefore \quad F'(t)\left(\frac{1}{r} - \frac{1}{\text{R}}\right) - \frac{1}{2}(v^2 - V^2) = -\frac{\mu c^3}{3} - \frac{\Pi}{\rho} \qquad \ldots(6)$$

From (1), $$F'(t) = \frac{d}{dt}(r^2 v) = \frac{d}{dr}(r^2 v)\cdot\frac{dr}{dt} \quad \text{or} \quad F'(t) = v\frac{d}{dr}(r^2 v) \qquad \ldots(7)$$

Again from (1), we get $$V = (r^2 v)/R^2 \qquad \ldots(8)$$

Using (7) and (8), (6) gives

$$v\frac{d}{dr}(r^2 v)\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-\frac{1}{2}\left(v^2-\frac{r^4 v^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}$$

Multiplying both sides by r^2, we get

$$2r^2 v\frac{d}{dr}(r^2 v)\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-v^2 r^4\left(\frac{1}{r^2}-\frac{r^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}r^2$$

or $$\frac{d}{dr}(vr^2)^2\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-(vr^2)^2\cdot\left(\frac{1}{r^2}-\frac{r^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}r^2 \qquad \ldots(9)$$

From (1), $r^2 v = R^2 V$ or $r^2\dfrac{dr}{dt}=R^2\dfrac{dR}{dt}$

i.e. $$r^2 dr = R^2 dR \qquad \ldots(10)$$

Integrating (9) and using (10), we have

$$r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2(\mu c^3\rho+3\Pi)}{3\rho}\int r^2 dr + C' = -\frac{2(\mu c^3\rho+3\Pi)}{9\rho}r^3 + C'$$

When $r = a$, $v = 0$ so that $C' = \dfrac{2(\mu c^3\rho+3\Pi)}{9\rho}a^3$

$$\therefore \qquad r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2(\mu c^3\rho+3\Pi)}{9\rho}(a^3-r^3)$$

i.e. $$r^4 v^2\left[\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right]=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)$$

Now, for the inner surface, $r = x$, $v=\dot{x}$. Hence, the above relation reduces to

$$\dot{x}^2 x^3[(x^3+c^3)^{1/3}-x]=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-x^3)(x^3+c^3)^{1/3}.$$

Ex. 9. *A sphere is at rest in an infinite mass of homogenous liquid of density* ρ, *the pressure at infinity being P. If the radius R of the sphere varies in such a way that R = a + b cos nt, where b > a, show that pressure at the surface of the sphere at any time is*

$$P+\frac{bn^2\rho}{4}(b-4a\cos nt-5b\cos 2nt).$$ **[Agra 2003, Himachal 2000]**

Sol. Let v' be the velocity at a distance r' at any time t and p' be the pressure there. Again, let v be the velocity on the surface of sphere of radius R, where

$$R = a + b\cos nt \qquad \ldots(1)$$

Then the equation of continuity is $$r'^2 v' = F(t) = R^2 v \qquad \ldots(2)$$

From (2), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(3)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \text{ using (3)}$$

Integrating, $\quad -\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = -\dfrac{p'}{\rho} + C$, C being an arbitrary constant

Given: when $r' = \infty$, $v = 0$, $p' = P$. So $C = P/\rho$. So the above equation gives

$$\therefore \qquad -\frac{F'(t)}{r} + \frac{1}{2}v'^2 = \frac{P - p'}{\rho} \qquad \ldots(4)$$

Let $p' = p$ when $r' = R$. Also, $v' = v$ when $r' = R$. Then, (4) yields

$$\therefore \qquad -\frac{F'(t)}{R} + \frac{1}{2}v^2 = \frac{P - p}{\rho} \qquad \text{or} \qquad p = P + \rho\left[\frac{F'(t)}{R} - \frac{1}{2}v^2\right] \qquad \ldots(5)$$

From (2), $\quad F'(t) = \dfrac{d}{dt}(vR^2) = 2R\dfrac{dR}{dt}\cdot v + R^2\dfrac{dv}{dt}$

$$= 2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2} \qquad \left[\because \ v = \frac{dR}{dt}\right]$$

Using the above value of $F'(t)$ and noting $v = dR/dt$, we have

$$\frac{F'(t)}{R} - \frac{1}{2}v^2 = 2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} - \frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2}$$

$= (3/2) \times (-bn \sin nt)^2 + (a + b\cos nt)(-bn^2 \cos nt)$, using (1)
$= (bn^2/2) \times (3b \sin^2 nt - 2b\cos^2 nt - 2a\cos nt)$
$= (bn^2/4) \times [3b(1 - \cos 2nt) - 2b(1 + \cos 2nt) - 4a\cos nt]$
$= (bn^2/4) \times (b - 4a\cos nt - 5b\cos 2nt)$

Hence (5) reduces to $\quad p = P + \dfrac{bn^2\rho}{4}(b - 4a\cos nt - 5b\cos 2nt)$.

Ex. 10. *For an inviscid, incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by*

$$u_r = V(1 - R^3/r^3)\cos\theta, \qquad u_\theta = -V(1 + R^3/2r^3)\sin\theta, \qquad u_\phi = 0.$$

Show that it is a possible solution of momentum equations (i.e. equations of motion). R and V are constants.

Sol. Here equations of motion in spherical polar coordinates are (refer Art. 14.12)

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(1)$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(2)$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\phi u_\theta\cot\theta}{r} = F_\phi - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} \qquad \ldots(3)$$

For steady flow $(\partial/\partial t = 0)$ with negligible body forces $(F_r = F_\theta = F_\phi = 0)$, the above equations reduces to

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(4)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(5)$$

$$0 = -\frac{1}{\rho r \sin\phi}\frac{\partial p}{\partial \phi} \qquad \ldots(6)$$

Equation (6) shows that p is function of r and θ only.

Given : $u_r = V\left(1 - \frac{R^3}{r^3}\right)\cos\theta,\qquad u_\theta = -V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta \qquad \ldots(7)$

From (7), $\frac{\partial u_r}{\partial r} = \frac{3VR^3}{r^4}\cos\theta,\qquad \frac{\partial u_r}{\partial \theta} = -V\left(1 - \frac{R^3}{r^3}\right)\sin\theta \qquad \ldots(8)$

Using (7) and (8), (4) reduces to

$$V\left(1-\frac{R^3}{r^3}\right)\cos\theta\cdot\frac{3VR^3}{r^4}\cos\theta - \frac{V}{r}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cdot\left[-V\left(1-\frac{R^3}{r^3}\right)\sin\theta\right] - \frac{1}{r}\left[V^2\left(1+\frac{R^3}{2r^3}\right)^2\sin^2\theta\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or
$$\frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta - \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta = -\frac{1}{\rho}\frac{\partial \rho}{\partial r} \qquad \ldots (9)$$

From (7), $\frac{\partial u_\theta}{\partial r} = \frac{3VR^3\sin\theta}{2r^4}$, and $\frac{\partial u_\theta}{\partial \theta} = -V\left(1+\frac{R^3}{2r^3}\right)\cos\theta \qquad \ldots (10)$

Using (7) and (10), (5) reduces to

$$V\left(1-\frac{R^3}{r^3}\right)\cos\theta\cdot\frac{3VR^3}{2r^4}\sin\theta + \frac{1}{r}\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\cos\theta\right]$$
$$+\frac{1}{r}\left[V\left(1-\frac{R^3}{r^3}\right)\cos\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right] = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

or
$$\frac{3V^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial \theta}$$

Differentiating (9) with respect to θ, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cdot 2\cos\theta\,(-\sin\theta) - \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^4}\right)\times 2\sin\theta\cos\theta$$

or
$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \left(-\frac{9V^2R^3}{r^4} + \frac{9V^2R^6}{2r^7}\right)\sin\theta\,\cos\theta \qquad \ldots (12)$$

Next, differentiating (11) with respect to r, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \frac{3V^2R^3}{2}\left(-\frac{3}{r^4}+\frac{6R^3}{r^7}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2}\left(-\frac{3}{r^4}-\frac{6R^3}{2r^7}\right)\times\sin\theta\cos\theta$$

or
$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \left(-\frac{9V^2R^3}{r^4}+\frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \text{... (13)}$$

Since (12) and (13) are identical, the equations of motion (*i.e.*, monemtum equations) are satisfied.

Ex. 11. *The velocity components* $u_r(r,\theta) = -V\left(1-\dfrac{a^2}{r^2}\right)\cos\theta,\, u_\theta(r,\theta) = u\left(1+\dfrac{a^2}{r^2}\right)\sin\theta$ *satisfy the equations of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. u and a are constants.*

Sol. The equations of motion for inviscid incompressible fluid in cylindrical polar coordinates are given by (Refer Art. 14.11)

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{... (1)}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \text{... (2)}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{... (3)}$$

For steady $(\partial/\partial t = 0)$ and two dimensional flow $(\partial/\partial z \equiv 0, u_z = 0)$ with negligible body forces $(F_r = F_\theta = F_z = 0)$, the above equations (1) to (3) reduces to

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{... (4)}$$

$$u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \text{... (5)}$$

and $0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}$, which implies that p is function of r and θ only.

Given
$$u_r = -U\left(1-\frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = U\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad \text{... (6)}$$

Using (6), (4) reduces to

$$\left[-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\right]\left(-\frac{2a^2U}{r^3}\right)\cos\theta + \frac{1}{r}U\left(1+\frac{a^2}{r^2}\right)\sin\theta\left[-U\left(1-\frac{a^2}{r^2}\right)\times(-\sin\theta)\right]$$
$$-\frac{1}{r}\cdot U^2\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or
$$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta + \frac{U^2}{r}\sin^2\theta\left[\left(1-\frac{a^4}{r^4}\right)-\left(1+\frac{a^2}{r^2}\right)^2\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta-\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{... (7)}$$

Again using (6), (5) reduces to

$$-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U\left(-\frac{2a^2}{r^3}\right)\sin\theta+\frac{U}{r}\left(1+\frac{a^2}{r^2}\right)\sin\theta\times U\left(1+\frac{a^2}{r^2}\right)\cos\theta$$

$$-\frac{1}{r}U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U\left(1+\frac{a^2}{r^2}\right)\sin\theta=-\frac{1}{r\rho}\frac{\partial p}{\partial\theta}$$

or $$\frac{2a^2U^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta=-\frac{1}{r\rho}\frac{\partial p}{\partial\theta}$$

or $$\frac{2a^2U^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2a^2U^2}{r^2}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial\theta}$$

or $$\frac{4U^2a^2}{r^2}\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial\theta} \qquad \text{... (8)}$$

Differentiating (7) with respect to θ, we have

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\partial r}=-\frac{4U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta-\frac{4U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta$$

or $$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r}=\frac{8U^2a^2}{r^3}\sin\theta\cos\theta \qquad \text{... (9)}$$

Differentiating (8) with respect to r, we have

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta}=\frac{8U^2a^2}{r^3}\sin\theta\cos\theta \qquad \text{... (10)}$$

Since (9) and (10) are identical, it follows that the given velocity components satisfy the equations of motion.

Since p is function of r and θ, we have

$$dp=(\partial p/\partial r)dr+(\partial p/\partial\theta)d\theta$$

Substituting the values of $\partial p/\partial r$ and $\partial p/\partial\theta$ given by (7) and (8) respectively in the above equation, we obtain

$$\therefore\quad dp=2\rho U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta-\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}dr-\frac{4\rho U^2a^2}{r^2}\sin\theta\cos\theta\,d\theta \qquad \text{... (11)}$$

Let $dp = Mdr + Nd\theta$. Then, by comparison, we have

$$M=2\rho U^2a^2\left\{(1/r^3+a^2/r^5)\sin^2\theta-(1/r^3-a^2/r^5)\cos^2\theta\right\}$$

and $$N=-(4\rho U^2a^2/r^2)\times\sin\theta\cos\theta$$

$$\therefore\quad \frac{\partial M}{\partial\theta}=\frac{\partial}{\partial\theta}\left[2\rho U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta-\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}\right]$$

$$=2\rho U^2a^2\left\{(1/r^3+a^2/r^5)\times 2\sin\theta\cos\theta+(1/r^3-a^2/r^5)\times 2\sin\theta\cos\theta\right\}$$

$$=(8/r^3)\times\rho U^2a^2\sin\theta\cos\theta$$

and $$\frac{\partial N}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{4\rho U^2 a^2}{r^2}\sin\theta\cos\theta\right) = \frac{8\rho U^2 a^2 \sin\theta\cos\theta}{r^3}$$

Thus, $$\partial M/\partial\theta = \partial N/\partial r.$$

Hence (11) must be exact and so its solution by the usual rule of an exact equation is

$$p = 2\rho U^2 a^2\left[\left(-\frac{1}{2r^2} - \frac{a^2}{4r^4}\right)\sin^2\theta - \left(-\frac{1}{2r^2} + \frac{a^2}{4r^4}\right)\cos^2\theta\right] + C$$

or $$p = 2\rho U^2 a^2\left(\frac{\cos 2\theta}{2r^2} - \frac{a^2}{4r^4}\right) + C, \ C \text{ being an arbitrary constant}$$

Ex. 12(*a*). *A steady inviscid incompressible fluid flow has a velocity field u = fx, v = –fy, w = 0, where if is a constant. Derive an expression for the pressure field p(x, y, z) if the pressure p(0, 0, 0) =* p_0 *and* $\boldsymbol{F} = -g\,\boldsymbol{i}\,z$. **[I.A.S. 2006]**

Sol. Given $u = fx, \quad v = -fy, \quad w = 0, \quad f$ being a constant ... (1)

Also, given that $p = p_0$, when $x = 0, \quad y = 0, \quad z = 0$ ... (2)

Again, $\boldsymbol{F} = -g\boldsymbol{i}z \Rightarrow X = 0, \quad Y = 0$ and $Z = -gz$ ... (3)

Equations of motion for steady motion $(\partial/\partial t) = 0$ of an incompressible fluid flow (see Art 3.1) are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = X - (1/\rho)\times(\partial p/\partial x) \quad \text{... (4)}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = Y - (1/\rho)\times(\partial p/\partial y) \quad \text{... (5)}$$

$$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = Z - (1/\rho)\times(\partial p/\partial z) \quad \text{... (6)}$$

Using (1) and (3), (4), (5) and (6) reduce to

$$f^2 x = -(1/\rho)\times(\partial p/\partial x), \qquad -f^2 y = -(1/\rho)\times(\partial p/\partial y), \qquad 0 = -gz - (1/\rho)\times(\partial p/\partial z)$$

$$\Rightarrow \quad \partial p/\partial x = -f^2\rho x, \qquad \partial p/\partial y = f^2\rho y \quad \text{and} \quad \partial p/\partial z = -\rho g z \quad \text{... (7)}$$

Now, $$dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy + (\partial p/\partial z)\,dz$$

$$\therefore \quad dp = -(f^2\rho x)\,dx + (f^2\rho\,y)dy - (\rho g\,z)\,dz, \quad \text{using (7)}$$

Integrating, $p = -(f^2\rho x^2)/2 + (f^2\rho y^2)/2 - (\rho g z^2)/2 + C$, C being a constant ... (8)

Putting $x = y = z = 0$ and $p = p_0$ (see condition (2)), in (8), we get $C = p_0$

Thus, the required expression for the pressure field is given by

$$p(x, y, z) = p_0 - \rho\,(f^2x^2 - f^2y^2 + g\,z^2)/2$$

Ex. 12(b). *For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the vorticity* $\boldsymbol{w}$ *and velocity* $\boldsymbol{q}$ *satisfies*

$$(\boldsymbol{q}\cdot\nabla)\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q}.$$ **[I.A.S. 1989]**

Sol. Vector equation of motion for invicid incompressible fluid is (refer Art. 3.1A)

$$\partial\boldsymbol{q}/\partial t + \nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times curl\ \boldsymbol{q} = \boldsymbol{F} - (1/\rho)\,\nabla p \quad \text{... (1)}$$

Since the motion is steady, $\partial\boldsymbol{q}/\partial t = \boldsymbol{0}$ ... (2)

Since ρ is uniform, $\quad (1/\rho)\,\nabla p = \nabla(p/\rho)$ ... (3)

Since **F** is conservative, $\quad \mathbf{F} = -\nabla\Omega$, where Ω is some scalar function. ... (4)

Again, by definition, $\quad$ vorticity vector = $\boldsymbol{w}$ = curl $\boldsymbol{q}$.

Using (2), (3), (4) and (5) in (1), we obtain

$$\nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times\boldsymbol{w} = -\nabla\Omega - \nabla(p/\rho) \qquad \text{or} \qquad \boldsymbol{q}\times\boldsymbol{w} = \nabla(\boldsymbol{q}^2/2 + \Omega + p/\rho)$$

Taking the curl of both sides of the above equation and using the vector identity curl gral $\phi = \mathbf{0}$, we have

$$\text{curl }(\boldsymbol{q}\times\boldsymbol{w}) = \mathbf{0} \qquad \text{or} \qquad (\nabla\cdot\boldsymbol{w})\,\boldsymbol{q} - (\boldsymbol{q}\cdot\nabla)\,\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q} - (\nabla\cdot\boldsymbol{q})\boldsymbol{w} = \mathbf{0}$$

or $$-(\boldsymbol{q}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{q} = \mathbf{0} \qquad \text{or} \qquad (\boldsymbol{q}\cdot\nabla)\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q}.$$

where we have used the following two results

$$\nabla\cdot\boldsymbol{w} = \nabla\cdot\nabla\times\boldsymbol{q} = 0 \qquad \text{and} \qquad \nabla\cdot\boldsymbol{q} = 0 \text{ (continuity equation).}$$

Ex. 13. *Show that if the velocity field*

$$u(x, y) = \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}, \qquad v(x, y) = \frac{2Bxy}{(x^2 + y^2)^2}, \qquad w(x, y) = 0$$

satisfies the equations of motion for inviscid incompressible flow, then determine the pressure associated with this velocity field, B being a constant.

[Kanpur 2002, 03, 05; Rohilkhand 2000, 05]

Sol. The equations of motion for steady inviscid incompressible flow are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = -(1/\rho)\,(\partial p/\partial x), \quad \text{... (1)}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = -(1/\rho)\,(\partial p/\partial y) \quad \text{... (2)}$$

and $$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = -(1/\rho)\,(\partial p/\partial z). \quad \text{... (3)}$$

From the given values of u, v and w, we have

$$\frac{\partial u}{\partial x} = B\frac{2x(x^2 + y^2)^2 - 4x(x^2 - y^2)\,(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{2Bx(3y^2 - x^2)}{(x^2 + y^2)^3},$$

$$\frac{\partial u}{\partial y} = B\frac{-2y(x^2 + y^2)^2 - 4y(x^2 - y^2)\,(x^2 + y^2)}{(x^2 + y^2)^4} = -\frac{2By(3x^2 - y^2)}{(x^2 + y^2)^3}, \qquad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial x} = 2B\frac{y(x^2 + y^2)^2 - 4x^2 y(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{2By(y^2 - 3x^2)}{(x^2 + y^2)^3},$$

$$\frac{\partial v}{\partial y} = 2B\frac{x(x^2 + y^2)^2 - 4xy^2(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{2Bx(x^2 - 3y^2)}{(x^2 + y^2)^3}, \qquad \frac{\partial v}{\partial z} = 0,$$

$$\partial w/\partial x = 0, \qquad \partial w/\partial y = 0 \qquad \text{and} \qquad \partial w/\partial z = 0.$$

Substituting the given values of u, v and w and also using the above relations, (1), (2) and (3) reduce to

$$\frac{B(x^2 - y^2)}{(x^2 + y^2)^2}\cdot\frac{2Bx(3y^2 - x^2)}{(x^2 + y^2)^3} - \frac{2Bxy}{(x^2 + y^2)^2}\cdot\frac{2By(3x^2 - y^2)}{(x^2 + y^2)^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2By(y^2-3x^2)}{(x^2+y^2)^3}+\frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}=-\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

and $$0=-(1/\rho)(\partial p/\partial z)$$

Simplifying the above equations, we have

$$\frac{2B^2x}{(x^2+y^2)^5}[(x^2-y^2)(3y^2-x^2)-2y^2(3x^2-y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{2B^2y}{(x^2+y^2)^5}[(x^2-y^2)(y^2-3x^2)+2x^2(x^2-3y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$0=\partial p/\partial z$$

Again simplifying the above equations, we have

or $$\frac{2B^2x}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial x}\qquad i.e.,\qquad \frac{2B^2x\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial x}\qquad \ldots(1)$$

$$\frac{2B^2y}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial z}\qquad i.e.,\qquad \frac{2B^2y\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial y}\qquad \ldots(2)$$

and $$0=\partial p/dz. \qquad \ldots(3)$$

Relation (3) shows that the pressure p is independent of z, *i.e.*, $p = p\ (x, y)$. Hence, we have

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy$$

or $$dp=\frac{2B^2x\rho}{(x^2+y^2)^3}dx+\frac{2B^2y\rho}{(x^2+y^2)^3}dy=B^2\rho(x^2+y^2)^{-3}(2xdx+2ydy)$$

or $$dp=B^2\rho(x^2+y^2)^{-3}\,d\,(x^2+y^2).$$

Integrating, $$p=C-(1/2)\times B^2\rho(x^2+y^2)^{-2}=C-\{B^2\rho/2(x^2+y^2)^2\},$$

where C is a constant of integratioon. It gives the required pressure distribution.

Ex. 14. *The particle velocity for a fluid motion referred to rectangular axes is given by the components* $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$, *where A is a constant. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube,* $-a \le x \le a$, $0 \le z \le 2a$. *Also, find the pressure associated with this velocity field.* **[I.A.S. 1994; Meerut 2003]**

Sol. Given $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$. ...(1)

From (1), $\partial u/\partial x=-(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a)$, $\partial v/\partial y=0$

and $\partial w/\partial z=(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a)$. ...(2)

$\therefore$ $$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0,$$

showing that the given velocity components represent a physically possible flow.

The equations of motion for steady inviscid incompressible flow under no body forces are

$$u(\partial u/\partial x)+v(\partial u/\partial y)+w(\partial u/\partial z)=-(1/\rho)(\partial p/\partial x), \qquad \ldots(3)$$

$$u(\partial v/\partial x)+v(\partial v/\partial y)+w(\partial v/\partial z)=-(1/\rho)(\partial p/\partial y) \qquad \ldots(4)$$

and $$u(\partial w/\partial x)+v(\partial w/\partial y)+w(\partial w/\partial z)=-(1/\rho)(\partial p/\partial z). \qquad \ldots(5)$$

From (1) $\partial u/\partial y=0$, $\partial u/\partial z=-(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)$

$\partial v/\partial x=\partial v/\partial z=0$, $\partial w/\partial x=(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)$

and $\partial w/\partial y=0.$ ...(6)

Using (1), (2) and (6), the equations of motion (3), (4) and (5) become

$$-A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}-A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$0 = -(1/\rho)\,(\partial p/\partial y)$$

$$A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}+A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Simplifying the above equations, we have

$$\partial p/\partial x = (\pi\rho A^2/2a)\cos(\pi x/2a)\sin(\pi x/2a), \qquad \text{...(7)}$$

$$\partial p/\partial y = 0 \qquad \text{...(8)}$$

and $$\partial p/\partial z = -(\pi\rho A^2/2a)\cos(\pi z/2a)\sin(\pi z/2a). \qquad \text{...(9)}$$

Equation (8) shows that the pressure p is independent of y so that $p = p(x, z)$. Then

$$dp = (\partial p/\partial x)dx + (\partial p/\partial z)dz$$

or $dp = (\pi\rho A^2/2a)\,[\cos(\pi x/2a)\sin(\pi x/2a)\,dx - \cos(\pi z/2a)\sin(\pi z/2a)\,dz]$, using (7) and (9)

Integrating, $\quad p = (\pi\rho A^2/2a)\,[(a/\pi)\sin^2(\pi x/2a) - (a/\pi)\sin^2(\pi z/2a)] + C$

or $\;p = (\rho A^2/2)\,[\sin^2(\pi x/2a) - \sin^2(\pi z/2a)] + C$, C being a constant of integration. ...(10)

(10) gives the required pressure associated with the velocity field given by (1).

Ex. 15. *Prove that if* $\lambda = (\partial u/\partial t) - v(\partial v/\partial x - \partial u/\partial y) + w(\partial u/\partial z - \partial w/\partial x)$ *and* μ, ν *are two similar expressions, then* $\lambda\, dx + \mu\, dy + \nu\, dz$ *is a perfect differential, if the external forces are conservative and the density is constant.* **[Agra 2006]**

Sol. Let (X, Y, Z) be the components of external forces. Since the external forces are conservative, there exists force potential $V(x, y, z)$ such that

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y \qquad \text{and} \qquad Z = -\partial V/\partial z. \qquad \text{...(1)}$$

Euler's dynamical equations of motion are

$$Du/Dt = X - (1/\rho)\,(\partial p/\partial x), \qquad \text{...(2)}$$

$$Dv/Dt = Y - (1/\rho)\,(\partial p/\partial y) \qquad \text{...(3)}$$

and $$Dw/Dt = Z - (1/\rho)\,(\partial p/\partial z), \qquad \text{...(4)}$$

where $p(x, y, z)$ is the pressure at any point (x, y, z).

Using (1), (2), (3) and (4) can be rewritten as

$$Du/Dt = -\partial V/\partial x - (1/\rho)\,(\partial p/\partial x), \qquad \text{...(5)}$$

$$Dv/Dt = -\partial V/\partial y - (1/\rho)\,(\partial p/\partial y) \qquad \text{...(6)}$$

and $$Dw/Dt = -\partial V/\partial z - (1/\rho)\,(\partial p/\partial z). \qquad \text{...(7)}$$

Multiplying (5), (6) and (7) by dx, dy and dz and then adding, we have

$$\frac{Du}{Dt}dx + \frac{dv}{Dt}dy + \frac{dw}{Dt}dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

or $$\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz = -dV - \frac{1}{\rho}dp. \qquad \text{...(8)}$$

Re-writing the given value of λ, we have

$$\lambda = \frac{\partial u}{\partial t} - v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - w\frac{\partial w}{\partial x}$$

$$= \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)$$

$$= \frac{Du}{Dt} - \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) = \frac{Du}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial x}. \quad \text{...(9)}$$

$$\left[\because \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad \text{and} \quad q^2 = u^2 + v^2 + w^2\right]$$

Similarly, $\mu = \dfrac{Dv}{Dt} - \dfrac{1}{2}\dfrac{\partial q^2}{\partial y}$ and $\nu = \dfrac{Dw}{Dt} - \dfrac{1}{2}\dfrac{\partial q^2}{\partial z}$. ...(10)

$\therefore$ Using (9) and (10), we have,

$$\lambda\, dx + \mu\, dy + \nu\, dz = \frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz - \frac{1}{2}\left[\frac{\partial q^2}{\partial x}dx + \frac{\partial q^2}{\partial y}dy + \frac{\partial q^2}{\partial z}dz\right]$$

$$= -dV - (1/\rho)\,dp - (1/2) \times dq^2 = -d\,[V + (p/\rho) + (1/2) \times q^2],$$

which is a perfect differential which is what we wished to prove.

Ex. 16. *A sphere whose radius at time t is b + a cos nt, is surrounded by liquid extending to infinity under no forces. Prove that the pressure at distance r from the centre is less than the pressure Π at infinity by*

$$\rho\frac{n^2 a}{r}(b + a\cos nt)\left\{a(1 - 3\sin^2 nt) + b\cos nt + \frac{a^3\sin^2 nt}{2r^3}(b + a\cos nt)^3\right\}.$$

Prove also that least pressure at the surface of the sphere during the motion is $\Pi - n^2\rho a(a+b)$.

Sol. Let v' be the velocity of the fluid at a distance r' from the origin at any time t and p be the pressure there. Let $r' = b + a\cos nt$ and let r be the radius of any concentric sphere and v be the velocity there. Then the equation continuity is

$$r'^2 v' = F(t) = r^2 v. \quad \text{...(1)}$$

From (1), $\partial v/\partial t = F(t)/r^2$ ...(2)

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{or} \quad \frac{F'(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \text{ using (2)}$$

Integrating it with respect to r, we have

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \quad \text{...(3)}$$

When $r = \infty$, $v = 0$, $p = \Pi$. So (3) gives $C = \Pi/\rho$. Hence (3) reduces to

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi - p}{\rho}. \quad \text{...(4)}$$

Now, $r' = b + a\cos nt \Rightarrow v' = dr'/dt = -an\sin nt$.

Then, (1) $\Rightarrow F(t) = r'^2 v' = (b + a\cos nt)^2(-an\sin nt)$

or $$F(t) = -an\,(b + a\cos nt)^2 \sin nt. \qquad \text{...(5)}$$

Defferentiating (5) with respect to 't', we have

$$F'(t) = 2a^2n^2\,(b + a\cos nt)\sin^2 nt - an^2(b + a\cos nt)^2 \cos nt$$

or $$F'(t) = an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] \qquad \text{...(6)}$$

Now, (4) $\Rightarrow \Pi - \rho = -(\rho/r)\,F'(t) + (1/2) \times \rho v^2.$ ...(7)

or $$\Pi - \rho = -(\rho/r)F'(t) + (\rho/2)\,\{F(t)/r^2\}^2, \text{ using (1)}$$

Using (5) and (6), the above equation becomes

$$\Pi - \rho = -(\rho/r) \times an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] + (\rho/2r^4) \times a^2n^2(b + a\cos nt)^4 \sin^2 nt$$

$$= (\rho an^2/r) \times (b + a\cos nt)\,\{a(1 - 3\sin^2 nt) + b\cos nt + (a/2r^3) \times \sin^2 nt\,(b + a\cos nt)^3\}$$

Second part : At surface $r = r' = b + a\cos nt$, $v = v' = = dr'/dt = -an\sin t$.

Also, using (6), (4) reduces to

$$\frac{\Pi - p}{\rho} = -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{1}{b + a\cos nt}\,.\,an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] + (1/2) \times a^2n^2\sin^2 nt$$

$$= n^2a\,[a(1 - 3\sin^2 nt) + b\cos nt + (1/2) \times a\sin^2 nt]. \qquad \text{...(8)}$$

For the maximum or minimum of p, we must have

$$\frac{d}{dt}\left(\frac{\Pi - p}{\rho}\right) = 0$$

i.e. $$n^2a\,[-6an\sin nt\cos nt - bn\sin nt + na\sin nt\cos nt] = 0,$$

giving $\sin nt = 0$ or $\cos nt = -(b/5a)$ *i.e.* $nt = 0$ or $nt = \cos^{-1}(-b/5a)$.

Now, $$\frac{d^2}{dt^2}\left(\frac{\Pi - p}{\rho}\right) = \frac{d}{dt}[n^2a\,\{-3an\sin 2nt - bn\sin nt + (1/2) \times an\sin 2nt\}$$

$$= n^2a\,[-6an^2\cos 2nt - bn^2\cos nt + an^2\cos 2nt]$$

$$= n^2a\,[-6an^2 - bn^2 + an^2], \text{ when } nt = 0$$

$$\therefore\ \frac{d^2}{dt^2}\left(\frac{\Pi - p}{\rho}\right) \text{ is negative when } nt = 0 \quad \Rightarrow \quad \frac{d^2p}{dt^2} \text{ is positive when } nt = 0.$$

Putting $nt = 0$ in (8), the least pressure p is given by $(\Pi - p)/\rho = n^2a(a + b)$

and hence the required least pressure $= p = \Pi - \rho n^2a(a + b)$.

Ex. 17. *A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external force. The sphere is forced to vibrate radially keeping its spherical shape, the radius r at any time being given by $r = a + b\cos nt$. Show that if Π is the pressure in the liquid at a great distance from the sphere, the least pressure (assumed positive) at the surface of the sphere during the motion is $\Pi - n^2\rho b(a + b)$.*

Hint. Refer Ex. 16.

Ex. 18. *A volume $(4/3) \times \pi c^3$ of gravitating liquid, of density ρ is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contract under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface is r, its velocity will be given by*

$$V^2 = (4\pi\gamma\rho R/15r^3)\,(2R^4 + 2R^3r + 2R^2r^2 - 3Rr^3 - 3r^4),$$

where γ is the constant of gravitation, and $R^3 = r^3 + c^3$.

Sol. Let r be the radius of the inner surface and r' be the distance of the point from the centre of the spherical shell, where at time t, p is the pressure and v' is the velocity.

Here attraction F at the point of which distance from the centre is r' is given by

$$F = \frac{(4/3)\times\pi\gamma\rho(r'^3 - r^3)}{r'^2} = \frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right). \qquad ...(1)$$

The equation of continuity is $\qquad r'^2 v' = F(t) = r^2 v. \qquad$ ...(2)

From (2), $\qquad \partial v'/\partial t = F'(t)/r'^2 \qquad$ ...(3)

The equation of motion is $\qquad \dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = -F - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ by (1) and (3)}$$

Integrating this with respect to r', we have

$$-\frac{F'(t)}{r'^2} + \frac{1}{2}v'^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \qquad ...(4)$$

Initially, at the inner surface, $r' = r$, $v' = v$ and $p = 0$.

$\therefore$ (4) $\Rightarrow$ $$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{4}{3}\pi\gamma\rho\times\frac{3}{2}r^2 + C. \qquad ...(5)$$

Also, initially, at the outer surface, $r' = R$, $v' = v_1$ (say) and $p = 0$.

$\therefore$ (4) $\Rightarrow$ $$-\frac{F'(t)}{R} + \frac{1}{2}v_1^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{R^2}{2} + \frac{r^3}{R}\right) + C. \qquad ...(6)$$

Subtracting (6) from (5), we have

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}(v^2 - v_1^2) = -\frac{4}{3}\pi\gamma\rho\left(\frac{3r^2}{2} - \frac{R^2}{2} - \frac{r^3}{R}\right). \qquad ...(7)$$

From equation of continuity, we have $\qquad r^2 v = R^2 v_1 = F(t). \qquad$ ...(8)

From (8), $\quad v = F(t)/r^2 \quad$ and $\quad v_1 = F(t)/R^2$. Then (7) becomes

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}\left[\frac{\{F(t)\}^2}{r^4} - \frac{\{F(t)\}^2}{R^4}\right] = -\frac{4}{3}\pi\gamma\rho\left(\frac{3r^2}{2} - \frac{R^2}{2} - \frac{r^3}{R}\right). \qquad ...(9)$$

Now, (8) $\Rightarrow$ $r^2(dr/dt) = R^2(dR/dt) = F(t)$ $\Rightarrow$ $r^2 dr = R^2 dR = F(t)\, dt$ ...(10)

Multiplying each term of (9) by $2r^2\, dr$ or $2R^2 dR$ or $2F(t)\, dt$ (all being equal by virtue of (10)), we have

$$-\left(\frac{1}{r} - \frac{1}{R}\right)\times 2F(t)\,F'(t)\,dt + \frac{\{F(t)\}^2}{2}\left[\frac{2r^2 dr}{r^4} - \frac{2R^2 dR}{R^4}\right] = -\frac{4}{3}\pi\gamma\rho[3r^4 dr - R^4 dR - \frac{r^3}{R}\times 2R^2\, dR]$$

or $$\left(\frac{1}{r} - \frac{1}{R}\right)d\{F(t)\}^2 + \{F(t)\}^2\, d\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{4}{3}\pi\gamma\rho[3r^4\, dr - R^4\, dR - 2R(R^3 - c^3)\, dR]$$

[$\because$ Given that $R^3 - r^3 = c^3$]

Integrating , $$\left(\frac{1}{r}-\frac{1}{R}\right)\{F(t)\}^2=\frac{4}{3}\pi\gamma\rho\left[\frac{3r^5}{5}-\frac{R^5}{5}-\frac{2R^5}{5}+c^3R^2\right],$$

where we have chosen the constant of integration to be zero

or $$\left(\frac{1}{r}-\frac{1}{R}\right)\{F(t)\}^2=\frac{4}{15}\pi\gamma\rho\{3(r^5-R^5)+5c^3R^2\}. \qquad ...(11)$$

Given that when $r = r$, $v = V$. So from (8), $r^2V = F(t)$.

$\therefore$ (11) gives $$\left(\frac{1}{r}-\frac{1}{R}\right)r^4V^2=\frac{4}{15}\pi\gamma\rho\{3(r^5-R^5)+5R^2(R^3-r^3)\}, \text{ as } R^3=r^3+c^3$$

or $$V^2=\frac{4}{15}\frac{\pi\gamma\rho R}{r^3}\left\{\frac{3(r^5-R^5)+5R^2(R^3-r^3)}{R-r}\right\}$$

or $$V^2=\frac{4\pi\gamma\rho R}{15r^3}\{-3(r^4+r^3R+r^2R^2+rR^3+R^4)+5R^2(R^2+Rr+r^2)\}$$

$\therefore$ $$V^2=(4\pi\gamma\rho R/15r^3)\,(2R^4+2R^3r+2R^2r^2-3Rr^3-3r^4).$$

Ex. 19. *A homogeneous liquid is contained between two concentric spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$ and a constant pressure Π is exerted at the outer surface. Suppose $\int\phi(r)\,dr=\psi(r)$ and $\psi(r)$ vanishes when r is infinite, show that if the inner surface is removed, the pressure at the distance r is suddenly diminished by*

$$\Pi(a/r)-(a\rho/r)\psi(a).$$

Find $\phi(r)$ so that the pressure immediately after the inner surface is removed may be the same as it would be if no attractive force existed. Also with this value of $\phi(r)$, find the velocity of the inner boundary of the fluid at any period of the motion.

Sol. Let v' be the velocity and p the pressure at a distance r' from the centre at any time t. Then the equation of continuity is $$r'^2v'=F(t). \qquad ...(1)$$

From (1), $$\partial v'/\partial t=F'(t)/r'^2. \qquad ...(2)$$

The equation of motion is

$$\partial v'/\partial t+v'(\partial v'/\partial r')=-\phi(r')-(1/\rho)(\partial p/\partial r'),$$

or $$\frac{F'(t)}{r'^2}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\phi(r')-\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ by (2)}$$

Integrating it with respect to r', we have

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\int\phi(r')\,dr'-\frac{p}{\rho}+C,\ C \text{ being an arbitrary constant}$$

or $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\psi(r')-\frac{p}{\rho}+C. \qquad ...(3)$$

[$\because$ given that $\int\phi(r')\,dr'=\psi(r')$]

When $r'=\infty$, $v'=0$, $p=\Pi$ and $\psi(\infty)=0$. So (3) $\Rightarrow$ $C=\Pi/\rho$.

$\therefore$ (3) becomes $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\psi(r')+\frac{\Pi-p}{\rho}. \qquad ...(4)$$

Given, initially when $t = 0$, $v' = 0$ and $p = p_0$ (say). Then (4) gives

$$-\frac{F'(0)}{r'}=-\psi(r')+\frac{\Pi-p_0}{\rho}. \qquad ...(5)$$

Again, when $r' = a$, $p_0 = 0$. So (5) reduces to

$$-(1/a)\, F'(0) = -\psi(a) + (\Pi/\rho). \qquad \text{...(6)}$$

Dividing (5) by (6) and re-writting, we have

$$\Pi - p_0 - \rho\psi(r') = (1/r')\Pi a - (1/r')a\, \rho\psi(a). \qquad \text{...(7)}$$

Initially the liquid was at rest. Then hydrostatic pressure is given by

$$dp' = -\rho\, \phi(r')\, dr' \qquad \text{so that} \qquad p' = C' - \rho\psi(r'). \qquad \text{...(8)}$$

$$[\because \ \textstyle\int \phi(r')\, dr' = \psi(r')]$$

But, when $r' = \infty$, $\psi(\infty) = 0$ and $p' = \Pi$. So (8) gives $C' = \Pi$ and hence (8) reduces to

$$p' = \Pi - \rho\psi(r'). \qquad \text{...(9)}$$

Then decrease in pressure

$= p' - p_0 = \Pi - \rho\psi(r') - [\Pi - \rho\psi(r') + (1/r')\Pi a + (1/r')\, a\rho\psi(a)]$, using (7) and (9)

$\therefore$ The required decrease in pressure at distance $r = \Pi(a/r) - (a\rho/r)\psi(a)$.

Second Part : In presence of attractive forces, (7) gives

$$p_0 = \Pi - \rho\psi(r') - (1/r')\Pi a + (1/r')a\rho\psi(a). \qquad \text{...(10)}$$

In absence of attractive forces, the terms containing ψ are zero and hence the corresponding pressure p_0' is given by (10) as

$$p_0' = \Pi - (1/r')\Pi a. \qquad \text{...(11)}$$

But, by the condition of the problem, $p_0 = p_0'$. Hence, using (10) and (11), we get

$$\therefore \qquad \Pi - \rho\psi(r') - (1/r')\Pi a + (1/r')a\rho\psi(a) = \Pi - (1/r')\Pi a.$$

$$\therefore \qquad \psi(r') = (a/r')\psi(a), \qquad \text{...(12)}$$

Given $\int \phi(r')\, dr' = \psi(r')$ so that $\phi(r') = \dfrac{d}{dr'}\psi(r')$.

$$\therefore \qquad \phi(r') = \frac{d}{dr'}\left(\frac{a}{r'}\psi(a)\right) = -\frac{a\psi(a)}{r'^2}.$$

Substituting the above value of $\psi(r')$ in (4), we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{a\psi(a)}{r'} + \frac{\Pi - p}{\rho}. \qquad \text{...(13)}$$

When $r' = r$, $v' = v$, $p = 0$. Then (13) becomes

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{a\psi(a)}{r} + \frac{\Pi}{\rho}. \qquad \text{...(14)}$$

The equation of continuity is $F(t) = r^2 v$.

$$\therefore \qquad F'(t) = \frac{d}{dt}(r^2 v) = \frac{d}{dr}(r^2 v)\cdot\frac{dr}{dt} = v\,\frac{d}{dr}(r^2 v), \qquad as \qquad v = \frac{dr}{dt}$$

$$= v\left[2rv + r^2\frac{dv}{dr}\right] = 2rv^2 + r^2 v\,\frac{dv}{dr}.$$

$\therefore$ Subtituting the above value of $F'(\text{t})$ in (14), we have

$$-\frac{1}{r}\left(2rv^2 + r^2\, v\,\frac{dv}{dr}\right) + \frac{1}{2}v^2 = -\frac{a\psi(a)}{r} + \frac{\Pi}{\rho}$$

or

$$rv\,\frac{dv}{dr} + 2v^2 - \frac{1}{2}v^2 = \frac{a\psi(a)}{r} - \frac{\Pi}{\rho}$$

or

$$2r^3 v dv + 3v^2 r^2 dr = [2ra\psi(a) - (2\Pi/\rho)r^2]\, dr$$

or $$d(r^3 v^2) = [2ra\psi(a) - (2\Pi/\rho)r^2]\, dr.$$

Integrating, $r^3 v^2 = a\psi(a) r^2 - (2\Pi/3\rho) r^3 + C'$, C' being an arbitrary constant ... (15)

When $r = a$, $v = 0$. So (15) gives $C' = (2\Pi/3\rho)a^3 - \psi(a)a^3$.

Putting this value of C' in (15), the required velocity is given by

$$r^3 v^2 = a\psi(a) r^2 + (2\Pi/3\rho)(a^3 - r^3).$$

Ex. 20. *A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b (a < b). The cavity is filled with gas the pressure of which varies according to Boyle's law and is initially equal to atmospheric pressure Π and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that particle moves along a line joining it to the centre, the time of small oscillation is*

$$2\pi a \left\{ \rho \frac{b-a}{3\Pi b} \right\}^{1/2}, \text{ where } \rho \text{ is the density of the fluid.}$$

Sol. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v will be radial and v will be function of x (the radial distance from the centre of the spherical shell which is taken as origin) and t only. Let p be pressure at a distance x. Then the equation of continuity is

$$x^2 v = F(t) \qquad \text{so that} \qquad \partial v/\partial t = F'(t)/x^2 \qquad \text{...(1)}$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{or} \qquad \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ using (1).}$$

Integrating it w.r.t. 'x', we get $$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \qquad \text{...(2)}$$

Let r and R be internal and external radii of the shell at any time t. Since the given shell contains gas, it follows that there will be pressure on the inner surface. Let $p = p_1$ when $x = r$.

Since the total mass of the liquid is constant, we have

$$\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho = \left(\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3\right)\rho \qquad \text{or} \qquad R^3 - r^3 = b^3 - a^3. \qquad \text{...(3)}$$

Since the initial pressure of the gas is equal to atmospheric pressure Π, Boyle's law gives

$$(4/3) \times \pi r^3 \times p_1 = (4/3) \times \pi a^3 \times \Pi \qquad \text{so that} \qquad p_1 = (a^3\Pi)/r^3. \qquad \text{...(4)}$$

Since the outer surface is exposed to atmospheric pressure Π, we have

when $x = R$, $v = dR/dt = U$, (say) and $p = \Pi$. So (2) gives

$$-\frac{F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\Pi}{\rho} + C. \qquad \text{...(5)}$$

Again, when $x = r$, $v = dr/dt = u$(say), $p = p_1 = (a^3\Pi)/r^3$, by (4)

So by (2), $$-\frac{F'(t)}{r} + \frac{1}{2}u^2 = -\frac{a^3\Pi}{\rho r^3} + C, \; C \text{ being an arbitrary constant.} \qquad \text{...(6)}$$

Subtracting (6) from (5), we have

$$\left(\frac{1}{r} - \frac{1}{R}\right)F'(t) + \frac{1}{2}(U^2 - u^2) = \frac{\Pi}{\rho}\left(\frac{a^3}{r^3} - 1\right). \qquad \text{...(7)}$$

Since we are to consider small oscillation, so U^2 and u^2 are small quantities and hence we neglect them. Then (7) reduces to

$$F'(t) = \frac{\Pi}{\rho} \times \frac{a^3 - r^3}{r^2} \times \frac{R}{R-r}. \qquad \ldots(8)$$

By continuity equation (1), $F(t) = r^2 u$.

$$\therefore \quad F'(t) = 2r\frac{dr}{dt}u + r^2\frac{du}{dt} = 2ru^2 + r^2\frac{d^2r}{dt^2}, \quad \text{as} \quad u = \frac{dr}{dt}$$

or $F'(t) = r^2(d^2r/dt^2)$, neglecting u^2 as before

Then (8) becomes
$$r^2\frac{d^2r}{dt^2} = \frac{\Pi}{\rho} \times \frac{a^3 - r^3}{r^2} \times \frac{R}{R-r}. \qquad \ldots(9)$$

Since the displacement is small. We choose small quantities x and x' such that

$$r = a + x \quad \text{and} \quad R = b + x'. \qquad \ldots(10)$$

Then (10) $\Rightarrow$ $\frac{d^2r}{dt^2} = \frac{d^2}{dt^2}(a + x)$ or $\frac{d^2r}{dt^2} = \frac{d^2x}{dt^2} = \ddot{x}$. (say)

$\therefore$ (9) becomes
$$(a + x)^2\,\ddot{x} = \frac{\Pi}{\rho} \times \frac{a^3 - (a+x)^3}{(a+x)^2} \times \frac{b + x'}{b + x' - (a + x)}$$

or
$$\ddot{x} = \frac{\Pi}{\rho} \times \frac{a^3 - a^3(1 + x/a)^3}{(a+x)^4} \times \frac{b + x'}{b - a + x' - x} \quad \text{or} \quad \ddot{x} = \frac{\Pi}{\rho} \times \frac{1 - (1 + x/a)^3}{(1 + x/a)^4} \times \frac{b + x'}{x' - x + b - a}$$

or
$$\ddot{x} = \frac{\Pi}{a\rho} \times \frac{1 - (1 + 3x/a)}{(1 + 4x/a)} \times \frac{b + x'}{x' - x + b - a}, \text{ to first order of approximation} \qquad \ldots(11)$$

Using (10), (3) reduces to

$(b + x')^3 - (a + x)^3 = b^3 - a^3$ or $b^3(1 + x'/b)^3 - a^3(1 + x/a)^3 = b^3 - a^3$

or $b^3(1 + 3x'/b) - a^3(1 + 3x/a) = b^3 - a^3$, to first order of approximation

or $3x'b^2 - 3a^2x = 0$ or $x' = a^2x/b^2$. ...(12)

Using (12), (11) reduces to

$$\ddot{x} = -\frac{\Pi}{a\rho}\frac{(3x/a)(b + a^2x/b^2)}{(1 + 4x/a)\,[(a^2x/b^2) - x + b - a]} = -\frac{\Pi}{a\rho}\frac{(3xb/a)}{(a^2x/b^2) - x + b - a + (4xb/a) - 4x}$$

[To first order approximation]

$$= -\frac{\Pi}{a\rho}\frac{(3xb/a)}{x(a^2/b^2 - 5 + 4b/a) + b - a} = -\frac{3b\Pi x}{a^2\rho(b-a)}\left[1 + \frac{(a^2/b^2) - 5 + (4b/a)}{b - a}x\right]^{-1}$$

$$= -\frac{3b\Pi x}{a^2\rho(b-a)}\left[1 - \frac{(a^2/b) - 5 + (4b/a)}{b - a}x + \ldots\right]$$

$$\therefore \quad \ddot{x} = -\frac{3b\Pi}{a^2\rho(b-a)}x, \text{ to first order of approximation}$$

which represents simple harmonic motion of time period

$$\frac{2\pi}{[3b\Pi/a^2\rho(b-a)]^{1/2}} \quad \text{or} \quad 2\pi a\left\{\frac{\rho(b-a)}{3\Pi b}\right\}^{1/2}.$$

EXERCISE 3 (A)

1. Obtain Euler's equation of motion in cartesian form. **[Kanpur 2002; 2004]**

2. Prove that the equation of motion of a homogeneous inviscid liquid moving under conservative forces may be written in the form $\dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\,\mathbf{q} = -\,\text{grad}\left(\dfrac{p}{\rho} + \dfrac{1}{2}q^2 + \Omega\right)$

[**Hint.** From Art. 3.1, we have $\dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\,\mathbf{q} = \mathbf{F} - \dfrac{1}{\rho}\nabla p - \dfrac{1}{2}\nabla q^2$ ...(1)

Since the forces form a conservative system, there exists a force potential Ω such that $\mathbf{F} = -\nabla\Omega$. Moreover, the fluid being homogeneous, we may write $(1/\rho)\,\nabla p = \nabla(p/\rho)$.

Hence (1) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\,\mathbf{q} = -\nabla\Omega - \nabla\left(\frac{p}{\rho}\right) - \nabla\left(\frac{1}{2}q^2\right) \quad \text{or} \quad \frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\,\mathbf{q} = -\,\text{grad}\left(\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega\right)$$

3. A mass of fluid of density ρ and volume $(4\pi c^3)/3$ is in the form of a spehericaI shell. There is a constant pressure p on the external surface, and zero pressure on the internal surface. Initially the fluid is at rest and the external radius $2nc$. Show that when the external radius becomes nc the velocity U of the external surface is given by

$$U^2 = \frac{14p}{3\rho}\frac{(n^3-1)^{1/3}}{n-(n^3-1)^{1/3}}$$

Ex. 4. *The particle velocity for a fluid motion referred to rectangular axes is given by* $(A\cos(\pi x/2a)\cos(\pi z/2a), 0, \; A\sin(\pi x/2a)\sin(\pi z/2a))$, *where A, a are constants. Show that this is a possible motion of an inconpressible fluid under no body forces in an infinite fixed rigid tube* $-a \le x \le a, \;\; 0 \le z \le 2a$. *Also find the pressure associated with this velocity field.*

Sol. Let u, v, w be the components of velocity referred to rectangular axes OX, OY, OZ. Then we have $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$.

Now do as in solved example 14 of Art 3.4

Ex. 5. *A sphere is at rest in an infinite mass of homogeneous liquid of density* ρ. *The pressure at infinity being* $\overline{w}$. *Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is* $\overline{w} + (\rho/2)\{d^2(R)^2/dt^2 + (dR/dt)^2\}$.

Sol. Refer solved Ex. 1 of Art 3.4 by taking $\Pi = \overline{w}$. **[I.A.S. 1996]**

3.5. Impulsive action.

Let sudden velocity changes be produced at the boundaries of an incompressible fluid or that impulsive forces be made to act to its interior. Then it is known that the impulsive pressure at any point is the same in every direction. Moreover the dstrubances produced in both cases are propagated instantaneously throughout the fluid.

3.6. Equation of motion under Impulsive forces (Vector form).

[Meerut 2007; Kanpur 2000, 03, 05, 09; Rohilkhand 2000, 05]

Let S be an arbitrary small closed surface drawn in the incompressible fluid enclosing a volume V. Let $\mathbf{I}$ be the impulsive body force per unit mass. Let this impulse change the velocity at

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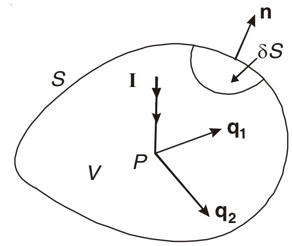


$P(\mathbf{r}, t)$ of V instantaneously from $\mathbf{q}_1$ to $\mathbf{q}_2$ and let it produce impulsive pressure on the boundary S. Let $\tilde{\omega}$ denote the inpulsive pressure on the element δS of S. Let $\mathbf{n}$ be the unit outward drawn normal at δS. Let ρ be density of the fluid.

We now apply Newton's second law for inpulsive motion to the fluid enclosed by S, namely,

Total impulse applied = Change of momentum

$$\therefore \qquad \int_V \mathbf{I}\rho dV - \int_S \mathbf{n}\tilde{\omega} dS = \int_V \rho(\mathbf{q}_2 - \mathbf{q}_1)\, dV \qquad ...(1)$$

But $$\int_S \mathbf{n}\,\tilde{\omega}\, dS = \int_V \nabla\tilde{\omega}\, dV \qquad \text{(by Gauss divergence theorem)}$$

$$\therefore \text{ From (1),} \qquad \int_V [\mathbf{I}\rho - \nabla\tilde{\omega} - \rho(\mathbf{q}_2 - \mathbf{q}_1)]\, dV = 0 \qquad ...(2)$$

Since V is an arbitrary small volume, (2) gives

$$\mathbf{I}\rho - \nabla\tilde{\omega} - \rho(\mathbf{q}_2 - \mathbf{q}_1) = 0 \qquad \text{or} \qquad \mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega} \qquad ...(3)$$

Cor. 1. Let $\mathbf{I} = \mathbf{0}$ (*i.e.* external impulsive body forces are absent) whereas impulsive pressures be present. Then (3) reduces to

$$\mathbf{q}_2 - \mathbf{q}_1 = -(1/\rho)\nabla\tilde{\omega} \qquad ...(4)$$

From (4), $$\nabla\cdot(\mathbf{q}_2 - \mathbf{q}_1) = \nabla\cdot[-(1/\rho)\nabla\tilde{\omega}]$$

or $$\nabla\cdot\mathbf{q}_2 - \nabla\cdot\mathbf{q}_1 = -(1/\rho)\nabla^2\tilde{\omega}, \qquad ...(5)$$

For the incompressible fluid, the equation of continuity gives

$$\nabla\cdot\mathbf{q}_2 = \nabla\cdot\mathbf{q}_1 = 0 \qquad ...(6)$$

Making use of (6), (5) reduces to

$$\nabla^2\tilde{\omega} = 0. \qquad \text{(Laplace's equation)} \qquad ...(7)$$

Cor. 2. Let $\mathbf{q}_1 = \mathbf{0}$ and $\mathbf{I} = \mathbf{0}$ so that the motion is started from rest by the application of impulsive pressure at the boundaries but without use of external impulsive body forces. Then, writing $\mathbf{q}_2 = \mathbf{q}$, (3) reduces to

$$\mathbf{q} = -\nabla(\tilde{\omega}/\rho), \qquad ...(8)$$

showing that there exists a velocity potential $\phi = \tilde{\omega}/\rho$ and that the motion is irrotational.

Cor. 3. Let $\mathbf{I} = \mathbf{0}$ *i.e.* let there be no extraneous impulses. Further, let ϕ_1 and ϕ_2 denote the velocity potential just before and just after the impulsive action. Then

$$\mathbf{q}_1 = -\nabla\phi_1 \qquad \text{and} \qquad \mathbf{q}_2 = -\nabla\phi_2 \qquad ...(9)$$

Then (3) reduces to

$$-\nabla\phi_2 + \nabla\phi_1 = -(1/\rho)\nabla\tilde{\omega} \qquad \text{or} \qquad \nabla\tilde{\omega} = \rho\nabla(\phi_2 - \phi_1)$$

Integrating, when ρ is constant $\qquad \tilde{\omega} = \rho(\phi_2 - \phi_1) + C.$

The constant C may be omitted by regarding as an extra pressure and constant throughout the fluid.

$$\therefore \qquad \tilde{\omega} = \rho\phi_2 - \rho\phi_1. \qquad ...(10)$$

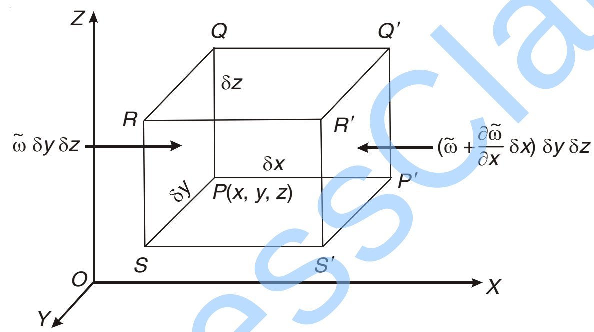
Cor. 4. Physical meaning of velocity potential.

Take $\phi_1 = 0$ and $\rho = 1$ in cor. 3. Then we find that any actual motion, for which a single valued velocity potential exists, could be produced instantaneously from rest by applying appropriate impulses. We then also note that the velocity potential is the impulsive pressure at any point.

It is also easily seen that when a state of rotational motion exists in a fluid, the motion could neither be created nor destroyed by impulsive pressures.

3.7. Equations of motion under Impulsive Force (Cartesian form). [Kanpur 2002, 05]

Let there by a fluid particle at $P(x, y, z)$ and let ρ be the density of the incompressible fluid. Let u_1, v_1, w_1 and u_2, v_2, w_2 be the velocity components at the point P just before and just after the impulsive action. Let I_x, I_y, I_z be the components of the external impulsive forces per unit mass of the fluid. Construct a small parallelepiped with edges of lengths δx, δy, δz parallel to their respective co-ordinate axes, having P at one of the angular points as shown in figure. Let $\tilde{\omega}$ denote the impulsive pressure at P. Then, we have



Force on the face $PQRS = \tilde{\omega}\delta y\,\delta z = f(x, y, z)$ say ...(1)

$\therefore$ Force on the face $P'Q'R'S' = f(x + \delta x, y, z) = f(x,y,z) + \delta x \dfrac{\partial}{\partial x} f(x,y,z) + \ldots$ ...(2)

(expanding by Taylor's theorem)

$\therefore$ The net force on the opposite faces $PQRS$ and $P'Q'R'S'$

$$= f(x,y,z) - [f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots]$$

$$= -\delta x \frac{\partial}{\partial x} f(x,y,z), \text{ to the first order of approximation}$$

$$= -\delta x \frac{\partial}{\partial x}(\tilde{\omega}\,\delta y\,\delta z), \text{ using (1)}$$

$$= -\delta x\,\delta y\,\delta z \frac{\partial \tilde{\omega}}{\partial x}, \text{ which will act along the } x\text{-axis.} \quad \text{...(3)}$$

Again, the impulse on the elementary parallelepiped along the x-axis due to external impulsive body force I_x $= \rho\,\delta x\,\delta y\,\delta z\, I_x$. ...(4)

Finally, the change in momentum along x-axis $= \rho\,\delta x\,\delta y\,\delta z\,(u_2 - u_1)$ ...(5)

We now apply Newton's second law for impulsive motion to the fluid enclosed by the parallelopiped, namely,

Total impulse applied along x-axis = Change of momentum along x-axis

$$\therefore \qquad -\delta x \delta y \delta z \, \frac{\partial \tilde{\omega}}{\partial x} + \rho \delta x \delta y \delta z \, I_x = \rho \delta x \delta y \delta z \, (u_2 - u_1)$$

or $$\rho (u_2 - u_1) = \rho I_x - (\partial \tilde{\omega} / \partial x) \qquad ...(6)$$

Similarly $$\rho (v_2 - v_1) = \rho I_y - (\partial \tilde{\omega} / \partial y) \qquad ...(7)$$

and $$\rho (w_2 - w_1) = \rho I_z - (\partial \tilde{\omega} / \partial z). \qquad ...(8)$$

Equations (6), (7) and (8) are the required equations of motion of an incompressible fluid under impulsive forces.

3.8. Illustrative solved examples.

Ex. 1. *A sphere of radius a is surrounded by infinite liquid of density* ρ, *the pressure at infinity being* Π. *The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre immediately falls to* $\Pi(1 - a/r)$. **[Purvanchel 2004, I.A.S. 1996]**

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius a/2, the impulsive pressure sustained by the surface of this sphere is $(7\Pi\rho^2/6)^{1/2}$.

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad ...(1)$$

From (1), $$\partial v' / \partial t = F'(t) / r'^2 \qquad ...(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial P}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2} v'^2\right) = -\frac{1}{\rho}\frac{\partial P}{\partial r'}, \text{ using (2)}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = -\frac{p}{r'} + C,$$ C being an arbitrary constant.

When $r' = \infty$, then $p = \Pi$ and $v' = 0$ so that $C = \Pi / \rho$.

$$\therefore \qquad -\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{\Pi - p}{\rho} \qquad ...(3)$$

When the sphere is suddenly annihilated, we have

$t = 0, \qquad r' = a, \qquad v' = 0 \qquad$ and $\qquad p = 0$

$$\therefore \text{ From (3),} \qquad -\frac{F'(0)}{a} = \frac{\Pi}{\rho} \qquad \text{so that} \qquad F'(0) = -\frac{a\Pi}{\rho}$$

Hence immediately after the annihilation of the sphere (with $t = 0$, $v' = 0$), (3) reduces to

$$\frac{a\Pi}{\rho r'} + 0 = \frac{\Pi - p}{\rho} \qquad \text{or} \qquad p = \Pi\left(1 - \frac{a}{r'}\right) \qquad ...(4)$$

Thus at the time of annihilation, when $r' = r$, the pressure is given by

$$p = \Pi(1 - a/r'). \qquad ...(5)$$

Second Part. If $\tilde{\omega}$ be the impulsive pressure at distance r', then we have

$$d\tilde{\omega} = -\rho v' \, dr' \qquad ...(6)$$

Let r be the radius of the inner surface and v the velocity there. Then by the equation of continuity, we have

$$F(t) = r^2 v = r'^2 v' \qquad \text{so that} \qquad v' = (r^2 v)/r'^2 \qquad ...(7)$$

$\therefore$ (6) gives $\qquad d\omega' = \rho v (r^2 / r'^2) \, dr'$

Integrating with respect to r', we get $\qquad \tilde{\omega} = \rho v (r^2 / r') + C' \qquad ...(8)$

When $\quad r' = \infty, \quad \tilde{\omega} = 0 \quad$ so that $\quad C' = 0.$

$$\therefore \qquad \tilde{\omega} = \rho v (r^2 / r'), \qquad ...(9)$$

which gives the impulsive pressure $\tilde{\omega}$ at a distance r'. Since $r = a/2$, (9) reduces to

$$\tilde{\omega} = \frac{1}{4} \rho v a^2 \cdot \frac{1}{r'} \qquad ...(10)$$

We now determine velocity v at the inner surface of the sphere. Setting $r' = r$, $v' = v$ and $p = 0$ in (3), we get

$$-\frac{F'(t)}{r} + \frac{1}{2} v^2 = \frac{\Pi}{\rho} \qquad ... (11)$$

From (7), $\qquad F'(t) = \frac{d}{dt}(r^2 v) = 2r \frac{dr}{dt} v + r^2 \frac{dv}{dt} = 2r \frac{dr}{dt} v + r^2 \frac{dv}{dt} \frac{dr}{dr}$

Thus, $\qquad F'(t) = 2rv^2 + r^2 v \frac{dv}{dr}, \qquad$ as $\qquad v = \frac{dr}{dt}$

$\therefore$ (11) gives $\qquad -\frac{1}{r}\left(2rv^2 + r^2 v \frac{dv}{dr}\right) + \frac{1}{2} v^2 = \frac{\Pi}{\rho}$

Multiplying both sides by ($-\, 2r^2 \, dr$), we get

$$2r^3 v dv + 3r^2 v^2 \, dr = -\frac{2\Pi r^2}{\rho} dr \qquad \text{or} \qquad d(r^3 v^2) = -\frac{2\Pi r^2}{\rho} dr$$

Integrating, $\qquad r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C'', \quad C''$ being an arbitrary constant

When $\quad r = a, \quad v = 0 \quad$ so that $\quad C'' = -\frac{2\Pi a^3}{3\rho}.$

$$\therefore \qquad r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) \qquad ...(12)$$

The velocity v on the surface of the sphere of radius $a/2$ (which would be the inner surface on which the liquid impinges) is given by (12) by replacing r by $a/2$

$$\therefore \qquad v^2 = \frac{2\Pi}{3\rho} \times \frac{a^3 - a^3/8}{a^3/8} = \frac{14}{3} \times \frac{\Pi}{\rho}$$

Putting this value of v in (10), the impulsive pressure at a distance r' is given by

$$\tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3}\times\frac{\Pi}{\rho}\right)^{1/2}\frac{a^2}{r'} \qquad ...(13)$$

Hence the desired impulsive pressure on the surface of the sphere of radius $a/2$ is given by setting $r' = a/2$ in (13).

$$\therefore \qquad \tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3}\times\frac{\Pi}{\rho}\right)^{1/2}\times\frac{a^2}{(a/2)} = \left(\frac{7\Pi\rho a^2}{6}\right)^{1/2}$$

Ex. 2. *A portion of homogeneous fluid is contained between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated and when the radii of the inner and outer surfaces of the fluid are r and R the fluid impinges on a solid ball concentric with these surfaces, prove that the impulsive pressure at any point of the ball for different values of R and r varies as*

$$\left\{(a^2 - r^2 - A^2 + R^2)\,(1/r - 1/R)\right\}^{1/2}$$ **[Agra 1996; Kanpur 1998]**

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad ...(1)$$

From (1),
$$\partial v'/\partial t = F'(t)/r'^2 \qquad ...(2)$$

Taking μ/r'^2 as the force towards the centre of the sphere, the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \quad \text{or} \quad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating,
$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \qquad ...(3)$$

Let r and R be the internal and external radii of the fluid at any time t and v and V be the velocities there. Thus, we have

When $r' = R$, $v' = V$, $p = 0$ and also when $r' = 0$, $v' = v$, $p = 0$

$\therefore$ (3) yields
$$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = C + \frac{\mu}{R} \qquad ...(4)$$

and
$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = C + \frac{\mu}{r} \qquad ...(5)$$

Subtracting (4) from (5), we have

$$-F'(t)\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{1}{2}(v^2 - V^2) = \mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad ...(6)$$

From the equation of continuity (1), we have

$$r^2 v = R^2 V = F(t) \qquad ...(7)$$

From (7),
$$r^2\frac{dr}{dt} = R^2\frac{dR}{dt} = F(t)$$

$\therefore$ $$r^2\, dr = R^2\, dR = F(t)\, dt \qquad ...(8)$$

Using (7), (6) reduces to

$$-F'(t)\left[\frac{1}{r}-\frac{1}{R}\right]+\frac{1}{2}\{F(t)\}^2\left[\frac{1}{r^4}-\frac{1}{R^4}\right]=\mu\left[\frac{1}{r}-\frac{1}{R}\right]$$

Multiplying both sides by $2F(t)\, dt$, we get

$$-2F(t)\,F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2F(t)}{r^4}-\frac{2F(t)}{R^4}\right]dt=\mu\left[\frac{2F(t)}{r}-\frac{2F(t)}{R}\right]dt$$

or $$-2F(t)\,F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2dr}{r^2}-\frac{2dR}{R^2}\right]=\mu(2rdr-2RdR), \text{ using (8)}$$

Integrating, $$-\{F(t)\}^2\left[\frac{1}{r}-\frac{1}{R}\right]=\mu(r^2-R^2)+C', \text{ being arbitrary constant} \qquad ...(9)$$

Since velocity is zero when $r = a$ and $R = A$, it follows that $F(t) = 0$. Then (9) reduces to

$$0=\mu(a^2-A^2)+C' \qquad i.e. \qquad C'=-\mu(a^2-A^2)$$

$\therefore$ (9) becommes $$-\{F(t)\}^2\left[\frac{1}{r}-\frac{1}{R}\right]=\mu(r^2-R^2-a^2+A^2) \qquad ...(10)$$

If $\tilde{\omega}$ be the impulsive pressure at a distance r', then we have

$$d\tilde{\omega}=-\rho v'\, dr'=-\rho\frac{F(t)}{r'^2}dr', \text{ using (1)}$$

Integrating, $$\tilde{\omega}=\frac{\rho F(t)}{r'^2}+C'', \quad C'' \text{ being an arbitrary constant}$$

But when, $r'=R$, $\tilde{\omega}=0$ so that $C''=[\rho F(t)]/R$. So the above equation gives

$\therefore$ $$\tilde{\omega}=\rho F(t)\,(1/r'-1/R)$$

Hence the impulsive pressure at any point of the ball where $r'=r$ is given by

$$\tilde{\omega}=\rho F(t)\,(1/r-1/R) \qquad ...(11)$$

From (10), $$F(t)=\left\{\frac{\mu(a^2-r^2-A^2+R^2)}{(1/r-1/R)}\right\}^{1/2}$$

$\therefore$ $$\tilde{\omega}=\rho\sqrt{\mu}\left\{(a^2-r^2-A^2+R^2)\,(1/r-1/R)\right\}^{1/2},$$

showing that the required impulsive pressure varies as $\left\{(a^2-r^2-A^2+R^2)(1/r-1/R)\right\}^{1/2}$

EXERCISE 3(B)

1. If a bomb shell explodes at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversely as the distance from the centre of the shell.

2. Prove that if $\tilde{\omega}$ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses, then $\tilde{\omega}+\rho V+\rho(\phi'-\phi)=\text{const.}$

3. Find the equations of motion of a perfect fluid under extraneous impulses and impulsive pressure. Deduce that any actual irrotational motion of a liquid can be produced instantaneously from rest by a set of impulses properly applied.

The energy equation.

[Kanpue 2007; Agra 2005; Banglore 2006; Patna 2003, 06; Garhwal 2005]

Statement : *The rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the extraneous forces is supposed to be independent of time.*

Proof. Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dV be the volume element surrounding P. Let $\mathbf{q}$ ($\mathbf{r}$, t) be the velocity of P. Then, the Euler's equation of motion is $d\mathbf{q}/dt = -(1/\rho)\nabla p + \mathbf{F}$. ... (1)

Let the external forces be conservative so that there exists a force potential Ω which is independent of time. Thus $\mathbf{F} = -\nabla\Omega$ and $\partial\Omega/\partial t = 0$.

Using the above results and then multiplying both sides of (1) scalarly by $\mathbf{q}$, we get

$$\rho\left(\mathbf{q}\cdot\frac{d\mathbf{q}}{dt}\right) = -\mathbf{q}\cdot\nabla p - \rho[\mathbf{q}\cdot\nabla\Omega] \quad \text{or} \quad \rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right) + (\mathbf{q}\cdot\nabla)\Omega\right] = -\mathbf{q}\cdot\nabla p \quad \text{... (2)}$$

But $$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (\mathbf{q}\cdot\nabla)\Omega = (\mathbf{q}\cdot\nabla)\Omega, \qquad \text{since} \quad \frac{\partial\Omega}{\partial t} = 0$$

Hence, equation (2) becomes $$\rho\frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) = -\mathbf{q}\cdot\nabla p \quad \text{... (3)}$$

Since the elementary mass remains invariant throughout the motion, so $d(\rho V)/dt = 0$ ...(4)

Integrating both sides of (3) over V, we have

$$\int_V \frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \int_V \frac{d}{dt}(\rho\Omega)\,dV = -\int_V (\boldsymbol{q}.\nabla p)\,dV$$

or $$\int_V \left\{\frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \frac{1}{2}q^2\frac{d}{dt}(\rho dV)\right\} + \int_V \frac{d}{dt}(\rho\Omega dV) = -\int_V (\mathbf{q}.\nabla p)dV$$

[Noting that , (4) $\Rightarrow (q^2/2)\times\{d(\rho dV)/dt)\} = 0$]

Thus, $$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho q^2\right)dV + \frac{d}{dt}\int_V (\rho\Omega)\,dV = -\int_V (\mathbf{q}.\nabla p)\,dV \quad \text{... (5)}$$

Let T, W and $\mathbf{I}$ denote the kinetic, potential and intrinsic (internal) energies respectively. Then, by definitions

---

* Here, we write d/dt for D/Dt so that $d/dt = D/Dt = \partial/\partial t + \mathbf{q}\cdot\nabla$ (Refer note of Art. 2.4)

$$T = \int_V \frac{1}{2}\rho q^2\, dV, \qquad W = \int_V \rho\Omega\, dV, \qquad \mathbf{I} = \int_V \rho E\, dV, \qquad ...(6)$$

where E is the intrinsic energy per unit mass,

Since $\nabla\cdot(p\mathbf{q}) = p\nabla\cdot\mathbf{q} + \mathbf{q}\cdot\nabla p$, we have $\mathbf{q}\cdot\nabla p = \nabla\cdot(p\mathbf{q}) - p\nabla\cdot\mathbf{q}$

$$\therefore \text{ R.H.S. of (4) } = -\int_V \nabla\cdot(p\mathbf{q})\, dV + \int_V p\nabla\cdot\mathbf{q}dV = \int_S p\mathbf{q}\cdot\mathbf{n}dS + \int_S p\nabla\cdot\mathbf{q}dV, \qquad ...(7)$$

[By Gauss divergence theorem]

where **n** is unit inward normal and dS is the element of the fluid surface S. We now prove that

$$\int_V p\nabla\cdot\mathbf{q}dV = -\frac{d\mathbf{I}}{dt} \qquad ...(8)$$

Now, E is defined as the work done by the unit mass of the fluid against external pressure p (assuming that there exists a relation between pressure and density) from its actual state to some standard state in which p_0 and ρ_0 are the values of pressure and density respectively.

$$\therefore \qquad E = \int_V^{V_0} pdV, \qquad \text{where } V\rho = 1, \qquad i.e., \qquad V = 1/\rho$$

or
$$E = \int_\rho^{\rho_0} p\, d\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2}\, d\rho = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\, d\rho \qquad ...(9)$$

From (9), $\dfrac{dE}{d\rho} = \dfrac{p}{\rho^2}$ and so $\dfrac{dE}{dt} = \dfrac{dE}{d\rho}\dfrac{d\rho}{dt} = \dfrac{p}{\rho^2}\dfrac{d\rho}{dt}$

Multiplying both sides by ρdV and then integrating over a volume V, we have

$$\int_V \frac{dE}{dt}\rho dV = \int_V \frac{p}{\rho}\frac{d\rho}{dt} dV \qquad ...(10)$$

But
$$\frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)$$

$$\therefore \qquad \frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV, \text{ using (4)} \qquad ...(11)$$

Also from the equation of continuity, $d\rho/dt = -\rho\nabla\cdot\mathbf{q}$ ...(12)

Using (11) and (12), (10) reduces to

$$\frac{d}{dt}\int_V E\rho dV = -\int_V p\nabla\cdot\mathbf{q}\, dV \qquad \text{or} \qquad \frac{d\mathbf{I}}{dt} = -\int_V p\nabla\cdot\mathbf{q}\, dV, \text{ by (6)}$$

which proves (8).

Again the rate of work done by the fluid pressure on an element δS of S is $p\,\delta S\ \mathbf{n}\cdot\mathbf{q}$.

Hence the net rate at which work is being done by the fluid pressure is

$$\int_S p\mathbf{q}\cdot\mathbf{n}\, dS = R, \text{ (say)} \qquad ...(13)$$

Using (8) and (13), (7) reduces to

$$\text{R.H.S. of (4) } = R - d\mathbf{I}/dt \qquad ...(14)$$

Hence using (6) and (14), (4) reduces to $\dfrac{d}{dt}(T + W + I) = R,$ ...(15)

which is the desired energy equation. It is also known as "the *Volume integral form of Bernoulli's equation*".

Re-writing (15), $$\frac{d}{dt}(T+W) = R - \frac{d\mathbf{I}}{dt} = \int_S p\mathbf{q}\cdot\mathbf{n}\,dS + \int_V p\nabla\cdot\mathbf{q}\,dV \qquad \ldots(15)'$$

[on putting values of R and dI/dt]

Corollary. Energy equation for incompressible fluids.

Since $I = 0$ for incompressible fluids, (15) reduces to

$$\frac{d}{dt}(T+W) = R. \qquad \ldots(16)$$

Remark. Many problems solved so far in this chapter may also be solved by using the energy equation. This principle is used to shorten the solution. In what follows, we will give two methods to solve many problems.

The energy equation is stated as follows : *The rate of increase of energy in the system is equal to the rate at which work is done on the system.*

3.10. Illustrative solved examples.

Ex. 1. *An infinite mass of fluid is acted on by a force* $\mu/r^{3/2}$ *per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere r = c in it, show that the cavity will be filled up after an interval of time* $(2/5\mu)^{1/2}\,c^{5/4}$.

[Kanpur 1999, 2009; Meerut 2005; I.A.S. 2003]

Sol. Method I At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2 v' = r^2 v \qquad \ldots(1)$$

When the radius of the cavity is r, then

$$\text{Kinetic energy} = \int_r^\infty \frac{1}{2}(4\pi r'^2 \rho\, dr')\cdot v'^2 \qquad \left[\because \text{Kinetic energy} = \frac{1}{2}\times\text{mass}\times(\text{velocity})^2\right]$$

$$= 2\pi\rho r^4 v^2 \int_r^\infty \frac{dr'}{r'^2}, \quad \text{using (1)}$$

$$= 2\pi\rho r^3 v^2.$$

The initial kinetic energy is zero.

Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{3/2}} \qquad \text{so that} \qquad V = \frac{2\mu}{r'^{1/2}}$$

$\therefore$ the work done $= \int_r^c V\,dm$, dm being the elementary mass

$$= \int_r^c \left(\frac{2\mu}{r'^{1/2}}\right)\cdot 4\pi r'^2\,dr'\rho = 8\pi\mu\rho\int_r^c r'^{3/2}\,dr' = \frac{16}{5}\pi\rho\mu(c^{5/2} - r^{5/2})$$

We now use energy equation, namely, Increase in kinetic energy = work done

This $\Rightarrow$ $$2\pi\rho r^3 v^2 - 0 = (16/5)\times\pi\rho\mu(c^{5/2} - r^{52})$$

$$\therefore \qquad v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2}\frac{(c^{5/2}-r^{5/2})^{1/2}}{r^{3/2}} \qquad \ldots(2)$$

wherein negative sign is taken because r decreases as t increases.

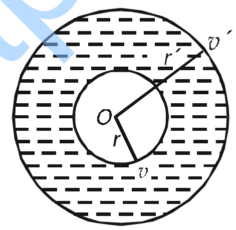
Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2} - r^{5/2})}} \qquad \text{or} \qquad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2} - r^{5/2})}}$$

Put $\quad r^{5/2} = c^{5/2}\sin^2\theta \quad$ so that $\quad (5/2)\times r^{3/2}\,dr = 2c^{5/2}\sin\theta\cos\theta\,d\theta.$

$$\therefore \qquad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/4} \sin\theta\,d\theta = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

Second Method. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin) and time t. Also, let v be the velocity at a distance r.



Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v. \qquad \text{...(1)}$$

From (1), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}. \qquad \text{...(2)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \text{...(3)}$$

Integrating (3) with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \qquad \text{...(4)}$$

When $r' = \infty$, $v' = 0$, $p = 0$. So from (4), $C = 0$. Then (4) becomes

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho}. \qquad \text{...(5)}$$

Now when $r' = r$, $v' = v$ and $p = 0$. So (5) reduces to

$$-\frac{F'(t)}{r} + \frac{1}{2}v'^2 = \frac{2\mu}{r^{1/2}}. \qquad \text{...(6)}$$

Now, $\quad$ (1) $\Rightarrow$ $F(t) = r^2 v$ $\Rightarrow$ $F'(t) = 2rv\,(dr/dt) + r^2(dv/dt)$

or $$F'(t) = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt} = 2rv^2 + r^2 v\frac{dv}{dr}, \qquad \text{as} \qquad \frac{dr}{dt} = v.$$

Hence (6) gives

$$-\frac{1}{r}\left[2rv^2 + r^2 v\frac{dv}{dr}\right] + \frac{v^2}{2} = \frac{2\mu}{r^{1/2}} \qquad \text{or} \qquad rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{2\mu}{r^{1/2}}.$$

Multiplying both sides by $2r^2$, the above equation can be written as

$$2r^3 v\,dv + 3r^2v^2\,dr = -4\mu r^{3/2}\,dr \qquad \text{or} \qquad d\,(r^3v^2) = -4\mu r^{3/2}\,dr.$$

Integrating, $r^3v^2 = -(8\mu/5)r^{5/2} + D$, D being an arbitrary constant ...(7)

When $r = c, v = 0$. So (7) gives $D = (8\mu/5)c^{5/2}$. Hence (7) reduces to

$$r^3v^2 = (8\mu/5) \times (c^{5/2} - r^{5/2})$$

or
$$v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2} \left(\frac{c^{5/2} - r^{5/2}}{r^3}\right)^{1/2},$$

taking negative sign for dr/dt since velocity increases as r decreases.

Let T be the time of filling up the cavity, then

$$T = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}\, dr}{(c^{5/2} - r^{5/2})^{1/2}}. \qquad ...(8)$$

Let $r^{5/2} = c^{5/2} \sin^2\theta$ so that $(5/2) \times r^{3/2}\, dr = c^{5/2} \sin\theta \cos\theta\, d\theta$.

$$\therefore \quad T = \frac{4}{5}\left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{c^{5/2} \sin\theta \cos\theta}{c^{5/4} \cos\theta}\, d\theta = \frac{4c^{5/4}}{5}\left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \sin\theta\, d\theta$$

or
$$T = (2/5\mu)^{1/2} \times c^{5/4}.$$

Ex. 4. *An infinite fluid in which a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure* Π *is applied at infinity, show that the time of filling up the cavity is* $a\left(\dfrac{\pi\rho}{6\Pi}\right)^{1/2} \dfrac{\Gamma(5/6)}{\Gamma(4/3)}$ **[Agra 2004, 05]**

and show that it is equivalent to $2^{5/6}\pi^2\, a(\rho/\Pi)^2\, \{\Gamma(1/3)\}^{-3}$ **[Meerut 2008, 11; Kanpur 2002]**

Sol. At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2v' = r^2v \qquad \ldots(1)$$

When the radius of the cavity is r, then

$$\text{Kinetic energy} = \int_r^\infty \frac{1}{2}(4\pi r'^2\, dr'\rho) \cdot v'^2 = 2\pi\rho r^4v^2 \int_r^\infty \frac{dr'}{r'^2}, \text{ using (1)}$$

$$= 2\pi\rho r^3v^2.$$

The initial kinetic energy is zero.

Again, the work done by the outer pressure $= \displaystyle\int_a^r 4\pi r^2\Pi(-dr) = \frac{4}{3}\rho\Pi(a^3 - r^3)$.

Then, by the energy equation, we get $2\pi\rho r^3v^2 - 0 = (4/3) \times \pi\Pi(a^3 - r^3)$

$$\therefore \quad v = \frac{dr}{dt} = -\left(\frac{2\Pi}{3\rho}\right)^{1/2} \frac{a^3 - r^3}{r^3} \qquad \ldots(2)$$

where negative sign is taken because r decreases as t increases.

Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\int_a^0 \left(\frac{3\rho}{2\Pi}\right)^{1/2} \frac{r^{3/2}dr}{\sqrt{(a^3 - r^3)}} \quad \text{or} \quad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^a \frac{r^{3/2}dr}{\sqrt{(a^3 - r^3)}}$$

Put $r^3 = a^3 \sin^2\theta$ *i.e.* $r = a \sin^{2/3}\theta$ and $dr = (2a/3) \times (\sin\theta)^{-1/3} \cos\theta\, d\theta$

$$\therefore \quad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta}\cdot\frac{2a}{3}(\sin\theta)^{-1/3}\cos\theta\, d\theta \;=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\sin^{2/3}\theta\, d\theta$$

$$=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\times\frac{\Gamma(5/6)\Gamma(1/2)}{2\Gamma(4/3)} \;= a\left(\frac{\pi\rho}{6\Pi}\right)^{1/2}\frac{\Gamma(5/6)}{\Gamma(4/3)}, \quad \text{as } \Gamma(1/2)=\sqrt{\pi} \qquad \ldots(3)$$

which is the required first part of the result.

From advanced Integral Calculus, we know that

$$\Gamma(n)\Gamma(n+1/2)=\frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad \text{(Duplication Formula)} \qquad \ldots(4)$$

and
$$\Gamma(n)\Gamma(1-n)=\pi/\sin n\pi \qquad \ldots(5)$$

Replacing n by 1/3 in (4), we get $\quad \Gamma(1/3)\Gamma(5/6)=\sqrt{\pi}\,2^{1/3}\,\Gamma(2/3)$

$$\therefore \quad \{\Gamma(1/3)\}^2\Gamma(5/6)=\sqrt{\pi}\,2^{1/3}\,\Gamma(1/3)\Gamma(2/3)=\sqrt{\pi}\,2^{1/3}\Gamma(1/3)\Gamma(1-1/3)$$
$$=\sqrt{\pi}\,2^{1/3}\times\{\pi/\sin(\pi/3)\}, \text{ by (5)}$$

Thus,
$$\{\Gamma(1/3)\}^2\,\Gamma(5/6)=\sqrt{\pi}\times 2^{1/3}\times(2\pi/\sqrt{3})$$

$$\therefore \quad \Gamma(5/6)=\sqrt{\pi}\times 2^{1/3}\times(2\pi/\sqrt{3})\times\{\Gamma(1/3)\}^{-2} \qquad \ldots(6)$$

Also
$$\Gamma(4/3)=\Gamma(1+1/3)=(1/3)\times\Gamma(1/3) \qquad \ldots(7)$$

Using (6) and (7), (3) reduces to

$$T=a\left(\frac{\rho}{\Pi}\right)^{1/2}\cdot\frac{\sqrt{\pi}}{\sqrt{6}}\times\frac{\sqrt{\pi}\times 2^{1/3}\times(2\pi/\sqrt{3})\times[\Gamma(1/3)]^{-2}}{(1/3)\times\Gamma(1/3)}$$

or
$$T=2^{5/6}\pi^2 a(\rho/\Pi)^{1/2}\,[\Gamma(1/3)]^{-3}.$$

Second Method. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and v' will be function of r' (the radial distance from the centre of the spherical shell which is taken as origin). and time t only. Let p be pressure at a distance r'. Then from the continuity equation, we have

$$r'^2\, v' = F(t) = r^2\, v. \qquad \ldots(1)$$

From (1),
$$\frac{\partial v'}{\partial t}=\frac{F'(t)}{r'^2}. \qquad \ldots(2)$$

The equation of motion is
$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}.$$

or
$$\frac{F'(t)}{r'^2}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating (3) with respect to r', we have

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=C-\frac{p}{\rho}, \text{ where } C \text{ is an arbitrary constant.} \qquad \ldots(4)$$

Initially, when $\quad r'=\infty, \quad v'=0 \quad$ and $\quad p=\Pi \quad$ so (4) $\Rightarrow C=\Pi/\rho$.

$\therefore$ (4) becomes $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p}{\rho}. \qquad ...(5)$$

Let v be the velocity and r be the radius of spherical cavity at any time t so that $v'=v$, $r'=r$ and $p=0$ (being hollow part of cavity). Then (5) reduces to

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=\frac{\Pi}{\rho}. \qquad ...(6)$$

Now, from (1), $$F(t)=r^2 v. \qquad ...(7)$$

Differentiating (7) with respect to t, we have

$$F'(t)=r^2\frac{dv}{dt}+2rv\frac{dr}{dt}=r^2\frac{dv}{dr}\frac{dr}{dt}+2rv\frac{dr}{dt}$$

or $$F'(t)=r^2v\,(dv/dr)+2rv^2, \qquad \text{as} \qquad dr/dt=v.$$

Substituting the above value of $F'(t)$ in (6), we have

$$-\frac{1}{r}\left[r^2v\frac{dv}{dr}+2rv^2\right]+\frac{1}{2}v^2=\frac{\Pi}{\rho} \qquad \text{or} \qquad rv\frac{dv}{dr}+\frac{3v^2}{2}=-\frac{\Pi}{\rho}.$$

Multiplying both sides by $2r^2dr$, we have

$$2r^3\,v\,dv+3r^2v^2\,dr=-(2\Pi/\rho)\,r^2dr.$$

Integrating, $r^3v^2=-(2\Pi/3\rho)\times r^3+D$, where D is an arbitrary constant. ...(8)

Initially, when radius of cavity $r=0$, then $v=0$. Hence (8) gives $D=2\Pi/3\rho$ and so (8) reduces to

$$r^3v^2=\frac{2\Pi}{3\rho}(a^3-r^3) \qquad \text{or} \qquad v=\frac{dr}{dt}=-\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left(\frac{a^3-r^3}{r^3}\right)^{1/3},$$

taking negative sign since v increases as r decreases.

Let T be the required time of filling up the cavity, then

$$T=-\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_a^0\left(\frac{r^3}{a^3-r^3}\right)^{1/2}dr. \qquad ...(9)$$

Putting $r=a\sin^{2/3}\theta$ so that $dr=(2a/3)\times\sin^{-1/3}\theta\cos\theta\,d\theta$, (9) gives

$$T=-\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_{\pi/2}^{0}\frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta}\cdot\frac{2a}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\sin^{2/3}\theta\,d\theta=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\Gamma(5/6)\Gamma(1/2)}{2\Gamma(4/3)}$$

$$=\frac{a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\sqrt{\pi}\,\Gamma(5/6)}{(1/3)\Gamma(1/3)}, \qquad \text{as} \qquad \Gamma\left(\frac{4}{3}\right)=\Gamma\left(1+\frac{1}{3}\right)=\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$\therefore \qquad T=a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\sqrt{\pi}\,\Gamma(5/6)}{\Gamma(1/3)}. \qquad ...(10)$$

From integral calculus, we know that

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad ...(11)$$

and

$$\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi. \qquad ...(12)$$

Putting $n = 1/3$ in (11), $\Gamma(1/3)\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(2/3).$ ...(13)

Multiplying both sides of (13) by $\Gamma(1/3)$, we get

$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(1/3)\,\Gamma(2/3)$$

or
$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(1/3)\,\Gamma(1-1/3)$$

or
$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}.\{\pi/\sin(\pi/3)\}, \text{ using (12)}$$

$$\therefore \qquad \Gamma(5/6) = \frac{\sqrt{\pi}\,2^{1/3}\,\pi}{(\sqrt{3}/2)\,[\Gamma(1/3)]^2}.$$

Substituting the above value of $\Gamma(5/6)$ in (10), we have

$$T = a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \times \frac{\sqrt{\pi}}{\Gamma(1/3)} \times \frac{\sqrt{\pi}\,2^{1/3}\,\pi}{\left(\sqrt{3}/2\right)[\Gamma(1/3)]^2}$$

or
$$T = \pi^2\,a\,(\rho/\Pi)^{1/2}\;2^{5/6}\{\Gamma(1/3)\}^{-3}.$$

Ex. 3. *A mass of fluid of density* ρ *and volume* $(4/3)\times\pi c^3$ *is in the form of a spherical shell. A constant pressure* Π *is exerted on the external surface of the shell. There is no pressure on the internal surface and no other forces act on the liquid. Initially the liquid is at rest and the internal radius of the shell is* $2c$. *Prove that the velocity of the internal surface when its radius is c, is*

$$\left(\frac{14\Pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3}-1}\right)^{1/2}$$ **[Kanpur 1997]**

Sol. At any time t, let v' be the velocity at distance r' from the centre. Let r and R be the radii and v and V the velocities of the internal and external surfaces of the shell. Then the equation of continuity yields $r'^2 v' = r^2 v$ ...(1)

Again from conservation of mass, we have

$$\frac{4}{3}\pi R^3\rho - \frac{4}{3}\pi r^3\rho = \text{constant} = \frac{4}{3}\pi c^3\rho$$

$\therefore \quad R^3 - r^3 = c^3 \quad$ so that $\quad R^3 = r^3 + c^3.$ ...(2)

Now, the initial kinetic energy is zero. Again the final kinetic energy

$$= \int_r^R \frac{1}{2}(4\pi r'^2 dr')v'^2 \;= 2\pi\rho r^4 v^2 \int_r^R \frac{dr'}{r'^2}, \text{ using (1)}$$

$$= 2\pi\rho r^4 v^2\,(1/r - 1/R) = 2\pi\rho r^4 v^2\left[\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right], \text{ using (2)}$$

Again, the work done by the external pressure Π in decreasing the shell from radius r to $2c$

$$= \int_{2c}^{r} 4\pi r^2 \Pi\,(-dr) = \frac{4\pi\Pi}{3}(8c^3 - r^3).$$

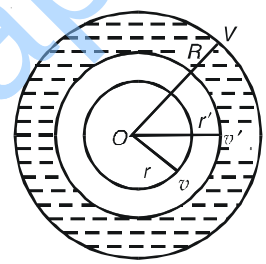
Then the energy equation yields

$$2\pi\rho r^4 v^2 \left[\frac{1}{r} - \frac{1}{(r^3 + c^3)^{1/3}}\right] = \frac{4}{3}\pi\Pi\,(8c^3 - r^3) \qquad \ldots(3)$$

The required value of velocity is given by setting $r = c$ in (3). Thus, we get

$$\rho c^4 v^2 \left(\frac{1}{c} - \frac{1}{c \times 2^{1/3}}\right) = \frac{2}{3} \times \Pi \times (7c^3) \qquad \text{or} \qquad v = \left(\frac{14\pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3} - 1}\right)^{1/2} \quad \text{as required.}$$

Second Method. Let r and R be the internal and external radii of the shell, r' be any radius, where the velocity is v' and the pressure p at any time t.



Volume of liquid $= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi c^3$ (given)

and so $\qquad R^3 = r^3 + c^3. \qquad \ldots(1)$

The equation of continuity is $r'^2 v' = F(t) = r^2 v = R^2 V \ \ldots(2)$

From (2), $\qquad \dfrac{\partial v'}{\partial t} = \dfrac{F'(t)}{r'^2}. \qquad \ldots(3)$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \quad \text{by (3)}$$

Integrating it with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad \ldots(4)$$

Initially, when $r' = R$, $v' = V$, $p = \Pi$ and when $r' = r$, $v' = v$, $p = 0$ (as there is no pressure on internal surface). Then (4) reduces to

$$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = -\frac{\Pi}{\rho} + C \qquad \ldots(5)$$

and

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = C. \qquad \ldots(6)$$

Subtracting (5) from (6), we have

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}(v^2 - V^2) = \frac{\Pi}{\rho} \qquad \text{or} \qquad -F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}\left[\frac{\{F(t)\}^2}{r^4} - \frac{\{F(t)\}^2}{R^4}\right] = \frac{\Pi}{\rho}$$

or

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{\{F(t)\}^2}{2}\left[\frac{1}{r^4} - \frac{1}{R^4}\right] = \frac{\Pi}{\rho}. \qquad \ldots(7)$$

From (2), $\qquad r^2 v = R^2 V = F(t)$

$\Rightarrow \qquad r^2(dr/dt) = R^2(dR/dt) = F(t) \qquad \Rightarrow \qquad r^2 dr = R^2 dR = F(t)\,dt. \qquad ..(8)$

Multiplying both sides of (7) by $2r^2 dr$, we get

$$-2F'(t)\, r^2\left(\frac{1}{r}-\frac{1}{R}\right)dr+\frac{\{F(t)\}^2}{2}\left[\frac{2r^2dr}{r^4}-\frac{2r^2dr}{R^4}\right]=\frac{2\Pi r^2 dr}{\rho}. \qquad ...(9)$$

Using relations (8), (9) may be rewritten as

$$-2F'(t)\, F(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{\{F(t)\}^2}{2}\left[\frac{2r^2dr}{r^4}-\frac{2R^2dR}{R^4}\right]=\frac{2\Pi r^2 dr}{\rho}$$

or $$\left(\frac{1}{r}-\frac{1}{R}\right)d\{F(t)\}^2+\{F(t)\}^2 d\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2\Pi r^2 dr}{\rho}. \qquad ...(10)$$

Integrating, $\quad \{F(t)\}^2\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2\Pi r^3}{3\rho}+D,$ D being an arbitrary constant

Initially, when $r = 2c$, $v = 0$ [so that $F(t) = 0$ by (2)]. Hence (10) gives

$0 = -(2\Pi/3\rho)c^3 + D \qquad$ or $\qquad D = (2\Pi/3\rho)c^3$.

$\therefore$ (10) reduces to $\qquad \{F(t)\}^2\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2\Pi}{3\rho}(8c^3-r^3)$

or $\qquad r^4v^2\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2\Pi}{3\rho}(8c^3-r^3)$, using (2).

or $\qquad r^4v^2\left\{\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right\}=\frac{2\Pi}{3\rho}(8c^3-r^3)$, using (1)

or $$v^2=\frac{2\Pi}{3\rho}\frac{8c^3-r^3}{r^4\left\{1/r-1/(r^3+c^3)^{1/3}\right\}}. \qquad ...(11)$$

giving velocity v at the inner surface of the cavity. Hence the velocity of the internal surface (where $r = c$) is given by

$$v^2=\frac{2\Pi}{3\rho}\frac{7c^3}{c^4\{1/c-1/(c\times 2^{1/3})\}} \qquad \text{or} \qquad v=\left[\frac{14\Pi}{3\rho}\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2}.$$

Ex. 4. *A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subjected to a given constant pressure* Π, *no other force being in action on the liquid. The solid, sphere, suddenly shrinks into a concentric sphere, determine the subsequent motion and the impulsive action on the sphere.*

[Allahabad 2000; Kerala 2004]

Sol. At time t, let v' be the velocity at distance r' from the centre. Again, let R, r be the radii of the external and internal boundaries at time t, and V, v their velocities. Then the equation of continuity yields. $\qquad r'^2v' = r^2v \qquad ...(1)$

Again from conservation of mass, we have

$$\frac{4}{3}\pi R^3\rho-\frac{4}{3}\pi r^3\rho=\frac{4}{3}\pi b^3\rho-\frac{4}{3}\pi a^3\rho$$

so that $$R^3 - r^3 = b^3 - a^3 = c^3, \text{ (say)} \qquad ...(2)$$

$$\therefore \qquad R = (r^3 + c^3)^{1/3}.$$

Now the initial kinetic energy is zero. Again the final kinetic energy

$$= \int_r^R \frac{1}{2}(4\pi r'^2\, dr'\, \rho)\, v'^2 = 2\pi\rho \int_r^R r'^2\, v'^2\, dr' = 2\pi\rho \int_r^R \frac{r^4 v^2}{r'^2} dr', \text{ using (1)}$$

$$= 2\pi r^4 \rho v^2\, (1/r - 1/R)$$

and the work done by the outer pressure

$$= \int_b^R 4\pi R^2\, \Pi\, (-dR) = \frac{4}{3}\pi\Pi\,(b^3 - R^3) = \frac{4}{3}\pi\Pi\,(a^3 - r^3), \quad \text{using (2)}$$

Therefore, using the energy equation, we have

$$2\pi r^4 \rho v^2\, (1/r - 1/R) = (4/3) \times \pi\Pi\,(a^3 - r^3)$$

$$\therefore \quad v = \left(\frac{2\Pi}{3\rho}\right)^{1/2} \frac{(a^3 - r^3)^{1/2}}{r^2\,(1/r - 1/R)^{1/2}} \quad \text{or} \quad v = \left(\frac{2\Pi}{3\rho}\right)^{1/2} \frac{(a^3 - r^3)^{1/2}}{r^2\left\{1/r - 1/(r^3 + c^3)^{1/3}\right\}} \qquad ...(3)$$

Expression for impulsive action on the sphere. Let r be the radius of the solid sphere and $\tilde{\omega}$ the impulsive pressure at distance r' from its centre. Then we have

$$d\tilde{\omega} = -\rho v'\, dr' = -\rho \frac{r^2 v dr'}{r'^2}, \quad \text{using (1)}$$

Integrating, $$\tilde{\omega} = \frac{\rho r^2 v}{r'} + C, \quad C \text{ being an arbitrary constant}$$

Given that $\tilde{\omega} = 0$ when $r' = R$. Hence $C = -(\rho r^2 v)/R$. So $\tilde{\omega} = \rho r^2 v\,(1/r - 1/R)$.

Thus the impulsive pressure when $r' = r$ is given by $\tilde{\omega} = \rho r^2 v (1/r - 1/R)$.

Hence the whole impulsive pressure on the sphere $= 4\pi r^2 \tilde{\omega} = 4\pi r^3 v\,(R - r)/R$,

and the whole momentum destroyed

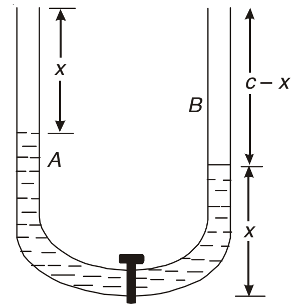
$$= \int_r^R (4\pi r'^2 dr'\, \rho)\, v' = 4\pi\rho \int_r^R v' r'^2 dr' = 4\pi\rho \int_r^R r^2 v\, dr, \quad \text{using (1)}$$

$$= 4\pi\rho\, r^2 v\,(R - r).$$

Ex. 5. *Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water and the other with air of such a density as to support a column h of water, h being lees than c. If a communication be opened between them at their bases, the height x, to which the water rises, is given by the equation.* $cx - x^2 + ch \log \{(c - x)/c\} = 0$.

[Meerut 1997; Rajasthan 2000]

Sol. Let A (shown on L.H.S.) and B (shown on R.H.S.) be two cylinders containing water and air respectively. Let α be the cross-section of each cylinder. The water and air are at rest before and after the communication is set up between the two cylinders. Hence the initial and final kinetic energies are zero.



Now, initial potential energy V_1 due to water height in A is given by

$$V_1 = \int_0^c g\rho\,\alpha x'\,dx' = \frac{1}{2}g\rho\alpha c^2.$$

After communication is set up, a height x of water rises in B and hence a height $(c - x)$ of water is left behind in A. Therefore, the final potential energy V_2 due to water in A and B is given by

$$V_2 = \int_0^{c-x} g\rho\,\alpha x'\,dx' + \int_0^x g\rho\alpha x'\,dx'$$

$$= \frac{1}{2}g\rho\,\alpha[(c-x)^2 + x^2] = \frac{1}{2}g\rho\alpha(c^2 - 2cx + 2x^2).$$

$\therefore$ The work done against gravity

$$= V_1 - V_2 = \text{loss in potential energy} = (1/2)\times g\rho\alpha(2cx - 2x^2) = g\rho\alpha x(c-x).$$

Again, work is also done against the compressions of air in B. Let p be the pressure of the air when the water stands to a height x'. Assume that temperature remains constant so that Boyle's law is applicable. Thus, we have

$$g\rho h\alpha c = \rho\alpha(c - x') \qquad \text{so that} \qquad p = (g\rho hc)/(c - x')$$

Thus the total work done by this pressure $= -\int_0^x \frac{g\rho hc\alpha}{c - x'}dx' = g\rho hc\alpha \log\frac{c-x}{c}.$

Now by energy equation, we have

Increase in K.E. = total work done so that total work done = 0

$$\therefore\ g\rho\alpha x(c-x) + g\rho hc\alpha \log\{(c-x)/c\} = 0 \qquad \text{or} \qquad cx - x^2 + ch\log\{(c-x)/c\} = 0.$$

Ex. 6. *Show that the rate per unit of time at which work is done by the internal pressures between the parts of a compressible fluid obeying Boyle's law is*

$$\iiint \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dx\,dy\,dz,$$

where p is the pressure and u, v, w the velocity components at any point and the integration extends through the volume of the fluid.

Sol. Let W be the work done in compressing the fluid, p is the pressure and dV an elementary volume. Then, we have $\quad W = \int p(-dV) = -\int p\,dV.$

Hence the rate per unit time of work done is given by

$$\frac{DW}{Dt} = -\iiint \frac{Dp}{Dt}dV. \qquad \text{...(1)}$$

The equation of continuity is

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \qquad \text{...(2)}$$

Since the compressible fluid obeys Boyle's law, hence we have

$$p = k\rho \qquad \text{so that} \qquad \rho = p/k. \qquad \text{...(3)}$$

Using (3), (2) becomes

$$\frac{D}{Dt}\left(\frac{p}{k}\right) + \frac{p}{k}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad \text{so that} \qquad \frac{Dp}{Dt} = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Hence (1) gives $$\frac{DW}{Dt} = \iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dV,$$

which gives the required rate per unit of time at which work is done.

Ex. 7. *A mass of perfect incompressible fluid of density ρ is bounded by concentric spherical surfaces. The outer surface is contained by a flexible envelope which exerts continuously uniform pressure Π and contracts from radius R_1 to radius R_2. The hollow is filled with a gas obeying Boyle's law, its radius contracts from c_1 to c_2 and the pressure of gas is initially, p_1. Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity v of the inner surface when the configuration (R_2, c_2) is reached is given by*

$$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\}\Big/\left(1-\frac{c_2}{R_2}\right)$$ **[I.A.S. 2005]**

Sol. Let p_2 be the pressure of the gas when the internal radius is c_2. Then, by Boyle's law,

$(4/3)\times \pi c_1^3 p_1 = (4/3)\times \pi c_2^3 p_2$ so that $p_2 = (c_1^3/c_2^3)p_1$. ...(1)

Equation of continuity is $r'^2 v' = F(t) = c_2^2 v$. ...(2)

From (2), $v' = c_2^2 v/r'^2$. ...(3)

Now, the initial kinetic energy (K.E) = 0.

and final K.E. $= \displaystyle\int_{c_2}^{R_2} \frac{1}{2}(4\pi r'^2 dr'\rho)v'^2 = 2\pi\rho c_2^4 v^2 \int_{c_2}^{R_2}\frac{dr'}{r'^2}$, by (3)

$$= 2\pi\rho c_2^4 v^2\left(\frac{1}{c_2}-\frac{1}{R_2}\right) = 2\pi\rho c_2^3 v^2\left(1-\frac{c_2}{R_2}\right). \quad ...(4)$$

Now, work done W by the external pressure Π and the internal pressure p_2 is given by

$$W = \int_{R_1}^{R_2} 4\pi R_2^2\,\Pi(-dR_2)+\int_{c_1}^{c_2} 4\Pi c_2^2 p_2 dc_2 = -\frac{4}{3}\pi\Pi\left[\frac{R_2^3}{3}\right]_{R_1}^{R_2} + 4\pi\int_{c_1}^{c_2} c_2^2 .\frac{c_1^3}{c_2^3}p_1 dc_2, \text{ using (1)}$$

$$= (4/3)\times\pi\Pi\,(R_1^3-R_2^3)+4\pi p_1 c_1^3\left[\log c_2\right]_{c_1}^{c_2}$$

$$\therefore \quad W = (4/3)\times\pi\Pi\,(R_1^3-R_2^3)+4\pi p_1 c_1^3 \log(c_2/c_1). \quad ...(5)$$

Since the mass of the fluid remains constant, we have

$$(4/3)\times\pi\,(R_2^3-c_2^3) = (4/3)\times\pi(R_1^3-c_1^3) \quad \text{or} \quad R_1^3-R_2^3 = c_1^3-c_2^3. \quad ...(6)$$

Using (6), (5) reduces to

$$\text{The work done} = W = (4/3)\,\pi\,\Pi(c_1^3-c_2^3)+4\pi p_1 c_1^3\log(c_2/c_1). \quad ...(7)$$

Now, from the energy equation, Increase in K.E. = total work done

or $$2\pi\rho c_2^3 v^2(1-c_2/R_2) = (4/3)\times\pi\Pi(c_1^3-c_2^3)+4\pi p_1 c_1^3\log(c_2/c_1)$$

or $$\frac{1}{2}v^2 c_2^3\left(1-\frac{c_2}{R_2}\right) = c_1^3\left[\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right]$$

or $$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\}\Big/\left(1-\frac{c_2}{R_2}\right).$$

Ex. 8. *A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle and the distances of its nearer and farther extremities from the vertex at the time t are r and r′, show that*

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left[3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right]=0.$$ **[Purvanchal 2005, Agra 2005]**

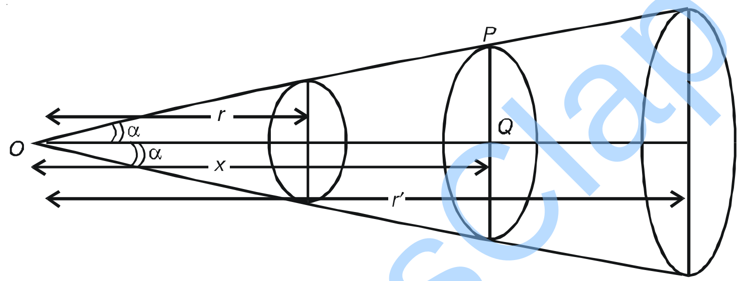
Show that it follows also by taking vis-viva of the mass of the liquid as constant; and that the velocity V of the inner surface is given by the equations

$V^2 = Cr'/(r'-r)r^3$, $\quad r'^3 - r^3 = c^3$, $\quad$ *C, c being constants.*

Sol. At any time t, let p' be the pressure at a distance x from the vertex and v' be the velocity there. Let α be the semi-vertical angle of the conical tube. Then the equation of continuity is given by

$v'(PQ)^2 = f(t)$ or $v'(x \tan \alpha)^2 = f(t)$

or $v'x^2 = F(t)$, where $F(t) = \cot^2\alpha\, f(t)$. ...(1)



The equation of motion is $$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial x}=-\frac{1}{\rho}\frac{\partial p'}{\partial x}.$$ ...(2)

From (1), $\partial v'/\partial t = (1/x^2)\, F'(t)$. ...(3)

Using (3), (2) $\Rightarrow$ $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p'}{\partial x}.$$ ...(4)

Integrating (4) with respect to x, we have

$$-\frac{F'(t)}{x}+\frac{1}{2}v'^2=C-\frac{p'}{\rho},$$ C being an arbitrary constant ...(5)

Let v and v' be the velocities when $x = r$ and $x = r'$ respectively and p be the pressure there. Then (5) gives

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=C-\frac{p}{\rho}$$ ...(6)

and $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=C-\frac{p}{\rho}.$$ ...(7)

Subtracting (6) from (7), $-F'(t)(1/r-1/r')+(1/2)\times(v^2-v'^2)=0.$ ...(8)

From the equation of continuity, $r^2v = r'^2v' = F(t)$, ...(9)

where $v = dr/dt$ and $v' = dr'/dt$. ...(10)

From (9), $v' = r^2v/r'^2$. Then (8) becomes

$$-F'(t)\left(\frac{1}{r}-\frac{1}{r'}\right)+\frac{1}{2}\left(v^2-\frac{r^4v^2}{r'^4}\right)=0 \quad\text{or}\quad -F'(t)\left(\frac{r'-r}{rr'}\right)+\frac{1}{2}v^2\left(\frac{r'^4-r^4}{r'^4}\right)$$

or $$F'(t)\left(\frac{r'-r}{rr'}\right)-\frac{v^2}{2}\frac{(r'-r)(r'^3+r'^2r+r'r^2+r^3)}{r'^4}=0$$

or $$\frac{2F'(t)}{r}-v^2\left(\frac{r'^3+r'^2r+r'r^2+r^3}{r'^3}\right)=0. \qquad \text{...(11)}$$

From (9), $$F'(t)=\frac{d}{dt}(r^2v)=\frac{d}{dt}\left(r^2\frac{dr}{dt}\right)=2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}$$

$\therefore$ (11) $\Rightarrow$ $$\frac{2}{r}\left[2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}\right]-\left(\frac{dr}{dt}\right)^2\left(1+\frac{r}{r'}+\frac{r^2}{r'^2}+\frac{r^3}{r'^3}\right)=0$$

or $$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left(3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right)=0.$$

Second part.

The vis-viva = 2 K.E. = $\int_r^{r'}\pi(x\tan\alpha)^2\,dx\,\rho\,v'^2=\pi\rho\tan^2\alpha\{F(t)\}^2\int_r^{r'}\frac{dx}{x^2}$, by (1)

$$=\pi\rho\tan^2\alpha\,\{F(t)\}^2\left[-\frac{1}{x}\right]_r^{r'}=\pi\rho\tan^2\alpha\{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right). \qquad \text{...(12)}$$

By the principal of conservation of vis-viva, we have

$\pi\rho\tan^2\alpha\,\{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right)$ = constant or $\{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right)$ = constant = C

or $(r^2v)^2\,(1/r-1/r')=C$ using (9)

or $v^2=Cr'/(r'-r)r^3$ or $V^2=Cr'/(r'-r)r^3$, taking $V=v$

Since the mass is constant, volume will also be constant.

Hence $(1/3)\times\pi(r'\tan\alpha)^2\,r'-(1/3)\times\pi\,(r\tan\alpha)^2\,r$ = constant

so that r'^3-r^3 = constant = c^3, say.

Ex. 9. *A spherical mass of liquid of radius b has a concentric spherical cavity of radius a, which contains gas at pressure p whose mass may be neglected; at every point of the external boundary of the liquid an impulsive pressure ω per unit area is applied. Assuming that the gas obeys Boyls' law, show that when the liquid first comes to rest, the radius of internal spherical surface will be a exp.*$\{\{-\omega^2b/(2p\rho a^2(b-a)\}\}$*], where exp x stands for* e^x.

Sol. Let v' be the velocity at a distance r' from the centre of the spherical cavity at any time t. Then the equation of continuity is

$$r'^2v'=F(t)=b^2V, \qquad \text{...(1)}$$

where we have assumed that $v'=V$ when $r'=b$.

Let ω' be the impulsive pressure at a distance r', then

$$d\omega'=-\rho v'\,dr'=-\rho(b^2V/r'^2)\,dr', \text{ by (1)}$$

Integrating, $\omega'=(\rho b^2V/r')+C$, C being an arbitrary constant. ...(2)

Given that, when $r'=b$, $\omega'=\omega$ and when $r'=a$, $\omega'=0$.

$\therefore$ (2) $\Rightarrow$ $\omega=(\rho b^2V/b)+C$ and $0=(\rho b^2V/a)+C$.

Subtracting, these give $\omega=(\rho bV/a)\,(a-b)$. ...(3)

The initial kinetic energy $= \int_a^b \frac{1}{2}(4\pi r'^2 \rho dr')v'^2 = 2\pi\rho \int_a^b r'^2 \cdot \frac{b^4 V^2}{r'^4} dr'$, by (1)

$$= 2\pi\rho b^4 V^2 \int_a^b \frac{dr'}{r'^2} = 2\pi\rho b^4 V^2 \left[-\frac{1}{r'}\right]_a^b$$

$$= 2\pi\rho b^4 V^2 \left[-\frac{1}{b}+\frac{1}{a}\right] = \frac{2\pi\rho b^3 V^2}{a}(b-a). \qquad ...(4)$$

Again, Final kinetic energy = 0. ...(5)

During the compression let r be the radius of the internal cavity and p_1 the pressure of the gas there. Since the gas obey Boyle's law, we have

$$(4/3) \times \pi r^3 \times p_1 = (4/3) \times \pi a^3 \times p \quad \Rightarrow \quad p_1 = a^3 p/r^3. \qquad ...(6)$$

Now, the work done by internal pressure, *i.e.*, work done in compression of the gas from a sphere of radius a to a sphere of radius r

$$= \int_a^r 4\pi r^2 \; p_1 dr = \int_a^r 4\pi r^2 (a^3 p/r^3)\, dr, \text{ by (6)}$$

$$= 4\pi a^3 p \int_a^r \frac{dr}{r} = 4\pi a^3 p \log\frac{r}{a}.$$

Now, by energy equation, increase in K.E. = work done

or Final K.E. − initial K.E. = work done

or $$0 - \frac{2\pi\rho b^3 V^2}{a}(b-a) = 4\pi a^3 p \log\frac{r}{a} \quad \text{or} \quad \log\frac{r}{a} = -\frac{2\pi\rho b^3 (b-a)}{4\pi a^4 p}\cdot V^2$$

or $$\log\frac{r}{a} = -\frac{2\pi\rho b^3 (b-a)}{4\pi a^4 p}\cdot\frac{\omega^2 a^2}{\rho^2 b^2 (a-b)^2}$$

or $$\log\frac{r}{a} = -\left\{\frac{\omega^2 b}{2p\rho^2 a^2 (b-a)}\right\} \quad \text{or} \quad r = a \exp\left\{-\frac{\omega^2 b}{2p\rho a^2 (b-a)}\right\}$$

EXERCISE 3(C)

1. An infinite mass of liquid acted upon by no forces is at rest and a spherical portion of radius c is suddenly annihilated; the pressure Π at an infinite distance being supposed to remain constant, prove that the pressure at a distance r from the centre of the space in instantaneously diminished in the ratio $(r-c)/r$ and that the cavity will be filled up in the time

$$\sqrt{\left(\frac{\pi\rho c^2}{6\Pi}\right)}\frac{\Gamma(5/6)}{\Gamma(4/3)}.$$

2. A spherical globule of gas of radius a and at pressure P extends in an infinite mass of liquid of density ρ in which the pressure at infinity is zero. The gas is initially at rest and its pressure and volume are governed by the equation $pv^{4/3}$ = const. Prove that the gas doubles its radius in time $(28a/15)\sqrt{2\rho P}$. **[I.A.S. 1999]**

3. An infinite fluid in which there is a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, find the rate at which the radius of the cavity diminishes.

3.11. Lagrange's hydrodynamical equations. [Ranchi 2010; Kanpur 2003, 04]

Let a, b, c be the initial co-ordinates of a particle and x, y, z the co-ordinates of the same particle at time t. Then (refer Lagrangian method in Art. 2.1 in chapter 2) we know that a, b, c, t are the independent variables. We wish to obtain x, y, z in terms of a, b, c and t and hence discuss completely the motion.

Now at time t the component accelerations of the fluid element $\delta x \delta y \delta z$ are $\partial^2 x/\partial t^2$, $\partial^2 y/\partial t^2$, $\partial^2 z/\partial t^2$. Let V be the force potential for the external forces. Then we have (noting that $X = -\partial V/\partial x$, $Y = -\partial V/\partial y$, and $Z = -\partial V/\partial z$) as in Art. 3.1

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(1)}$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(2)}$$

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{...(3)}$$

We now try to get equations containing only differentiations with respect to a, b, c and t. To this end, we multiply (1), (2) and (3) by $\partial x/\partial a$, $\partial y/\partial a$ and $\partial z/\partial a$ then add. Thus, we get*

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \qquad \text{...(4)}$$

Similarly,
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial V}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \qquad \text{...(5)}$$

and
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial V}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \qquad \text{...(6)}$$

These equations, together with the equation of continuity

$$\rho \frac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_0, \qquad \text{...(7)}$$

are known as *Lagrange's Hydrodynamical Equations*.

3.12. Cauchy's integrals.

Let a, b, c be the initial co-ordinates of a particle and x, y, z the coordinates of the same particle at time t. Then (refer Lagrangian method in Art. 2.1 in chapter 2) we know that a, b, c, t are the independent variables.

Now at time t the component accelerations of the fluid element $\delta x \delta y \delta z$ are $\partial^2 x/\partial t^2$, $\partial^2 y/\partial t^2$, $\partial^2 z/\partial t^2$. Let V be the force potential for the external forces. Then we have (noting that $X = -\partial V/\partial x$, $Y = -\partial V/\partial y$ and $Z = -\partial V/\partial z$) as in Art. 3.1

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(1)}$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(2)}$$

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* Use results: $\dfrac{\partial V}{\partial a} = \dfrac{\partial V}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial V}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial V}{\partial z}\dfrac{\partial z}{\partial a}$ etc.; $\dfrac{\partial p}{\partial a} = \dfrac{\partial p}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial p}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial p}{\partial z}\dfrac{\partial z}{\partial a}$ etc.

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(3)$$

Taking ρ as a function of p, we take $$Q = V + \int \frac{dp}{\rho} \qquad ...(4)$$

Then from (4), we have

$$-\frac{\partial Q}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \qquad ...(5)$$

$$-\frac{\partial Q}{\partial b} = -\frac{\partial V}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \qquad ...(6)$$

$$-\frac{\partial Q}{\partial c} = -\frac{\partial V}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \qquad ...(7)$$

Multiplying (1), (2), (3) by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$ respectively, and adding, we have

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a}$$

or $$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a}, \text{ using (5)} \qquad ...(8)$$

Similarly $$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(9)$$

and $$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(10)$$

Writing u, v, w for $\partial x/\partial t$, $\partial y/\partial t$, $\partial z/\partial t$, (8), (9) and (10) may be re-written as

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial a} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \qquad ...(11)$$

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(12)$$

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(13)$$

Differentiating (12) and (13) partially w.r.t. to c and b respectively, subtracting and noting that $\frac{\partial}{\partial c}\left(\frac{\partial Q}{\partial b}\right) = \frac{\partial}{\partial b}\left(\frac{\partial Q}{\partial c}\right)$ etc., we get

$$\left(\frac{\partial^2 u}{\partial b \partial t}\frac{\partial x}{\partial c} - \frac{\partial^2 u}{\partial c \partial t}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial^2 v}{\partial b \partial t}\frac{\partial y}{\partial c} - \frac{\partial^2 v}{\partial c \partial t}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial^2 w}{\partial b \partial t}\frac{\partial z}{\partial c} - \frac{\partial^2 w}{\partial c \partial t}\frac{\partial z}{\partial b}\right) = 0$$

or $$\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) - \frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t \partial c} + \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t \partial b}\right\} + \text{two similar terms} = 0$$

or $$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \text{two similar terms} = 0, \text{ as } \frac{\partial^2 x}{\partial c \partial t} = \frac{\partial u}{\partial c}, \quad \frac{\partial^2 x}{\partial t \partial b} = \frac{\partial u}{\partial b} \quad \text{and} \quad u = \frac{\partial x}{\partial t}$$

or $$\frac{\partial}{\partial t}\left\{\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right)+\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right)+\left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right)\right\}=0$$

Integrating the above equation with respect to t and taking u_0, v_0, w_0 as initial values, we get

$$\frac{\partial u}{\partial b}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\frac{\partial x}{\partial b}+\frac{\partial v}{\partial b}\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\frac{\partial y}{\partial b}+\frac{\partial w}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\frac{\partial z}{\partial b}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c}, \quad \text{...(14)}$$

where we have used the following results :

Initially : $x = a$, $y = b$ and $z = c$ so that $\partial x/\partial a = 1$, $\partial x/\partial b = 0$, $\partial x/\partial c = 0$ etc.

Now, $$\frac{\partial u}{\partial a}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial a} \text{ etc.} \quad \text{...(15)}$$

Making use of relations of the type (15), (14) may be re-wirtten as

$$\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\frac{\partial(y,z)}{\partial(b,c)}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\frac{\partial(z,x)}{\partial(b,c)}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial(x,y)}{\partial(b,c)}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c} \quad \text{...(14)'}$$

Let ξ, η, ζ be the vorticity components. Then, we have [refer Art. 2.27 of chapter 2]

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \qquad \eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \qquad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

Then the above equation (14) becomes

$$\xi\frac{\partial(y,z)}{\partial(b,c)}+\eta\frac{\partial(z,x)}{\partial(b,c)}+\zeta\frac{\partial(x,y)}{\partial(b,c)}=\xi_0 \quad \text{...(16)}$$

Similarly, $$\xi\frac{\partial(y,z)}{\partial(c,a)}+\eta\frac{\partial(z,x)}{\partial(c,a)}+\zeta\frac{\partial(x,y)}{\partial(c,a)}=\eta_0 \quad \text{...(17)}$$

and $$\xi\frac{\partial(y,z)}{\partial(a,b)}+\eta\frac{\partial(z,x)}{\partial(a,b)}+\zeta\frac{\partial(x,y)}{\partial(a,b)}=\zeta_0, \quad \text{...(18)}$$

where ξ_0, η_0, ζ_0 are the initial vorticity components.

The equation of continuity in Lagrangian system is

$$\rho\frac{\partial(x,y,z)}{\partial(a,b,c)}=\rho_0. \quad \text{...(19)}$$

Multiplying (16), (17), (18) by $\partial x/\partial a$, $\partial x/\partial b$, $\partial x/\partial c$ respectively, adding and using (19), we obtain

$$\frac{\xi}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c} \quad \text{...(20)}$$

Similarly, $$\frac{\eta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \quad \text{...(21)}$$

and $$\frac{\zeta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \quad \text{...(22)}$$

These are known as *Cauchy's integrals.*

We now state and prove the following theorem.

Statement : *The motion of a inviscid fluid under conservative forces, if once irrotational, is* *always irrotational.*

OR

When the external forces are conservative and are derived from a single valued potential and pressure is a function of density only, then if once the motion of a non-viscous fluid is irrotational, it remains irrotational even afterwards.

Proof. Let the motion be initially irrotational so that $\xi_0 = \eta_0 = \zeta_0 = 0$. Then (20), (21) and (22) show that $\xi = \eta = \zeta = 0$ are always zero. Thus if once the motion is irrotational, it remains irrotational even afterwards.

3.13. Helmholtz equations or Helmholtz vorticity equations

[G.N.D.U. Amritsar 1999, Kanpur 2000]

The Euler's equations of motion are :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(1b)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(1c)$$

Let V be the potential function of the external forces and let ρ be a function of p. Then $1(a)$ may be re-written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or
$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(2)$$

Let $\Omega = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$ be the vorticity vector so that (ξ, η, ζ) are the vorticity components or the components of spin. These are given by

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad ...(3)$$

Let
$$q^2 = u^2 + v^2 + w^2$$

Then
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = \frac{1}{2}\frac{\partial q^2}{\partial x} \qquad ...(4)$$

Using (3) and (4), (2) reduces to

$$\frac{\partial u}{\partial t} - 2v\zeta + 2w\eta = -\frac{\partial}{\partial x}\left(V + \frac{1}{2}q^2 + \int\frac{dp}{\rho}\right) \qquad ...(5)$$

Let
$$Q = V + \frac{1}{2}q^2 + \int\frac{dp}{\rho} \qquad ...(6)$$

Then (5) reduces to

$$\frac{\partial u}{\partial t} - 2v\zeta + 2w\eta = -\frac{\partial Q}{\partial x} \qquad ...(7)$$

Similarly, 1(b) and 1(c) may be re-written as :

$$\frac{\partial v}{\partial t} - 2w\xi + 2u\zeta = -\frac{\partial Q}{\partial y} \qquad ...(8)$$

and
$$\frac{\partial w}{\partial t} - 2u\eta + 2v\xi = -\frac{\partial Q}{\partial z} \qquad ...(9)$$

Differentiating (8) and (9) partially w.r.t. 'z' and 'y' and using the fact $\partial^2 Q/\partial z\partial y = \partial Q/\partial y\partial z$, we obtain

$$\frac{\partial^2 v}{\partial z\partial t} - 2w\frac{\partial \xi}{\partial z} - 2\xi\frac{\partial w}{\partial z} + 2u\frac{\partial \zeta}{\partial z} + 2\zeta\frac{\partial u}{\partial z} = \frac{\partial^2 w}{\partial y\partial t} - 2u\frac{\partial \eta}{\partial y} - 2\eta\frac{\partial u}{\partial y} + 2v\frac{\partial \xi}{\partial y} + 2\xi\frac{\partial v}{\partial y}$$

or
$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - 2u\left(\frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) + 2v\frac{\partial \xi}{\partial y} + 2w\frac{\partial \xi}{\partial z} + 2\xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - 2\eta\frac{\partial u}{\partial y} - 2\zeta\frac{\partial u}{\partial z} = 0 \qquad ...(10)$$

From (3), it easily follows that
$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 \qquad ...(11)$$

Using (11), (10) reduces to

$$2\frac{\partial \xi}{\partial t} + 2u\frac{\partial \xi}{\partial x} + 2v\frac{\partial \xi}{\partial y} + 2w\frac{\partial \xi}{\partial z} + 2\xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - 2\xi\frac{\partial u}{\partial x} - 2\eta\frac{\partial u}{\partial z} - 2\zeta\frac{\partial u}{\partial z} = 0$$

or
$$\frac{D\xi}{Dt} + \xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right) \qquad ...(12)$$

Now the equation of continuity is

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad \text{so that} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{D\rho}{Dt}. \qquad ...(13)$$

Using (13), (12) becomes

$$\frac{D\xi}{Dt} - \frac{\xi}{\rho}\frac{D\rho}{Dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} \quad \text{or} \quad \frac{1}{\rho}\frac{D\xi}{Dt} - \frac{\xi}{\rho^2}\frac{D\rho}{Dt} = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

or
$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} \qquad ...(14a)$$

Similarly, we have

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z} \qquad ...(14b)$$

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad ...(14c)$$

We now re-write (14a), (14b), (14c) in another form. Using (3), we observe that

$$\frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} = \frac{\eta}{\rho}\left[\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\right] + \frac{\zeta}{\rho}\left[\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial x}\right]$$

$$= \frac{\eta}{\rho}\cdot(-2\zeta) + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\cdot(2\eta) + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} = \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta\partial w}{\rho\partial x} \qquad ...(15)$$

Using (15), (14a) reduces to

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} \qquad \text{...(16}a\text{)}$$

Similarly, (14*b*) and (14*c*) reduce to

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial y} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial y} \qquad \text{...(16}b\text{)}$$

and

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial z} + \frac{\eta}{\rho}\frac{\partial v}{\partial z} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z}. \qquad \text{...(16}c\text{)}$$

Equations (16*a*), (16*b*), (16*c*) are known as *Helmholtz's equations*. Let at any instant $t, \xi = \eta = \zeta = 0$. Then the above equations reduce to

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = 0 \qquad \text{...(17)}$$

or

$$\frac{D\xi}{Dt} = \frac{D\eta}{Dt} = \frac{D\zeta}{Dt} = 0, \quad \text{if } \rho = \text{constant} \qquad \text{...(18)}$$

Equation (18) shows that ξ, η, ζ must be constant. Since these are zero at time $t = 0$, it follows that $\xi = \eta = \zeta = 0$ at all time afterwards.

Thus those elements of fluid which at any instant have no rotation remain during the motion without rotation.

We discuss the general case by taking $\rho \neq$ constant. Let $\partial u/\partial x, \partial v/\partial x,$ etc. be all finite and let L denote their superior limit. Then $\xi/\rho, \eta/\rho, \zeta/\rho$ cannot increase faster than if they satisfied the equations

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = L(\xi + \eta + \zeta)/\rho \qquad \text{...(19)}$$

Let

$$\xi + \eta + \zeta = \rho W \qquad \text{...(20)}$$

Then (19) reduces to

$$\frac{D}{Dt}\left(\frac{\xi}{\rho} + \frac{\eta}{\rho} + \frac{\zeta}{\rho}\right) = \frac{DW}{Dt} = 3LW,$$

so that if W is not zero, by dividing by W and integrating, we have

$$W = Ce^{3Lt}, \; C \text{ being an arbitrary constant} \qquad \text{...(21)}$$

But $\xi = \eta = \zeta = 0$ at $t = 0$. So $W = 0$ at $t = 0$. Hence (21) shows that $C = 0$ and so W is always zero. But W is the sum of three quantities ξ, η, ζ which evidently cannot be negative. It follows that $\xi = \eta = \zeta = 0$. Moreover as ξ, η, ζ remain zero when they satisfy (19), still more will they do so when they satisfy (16*a*) to (16*c*)

Thus, in general, if the motion is irrotational at any instant, it must be so for all time. In other words, if once, the velocity potential exists, it exists for all time. **T***his is known as the principle of permanence of irrotational motion.*

An illustrative solved example

Prove that in the steady motion of an incompressible liquid, under the action of conserrative forces, we have $\xi(\partial u/\partial x) + \eta(\partial u/\partial y) + \zeta(\partial u/\partial z) = 0$ *and two similar equation in v and w.*

Sol. Helmholtz equations are given by

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} \qquad ...(1a)$$

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z} \qquad ...(1b)$$

and

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad ...(1c)$$

For the steady incompressible liquid, $\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = 0$

$\therefore$ $(1a) \Rightarrow$ $\xi(\partial u/\partial x) + \eta(\partial u/\partial y) + \zeta(\partial u/\partial z) = 0$

$(1b) \Rightarrow$ $\xi(\partial v/\partial x) + \eta(\partial v/\partial y) + \zeta(\partial v/\partial z) = 0$

and $(1c) \Rightarrow$ $\xi(\partial w/\partial x) + \eta(\partial w/\partial y) + \zeta(\partial w/\partial z) = 0$

OBJECTIVE QUESTIONS ON CHAPTER 3

Multiple choice questions

Choose the correct alternative from the following questions

1. The equation for impulsive action is

(*i*) $\mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} + (1/\rho)\nabla\tilde{\omega}$ (*ii*) $\mathbf{q}_2 + \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega}$

(*iii*) $\mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega}$ (*iv*) $\mathbf{q}_1 - \mathbf{q}_2 = \mathbf{I} + \rho\nabla\tilde{\omega}$ **[Kanpur 2001]**

2. The motion of a inviscid fluid under conservative forces, if once irrorational, is always

(*i*) rotational (*ii*) irrotational (*iii*) laminar (*iv*) None of these

3. If $\tilde{\omega}$ denotes the impulsive pressure and external impulsive body forces are absent, then

(*i*) $\nabla^2\tilde{\omega} = 0$ (*ii*) $\nabla\tilde{\omega} = 0$ (*iii*) $\nabla^2\tilde{\omega} \neq 0$ (*iv*) None of these

4. Euler's equation of motion in *x*-direction is

(*i*) $Du/Dt = X - (1/\rho)\times(\partial p/\partial x)$ (*ii*) $Du/Dt = X + (1/\rho)\times(\partial p/\partial x)$

(*iii*) $\partial u/\partial t = X - (1/\rho)\times(\partial p/\partial x)$ (*iv*) $\partial u/\partial t = X + (1/\rho)\times(\partial p/\partial x)$

Answers/Hints to objective type questions

1. (*iii*). See Eq. (3), Art. 3.6 **2.** (*ii*). Refer Art. 3.1.2

3. (*i*). See Cor. 1, Art. 3.6 **4.** (*i*). See Eq. (1), Art. 3.1

One-Dimensional Inviscid Incompressible Flow (Bernoulli's Equation and its Applications)

4.1. Integration of Euler's equations of motion. Bernoulli's equation. Pressure equation.

[I.A.S. 2005; Kanpur 2002, 04, 05, 09; Meerut 2000, 02, 08]

When a velocity potential exists (so that the motion is irrotational) and the external forces are derivable from a potential function, the equations of motion can always be integrated. Let ϕ be the velocity potential and V be the force potential. Then, by definition, we get

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z, \qquad ...(1)$$

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y, \qquad Z = -\partial V/\partial y, \qquad ...(2)$$

and
$$\partial u/\partial y = \partial v/\partial x, \qquad \partial v/\partial z = \partial w/\partial y, \qquad \partial w/\partial x = \partial u/\partial z. \qquad ...(3)$$

Then well known Euler's dynamical equation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using (1) (2) and (3), these can be re-written as

$$\left.\begin{aligned} -\frac{\partial^2\phi}{\partial t\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} &= -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{\partial^2\phi}{\partial t\partial y} + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} &= -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ -\frac{\partial^2\phi}{\partial t\partial z} + u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} &= -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad ...(4)$$

Re-writing equations (4), we get

$$-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(5)$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2)=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ...(6)$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2)=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \quad ...(7)$$

Now
$$d\left(\frac{\partial \phi}{\partial t}\right)=\frac{\partial}{dx}\left(\frac{\partial \phi}{\partial t}\right)dx+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy+\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz \quad ...(8)$$

$$dV=(\partial V/\partial x)\,dx+(\partial V/\partial y)\,dy+(\partial V/\partial z)\,dz \quad ...(9)$$

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy+(\partial p/\partial z)\,dz \quad ...(10)$$

$$d(u^2+v^2+w^2)=\frac{\partial}{\partial x}(u^2+v^2+w^2)\,dx+\frac{\partial}{\partial y}(u^2+v^2+w^2)\,dy+\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dz \quad ...(11)$$

Multiplying (5), (6) and (7) by dx, dy and dz respectively, then adding and using (8), (9), (10) and (11), we have

$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}d(u^2+v^2+w^2)=-dV-\frac{1}{\rho}dp$$

or
$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}dq^2+dV+\frac{1}{\rho}dp=0 \quad ...(12)$$

where $q^2=u^2+v^2+w^2=$ (velocity of fluid particle)^2

If ρ is a function of p, integration of (12) gives

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=F(t), \quad ...(13)$$

where F(t) is an arbitrary function of t arising from integration in which t is regarded as constant. (13) is *Bernoulli's equation* in its most general form. Equation (13) is also known as *pressure equation.*

Special Case I. Let the fluid be homogeneous and inelastic (so that ρ = constant *i.e.*, fluid is incompressible). Then Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\frac{p}{\rho}=F(t) \quad ...(14)$$

Special Case II. If the motion be steady $\partial\phi/\partial t=0$, the Bernoulli's equation for steady irrotational motion of an incompressible fluid is given by

$q^2/2+V+p/\rho=C$, where C is an absolute constant. **(Kanpur 2010)** ...(15)

4.2. Bernoulli's theorem. (Steady motion with no velocity potential and conservative field of force).

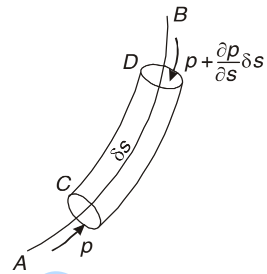
[Agra 2009; Meerut 2009, 2010; Kanpur 2004; Purvanchel 2005; G.N.D.U. Amritsar 2002, 05]

When the motion is steady and the velocity potential does not exist, we have

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C,$$

where V is the force potential from which the external forces are derivable. **[Meerut 2011]**

Proof. Consider a streamline AB in the fluid. Let δs be an element of this stream line and CD be a small cylinder of cross-sectional area α and δs as axis. If q be the velocity and S be the component of external force per unit mass in direction of the streamline, then by Newton's second law of motion, we have



$$\rho\,\alpha\,\delta s.\frac{Dq}{Dt} = \rho\,\alpha\,\delta s \cdot S + p\,\alpha - \left(p + \frac{\partial p}{\partial s}\delta s\right)\alpha$$

or $$\frac{Dq}{Dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

or $$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad ...(1)$$

If the motion be steady $\partial q/\partial t = 0$, and if the external forces have a potential function V such that $S = -\partial V/\partial s$, (1) reduces to

$$\frac{1}{2}\frac{\partial q^2}{\partial s} + \frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0 \qquad ...(2)$$

If ρ is a function of p, integration of (2) along the streamline AB yields

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = C, \qquad ...(3)$$

where C is constant whose value depends on the particular chosen streamline.

Special Case I. If the fluid be homogeneous and incompressible, ρ = constant and hence (3) reduces to

$$q^2/2 + V + p/\rho = C. \qquad \textbf{(Kanpur 2008)} \qquad ...(4)$$

Special Case II. Let S be a graviational force per unit mass. Let δh be the vertical distance between C and D. Then we have

$$S = -g\frac{\partial h}{\partial s} = -\frac{\partial}{\partial s}(gh), \qquad \text{as} \qquad V = gh$$

Hence, if the fluid be incompressible, (3) reduces to

$$q^2/2 + gh + p/\rho = C. \qquad ...(5)$$

4.3. Illustrative solved examples.

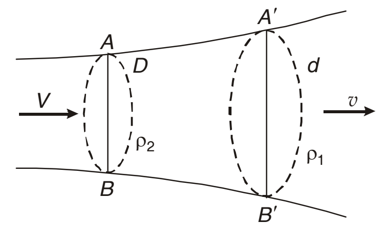
Ex. 1. *A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion be supposed to be that of the divergence from the vertex of the cone, prove that*

$$v/V = (D^2/d^2)e^{(v^2 - V^2)/2k} \qquad \textbf{[I.A.S. 1993, 98]}$$

where k is the pressure divided by the density and supposed constant.

Sol. Let AB and $A'B'$ be the ends of the conical pipe such that $A'B'$ = d and $AB = D$. Let ρ_1 and ρ_2 be densities of the stream at $A'B'$ and AB. *By principle of conservation of mass, the mass of the streem that entres the end AB and leaves at the end A'B'* must be the same. Hence the equation of continuity is

$$\pi(d/2)^2 v\rho_1 = \pi(D/2)^2 V\rho_2$$



so that $$\frac{v}{V}=\frac{D^2}{d^2}\times\frac{\rho_2}{\rho_1} \qquad ...(1)$$

By Bernoulli's theorem (in absence of external forces like gravity), we have

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2=C \qquad ...(2)$$

Given that $p/\rho = k$ so that $dp = k\,d\rho$. ...(3)

$\therefore$ (2) reduces to $k\int\frac{d\rho}{\rho}+\frac{1}{2}q^2=C$, using (3)

Integrating, $k\log\rho+q^2/2=C$, C being an arbitrary constant ...(4)

When $q = v$, $\rho=\rho_1$ and when $q = V$, $\rho=\rho_2$. Hence, (4) yields

$$k\log\rho_1+v^2/2=C \qquad \text{and} \qquad k\log\rho_2+V^2/2=C$$

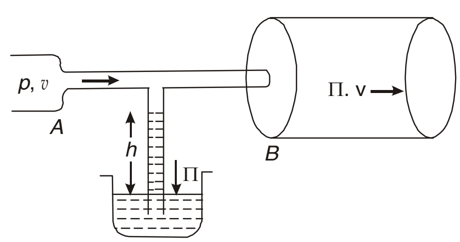
Subtracting, $k(\log\rho_2-\log\rho_1)+(V^2-v^2)/2=0$

or $\log(\rho_2/\rho_1)=(v^2-V^2)2k$ or $\rho_2/\rho_1=e^{(v^2-V^2)2k}$. ...(5)

Using (5), (1) reduces to $v/V=(D^2/d^2)\times e^{(v^2-V^2)/2k}$.

Ex. 2. *A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth* $(s^2/2g)\times(1/A^2-1/B^2)$ *below the pipe, s being the delivery per second.* **[I.A.S. 1997]**

Sol. Let v be the velocity in the tube of smaller section A and p the pressure at that section. Further, let V and Π be the corresponding quantities at the bigger section B of the figure. Then, by Bernoulli's Theorem (in absence of external forces like gravity) for incompressible fluid, namely



$$p/\rho+q^2/2 = \text{constant},$$

we obtain $$p/\rho+v^2/2=\Pi/\rho+V^2/2$$

so that $$(\Pi-p)/\rho=(v^2-V^2)/2 \qquad ..(1)$$

Let h be the height through which water is sucked up. Then

$$(\alpha h)\rho g=\alpha\Pi-\alpha p,\ \alpha \text{ being area of cross-section of the tube}$$

or $$g\rho h=\text{difference of pressure}=\Pi-p. \qquad ...(2)$$

The equation of continuity is $Av = BV = s$ (delivery per second)

so that $v = s/A$ and $V=s/B$ ...(3)

Using (2) and (3), (1) reduces to

$$\frac{1}{\rho}\times g\rho h=\frac{1}{2}\left(\frac{s^2}{A^2}-\frac{s^2}{B^2}\right) \qquad \text{or} \qquad h=\frac{s^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right).$$

Ex. 3. *A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time and that the pressure is given by*

$$p/\rho = \mu x y z - (t^2/2) \times (y^2z^2 + z^2x^2 + x^2y^2).$$

Prove that this motion may have been generated from rest by natural forces independent of the time and show that, if the direction of motion at every point coincides with the direction of the acting force, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Sol. Given that velocity q is proportional to time. So $\qquad q = \lambda t \qquad$ ...(1)

Also, given $\qquad p/\rho = \mu x y z - (t^2/2) \times (y^2z^2 + z^2x^2 + x^2y^2) \qquad$ ...(2)

Suppose that the motion is produced by finite natural forces (conservative forces) which are derivable from the potential function V. Then by Bernoulli's equation, we get

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + V = F(t)$$

or $$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}\lambda^2 t^2 - V + F(t), \text{ using (1)} \qquad \text{...(3)}$$

Since (2) and (3) must be identical, equating the coefficients of t^2 on R.H.S. of (2) and (3), we get

$$\lambda^2 = y^2z^2 + z^2x^2 + x^2y^2 \qquad \text{...(4)}$$

Using (4), (1) reduces to $\qquad q^2 = t^2(y^2z^2 + z^2x^2 + x^2y^2) \qquad$ ...(5)

But $\qquad q^2 = (\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2 \qquad$ ...(6)

Comparing (5) and (6), an appropriate value of ϕ is given by

$$\phi = t\, xyz \qquad \text{...(7)}$$

Using (7) and (4), (3) reduces to

$$p/\rho = xyz - (t^2/2) \times (y^2z^2 + z^2x^2 + x^2y^2) - V + F(t) \qquad \text{...(8)}$$

Comparing (2) and (8), we find $\qquad F(t) = 0 \qquad$ and $\qquad xyz - V = \mu\, xyz$

Thus, $\qquad V = xyz\,(1 - \mu) \qquad$ ...(9)

If u, v, w are the components of velocities and X, Y, Z are the components of forces, then

$$u = -(\partial\phi/\partial x) = -tyz, \qquad v = -(\partial\phi/\partial y) = -txz, \qquad w = -(\partial\phi/\partial z) = -txy$$

and $X = -(\partial V/\partial x) = (\mu - 1)yz,\ Y = -(\partial V/\partial y) = (\mu - 1)xz,\ Z = -(\partial V/\partial z) = (\mu - 1)xy \qquad$ ... (9)′

Given that the direction of motion coincides with that of the acting forces. Hence, we have

$$u/X = v/Y = w/Z$$

Again, the equations of the path

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \text{reduce to} \qquad \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

i.e., $$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}, \text{ using (9)}' \qquad \text{...(10)}$$

Taking the first two members of (10), we get

$$xdx - ydy = 0 \qquad \text{so that} \qquad x^2 - y^2 = C_1 \qquad \text{...(11)}$$

Taking the last two members of (10), we get

$$ydy - zdz = 0 \qquad \text{so that} \qquad y^2 - z^2 = c_2 \qquad ...(12)$$

Thus each particle of the fluid will be on the curve which is the intersection of two hyperbolic cylinders $x^2 - y^2 = C_1$ and $y^2 - z^2 = C_2$, C_1 and C_2 being arbitrary constants

Ex. 4. A *quantity of liquid occupies a length 2l of a straight tube of uniform small bore under the action of a force to a point in the tube varying as a distance from that point. Determine the pressure at any point.*

OR

A quantity of liquid occupies a length 2l of a straight tube of uniform bore under the action of force which is equal to μx to a point O in the tube, where x is the distance from O. Find the motion and show that if z be the distance of the nearer free surface from O, pressure at any point is given by $p/\rho = -(\mu/2)\times(x^2 - z^2) + \mu(x-z)(z+l)$.

Sol. Let p be the pressure and u the velocity at a distance x from the fixed point O; and let z be the distance of the nearer free surface from O. Then the equation of continuity is

$$\partial u/\partial x = 0 \qquad ...(1)$$

Let μx be the external force at a distance x which acts towards O. Then equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{reduces to} \qquad \frac{\partial u}{\partial t} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(2)$$

Integrating (2) w.r.t. 'x', we get

$$x\frac{\partial u}{\partial t} = -\frac{1}{2}\mu x^2 - \frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad ...(3)$$

But $p = 0$ when $x = z$ and $x = z + 2l$. So (3) gives

$$z\frac{\partial u}{\partial t} = -\frac{1}{2}\mu z^2 + C \qquad ...(4)$$

$$(z+2l)\frac{\partial u}{\partial t} = -\frac{1}{2}\mu(z+2l)^2 + C \qquad ...(5)$$

Subtracting (4) from (5), we get

$$2l\frac{\partial u}{\partial t} = -\frac{1}{2}\mu[(z+2l)^2 - z^2] \qquad \text{or} \qquad \frac{\partial u}{\partial t} = -\mu(z+l) \qquad ...(6)$$

or $$d^2z/dt^2 = -\mu(z+l) \qquad [\because\ u = dz/dt] \qquad ...(7)$$

Putting $z + l = y$ so that $z = y - l$, (7) gives

$$d^2y/dt^2 + \mu y = 0$$

whose solution is $y = A\cos(t\sqrt{\mu} + B)$, A and B being arbitrary constants.

Since $y = z + 1$, it yields $$z = A\cos(t\sqrt{\mu} + B) - l \qquad ...(8)$$

in which A and B may be determined from the knowledge of initial position and velocity.

We now determine pressure. From (4), we get

$$C = z\frac{\partial u}{\partial t} + \frac{1}{2}\mu z^2$$

Putting this value of C in (3), we get

$$\frac{p}{\rho} = -\frac{1}{2}\mu(x^2 - z^2) - (x - z)\frac{\partial u}{\partial t} \quad \text{or} \quad \frac{p}{\rho} = -\frac{1}{2}\mu(x^2 - z^2) + \mu(x - z)(z + l), \quad \text{using (6)}$$

which gives the pressure at any point.

Ex. 5. *A horizontal pipe gradually reduces in diameter from 24 in. to 12 in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 50 lbf/in^2 and the velocity of the water is 8 ft./sec.*

Sol. Let S_1 and S_2 be the cross-sections of the larger and the smaller ends. Let q_1 and q_2 be the velocities and p_1 and p_2 be the pressures at the larger and the smaller ends of the pipe. Given

$$S_1 = \pi(12)^2 \text{in}^2. \quad \text{and} \quad S_2 = \pi(6)^2 \text{in}^2.,$$

Also, $q_1 = 8 \times 12 = 96$ in/sec. and $p_1 = 50$ lbf/in^2.

The equation of continuity $S_1 q_1 = S_2 q_2$ gives $\pi(12)^2 q_1 = \pi(6)^2 q_2$

so that
$$q_2 = 4q_1 \qquad \text{...(1)}$$

By Bernouill's equation, we have

$$\frac{p_1}{\rho} + \frac{1}{2}q_1^2 = \frac{p_2}{\rho} + \frac{1}{2}q_2^2 \quad \text{or} \quad p_1 - p_2 = \frac{1}{2}\rho(q_2^2 - q_1^2)$$

$$p_1 - p_2 = (1/2) \times \rho(16q_1^2 - q_1^2), \text{ using (1)}$$

or
$$p_2 = p_1 - (15/2) \times \rho q_1^2 \qquad \text{...(2)}$$

$\therefore$ The required longitudinal thrust on the pipe

$$= p_1 S_1 - p_2 S_2 = \pi(12)^2 p_1 - \pi(6)^2 p_2 = 36\pi(4p_1 - p_2)$$

$$= 36\pi[4p_1 - p_1 + (15/2) \times \rho q_1^2], \text{ using (2)}$$

$$= 36\pi(3p_1 + \frac{15}{2}\rho q_1^2) = 36\pi\left(150 + \frac{15}{2} \times \frac{62.4}{12^3} \times 96^2\right) = 95040\pi$$

$$(\because \rho = 62.4 \text{ lb/ft}^3 = (62.4)/12^3 \text{ lb/in}^3.)$$

Ex. 6. *An elastic fluid, the weight of which is neglected, obeying Boyle's law is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation*

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t} + v^2\frac{\partial v}{\partial r}\right) = k\frac{\partial^2 v}{\partial r^2}.$$

[Kanpur 2006; Rohilkhand 2005]

Sol. The equation of continuity is
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v) = 0. \qquad \text{...(1)}$$

The equation of motion is
$$\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \qquad \text{...(2)}$$

Since the given elastic fluid obeys Boyle's law, we have

$$p = k\rho \quad \text{so that} \quad \frac{\partial p}{\partial r} = k\frac{\partial \rho}{\partial r}. \qquad \text{...(3)}$$

Using (3), (2) becomes $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{k}{\rho}\frac{\partial \rho}{\partial r}. \qquad ..(4)$$

Differentiating (4) partially with respect to t, we have

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial r}\right)=-\frac{\partial}{\partial t}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right). \qquad ...(5)$$

Now, $$\frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right)\right]=\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right)\right]=\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}\right)$$

and $$\frac{\partial}{\partial t}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right)=\frac{\partial}{\partial t}\frac{\partial}{\partial r}(k\log\rho)=\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}(k\log\rho)\right]=\frac{\partial}{\partial r}\left[\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right].$$

Hence (5) reduces to

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}\right)=-\frac{\partial}{\partial r}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right) \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right)=0$$

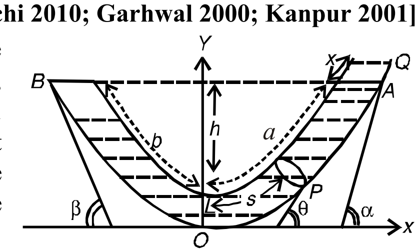
or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}+\frac{k}{\rho}\left\{-\frac{\partial(\rho v)}{\partial r}\right\}\right]=0, \text{ using (1)}$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-\frac{k}{\rho}\left(\rho\frac{\partial v}{\partial r}+v\frac{\partial \rho}{\partial r}\right)\right\}=0 \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}-v\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right)=0$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}+v\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}\right)\right\}, \text{ using (4)}$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}-k\frac{\partial v}{\partial r}\right)=0 \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2 v}{\partial r^2}.$$

Ex. 7. *Water oscillates in a bent uniform tube in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of the water, α, β the inclinations of the tube to the horizontal at A, B and $OA = a$, $OB = b$, the period of oscillation is given by* $2\pi\{(a+b)/g(\sin\alpha+\sin\beta)\}^{1/2}$ **[Ranchi 2010; Garhwal 2000; Kanpur 2001]**



Sol. Let O be the lowest point of the tube, AB the equilibrium level of water, h the height of AB above O, α, β, the inclinations of the tube to the horizontal at A and B and θ its inclination at P at distance s from O. Let a, b, denote the lengths OA, OB and suppose that at time t the water is displaced a small distance x along the tube from its equilibrium position.

If q is the velocity, the equation of continuity is $\partial q/\partial s = 0$. ...(1)

Again, the equation of motion is $$\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}. \qquad ...(2)$$

Using (1), (2) becomes $$\frac{\partial q}{\partial t}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

or $$\frac{\partial q}{\partial t}=-g\frac{\partial y}{\partial s}-\frac{1}{\rho}\frac{\partial p}{\partial s}, \quad \textit{as} \quad \sin\theta=\frac{\partial y}{\partial s} \qquad ...(3)$$

Integrating (3) with respect to s, we have

$$s\frac{\partial q}{\partial t} = -gy - \frac{p}{\rho} + f(t), \quad \text{where } f(t) \text{ is an arbitrary function of } t. \quad ...(4)$$

Let Π be the atmospheric pressure. Then, the conditions at the ends of the tube are :
When $y = h + x \sin\alpha$, $s = a + x$, $p = \Pi$ and when $y = h - x\sin\beta$, $s = -(b - x)$, $p = \Pi$.
Hence, (4) yields

$$(a + x)\frac{\partial q}{\partial t} = -g(h + x\sin\alpha) - \frac{\Pi}{\rho} + f(t) \quad ...(5)$$

and
$$-(b - x)\frac{\partial q}{\partial t} = -g(h - x\sin\beta) - \frac{\Pi}{\rho} + f(t). \quad ...(6)$$

Subtracting (6) from (5), we have $\quad (a + b)\dfrac{\partial q}{\partial t} = -gx(\sin\alpha + \sin\beta)$

or $\quad \dfrac{d^2x}{dt^2} = -\dfrac{g}{a+b}(\sin\alpha + \sin\beta)x, \quad$ as $\quad q = \dfrac{dx}{dt} \quad$ and $\quad \dfrac{\partial q}{\partial t} = \dfrac{d^2x}{dt^2}$

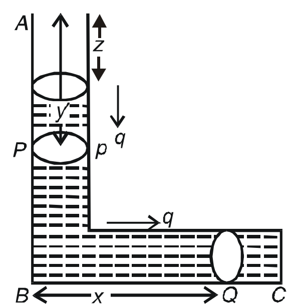
or $\quad \ddot{x} = -\mu x, \quad$ where $\quad \mu = g(\sin\alpha + \sin\beta)/(a + b)$.

This represents a simple harmonic motion. If T be its time period, then we have

$$T = 2\pi/\sqrt{\mu} = 2\mu l[g(\sin\alpha + \sin\beta)/(a+b)]^{1/2} = 2\pi\{(a+b)/g(\sin\alpha + \sin\beta)\}^{1/2}$$

Ex. 8. *A straight tube of small bore, ABC, is bent so as to make the angle ABC a right angle and AB equal to BC. The end C is closed and the tube is placed with end A upwards and AB vertical and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half and find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmosphere being neglected.*

[Aligarh 2005; Bangalore 2003, Nagpur 1999]



Sol. Let ABC be the given tube in which AB is vertical and BC is horizontal. Let $AB = BC = a$.

Let at time t the liquid fall through a depth z and at that instant let q be the downward velocity and p be the pressure at a point P at depth y in the vertical tube. Since the only external force acting downwards is g, the equation of motion is

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial y} = g - \frac{1}{\rho}\frac{\partial p}{\partial y}. \quad ...(1)$$

Since the motion is one-dimensional, the equation of continuity is

$$\partial q/\partial y = 0. \quad ...(2)$$

Using (2), (1) becomes $\quad \partial q/\partial t = g - (1/\rho)(\partial p/\partial y). \quad ...(3)$

Integrating (3) with respect to y, we have

$$(\partial q/\partial t)\, y = gy - (p/\rho) + C, \quad C \text{ being an arbitrary constant.} \quad ...(4)$$

Initially, when $y = z$, $p = 0$. So $\quad (4) \Rightarrow \quad C = z(\partial q/\partial t - g)$.

So (4) gives $\quad \dfrac{\partial q}{\partial t}y = gy - \dfrac{p}{\rho} + z\dfrac{\partial q}{\partial t} - gz$

so that
$$\frac{p}{\rho} = \left(g - \frac{\partial q}{\partial t}\right)(y - z). \quad ...(5)$$

At B, where $y = a$, let $p = p_1$ so that from (5), we have

$$p_1 = \rho\left(g - \frac{\partial q}{\partial t}\right)(a - z). \qquad ...(6)$$

The cross-section of the tube being the same everywhere, let a point Q at a distance x from the point B have the velocity q and let p' be the pressure there.

The equation of continuity is $\qquad \partial q / \partial x = 0. \qquad ...(7)$

and the equation of motion is $\qquad \dfrac{\partial q}{\partial t} + q\dfrac{\partial q}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p'}{\partial x}$

or $\qquad \partial q / \partial t = -(1/\rho)(\partial p'/\partial x)$, using (7). $\qquad ...(8)$

Integrating (8) with respect to x, $\qquad x\,(\partial q/\partial t) = -(p'/\rho) + C'. \qquad ...(9)$

At C, when $\quad x = a, \quad p' = 0 \quad$ so $\quad (9) \quad \Rightarrow \quad C' = a(\partial q / \partial t)$.

Hence (9) gives $\qquad p' = \rho(-\partial q/\partial t)(x - a). \qquad ...(10)$

At B, where $x = 0$, $p' = p_1$ so that from (10), we have

$$p_1 = \rho a\,(\partial q/\partial t). \qquad ...(11)$$

Now, $\qquad$ (6) and (11) $\qquad \Rightarrow \qquad \rho\left(g - \dfrac{\partial q}{\partial t}\right)(a - z) = \rho a \dfrac{\partial q}{\partial t}$

or $\qquad (\partial q/\partial t)(2a - z) = g(a - z). \qquad ...(12)$

Initially, when $z = 0$, (12) gives $\qquad \left(\dfrac{\partial q}{\partial t}\right)_0 = \dfrac{1}{2}g. \qquad ...(13)$

If the initial pressure at P be p_0, then putting $z = 0$ in (5), we get

$$\frac{p_0}{\rho} = \left\{g - \left(\frac{\partial q}{\partial t}\right)_0\right\}y = \left(g - \frac{1}{2}g\right)y = \frac{1}{2}gy.$$

Thus, $\qquad p_0 = (1/2) \times g\rho y. \qquad ...(14)$

When end C is closed, the hydrostatic pressure p_H at P is $\qquad p_H = g\rho y. \qquad ...(15)$

Now, $\qquad$ (14) and (15) $\qquad \Rightarrow \qquad p_0 = (1/2) \times p_H$,

showing that the pressure at any point of the vertical tube is instantaneously diminished by half.

If p'_0 be the initial pressure at Q, then from (10), we get

$$p'_0 = -\rho(x - a)\left(\frac{\partial q}{\partial t}\right)_0 = -\rho \times \frac{1}{2}(x - a)g, \text{ using (13)}$$

When end C is closed, let p_2 be the initial pressure at Q.

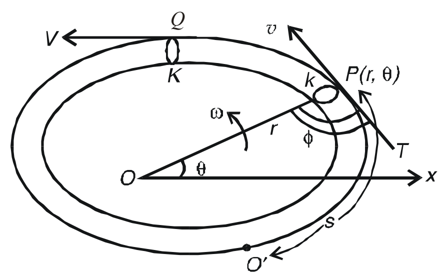
$\therefore \qquad p_2 =$ the initial pressure at $B = g\rho a$.

Hence the change in pressure

$$= p_2 - p'_0 = g\rho a + (1/2) \times \rho\,(x - a)g = (1/2) \times g\rho(a + x).$$

Ex. 9. *A fine tube whose section k is a function of its length s, in the form of a closed plane curve of area A, filled with ice, is moved in any manner. When the component angular velocity of the tube about a normal to its plane is* Ω, *the ice melts without change of volume. Prove that the velocity of the fluid relatively to the tube at a point where the section is K at any subsequent time when* ω *is the anguler velocity is*

$2A(\Omega - \omega)\Big/\left\{K\displaystyle\int \frac{ds}{k}\right\}$, *the integral being taken once round tube.*



Sol. With O as pole (origin) and OX as initial line, let polar coordinates of an arbitrary point P on the tube be (r, θ). Let O' be taken as a fixed point on the tube and the length of the arc $O'P$ of the tube be s. Let the tube rotate about a normal through O and let at any subsequent time, after the ice melts, the component angular velocity be Ω. Let the cross-section of the tube be k at P and let v be velocity there. Let Q be the point where the cross-section of the tube is K and velocity V. Let p be pressure at P.

The equation of continuity is $\qquad vk = VK.$ ...(1)

Relative to the tube, the acceleration of the fluid particle along the tangent at P is

$$(\partial V/\partial t) + V(\partial V/\partial s).$$

Let ϕ be the angle between radius vector OP and tangent PT. Then we know that

$$\sin\phi = r\,(d\theta/ds) \qquad \text{and} \qquad \cos\phi = dr/ds. \qquad ...(2)$$

Now, acceleration of the point P of the tube along the tangent at P

$$= r\dot{\omega}\sin\phi - r\omega^2\cos\phi = r\dot{\omega}\frac{rd\theta}{ds} - r\omega^2\frac{dr}{ds}, \qquad \text{where} \qquad \dot{\omega} = \frac{d\omega}{dt}$$

Hence the equation of motion is given by $\dfrac{\partial V}{\partial t} + V\dfrac{\partial V}{\partial s} + r^2\dot{\omega}\dfrac{d\theta}{ds} - r\omega^2\dfrac{dr}{ds} = -\dfrac{1}{\rho}\dfrac{dp}{ds}.$

Integrating both sides of the above equation w.r.t. 's' once round the tube, we have

$$\int\frac{\partial V}{\partial t}ds + \int V\frac{\partial V}{\partial s}ds + \int r^2\dot{\omega}\frac{d\theta}{ds}ds - \int r\omega^2\frac{dr}{ds}ds = -\frac{1}{\rho}\int\frac{dp}{ds}ds \qquad ...(3)$$

But $\int V\dfrac{\partial V}{\partial s}ds = \int V\,dV = [V^2/2]_Q^Q = 0,$ Similarly, $\int r\omega^2\dfrac{dr}{ds}ds = \omega^2\left[r^2/2\right]_Q^Q = 0$ and $\int (dp/ds)ds = [p]_Q^Q = 0$. Hence, (3) reduces to

$$\int\frac{\partial V}{\partial t}ds + \int r^2\dot{\omega}d\theta = 0 \qquad \text{or} \qquad \int\frac{\partial V}{\partial t}ds = -\dot{\omega}\int r^2 d\theta$$

or $$\int\frac{\partial V}{\partial t}ds = -2A\dot{\omega}, \qquad \text{as} \qquad A = \int\frac{1}{2}r^2\,d\theta \qquad ...(4)$$

Integrating both sides of (4) w.r.t. 't', we get

$$\iint\frac{\partial V}{\partial t}ds\,dt = -2A\int_\Omega^\omega\frac{d\omega}{dt}dt \qquad \text{or} \qquad \int v\,ds = 2A(\Omega - \omega)$$

or $$\int\frac{VK}{k}ds = 2A(\Omega - \omega), \qquad \text{since from (1),} \qquad v = \frac{VK}{k}$$

or $$VK\int\frac{ds}{k} = 2A(\Omega - \omega) \qquad \text{or} \qquad V = 2A(\Omega - \omega)\Big/\left\{K\int\frac{ds}{k}\right\},$$

Ex. 10. *The water is flowing through a tapering pipe having diameters* 300 mm *and* 150 mm *at sections AA′ and BB′ respectively. The discharge through the pipe is* 40 litres/s. *The section AA′ is* 10 m *above datum and section BB′ is* 6 m *above datum. Find the intensity of pressure at section BB′ if that at section AA′ is* 400 KN/m^2.

Sol. At section AA', we have $\qquad$ diameter $= d_1 = 300$ mm $= 0.3$ m

$\therefore$ Area of cross-section $= S_1 = (\pi/4) \times (0{\cdot}3)^2 = 0{\cdot}0707$ m^2

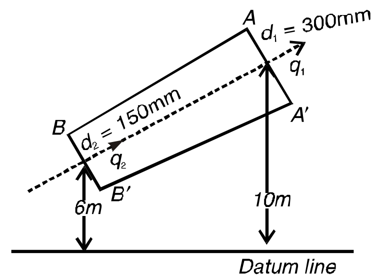
Pressure = p_1 = 400 kN/m^2

Height of upper end above the datum = h_1 = 10 m

At section BB', we have

diameter = d_2 = 150 mm = 0.15 m

$\therefore$ Area of crosssection = $S_2 = (\pi/4) \times (0.15)^2$
$= 0.01767$ m^2.



Height of the lower end above the datum = h_2 = 6 m.

Let pressure at section BB' be p_2.

Rate of flow (*i.e.* discharge) = Q

$$= 40 \text{ liters/s} = \frac{40 \times 10^3}{10^6} = 0.04 \text{ m}^3\text{/s}.$$

Let velocity of flow at sections AA' and BB' be q_1 and q_2 respectively. Then, we have

$$q_1 = \frac{Q}{S_1} = \frac{0.04}{0.0707} = 0.566 \text{ m/s} \quad \text{and} \quad q_2 = \frac{Q}{S_2} = \frac{0.04}{0.01767} = 2.264 \text{ m/s}.$$

Applying Bernaulli's equation at sections AA' and BB', we get

$$\frac{1}{2}q_1^2 + gh_1 + \frac{p_1}{\rho} = \frac{1}{2}q_2^2 + gh_2 + \frac{p_2}{\rho} \quad \text{or} \quad \frac{p_2}{\rho} = \frac{p_1}{\rho} + \frac{1}{2}(q_1^2 - q_2^2) + g(h_1 - h_2)$$

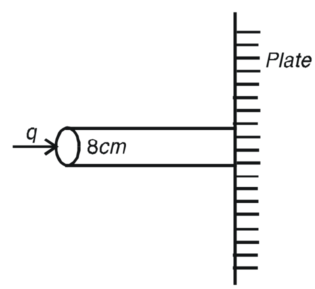
or, $$\frac{p_2}{w} = \frac{p_1}{w} + \frac{1}{2g}(q_1^2 - q_2^2) + (h_1 - h_2), \quad \text{as} \quad \rho g = w$$

$$\Rightarrow \frac{p_2}{w} = \frac{400}{9.81} + \frac{1}{2 \times 9.81}\left[(0.566)^2 - (2.264)^2\right] + (10 - 6) = 40.77 - 0.245 + 4 = 44.525 \text{ m}$$

$\Rightarrow$ $p_2 = 44.525 \times w = 44.525 \times 9.81 = 436.8$ kN/m^2 as $w = 9.81$kN/m^3

Ex. 11. *A jet of water* 8 *cm in diameter impinges on a plate held normal to its axis. For a velocity of* 4 *m/s, what force will keep the plate in equilibrium?*

Sol. Diameter of jet = d = 8 cm = 0.08 m.



$\therefore$ Area of cross section of the jet

$$= S = (\pi/4) \times d^2 = (\pi/4) \times (0.08)^2 m^2.$$

q = velocity of jet = 4 m/s

w = weight per unit cubic meter of water = 10^3 kg/m^3

F = Force acting on the jet.

Now, force on the plate = charge in momentum

$$\Rightarrow \quad F = \frac{w(Sq)q}{g} = \frac{wSq^2}{g} = \frac{1000 \times (\pi/4) \times (0.08)^2 \times 4^2}{9.81} = 32.8\,kg.$$

Ex. 12. *Air, obeying Boyle's law, is in motion in a uniform tube of a small section prove that if* ρ *be the density and* v *the velocity at a distance* x *from a fixed point at time* t*, then*

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2 + k)\}, \quad \text{where} \quad k = \frac{p}{\rho}.$$ **[Garhwal 2005. Kanpur 2004, Meerut 2003, 2011]**

Sol. Given that $p/\rho = k$ that is, $p = \rho k$ ... (1)

Since the motion is one-densional, the equations of continuity and motion (refer equation (1) with $S = 0$ in Art. 4.2) are respectively

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad ...(2)$$

and $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(3)$$

From (1), we have $$\partial p/\partial x = k(\partial \rho/\partial x) \qquad ...(4)$$

Using (4), (3) reduces to $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad ...(5)$$

Differentiating (2) partialy w.r.t. 't', we get

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}(\rho v)\right\} = 0 \qquad \text{or} \qquad \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial t}(\rho v)\right\} = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t}\right\} = 0 \qquad \text{or} \qquad \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\rho\left(-v\frac{\partial v}{\partial x} - \frac{k}{\rho}\frac{\partial \rho}{\partial x}\right) - v\frac{\partial}{\partial x}(\rho v)\right\} = 0$$

[on using (2) and (5)]

or $$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left\{\rho v\frac{\partial v}{\partial x} + k\frac{\partial \rho}{\partial x} + v\frac{\partial}{\partial x}(\rho v)\right\} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v \cdot v) + k\frac{\partial \rho}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v^2 + k\rho)\right\}$$

or $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2 + k)\}.$$

Ex. 13. *If the body force* $\boldsymbol{F}$ *form a conservative system, density* ρ *is a function of p only and the flow is steady, prove that* $\Omega + P + \mathbf{q}^2/2$ *is constant along every streamline and vortex line, where* $\boldsymbol{F} = -\nabla\Omega$, $P = \int\left(\frac{1}{\rho}\right)dp$ and $\boldsymbol{q}$ is velocity. **[I.A.S. 1989]**

Sol. Vector equation of motion for invicid incompressible fluid is

$$\partial \boldsymbol{q}/\partial t + \nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times \text{curl}\,\boldsymbol{q} = \boldsymbol{F} - (1/\rho)\nabla p \qquad ...(1)$$

Since the flow in steady, $\partial \boldsymbol{q}/\partial t = \mathbf{0}$. ...(2)

Since ρ in function of p only, $(1/\rho)\nabla p = \nabla(p/\rho)$. ...(3)

Also given that $\boldsymbol{F} = -\nabla\Omega$. ...(4)

By definition, vorticity vector = $\boldsymbol{w}$ = curl $\boldsymbol{q}$. ...(5)

Using (2), (3), (4) and (5), (1) reduces to

$$\nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times\boldsymbol{w} = -\nabla\Omega - \nabla(p/\rho) \qquad \text{or} \qquad \nabla\left(\frac{1}{2}\boldsymbol{q}^2 + \Omega + \int\frac{dp}{\rho}\right) = \boldsymbol{q}\times\boldsymbol{w}$$

or $$\nabla\left(\Omega + P + \frac{1}{2}\boldsymbol{q}^2\right) = \boldsymbol{q}\times\boldsymbol{w}, \qquad \text{as} \qquad \int\left(\frac{1}{\rho}\right)dp = P \text{ (given)} \qquad ...(6)$$

Taking scalar product of (6) with $d\,\boldsymbol{r}$, *a* time independent variation in the position vector $\boldsymbol{r}$ of the fluid particle, we get

$$d(\boldsymbol{\Omega} + P + \boldsymbol{q}^2/2) = d\boldsymbol{r}\cdot(\boldsymbol{q}\times\boldsymbol{w}). \qquad ...(7)$$

Two cases arise:

Case I. Let $\boldsymbol{q}\times\boldsymbol{w} = \mathbf{0}$. Then have

either (*i*) when $\boldsymbol{q}$ and $\boldsymbol{w}$ are parallel, *i.e.*, when the streamlines and vortex lines coincide. For such motion, $\boldsymbol{q}$ is known as *Beltrami vector.*

or (*ii*) when $\boldsymbol{w} = \mathbf{0}$, *i.e.* the motion is irrotational.

In both cases, (7) gives $$d(\Omega + P + \boldsymbol{q}^2/2) = 0 \qquad ...(8)$$
at all times throughout the entire flow field.

Integrating (8), $$\Omega + P + \boldsymbol{q}^2/2 = \text{constant} \qquad ...(9)$$
throughout the entire field of flow. The constatnt in (9) will remain unchanged throughout the entire field because the differential $d\boldsymbol{r}$ in (7) is an arbitrary small variation of position vector $\boldsymbol{r}$ in the field.

Case II. When $\mathbf{q} \times \mathbf{w} \neq \mathbf{0}$. Since $\boldsymbol{q} \times \boldsymbol{w}$ is perpendicular to the vectors $\boldsymbol{q}$ and $\boldsymbol{w}$, it follows that if $d\boldsymbol{r} \neq \mathbf{0}$, then $d\boldsymbol{r} \cdot (\boldsymbol{q} \times \boldsymbol{w}) = 0$ whenever $d\boldsymbol{r}$ lies in the plane of $\boldsymbol{q}$ and $\boldsymbol{w}$. Therefore, if we take the variation $d\boldsymbol{r}$ in the surface containing both the streamlines and vortex lines, then (7) shows that $$d[\Omega + P + \boldsymbol{q}^2/2] = 0 \quad \text{over such a surface}$$
and hence $$\Omega + P + \boldsymbol{q}^2/2 = \text{constant} \qquad ...(10)$$
over a surfce containing the streamlines and vortex lines. It may be observed that the constant in (10) is the same everywhere on any one such surface, but that its value varies from one surface to another. It may be noted that (10) holds for steady rotational as well as irrotational motious.

Ex. 14. *Prove that in a steady motion of a liquid,* $H = p/\rho + q^2/2 + V = \text{constant}$ *along a streamline. If this constant has the same value everywhere in the liquid, then prove that the motion must be either irrotational or the vertex lines must coincide with the streamlines.*

In two dimensional motion of a liquid with constant vorticity ζ, *prove that*
$$\nabla(H - 2\zeta\psi) = 0.$$
Show that if the motion be steady, the pressure is given by $p/\rho + q^2 + V - 2\zeta\psi = \text{const.}$, *where* ∇ *is the Laplace's operator.* **[Agra 2000; I.A.S. 1992; Rohilkhand 1999]**

Sol. For the first part, refer Art. 4.2 Thus, we have
$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 + V = \text{const. along a streamline} \qquad ...(1)$$
If the fluid be homogeneous so that $\rho = \text{const}$, then (1) becomes
$$H = p/\rho + q^2/2 + V = \text{constant along a streamline} \qquad ...(2)$$
Second part. We know that for steady motion, Euler's equations of motion for homogeneous liquid moving under conservative forces (so that $X = -\partial V/\partial x$, $y = -\partial V/\partial y$, $Z = -\partial V/\partial z$, where V is force potential and $\boldsymbol{F} = X\,\boldsymbol{i} + Y\,\boldsymbol{j} + Z\,\boldsymbol{k}$ and $\boldsymbol{q} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$) (refer Art. of 3.1 chapter3).
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial y}\right)u = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(3A)$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(3B)$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(3C)$$
Let ξ, η, ζ be the vorticity components, then
$$2\xi = \partial w/\partial y - \partial v/\partial z, \qquad 2\eta = \partial u/\partial z - \partial w/\partial x, \qquad 2\zeta = \partial v/\partial x - \partial u/\partial y, \qquad ...(4)$$

Re-writing (3*A*), we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V}{\partial x}=0$$

or $$\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)+v(-2\zeta)+w(2\eta)+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V}{\partial x}=0,\ \text{using (4)}$$

or $$\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{q^2}{2}+V\right)=2(v\zeta-w\eta), \qquad \text{where} \qquad q^2=u^2+v^2+w^2$$

or $$\partial H/\partial x=2(v\zeta-w\eta),\ \text{using (2)} \qquad \text{...(5A)}$$

Similarly (3B) and (3C) reduce to

$$\partial H/\partial y=2(w\xi-u\zeta) \qquad \text{...(5B)}$$

$$\partial H/\partial z=2(u\eta-v\xi) \qquad \text{...(5C)}$$

Multiplying (5A), (5B), (5C) by u, v, w respectively and then adding, we get

$$u(\partial H/\partial x)+v(\partial H/\partial y)+w(\partial H/\partial z)=0 \qquad \text{...(6)}$$

Multiplying (5A), (5B), (5C) by ξ, η, ζ and the adding, we get

$$\xi(\partial H/\partial x)+\eta(\partial H/\partial y)+\zeta(\partial H/\partial z)=0 \qquad \text{...(7)}$$

From (6) and (7), it follows that the surface H = const. contains the streamlines (whose direction cosines are proportional to u, v, w) and vertex lines (whose direction cosines are proportional to ξ, η, ζ).

If H has the same value everywhere in the liquid, we have

$$\partial H/\partial x=0, \qquad \partial H/\partial y=0 \qquad \text{and} \qquad \partial H/\partial z=0$$

$\Rightarrow$ $v\zeta-w\eta=0$, $\quad w\xi-u\zeta=0$, $\quad u\eta-v\xi=0$, by (7A) (7B) and (7C)

$\Rightarrow$ either $\quad \xi=\eta=\zeta=0 \quad$ or $\quad u/\xi=v/\eta=w/\zeta$

Now, $\quad \xi=\eta=\zeta=0 \quad \Rightarrow$ the motion is irrotational.

and $\quad u/\xi=v/\eta=w/\zeta \quad \Rightarrow$ streamlines given by $\quad (dx)/u=(dy)/v=(dz)/w$

coincide with vortex lines $\quad (dx)/\xi=(dy)/\eta=(dz)/\zeta$

Third part. Consider two dimensional motion such that $\quad \zeta=$ constnat.

Then, $w=0$ and $2\zeta=\partial v/\partial x-\partial u/\partial y$. Also, if ψ be stream function, then we have (refer Art. 5.4, chapter 5)

$$2\zeta=\partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2=\nabla\psi, \qquad \text{...(8)}$$

where $\quad \nabla\ (\equiv\partial^2/\partial x^2+\partial^2/\partial y^2)$ is Laplace's operater ...(8)'

As before, Euler's equations motion (refer Art. 3.1, chapter 3), for two dimensional notion (so that $w=0$), we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(9A)}$$

and $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(9B)}$$

Re-writing (9A), $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)+v\times(-2\zeta)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t}-2v\zeta=-\left[\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right)+\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\right], \qquad \text{where} \qquad q^2=u^2+v^2$$

or $$\partial u/\partial t-2v\zeta=-(\partial H/\partial x). \qquad \text{...(10 A)}$$

Similarly, (9B) gives $$\partial v/\partial t+2u\zeta=-(\partial H/\partial y) \qquad \text{...(10B)}$$

Differentiating (10A) and (10B) w.r.t.'x' and 'y' respectively and then adding, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-2\zeta\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\left(\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right) \qquad \text{...(11)}$$

For incompressible fluid in two dimensions, equations of continuity is

$$\partial u/\partial x+\partial v/\partial y=0 \qquad \text{...(12)}$$

Also, for a two dimensinonal, if ψ be velocity potential, then we have

$$u=-\partial\psi/\partial y, \quad v=\partial\psi/\partial x \quad \text{so that} \quad \partial u/\partial y=-\partial^2\psi/\partial y^2, \quad \partial v/\partial x=\partial^2\psi/\partial x^2$$

$$\therefore \qquad \partial v/\partial x-\partial u/\partial y=\partial^2\psi/\partial x^2+\partial^2\psi/\partial^2 y^2=\nabla\psi, \qquad \text{...(13)}$$

where ∇ is given by (8)'.

Using (12), (13) and (8)', (11) reduces to

$$-2\zeta\nabla\psi=-\nabla H \qquad \text{or} \qquad \nabla(H-2\zeta\psi)=0 \qquad \text{...(14)}$$

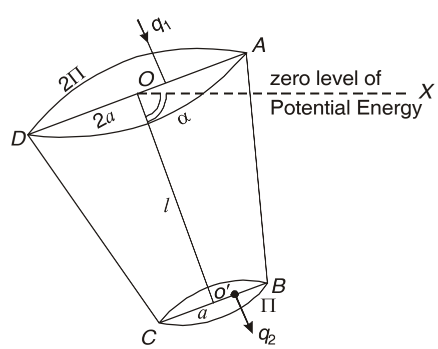
Fourth Part. Let the motion be steady so that $\partial u/\partial t=\partial v/\partial t=0$ and so again (11) reduces to (15),

Integrating (14), $$H-2\zeta\psi=\text{const.}$$

or $$P/\rho+q^2/2+V-2\zeta\psi=\text{const.}, \text{ using (2)}$$

Ex. 15. *A long pipe is of length l and has slowly tapering cross-section. It is inclined at angle α to the horizontal and water flows steadily through it from the upper to the lower end. The section at the upper end has twice the radius of the lower end. At the lower end, the water is delivered at atmospheric pressure. If the pressure at the upper end is twice atmospheric, find the velocity of* delivery.

Sol. Let $ABCD$ be the given pipe of length $l(=OO')$. Let OX be horizontal line which is taken as the zero level of potential energy. Let radii of the ends AD and BC of the given pipe be $2a$ and a respecively Let Π be the atmopsheric pressure. Then, the pressures at upper and lower ends are 2Π and Π respectively. Let q_1 and q_2 be velocities at the entry end AD and exist end BC respectively.



Now, Bernoulli's equation for steady motion is

$$p/\rho+q^2/2+V=C, \qquad \text{...(1)}$$

where V is the force potential and C is a absolute constant.

At end AD, $V = 0$ and at end BC, $V = -gl\sin\alpha$, where $-gl\sin\alpha$ is the potential ernergy per unit mass of the gravitational force at the end BC. Hence, using (1) at the ends AD and BC,

$$\frac{2\Pi}{\rho}+\frac{1}{2}q_1^2+0=C \qquad \text{and} \qquad \frac{\Pi}{\rho}+\frac{1}{2}q_2^2-ql\sin\alpha=C \qquad \text{...(2)}$$

Since the fluid is incompressible, equation of continuity of water flowing through the given pipe is (refer Art. 2.14) given by

$$q_1\times(4\pi a^2)=q_2\times(\pi a^2) \qquad \text{or} \qquad q_1=q_2/4 \qquad \text{...(3)}$$

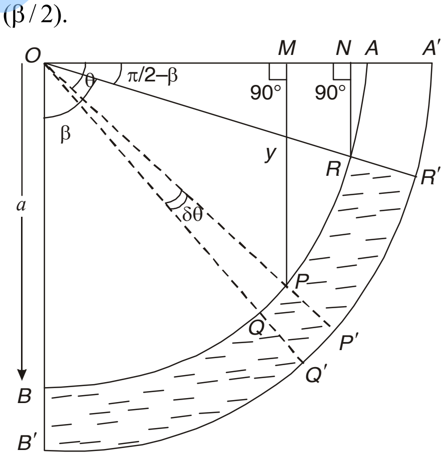
From (2), $\dfrac{2\Pi}{\rho}+\dfrac{q_1^2}{2}=\dfrac{\Pi}{\rho}+\dfrac{q_2^2}{2}-gl\sin\alpha$ or $\dfrac{\Pi}{\rho}+gl\sin\alpha=\dfrac{q_2^2}{2}-\dfrac{q_2^2}{32}$, using (3)

or $\dfrac{15}{32}q_2^2=\dfrac{\Pi}{\rho}+gl\sin\alpha$ or $q_2=\left\{\dfrac{32}{15}\left(\dfrac{\Pi}{\rho}+gl\sin\alpha\right)\right\}^{1/2}$,

which yields the desired velocity of delivery at exist BC.

Ex. 16. *AB is a tube of small uniform, bore forming a quadrant arc of a circle of radius a and centre O, OA being horizontal and OB vertical with B below O. The tube is full of liquid of density* ρ*, the end B being closed. If B is suddeenly opened, show that intially du/dt = 2g/*π*, where u = u(t) is the velocity and that the pressure at a point whose angular distance from A is* θ *immediately drops to* $\rho g a(\sin\theta-2\theta/\pi)$ *above atmospheric pressure. Prove further that when the liquid contained in the tube subtends an angle* β *at the centre,*

$$d^2\beta/dt^2=-(2g/a\beta)\times\sin^2(\beta/2).$$



Sol. As shown in the figure, AB is a tube of small uniform bore $(= AA' = BB' = PP' = QQ')$. Let $\angle AOB = 90°$. Let P be any point of the tube such that $\angle AOP = \theta$. Let Q be another point on the tube such that $\angle AOQ = \theta+\delta\theta$. Also, let arc $AP = s$ and arc $AQ = s+\delta s$, where s is measured from A.

Let $u(t)$ be the velocity of the liquid along arc AB so that $\partial u/\partial s = 0$. Then equation of motion of the element $PP'Q'Q$ is given by (refer equation (1) of Art. 4.2)

$$\frac{du}{dt}=g\cos\theta-\frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \text{...(1)}$$

But, $\dfrac{du}{dt}=\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial s}=\dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial s}\left(\dfrac{1}{2}u^2\right)$ and $\cos\theta=\dfrac{dy}{ds}$,

where y is the depth of P blow OA. Therefore, (1) reduces to

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial s}\left(\frac{u^2}{2}\right)=g\frac{dy}{ds}-\frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \text{...(2)}$$

Integrating (2) w.r.t. 's' while treating t as constant, we get

$$s(\partial u/\partial t)+u^2/2=gy-(p/\rho)+f(t), \qquad \text{...(3)}$$

where $f(t)$ is the arbitary function of t. Note that while integrating w.r.t. 's' we have assumed that at any time t, $\partial u/\partial t$ is the same at all points of the liquid.

Let Π be the atmospheric pressure.

Now, initially at A, when $t = 0$, $s = 0$, $u = 0$, $p = \Pi$ and $y = 0$.

$\therefore$ (3) reduces to $\quad 0 = -(\Pi/\rho) + f(0) \quad$ or $\quad f(0) = \Pi/\rho$ ...(4)

Again, at B, when $t = 0$, $s = \text{arc } AB = (\pi a)/2$, $u = 0$, $p = \Pi$, $y = a$

$\therefore$ (3) reduces to $$\frac{\pi a}{2} \times \left(\frac{\partial u}{\partial t}\right)_{t=0} = ga - \frac{\Pi}{\rho} + f(0) \qquad ...(5)$$

Subtracting (4) from (5), we have

$$\frac{\pi a}{2} \times \left(\frac{\partial u}{\partial t}\right)_{t=0} = ga \qquad \text{or} \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{2g}{\pi} \qquad ...(6)$$

We now apply (3) at P. Initialy at P, $t = 0$, $u = 0$, $y = OM = a \sin\theta$, s = arc $AP = a\theta$. Also, let $p = p_0$. Then (3) reduces

$$a\theta \times (\partial u/\partial t)_{t=0} = ga \sin\theta - (p_0/\rho) + f(0)$$

or $$a\theta \times (2g/\pi) = ga \sin\theta - (p_0/\rho) + \Pi/\rho, \text{ by (4) and (6)}$$

or $$(p_0 - \Pi)/\rho = ga \sin\theta - (2ga\theta)/\pi \qquad \text{or} \qquad p_0 - \Pi = \rho ga(\sin\theta - 2\theta/\pi),$$

showing that if B is suddenly opened, the pressure at P immediately drops to $\rho ga(\sin\theta - 2\theta/\pi)$.

Let us now consider the situation when the liquid contained in the tube AB subtends an angle β at the centre. In this case liquid is shown in part $RBB'R'$ of the tube such that $\angle ROB = \beta$. Then $\angle NOR = \pi/2 - \beta$. Let RN be perpendicular to OA. Thus $RN = OR \sin(\pi/2 - \beta)$ $= a\cos\beta$. Also, arc $AR = a(\pi/2 - \beta)$. For this situation, $P = \Pi$ at the surface RR'. So (3) gives

$$a\left(\frac{\pi}{2} - \beta\right)\frac{\partial u}{\partial t} + \frac{u^2}{2} = ga\cos\beta - \frac{\Pi}{\rho} + f(t) \qquad ...(7)$$

Again, using (3) at B, where $s = (a\pi)/2$, $y = a$, $p = \Pi$, we get

$$\therefore \qquad \frac{a\pi}{2} \times \frac{\partial u}{\partial t} + \frac{u^2}{2} = ga - \frac{\Pi}{\rho} + f(t) \qquad ...(8)$$

Subtracting (7) from (8), we have

$$a\beta\frac{\partial u}{\partial t} = ga(1 - \cos\beta) \qquad \text{or} \qquad \frac{\partial u}{\partial t} = \frac{2g}{\beta}\sin^2\frac{\beta}{2} \qquad ...(9)$$

Since arc $AR = a(\pi/2 - \beta)$, we have $u = \dfrac{\partial}{\partial t}\left(\dfrac{a\pi}{2} - a\beta\right)$

or $$u = -a\frac{d\beta}{dt} \qquad \text{so that} \qquad \frac{\partial u}{\partial t} = -a\frac{d^2\beta}{dt^2} \qquad ...(10)$$

From (9) and (10), $$-a\frac{d^2\beta}{dt^2} = \frac{2g}{\beta}\sin^2\frac{\beta}{2} \qquad \text{or} \qquad \frac{d^2\beta}{dt^2} = -\frac{2g}{a\beta}\sin^2\frac{\beta}{2}$$

EXERCISE 4 (A)

1. Liquid of density ρ is flowing along a horizontal pipe of variable cross-section, and the pipe is connected with a differential pressure gauge at two points A and B. Show that if $p_1 - p_2$ is the pressure indicated by the gauge, the mass m of liquid through the pipe per second is given by

$m = \sigma_1\sigma_2\sqrt{\dfrac{2\sigma(p_1 - p_2)}{\sigma_1^2 - \sigma_2^2}}$, where σ_1, σ_2 are the cross-sections at A, B respectively.

2. The diameter of a pipe changes from 20 cm at a section 5 metres above datum, to 5 cm at a section 3 metres above datum. The pressure of water at first section is 5 kg/cm^2. If the velocity of flow at the first section is 1 m/sec, determine the intensity of pressure at the second section. [**Ans.** 3.9 kg/cm^2]

3. A pipe 300 metres long has a slope of 1 in 100 and tapers from 1 metre diameter at the high end to 0.5 metre at the low end. Quantity of water flowing is 5,400 litres per minute. If the pressure at the high end is 0.7 kg/cm^2, find the pressure at the low end. [**Ans.** 0.999 kg/cm^2]

4. Find out Bernoulli's equation for unsteady irratational. **[Kanpur 2005; Meerut 2002]**

5. Obtain the well known equation $-(\partial\phi/\partial t) + p/\rho + (q^2/2) + V = C$ **[Kanpur 2001]**

6. Obtain Bernoulli's equation for steady motion. **[Kanpur 2002, 03]**

7. Define pressure equation. **[Kanpur 2000]**

8. Define pressure equation in its most general form by integrating Euler's equation of motion. **[I.A.S. 2005; Meerut 2000]**

9. A vertical tube AB of small section has two apertures close to its base B in which horizontal tubes are fitted and the apertures are closed by valves; a given height a of the tube AB is filled with water and the valves are then opened. The areal section of each horizontal tube being half that of the vertical tube and the length of each greater than AB, prove that the motion is of the simple harmonic type until the vertical tube is emptied which will take place after time $(\pi/2)\sqrt{(a/g)}$.

10. In the case of a steady motion of an ineladtic fluid under no forces the velocities parallel to the axes at the point (x, y, z) are proportional to $y + z$, $z + x$, $x + y$; prove that the surfaces of equal pressure are oblate spheroids, the eccentricity of the generating ellipse being $\sqrt{2/3}$.

11. Derive Bernoulli's equation for unsteady motion of an incompressible fluid and hence derive expression for steady motion.

12. (*a*) State the condition under which Euler's equation of motion can be integrated. Show that

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + V + \int\frac{dp}{\rho} = F(t),$$

where the symbols have this usual meaning. **[I.A.S. 2005]**

[**Hint:** Proceed as in Art. 4.1 upto equation (13)]

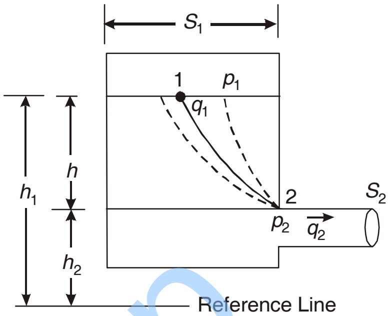
(*b*) When the velocity potential exists and the forces are conservative, show that the Euler's dynamical equations can always be integrated in the form $\displaystyle\int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} + V = f(t)$, where the symbols have their usual meaning. **[Kanpur 2008]**

4.4. Applications of Bernoulli's equation and theorem. [Meerut 2008]

Bernoulli's equation is of fundamental importance in fluid dynamics, especially in hydraulics. It is employed to handle some complicated situations of fluid flow problems in a simple manner. We now discuss some practical applications of the Bernoulli's equation. In each case the fluid will be assumed inviscid and incompressible

4.4A. Flow from a tank through a small orifice. Torrieclli's theorem.

Consider a tank containing a liquid. Let the tank be sealed except for a small orifice near its base. We wish to determine the velocity of efflux from the tank when the orifice is opened. Let S_1 and S_2 be the areas of cross-section of the tank and the orifice respectively.



Now the water will move out steadily in the form of a smooth jet. Let the line connecting point 1 on the liquid surface with the point 2 in the jet represents a streamline of the flow. Then the Bernoulli's theorem [refer equation (5) of Art 4.2] yields

$$\frac{1}{2}q_1^2 + gh_1 + \frac{p_1}{\rho} = \frac{1}{2}q_2^2 + gh_2 + \frac{p_2}{\rho} \qquad ...(1)$$

But from figure $\qquad h_1 - h_2 = h \qquad ...(2)$

Now, from the equation of continuity, we have

$$q_1 S_1 = q_2 S_2 \quad \text{or} \quad q_1 = (S_2/S_1) \times q_2 \qquad ...(3)$$

Using (2) and (3), (1) reduces to

$$\frac{1}{2}q_2^2 - \frac{1}{2}\frac{S_2^2}{S_1^2}q_2^2 = g(h_1 - h_2) + \frac{1}{\rho}(p_1 - p_2) \quad \text{or} \quad (q_2^2/2) \times (1 - S_2^2/S_1^2) = gh + (p_1 - p_2)/\rho$$

or

$$q_2 = \sqrt{\frac{2}{(1 - S_2^2/S_1^2)}\left(\frac{p_1 - p_2}{\rho} + gh\right)} \qquad ...(4)$$

which gives the desired velocity of efflux from the tank through the orifice.

We now discuss two special cases of (4) :

Case I. Suppose the tank is vented to the atmosphere or has an open surface, so that $p_1 = p_2$. Further, let $S_2 << S_1$. Then (4) reduces to

$$q_2 = \sqrt{2gh}. \qquad ...(5)$$

Hence the velocity of efflux from the vented tank is equal to that of a rigid body falling freely from a height h.

The above result is known as **Torricelli's theorem.**

Case II. Let $S_2 << S_1$ and $(p_1 - p_2)/\rho >> gh$. Then (4) reduces to

$$q_2 = \sqrt{(2/\rho) \times (p_1 - p_2)}.$$

Illustrative Examples.

Ex. 1. *Calculate the velocity of the water jet in above problem if* $p_2 = 14.7\ lb/in^2.$, $p_1 = 30\ lb/in^2.$, $S_2/S_1 = 0.01$ and $h = 10 ft.$, $\rho = 1.94\ lb/ft^3$.

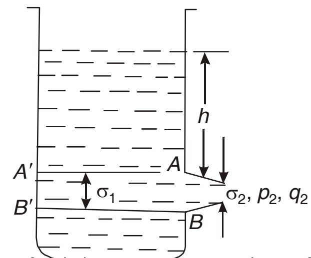
Sol. From (4), we have

$$q_2 = \sqrt{\frac{2}{1-(0.01)^2}\left[\frac{(30-14.7)\times 144}{1.94} + (32.2\times 10)\right]} = \sqrt{2917.8} = 54.01\,\text{ft/sec}.$$

If $p_1 = p_2$, the discharge velocity is given by $\qquad q_2 = \sqrt{2\times(32.2)\times(10)} = 25.38\,\text{ft/sec}.$

Ex. 2. *Fluid is coming out from a small hole of cross-section* σ_1 *in a tank. If the minimum cross-section of the stream coming out of the hole is* σ_2, *then show that* $\sigma_2/\sigma_1 = 1/2.$

Sol. Let AB be the hole and $A'B'$ be its image on the opposite wall of the tank. Let h be the height of the fluid level in the tank above the orifice. Again, let p_1 be the pressure at AB when the hole is closed. Since the velocity of the fluid coming out from minimum cross-section is at right angles to the hole, the direction of velocity will be horizontal there. If the velocity at the minimum cross-section is q_2 and pressure is p_2 there, then the principle of the conservation of momentum yields



$$\sigma_1(p_1 - p_2) = \sigma_2 \rho q_2^2 \qquad \text{or} \qquad p_1 - p_2 = (\sigma_2/\sigma_1)\rho q_2^2 \qquad ...(1)$$

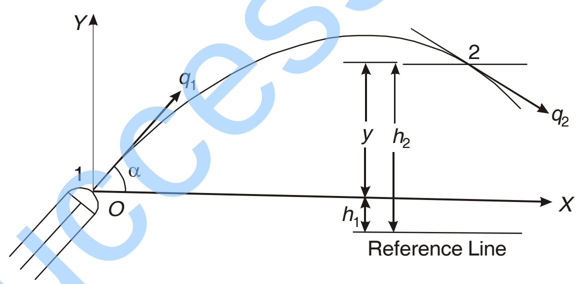
Consider a streamline connecting a point of $A'B'$ and a point of minimum cross-section of the jet. Then the Bernoulli's equation for the above streamline gives

$$\frac{p_1}{\rho} = \frac{p_2}{\rho} + \frac{1}{2}q_2^2 \qquad \text{or} \qquad p_1 - p_2 = \frac{1}{2}\rho q_2^2 \qquad ...(2)$$

Comparing (1) and (2), we find that $\sigma_2/\sigma_1 = 1/2.$

4.4B. Trajectory of a free jet.

Consider a liquid jet which is coming out from a small hole of area of cross-section S with velocity q_1 and making an angle α with the horizon. Since the entire jet is in the atmosphere, $p_1 = p_2$. Then the Bernoulli's equation between the jet exit 1 and an arbitrary point 2 on the stream line yields



$$gh_1 + q_1^2/2 = gh_2 + q_2^2/2 \qquad \text{or} \qquad q_2^2 = q_1^2 - 2gy, \qquad ...(1)$$

where $y = h_2 - h_1$. Let $Q = q_1 S$ so that $q_1 = Q/S$. Then (1) reduces to

$$q_2^2 = (Q/S)^2 - 2gy \qquad ...(2)$$

To determine the trajectory of the jet, consider the equations of motion for the jet along the horizontal line (x-axis) and the vertical line (y-axis).

$$dx/dt = q_1 \cos\alpha \qquad ...(3)$$

and
$$dy/dt = q_1 \sin\alpha - gt, \qquad ...(4)$$

where (x, y) are coordinates of point 2 with point 1 as the origin. Integrating (3) and (4), we get

$$x = q_1 \cos\alpha \cdot t + c_1,\ c_1 \text{ being an arbitrary constant} \qquad ...(5)$$

and
$$y = q_1 \sin\alpha \cdot t - (1/2) \times gt^2 + c_2,\ c_2 \text{ being an arbitrary constant} \qquad ...(6)$$

Initially at point 1, $t = 0$, $x = 0$, $y = 0$ so that $c_1 = 0$ and $c_2 = 0$. Then (5) and (6) give

$$x = q_1 \cos\alpha \cdot t \qquad ...(7)$$

$$y = q_1 \sin\alpha \cdot t - (1/2) \times gt^2 \qquad ...(8)$$

Eliminating t from (7) and (8), the equation of the jet (or streamline) is given by

$$y = x\tan\alpha - (1/2)\times(g/q_1^2)\sec^2\alpha \cdot x^2 \qquad ...(9)$$

or
$$y = x\tan\alpha - (1/2)\times g(S/Q)^2 \sec^2\alpha \cdot x^2 \qquad ...(10)$$

Putting this value of y in (2), the velocity of the liquid at any point of the jet is given by

$$y = (Q/S)^2 - 2gx\tan\alpha + (S/Q)^2 \times g^2 x^2 \sec^2\alpha \qquad ...(11)$$

Illustrative Solved Examples

Ex. 1. *Calculate the horizontal distance required for a jet striking the ground which is* 3 *feet below the horizontal line of the nozzle. The jet is inclined at an angle of* 60° *with the horizontal at a velocity of* 20 ft./sec. *What is the velocity of the jet just before reaching the ground.*

Sol. Refer equation (9) in above article. Here $\alpha = 60°, q_1 = 20$ ft./sec., $y = -3$ ft. Then we get

$-3 = x\tan 60° - (1/2)\times(g/20^2)\sec^2 60° \times x^2$ or $x^2 - 10.74x - 18.63 = 0$, on simplification

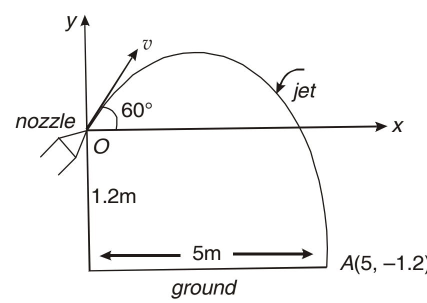
or
$$x = \{10.74 \pm \sqrt{(10.74)^2 + (4\times 18.62)}\}/2$$

Since x cannot be – ve, reject – ve sign in the above value of x. Then (1) gives x = 12.13 ft, which is the required horizontal distance between the nozzle and jet striking the ground.

The velocity q_2 of the jet just before reaching the ground can be obtained by using (1) of above article. Thus,
$$q_2 = \sqrt{(20)^2 - 2g(-3)} = \sqrt{593} = 24.24 \text{ft/sec}.$$

Ex 2. *A nozzle is situated at a distance of* 1.2 m *above the ground level and is inclined at* 60° *to the horizontal. The diameter of the nozzel is* 40 mm *and the jet of water from the nozzle strikes the ground at a horizontal distance of* 5m.*Find the flow rate.*

Sol. Here the co-ordinates of A, which is on the centre line of the jet of water and is situated on the ground with respect to O as origin are (5, –1.2). Let v be the velocity of the jet. Then the equation of the jet is given by



$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$$

or
$$-1.2 = 5\tan 60° - \frac{9.81\times 5^2}{2v^2\cos^2 60°}$$

or
$$-1.2 = 5\times 1.732 - (122.62\times 4)/v^2$$

or
$$498.48/v^2 = 8.66 + 1.2$$

or
$$v^2 = 49.74 \quad \text{or} \quad v = 7.05 \text{ m/s}.$$

Area of cross-section of nozzle = $(\pi/4)\times(\text{diameter})^2 = (\pi/4)\times(0.04)^2 \text{m}^2 = 0.001256 \text{m}^2$.

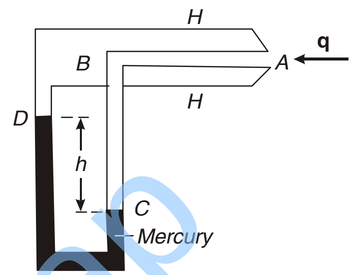
Hence, flow rate = $Q = Sv$ = $0.001256 \times 7.05 = 0.00885 \text{m}^3/\text{s}$.

Ex. 2. *Calculate the horizontal distance required for the jet striking the ground which is* 2 *ft. above the horizontal line of the nozzle. The jet is inclined at a* 30° *angle with the horizontal at a velocity of* 25 *ft/sec.*

Hint: Do just like Ex. 1.

4.4C. Pitot tube. **[Garhwal [1994, 96]**

A Pitot tube is an instrument to measure the velocity of flow at the required point in a pipe or a stream. Suppose we wish to determine the velocity q of a stream of water. The inner tube BA is kept so as to face the direction of the flow as shown in figure. The outer tube of the Pitot tube has holes such as H. If p is the pressure in the stream where the fluid velocity is q then p is also the pressure on the inside and outside of the hole and therefore p is also the pressure at the meniscus D of the mercury in the U-tube (manometer). Let the steam enter the tube AB and let it be brought to rest at meniscus C. C is called a stagnation point. Let p_0 be pressure at C. Applying the Bernoulli's equation to the steamline passing through A and C, we have



$$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{p}{\rho} \qquad \text{or} \qquad q=\sqrt{\left\{\frac{2(p_0-p)}{\rho}\right\}}, \qquad \text{...(1)}$$

where ρ is the density of the water.

Let h be the difference in level of the mercury in the U-tube and let σ be the density of the mercury. Then we have
$$p_0-p=\sigma gh \qquad \text{...(2)}$$

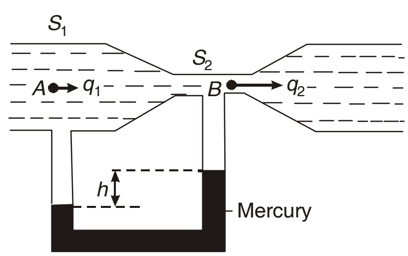
Using (2), (1) reduces to
$$q=(2\sigma gh/\rho)^{1/2} \qquad \text{...(3)}$$

which determines the fluid velocity at a point in the flow region.

4.4D. Venturi meter (or tube).

A venturi meter is an instrument to measure the fluid velocity in pipes. The flow rate of a fluid in conduit and the discharge of a fluid flowing in a pipe may also be measured. The venturi meter is made up of a constant cross-section S_1 tapering to a section of smaller cross-section S_2 (also known as throat) and then gradually expanding to the original cross-section. A U-tube serving as a mercury manometer is attached to connect the broad and narrow sections at A and B.

Let q_1, q_2 be the fluid velocities at A, B and p_1, p_2 the pressures. Then by the equation of continuity, we have



$$q_1S_1=q_2S_2 \qquad \text{or} \qquad q_2=(q_1S_1)/S_2 \qquad \text{...(1)}$$

Applying the Bernoulli's equation to the central streamline passing through A and B, we get
$$p_1/\rho+q_1^2/2=p_2/\rho+q_2^2/2, \qquad \text{...(2)}$$

where ρ is the density of the fluid. Eliminating q_2 from (1) and (2) we have
$$q_1=\left\{\frac{2(p_1-p_2)S_2^2}{\rho(S_1^2-S_2^2)}\right\}^{1/2} \qquad \text{...(3)}$$

Let h be the difference in levels of the mercury in the U-tube and let σ be the density of the density of the mercury. Then we have
$$p_1-p_2=\sigma gh \qquad \text{...(4)}$$

Using (4), (3) reduces to $$q_1 = \left\{\frac{2\sigma g h S_2^2}{\rho(S_1^2 - S_2^2)}\right\}^{1/2} \qquad \ldots(5)$$

Let Q be flow rate of the fluid flowing through the broad section at A. Then

$$q_1 = \rho q_1 S_1 = \rho S_1 \left\{\frac{2\sigma g h S_2^2}{\rho(S_1^2 - S_2^2)}\right\}^{1/2} \qquad \ldots(6)$$

Remarks. Let the venturi meter be kept inclined at a certain angle to the horizon. With reference to a fixed horizontal line, let vertical heights of A and B be h_1 and h_2 ($h_2 > h_1$) and let $h_2 - h_1 = \Delta h$. Then equation (2) modifies in the following form:

$$p_1/\rho + q_1^2/2 + gh_1 = p_2/\rho + q_2^2/2 + gh_2 \qquad \ldots(7)$$

Eliminating q_1 from (1) and (7), we get

$$q_2 = \left\{\frac{2[(p_1 - p_2)/\rho - g(h_2 - h_1)]}{1 - (S_2/S_1)^2}\right\}^{1/2}$$

and hence the flow rate at either sections is given by

$$Q = S_2 q_2 = S_2 \left\{\frac{2[(\sigma g h)/\rho - g\Delta h]}{1 - (S_2/S_1)^2}\right\}^{1/2} \qquad \ldots(8)$$

Let C be the *coefficient of venturi meter* (or the *coefficient of discharge*). Let Q be the discharge through the venturi meter. Then we know that

$$Q = CS_2 q_2 = CS_2 \left\{\frac{2[(\sigma g h/\rho) - g\Delta h]}{1 - (S_2/S_1)^2}\right\}^{1/2} \qquad \ldots(9)$$

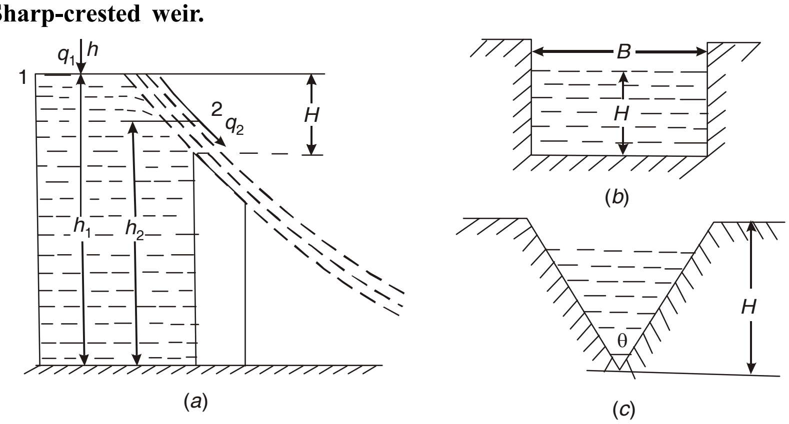
If $\Delta h = 0$ (*i.e.* the venturi meter is horizontal), then (9) reduces to

$$Q = \frac{CS_1 S_2}{\sqrt{S_1^2 - S_2^2}} \sqrt{\frac{\sigma}{\rho}} \sqrt{2gh}. \qquad \ldots(10)$$

4.4 E. Weir.

A structure, used to measure the flow rate of fluid with a free surface (as in an open channel or river), is known as a weir. There are two types of weirs depending on the common physical principle. One type of weir, known as sharp-crested weir, is made up of a sharp edged plate mounted normal to the direction of the flow so as to span the fluid stream. The opening is generally either a rectangle or a triangle (called V-notch also). The other type of weir, known as broad-crested weir, is made up of a obstacle with broad edge. We now discuss these in details.

(*i*) **Sharp-crested weir.**



Consider a streamline lying entirely in the free surface and joining points 1 and 2 as shown in Fig. (*a*). Then the Bernoulli's equation for this streamlines yields

$$\frac{p_1}{\rho}+\frac{1}{2}q_1^2+gh_1=\frac{p_2}{\rho}+\frac{1}{2}q_2^2+gh_2 \quad ...(1)$$

Since the entire streamline lies in free surface, we have $p_1 = p_2$. Let $h_1 - h_2 = h$ Then, (1) reduces to

$$q_2=\sqrt{q_1^2+2gh}, \quad ...(2)$$

which gives the velocity at the weir plane. Again the flow rate over the weir is given by

$$Q=\int_o^H q_2 dS_2, \quad ...(3)$$

where dS_2 is the cross-sectional element parallel to the width lying in the plane of the weir.

For rectangular weir [see Fig. (*b*)] dS_2 = Bdh and hence

$$Q=\int_o^H q_2 B\,dh=\frac{B}{3g}[(2gH+q_1^2)^{3/2}-q_1^3] \quad ...(4)$$

When the body of water being measured is very large in comparison with the weir opening, q_1 may be neglected. Than, we have

$$Q=(2B/3)\times\sqrt{2g}\;H^{3/2} \quad ...(5)$$

If C_{dw} is a discharge coefficient, for the actual case, we have

$$Q_{\text{actual}}=\frac{2BC_{dw}}{2}\sqrt{2g}\;H^{3/2} \quad ...(6)$$

In a similar manner, for the triangular weir the flow rate is given by

$$Q_{\text{actual}}=\frac{8}{15}C_{dw}\sqrt{2g}\;H^{5/2}\tan(\theta/2) \quad ...(7)$$

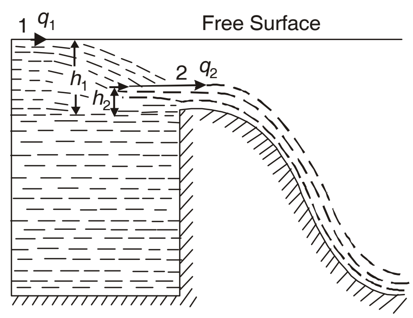
(*ii*) **Broad-crested Wear.**

We again proceed as in (*i*) above. Let there be uniform parallel flow at point 2 and let $q_1 << q_2$. Then Bernoulli's equation for the streamline joining points 1 and 2 gives

$$gh_1=q_2^2/2+gh_2$$

$\therefore$ The flow rate over a weir of width B is given by

$$Q=Bh_2\sqrt{2g(h_1-h_2)} \quad ...(8)$$

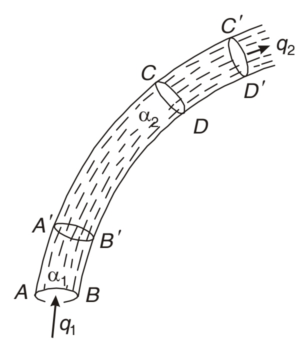


Then, we have

$$Q_{\text{actual}}=C_{dw}B\,h_2\sqrt{2g\,(h_1-h_2)} \quad ...(9)$$

4.5. Euler's momentum theorem.

Consider steady motion of a non-viscous liquid contained between, AB and CD of the filament at a given time t. The surrounding fluid will produce a force on the walls and ends of the filament. By Newton's second law of motion, the net force will be equal to the rate of change of momentum of the fluid in the filament $ABCD$ at time t. At time $t+\delta t$, let the new position of the fluid be $A'B'C'D'$. Then notice that the momentum of the given fluid has increased by the momentum of the fluid between CD and $C'D'$ and has decreased by the momentum of the fluid between AB and $A'B'$.



$\therefore$ Gain of momentum at $CD = (\rho\alpha_2 q_2 \delta t)q_2$

and loss of momentum at $AB = (\rho\alpha_1 q_1 \delta t)q_1,$

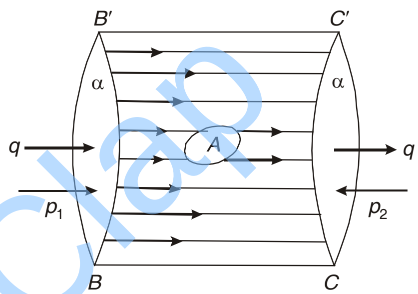
where q_1 and q_2 are the velocities at AB and CD respectively.

Hence the net gain $= \rho\delta t(\alpha_2 q_2^2 - \alpha_1 q_1^2)$ or the net rate of gain $= \rho(\alpha_2 q_2^2 - \alpha_1 q_1^2)$.

This gives the resultant force due to pressure of the surrounding liquid on the walls and ends of the filament. This result is known as *Euler's momentum theorem*.

4.6. D'Alembert's paradox. **[Meerut 2000, Kanpur 2001]**

Consider a long straight channel of uniform cross section in which a liquid is flowing with a uniform speed q. Let the ends of the tube be bounded by equal cross-sectional area α. If an obstacle A is placed in the middle of the channel, the flow in the immediate neighbourhood of A will be disturbed whereas the flow at a great distance either up-stream or down-stream will remain undisturbed. Suppose F is the force required to hold the obstacle to rest, in the direction of uniform flow.



Let BB' and CC' be two sections at a great distance from A and let the fluid between these sections be split up into stream filaments. Since the outer filaments are bounded by the walls of the channel, the thrust components are normal to the direction of flow. Moreover, the obstacle A acts on those filaments which are in contact with it by a force $-F$.

By Euler's momentum theorem the resultant of all the thrusts on the fluid is $\rho\alpha q^2 - \rho\alpha q^2$.

Let p_1 and p_2 be the pressures on BB' and CC' respectively. Then Bernoulli's theorem gives

$$\frac{p_1}{\rho} + \frac{1}{2}q^2 = C = \frac{p_2}{\rho} + \frac{1}{2}q^2 \quad \text{so that} \quad p_1 = p_2$$

Now, the thrust due to pressure p_1 and p_2 is $p_1\alpha - p_2\alpha$.

Thus the equation of motion becomes

$$p_1\alpha - p_2\alpha - F = \rho\alpha q^2 - p\alpha q^2 \quad \text{so that} \quad F = 0, \quad \text{as} \quad p_1 = p_2$$

Let the diameter of the channel increase indefinitely. Then the above problem reduces to that of a obstacle immersed in an infinite uniform stream. As before, again the resultant force exerted by the liquid on the obstacle is zero.

Now let us superimpose a velocity u in the opposite direction on the entire system (the body A and the liquid). Then the body A can be thought as moving with uniform velocity u and the liquid at great distance is reduced to rest.

Thus a body moving with uniform velocity through an infinite liquid, otherwise at rest, will experience no resistance at all. This result is known as *D'Alembert's paradox*.

EXERCISE 4 (B)

1. A venturi meter has its axis vertical, the inlet and throat diameter ratio being 2.5. The throat is 12 in. above the inlet and the coefficient of discharge is 0.97. Determine the pressure difference between the inlet and throat when the velocity of water at the inlet is 6 ft/sec.

2. A jet of water 1 in. in diameter strikes a flat plate at an angle 30° to the normal of the plate with a velocity of 30 ft/sec. Determine the velocity of the plate, moving parallel to itself, if the normal force exerted by the jet is 2.5 Ib**.**

3. A rectangular plate 4 in. wide and 10 in. long hangs vertically from hinges at its top edge. A jet of water 1 in. in diameter with a velocity of 30 ft/sec strikes the plate at its centre.

Determine the weight of the plate if the plate stays in equilibrium after deflecting 20° from its original position.

4. A jet of water 2 in. in diameter discharges 1.0 ft^3/sec. Calculate the force required to move a flat plate towards the jet with velocity of 25 ft/sec. The jet is perpendicular to the plate.

5. Calculate the force exerted by a jet of water 3/4 in. in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 30 ft/sec if (*a*) the plate is stationary, (*b*) the plate is moving in the direction of the jet with a velocity of 10 ft/sec.

[**Ans.** (*a*) 4.63 Ibf. (*b*) 3.09 Ibf.]

6. State and prove Bernoulli's theorem for steady inviscid flow in a conservative field of force and discuss the nature of the constant.

7. Fluid enters a contracting pipe with velocity V_1 through area A_1 and leaves with velocity V_2 through area A_2 after having been turned through an angle α. Determine the force required to hold the pipe in equilibrium against the pressure of the fluid.

8. A water venturi meter has a throat diameter of 3 in. and a pipe diameter of 6 in. Calculate the velocity at the throat if the deflection of mercury manometer which connects the pipe and the throat is 9.5 in. The coefficient of discharge is 0.96.

9. Define stream tube. Using this obtain the Bernoulli's equation for a steady flow.

10. The venturi meter has an entrance of 6 inch diameter and a throat of 3 inch diameter whose centre is 18 inch above the centre of the entrance. Find the velocity of water at the throat when $p_1 - p_2 = 5$ Ib/in^2. The coefficient of discharge C_d is 0.97. Find the value of the constant for the venturimeter. [**Ans.** 25.4 ft/sec]

11. Calculate the horizontal distance required for the jet striking ground which is 0.5 m above the horizontal line of the nozzle. The jet is inclined at a 30° angle with the horizontal at a velocity of 10 m/s.

12. If the water jet is discharged from a nozzle (inclined at a 60° angle with the horizontal) at 5 m/s, calculate the horizontal distance required for the jet striking the ground which is 1 m below the horizontal line of the nozzle. What is the velocity of the jet just before reaching the ground ?

13. A horizontal straight pipe gradually reduces in diameter from 0.5 m to 0.25 m. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 0.4 MN/m^2 and the velocity of the water is 2 m/s.

14. A jet of water 0.05 m in diameter strikes a plate at an angle 30° to the normal of the plate with a velocity of 10 m/s. Determine the velocity of the plate, moving parallel to itself, if the normal force exerted by the jet is 10 N.

15. Calculate the force exerted by a jet of water 10 mm in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 10 m/s if (*a*) the plate is stationary, (*b*) the plate is moving in the direction of the jet with a velocity of 2 m/s.

16. A venturi meter has its axis vertical, the inlet and throat diameter ratio being 2.5. The throat is 0.3 m above the inlet and the coefficient of discharge is 0.97. Determine the pressure difference between the inlet and throat when the velocity of water at the inlet is 2 m/s.

17. A water venturi meter has a throat diameter of 0.1 m and a pipe diameter of 0.2 m. Calculate the velocity at the throat if the deflection of the mercury manometer which connects the pipe and the throat is 0.15 m. The coefficient of discharge is 0.96.

18. State and prove D'Alembert's Paradox. **[Meerut 1999, 2000]**

19. Briefly explain the application of Bernoulli's theorem. **[Kanpur 2000]**

Hint. Refer Art. 4.4 and Art 4.4A

OBJECTIVE QUESTIONS ON CHAPTER 4

Multiple choice questions

Choose the correct alternative from the following questions

1. If the motion is steady, velocity potential does not exist and V be the potential function from which the external forces are derivable, then Bernoulli's theorem is

(*i*) $-\dfrac{\partial\phi}{\partial t}+\dfrac{1}{2}q^2+V+\int\dfrac{dp}{\rho}=C$ (*ii*) $\int\dfrac{dp}{\rho}+\dfrac{1}{2}q^2+V=C$

(*iii*) $p/\rho+q^2/2+V=C$ (*iv*) None of these

2. The Bernoulli's equation for unsteady and irrotational motion is given by

(*i*) $-\partial\phi/\partial t+q^2/2+V+p/\rho=F(t)$ (*ii*) $-\partial\phi/\partial t+q^2/2+V=F(t)$

(*iii*) $-\partial\phi/\partial t-q^2/2+V-p/\rho=F(t)$ (*iv*) $q^2/2+V+p/\rho=F(t)$

3. A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. If a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth h below the pipe, s being the delivery per second, where h is given by

(*i*) $(s^2/2g)\times(1/A^2+1/B^2)$ (*ii*) $(s^2/2g)\times(1/A^2-1/B^2)$

(*iii*) $(2g/s^2)\times(1/A^2-1/B^2)$ (*iv*) $(2g/s^2)\times(1/A^2+1/B^2)$

4. The horizontal distance required for a jet striking the ground which is 3 feet below the horizontal line of the nozzle (given that the jet is inclined at a an angle 60° with the horizontal at a velocity of 20 ft/sec.) is

(*i*) 12.20 feet (*ii*) 12.15 feet (*iii*) 12.13 feet (*iv*) None of these

5. A body moving with uniform velocity through an infinite liquid otherwise at rest, will experience no resistance at all. This result is known as

(*i*) Euler's paradox (*ii*) Lagrange's paradox

(*iii*) D'Alembert's paradox (*iv*) none of these

6. If the fluid be homogeneous and incompressible, then in usual, symbols, the Bernoulli's theorem bcomes

(*i*) $q^2/2+V+p=C$ (*ii*) $q^2+V+p/\rho=C$

(*iii*) $q^2/2+V+p/\rho^2=C$ (*iv*) $q^2/2+V+p/\rho=C$

7. The most general form of Bernoulli's equation for motion of fluid is

(*a*) $\dfrac{1}{2}q^2+\Omega+\int\dfrac{dp}{\rho}-\dfrac{\partial\phi}{\partial t}=f(t)$ (*b*) $\dfrac{1}{2}q^2+\Omega-\int\dfrac{dp}{\rho}-\dfrac{\partial\phi}{\partial t}=f(t)$

(*c*) $\dfrac{1}{2}q^2+\Omega-\int\dfrac{dp}{\rho}+\dfrac{\partial\phi}{\partial t}=f(t)$ (*d*) $\dfrac{1}{2}q^2+\Omega+\int\dfrac{dp}{\rho}+\dfrac{\partial\phi}{\partial t}=f(t)$ **[Agra 2005]**

8. The equation $q^2/2+\Omega+p/\rho=$ constant is known as (*a*) Navier equation

(*b*) Bernonilli equation (*c*) Euler equation (*d*) Stokes equation **[Agra 2007]**

9. The Bernoulli's equation for steady motion with the *velocity* potential and conservative field of force is.......... (Fill up the gap) **[Agra 2008]**

Answers/Hints to objective type questions

1. (*ii*). See Eq. (3), Art. 4.2
2. (*i*). See Eq. (14), Art. 4.1
3. (*ii*). See Ex. 2, Art. 4.3
4. (*iii*). See Ex. 1, Art. 4.4B
5. (*iii*). Refer Art. 4.6
6. (*iv*). See Eq. (4), Art. 4.2
7. (*a*). See Art. 4.1
8. (*b*). See Eq. (15), Art. 4.1
9. $q^2/2 + V + \int (1/\rho) dp = C.$ Refer Eq. (3), Art. 4.2.

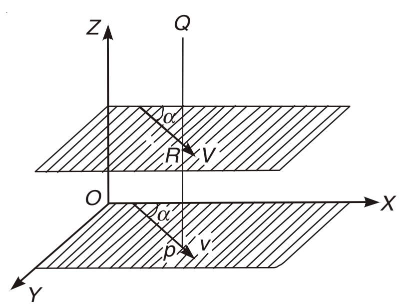
CHAPTER

5 Motion in Two-Dimensions Sources and Sinks

5.1. Motion in two-dimensions.

Let a fluid move in such a way that at any given instant the flow pattern in a certain plane (say XOY) is the same as that in all other parallel planes within the fluid. Then the fluid is said to have two-dimensional motion. If (x, y, z) are coordinates of any point in the fluid, then all physical quantities (velocity, density, pressure etc.) associated with the fluid are independent of z. Thus u, are functions of x, y and t and $w = 0$ for such a motion.

To make the concept of two-dimensional motion more clear, suppose the plane under consideration be xy-plane. Let P be an arbitrary point on that plane. Draw a straight line PQ parallel to OZ (or perpendicular to the xy-plane). Then all points on the line PQ are said to correspond to P. Draw a plane (in the fluid) parallel to the xy-plane and meeting PQ in R. Then, if the velocity at P is V in the xy-plane in a direction making an angle α with OX, the velocity at R is also V in magnitude and parallel in direction to the velocity at P as shown in the figure. It follows that the velocity at corresponding points is a function of x, y and the time t, but not of z.



In order to maintain physical reality, we assume that the fluid in two-dimensional motion is confined between two planes parallel to the plane of motion and at a unit distance apart. The reference plane of motion is taken parallel to and midway between the assumed fixed planes. Thus while studying the flow of a fluid past a cylinder in a two-dimensional motion in planes perpendicular to the axis of the cylinder, it is useful to restrict attention to a unit length of cylinder confined between the said planes in place of worrying over the cylinder of infinite length.

Suppose we are dealing with a two-dimensional motion in xy plane. Then by flow across a curve in this plane, we mean the flow across unit length of a cylinder whose trace on the plane xy is the curve under consideration, the generators of the cylinder being parallel to the z-axis. By a point in a flow, we mean a line through that point parallel to z-axis.

5.2. Stream function or current function.

[Agra 2005; Rohilkhand 2002, 03; Meerut 1999, 2010; Kanpur 2010, 09]

Let u and be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$dx/u = dy/ \qquad \text{or} \qquad dx - udy = 0 \qquad ...(1)$$

and the equation of continuity is

Motion in Two-Dimensions Sources and Sinks

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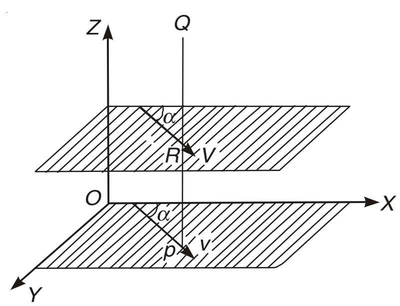

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and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}=0 \qquad \text{or} \qquad \frac{\partial}{\partial y}=\frac{\partial(-u)}{\partial x} \qquad ...(2)$$

(2) shows that *L.H.S.* of (1) must be an exact differential, $d\psi$ (say). Thus, we have

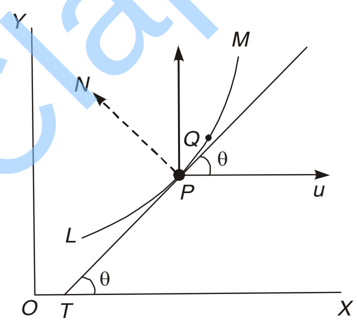
$$dx-udy=d\psi=(d\psi/\partial x)dx+(\partial\psi/\partial y)dy \qquad ...(3)$$

so that $\qquad u=-\partial\psi/\partial y \qquad$ and $\qquad =\partial\psi/\partial x \qquad ...(4)$

This function ψ is known as the *stream function*. Then using (1) and (3), the streamlines are given by $d\psi=0$ *i.e.*, by the equation $\psi=c$, where c is an arbitrary constant. Thus the stream function is constant along a streamline. Clearly the current function exists by virtue of the equation of continuity and incompressibility of the fluid. Hence the current function exists in all types of two-dimensional motion whether rotational or irrotational.

5.3. Physical significance of stream function. **[Kanpur 2005; Rohilkhand 2002, 03]**

Let LM be any curve in the x-y plane and let ψ_1 and ψ_2 be the stream functions at L and M respectively. Let P be an arbitrary point on LM such that arc $LP=s$ and let Q be a neighbouring point on LM such that arc $LQ=s+\delta s$. Let θ be the angle between tangent at P and the x-axis. If u and be the velocity-components at P, then



velocity at P along inward drawn normal PN

$$= \cos\theta-u\sin\theta \qquad ...(1)$$

When ψ is the stream function, then we have

$$u=-\partial\psi/\partial y \qquad \text{and} \qquad =\partial\psi/\partial x \qquad ...(2)$$

Also from Calculus, $\qquad \cos\theta=dx/ds \qquad$ and $\qquad \sin\theta=dy/ds \qquad ...(3)$

Using (1), we get $\qquad$ flux across PQ from right to left $=(\ \cos\theta-u\sin\theta)\delta s$

$\therefore$ Total flux across curve LM from right to left

$$=\int_{LM}(\ \cos\theta-u\sin\theta)\,ds=\int_{LM}\left(\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}\right)ds, \text{ using (2) and (3)}$$

$$=\int_{LM}\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy\right)=\int_{\psi_1}^{\psi_2}d\psi=\psi_2-\psi_1$$

Thus a *property of the current function is that the difference of its values at two points represents the flow across any line joining the points.*

Remark 1. The current function ψ at any point can also be defined as the flux (*i.e.* rate of flow of fluid) across a curve LP where L is some fixed point in the plane.

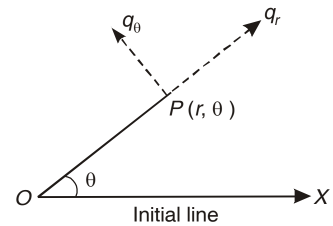
Remark 2. Since the velocity normal to δs will contribute to the flux across δs whereas the velocity along tangent to δs will not contribute towards flux across δs, we have

$$\text{flux across } \delta s = \delta s\times \text{ normal velocity}$$

or $\qquad (\psi+\delta\psi)-\psi=\delta s\ \times$ velocity from right to left acrss δs

or $\qquad$ Velocity from right to left across $\delta s=\partial\psi/\partial s \qquad ...(4)$

Remark 3. Velocity components in terms of ψ in plane-polar coordinates (r, θ) can be obtained by using the method outlined in remark 2 above. Let q_r and q_θ be velocity components in the directions of r and θ increasing respectively. Then



$$q_r = \text{velocity from right to left across } r\,\delta\theta$$

$$= \lim_{\delta\theta \to 0} \frac{\delta\psi}{r\delta\theta} = \frac{1}{r}\frac{\partial\psi}{\partial\theta},$$

and $\quad q_\theta$ = velocity from right to left across δr

$$= \lim_{\delta r \to 0} \frac{\delta\psi}{\delta r} = \frac{\partial\psi}{\partial r}.$$

Thus, $\qquad q_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \qquad$ and $\qquad q_\theta = \dfrac{\partial\psi}{\partial r}. \qquad$ ...(5)

5.4. Spin components in terms of ψ.

We know that the velocity components u and are functions of x, y and t and $w = 0$ in two-dimensional flow. Hence the spin components (ξ, η, ζ) are given by

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial}{\partial z} = 0, \qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

and
$$2\zeta = \frac{\partial}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

Let the motion be irrotational so that $\zeta = 0$ also. Then we obtain

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 \qquad \text{or} \qquad \nabla^2\psi = 0$$

showing that ψ satisfies Laplace's equation.

5.5. Some aspects of elementary theory of functions of a complex variables.

Suppose that $\quad z = x + iy \quad$ and that $\quad w = f(z) = \phi(x, y) + i\psi(x, y)$,

where x, y, ϕ, ψ are all real and $i = \sqrt{-1}$. Also, suppose that ϕ and ψ and their first derivatives are everywhere continuous within a given region. If at any point of the region specified by z the derivative $dw/dz\ (= f'(z))$ is unique, then w is said to be *analytic* or *regular* at that point. If the derivative is unique throughout the region, then w is said to be analytic or regular throughout the region. It can be shown that the necessary and sufficient conditions for w to be analytic at z are

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x,$$

which are known as the *Cauchy-Riemann equations*. The functions ϕ, ψ are known as *conjugate functions*.

5.6. Irrotational motion in two-dimensions. [Meerut 2007; Purvanchal 2004, 05]

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$u = -\,\partial\phi/\partial x \qquad$ and $\qquad = -\,\partial\phi/\partial y \qquad$ ....(1)

In two-dimensional flow the stream function ψ always exists such that

$u = -\,\partial\psi/\partial y \qquad$ and $\qquad = \partial\psi/\partial x \qquad$ ...(2)

From (1) and (2), we have

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x \qquad ...(3)$$

which are well known *Cauchy-Riemann's equations*. Hence $\phi + i\psi$ is an analytic function of $z = x + i\,y$. Moreover ϕ and ψ are known as *conjugate functions.*

On multiplying and re-writing, (3) gives

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y} = 0\,, \qquad ...(4)$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally.

Thus the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equations given in (3) with respect to x and y respectively, we get

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial y^2} = -\frac{\partial^2\psi}{\partial x\partial y}. \qquad ...(5)$$

Since $\dfrac{\partial^2\psi}{\partial x\partial y} = \dfrac{\partial^2\psi}{\partial y\partial x}$, adding (5) gives

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0. \qquad ...(6)$$

Again, differentiating the equations given in (3) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\psi}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2\psi}{\partial x^2}$$

Since $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\,\partial x$, subtracting these, we get $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$ ...(7)

Equations (6) and (7) show that *ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.* **[Meerut 2010]**

5.7. Complex potential. **[Meerut 2011]**

[G.N.D.U. Amritsar 2003; Rohilkhand 2001; Kanpur 2001, 05; Agra 2005]

Let $w = \phi + i\psi$ be taken as a function of $x + iy$ *i.e.*, z. Thus, suppose that $w = f(z)$ *i.e.*

$$\phi + i\psi = f(x + iy) \qquad ...(1)$$

Differentiating (1) w.r.t x and y respectively, we get

$$\partial\phi/\partial x + i(\partial\psi/\partial x) = f'(x + iy) \qquad ...(2)$$

and $$\partial\phi/\partial y + i(\partial\psi/\partial y) = if'(x + iy)$$

or $$\partial\phi/\partial y + i(\partial\psi/\partial y) = i\{\partial\phi/\partial x + i(\partial\psi/\partial x)\}, \text{ by (2)}$$

Equating real and imaginary parts, we get

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x \qquad ...(3)$$

which are *Cauchy-Riemann equations*. Then w is an analytic function of z and w is known as the *complex potential.*

Conversely, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion.

Remarks. If $\phi + i\psi = f(x + iy)$, then $i\phi - \psi = if(x + iy)$

Thus, $\psi - i\phi = -if(x + iy) = g(x + iy)$, say

Hence proceeding as before, we get (3). Hence another irrotational motion is also possible in which lines of equi - velocity potential are given by ψ = constant and the streamlines by ϕ = constant.

5.7A. Cauchy-Riemann equations in polar form. **[Kanpur 2003]**

Let $$\phi + i\psi = f(z) = f(re^{i\theta}) \qquad \ldots(1)$$

Differentiating (1) w.r.t. r and θ, we get

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \qquad \ldots(2)$$

and $$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta}) \cdot rie^{i\theta} \qquad \ldots(3)$$

From (2) and (3), we easily obtain $$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = ir\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right)$$

Equating real and imadginary parts,we get

$$\frac{\partial \phi}{\partial \theta} = -r\frac{\partial \psi}{\partial r} \qquad \text{and} \qquad \frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r}$$

Thus, $$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}, \qquad \ldots(4)$$

which are *Cauchy-Riemann equations in polar form*.

5.8. Magnitude of velocity. **[G.N.D.U. Amritsar 2002; Kanpur 1997]**

Let $w = f(z)$ be the complex potential. Then

$$w = \phi + i\psi \qquad \text{and} \qquad z = x + iy \qquad \ldots(1)$$

Also $$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x \qquad \ldots(2)$$

For two-diensional irrotational motion, we have (see Art. 5.1.)

$$u = -\partial\phi/\partial x \qquad \text{and} \qquad = -\partial\phi/\partial y \qquad \ldots(3)$$

From (1), $$\frac{dw}{dz}.\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \qquad \text{and} \qquad \frac{\partial z}{\partial x} = 1$$

$\therefore$ $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}, \quad \text{using (2)} \qquad \ldots(4)$$

or $$dw/dz = -u + i \quad \text{using (3)} \qquad \ldots(5)$$

which is called the *complex velocity*.

From (4) and (5), we see that the magnitude of velocity q at any point in a two-dimensional irrotational motion is given by | dw/dz |, where

$$|\,dw/dz\,| = \{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\}^{1/2} = (u^2 + \;^2)^{1/2} = q \qquad \ldots(5)$$

Remarks. The points where velocity is zero are known as *stagnation points*.

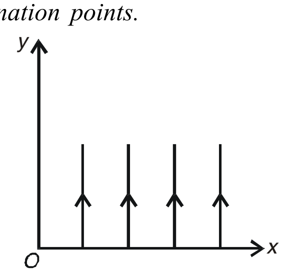
5.9. Complex potential for some uniform flows

(*i*) Consider $$w = ikz, \qquad \ldots(1)$$

where k is a real and positive constant

Now, (1) $\Rightarrow$ $dw/dz = -u + i \;= ik$ $\Rightarrow$ $u = 0$ and $\;= k$,

which is clearly a uniform flow parallel to y-axis.



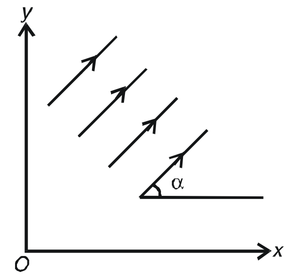
Hence the complex potential for a uniform flow whose magnitude of the stream is V in the positive y-direction is given by $$w = iVz.$$

(*ii*) Consider $$w = -ke^{-i\alpha}z, \qquad \text{...(2)}$$
where k and α are real constants.

From (2), $dw/dz = -u + i \quad = -ke^{-i\alpha}$

$\Rightarrow \quad -u + i \quad = -k(\cos\alpha - i\sin\alpha)$

$\Rightarrow \quad u = k\cos\alpha$ and $= k\sin\alpha$, which corresponds to a uniform flow inclined at an angle α to the x-axis.



Hence the complex potential for a uniform flow whose magnitude is V and which is inclined is at an angle α to x-axis is given by $w = -Ve^{-i\alpha}z$.

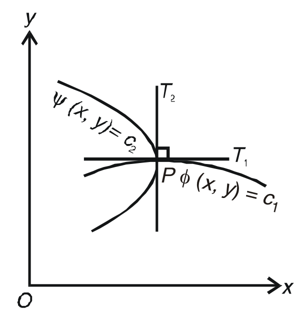
5.10. Illustrative Solved Exaples

Ex. 1. *To show that the curves of constant velocity potential and constant stream functions cut orthogonally at their points of intersection.* **[Meerut 2007; Garhwal 2005]**

OR

To shows that the family of curves $\phi(x, y) = c_1$ *and* $\psi(x, y) = c_2$, c_1, c_2 *being constants, cut orthogonally at their points of intersection.*

Proof. Let the curves of constant velocity potential and constant stream function be given by
$$\phi(x, y) = c_1 \qquad \text{...(1)}$$
and $$\psi(x, y) = c_2, \qquad \text{...(2)}$$
where c_1 and c_2 are arbitrary constants. Let m_1 and m_2 be gradients of tangents PT_1 and PT_2 at point of intersection P of (1) and (2). Then, we have



$$m_1 = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} \quad \text{and} \quad m_2 = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} \qquad \text{...(3)}$$

We know that ϕ and ψ satisfy the Cauchy-Riemann equations, namely,
$$\partial\phi/\partial x = \partial\psi/\partial y \quad \text{and} \quad \partial\phi/\partial y = -\partial\psi/\partial x. \qquad \text{...(4)}$$

Now, from (3), $$m_1 m_2 = \frac{(\partial\phi/\partial x)(\partial\psi/\partial x)}{(\partial\phi/\partial y)(\partial\psi/\partial y)} = \frac{(\partial\psi/\partial y)(\partial\psi/\partial x)}{-(\partial\psi/\partial x)(\partial\psi/\partial y)}, \text{ by (4)}$$

Hence $m_1 m_2 = -1$, showing that the curves (1) and (2) cut each other orthogonally.

Ex. 2. *If* $\phi = A(x^2 - y^2)$ *represents a possible flow phenomenon, determine the stream function.*

Sol. Here $$\phi = A(x^2 - y^2) \qquad \text{...(1)}$$

$$\therefore \quad \partial\psi/\partial y = \partial\phi/\partial x = 2Ax, \text{ using (1)}$$

Integrating it w.r.t. 'y', $$\psi = 2Axy + f(x), \qquad \text{...(2)}$$
where $f(x)$ is an arbitrary function of x. (2) gives the required stream function.

Ex. 3. *Determine the stream function* $\psi(x, y, t)$ *for the given velocity field* $u = Ut$, $= x$.

Sol. We know that $u = -(\partial\psi/\partial y)$ and $= \partial\psi/\partial x$.

$$\therefore \quad \partial\psi/\partial y = -Ut. \qquad \text{...(1)}$$
and $$\partial\psi/\partial x = x. \qquad \text{...(2)}$$

Integrating (1), $$\psi(x, y, t) = -Uty + f(x, t), \qquad \text{...(3)}$$
where $f(x, t)$ is an arbitrary function of x and t.

From (3), $\partial\psi/\partial x = \partial f/\partial x$. ...(4)

Then (2) and (4) $\Rightarrow$ $\partial f/\partial x = x$. ...(5)

Integrating (5), $f(x, t) = x^2/2 + F(t)$, ...(6)

where $F(t)$ is an arbitrary function of t.

From (3) and (6), $\psi(x, y, t) = -Uty + x^2/2 + F(t)$.

Ex. 4. *The velocity potential function for a two-dimensional flow is $\phi = x(2y - 1)$. At a point $P(4, 5)$ determine : (i) The velocity and (ii) The value of stream function.*

Sol. Given $\phi = 2xy - x$. ...(1)

(*i*) The velocity components u and   in x and y directions are given by

$$u = -\partial\phi/\partial x = -2y + 1 \quad \text{and} \quad = -\partial\phi/\partial y = -2x. \qquad ...(2)$$

At the point $P(4, 5)$, $u = -10 + 1 = 9$ and $= -8$.

$\therefore$ Resultant velocity $= V = (u^2 + \ ^2)^{1/2} = (81 + 64)^{1/2} = 12{\cdot}04$ units.

(*ii*) Now, $u = -\partial\psi/\partial y$ and $= \partial\psi/\partial x$. ...(3)

From (2) and (3), $\partial\psi/\partial x = -2x$ and $\partial\psi/\partial y = 2y - 1$.

Now, $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = -2x\,dx + (2y - 1)dy$.

Integrating, $\psi = -x^2 + y^2 - y + C$, C being constant of integration.

For $\psi = 0$ at the origin, we have $0 = 0 + C$ or $C = 0$.

Hence $\psi = -x^2 + y^2 - y$.

At the point $P(4, 5)$, $\psi = -4^2 + 5^2 - 5 = 4$ units.

Ex. 5. *The streamlines are represented by (a) $\psi = x^2 - y^2$ and (b) $\psi = x^2 + y^2$. Then (i) determine the velocity and its direction at (2, 2) (ii) sketch the streamlines and show the direction of flow in each case.*

Part (a) Given that $\psi = x^2 - y^2$.

Now, $u = \partial\psi/\partial y = -2y$ and $= -\partial\psi/\partial x = -2x$.

At (2, 2), $u = -4$ and $= -4$.

$\therefore$ The resultant velocity $= (u^2 + \ ^2)^{1/2} = (16 + 16)^{1/2} = 4\sqrt{2}$ units

and its direction has a slope $= \ /u = 1$ showing that the velocity vector is inclined at 45° to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, *i.e.* $x^2 - y^2 = c$, which represents a family of hyperbolas. In figure, we have sketched the steamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.

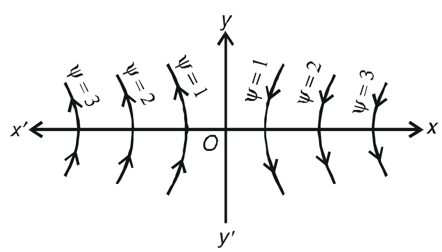


Fig. Pattern of Streamlines for $y = x^2 - y^2$

Part (b) Given that $\psi = x^2 + y^2$

Now, $u = \partial\psi/\partial y = 2y$, $= -\partial\psi/\partial x = -2x$.

At (2, 2), $u = 4$ and $= -4$.

$\therefore$ The resultant velocity

$$= (u^2 + \ ^2)^{1/2} = (16 + 16)^{1/2} = 4\sqrt{2} \text{ units.}$$

and its direction has a slope $= \ /u = -1$, showing that the velocity vector is inclined at 135^o to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, *i.e.* $x^2 + y^2 = c$, which represents a family of circles. In figure, we have sketched the streamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.

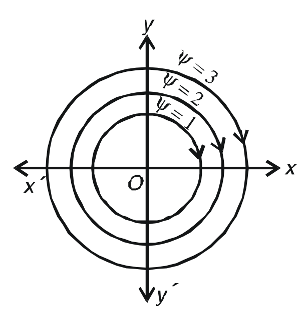


Fig. Pattern of Streamlines for $\psi = x^2 + y^2$

Ex. 6. If $\phi = 3xy$, find x and y components of velocity at (1, 3) and (3, 3). Determine the discharge passing between streamlines passing through these points.

Sol. The velocity components u and in x and y directions are given by

$$u = -\partial\phi/\partial x = -3y \qquad \text{and} \qquad = -\partial\phi/\partial y = -3x. \qquad ...(1)$$

Hence the velocity components at (1, 3) are $\quad u = -9, \quad = -3.$

and the velocity components at (3, 3) are $\quad u = -9, \quad = -9.$

Now, we have $\quad u = \partial\psi/\partial y \quad$ and $\quad = -\partial\psi/\partial x. \qquad ...(2)$

Then, (1) and (2) $\Rightarrow \quad \partial\psi/\partial y = -3y \quad$ and $\quad \partial\psi/\partial x = 3x. \qquad ...(3)$

$$d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = 3xdx - 3y\, dy.$$

Integrating, $\quad y = (3x^2/2) - (3y^2/2) + C$, where C is constant of integration. $\qquad ...(4)$

Discharge between the streamlines passing through (1, 3) and (3, 3)

$$= \psi(1, 3) - \psi(3, 3) = (3/2) \times (1 - 9) - (3/2) \times (9 - 9) = -12 \text{ units}.$$

Ex. 7. *If the expression for stream function is described by* $\psi = x^3 - 3xy^2$, *determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential.* $\quad$ (*a*) $\phi = y^3 - 3x^2y$. $\quad$ (*b*) $\phi = -3x^2y$.

Sol. Now $\quad u = \partial\psi/\partial y = -6xy, \quad = -\partial\psi/\partial x = -3(x^2 - y^2). \qquad ...(1)$

Hence, $\quad \partial\ /\partial x = -6x \quad$ and $\quad \partial u/\partial y = -6x. \qquad ...(2)$

A two-dimensional flow in xy-plane will be irrotational if the vorticity vector component Ω_z in the z-direction is zero.

Here $\quad \Omega_z = (\partial\ /\partial x) - (\partial u/\partial y) = -6x - (-6x) = 0$, by (2)

Hence the flow is irrotational.

Now, $\quad u = -\partial\phi/\partial x \quad$ and $\quad = -\partial\phi/\partial y. \qquad ...(3)$

For an irrotational flow Laplace equation in ϕ must be satisfied, *i.e.* $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$.

We now check the validity of each given value of ϕ.

(*a*) Given $\quad \phi = y^3 - 3x^2y \quad \Rightarrow \quad \partial^2\phi/\partial x^2 = -6y \quad$ and $\quad \partial^2\phi/\partial y^2 = 6y$

$\therefore \quad (\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 6y = 0.$

(*b*) Given $\quad \phi = -3x^2y \quad \Rightarrow \quad \partial^2\phi/\partial x^2 = -6y \quad$ and $\quad \partial^2\phi/\partial y^2 = 0$

$\therefore \quad (\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 0 \neq 0.$

Hence the correct value of ϕ is given by $\quad \phi = y^3 - 3x^2y.$

Ex. 8. *Show that the velocity vector* **q** *is everywhere tangent to lines in the xy-plane along which* $\psi\ (x, y) = const.$

Sol. We have $\quad d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy$

or $\quad (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = 0 \qquad [\because\ \psi(x, y) = \text{const.} \Rightarrow d\psi = 0]$

or $\quad dx - udy = 0, \quad$ as $\quad u = -\partial\psi/\partial y \quad$ and $\quad = \partial\psi/\partial x$

or $\quad (dx)/u = (dy)/\ ,$

showing that the velocity vector **q** = u**i** + **j** is tangent to the streamlines $\psi(x, y)$ = const.

Ex. 9. *Find the stream function* ψ *for a given velocity potential* $\phi = cx$, where *c is a constant. Also, draw a set of steamlines and equipotential lines.* **[Rohilkhand 2003]**

Sol. The velocity components u and in x and y directions are given by

$u = -\partial\phi/\partial x = -c \quad$ and $\quad = -\partial\phi/\partial y = 0. \quad ...(1)$

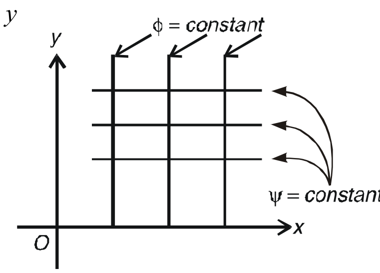
$\therefore \quad u = -\partial\psi/\partial y \quad$ and $\quad = \partial\psi/\partial x$

$\Rightarrow \quad \partial\psi/\partial y = c \quad$ and $\quad \partial\psi/\partial x = 0. \quad ...(2)$

Then, $\quad d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = c\ dy.$

Integrating, $\quad \psi = cy + d, \quad ...(3)$

where d is constant of integration.



Now, $\quad \phi$ = constant $\quad \Rightarrow \quad cx$ = constant $\quad \Rightarrow \quad x$ = constant,

showing that the lines of equipotential are parallel to y-axis.

Next, ψ = constant $\Rightarrow$ $cy + d$ = constant $\Rightarrow$ y = constant, showing that the streamlines are parallel to x-axis as shown in the figure.

Ex. 10. *In a two-dimensional incompressible flow, the fluid velocity components are given by* $u = x - 4y$ *and* $= -y - 4x$. *Show that velocity potential exists and determine its form as well as stream function.*

Sol. Given $u = x - 4y$ and $= -y - 4x$. ...(1)

The velocity potential will exist if flow is irrotational. Therefore, the vorticity component Ω_z in the z-direction must be zero.

Here $\Omega_z = (\partial\ /\partial x) - (\partial u/\partial y) = -4 - (-4) = 0$, using (1).

Here the vorticity being zero, the flow is irrotational and so the velocity potential ϕ exists.

Now, we have $d\phi = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy = -\,udx - \ \ dy$

or $d\phi = -(x - 4y)dx - (-y - 4x)dy = -xdx + ydy + 4(ydx + xdy)$.

Integration, $\phi = -(x^2/2) + y^2/2 + 4xy + C$, where C is constant of integration. ...(2)

If $\phi = 0$ at the origin, then from (2), we find $C = 0$. Hence (2) reduces to

$$\phi = (y^2 - x^2)/2 + 4xy.$$

Ex. 11. *For a two-dimensional flow the velocity function is given by the expression,* $\phi = x^2 - y^2$. *Then* (*i*) *Determine velocity components in x and y directions* (*ii*) *Show that the velocity components satisfy the conditions of flow continuity and irrotationality* (*iii*) *Determine stream function and flow rate between the streamlines* (2, 0) *and* (2, 2) (*iv*) *Show that the streamlines and potential lines intersect orthogonally at the point* (2, 2).

Sol. (*i*) The velocity components in x and y directions are

$u = -\partial\phi/\partial x = -2x$ and $= -\partial\phi/\partial y = 2y$. ...(1)

(*ii*) Here $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(2y) = -2 + 2 = 0,$$

showing that the velocity components satisfy the flow continuity conditions.

Here $$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & & w \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x & 2y & 0 \end{vmatrix}$$

or $$\text{curl } \mathbf{q} = \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2y)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-2x)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(-2x)\right]\mathbf{k}$$

$\Rightarrow$ curl $\mathbf{q} = \mathbf{0}$ $\Rightarrow$ flow is irrotational.

(*iii*) We know that $u = \partial\psi/\partial y$ and $= -\partial\psi/\partial x$. ...(2)

Then (1) and (2) $\Rightarrow$ $\partial\psi/\partial y = -2x$ and $\partial\psi/\partial x = -2y$

$\therefore$ $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = -2(y\,dx + xdy)$.

Integrating, $\psi = -2xy + C$, C being constant of integration.

The required flow between the streamlines through (2, 0) and (2, 2)

$= \psi(2, 0) - \psi(2, 2) = 0 - (-8) = 8$ m^3/s.

Now, we have $\phi = x^2 - y^2$ and $\psi = -2xy + C$. ...(3)

m_1 = The slope of tangent at (x, y) to potential lines $\phi = c_1$

$$= -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{2x}{-2y} = \frac{x}{y}, \quad \text{using (3)}$$

$\therefore$ m_1 = The slope of tangent to $\phi = c_1$ at (2, 2) = 2/2 = 1.

Next, m_2 = the slope of tangent to $\psi = c_2$ at $(x, y) = -\dfrac{\partial\psi/\partial x}{\partial\psi/\partial y} = -\dfrac{-2y}{-2x} = -\dfrac{y}{x}$, by (3)

$\therefore$ m_2 = slope of tangent to streamlines $\psi = c_2$ at $(2, 2) = -(2/2) = -1$

Here $m_1 m_2 = -1$ showing that the streamlines and the potential lines intersect orthogonally

Ex. 12. *Find the lines of flow in the two dimensional fluid motion given by* $\phi + i\psi = -(n/2) \times (x + iy)^2 e^{2int}$. *Prove or verify that the paths of the particles of the fluid (in polar coordinates) may be obtained by eliminating t from the equations.*

$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt\,(x_0 - y_0)$. **[Banaras 2003; I.A.S. 1992]**

Sol. Given $\qquad \phi + i\psi = -(n/2) \times (x + iy)^2 e^{2int}$. ...(1)

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

So (1) becomes $\qquad \phi + i\psi = -(n/2) \times (re^{i\theta})^2 e^{2int} = -(n/2) \times r^2 e^{2i(\theta + nt)}$

or $\qquad \phi + i\psi = -(n/2) \times r^2 [\cos 2(\theta + nt) + i \sin 2(\theta + nt)]$.

Equating the real and imaginary parts on both sides of (2), we get

$\phi = -(n/2) \times r^2 \cos 2(\theta + nt)$ and $\psi = -(n/2) \times r^2 \sin 2(\theta + nt)$. ...(2)

The lines of flow are given by ψ = constant, namely,

$-(n/2) \times r^2 \sin 2(\theta + nt)$ = constant or $r^2 \sin 2(\theta + nt)$ = constant.

We now proceed to find the path of the particles. We have

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = nr\cos 2(\theta + nt) = nr\cos 2\lambda, \text{ by (2)} \qquad \text{...(3)}$$

and $$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -nr\sin 2(\theta + nt) = -nr\sin 2\lambda, \text{ by (2)} \qquad \text{...(4)}$$

where $$nt + \theta = \lambda. \qquad \text{...(5)}$$

Now, $\quad$ (3) $\Rightarrow$ $\quad nr\cos 2\lambda = \dfrac{dr}{dt} = \dfrac{dr}{d\lambda}\dfrac{d\lambda}{dt} = \dfrac{dr}{d\lambda}\left(\dfrac{d\theta}{dt} + n\right)$, by (5)

or $$nr\cos 2\lambda = \frac{dr}{d\lambda}(-n\sin 2\lambda + n), \text{ using (4)}$$

or $$(2/r)\,dr - [2\cos 2\lambda/(1 - \sin 2\lambda)]\,d\lambda = 0.$$

Integrating, $\quad 2\log r + \log(1 - \sin 2\lambda) = \log C \quad$ or $\quad r^2(1 - \sin 2\lambda) = C$

or $\quad r^2(\sin^2\lambda + \cos^2\lambda - 2\sin\lambda\cos\lambda) = C \quad$ or $\quad [r(\cos\lambda - \sin\lambda)]^2 = C$

or $$r(\cos\lambda - \sin\lambda) = C', \text{ where } C' \,(= \sqrt{C}) \text{ is an arbitrary constant.} \qquad \text{...(6)}$$

Initially, let $\lambda = \theta_0$ and $r = r_0$ when $t = 0$. Then (6) gives

$C' = r_0(\cos\theta_0 - \sin\theta_0) = x_0 - y_0$, where $x_0 = r_0\cos\theta_0$, $y_0 = r_0\sin\theta_0$.

$\therefore$ (6) becomes $\qquad r\cos\lambda - r\sin\lambda = x_0 - y_0$ ...(7)

or $$r\cos(\theta + nt) - x_0 = r\sin(\theta - nt) - y_0, \text{ using (5).} \qquad \text{...(8)}$$

Now, from (5), $\quad d\lambda/dt = n + (d\theta/dt) \quad$ or $\quad d\lambda/dt = n - n\sin 2\lambda$, using (4)

or $\quad \dfrac{d\lambda}{1 - \sin 2\lambda} = ndt \quad$ or $\quad \dfrac{d\lambda}{(\cos\lambda - \sin\lambda)^2} = n\,dt$

$\therefore \quad \displaystyle\int \frac{\sec^2\lambda\, d\lambda}{(1 - \tan\lambda)^2} = \int n\,dt \quad$ or $\quad -\displaystyle\int \frac{du}{u^2} = nt + D$

(Putting $1 - \tan\lambda = u$ so that $-\sec^2\lambda\, d\lambda = du$)

or $\qquad \frac{1}{u} = nt + D \qquad$ or $\qquad \frac{1}{1 - \tan\lambda} = nt + D$

or $\qquad \cos\lambda/(\cos\lambda - \sin\lambda) = nt + D.$ ...(9)

As before, initially $\lambda = \theta_0$ and $t = 0$. Hence (9) gives

$$D = \frac{\cos\theta_0}{\cos\theta_0 - \sin\theta_0} = \frac{r_0 \cos\theta_0}{r_0\cos\theta_0 - r_0\sin\theta_0} = \frac{x_0}{x_0 - y_0}, \text{ as before}$$

Then, (9) becomes $\qquad \frac{r\cos\lambda}{r\cos\lambda - r\sin\lambda} = nt + \frac{x_0}{x_0 - y_0}$

or $\qquad \frac{r\cos(\theta + nt)}{x_0 - y_0} = nt + \frac{x_0}{x_0 - y_0} \qquad$ or $\qquad r\cos(\theta + nt) = nt(x_0 - y_0) + x_0$

or $\qquad r\cos(nt + \theta) - x_0 = nt(x_0 - y_0).$ ...(10)

$\therefore$ Then, from (8) and (10), we have

$$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt(x_0 - y_0)$$

Ex. 13. *A single source is placed in an infinite perfectly elastic fluid, which is also a perfect conductor of heat. Show that if the motion be steady, the velocity at a distance r from the source satisfies the equation* $\left(\;\; - \frac{k}{\;}\right) \frac{\partial}{\partial r} = \frac{2k}{r}$ and hence that $\qquad r = \frac{1}{\sqrt{\;}} e^{\;\;^2/4k}$.

Sol. Since we have an infinite perfectly elastic fluid, there would be hardly any change in temperature, and hence Boyle's law would be obeyed and so $\qquad p = k\rho$ ... (1)

Since the motion is symmetrical about the source, the equation of continuity may be written as

$$\rho r^2 \;\; = \text{constant},$$ ...(2)

where is the velocity at a distance r and ρ is the density of fluid. The pressure equation takes the form

$\int \frac{dp}{\rho} + \frac{\;\;^2}{2} = \text{constant} \qquad$ or $\qquad k\int \frac{\partial\rho}{\rho} + \frac{\;\;^2}{2} = \text{constant, by (1)}.$ ...(3)

Differentiating (2) and (3) w.r.t. 'r', we have

$$r^2 \frac{\partial\rho}{\partial r} + \rho\left[r^2 \frac{\partial}{\partial r} + 2r \;\;\right] = 0$$ ...(4)

and $\qquad \frac{k}{\rho}\frac{\partial\rho}{\partial r} + \;\frac{\partial}{\partial r} = 0, \qquad$ *i.e.*, $\qquad \frac{\partial\rho}{\partial r} = -\frac{\rho}{k}\frac{\partial}{\partial r}.$ ...(5)

Substituting the value of $\partial\rho/\partial r$ given by (5) in (4), we get

$$r^2\left(-\frac{\rho}{k}\frac{\partial}{\partial r}\right) + \rho\left(r^2\frac{\partial}{\partial r} + 2r \;\;\right) = 0$$

or $\qquad \frac{r^2}{k}\frac{\partial}{\partial r}(k - \;\;^2) = -2r \qquad$ or $\qquad \left(\;\; - \frac{k}{\;}\right)\frac{\partial}{\partial r} = \frac{2k}{r},$ ...(6)

which proves the first part of the problem.

Integrating (6), $(\;\;^2/2) - k\log \;\; = 2k\log r - 2k\log C,$ C being an arbitrary constant.

or $\qquad (1/2)\times\log \;\; + \log r - \log C = \;\;^2/4k \qquad$ or $\qquad r\sqrt{\;} = Ce^{\;\;^2/4k}$

or $\qquad r = (1/\sqrt{\;})e^{\;\;^2/4k}$, taking $C = 1$.

Ex.14. *Prove that the radius of curvature R at any point of a streamline* ψ = *constant is given by* $R = (u^2 + \ ^2)^{3/2} / | u^2 (\partial \ / \partial x) - 2u \ (\partial u / \partial x) - \ ^2 (\partial u / \partial y) |$,

where u,   are respectively the velocity components of a fluid motion along OX and OY.

Sol. From Differential Calculus, we know that the radius of curvature R at a point (x, y) of streamline $\psi (x, y) = \text{constant}$ is given by

$$R = [1 + (dy/dx)^2]^{3/2} / (d^2 y / dx^2). \qquad \text{...(1)}$$

Given streamline is $\qquad \psi(x, y) = 0. \qquad$ ...(2)

Also, we have $\qquad u = -\partial\psi/\partial y \qquad$ and $\qquad = \partial\psi/\partial x. \qquad$ ...(3)

Differentiating (2) w.r.t. x, $\qquad (\partial\psi/\partial x) + (\partial\psi/\partial y)(dy/dx) = 0$

or $\qquad - u\,(dy/dx) = 0 \qquad$ or $\qquad dy/dx = \ /u. \qquad$ ...(4)

Differentiating (4) w.r.t. x, $\qquad \dfrac{d^2 y}{dx^2} = \dfrac{\partial}{\partial x}\left(\dfrac{}{u}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{}{u}\right)\dfrac{dy}{dx}$

or $\qquad \dfrac{d^2 y}{dx^2} = \dfrac{u(\partial \ / \partial x) - \ (\partial u / \partial x)}{u^2} + \dfrac{u(\partial \ / \partial y) - \ (\partial u / \partial y)}{u^2} \cdot \dfrac{}{u}$, uning (4)

or $\qquad \dfrac{d^2 y}{dx^2} = \dfrac{u[u(\partial \ / \partial x) - \ (\partial u / \partial x)] + \ [u(\partial \ / \partial y) - \ (\partial u / \partial y)]}{u^3}$

or $\qquad \dfrac{d^2 y}{dx^2} = \dfrac{u^2(\partial \ / \partial x) - 2u \ (\partial u / \partial x) + \ (\partial u / \partial y)}{u^3} \qquad$ ...(5)

$$\left[\because \ \frac{\partial}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) = -\frac{\partial u}{\partial x}, \text{ by (3)}\right]$$

Putting the values of dy/dx and d^2y/dx^2 from (4) and (5) in (1), we get

$$R = \frac{(1 + \ ^2/u^2)^{3/2}}{|\{u^2(\partial \ / \partial x) - 2u \ (\partial u / \partial x) - \ ^2(\partial u / \partial y)\}/u^3|} = \frac{(u^2 + \ ^2)^{3/2}}{|u^2(\partial \ / \partial x) - 2u \ (\partial u / \partial x) - \ ^2(\partial u / \partial y)|}.$$

Ex.15 *Show that* $u = 2cxy$, $\ = c(a^2 + x^2 - y^2)$ *are the velocity components of a possible fluid motion. Determine the stream function.* **[Rohilkhand 1999]**

Sol. Given $\qquad u = 2cxy, \qquad = c(a^2 + x^2 - y^2) \qquad$ ...(1)

Equation of continuity in xy-plane is given by

$$\partial u / \partial x + \partial \ / \partial y = 0 \qquad \text{...(2)}$$

From (1), $\partial u/\partial x = 2cy$ and $\partial \ /\partial y = -2cy$. Putting these values in (2) we get 0 = 0, showing (2) is satisfied by u,   given by (1). Hence u and   constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have

$u = -(\partial\psi/\partial y) \qquad$ or $\qquad \partial\psi/\partial y = -2cxy \qquad$ ...(3)

and $\qquad = \partial\psi/\partial x \qquad$ or $\qquad \partial\psi/\partial x = c(a^2 + x^2 - y^2) \qquad$ ...(4)

Integrating (3) partially w.r.t. 'y' $\qquad \psi = -cxy^2 + \phi(x,t),$ ...(5)

where $\phi(x,t)$ is an arbitrary function of x and t.

Differentiating (5) partially w.r.t. 'x', $\qquad \partial\psi/\partial x = -cy^2 + \partial\phi/\partial x$ ...(6)

(4) and (6) $\Rightarrow \quad -cy^2 + \partial\phi/\partial x = c(a^2 + x^2 - y^2) \quad$ or $\quad \partial\phi/\partial x = c(a^2 + x^2)$ ...(7)

Integrating (7) partially w.r.t. 'x', $\qquad \phi(x,t) = c(a^2 x + x^3/3) + \psi(y,t),$

where $\psi(y,t)$ is an arbitrary function of y and t.

Substituting the above value of $\phi(x,t)$ in (5), we get

$\psi = c(ax^2 + x^3/3 - xy^2) + \psi(y,t),$ which is the required stream function.

Ex. 16. *Show that $u = -\omega y$, $\quad = \omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between this motion and one represented by the potential* $\phi = A\log r$, where $r = (x^2 + y^2)^{1/2}$.

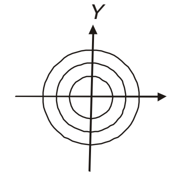
Sol. Given $\qquad u = -\omega y, \qquad = \omega x \qquad$ and $\qquad w = 0$ ...(1)

(1) $\Rightarrow \quad \partial u/\partial x = 0 = \partial \quad /\partial y$. Hence the equation of continuity $\partial u/\partial x + \partial \quad /\partial y = 0$ is satisfied. Hence these exist a two dimensional motion defined by (1).

Now, $\qquad \partial\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy$ ...(2)

But $\qquad \dfrac{\partial\psi}{\partial x} = -\dfrac{\partial\phi}{\partial y} = \quad = \omega x \qquad$ and $\qquad \dfrac{\partial\psi}{\partial y} = \dfrac{\partial\phi}{\partial x} = -u = \omega y$

$\therefore$ (3) reduces to $\qquad d\psi = \omega x\,dx + \omega y\,dy = d\{\omega(x^2 + y^2)/2\}$



Integrating, $\psi = \omega(x^2 + y^2)/2 + c$, where c is an arbitrary constant.

The required streamlines are given by ψ = constant = c', say

i.e. $\quad c' = \omega(x^2 + y^2)/2 + c \quad$ or $\quad x^2 + y^2 = 2(c' - c)/w = a^2$, say

Hence the required streamlines are concentric circles with centres at origin as shown in the adjoining figure.

Second part: Given

$$\phi = A\log r = A\log(x^2 + y^2)^{1/2} = (A/2)\times\log(x^2 + y^2) \qquad ...(3)$$

$\therefore \quad u = -\dfrac{\partial\phi}{\partial x} = -\dfrac{Ax}{x^2 + y^2} \qquad$ and $\qquad = -\dfrac{\partial\phi}{\partial y} = -\dfrac{Ay}{x^2 + y^2}$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = -A\dfrac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = A\dfrac{x^2 - y^2}{(x^2 + y^2)^2}, \qquad \dfrac{\partial}{\partial y} = -A\dfrac{x^2 - y^2 - 2y^2}{(x^2 + y^2)^2} = -\dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

$\therefore \partial u/\partial x + \partial \quad /\partial y = 0$ so that the equation of continuity is satisfied.

Hence there exists exists a motion for the given value of ϕ.

Third part. Difference between the two given motions.

For the fluid motion given by (1), we have

curl $\boldsymbol{q} = \boldsymbol{i}(\partial w/\partial y - \partial \quad /\partial z) + \boldsymbol{j}(\partial u/\partial z - \partial w/\partial x) + \boldsymbol{k}(\partial \quad /\partial x - \partial u/\partial y)$

$= \boldsymbol{i}(0 - 0) + \boldsymbol{j}(0 - 0) + \boldsymbol{k}(\omega + \omega) \neq \boldsymbol{0},$

showing that curl $\boldsymbol{q} \neq 0$. Hence velocity potential does not exist for the fluid motion defined by (1) (refer Art. 2.26), whereas velocity potential exist for the second fluid motion.

Ex. 17. *In irrotational motion in two dimensions, prove that*

$(\partial q/\partial x)^2 + (\partial q/\partial y)^2 = q\nabla^2 q$. **(Agra 2012; Kanpur 2002; Meerut 2002,05)**

Sol. Since the motion is irrotational, the velocity potential ϕ exists such that

$$\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \quad ...(1)$$

Again, $$q^2 = (\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 \quad ...(2)$$

Differentiating (2) partially w.r.t. x and y respectively, we get

$$q\frac{\partial q}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} \quad ...(3)$$

and $$q\frac{\partial q}{\partial y} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} \quad ...(4)$$

Differentiating (3) and (4) partially *w.r.t.* x and y respectively, we get

$$q\frac{\partial^2 q}{\partial x^2} + \left(\frac{\partial q}{\partial x}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x^3} + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial x^2\partial y} \quad ...(5)$$

and $$q\frac{\partial^2 q}{\partial y^2} + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x\partial y^2} + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial y^3} \quad ...(6)$$

Adding (5) and (6) and simplifying, we get

$$q\nabla^2 q + \left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

$$= 2(\partial^2\phi/\partial x^2)^2 + 2(\partial^2\phi/\partial x\partial y)^2 \quad ...(7)$$

$$\left[\because \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2} \Rightarrow \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 = \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$

Next, squaring and adding (3) and (4), we get

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$

$$+2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

$$= \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right], \text{ using (1)}$$

$$= q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right], \text{ using (2)}$$

Thus, $$(\partial^2\phi/\partial x^2)^2 + (\partial^2\phi/\partial x\partial y)^2 = (\partial q/\partial x)^2 + (\partial q/\partial y)^2 \quad ...(8)$$

From (7) and (8), we find

$$q\nabla^2 q + (\partial q/\partial x)^2 + (\partial q/\partial y)^2 = 2[(\partial q/\partial x)^2 + (\partial q/\partial y)^2]$$

or $$q\nabla^2 q = (\partial q/\partial x)^2 + (\partial q/\partial y)^2.$$

Ex. 18. *λ denoting a variable parameter, and f a given function, find the condition that $f(x, y, \lambda) = 0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.* **[Kurukshetra 1998]**

Sol. If ψ is the stream function, then streamlines are given by

$$\psi = C \text{ (constant)} \quad ...(1)$$

Given that $$f(x, y, \lambda) = 0 \quad ...(2)$$

represents a system of streamlines, λ being parameter. Then for $\lambda = \lambda'$ (say), (2) must give a streamline which corresponds with (1) for $C = C'$. Hence ψ is a function of λ alone. Moreover λ is a function of x and y from (2). Hence, we obtain

$$\frac{\partial\psi}{\partial x} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial x} \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial y}$$

Again, $$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{dx}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\cdot\frac{\partial\lambda}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\right)\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial}{\partial x}\left(\frac{\partial\lambda}{\partial x}\right)$$

so that $$\frac{\partial^2\psi}{\partial x^2} = \left\{\frac{d}{d\lambda}\left(\frac{d\psi}{d\lambda}\right)\right\}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2}$$

Thus, $$\frac{\partial^2\psi}{\partial x^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2} \quad ...(3)$$

Similarly, $$\frac{\partial^2\psi}{\partial y^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial y^2} \quad ...(4)$$

For the irrotational motion, $$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0. \quad ...(5)$$

Adding (3) and (4) and using (5), we get

$$\frac{d^2\psi}{d\lambda^2}\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) = 0$$

or $$\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right] \Big/ \left[\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right] = -\frac{d\psi/d\lambda}{d^2\psi/d\lambda^2} \quad ...(6)$$

Since the R.H.S. of (6) is a function of λ alone, the required condition is that the L.H.S. of (6) should be a function of λ alone.

Ex. 19. *In two-dimensional motion show that, if the streamlines are confocal ellipses*

$$x^2/(a^2+\lambda) + y^2/b^2+\lambda) = 1, \quad \text{then} \quad \psi = A\log(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}) + B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point. **[Rajasthan 1998]**

Sol. Take $z = C\cos w$ then $x + iy = C\cos(\phi + i\psi)$ ...(1)

or $$x + iy = C\,(\cos\phi \cos i\psi - \sin\phi \sin i\psi)$$

or $$x + iy = C\cos\phi\cosh\psi - iC\sin\phi\sinh\psi \qquad \text{...(2)}$$

Equating real and imaginary parts, (2) gives

$$x = C\cos\phi\cosh\psi \qquad \text{and} \qquad y = -C\sin\phi\sinh\psi$$

so that $$\cos\phi = \frac{x}{C\cosh\psi} \qquad \text{and} \qquad \sin\phi = -\frac{y}{C\sinh\psi}$$

Squaring and adding these, we obtain

$$\frac{x^2}{C^2\cosh^2\psi} + \frac{y^2}{C^2\sinh^2\psi} = 1 \qquad \text{...(3)}$$

which give the streamlines in two-dimensions.

Again, given that the streamlines are confocal ellipses

$$x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1 \qquad \text{...(4)}$$

Since (3) and (4) must be identical, we have

$$C^2\cosh^2\psi = a^2+\lambda \qquad \text{and} \qquad C^2\sinh^2\psi = b^2+\lambda$$

$$\therefore \quad C(\cosh\psi + \sinh\psi) = \sqrt{a^2+\lambda} + \sqrt{b^2+\lambda} \quad \text{or} \quad Ce^{\psi} = \sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}$$

$[\because \cosh\psi = (e^{\psi} + e^{-\psi})/2 \quad \text{and} \quad \sinh\psi = (e^{\psi} - e^{-\psi})/2\,]$

or $$\psi = \log(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}) - \log C \qquad \text{...(5)}$$

If ϕ, ψ are velocity potential and stream function, so also will be $A\phi$ and $A\psi$ where A is a constant. Hence (5) may be re-written as

$$\psi = A\log(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}) + B$$

From (1), $$\frac{dz}{dw} = -C\sin w = -C\sqrt{1-\cos^2 w} = -C(1 - z^2/C^2)^{1/2}$$

$$= -\sqrt{C^2 - z^2} = -\sqrt{(C+z)(C-z)} = -\sqrt{r_1 r_2}$$

where r_1 and r_2 are the focal distances (radii) of any point $P(z)$ from the foci $S\,(C, 0)$ and $S'\,(-C, 0)$ of the ellipses.

Thus $$q = |dw/dz| = 1/\sqrt{r_1 r_2}.$$

Ex. 20. *Show that the velocity potential* $$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

gives a possible motion. Determine the streamlines and show also that the curves of equal speed are the ovals of Cassini given by $rr' = $ *const.* **[Rajasthan 2000; I.A.S. 1990]**

Sol. Given $$\phi = (1/2)\times\log[(x+a)^2+y^2] - (1/2)\times\log[x-a)^2+y^2]$$

$$\therefore \quad u = -\frac{\partial\phi}{\partial x} = -\frac{x+a}{(x+a)^2+y^2} + \frac{x-a}{(x-a)^2+y^2} \qquad \text{...(1)}$$

and $$= -\frac{\partial\phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \qquad \text{...(2)}$$

From (1) $$\frac{\partial u}{\partial x} = -\frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2} + \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \qquad \text{...(3)}$$

From (2) $$\frac{\partial}{\partial y} = -\frac{(x+a)^2 - y^2}{[(x+a)^2 + y^2]^2} + \frac{(x-a)^2 - y^2}{[(x-a)^2 + y^2]^2} \qquad \text{...(4)}$$

Adding (3) and (4), we see that the equation of continuity $\partial u/\partial x + \partial \ /\partial y = 0$ is satisfied. Hence there exists a motion for the given ϕ.

To determine the streamlines, we use the fact that velocity potential ϕ and the stream function ψ satisfy the Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \quad \text{and} \quad \partial\phi/\partial y = -\partial\psi/\partial x \qquad \text{...(5)}$$

From (1) and (5), we have $$\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2 + y^2} - \frac{x-a}{(x-a)^2 + y^2}$$

Integrating it w.r.t. y, we get

$$\psi = \tan^{-1}\frac{y}{x+a} \tan^{1}\frac{y}{x-a} + f(x),\ f(x) \text{ being an arbitrary function of } x \qquad \text{...(6)}$$

$$\therefore \quad \frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} + f'(x) \qquad \text{...(7)}$$

Again from (5) and (2), we get

$$\frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} \qquad \text{...(8)}$$

Comparing (7) and (8) $f'(x) = 0$ so that $f(x)$ = constant. Omitting the additive constant, (6) gives

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} = \tan^{-1}\frac{[y/(x+a)] - [y/(x-a)]}{1 - [y/(x+a)][y/(x-a)]}$$

$$\therefore \quad \psi = \tan^{-1}\frac{(-2ay)}{x^2 + y^2 - a^2}$$

Hence the streamlines are given by ψ = const. = $\tan^{-1}$ ($-$ $2a/C$), that is,

$$x^2 + y^2 - Cy = a^2 \qquad \text{....(9)}$$

which are circles. When $C = 0$, the stream line is the circle passing through $(a, 0)$ and $(-a, 0)$. Again, if C is infinite then stream line $y = 0$ [divide (9) by C and then let $C \to \infty$]

Now, $$w = \phi + i\psi = \frac{1}{2}\log\left[(x+a)^2 + y^2)\right] - \frac{1}{2}\log\left[(x-a)^2 + y^2\right] + i\tan^{-1}\frac{y}{x+a} - i\tan^{-1}\frac{y}{x-a}$$

$$= \log[(x+a) + iy] - \log[(x-a) + iy] = \log(z+a) - \log(z-a), \text{ as } z = x + iy$$

$$\therefore \quad q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a| \cdot |z-a|} = \frac{2a}{rr'},$$

where r, r' are the distances of the point from the points P (x, y) from the points $(a, 0)$ and $(-a, 0)$. The curves of equal speed are given by q = constant or rr' = constant, which are Cassini ovals.

Ex. 21. *A velocity field is given by* **q** = $-$ x**i** + $(y + t)$**j**. *Find the stream function and the streamlines for this field at* $t = 2$. **[Agra 2005; Garhwal 2000; Rohilkhand 2002]**

Sol. We have $$-\partial\psi/\partial y = u = -x \quad \text{...(1)}$$

and $$\partial\psi/\partial x = \quad = y + t \quad \text{...(2)}$$

Integrating (1) and (2), we get $$\psi = xy + f_1(x, t) \quad \text{...(3)}$$

and $$\psi = xy + tx + f_2(x, t) \quad \text{...(4)}$$

Note that f_2 must be a function of t alone, otherwise (4) will not be satisfied; and then $f_1 = tx + f_2$. Thus

$$\psi = xy + tx + f_2(t) \quad \text{...(5)}$$

The function f_2 cannot be obtained from the given data. However since we deal only with differences in ψ values at a given t or with the derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$, the determination of f_2 is not necessary. At $t = 2$, (5) becomes

$$\psi = xy + 2x + f_2(2) \quad \text{...(6)}$$

The stream lines (ψ = constant) are given by $x(y + 2)$ = constant, which are rectangular hyperbolas.

Ex. 22. *A two-dimensional flow field is given by* $\psi = xy$. (*a*) Show that the flow is irrotational. (*b*) Find the velocity potential. (*c*) Verify that ψ and ϕ satisfy the Laplace equation. (*d*) Find the streamlines and potential lines. **[Agra 2005, 2011; Garhwal 2005]**

Sol. (*a*) The velocity components are given by $u = -\partial\psi/\partial y = -x$, $= \partial\psi/\partial x = y$ so that

$$\mathbf{q} = u\mathbf{i} + \quad \mathbf{j} \qquad \text{or} \qquad \mathbf{q} = -x\mathbf{i} + y\mathbf{j}$$

and
$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = \mathbf{0}.$$

Hence the flow is irrotational.

(*b*) We have $$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$$\therefore \quad \phi = \int(\partial\psi/\partial y)dx + f_1(y) = x^2/2 + f_1(y), \quad \text{...(1)}$$

and $$\phi = -\int(\partial\psi/\partial x)\,dy + f_2(x) = -(y^2/2) + f_2(x). \quad \text{...(2)}$$

(1) and (2) show that

$$f_1(y) = -y^2/2 + \text{constant} \qquad \text{and} \qquad f_2(x) = x^2/2 + \text{constant},$$

so that $$\phi = (x^2 - y^2)/2 + \text{constant}$$

(*c*) $\nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 + 0 = 0$ and $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 1 - 1 = 0$

Hence ψ and ϕ satisfy the Laplace equation.

(*d*) The streamlines (ψ = constant) and the potential lines (ϕ = constant) are given by $xy = C_1$ and $x^2 - y^2 = C_2$, respectively, where C_1 and C_2 are constants.

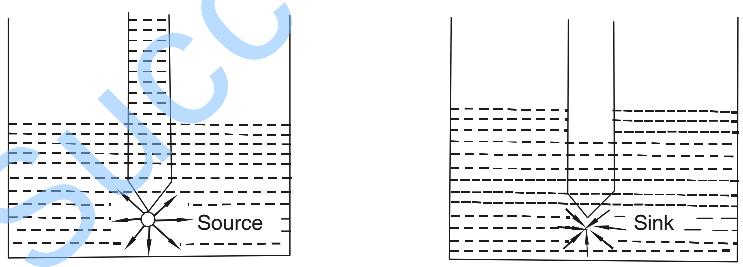
EXERCISE 5 A

1. Show that the difference of the values of ψ at two points represents the flux of the fluid across any curve joining the two points. **[Kanpur 2005]**

Sources and sinks.

[Aga 2005; Kanpur 2000, 02; Purvanchal 2004; Meerat 2009; Rohilkhand 2005]

If the motion of a fluid consists of symmetrical radial flow in all directions proceeding from a point, the point is known as a *simple source*. If, however, the flow is such that the fluid is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as a *simple sink*.



(*i*) (*ii*)

Obviously a source implies the creation of fluid at a point whereas a sink implies the annihilation of fluid at a point. Sources and sinks are not readily obtained by some dynamical effects of the motion of fluid but may occur due to some external causes. For example, consider a simple source in a tank filled with a fluid. This source may be created by taking a long tube of very small cross-section and injecting fluid through it into the tank as shown in figure (*i*). In such a situation, we find that the fluid is coming out from the tube radially into the tank, Again, a sink can be created by taking a long tube of very small cross-section and sucking fluid through the tube from the tank as shown in figure (*ii*).

Consider a source at the origin. Then the mass m of the fluid coming out from the origin in a unit time is known as the *strength of the source*. Similarly, in a tank at the origin, the amount of fluid going into the sink in a unit time is called the *strength of the sink*.

Remark. Since the velocity is unique at a point, so usually no two streamlines intersect each other. But some flow fields may have singularities, where the velocity vector is not unique. Sources and sinks are examples of singularities of a flow field because infinitely many stream lines meet at such points as indicated in the figures (*i*) and (*ii*).

5.12. Source and sinks in two-dimensions.

[Garhwal 2002; Kanpur 1999; Meerut 2010]

In two-dimensions a source of strength m is such that the flow across any small curve surrounding is $2\pi m$. Sink is regarded as a source of strength $-m$.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{...(1)}$$

or $$q_r = -\frac{\partial \phi}{\partial r}, \qquad \text{as} \qquad \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{...(2)}$$

Then the flow across the circle is $2\pi r q_r$. Hence we have

$$2\pi r q_r = 2\pi m \qquad \text{or} \qquad r q_r = m \qquad \text{...(3)}$$

or $$r\left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = m, \text{ by (1)}$$

Integrating and omitting constant of integration, we get

$$\psi = -m\theta \qquad \text{...(4)}$$

Using (2) and (3), we obtain as before

$$\phi = -m \log r \qquad \text{...(5)}$$

Equation (4) shows that the streamlines are θ = constant, *i.e.*, straight lines radiating from the source. Again (5) shows that the curves of equi-velocity potential are r = constant, *i.e.*, concentric circles with centre at the source.

5.13. Complex potential due to a source.

[Kurukshetra 2004; Meerut 2001, 2012; Kanpur 2009]

Let there be a source of strength m at origin. Then

$$w = \phi + i\psi = -m \log r - im\theta = -m\,(\log r + i \log e^{i\theta}) = -m \log (re^{i\theta}) = -m \log z.$$

If, however, the source is at z', then the complex potential is given by $w = -m \log (z - z')$

The relation between w and z for sources of strengths $m_1, m_2, m_3, \ldots$ situated at the points $z = z_1, z_2, z_3, \ldots$ is given by

$$w = -m_1 \log (z - z_1) - m_2 \log (z - z_2) - m_3 \log (z - z_3) - \ldots$$

leading to $$\phi = -m_1 \log r_1 - m_2 \log r_2 - m_3 \log r_3 - \ldots$$

and $$\psi = -m_1\theta_1 - m_2\theta_2 - m_3\theta_3 - \ldots$$

where $r_n = |z - z_n|$ and $\theta_n = \arg (z - z_n)$, $n = 1, 2, 3, \ldots$

5.14. Doublet (or dipole) in two dimensions

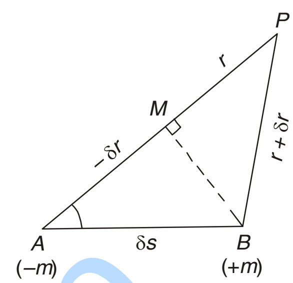
[Agra 2005; Garhwal 2000, 04; Rohilkhand 2000; Kanpur 1999, 2002, 07]

A combination of a source of strength m and a sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely great and δs infinitely small but so that the product $m\delta s$ remains finite and equal to μ, is called a *doublet of strength* μ, and the line δs taken in the sense from $-m$ to $+m$ is taken as the *axis of the doublet*.

Complex potential due to a doublet in two-dimensions

[G.N.D.U. Amritsar 2004, 06; Kanpur 2000, 05, 07; Meerut 2002, 09, 10; Purvanchal 2004, 05]

Let A, B denote the positions of the sink and source and P be any point. Let $AP = r$, $BP = r + \delta r$ and $\angle PAB = \theta$. Let ϕ be the velocity potential due to this doublet.



Then $$\phi = m \log r - m \log (r + \delta r) = -m \log \frac{r + \delta r}{r}$$

or $$\phi = -m \log \left(1 + \frac{\delta r}{r}\right)$$

$\therefore$ $\phi = -m \dfrac{\delta r}{r}$, to first order of approximation. ...(1)

Let BM be perpendicular drawn from B on AP. Then,

$$AM = AP - MP = r - (r + \delta r) = -\delta r$$

$\therefore$ $\cos\theta = AM / AB = -\delta r / \delta s$ so that $\delta r = -\delta s \cos\theta$

$\therefore$ From (1), $$\phi = m\delta s \cdot \frac{\cos\theta}{r} = \frac{\mu \cos\theta}{r} \qquad ...(2)$$

where $\mu = m\delta s =$ strength of the doublet.

From (2), $$\frac{\partial\phi}{\partial r} = -\frac{\mu\cos\theta}{r^2}$$

or $\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = -\dfrac{\mu\cos\theta}{r^2}$, as $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$

or $$\frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r}$$

Integrating it with respect to θ, we get

$$\psi = -\frac{\mu\sin\theta}{r} + f(r) \qquad ...(3)$$

Now, $$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} \qquad ...(4)$$

Using (2) and (3), (4) reduces to

$$\frac{1}{r}\left(-\frac{\mu\sin\theta}{r}\right) = -\left[\frac{\mu\sin\theta}{r^2} + f'(r)\right]$$

or $f'(r) = 0$ so that $f(r)$ = constant Hence omitting the additive constant, (3) reduces to

$$\psi = -\frac{\mu\sin\theta}{r} \qquad ...(5)$$

Using (2) and (5), the complex potential due to a doublet is given by

$$w = \phi + i\psi = \frac{\mu}{r}(\cos\theta - i\sin\theta) = \frac{\mu}{r}e^{-i\theta} = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z}.$$

Note 1. Equi-potential curves are given by ϕ = constant, *i.e.*, by

$(\mu\cos\theta)/r =$ constant or $(\cos\theta)/r = C$

$\therefore$ $r\cos\theta = Cr^2$ or $x = C(x^2 + y^2)$,

which represent circles touching the y-axis at the origin.

Note 2. Streamlines are given by ψ = constant *i.e.*, by

$$(-\mu \sin\theta)/r = \text{constant} \quad \text{or} \quad (\sin\theta)/r = C'$$

or $$r\sin\theta = C'r^2 \quad \text{or} \quad y = C'(x^2+y^2),$$

which represent circles touching the x-axis at the origin.

Note 3. If the doublet makes an angle θ with x-axis, we have to write $\theta - \alpha$ for θ so that

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}.$$

If the doublet be at the point $A(x', y')$ where $z' = x' + iy'$ [in place of A being origin (0, 0)]

then we have $$w = \frac{\mu e^{i\alpha}}{z - z'}$$

Note 4. If doublets of strengths $\mu_1, \mu_2, \mu_3, \ldots$ are situated at $z = z_1, z_2, z_3, \ldots$ and their axes making angles $\alpha_1, \alpha_2, \alpha_3, \ldots$ with x-axis, then the complex potential due to the above system is given by

$$w = \frac{\mu_1 e^{i\alpha_1}}{z - z_1} + \frac{\mu_2 e^{i\alpha_2}}{z - z_2} + \frac{\mu_3 e^{i\alpha_3}}{z - z_3} + \ldots$$

5.15. Illustrative solved examples.

Ex. 1. *What arrangement of sources and sinks will give rise to the function* $w = \log (z - a^2/z)$. *Draw a rough sketch of the streamlines. Prove that two of the streamlines subdivide into the circle $r = a$ and axis of y.* **[Kanpur 2003, 04; Meerut 2001, 03, 10, 11; [Agra 2005; Garhwal 2004; GNDU Amritsar 2003, 05; Rohilkhand 2000, 03, 05]**

Sol. Given $$w = \log\left(z - \frac{a^2}{z}\right) = \log\left[\frac{(z-a)(z+a)}{z}\right]$$

or $$w = \log (z - a) + \log (z + a) - \log z$$

which shows that there are two sinks of unit strength at the points $z = a$ and $z = -a$ and a source of unit strength at origin. Since $w = \phi + i\psi$ and $z = x + iy$, we obtain

$$\phi + i\psi = \log(x + iy - a) + \log(x + iy + a) - \log(x + iy)$$

$$\therefore \quad \phi + i\psi = \log[(x-a) + iy] + \log[(x+a) + iy] - \log(x + iy)$$

Equating imaginary parts on both sides, we have

$$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}, \text{ as } \log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= \tan^{-1}\frac{\dfrac{y}{x-a} + \dfrac{y}{x+a}}{1 - \dfrac{y}{x-a}\cdot\dfrac{y}{x+a}} - \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} - \tan^{-1}\frac{y}{x}$$

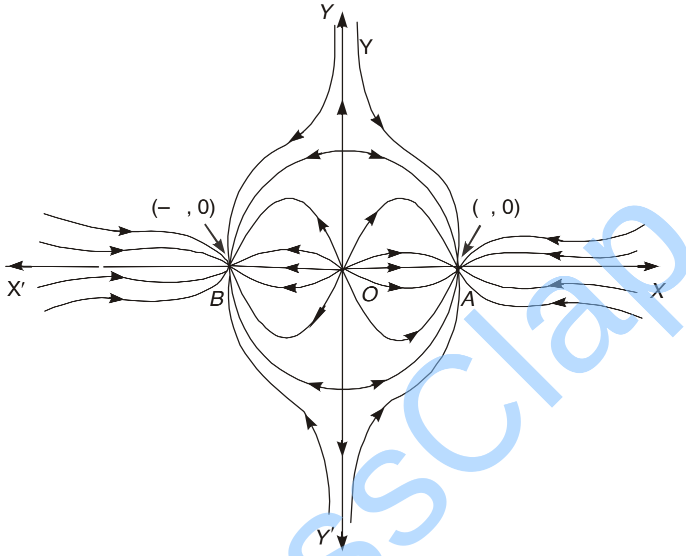
$$= \tan^{-1}\frac{\dfrac{2xy}{x^2-y^2-a^2} - \dfrac{y}{x}}{1 + \dfrac{2xy}{x^2-y^2-a^2}\cdot\dfrac{y}{x}} = \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}.$$

The desired streamlines are given by ψ = constant = $\tan^{-1}(C)$, *i.e.*

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C. \qquad \ldots(1)$$

When $C = 0$, (1) reduces to $y = 0$. Thus x-axis is a streamline. Again, when $C \to \infty$, (1) reduces to $x(x^2 + y^2 - a^2) = 0$, *i.e*, $x = 0$ and $x^2 + y^2 = a^2$ or $r = a$, which are streamlines.

Hence the rough sketch of the streamlines is as shown in the following figure. In this figure there is a source of unit strength at origin O and there are two sinks each of unit strength at A $(a, 0)$ and $(-a, 0)$.



Ex. 2. *There is a source of strength m at* (0, 0) *and equal sinks at* (1, 0) *and* (–1, 0). *Discuss two-dimensional motion. Also draw the stream lines.* **[Meerut 2002, 09]**

Sol. Proceed just like Ex. 1. Here, we have

$$w = m \log (z - 1) + m \log (z + 1) - m \log (z - 0)$$

$$\phi + i\psi = m\,[\log (x + iy - 1) + \log (x + iy + 1) - \log (x + iy)]$$

$$\therefore \quad \psi = m\left[\tan^{-1}\frac{y}{x-1} + \tan^{-1}\frac{y}{x+1} - \tan^{-1}\frac{y}{x}\right] \quad \text{or} \quad \frac{\psi}{m} = \tan^{-1}\frac{y(x^2+y^2+1)}{x(x^2+y^2-1)}$$

[As in Ex. 1, here note that $a = 1$]

The desired streamlines are given by $\psi/m = \text{constant} = \tan^{-1} C$ *i.e.*

$$\frac{y(x^2+y^2+1)}{x(x^2+y^2-1)} = C. \qquad \text{...(1)}$$

Now give the same discussion and figure as given in Ex. 1 noting that here $a = 1$.

Ex. 3. *Two sources, each of strength m are placed at the points* $(-a, 0)$, $(a, 0)$ *and a sink of strength 2m at the origin. Show that the streamlines are the curves* $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ *where* λ *is a variable parameter.* **[U.P. P.C.S. 1999; I.A.S. 1999, 2003]**

Show *also that the fluid speed at any point is* $(2ma^2)/(r_1 r_2 r_3)$ *where* r_1, r_2, r_3, *are the distances of the points from the sources and the sink.*

[I.A.S. 1999, 2003; Meerut 2000; Garhwal 2005; Rohilkhand 2002]

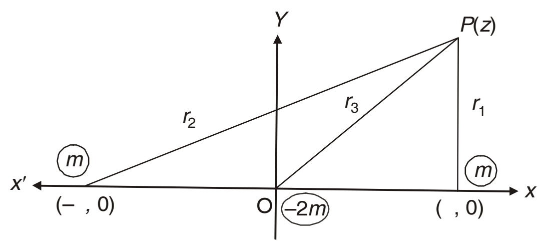
Sol. First Part.

The complex potential w at any point $P(z)$ is given by

$$w = -m \log (z - a) - m \log (z + a) + 2m \log z \qquad \text{...(1)}$$

or $$w = m\,[\log z^2 - \log (z^2 - a^2)]$$

or $$\phi + i\psi = m[\log(x^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)], \text{ as } \quad z = x + iy$$



Equating the imaginary parts, we have

$$\psi = m\left[\tan^{-1}\left\{2xy/(x^2 - y^2)\right\} - \tan^{-1}\left\{2xy/(x^2 - y^2 - a^2)\right\}\right]$$

$\therefore$ $$\psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right], \text{ on simplification.}$$

The desired streamlines are given by ψ = constant = m tan^{-1} $(-2/\lambda)$. Then we obtain

$$(-2/\lambda) = (-2a^2xy)/[(x^2+y^2)^2 - a^2(x^2-y^2)] \quad \text{or} \quad (x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy).$$

Second Part. From (1), we have

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)(z+a)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where $r_1 = |z - a|,$ $r_2 = |z + a|$ and $r_3 = |z|.$

Ex. 4. *An area A is bounded by that part of the x-axis for which x > a and by that branch of $x^2 - y^2 = a^2$ which is in the positive quadrant. There is a two-dimensional unit source at (a, 0) which sends out liquid uniformly in all directions. Show by means of the transformation $w = \log (z^2 - a^2)$ that in steady motion the streamlines of the liquid within the area A are portions of rectangular hyperbola. Draw the streamlines corresponding to ψ = 0, π/4, π/2. If ρ_1 and ρ_2 are the distances of a point P within the fluid from the points (± a, 0), show that the velocity of the fluid at P is measured by $2OP/\rho_1\rho_2$, O being the origin.* **[Grahwal 2001]**

Sol. Given $$w = \log (z^2 - a^2) \qquad ...(1)$$

or $w = \log [(x + iy)^2 - a^2]$ or $\phi + i\psi = \log[(x^2 - y^2 - a^2) + 2ixy]$

Equating the imaginary parts, we have

$$\psi = \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} \qquad ...(2)$$

The streamlines are given by ψ = constant = tan^{-1}C, *i.e.*,

$$(2xy)/(x^2 - y^2 - a^2) = C \qquad ...(3)$$

When $C = 0$, stream lines (3) reduce to $xy = 0$ *i.e.*, $x = 0$, and $y = 0$. Again, when $C \to \infty$, (3) reduces to $x^2 - y^2 - a^2 = 0$, *i.e.* $x^2 - y^2 = a^2$.

Hence the liquid flows in the area A bounded by $x = 0$, $y = 0$ and $x^2 - y^2 = a^2$ in the positive quadrant.

From (1), $\quad w = \log(z - a) + \log(z + a),$

which shows that there is a source of unit strength at $(a, 0)$ and an equal source at $(-a, 0)$. Here the source at $(-a, 0)$ is the image of $(a, 0)$ with respect to y-axis.

From (1), $$\frac{dw}{dz} = \frac{2z}{z^2 - a^2} = \frac{2z}{(z-a)(z+a)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2|z|}{|z-a||z+a|} = \frac{2OP}{\rho_1 \rho_2}$$

From (2), the streamline corresponding to $\psi = 0$ is

$$\frac{2xy}{x^2 + y^2 - a^2} = 0 \qquad \text{giving} \qquad x = 0 \qquad \text{and} \qquad y = 0.$$

From (2), the streamline corresponding to $\psi = \pi/4$ is

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{4} = 1 \qquad \text{or} \qquad x^2 - y^2 - a^2 = 2xy.$$

From (2), the streamline corresponding to $\psi = \pi/2$ is

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{2} = \infty \qquad \text{or} \qquad x^2 - y^2 = a^2.$$

Ex. 5. *Find the stream function of the two-dimencial motion due to two equal sources and an equal sink situated midway between them.* **[Kanpur 2008; I.A.S. 1996]**

Sol. Let there be two sources of strength m at the points $z = a$ and $z = -a$ and a sink of same strength at $z = 0$ (origin). Then complex potential w due to these sources and sink is given by

$$w = -m\log(z - a) - m\log(z + a) + m\log(z - 0)$$

or $$\phi + i\psi = m\log(x + iy) - m\log(x + iy - a) - m\log(x + iy + a)$$

or $$\phi + i\psi = m\log(x + iy) - m\log\{(x - a) + iy\} - m\log\{(x + a) + iy\}$$

or $$\phi + i\psi = m\{(1/2)\times\log(x^2 + y^2) + i\tan^{-1}(y/x)\} - m[(1/2)\times\log\{(x-a)^2 + y^2\}$$
$$+ i\tan^{-1}\{y/(x - a)\}] - m\,[(1/2)\times\log\{(x + a)^2 + y^2\} + i\tan^{-1}\{y/(x + a)\}]$$

Equating imaginars parts on both sides, we get

$$\psi = m\tan^{-1}(y/x) - m[\tan^{-1}\{y/(x-a)\} + \tan^{-1}\{y/(x+a)\}]$$

or $$\frac{\psi}{m} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{\{y/(x-a)\} + \{y/(x+a)\}}{1 - \{y/(x-a)\}\{y/(x+a)\}} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2}$$

or $$\frac{\psi}{m} = \tan^{-1}\frac{(y/x) - \{2xy/(x^2 - y^2 - a^2)\}}{1 + (y/x)\{2xy/(x^2 - y^2 - a^2)\}} \qquad \text{or} \qquad \psi = m\tan^{-1}\frac{y(x^2 + y^2 + a^2)}{x(a^2 - x^2 - y^2)}$$

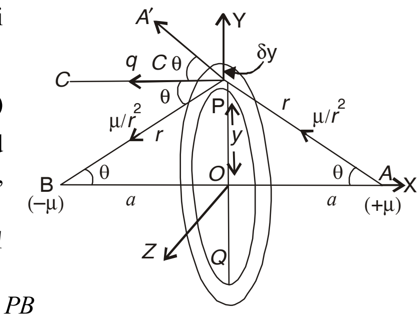
Ex. 6. *An infinite mass of liquid is moving irrotationally and steadily under the influence of a source of strength* μ *and an equal sink at a distance* $2a$ *from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles the line joining the source and sink is* $(8\pi\rho\mu^3)/7a^4$, ρ *being the density of the liquid.*

Sol. Let a aource of strength μ and a sink of strength $-\mu$ be situated at A and B such that $AB = 2a$. Let O be the middle point of AB so that $OA = OB = a$. Let OYZ be the plane which bisects AB at right angles. Hence $\angle POA = \angle POB = 90°$. Let $\angle PAB = \angle PBA = \theta$. Let PC be parallel to AB such that $\angle A'PC = \angle BPC = \theta$. Also let $AP = BP = r$. From ΔPAO, we have

$$\cos\theta = a/r \qquad \text{and} \qquad y = a\tan\theta, \qquad \text{where} \qquad OP = y. \qquad \text{Also,} \qquad r = (a^2 + y^2)^{1/2}.$$

Consider an annular strip bounded by circles of radii y and $y+\delta y$. It δS be the area of this strip, then

$$\delta S = 2\pi y\, \delta y. \qquad \ldots(1)$$



At any point inside the circular ring all the fluid particles have the same velocity q in the same direction, namely normal to the plane.

Now, velocity at P due to source $+\mu$ at A $= \mu/r^2$ along AP

and velocity at P due to sink $-\mu$ at $B = \mu/r^2$ along PB

$\therefore$ q = the resultant of the above velocites along $PC = \dfrac{2\mu}{r^2}\cos\theta = \dfrac{2\mu a}{r^3} = \dfrac{2\mu a}{(a^2+y^2)^{3/2}}$.

In unit time the mass δm of the liquid crossing the strip is given by

$$\delta m = \rho(\delta S)q = \rho(2\pi y\delta y)q, \text{ by (1)} \qquad \ldots(2)$$

Hence the required K.E. of the liquid which passes across the plane OYZ in unit time

$$= \int \frac{1}{2}\delta m q^2 = \int_0^\infty \frac{1}{2} q^2 (2\pi\rho\, y\, q\, dy) = \pi\rho \int_0^\infty q^3 y\, dy, \text{ by (2)}$$

$$= \pi\rho \int_0^\infty \left[\frac{2\mu a}{(a^2+y^2)^{3/2}}\right]^3 y\, dy = 8\pi\rho\mu^3 a^3 \int_0^\infty \frac{y\, dy}{(a^2+y^2)^{9/2}}.$$

$$= 8\pi\rho\mu^3 a^3 \int_0^{\pi/2} \frac{a\tan\theta \cdot a\sec^2\theta\, d\theta}{a^9 \sec^9\theta}, \text{ putting } y = a\tan\theta \text{ and } dy = a\sec^2\theta\, d\theta$$

$$= \frac{8\pi\rho\mu^3}{a^4}\int_0^{\pi/2} \cos^6\theta \sin\theta\, d\theta = \frac{8\pi\rho\mu^3}{a^4}\left[-\frac{\cos^7\theta}{7}\right]_0^{\pi/2} = \frac{8\pi\rho\mu^3}{7a^4}.$$

Ex. 7. In a two dimensinal liquid motion ϕ and ψ are the velocitly and current functions, show that a second fluid motion exists in which ψ is the velocity potential and $-\phi$ the current function; and prove that if the first motion be due to sources and sinks, the second motion can be built up by replacing a source and an equal sink by a line of doublets uniformly distributed along any curve joining them.

Sol. Since ϕ and ψ are the velocity potential and stream function respectively for the two-dimensional motion, we have

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -(\partial\psi/\partial x) \qquad \ldots(1)$$

Again if ψ and $-\phi$ be the velocity potential and strem function respectively for another fluid motion in two-dimensions, then the conditions of the type (1) must be satisfied by ψ and $-\phi$ *i.e.*, we must have

$$\frac{\partial\psi}{\partial x} = \frac{\partial(-\phi)}{\partial y} \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial(-\phi)}{\partial x}$$

i.e., $\quad \partial\phi/\partial y = -(\partial\psi/\partial x) \quad$ and $\quad \partial\phi/\partial x = \partial\psi/\partial y$

which is true by virtue of (1).

If follows that if $w = \phi + i\psi$ exists, then $w' = \psi - i\phi = -i(\phi + i\psi) = -iw$, also exists.

Second part. Consider a source of strength m at A $(a, 0)$ and a sink of strength $-m$ at B $(-a, 0)$. Then, the complex potential function w due to them is given by

$$w = -m\log(z-a) + m\log(z+a) = m\log\{(z+a)/(z-a)\} \qquad \ldots(2)$$

Join A, B by an arbitrary curve. Then the axis of the doublet on this curve is normal to AB. If w'' be the complex potential due this line of doublets then

$$w'' = \int_A^B \frac{me^{i\pi/2}}{z-t}\,dt = m\,e^{i\pi/2}\log\frac{z-a}{z+a} = mi\log\frac{z-a}{z+a} = -iw$$

The required result now follows from the first part.

EXERCISE 5 (B)

1. Find the cartesian equation of the lines of plane flow when fluid is streaming from three equal sources situated at the corners of an equilateral triangle.

2. Let there be a source of strength m at $(a, 0)$ and a sink $-m$ at $(-a, 0)$. Find ϕ, ψ, w and velocity q.

3. Let there be a source of strength m at $(a, 0)$ and a sink $-m$ at $(0, a)$. Find ϕ, ψ, w and velocity q.

4. If there are sources at $(a, 0)$ and $(-a, 0)$ and sinks at $(0, a)$, $(0, -a)$ all of equal strengths, show that the circle through these four points is a streamline. **[I.A.S. 1990]**

5. A source of strength m at A $(a, 0)$ and a sink of strength $-m$ at B $(-a, 0)$ are in the xy plane and in the presence of a uniform stream U-parallel to the x-axis. The stream is directed from the source to the sink. Derive the stream function of the resulting motion.

6. A source and a sink of the same strength are placed at a given distance apart in an infinite fluid which is otherwise at rest. Show that the streamlines are circles and that the fluid speed along any streamline varies inversely as the distance from the line joining the source and sink.

7. Define sources and sinks and explain their utility in hydrodynamics. **[Kanpur 2002]**

8. There is a source at A and an equal sink at B. AB is the direction of a uniform stream. If A is $(a, 0)$, B is $(-a, 0)$ and the ratio of the flow issuing from A in unit time to the speed of the stream is $2\pi b$, show the stream function is $\psi = Vy - Vb\tan^{-1}[2ay/(x^2+y^2-a^2)]$ and that the length $2l$, and the breadth $2d$, of the closed wall that forms part of the dividing streamline is given by $l^2 = a^2 + 2ab$, $\tan(d/b) = 2ad/(d^2 - a^2)$ and the locus of the points at which the speed is equal to that of the stream is $x^2 - y^2 = a^2 + ab$.

9. Sources of equal strength are placed at the points $z = nia$ where $n = \ldots, -2, -1, 0, 1, 2, \ldots$. Prove that the complex potential is $w = -m\log\sinh(\pi z/a)$. Hence show that the complex potential for doublets, parallel to x-axis of strength μ at the same points is given by $w = \mu\coth(\pi z/a)$.

If the row of doublets is placed in a uniform stream $-$ U parallel to x-axis, prove that the streamline $\psi = 0$ is
$$\frac{ay}{\pi b^2} = \frac{\sin(2\pi y/a)}{\cosh(2\pi x/a) - \cos(2\pi y/a)}$$
and show that this consists of part of the x-axis and part of an oval curve which is nearly circular (diameter $2b$) if $b << a$.

5.16. Images.

If in a liquid a surface S can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface. Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unchanged.

As there is no flow across the surface, it must be a streamline. Thus the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

Images in two dimensions.

If in a liquid a curve C can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this curve is known as the image of the system with regard to the curve.

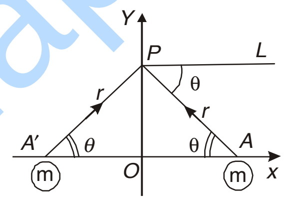
5.17. Advantages of images in fluid dynamics. [Kanpur 2002]

The method of images is used to determine the complex potential due to sources, sinks and doublets in the presence of rigid boundaries. Suppose we wish to determine the flow field outside a rigid boundary due to sources, sinks, doublets lying outside the boundary. To this end we assume the existence of some hypothetical image sources, sinks, doublets within the boundary in such a manner so that the boundary behaves as a streamline or surface. Then the given system of sources, sinks and doublets together with the hypothetical one will be equivalent to the given sources and the rigid boundaries for the region outside the rigid boundary.

5.18. Image of a source with respect to a line.

[Agra 2006; Kanpur 2003, 04, 07, 08; Meerut 2003]

Suppose that image of the source m at A $(a, 0)$ on x-axis is required with respect to OY. Take an equal source at $A'(-a, 0)$. Let P be any point on OY such that $AP = A'P = r$. Then the velocity at P due to source at A is m/r along AP and velocity at P due to source A' is m/r along $A'P$. Let PL be perpendicular to OY. Then, we see that



Resultant velocity at P due to sources at A and A' along PL

$$= (m/r)\cos\theta - (m/r)\cos\theta = 0,$$

showing that there will be no flow across OY. Hence by definition, the image of a simple source with respect to a line in two-dimensions is an equal source equidistant from the line opposite to the source.

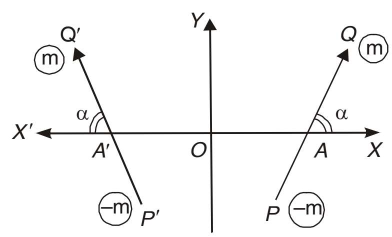
Remark 1. Proceeding as above we can prove that the image of a sink with respect to a line in two-dimensions is an equal sink equidistant from the line opposite to the sink.

Remark 2. The result of Art. 5.18 will still hold good if a line is replaced by a plane.

Corollary. Image of a doublet with respect to a line.

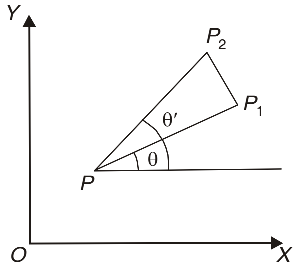
[Kanpur 2002, Rohilkhand 2003]

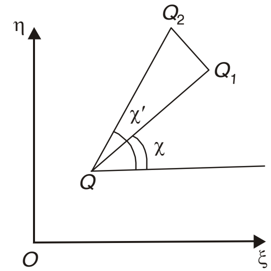
Let PQ be a doublet with its axis inclined at an angle α to OX. Then by using the above result for finding the images of source and sink with respect to OY, we see that the image of the doublet PQ is again an equal doublet $P'Q'$ symmetrically placed as shown in the adjoining figure.



5.19A. Conformal representation (or transformation or mapping.)

Let $f(z)$ be a function of the complex variable $z = x + iy$ and let $f(z)$ be single-valued and differentiable within a closed contour C in the z-plane (*i.e.* xy- plane). Let $\zeta = \xi + i\eta$ be another complex variable in ζ-plane (*i.e* $\xi\eta$-plane) and let there be a relation $\zeta = f(z)$. Then corresponding to each point in the z-plane within or on C, there will be a point ζ in the ζ-plane and points on C or within C will lie on or within a certain contour C' in the ζ-plane. The necessary condition for existence of such a mapping of z-plane into ζ-plane is that $f'(z)$ should never vanish at any point on or within C, or in other words, $d\zeta/dz$ must exist independent of the directions of δz. Thus, let P, P_1, P_2 be neighbouring points z, z_1, z_2 and Q, Q_1, Q_2 the corresponding points ζ, ζ_1, ζ_2. Then, we have





$$\frac{\zeta_1-\zeta}{z_1-z}=\frac{f(z_1)-f(z)}{z_1-z}, \qquad \text{and} \qquad \frac{\zeta_2-\zeta}{z_2-z}=\frac{f(z_2)-f(z)}{z_2-z}$$

In the limit when P_1, P_2 approach P, we have

$$\frac{\zeta_1-\zeta}{z_1-z}=f'(z), \qquad \text{and} \qquad \frac{\zeta_2-\zeta}{z_2-z}=f'(z), \text{ very nearly}$$

$$\therefore \qquad \frac{\zeta_1-\zeta}{z_1-z}=\frac{\zeta_2-\zeta}{z_2-z}=f'(z)=\frac{d\zeta}{dz} \qquad \ldots(1)$$

or $$\frac{QQ_1e^{i\chi}}{PP_1e^{i\theta}}=\frac{QQ_2e^{i\chi'}}{PP_2e^{i\theta'}} \qquad \text{or} \qquad \frac{QQ_1}{PP_1}e^{i(\chi-\theta)}=\frac{QQ_2}{PP_2}e^{i(\chi'-\theta')}$$

$\therefore\ \chi-\theta=\chi'-\theta'$ or $\chi'-\chi=\theta'-\theta$ *i.e.* $\angle Q_1QQ_2=\angle P_1PP_2$

and $$\frac{QQ_1}{PP_1}=\frac{QQ_2}{PP_2}=|f'(z)|=\left|\frac{d\zeta}{dz}\right|$$

Hence the triangles P_1PP_2 and Q_1QQ_2 are similar. This establishes the similarity of the correspoding infitiesimal elements of the two planes. Such a relation between the two planes is called the *conformal represenation* of either plane on the other.

Again $$\frac{\Delta\, Q_1QQ_2}{\Delta P_1\, PP_2}=\frac{(1/2)\times QQ_1\times QQ_2\,\sin\angle Q_1OQ_2}{(1/2)\times PP_1\times PP_2\,\sin\angle P_1OP_2}=\frac{QQ_1}{PP_1}\times\frac{QQ_2}{PP_2}=|f'(z)|^2 \qquad \ldots(2)$$

From $\zeta=f(z)$ *i.e.,* $\xi+i\eta=f(x+iy)$, we have

$$\frac{\delta\xi}{\delta z}=\frac{\delta(\xi+i\eta)}{\delta(x+iy)}=\frac{\frac{\partial\xi}{\partial x}\delta x+\frac{\partial\xi}{\partial y}\delta y+i\left(\frac{\partial\eta}{\partial x}\delta x+\frac{\partial\eta}{\partial y}\delta y\right)}{\delta x+i\delta y}=\frac{\left(\frac{\partial\xi}{\partial x}+i\frac{\partial\eta}{\partial x}\right)\delta x+\left(\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y}\right)\delta y}{\delta x+i\delta y}$$

Since $\delta\xi/\delta z$ is independent of $\delta x/\delta y$, we must have

$$\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y}=i\left(\frac{\partial\xi}{\partial x}+i\frac{\partial\eta}{\partial x}\right)$$

so that $$\frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial y} \qquad \text{and} \qquad \frac{\partial\xi}{\partial y}=-\frac{\partial\eta}{\partial x} \qquad \ldots(3)$$

Also $$\frac{d\zeta}{dz}=\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y} \qquad \ldots(4)$$

$$\therefore \qquad |f'(z)|^2=\left|\frac{d\xi}{dz}\right|^2=\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2, \text{ using (1) and (4)}$$

or $$|f'(z)|^2 = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 = h^2, \text{ (say), using (3)} \qquad \ldots(5)$$

$\therefore$ From (2), $\Delta Q_1 Q Q_2 / \Delta P_1 P P_2 = h^2$ so that $d\xi\, d\eta = h^2 dxdy$,

showing that the corresponding areas in the ζ and z planes are in the ratio $h^2 : 1$.

Let ϕ and ψ be the velocity and current functions of any motion within the contour C' in ζ-plane. Then, within the contour C', we have

$$\phi + i\psi = F_1(\xi + i\eta) \qquad \ldots(6)$$

and C' is given by $$\psi = f_1(\xi, \eta) = \text{const.} \qquad \ldots(7)$$

Substituting the values of ξ, η in terms of x, y, (6) and (7) respectively reduce to

$$\phi + i\psi = F_2(x + iy) \qquad \ldots(8)$$

and $$\psi = f_2(x, y) = \text{constant}, \qquad \ldots(9)$$

where $f_2(x, y)$ is the new value of $f_1(\xi, \eta)$. Thus we fund that ϕ and ψ are the same in the two cases. In other words, $w = \phi + i\psi$ is the same in both the motions so that if q_1 and q_2 be velocities at P and Q respectively, then

$$q_1 = |dw/dz|^2 \qquad \text{and} \qquad q_2^2 = |dw/d\zeta|^2$$

so that $$q_2^2 = q_1^2\, |dz/d\zeta|^2 = q_1^2/h^2, \text{ using (5)}$$

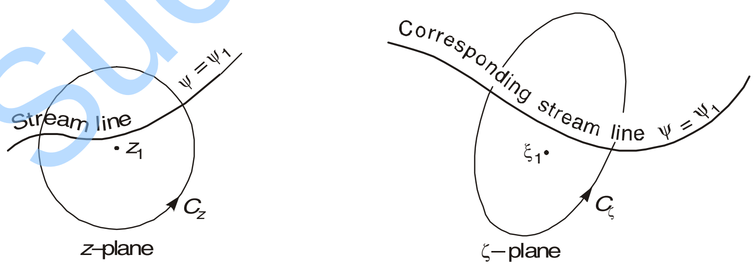
$\therefore$ $$q_2^2\, d\xi d\eta = (q_1^2/h^2) \times h^2\, dx\, dy = q_1^2 dx\, dy,$$

so that $$\frac{1}{2}\int \rho q_2^2\, d\xi\, d\eta = \frac{1}{2}\int \rho\, q_1^2 dx\, dy,$$

showing that the kinetic energies of the two fields are equal.

5.19B. Two important transformation.

(*i*) **Transformation of a source.**



Let there be a source of strength m at z_1 and ζ_1 be the corresponding point in the ζ-plane. Let these be regular points of the transformation. Then a small closed curve C_z may be drawn to enclose z_1 which will transform into a small closed curve C_ζ enclosing ζ_1. Since the value of the stream function is independent of the domain considered, we obtain

$$\int_{C_z} d\psi = \int_{C_\zeta} d\psi, \qquad \ldots(1)$$

But $$\int_{C_z} d\psi = \int_{C_z} (udy - \ dx) = \text{total flow across the contour } C_z$$

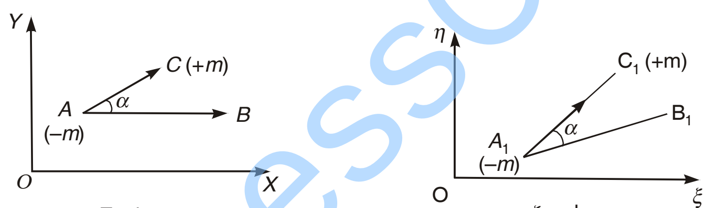
$$= \text{sum of source strength within } C_z$$

Let C_z be chosen sufficiently small so as to isolate the single source of strength m at z_1. It follows from (1) that this will transform into a source at ζ_1 of equal strength. Thus, a source will always transform into a source but the strengths will be equal only if the position of the source is a regular point of the transformation *i.e.* when it is possible to draw a closed contour surrounding the point. However care has to be taken at a zero, infinity or branch point of the function that ζ is of z or that z is of ζ. For example, in the case $\zeta = z^{1/2}$, since a semi circle with centre $\zeta = 0$ transforms into a circle with centre $z = 0$ (as arg $\zeta = (1/2) \times \arg z$). Hence if there be a source of strength m at $z = 0$ the corresponding source at $\zeta = 0$ must be of strength $2m$ as the mass efflux is unchanged by the transformation. (see the following figure).

Fluid produced = m × 2π
z - plane

fluid produced = 2 m × π
ζ - plane

(*ii*) **Transformation of a Doublet.**



Z–plane

ζ – plane

Let there be doublet of strength μ at A. Then by above case (*i*), it follows that there will be doublet of strength μ' at the corresponding point A_1 in the ζ-plane.

Also, $A_1C_1 = h\, AC$, $\mu = m \cdot AC$, $\mu' = m \cdot A_1C_1$

$\therefore$ $\dfrac{\mu'}{\mu} = \dfrac{A_1C_1}{AC} = h$, *i.e.*, $\mu' = \mu h$

If the axis AC of the doublet at A makes an angle α with a given direction, the axis A_1C_1 of doublet at A_1 will make angle α with the corresponding direction in ζ-plane.

5.19C. Same theorems concerning conformal transformation of line distribution

[Bangalore 2005, Kurukshetra 1997]

Theorem 1. *Under conformal transformation a uniform line source maps into another uniform line source of the same strength.* **[Bangalore 2005; K.U. Kurukshetra 1997]**

Proof. Let there be a uniform line source of strength m per unit length through the point $z = z_0$ and suppose the conformal transformation $\zeta = f(z)$ is made from the z-plane to the ζ-plane so that the point $z = z_0$ maps into the point $\zeta = \zeta_0$. Let C_{z_0} be a closed curve in the z-plane containing the point $z = z_0$ and C_{z_0} maps into C_{ζ_0} is the ζ-plane. Then $\zeta = \zeta_0$ lies within C_{ζ_0}. The complex potential w is clearly the same for both the systems and has the forms

$$\left.\begin{aligned} w &= \phi + i\psi, \text{ for the } z\text{-plane} \\ &= \phi' + i\psi', \text{ for the } \zeta\text{-plane.} \end{aligned}\right\} \quad \ldots(1)$$

From (1), $\phi = \phi'$ and $\psi = \psi'$. Since ψ is the same at corresponding points of C_{z_0} and C_{ς_0} we have,

$$\oint_{C_{z_0}} d\psi = \oint_{C_{\varsigma_0}} d\psi' \qquad \ldots(2)$$

But in the z-plane, $w = -m \log(z - z_0)$ and $dw = -m dz/(z - z_0)$. Then, using Cauchy-Residue theorem, we have

$$\therefore \qquad \oint_{C_{z_0}} dw = -m \oint_{C_{z_0}} \frac{dz}{z - z_0} = -m \times (2\pi i), \qquad \ldots(3)$$

since the integrand has a residue of 1 at $z = z_0$. Also, $w = \phi + i\psi \;\Rightarrow\; dw = d\phi + i\, d\psi$. So, (3) reduces to

$$\oint_{C_{z_0}} (d\phi + i d\psi) = -m \times (2\pi i) \qquad \Rightarrow \qquad \oint_{C_{z_0}} d\psi = -2\pi m \qquad \ldots(4)$$

The numerical value of this is clearly the volume of fluid crossing unit thickness of C_{z_0} per unit time. Thus, (2) and (4) show that the same volume crosses unit thickness of C_{ζ_0} per unit time which implies an equal line source of strength m per unit length at $\zeta = \zeta_0$.

Theorem II. *Under conformal transformation a uniform line vortex maps into another uniform line vertex of the same strength.* **[Nagpur 2003, 05]**

Proof. Let there be a uniform line vortex of strength k per unit length through $z = z_o$. Also assume that the conformed transformation $\zeta = f(z)$ is made from the z-plane to the ζ-plane so that the point $z = z_o$ maps into $\zeta = \zeta_0$. Let C_{z_0} be a closed curve containing $z = z_o$ and let C_{ζ_0} be its map in the ζ-plane. Then C_{ζ_0} contains $\zeta = \zeta_0$. The complex potential w is clearly the same for both the systems and has the forms

$$\left.\begin{aligned} w &= \phi + i\psi, \text{ for the z-plane} \\ &= \phi' + i\psi', \text{ for the } \varsigma\text{-plane}. \end{aligned}\right\} \qquad \ldots(1)$$

From (1), $\phi = \phi'$ and $\psi = \psi'$. Since ψ is the same at corresponding points of C_{z_0} and C_{ζ_0}, we have

$$\oint_{C_{z_0}} d\psi = \oint_{C_{\zeta_0}} d\psi' \qquad \ldots(2)$$

Now, in the z-plane, $\quad w = \dfrac{ik}{2\pi} \log(z - z_o) \quad$ so that $\quad dw = \dfrac{ik}{2\pi(z - z_o)} dz$

[Using result (14) of Art. 11.4 of chapter 11]

$$\therefore \qquad \oint_{C_{z_0}} dw = \frac{ik}{2\pi} \oint_{C_{z_0}} \frac{dz}{z - z_0} = \frac{ik}{2\pi} \times 2\pi i = -k, \qquad \ldots(3)$$

since the integrand has a residue of 1 at $z = z_0$.

Also, $\quad w = \phi + i\psi \quad \Rightarrow \quad dw = d\phi + i\, d\psi. \quad$ Hence, (3) reduces to

$$\oint_{C_{z_0}} (d\phi + i\, d\psi) = -k \qquad \Rightarrow \qquad -\oint_{C_{z_0}} d\phi = k \qquad \ldots(4)$$

The integral on the L.H.S. of (4) is the circulation round C_{z_0}. Equations (2) and (4) show that the circulation round C_{ζ_0} is also k. Since C_{z_0} and C_{ζ_0} are arbitrary, it follows that the line source through $z = z_0$ of strength k per unit length maps into an equal line source thorugh $\zeta = \zeta_0$.

Theorem III. *Under conformal transformation a uniform line doublet maps into another uniform line doublet of different strength.* **[K.U.Kurukshetra 2003]**

Proof. Let there be a uniform doublet of strength μ per unit length through P where $z = z_0$. Also assume that under conformal transformation $\zeta = f(z)$, P maps into Q where $\zeta = \zeta_0$.

Let the doublet be replaced by equivalent line sources of strengths $-m$, $+m$ per unit length through P, P' where $\overline{PP'} = \delta z$, $\mu = m|\delta z|$ and $\overline{PP'}$ is in the direction of the axis of the line doublet. Suppose P' maps into Q'. Then by the theorem I, the line sources of strengths $-m$, $+m$ per unit length through P, P' map into ones of strengths $-m$, $+m$ per unit length through Q, Q'. If $\overline{QQ'} = \delta\zeta$, then $\delta\zeta = f'(z)\delta z$, so that $|\delta\zeta| = |f'(z)|\cdot|\delta z|$ and arg $\delta\zeta = \arg f'(z) + \arg \delta z$, showing that the two line sources through Q, Q' give a line doublet at Q of strength μ' where $\mu' = m|\delta\zeta| = \mu|f'(z)|$. The inclination of the axis of the line doublet to the real axis is increased by $\arg f'(z)$.

5.19D. Summery of important results regarding applications of conformal transformations in fluid dynamics [Kanpur 2000]

(*i*) In a conformal transformation a source is transformed into an equal source, a sink into an equal sink and a doublet into an equal doublet.

(*ii*) The complex potential $w = \phi + i\psi$ is invariant under a conformal transformation.

(*iii*) Let $\zeta = f(z)$ be the conformal transformal transformation. Then
total K.E. of fluid in z-plane (per unit depth) = Total K.E. of the liquid in ζ-plane (per unit depth)

(*iv*) Under a conformal transformation, a stream line in z-plane is transtormed into, a stream line in ζ-plane.

(*v*) While using conformal transformation $\zeta = z^n$, n is found by dividing $\pi/2$ by half the angle contained between the rigid boundaries.

5.20. Illustrative solved examples.

Ex. 1. *Between the fixed boundaries* $\theta = \pi/6$ *and* $\theta = -\pi/6$ *there is a two-dimensional liquid motion due to a source at the point* $(r = c, \theta = \alpha)$ *and a sink at the origin absorbing water at the same rate as the source produces. Find the stream function and show that one of the stream lines is a part of the curve* $r^3 \sin 3\alpha = c^3 \sin 3\theta$.

[Kanpur 2000, 07; Meerut 2002 Garhwal 2003, 04; Rohilkhand 2003, 04]

Sol. Consider the following conformal transformation from z-plane (xy-plane) to ζ-plane ($\xi\eta$-plane) :

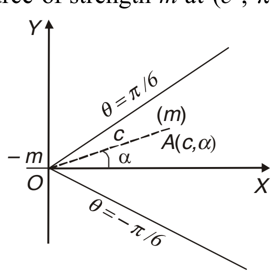
$$^*\zeta = z^3 \qquad \text{where} \qquad z = re^{i\theta} \qquad \text{and} \qquad \zeta = Re^{i\Theta}$$

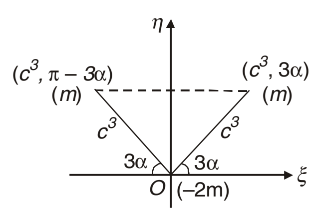
This $\Rightarrow$ $Re^{i\Theta} = r^3 e^{3i\theta}$ $\Rightarrow$ $R = r^3$ and $\Theta = 3\theta$.

Hence the boundaries $\theta = \pm\pi/6$ in z-plane transform to $\Theta = \pm\pi/2$, *i.e.*, imaginary axis in ζ-plane. The point (c, α) in z-plane transforms to $(c^3, 3\alpha)$ in ζ-plane. Hence the image system with respect to imaginary axis $(\Theta = \pm\pi/2)$ in ζ-plane consists of

(*i*) a source of strength m at $(c^3, 3\alpha)$, (*ii*) a sink of strength $-m$ at (0, 0),

(*iii*) a source of strength m at $(c^3, \pi - 3\alpha)$ (*iv*) a sink of strength $-m$ at (0, 0)





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* Refer result (*v*) of Art. 5.19D. Take $\zeta = z^n$. Here, half the angle contained by boundaries = $\pi/6$. Therefore, $n = (\pi/2)/(\pi/6) = 3$ and hence we take $\zeta = z^3$.

$$\therefore \quad w = -\log(\zeta - c^3 e^{3i\alpha}) - m\log\{\zeta - c^3 e^{i(\pi-3\alpha)}\} + 2m\log(\zeta - 0)$$

$$= -\log(z^3 - c^3 e^{3i\alpha}) - m\log(z^3 + c^3 e^{-3i\alpha}) + 2m\log z^3$$

$$[\because \ \zeta = z^3 \ \text{and} \ e^{i\pi} = \cos\pi + i\sin\pi = -1]$$

$$= -m\log[(z^3 - c^3 e^{3i\alpha})(z^3 + c^3 e^{-3i\alpha})] + 6m\log z = -m\log(z^6 - c^6 - 2ic^3 z^3 \sin 3\alpha) + 6m\log z$$

$$= -m\log\left(\frac{z^6 - c^6 - 2ic^3 z^3 \sin 3\alpha}{z^6}\right) = -m\log(1 - c^6 z^{-6} - 2ic^3 z^{-3}\sin 3\alpha)$$

$$= -m\log(1 - c^6 r^{-6} e^{-6i\theta} - 2ic^3 r^{-3} e^{-3i\theta}\sin 3\alpha)$$

$$\therefore \quad w = -m\log[1 - c^6 r^{-6}\cos 6\theta - 2c^3 r^{-3}\sin 3\alpha \sin 3\theta + i(c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta)]$$

Writing $w = \phi + i\psi$ and equating imaginary parts, we get*

$$\psi = -m\tan^{-1}\frac{c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta}{1 - c^6 r^{-6}\cos 6\theta - 2c^3 r^{-3}\sin 3\theta\sin 3\theta}, \qquad \ldots(1)$$

which is the required stream function. The stream lines are given by ψ = constant. The stream line corresponding to $\psi = 0$ is given by [putting $\psi = 0$ in (1) and noting that $\tan^{-1} = 0$]

$$c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta = 0 \qquad \text{or} \qquad c^3\sin 6\theta = 2r^3\sin 3\alpha\cos 3\theta$$

or $$2c^3\sin 3\theta\cos 3\theta = 2r^3\sin 3\alpha\cos 3\theta \qquad \text{or} \qquad c^3\sin 3\theta = r^3\sin 3\alpha.$$

Ex.2. *Between two fixed boundaries $\theta = \pi/4$ and $\theta = -\pi/4$, there is two-dimensional liquid motion due to a source of strength m at the point ($r = a$, $\theta = 0$) and an equal sink at the point ($r = b$, $\theta = 0$). Show that the stream function is*

$$-m\tan^{-1}\frac{r^4(a^4 - b^4)\sin 4\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta + a^4 b^4}$$

[I.A.S. 1998; Garhwal 1996; Meerut 2006; Rohilkhand 2000; U.P.P.C.S 2000; Kanpur 1999]

and show that the velocity at (r, θ) is

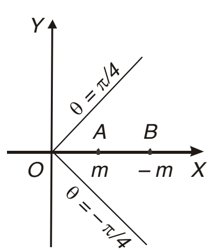
$$\frac{4m(a^4 - b^4)r^3}{(r^8 - 2a^4 r^4\cos 4\theta + a^8)^{1/2}(r^8 - 2b^4 r^4\cos 4\theta + b^8)^{1/2}}$$

[I.A.S. 1991, 94]

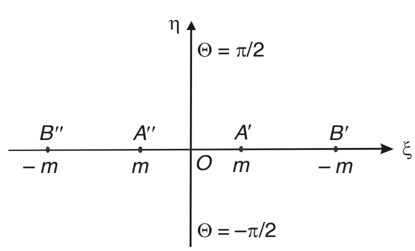
Sol. Consider the following conformal transformation from z-plane (*xy*-plane) to ζ-plane ($\xi\eta$-plane).

$$\zeta = z^2, \qquad \text{where} \qquad z = re^{i\theta} \qquad \text{and} \qquad \zeta = Re^{i\Theta}$$

This $\Rightarrow$ $Re^{i\Theta} = r^2 e^{2i\theta}$ $\Rightarrow$ $R = r^2$ and $\Theta = 2\theta$.



z - PLANE



ζ - PLANE

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*$\log(x + iy) = (1/2) \times \log(x^2 + y^2) + i\tan^{-1}(y/x)$, $\quad \log(x - iy) = (1/2) \times \log(x^2 + y^2) - i\tan^{-1}(y/x)$

Hence the boundaries $\theta = \pm\pi/4$ in z-plane transform to $\Theta = \pm\pi/2$ *i.e.*, imaginary axis of ζ-plane. The points $A\,(a, 0)$ and $B\,(b, 0)$ in z-plane transform to $A'\,(a^2,0)$ and $B'\,(b^2,0)$ respectively in ζ-plane. Then, the image system with respect to imaginary axis $(\Theta = \pm\pi/2)$ in ζ-plane consists of

(*i*) a source of strength m at $A'\,(a^2, 0)$ (*ii*) a sink of strength $-m$ at $B'\,(b^2, 0)$
(*iii*) a source of strength m at $A''\,(-a^2, 0)$ (*iv*) a sink of strength $-m$ at $B''\,(-b^2, 0)$

$$\therefore\quad w = -m\log(\zeta - a^2) + m\log(\zeta - b^2) - m\log(\zeta + a^2) + m\log(\zeta + b^2)$$

$$= -m\log(\zeta^2 - a^4) + m\log(\zeta^2 - b^4) = -m\log(z^4 - a^4) + m\log(z^4 - b^4),\ \text{as}\ \zeta = z^2 \quad \ldots(2)$$

$$= -m\log(r^4 e^{4i\theta} - a^4) + m\log(r^4 e^{4i\theta} - b^4),\ \text{as}\ z = re^{i\theta}$$

$$\therefore\quad w = -m\log(r^4\cos 4\theta - a^4 + ir^4\sin 4\theta) + m\log(r^4\cos 4\theta - b^4 + ir^4\sin 4\theta) \quad \ldots(3)$$

Writing $w = \phi + i\psi$ in (3) and equating imaginary parts, we get

$$\psi = -m\left[\tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta - a^4} - \tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta + b^4}\right]$$

or $\psi = -m\tan^{-1}\dfrac{r^4(a^4 - b^4)\sin 4\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta - a^4 b^4}$, on using $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$

From (2), $$\frac{dw}{dz} = -m\cdot\frac{4z^3}{z^4 - a^4} + m\frac{4z^3}{z^4 - b^4} = \frac{-4mz^3(a^4 - b^4)}{(z^4 - a^4)(z^4 - b^4)}$$

$$= \frac{-4mr^3 e^{3i\theta}(a^4 - b^4)}{(r^4 e^{4i\theta} - a^4)(r^4 e^{4i\theta} - b^4)} = \frac{-4mr^3(\cos 3\theta + i\sin 3\theta)(a^4 - b^4)}{(r^4\cos 4\theta - a^4 + 4ir^4\sin 4\theta)(r^4\cos 4\theta - b^4 + 4ir^4\sin^4\theta)}$$

Hence the required velocity $q = |dw/dz|$ is given by

$$q = \frac{4mr^3(a^4 - b^4)}{(r^8 - 2a^4 r^4\cos 4\theta + a^8)^{1/2}(r^8 - 2b^4 r^4\cos 4\theta + b^8)^{1/2}}$$

Ex. 3. *Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \pi/3$, the solution is*

$$\phi + i\psi = -m\log\{(z^3 - z_0^3)(z^3 - z_0'^3)\} \quad \text{where} \quad z_0 = x_0 + iy_0 \quad \text{and} \quad z_0' = x_0 - iy_0.$$

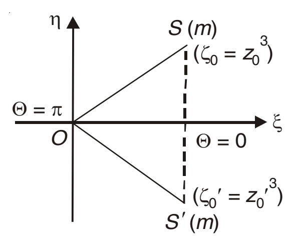
[Agra 2000; Garhwal 2005; Kanpur 2002, 04; I.A.S. 1997]

Sol. Consider the following conformal transformation from z-plane (xy-plane) to ζ-plane ($\xi\eta$-plane) :

$$\zeta = z^3 \quad \text{where} \quad z = re^{i\theta} \quad \text{and} \quad \zeta = Re^{i\Theta} \quad \ldots(1)$$

This $\Rightarrow$ $Re^{i\Theta} = r^3 e^{3i\theta}$ $\Rightarrow$ $R = r^3$ and $\Theta = 3\theta$.

Hence the boundaries $\theta = 0$ and $\theta = \pi/3$ in z-plane transform to $\Theta = 0$ and $\Theta = \pi$ *i.e.*, real axis in ζ-plane. The point z_0 in z-plane transforms to point ζ_0 in z-plane such that $\zeta_0 = z_0^3$. Hence the image system with respect to real axis in ζ-plane consists of



(*i*) a source m at $\zeta_0 = z_0^3$ (*ii*) a source m at $\zeta_0' = z_0'^3$

Hence, $$w = -m\log(\zeta - \zeta_0) - m\log(\zeta - \zeta_0')$$

or $$w = -m\log(z^3 - z_0^3) - m\log(z^3 - z_0'^3)$$

or $$\phi + i\psi = -m\log\{(z^3 - z_0^3)(z^3 - z_0'^3)\}.$$

Ex. 4. *If fluid fills the region of space on the positive side of the x-axis, which is a rigid boundary and if there be a source m at the point* (0, *a*) *and an equal sink at* (0, *b*) *and if the pressure on the negative side be the same as the pressure at infunity, show that the resultant pressure on the boundary is* $\pi\rho m^2 (a-b)^2 / 2ab(a+b)$, *where* ρ *is the density of the fluid.*

[U.P.P.C.S. 1995; I.A.S. 1995, 2008]

Sol. Here the image system with respect to *x*-axis in *z*-plane consists of

(*i*) a source *m* at (0, *a*) *i.e.*, at $z = ai$ (*ii*) a sink $-m$ at (0, *b*) *i.e.*, at $z = bi$

(*iii*) a source *m* at (0, $-a$) *i.e.*, at $z = -ai$ (*iv*) a sink $-m$ at (0, $-b$) *i.e.*, at $z = -bi$

Clearly this image system does away with the boundary $y = 0$ (*i.e.*, *x*-axis). Thus, the complex potential of this entire system is given by

$\therefore$ $$w = -m \log(z - ai) + m\log(z - bi) - m\log(z + ai) + \log(z + bi)$$

or $$w = -m\log(z^2 + a^2) + m\log(z^2 + b^2)$$

$\therefore$ $$\text{velocity} = \left|\frac{dw}{dz}\right| = \left|-\frac{2zm}{z^2+a^2} + \frac{2zm}{z^2+b^2}\right|$$

The velocity *q* at a point on the boundary (*i.e.*, $y = 0$) is given by (setting $z = x + iy = x$ as $y = 0$)

$$q = \left|-\frac{2xm}{x^2+a^2} + \frac{2xm}{x^2+b^2}\right| = \frac{2xm(a^2-b^2)}{(x^2+a^2)(x^2+b^2)} \qquad \text{... (1)}$$

Let p_0 be the pressure at infinity. Then by Bernoulli's theorem, the pressure *p* at any piont is given by

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \frac{1}{2}\times 0^2 + \frac{p_0}{\rho} \quad \text{or} \quad \frac{p_0 - p}{\rho} = \frac{1}{2}q^2. \qquad \text{... (2)}$$

$\therefore$ The resultant pressure on the boundary

$$= \int_0^\infty (p_0 - p)\,dx = \frac{1}{2}\rho\int_0^\infty q^2\,dx = 2\rho m^2 \int_0^\infty \frac{x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}dx, \text{ by (1) and (2)}$$

$$= 2\rho m^2 \int_0^\infty \left[-\frac{a^2+b^2}{a^2-b^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right) - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right]dx$$

(on resolving into partial fractions)

$$= 2\rho m^2\left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) - \frac{\pi}{4a} - \frac{\pi}{4b}\right\}, \text{ on simplification}$$

$$= \frac{\pi\rho m^2}{2ab}\left[\frac{2(a^2+b^2) - (a+b)^2}{(a+b)}\right] = \frac{\pi\rho m^2(a-b)^2}{2ab(a+b)}.$$

Ex. 5. *Parallel line sources* (*perpendicular to xy-plane*) *of equal strength m are parallel at the points* $z = nia$ *where* $n = \ldots, -2, -1, 0, 1, 2, \ldots$. *Prove that the complex potential is* $w = -m \log \sinh(\pi z/a)$. *Hence, show that the complex potential for two dimensional doublets* (*lines doublets*), *with their axes parallel to the x-axis, of strength* μ *at the same points is given by* $w = \mu \coth(\pi z/a)$.

Sol. The complex potential due to sources of strength *m* situated at the points $z = 0, ia, -ia, 2ia, -2ia, \ldots$ is given by

$$w = -m\log(z-0) - m\log(z-ia) - m\log(z+ia) - m\log(z-2ia) - m\log(z+2ia) - \ldots$$

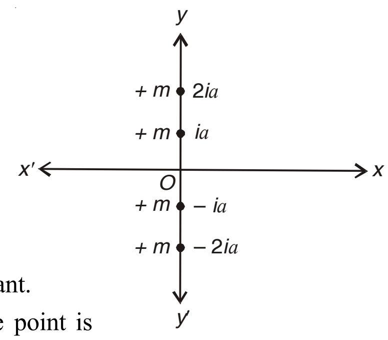
$$= -m\log z - m\log\{(z-ia)(z+ia)\} - m\log\{(z-2ia)(z+2ia)\} - \ldots$$

$= -m\log z - m\log(z^2 + a^2) - m\log(z^2 + 2^2a^2) - \ldots$

$= -m\log[z(z^2 + a^2)(z^2 + 2^2a^2)(z^2 + 3^2a^2)\ldots]$

$$= -m\log\left\{\frac{\pi}{a}z\left(1+\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{2^2a^2}\right)\left(1+\frac{z^2}{3^2a^2}\right)\ldots\right\}$$

$$-m\log\left[\left(\frac{a}{\pi}\right)a^2(2^2a^2)(3^2a^2)\ldots\right]$$



$\therefore$ $\qquad w = -m\sinh(\pi z/a) + \text{constant}.$

The complex potential w_1 for the doublets at the same point is

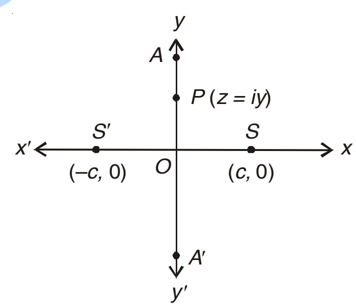
$$w_1 = -\frac{\partial w}{\partial z} = \frac{m\pi}{a}\coth\left(\frac{\pi z}{a}\right) = \mu\coth\left(\frac{\pi z}{a}\right), \qquad \text{where} \qquad \mu = \frac{m\pi}{a}.$$

Ex. 6. *In the case of the motion of liquid in a part of a plane bounded by a straight line due to a source in the plane, prove that if* $m\rho$ *is the mass of fluid (of density* ρ*) generated at the source per unit of time the pressure on the length 2l of the boundary immediately oppostite to the source is less than that on an equal length at a great distance by*

$\dfrac{1}{\rho}\dfrac{m^2\rho}{\pi^2}\left[\dfrac{1}{c}\tan^{-1}\dfrac{l}{c} - \dfrac{l}{l^2+c^2}\right]$, *where c is the distance of source to the boundary.*

[Rohilkhand 2005; Indore 1998; Meerut 1996; Kanpur 2000]

Sol. Let y-axis be the bounding line and let the given source of strength (μ, say) be situated at S where $OS = c$. Now, by the definition of strength μ of the source, we have $2\pi\mu\rho = m\rho$ so that $\mu = m/2\pi$. Now, the image system consists of



(*i*) a source of strength $m/2\pi$ at $S(c, 0)$

(*ii*) a source of strength $m/2\pi$ at $S'(-c, 0)$

Here S' is image of S such that $OS = OS' = c$.

The complex potential w is given by

$$w = -(m/2\pi)\log(z-c) - (m/2\pi)\log(z+c) = -(m/2\pi)\log(z^2 - c^2).$$

The velocity is given by $\qquad \left|\dfrac{dw}{dz}\right| = \left|-\dfrac{m}{2\pi}\cdot\dfrac{2z}{z^2-c^2}\right| = \dfrac{m}{\pi}\left|\dfrac{z}{z^2-c^2}\right|.$

Hence velocity q at any point P (where $z = iy$) is given by

$$q = \frac{m}{\pi}\left|\frac{iy}{-y^2-c^2}\right| = \frac{my}{\pi(y^2+c^2)}. \qquad \ldots(1)$$

Bernoulli's equation for steady motionis given by

$$p/\rho + q^2/2 = \text{constant} = c, \text{ say.} \qquad \ldots(2)$$

Let p_0 be the pressure on y-axis at great distance from O so that $p = p_0$ and $q = 0$ when $y = \infty$. Then (2) reduces to $p_0/\rho = c$ and hence (2) becomes

$$\frac{p}{\rho} + \frac{q^2}{2} = \frac{p_0}{\rho} \qquad \text{or} \qquad \frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

or $$p_0 - p = \frac{\rho}{2} q^2 = \frac{\rho}{2} \cdot \frac{m^2 y^2}{\pi^2 (y^2 + c^2)^2}, \text{ using (1).} \qquad ...(3)$$

Let $AA' = 2l$, where $OA = OA' = l$. Then pressure on the length AA' of the boundary (*i.e.* *y*-axis) $= \int_{-l}^{l} (p_0 - p)\, dy = \frac{m^2 \rho}{2\pi^2} \int_{-l}^{l} \frac{y^2 dy}{(y^2 + c^2)^2} = \frac{m^2 \rho}{\pi^2} \int_0^l \frac{y^2 dy}{(y^2 + c^2)^2}$

[$\because$ The integrand is an even function of *y*]

$$= \frac{m^2 \rho}{\pi^2} \int_0^{\alpha} \frac{c^2 \tan^2 \theta \cdot c \sec^2 \theta \, d\theta}{c^4 \sec^4 \theta}$$

[Putting $y = c \tan \theta$ so that $dy = c \sec^2 \theta \, d\theta$.
Here let $\theta = \alpha$ when $y = l$ so that $l = c \tan \alpha$]

$$= \frac{m^2 \rho}{\pi^2 c} \int_0^{\alpha} \sin^2 \theta \, d\theta = \frac{m^2 \rho}{2\pi^2 c} \int_0^{\alpha} (1 - \cos 2\theta)\, d\theta = \frac{m^2 \rho}{2\pi^2 c} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\alpha} = \frac{m^2 \rho}{2\pi^2 c} [\theta - \sin\theta \cos\theta]_0^{\alpha}$$

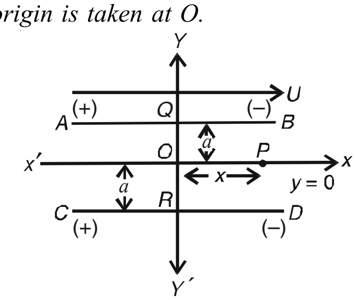
$$= \frac{m^2 \rho}{2\pi^2 c} [\alpha - \sin\alpha \, \cos\alpha] = \frac{m^2 \rho}{2\pi^2 c} \left[\tan^{-1} \frac{l}{c} - \frac{lc}{l^2 + c^2} \right] = \frac{m^2 \rho}{2\pi^2} \left[\frac{1}{c} \tan^{-1} \frac{l}{c} - \frac{l}{l^2 + c^2} \right].$$

$$\left[\because \tan\alpha = l/c \quad \Rightarrow \quad \sin\alpha = l/(l^2 + c^2)^{1/2} \quad \text{and} \quad \cos\alpha = c/(l^2 + c^2)^{1/2} \right]$$

Ex. 7. *The space on one side of an infinite plane wall y = 0, is filled with inviscid, incompressible fluid, moving at infinity with velocity U in the direction of the axis of X. The motion of the fluid is wholly two dimensional, in the (x, y) plane. A doublet of strength* μ *is at a distance a from the wall and points in the negative direction of the axis of X. Show that if* μ *is less than* $4a^2U$*, the pressure of the fluid on the wall is a maximum at points distant* $a\sqrt{3}$ *from O, the foot of the perpendicular from the doublet on the wall, and is minimum at O.*

If μ *is equal to* $4a^2U$*, find the point where the velocity of the fluid is zero, and show that the streamlines include the circle* $x^2 + (y - a)^2 = 4a^2$*, where the origin is taken at O.*

Sol. We know that the complex potential for a doublet of strength μ at $z = z_0$ inclined at an angle α to the *x*-axis is given by $\mu\, e^{i\alpha}/(z - z_0)$.



Here the given doublet *AB* points in the negative direction of *x*-axis and so the given doublet makes an angle π with *OX*. Now the given doublet is situated at *Q* where $z = ia$. Hence the image of the given doublet *AB* will be an equal doublet of strength μ, similarly oriented, and will be situated at *R* where $z = -ia$ (note that here $OR = OQ = a$).

Let *w* be the complex potential for the system, consisting of original doublet *AB*, image doublet *CD* and the stream *U*.

$$\therefore \qquad w = \frac{\mu e^{i\pi}}{z - ia} + \frac{\mu e^{i\pi}}{z + ia} - Uz = -\frac{2\mu z}{z^2 + a^2} - Uz. \qquad ...(1)$$

From (1), $$\frac{dw}{dz} = -2\mu \frac{1 \times (z^2 + a^2) - z \times 2z}{(z^2 + a^2)^2} - U = -\frac{2\mu(a^2 - z^2)}{(z^2 + a^2)^2} - U. \qquad ...(2)$$

Let *P* be any point on the wall *X′X*. Then, at *P*, $z = x$. Hence the velocity *q* on the wall is given by

$$q = \left| \frac{dw}{dz} \right| = \left| -\frac{2\mu(a^2 - x^2)}{(x^2 + a^2)^2} - U \right| = \left| \frac{2\mu(a^2 - x^2)}{(x^2 + a^2)^2} + U \right|. \qquad ...(3)$$

By Bernouilli's theorem, we have

$$p/\rho + q^2/2 = \text{constant} = C, \text{ say.} \quad ...(4)$$

Now, when $z = \infty$, $p = \Pi$ and $q = U$. Then (4) becomes

$$\Pi/\rho + U^2/2 = C. \quad ...(5)$$

Subtracting (4) from (5), $(\Pi - p)/\rho + (U^2 - q^2)/2 = 0$

or $$\Pi - p = (\rho/2) \times (q^2 - U^2)$$

or $$\Pi - p = \frac{\rho}{2}\left[\left\{\frac{2\mu(a^2 - x^2)}{(a^2 + x^2)^2} + U\right\}^2 - U^2\right], \text{ using (3)}$$

or $$\Pi - p = 2\mu\rho\left[\frac{\mu(a^2 - x^2)^2}{(a^2 + x^2)^4} + U\frac{a^2 - x^2}{(a^2 + x^2)^2}\right].$$

or $$p = \Pi - 2\mu^2\rho\frac{(a^2 - x^2)^2}{(a^2 + x^2)^4} - 2\mu\rho U\frac{a^2 - x^2}{(a^2 + x^2)^2}. \quad ...(6)$$

$$\therefore \quad \frac{dp}{dx} = -2\mu^2\rho\left[\frac{4x(a^2 - x^2)}{(a^2 + x^2)^4} + \frac{8x(a^2 - x^2)^2}{(a^2 + x^2)^5}\right] - 2\mu\rho U\left[\frac{2x}{(a^2 + x^2)^2} + \frac{4x(a^2 - x^2)}{(a^2 + x^2)^3}\right]$$

or $$\frac{dp}{dx} = -\frac{4\mu x\rho\{2\mu(a^2 - x^2) + U(a^2 + x^2)^2\}(3a - x^2)}{(a^2 + x^2)^5} \quad ...(7)$$

The pressure will be maximum or minimum if $dp/dx = 0$ *i.e.*, if

$x(3a^2 - x^2) = 0$ or $2\mu(a^2 - x^2) + U(a^2 + x^2)^2 = 0$, by (7)

If $x(3a^2 - x^2) = 0$, then we have $x = 0$, $x = a\sqrt{3}$ and $x = -a\sqrt{3}$.

Now on the wall XX' $(y = 0)$ at $x = a\sqrt{3}$, by (2), we have

$$\frac{dw}{dz} = -U - \frac{2\mu(a^2 - 3a^2)}{(a^2 + 3a^2)^2} = -U + \frac{\mu}{4a^2} = \frac{\mu - 4a^2U}{4a^2}. \quad ...(8)$$

If $\mu < 4a^2 U$, the value of d^2p/dx^2 at $x = a\sqrt{3}$ is negative and hence the pressure of the fluid at the wall is a maximum when $x = a\sqrt{3}$.

Again, d^2p/dx^2 at $x = 0$ is positive and hence the pressure of the fluid at the wall is a minimum when $x = 0$.

Now, if $\mu = 4a^2U$, then $dw/dz = 0$ from (8). So (2) reduces to

$$-8a^2 U(a^2 - z^2)/(a^2 + z^2)^2 - U = 0$$

or $z^4 - 6a^2z^2 + 9a^4 = 0$, so that $(z^2 - 3a^2)^2 = 0$ or $z = \pm a\sqrt{3}$.

Hence the stragnation points are given by $(a\sqrt{3}, 0)$ and $(-a\sqrt{3}, 0)$.

Writing $\mu = 4a^2U$, $w = \phi + i\psi$ and $z = x + iy$, (1) may be re written as

$$\phi + i\psi = -\frac{8a^2U(x + iy)}{(x^2 + a^2 - y^2) + 2ixy} - U(x + iy)$$

or $$\phi + i\psi = -\frac{8a^2U(x + iy)[(x^2 + a^2 - y^2) - 2ixy]}{[(x^2 + a^2 - y^2) + 2ixy][(x^2 + a^2 - y^2) - 2ixy]} - U(x + iy). \quad ...(9)$$

Equating the imaginary parts on both sides of (9), we have

$$\psi = -\left[\frac{8a^2U\{y(x^2 + a^2 - y^2) - 2x^2y\}}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + Uy\right]. \quad ...(10)$$

The streamlines are given by ψ = constant. Taking constant = 0, the streamlines given by $\psi = 0$ are

$$\frac{8a^2Uy[(x^2+a^2-y^2)-2x^2]}{(x^2+a^2-y^2)^2+4x^2y^2}+Uy=0$$

or $$8a^2[a^2-(x^2+y^2)]+[(x^2-y^2)+a^2]^2+4x^2y^2=0$$

or $$8a^4-8a^2(x^2+y^2)+(x^2-y^2)^2+2a^2(x^2-y^2)+a^4+4x^2y^2=0$$

or $$9a^4-8a^2(x^2+y^2)+(x^2+y^2)^2+2a^2(x^2-y^2)=0$$

or $$9a^4+(x^2+y^2)^2-6a^2x^2-6a^2y^2-4a^2y^2=0 \quad \text{or} \quad [(x^2+y^2)-3a^2]^2-4a^2y^2=0$$

or $$(x^2+y^2-3a^2-2ay)(x^2+y^2-3a^2+2ay)=0,$$

which includes the circle $x^2+y^2-3a^2-2ay=0$ or $x^2+(y-a)^2=4a^2$.

EXERCISE 5 (C)

1. Obtain the image of a simple source with respect to a plane (or a straight line). **[Kanpur 2003]**

Hint. Proceeed as in Art 5.18 by replacing line by plane to get the same result.

2. A two-dimensional source of strength m is situated at the point $(a, 0)$, the axis of y being a fixed boundary. Find the points in the boundary at which the fluid velocity is a maximum. Show that the resultant thrust on the part of the axis of y which lies between $y = \pm b$ is

$$2p_0b-2m^2\rho\left[\frac{1}{a}\tan^{-1}\frac{b}{a}-\frac{b}{a^2+b^2}\right],$$ where p_0 is the pressure at infinity.

Particular Case. When $b = a$, the thrust $= 2p_0a-(m^2\rho/a)\times(\pi/2-1)$

3. OX and OY are fixed rigid boundaries and there is a source at (a, b). Find the stream lines and show that the dividing line is $xy\,(x^2-y^2-a^2+b^2)=0$.

4. The irrotational motion in two-dimensions of a fluid bounded by the lines $y = \pm b$ is due to a doublet of strength μ at the origin, the axis of the doublet being in the positive direction of the axis of x. Prove that the motion is given by

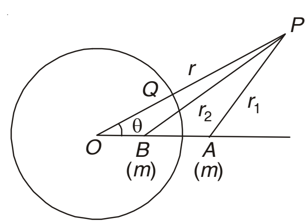
$$\phi+i\psi=\frac{\pi\mu}{2b}\coth\left\{\frac{\pi(x+iy)}{2b}\right\}$$

Show also that the points where the fluid is moving parallel to the axis of y lie on the curve $\coth(\pi x/b) = \sec(\pi y/b)$.

5. In liquid bounded by the axes of x and y in the first quadrant there is a source of strength m at distance a from the origin on the bisector of the angle XOY. Prove that the complex potential is $-m\log(x^4+a^4)$.

5.21. Image of a source with regard to a circle. **[Meerut 2000, 06; Rajasthan 2000, 2010; Agra 2005; Garhwal 1999; G.N.D.U. Amritsar 2000, Kanpur 2002, 2010]**

Let us determine the image of a source of strength m at a point A with respect to the circle with O as centre. Let $OA = f$ and let B be inverse point of A with respect to the circle. If a be the radius of the circle, then $OA\cdot PB = a^2$ so that $OB = a^2/f$. Let $P(z)$ be an arbitrary point in the plane of the circle.

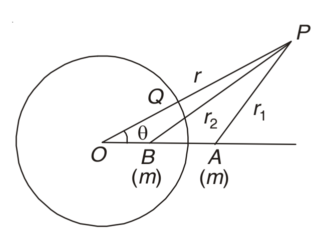


Let there be a source of strength m at B. If w be the complex potential due to sources at A and B, then we get

$$w=-m\log(z-f)-m\log(z-a^2/f)$$
$$=-m[\log(r\cos\theta-r+ir\sin\theta)+\log(r\cos\theta-a^2/f+ir\sin\theta)]$$
$$[\because\ z=re^{i\theta}=r(\cos\theta+i\sin\theta)=r\cos\theta+ir\sin\theta]$$

Image of a source with regard to a circle. **[Meerut 2000, 06; Rajasthan 2000, 2010; Agra 2005; Garhwal 1999; G.N.D.U. Amritsar 2000, Kanpur 2002, 2010]**

Let us determine the image of a source of strength m at a point A with respect to the circle with O as centre. Let $OA = f$ and let B be inverse point of A with respect to the circle. If a be the radius of the circle, then $OA \cdot PB = a^2$ so that $OB = a^2/f$. Let $P(z)$ be an arbitrary point in the plane of the circle.



Let there be a source of strength m at B. If w be the complex potential due to sources at A and B, then we get

$$w = -m \log (z - f) - m \log (z - a^2/f)$$

$$= -m[\log (r\cos\theta - r + ir\sin\theta) + \log (r\cos\theta - a^2/f + ir\sin\theta)]$$

$$[\because\ z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta]$$

Writing $w = \phi + i\psi$ and equating real parts, we get

$$\phi = -(m/2)\times\left[\log\left\{(r\cos\theta - f)^2 + (r\sin\theta)^2\right\} + \log\left\{(r\cos\theta - a^2/f)^2 + (r\sin\theta)^2\right\}\right]$$

$$= -\frac{m}{2}\left[\log(r^2 + f^2 - 2fr\cos\theta) + \log\left(r^2 + \frac{a^4}{f^2} - \frac{2ra^2}{f}\cos\theta\right)\right]$$

$$\therefore \qquad \frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2fr\cos\theta} + \frac{2\{r - (a^2/f)\cos\theta\}}{r^2 + a^4/f^2 - 2r(a^2/f)\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle

$$= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = m\left[\frac{a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \frac{(a/f)(f - a\cos\theta)}{(a^2/f^2)(f^2 + a^2 - 2af\cos\theta)}\right]$$

$$= m\left[\frac{a - f\cos\theta + f^2/a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \right] = \frac{m}{a}.$$

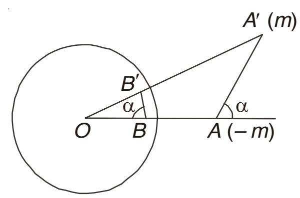
Now, if we place a source of strength $-m$ at O, the normal velocity due to it at Q will be $-(m/a)$ and hence the normal velocity of the system will reduce to zero.

Hence *the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.* **[Kanpur 2002, 08]**

5.22. Image of a doublet with regard to a circle.

[Kanpur 2003, 06; Kurukshetra 2000; Rajasthan 1998]

Let us determine the image of a doublet AA' with its axis making an angle α with OA, outside the circle, there being a sink $-m$ at A and a source m at A′. Join OA and OA'. Let B and B' be the inverse points of A and A' with regard to the circle with O as centre.



Then $\qquad OA\cdot OB = OA'\cdot OB' = a^2, \qquad \ldots(1)$

where a is the radius of the circle.

Now the image of source m at A' consists of a source m at B' and and a sink $-m$ at O. Similarly, the image of sink $-m$ at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink $-m$ at O cancel each other and hence the image of the given doublet AA' is another doublet BB'.

Let the strength of the given doublet AA' be μ.

Then $$\mu = \lim_{A\to A'}(m\cdot AA') \qquad \ldots(2)$$

From (1) $$OA/OA' = OB'/OB, \qquad \ldots(3)$$

showing that triangles OAA' and $OB'B$ are similar. From these similar triangles, we have

$$\frac{BB'}{AA'} = \frac{OB'}{OA} = \frac{OB'}{OA}\cdot\frac{OA'}{OA'} = \frac{a^2}{OA\cdot OA'} \qquad \ldots(4)$$

$\therefore \quad \mu'$ = strength of doublet $B'B$ $= \lim_{B'\to B}(m\cdot B'B) = \lim_{A\to A'}\frac{a^2}{OA\cdot OA'}\cdot(m\cdot AA')$, by (4)

$= \mu a^2/f^2$, using (2) *and taking* $OA = OA' = f$

Thus *the image of a two-dimensional doublet at A with regard to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with the radius OBA.*

5.23A. The Milne-Thomson circle theorem or simply the circle theorem.

Statement : *Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle* $|z| = a$. *Then, on introducing the solid circular cylinder* $|z| = a$ *into the flow, the new complex potential is given by* $w = f(z) + \overline{f}(a^2/z)$ *for* $|z| \geq a$.

[Rohilkhand 2002, 03, 05; Kanpur 2000, 09; Garhwal 2003, 05; Meerut 1998]

Proof. Let C be the cross-section of the circular cylinder $|z| = a$. Then on C, $z\overline{z} = a^2$ or $\overline{z} = a^2/z$. Hence for points on the circle, we have

$$w = f(z) + \overline{f}(a^2/z) = f(z) + \overline{f}(\overline{z}) \quad \text{or} \quad \phi + i\psi = f(z) + \overline{f}(\overline{z}) \qquad \ldots(1)$$

Since the quantity on R.H.S. of (1) is purely real, equating imaginary parts (1) gives $\psi = 0$ on C. Hence C is a streamline in the new flow.

By hypothesis all the singularites of $f(z)$ (at which sources, sinks, doublets or vortices may be present) lie outside the circle $|z| = a$ and so the singularities of $f(a^2/z)$ lie inside the circle $|z| = a$. Hence the singularities of $\overline{f}(a^2/z)$ also lie inside the circle $|z| = a$. Thus we find that the additional term $\overline{f}(a^2/z)$ introduces no new singularities into the flow outside the circle $|z| = a$.

Hence $|z| = a$ is a possible boundary for the new flow and $w = f(z) + \overline{f}(a^2/z)$ is the appropriate complex potential for the new flow.

Remark 1. In the above proof of circle theorem we have used the following important results :

Let $u(t)$ and $v(t)$ be real functions of a real variable t. Let $f(t) = u(t) + iv(t)$ so that $f(t)$ is a complex function of the real variable t. Then conjugate of $f(t)$ is denoted and defined as

$$\overline{f}(t) = u(t) - iv(t).$$

On replacing real variable t by the complex variable z $(= x + iy)$, $f(z)$ and $\overline{f}(z)$ are defined as follows :

$$f(z) = u(z) + iv(z), \qquad \overline{f}(z) = u(z) - iv(z)$$

Again, $$f(\overline{z}) = u(\overline{z}) + iv(\overline{z}), \qquad \overline{f}(\overline{z}) = u(\overline{z}) - iv(\overline{z})$$

On comparing the forms of $f(z)$ and $f(\overline{z})$, we find that, since $z = x + iy$, $\overline{z} = x - iy$, the value of $\overline{f}(\overline{z})$ is obtained from $f(z)$ by replacing i throughout by $-i$. It then follows that $\overline{f}(\overline{z})$ is merely the complex conjugate of $f(z)$ and accordingly, we write $\overline{f}(\overline{z}) = \overline{f(z)}$.

Remark 2. When a circular cylinder is present in the field of sources, sinks, doublets or vortices, the above theorem provides an easy method for determining the image system. Furthermore the theorem can also be used to determine modified flows when a long circular cylinder is introduced into a given two-dimensional flow. Consider the following application of "Circle theorem".

523B. To determine image system for a source outside a circle (or a circular cylinder) of radius *a* with help of the circle theorem.

Refer figure of Art. 5.21. Let $OA = f$. Suppose there is a source of strength m at A where $z = f$, outside the circle of radius a whose centre is at O. When the source is alone in the fluid the complex potential at a point $P(z)$ is given by

$$f(z) = -m \log(z - f) \qquad \text{Then} \qquad \overline{f}(z) = -m \log(z - f)$$

$$\therefore \qquad \overline{f}(a^2/z) = -m \log(a^2/z - f)$$

When the circle of section $|z| = a$ is introduced, then the complex potential in the region $|z| \geq a$ is given by

$$w = f(z) + \overline{f}(a^2/z) = -m \log(z-f) - m \log(a^2/z - f)$$

$$= -m \log(z-f) - m \log\left(\frac{a^2 - zf}{z}\right)$$

$$= -m \log(z-f) - m \log(a^2 - zf) + m \log z$$

$$= -m \log(z-f) - m \log[(-f)(z - a^2/f)] + m \log z$$

$$= -m \log(z-f) - m \log(z - a^2/f) + m \log z - m \log(-f)$$

$$\therefore \qquad w = -\log(z-f) - m \log(z - a^2/f) + m \log z + \text{constant}, \qquad \ldots(1)$$

the constant (real or complex, $-m \log(-f)$) being immaterial from the view point of analysing the flow. (1) shows that w is the complex potential of

(*i*) a source m at A, $z = f$ (*ii*)a source m at B, $z = a^2/f$ (*iii*) a sink $-m$ at the origin

Since $OA \cdot OB = a^2$, A and B are the inverse points with respect to the circle $|z| = a$ and so B is inside the circle.

Thus *the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.*

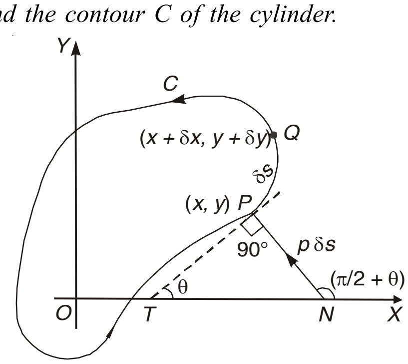
5.24. The Theorem of Blasius. **[Agra 2005, 08, 09, 11; I.A.S. 1985; Kanpur 2000; Meerut 2001, 02, 04, 08, 09, 10, 11, 12; G.N.D.U. Amritsar 2003, 05; Rohilkhand 2003]**

In a steady two-dimensional irrotational motion of an incompressible fluid under no external forced given by the complex potential $w = f(z)$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz, \qquad M = \text{Real part of } \left\{-\frac{1}{2} i\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz\right\},$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

Proof. Figure shows the section C of the cylinder in plane XOY. Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on C such that arc $PQ = \delta s$. If θ be the angle which the tangent PT at P on the contour C makes with x-axis, then



$$\cos\theta = dx/ds, \qquad \sin\theta = dy/ds, \qquad \ldots(1)$$

and the normal at P makes an angle $(\theta + \pi/2)$ with the x-axis. Now, if p denotes the pressure at p, the force on unit length of the section δs is $p\delta s$ normal to C. Then using (1), we have

$$X = \int_C p\cos(\theta + \pi/2)ds = -\int_C p\sin\theta\, ds = -\int_C p\, dy, \text{ using (1)} \qquad \ldots(2)$$

$$Y = \int_C p\sin(\theta + \pi/2)ds = \int_C p\cos\theta\, ds = \int_C p\, dx, \text{ using (1)} \qquad \ldots(3)$$

$$M = \int_C [x \cdot p\sin(\theta + \pi/2)ds - y \cdot p\cos(\theta + \pi/2)ds] = \int_C p(x\cos\theta\, ds + y\sin\theta\, ds)$$

or

$$M = \int_C p(x\,dx + y\,dy), \text{ using (1)} \qquad \ldots(4)$$

Now Bernoulli's equation in this context is

$$\frac{1}{2}q^2 + \frac{p}{\rho} = B \qquad \text{so that} \qquad p = \rho B - \frac{1}{2}\rho q^2, \qquad \ldots(5)$$

where q is the fluid velocity, ρ the density. Since ρ is constant for an incompressible fluid, take $\rho B = A$ (a contant). Again $q^2 = u^2 + v^2$ where u and v are the velocity components. Then (5) reduces to

$$p = A - (\rho/2) \times (u^2 + v^2) \qquad \ldots(6)$$

Also, $\quad dw/dz = -u + iv \quad$ or $\quad -dw/dz = u - iv \qquad \ldots(7)$

Using (6), (2), (3) and (4) reduce to

$$X = -\int_C \left[A - \frac{1}{2}\rho(u^2+v^2)\right]dy = \frac{1}{2}\rho\int_C (u^2+v^2)\,dy \qquad \ldots(8)$$

$$Y = \int_C \left[A - \frac{1}{2}\rho(u^2+v^2)\right]dx = -\frac{1}{2}\rho\int_C (u^2+v^2)\,dx, \qquad \ldots(9)$$

and $$M = \int_C \left[A - \frac{1}{2}\rho(u^2+v^2)\right](x\,dx + y\,dy) = -\frac{1}{2}\rho\int_C (u^2+v^2)\,(xdx+ydy) \qquad \ldots(10)$$

While simplifying (8), (9) and (10), we have to use the following results

$$\int_C dy = \int_C dx = \int_C x\,dx = \int_C x\,dy = 0$$

which hold good because C is a closed contour.

Now the contour of the cylinder is a streamline. Hence we have $\quad dx/u = dy/v$.

Now, $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dx+idy}{u+iv} = \dfrac{dx-idy}{u-iv}$ or $\dfrac{dx-idy}{dx+idy} = \dfrac{u-iv}{u+iv} = \dfrac{(u-iv)^2}{(u+iv)(u-iv)} = \dfrac{(u-iv)^2}{u^2+v^2}$

$$\therefore \qquad (u-iv)^2\,(dx+idy) = (u^2+v^2)\,(dx-idy) \qquad \ldots(11)$$

From (8) and (9), we have

$$X - iY = \frac{1}{2}\rho\int_C (u^2+v^2)\,(dy+idx) = \frac{1}{2}\rho i\int_C (u^2+v^2)\left(dx + \frac{1}{i}dy\right)$$

$$= \frac{1}{2}\rho i\int_C (u^2+v^2)(dx-idy) = \frac{1}{2}\rho i\int_C (u-iv)^2(dx+idy), \text{ by (11)}$$

$$= \frac{1}{2}\rho i\int_C \left(\frac{dw}{dz}\right)^2 dz, \text{ using (7) and the fact } z = x+iy \Rightarrow dz = dx+idy;$$

Re-writing (10), we have

$$M = \text{Reat part of } -\frac{1}{2}\rho\int_C (x+iy)\,(dx-idy)\,(u^2+v^2)$$

$$= \text{Real part of } -\frac{1}{2}\rho\int_C (x+iy)\,(u-iv)^2\,(dx+idy), \text{ using (11)}$$

$$= \text{Real part of } \left\{-\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz\right\}, \text{ using (7)}$$

Remark 1. The above integrals are to be taken over the contour of the cylinder. If however, we take a large contour surrounding the cylinder such that between this contour and the cylinder there is no singularity of the integrand, then we can take the integrals round such large contours. The singularities of the integrand occur at sources, sinks, doublets etc.

Remark 2. In what follows, we shall often use the following important definitions and results of functions of complex variables.

A point at which a function $f(z)$ ceases to be analytic is known as a *singular point* or *singularity* of the function. If in the neighbourhood of the point $z = a$, $f(z)$ can be expanded in positive and negative powers of $(z - a)$, say

$$f(z) = \dots + A_2 (z-a)^2 + A_1 (z-a) + A_0 + \frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \dots$$

then the point $z = a$ is a *singular point* of $f(z)$. If only a finite number of terms contain negative powers of $z - a$, the point $z = a$ is called a *pole*. In this case the coefficient of $1/(z-a)$ is called the *residue* of the function at $z = a$.

Cauchy's Residue theorem. If $f(z)$ is analytic, except at a finite number of poles within a closed contour C and continuous on the boundary C, then

$$\int_C f(z)dz = 2\pi i \times [\text{sum of the residues of } f(z) \text{ at its poles within } C]$$

5.25. Illustrative solved examples.

Ex. 1(*a*). *In the region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle* α *with the radius is* $r^2 \sin(\alpha+\theta) = a^2 \sin(\alpha - \theta)$.

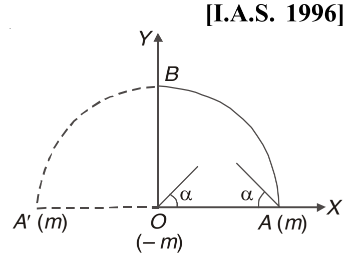
[Kanpur 2011; P.C.S. (U.P.) 2000; Rohilkhand 2003; I.A.S. 1986; Meerut 2002, 07]

(*b*) *In a region bounded by a fixed quadrant arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle* $\pi/6$ *with radius is* $r^2 \sin(\pi/6+\theta) = a^2 \sin(\pi/6-\theta)$, where a is radius of the quadrant. **[I.A.S. 1996]**

Sol. (*a*). Let AOB be the circular quadrant of radius a with OA and OB as bounding radii. Consider a source of strength m at A and a sink of strength $-m$ at O. Then the image system consists of (*i*) a source m at A $(a, 0)$

(*ii*) a source m at A' $(-a, 0)$

(*iii*) a sink $-m$ at O $(0, 0)$.



Hence the complex potential w for the motion of the fluid at any point $P(z+x+iy=re^{i\theta})$ is given by

$$w = -m\log(z-a) - m\log(z+a) + m\log z = -m\log\frac{z^2-a^2}{z} = -m\log(z - a^2 z^{-1})$$

or $$w = -m\log(re^{i\theta} - a^2 r^{-1} e^{-i\theta}), \qquad \text{as} \qquad z = re^{i\theta}$$

or $$w = -m\log[r(\cos\theta + i\sin\theta) - a^2 r^{-1}(\cos\theta - i\sin\theta)]$$

or $$\phi + i\psi = -m\log[(r - a^2/r)\cos\theta + i(r + a^2/r)\sin\theta]$$

Equating imaginary parts, we obtain

$$\psi = -m\tan^{-1}\frac{(r+a^2/r)\sin\theta}{(r-a^2/r)\cos\theta} = -m\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\}$$

The streamline leaving the end A and O at an angle α is given by

$$\psi = -m(\pi - \alpha) \qquad i.e., \qquad -m\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\} = -m(\pi-\alpha)$$

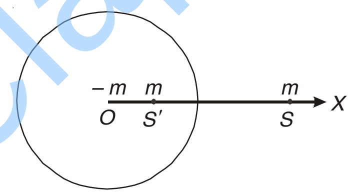
or $$\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta} = \tan(\pi-\alpha) = -\tan\alpha = -\frac{\sin\alpha}{\cos\alpha}$$

or $$(r^2+a^2)\sin\theta\cos\alpha = -(r^2-a^2)\cos\theta\sin\alpha \qquad \text{or} \qquad r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta).$$

(*b*) Hint. Proceed as in part (*a*) by taking $\alpha = \pi/6$.

Ex. 2. *In the case of the two-dimensional fluid motion produced by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS meet the circle, prove that its magnitude at these points is* $(2m \times OS)/(OS^2 - a^2)$

Sol. Let S' be the inverse point of S with respect to the circular disc, with O as its centre. Let $OS = c$. Then $OS \times OS' = a^2$ so that $OS' = a^2/c$.



The equivlent image system consists of

(*i*) a source of strength m at $S\,(c, 0)$,

(*ii*) a source of strength m at $S'\,(a^2/c, 0)$,

(*iii*) a sink of strength $-m$ at $O\,(0, 0)$.

Let OS be taken as x-axis. Then the complex potential for the motion of the fluid at any point $z(=x+iy=re^{i\theta})$ is given by

$$w = -m\log(z-c) - m\log(z-a^2/c) + m\log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-a^2/c} + \frac{m}{z}$$

Let $q\ (= |dw/dz|)$ be the velocity at any point z. Then

$$q = m\left|\frac{1}{z-c} + \frac{1}{z-a^2/c} - \frac{1}{z}\right| = m\left|\frac{(z-a)(z+a)}{z(z-c)(z-a^2/c)}\right|$$

Hence the velocity at any point $z = ae^{i\theta}$ on the boundary of the circular disc is given by

$$q = m\left|\frac{(ae^{i\theta}-a)(ae^{i\theta}+a)}{ae^{i\theta}(ae^{i\theta}-c)(ae^{i\theta}-a^2/c)}\right| = m\left|\frac{c(e^{i\theta}-1)(e^{i\theta}+1)}{e^{i\theta}(ae^{i\theta}-c)(ce^{i\theta}-a)}\right|$$

or $$q = mc\left|\frac{(1-e^{-i\theta})(1+e^{i\theta})}{(ae^{i\theta}-c)(ce^{i\theta}-a)}\right| = \frac{2mc\sin\theta}{a^2+c^2-2ac\cos\theta} \qquad \ldots(1)$$

For maximum q, $dq/d\theta = 0$. Hence (1) gives

$$2mc\frac{(a^2+c^2-2ac\cos\theta)\cos\theta - \sin\theta(2ac\sin\theta)}{(a^2+c^2-2ac\cos\theta)^2} = 0$$

or $$(a^2+c^2)\cos\theta - 2ac = 0 \qquad \text{or} \qquad \cos\theta = (2ac)/(a^2+c^2) \qquad \ldots(2)$$

Since $\theta = 0$ gives the minimum velocity [q becomes zero at $\theta = 0$ by (1)], the value of θ given by (2) must correspond to the maximum value of velocity q. Moreover (2) gives the same angles which the diameter through the point where the line joining S to the end of the diameter at right angle to OS cuts the circle, will make with OS.

From (2), $$\sin\theta = \sqrt{1-\cos^2\theta} = (c^2 - a^2)/(c^2 + a^2) \qquad \ldots(3)$$

Using (1), (2) and (3), the maximum value of q is given by

$$q = \frac{2mc \cdot \left(\dfrac{c^2 - a^2}{c^2 + a^2}\right)}{a^2 + c^2 - \dfrac{4a^2c^2}{a^2 + c^2}} = \frac{2mc\,(c^2 - a^2)}{(a^2 + c^2)^2 - 4a^2c^2} \qquad \text{or} \qquad q = \frac{2mc}{c^2 - a^2} = \frac{2m \cdot OS}{OS^2 - a^2}$$

Since the boundary of the circular disc is a streamline, the velocity on the boundary is the velocity of the slip.

Ex. 3. *A source S and a sink T of equal strengths m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, show that the velocity of the liquid at any point P is*

$$2m \frac{OS^2 + OA^2}{OS} \frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'},$$

where S′ and T′ are the inverses of S and T with respect to the circle. **[Rohilkhand 2001]**

Sol. Let $OS = OT = c$. Then, we have $OA = a$, $OS \cdot OS' = a^2$ and $OT \cdot OT' = a^2$ so that

$$OS' = a^2/c \qquad \text{and} \qquad OT' = a^2/c. \qquad \ldots(1)$$

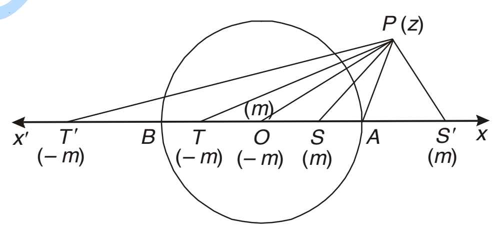
Now the image system of source m at S consists of a source m at S' and a sink $-m$ at O. Again the image system of sink $-m$ at T consists of a sink $-m$ at T' and a source m at O. Compounding these, we find that source m and sink $-m$ at O cancel each other. Hence the equivalent image system finally consists of

(*i*) a source of strength m at $S\,(c, 0)$

(*ii*) a source of strength m at $S'\,(a^2/c, 0)$

(*iii*) a sink of strength $-m$ at $T\,(-c, 0)$

(*iv*) a sink of strength $-m$ at $T'\,(-a^2/c, 0)$



Taking OS as the x-axis, the complex potential at any point $z\ (= x + iy)$ is given by

$$w = -m\log(z-c) - m\log\left(z - \frac{a^2}{c}\right) + m\log(z+c) + m\log\left(z + \frac{a^2}{c}\right)$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z - a^2/c} + \frac{m}{z+c} + \frac{m}{z + a^2/c}$$

The velocity $q\ (= |\,dw/dz\,|)$ at any point is given by

$$q = m\left|-\frac{2c}{z^2 - c^2} - \frac{(2a^2/c)}{z^2 - (a^4/c^2)}\right| = 2m\left|\frac{c(z^2 - a^2) + (a^2/c)(z^2 - a^2)}{(z^2 - c^2)(z^2 - a^4/c^2)}\right|$$

$$= 2m\frac{c^2 + a^2}{c}\left|\frac{z^2 - a^2}{(z^2 - c^2)\left(z^2 - a^4/c^2\right)}\right| = 2m\frac{c^2 + a^2}{c}\,\frac{|z-a|\,|z+a|}{|z-c|\,|z+c|\,|z - \frac{a^2}{c}|\,|z + \frac{a^2}{c}|}$$

$$= 2m\frac{OS^2 + OA^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'}$$

Ex. 4. *In the part of an infinite plane bounded by a circular quadrant AB and the production of the radii OA, OB, there is a two-dimensional motion due to the production of the liquid at A and its absorption at B, at the uniform rate m. Find the velocity potential of the motion and show that the fluid which issues from A in the direction making an angle μ with OA follows the path whose polar equation is*

$$r = a\sqrt{\sin 2\theta}\,[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2},$$

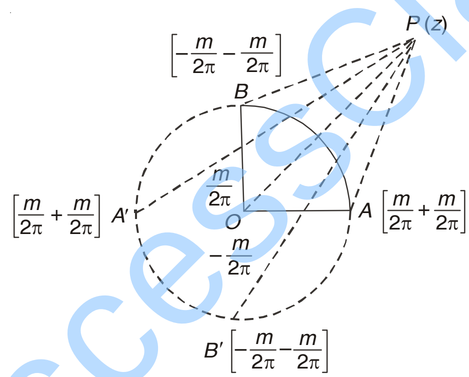
the positive sign being taken for all square roots.

Sol. The image system of source $m/2\pi$ at A with respect to the circular boundary consists of a source $m/2\pi$ at A (since A is the inverse point of itself) and a sink $-m/2\pi$ at O, the centre of the circle. Next, the image of system of the above mentioned image system with respect to the line OA and OB consists of

(*i*) a source of strength $m/2\pi + m/2\pi$ *i.e.* m/π at A $(a, 0)$

(*ii*) a source of strength $m/2\pi + m/2\pi$ *i.e.* m/π at A' $(-a, 0)$

(*iii*) a sink of strength $-\dfrac{m}{2\pi}$ at O $(0, 0)$



Again there is a sink of strength $-m/2\pi$ at B. The image system of this sink with respect to the circular boundary consists of a sink $-m/2\pi$ at B (since B is the inverse point of itself) and a source $m/2\pi$ and O. Again the image of the system of the above mentioned image system with respect to lines OA and OB as before consists of

(*i*) a sink of strength $-(m/2\pi)-(m/2\pi)$ *i.e.* $-(m/\pi)$ at B $(0, a)$

(*ii*) a sink of strength $-(m/2\pi)-(m/2\pi)$ *i.e.* $-(m/\pi)$ at B' $(0, -a)$

(*iii*) a source of strength $m/2\pi$ at O $(0, 0)$

Compounding these we find that source $m/2\pi$ and sink $-m/2\pi$ at O cancel each other. Taking OA as the x-axis, the complex potential at any point $P\,(z = x + iy = re^{i\theta})$ is given by

$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ai) + \frac{m}{\pi}\log(z+ai)$$

$$\therefore \qquad \phi + i\psi = -\frac{m}{\pi}\log(z^2-a^2) + \frac{m}{\pi}\log(z^2+a^2) \qquad \ldots(1)$$

Equating real parts, (1) gives

$$\phi = -\frac{m}{\pi}\log\left|z^2-a^2\right| + \frac{m}{\pi}\log\left|z^2-i^2a^2\right| = -\frac{m}{\pi}\{|z-a|\cdot|z+a|\} + \frac{m}{\pi}\log\{|z-ia||z+ia|\}$$

or $$\phi = -\frac{m}{\pi}\log(AP\cdot A'P) + \frac{m}{\pi}\log(BP\cdot B'P) = \frac{m}{\pi}\log\frac{BP\cdot B'P}{AP\cdot A'P}$$

Putting $z = e^{i\theta}$ in (1) and equating imaginary parts, we get

$$\psi = -\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} + \frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}$$

$$= -\frac{m}{\pi}\tan^{-1}\frac{\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} - \dfrac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}}{1 + \dfrac{r^4\sin^2 2\theta}{r^4\cos^2 2\theta - a^4}} = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4 - a^4}$$

The required streamline that leaves A at an inclination μ is given by $\psi = -(m/\pi)\mu$, *i.e.*,

$$-\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4 - a^4} \qquad \text{or} \qquad r^4 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0$$

$$\therefore \qquad r^2 = [2a^2\sin 2\theta\cot\mu + \sqrt{(4a^4\sin^2 2\theta\cot^2\mu + 4a^4)}]/2$$

wherein negative sign has been omitted because r^2 is non-negative quantity. Thus, we have

$$r = a\sqrt{\sin 2\theta}\,[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2}.$$

Ex. 5. *Prove that in the two-dimensional liquid motion due to any number of sources at points on a circle, the circle is a streamline provided that there is no boundary and that the algebraic sum of the strengths of sources is zero. Show that the same is true if the region of flow is bounded by a circle which cuts orthogonally the circle in question.*

[Kanpur 1997, 2000; Rohilkhand 2000]

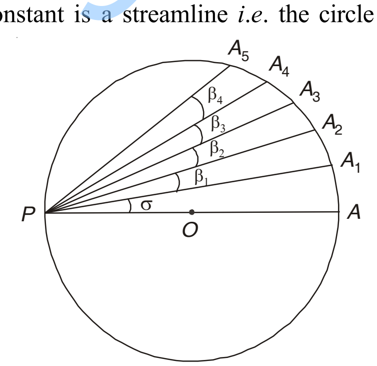
Sol. Let $A_1, A_2, A_3, \ldots$ be the positions of the sources of strengths $m_1, m_2, m_3, \ldots$ respectivly. Let P be any point on the circle and let the diameter through P be taken as the initial line.

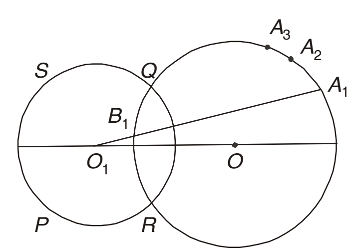
Let $\angle A_1PA = \delta$, $\angle A_2PA_1 = \beta_1$, $\angle A_3PA_2 = \beta_2$ and so on. Then the stream function ψ of the system is given by

$$\psi = -m_1\delta - m_2(\delta + \beta_1) - m_3(\delta + \beta_1 + \beta_2) - \ldots$$

$$= -\delta(m_1 + m_2 + m_3 + \ldots) - [m_2\beta_1 + m_3(\beta_1 + \beta_2) + \ldots] = -\delta\,(m_1 + m_2 + m_3 + \ldots) - \text{constant},$$

since $\beta_1, \beta_2, \beta_3, \ldots$ do not depend on the position of P. If we take $m_1 + m_2 + m_3 + \ldots = 0$, then ψ = constant is a streamline *i.e.* the circle is a stream line.





Second Part. Let O_1 be the centre of a circle which cuts the above circle (with centre O) orthogonally. The image of m_1 at A is m_1 at B_1, the inverse point of A and a sink $-m_1$ at O_1. If the barriers are omitted, we see that the system reduces to a source 2 $(m_1 + m_2 + \ldots)$ on the boundary of the given circle and a sink $-(m_1 + m_2 + \ldots)$ at O_1. Since $m_1 + m_2 + \ldots = 0$, the result follows.

Ex. 6. *A line source is in the presence of an infinite plane on which is placed a semi-circular cylindrical boss, the direction of the source is parallel to the axis of the boss, the source is at a disdtance c from the plane and the axis of the boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle* θ *with the radius to the source, where*

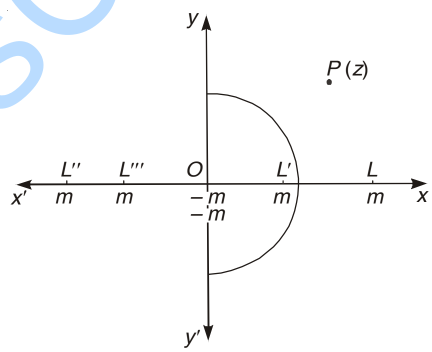
$$\theta = \cos^{-1}\frac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}$$ **[Agra 1999, 2000]**

OR *If the axis of y and the circle* $x^2 + y^2 = a^2$ *are fixed boundaries and there is a two-dimensional source at the point (c, 0) where c > a, show that the radius drawn from the origin to the point on the circle, where the velocity is a maximum, makes with the axis of x an angle*

$$\cos^{-1}\frac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}.$$

When c = 2a, show that the required angle is $\cos^{-1}(5/\sqrt{34})$.

Sol. Let there be a source of strength m at L $(c, 0)$. Let L' be the inverse point of L with respect to the circular boundary so that $OL \times OL' = a^2$ *i.e.* $OL' = a^2/c$. The image of source m at L in the circular boundary (cylindrical boundary) is a source m at L' and a sink $-m$ at O.



For the above system the equivalent image system with respect to the y-axis (*i.e.* the line $x = 0$) consists of

(*i*) a source m at L $(c, 0)$ and $L''(-c, 0)$

(*ii*) a source m at L' $(a^2/c, 0)$ and $L'''(-a^2/c, 0)$

(*iii*) a sink $-m - m$ *i.e.* $-2m$ at O $(0, 0)$

Thus, if P $(z = x + iy = re^{i\theta})$ is any point in the fluid, the complex potential at P due to the above system is given by

$$w = -m \log(z - c) - m \log(z + c) - m \log(z - a^2/c) - m \log(z + a^2/c) + 2m \log z$$

or $$w = 2m \log z - m \log(z^2 - c^2) - m \log(z^2 - a^4/c^2)$$

$$\therefore \quad \frac{dw}{dz} = \frac{2m}{z} - \frac{2mz}{z^2-c^2} - \frac{2mz}{z^2-a^4/c^2} \qquad \text{or} \qquad \frac{dw}{dz} = -\frac{2m(z^4-a^4)}{z(z^2-c^2)(z^2-a^4/c^2)}$$

The velocity $q\,(=|dw/dz|)$ at any point $P(z = ae)^{i\theta}$ on the circular boundary is given by

$$q = \frac{2m|a^4e^{4i\theta}-1|}{|ae^{i\theta}(a^2e^{2i\theta}-c^2)(a^2e^{2i\theta}-a^4/c^2)|} \qquad \text{or} \qquad q = \frac{4mac^2\sin 2\theta}{a^4+c^4-2a^2c^2\cos 2\theta}$$

or $$(4mac^2/q) = (a^4 + c^4 - 2a^2c^2\cos 2\theta)/\sin 2\theta \qquad \ldots(1)$$

Let $f = 4mac^{2\prime}/q$. When q is maximum, then f will be minimum. From (1), we have

$$f = (a^4 + c^4)\operatorname{cosec} 2\theta - 2a^2c^2 \cot 2\theta \qquad \ldots(2)$$

$\therefore \qquad df/d\theta = -2(a^4 + c^4)\operatorname{cosec} 2\theta \cot 2\theta + 4a^2c^2 \operatorname{cosec}^2 2\theta \qquad \ldots(3)$

$$d^2 f/d\theta^2 = 4(a^4 + c^4)\operatorname{cosec} 2\theta(\operatorname{cosec}^2 2\theta + \cot^2 2\theta) - 8a^2c^2 \operatorname{cosec}^2 2\theta \cot 2\theta$$

$$= 4\operatorname{cosec} 2\theta[(a^2 \operatorname{cosec} 2\theta - c^2 \cot 2\theta)^2 + a^4 \cot^2 2\theta + c^4 \operatorname{cosec}^2 2\theta]$$

Since $\theta \leq \pi/2$, clearly $d^2 f/d\theta^2$ is positive and hence f will be minimum and consequently q will be maximum. From (3), setting $df/d\theta = 0$, we get

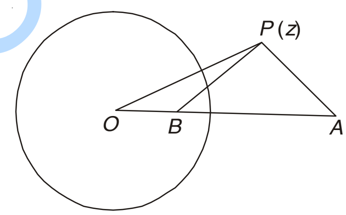
$$(a^4 + c^4)\operatorname{cosec} 2\theta \cot 2\theta = 4a^2c^2 \operatorname{cosec}^2 2\theta \qquad \text{or} \qquad \cos 2\theta = 2a^2c^2/(a^4 + c^4)$$

$\therefore \qquad 2\cos^2\theta - 1 = 2a^2c^2/(a^4 + c^4), \qquad \text{or} \qquad \cos^2\theta = (a^2 + c^2)^2/2(a^4 + c^4)$

so that $\qquad \cos\theta = (a^2 + c^2)/\sqrt{\{2(a^4 + c^4)\}}.$

Ex. 7. *A source of fluid situated in space of two dimensions, is of such strength that* $2\pi\rho\mu$ *represents the mass of fluid of density* ρ *emitted pet unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is* $2\pi\rho\mu^2 a^2/r(r^2 - a^2)$, *where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ?* **[Kanpur 2005, 06; Meerut 2005, 11; Rohilkhand 2002]**

Sol. Since the mass of fluid emitted is $2\pi\rho\mu$ per unit of time, by definition the strength of the given source is μ. Let this source be situated at A such that $OA = r$ and let B be the inverse point of A. Then, $OA \cdot OB = a^2$ so that $OB = a^2/r$. Here the equivalent image system consists of (taking OA as x-axis and using Art. 5.21)



(*i*) a source of strength μ at A $(r, 0)$

(*ii*) a source of strength μ at B $(a^2/r, 0)$

(*iii*) a sink of strength μ at O $(0, 0)$

Hence the complex potentioal at any point P $(z = x + iy)$ is given by

$$w = -\mu \log(z - r) - \mu \log(z - a^2/r) + \mu \log z$$

$\therefore \qquad \dfrac{dw}{dz} = -\dfrac{\mu}{z - r} - \dfrac{\mu}{z - a^2/r} + \dfrac{\mu}{z} \qquad \ldots(1)$

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots(2)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}], \qquad \ldots(3)$$

wherein the indicated sum of the residues is calcualated at poles of $(dw/dz)^2$ lying within the circular boundary. Using (3), (2) reduces to

$$X - iY = -\pi\rho \times [\text{sum of the residues}] \qquad \ldots(4)$$

We proceed to find the residues of $(dw/dz)^2$. From (1), we have

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z - a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} - \frac{2}{z(z - a^2/r)} + \frac{2}{(z-r)(z - a^2/r)}\right]$$

$$= \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} + \frac{2}{rz} - \frac{2}{(a^2/r)(z-a^2/r)} \right.$$

$$\left. + \frac{2}{(a^2/r)z} + \frac{2}{(r-a^2/r)(z-r)} + \frac{2}{(a^2/r-r)(z-a^2/r)} \right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fractions]

From (5), we find that the poles inside the circular contour C are $z = 0$ and $z = a^2/r$.

$\therefore$ The required sum of the residues

= the sum of the coffecients of z^{-1} and $(z - a^2/r)^{-1}$ in R.H.S. of (5)

$$= \frac{2\mu^2}{r} + \frac{2\mu^2}{a^2/r} - \frac{2\mu^2}{a^2/r} + \frac{2\mu^2}{a^2/r-r} = \frac{2\mu^2 a^2}{r(a^2-r^2)} \qquad \ldots(6)$$

Using (6) in (4) and then equating real and imaginary parts, we have

$$X = 2\pi\rho\mu^2 a^2 / r(r^2 - a^2) \qquad \text{and} \qquad Y = 0.$$

Thus the disc is attracted towards the source along OA. Hence the disc will be urged to move along OA.

Ex. 8. *Within a circular boundary of radius a there is a two-dimensional liquid motion due to source producing liquid at the rate m, at a distance f from the centre, and an equal sink at the centre. Find the velocity potential and show that the resultant pressure on the boundary is* $\rho m^2 f^3 / 2a^2 (a^2 - f^2)$, *where* ρ *is the density. Deduce as a limit velocity potential due to a doublet at the centre.* **[Rohilkhand 2000; Agra 2004; Kanpur 1997; Meerut 1999, 2005]**

Sol. Since the rate of production of liquid is m, by definition the strength of the given source is $m/2\pi$. Let this sources be situated at B such that $OB = f$ (refer figure of Ex. 7). Let A be the inverse point of B. Then $OA \cdot OB = a^2$ so that $OA = a^2/f$.

Taking OA as x-axis, the equivalent image system consists of

(*i*) a source of strength $m/2\pi$ at B $(f, 0)$ (*ii*) a source of length $m/2\pi$ at A $(a^2/f, 0)$

(*iii*) a sink of strength $-\, m/2\pi$ at O $(0, 0)$

Hence the complex potential w at any point $P\,(z = x + iy)$ is

$$w = -(m/2\pi)\log(z-f) - (m/2\pi)\log(z - a^2/f) + (m/2\pi)\log z \qquad \ldots(1)$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{2\pi}\left[\frac{1}{z-f} + \frac{1}{z-a^2/f} - \frac{1}{z}\right]$$

$$\therefore \left(\frac{dw}{dz}\right)^2 = \frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(z-f)(z-a^2/f)} - \frac{2}{z(z-a^2/f)} - \frac{2}{z(z-f)}\right]$$

$$= \frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(f-a^2/f)(z-f)} + \frac{2}{(a^2/f-f)(z-a^2/f)} \right.$$

$$\left. + \frac{2}{za^2/f} - \frac{2}{(a^2/f)(z-a^2/f)} - \frac{2}{f(z-f)} + \frac{2}{fz}\right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fraction]

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots(6)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}] \qquad \ldots(7)$$

Using (7), (6) reduces to $\qquad X - iY = \pi\rho \times [\text{sum of the residues}] \qquad \ldots(8)$

From (5), we find that poles inside the circular contour C are at $z = 0$ and $z = f$.

$\therefore$ The required sum of the residues

= the sum of the coefficients of z^{-1} and $(z - f)^{-1}$ in R.H.S. of (5)

$$= \frac{m^2}{4\pi^2}\left[\frac{2}{(f - a^2/f)} + \frac{2}{a^2/f} - \frac{2}{f} + \frac{2}{f}\right] = \frac{m^2 f^3}{2a^2\pi^2(f^2 - a^2)}$$

Using this in (8) and equating real and imaginary parts, we have

$$X = \rho m^2 f^3 / 2a^2 (a^2 - f^2) \qquad \text{and} \qquad Y = 0, \qquad \ldots(9)$$

giving the required pressure.

We now obtain the velocity potential. Note that real part of $\log z = \log |z|$. Writing $w = \phi + i\psi$ (1) and equating real parts, we have

$$\phi = -\frac{m}{2\pi}[\log|z - f| + \log|z + f| - \log|z|]$$

$$= -\frac{m}{2\pi}[\log PB + \log PA - \log PO] = -\frac{m}{2\pi}\log\frac{PB \cdot PA}{PO}.$$

Second part. In order to obtain the doublet at the centre, make $f \to 0$, $(m/2\pi)f \to \mu$ so that $a^2/f \to \infty$. Then, (1) reduces to

$$w = -\frac{m}{2\pi}\left[\log\left(1 - \frac{f}{z}\right) + \log\left(1 - \frac{fz}{a^2}\right)\right],$$

where we have rejected the constant term. Using the expansions of log $(1 \pm x)$ and rejecting powers of f higher than the first, we get

$$w = \frac{m}{2\pi}\left(\frac{f}{z} + \frac{fz}{a^2}\right) \qquad \text{or} \qquad \phi + i\psi = \frac{\mu}{z} + \frac{\mu z}{a^2} \qquad \left[\because \frac{mf}{2\pi} = \mu\right]$$

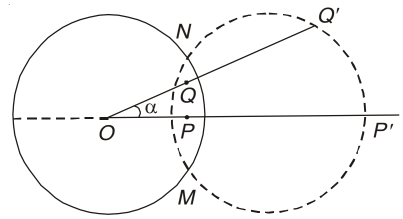
or $$\phi + i\psi = \frac{\mu}{r}e^{-i\theta} + \frac{\mu r}{a^2}e^{i\theta}, \qquad \text{as} \qquad z = re^{i\theta}$$

Equating real parts, the velocity potential due to doublet at centre is given by

$$\phi = \mu(r^{-1} + a^{-2}r)\cos\theta$$

Ex. 9. *Find the velocity potential when there is a source and an equal sink within a circular cavity and show that one of the stream lines is an arc of the circle which passes through the source and the sink and cuts orthogonally the boundary of the cavity.*

Sol. Let the source m and the sink $-m$ be situated at the P and Q within the circular cavity with centre at O (0, 0). Let P' and Q' be the inverse points of P and Q respectively. Now, to do away with the circular cavity, we proceed to get an equivalent system. The image system of source m at P consists of a source m at P' and a sink $-m$ at O. Similarly, the image system of sink $-m$ at Q consists of a sink $-m$ at Q' and a source m at O. The source and sink at O cancel each other and then the resulting equivalent system consists of



(i) a source of strength m at P $(z = c)$

(ii) a source of strength m at P' $(z = a^2/c)$

(iii) a sink of strength $-m$ at $Q(z = be^{i\alpha})$

(iv) a sink of strength $-m$ at $Q'[z = (a^2/b)e^{i\alpha}]$

wherein $OP = c$, $OQ = b$, a = radius of circle with centre O, $OP \cdot OP' = a^2$ so that $OP' = a^2/c$, $OQ \cdot OQ' = a^2$ so that $OQ' = a^2/b$ and $\angle QOP = \alpha$.

Hence the complex potential at any point $P(z)$ is

$$w = -m\log(z-c) - m\log(z - a^2/c) + m\log(z - be^{i\alpha}) + m\log[z - (a^2/b)e^{i\alpha}]$$

$$\phi + i\psi = m\log\frac{(z - be^{i\alpha})[z - (a^2/b)e^{i\alpha}]}{(z-c)(z - a^2/c)} \qquad \ldots(1)$$

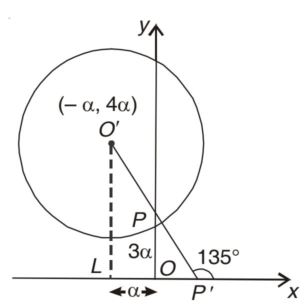
The desired velocity potential ϕ and stream function ψ may be obtained by equating real and imaginary parts in (1).

Since $OP \cdot OP' = OQ \cdot OQ' = a^2$, the points P, Q, P', Q' are concyclic. Let the circle passing through P, Q, P', Q' cut given circle (with centre O) at N and M. Since $OP \cdot OP' = a^2 = ON^2$, ON must be tangent to the circle through N. Hence the two circles cut orthogonally. Again the circle $PQMN$ passes through P and Q ($i.e.$ source and sink) and so it is a streamline.

Ex. 10. *With a rigid boundary in the form of the circle* $(x+\alpha)^2 + (y-4\alpha)^2 = 8\alpha^2$, *there is a liquid motion due to a doublet of strength* μ *at the point* $(0, 3\alpha)$ *with its axis along the axis of y. Show that the velocity potential is*

$$\mu\left\{\frac{4(x-3\alpha)}{(x-3\alpha)^2 + y^2} + \frac{y-3\alpha}{x^2 + (y-3\alpha)^2}\right\}. \qquad \textbf{[Meerut 2008]}$$

Sol. The given circle has centre $O'(-\alpha, 4\alpha)$ and radius $= \sqrt{(8\alpha^2)} = 2\sqrt{2}\alpha$. Let the given doublet be at P $(0, 3\alpha)$.



Gradient of $O'P = \dfrac{3\alpha - 4\alpha}{0 - (-\alpha)} = -1 = \tan\dfrac{3\pi}{4}$.

Hence $O'P$ makes an angle $\pi/4$ with OY. Let P' be the image of P. Then the axis of doublet at P' will make an angle of 45°with PP' and hence it will be parallel to x-axis.

We now show that P' lies on x-axis. We have

$$O'P \cdot O'P' = 8\alpha^2 \quad \text{or} \quad O'P(O'P + PP') = 8\alpha^2 \qquad \ldots(1)$$

But $$O'P = \sqrt{[(-\alpha - 0)^2 + (4\alpha - 3\alpha)^2]} = \alpha\sqrt{2} \qquad \ldots(2)$$

$\therefore$ From (1), $\alpha\sqrt{2}(\alpha\sqrt{2} + PP') = 8\alpha^2$ or $PP' = 3\alpha\sqrt{2} = 3\alpha\sec 45° = OP\sec 45°$

This shows that P' lies on the x-axis and that co-ordinates of P' are $(3\alpha, 0)$. Let μ be the strength of P.

Then, the strength of doublet at $P' = \mu \cdot \dfrac{(\text{radius})^2}{(O'P)^2} = \mu \dfrac{8\alpha^2}{2\alpha^2} = 4\mu \cdot$

Thus the equivalent image system consists of doublets at P and P'. Hence the complex potential of motion at point z $(= x + iy)$ is given by

$$w = \frac{\mu e^{\pi i/2}}{z - 3i\alpha} + \frac{4\mu e^{0.i}}{z - 3\alpha}, \qquad \textit{where} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

or $$\phi + i\psi = \mu\left[\frac{4}{x + iy - 3\alpha} + \frac{i}{x + iy - 3i\alpha}\right] = \mu\left[4\frac{(x - 3\alpha) - iy}{(x - 3\alpha)^2 + y^2} + \frac{i\{x - i(y - 3\alpha)\}}{x^2 + (y - 3\alpha)^2}\right] \quad \text{... (3)}$$

Equating real parts, (3) gives $$\phi = \mu\left[\frac{4(x - 3\alpha)}{(x - 3\alpha)^2 + y^2} + \frac{y - 3\alpha}{x^2 + (y - 3\alpha)^2}\right].$$

Ex. 11. *Find image of a line source in a circular cylinder.*

Sol. Let there be a uniform line source of strength m per unit length through the point $z = c$, where $z > a$. Then the complex potential at a point z is given by

$$f(z) = -m \log (z - c)$$

Then $$\bar{f}(z) = -m \log (z - c)$$

and so $$\bar{f}(a^2/z) = -m \log \{(a^2/z) - c\}$$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thomson's circle theorem is given by

$$w = f(z) + \bar{f}(a^2/z) \qquad \text{for} \qquad |z| \geq a$$

i.e. $$w = -m \log (z - c) - m \log \{(a^2/z) - c\}$$

or $$w = -m \log (z - c) - m \log \{z - (a^2/c)\} + m \log z + \text{constant}, \qquad \text{...(1)}$$

the constant (real or complex) being immaterial for the discussion of the flow. The point $z = a^2/c$ is the inverse point of the point $z = c$ with regard to the circle $|z| = a$. Hence (1) shows that the image of a line source in a right circular cylinder is an equal line source through the inverse point in the circular section in the plane of flow together with an equal line sink through the centre of the section.

Ex. 12. *Determine image of a line doublet parallel to the axis of a right circular cylinder.*

Sol. Let there be a uniform line doublet of strength μ per unit length through the point $z = c > a$. Furthermore let the axis of the line doublet be inclined at an angle α to x-axis. Then the complex potential at a point z is given by

$$f(z) = (\mu e^{i\alpha})/(z - c)$$

Then $$\bar{f}(z) = (\mu e^{-i\alpha})/(z - c)$$

and so $$\bar{f}(a^2/z) = \frac{\mu e^{-i\alpha}}{(a^2/z) - c}.$$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thomson's circle theorem is given by $$w = \frac{\mu e^{-i\alpha}}{z - c} + \frac{\mu e^{-i\alpha}}{(a^2/z) - c}.$$

Ex. 13. A source and sink of equal strength are placed at the points ($\pm a/2$, 0) within a fixed circular boundary $x^2 + y^2 = a^2$. Show that the streamlines are given by
$(r^2 - a^2/4)(r^2 - 4a^2) - 4a^2 y^2 = ky(r^2 - a^2)$. **[Bhopal 1999, 2000; I.A.S. 1984, 86]**

Sol. Corresponding to a source of strength m, say at $(a/2, 0)$ and an equal sink of strength $-m$ at $(-a/2, 0)$, the complex potential $f(z)$ in absence of the given circular boundary $x^2 + y^2 = a^2$, is given by
$$f(z) = -m \log(z - a/2) + m \log(z + a/2) \quad \ldots(1)$$
(1) $\Rightarrow$
$$f(a^2/z) = -m \log(a^2/z - a/2) + m \log(a^2/z + a/2)$$

When the circular boundary $x^2 + y^2 = a^2$ is inserted, the complex potential w at any interior point of the boundary is given by
$$w = f(z) + f(a^2/z), \qquad \text{that is,}$$
$$w = m \log(z + a/2) - m \log(z - a/2) + m \log\{(z + 2a)/2z\} - m \log\{(2a - z)/2z\}$$
or $w = m \log(z + a/2) - m \log(z - a/2) + m\{\log(2a + z) - \log 2z\} - m\{\log(2a - z) - \log 2\}$

or $\phi + i\psi = m \log(x + a/2 + iy) - m \log(x - a/2 + iy) + m \log(2a + x + iy) - m \log(2a - x - iy)$

Using results $\log(x + iy) = (1/2) \times \log(x^2 + y^2) + i \tan^{-1}(y/x)$ and $\log(x - iy) = (1/2) \times \log(x^2 + y^2) - i \tan^{-1}(y/x)$ on R.H.S. of the above equation and then equation imaginary parts on both sides, we obtain

$$\frac{\psi}{m} = \tan^{-1}\frac{y}{x + a/2} - \tan^{-1}\frac{y}{x - a/2} + \tan^{-1}\frac{y}{2a + x} - \left(-\tan^{-1}\frac{y}{2a - x}\right)$$

$$= \tan^{-1}\frac{\dfrac{y}{x + a/2} - \dfrac{y}{x - a/2}}{1 + \dfrac{y^2}{x^2 - a^2/4}} + \tan^{-1}\frac{\dfrac{y}{2a + x} + \dfrac{y}{2a - x}}{1 - \dfrac{y^2}{4a^2 - x^2}}$$

$$= \tan^{-1}\left(\frac{4ay}{4a^2 - r^2}\right) - \tan^{-1}\left(\frac{ay}{r^2 - a^2/4}\right), \quad \text{where} \quad r^2 = x^2 + y^2$$

$$= \tan^{-1}\frac{\dfrac{4ay}{4a^2 - r^2} - \dfrac{ay}{r^2 - a^2/4}}{1 + \dfrac{4a^2y^2}{(4a^2 - r^2)(r^2 - a^2/4)}} = \tan^{-1}\frac{5ay(r^2 - a^2)}{(4a^2 - r^2)(r^2 - a^2/4) + 4a^2y^2}$$

The required streamlines in the desired form can be obtained by choosing ψ = constant $= m \tan^{-1}(-5a/k)$. Thus, the required streamlines are given by

$$-\frac{5a}{k} = \frac{5ay(r^2 - a^2)}{(4a^2 - r^2)(r^2 - a^2/4) + 4a^2y^2} \qquad \text{or} \qquad \frac{1}{k} = \frac{y(r^2 - a^2)}{(r^2 - a^2/4)(r^2 - 4a^2) - 4a^2y^2}$$

or
$$(r^2 - a^2/4)(r^2 - 4a^2) - 4a^2y^2 = ky(r^2 - a^2).$$

Ex. 14. *Verify that $w = ik \log\{(z - ia)/(z + ia)\}$ is the complex potential of a steady flow of liquid about a circular cylinder the plane $y = 0$ being a rigid boundary. Find the force exerted by the liquid on unit length of the cylinder.* **[Bhopal 1993; Rohilkhand 1998]**

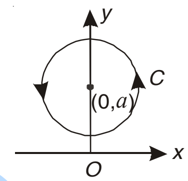
Sol. We have
$$w = \phi + i\psi = ik \log|(z - ia)/(z + ia)| \quad \ldots(1)$$
Hence,
$$\psi = k \log|(z - ia)/(z + ia)|$$
and so the streamlines are given by ψ = constant $= k\lambda$, say *i.e.*,
$$|(z - ia)/(z + ia)| = \lambda, \qquad \text{or} \qquad |z - ia| = \lambda|z + ia|, \quad \ldots(2)$$

which are non-intersecting co-axial circles having $z = \pm ia$ as the limiting points. In particular, for $\lambda = 1$, (2) represents the straight line $|z - ia| = |z + ia|$, *i.e.*, $|x + i(y - a)| = |x + i(y + a)|$, *i.e.*, $x^2 + (y - a)^2 = x^2 + (y + a)^2$, *i.e.*, $y = 0$, showing that $y = 0$ is a rigid boundary.

From (1),
$$w = ik\,\{\log(z - ia) - \log(z + ia)\}$$

so that
$$\frac{dw}{dz} = ik\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right) = \frac{2ka}{z^2 + a^2} \qquad \ldots(3)$$

The adjoining figure shows that circular section C of the cylinder and the rigid plane.



If the pressure thrusts on the given circular disc are represented by (X, Y) then by Blasius theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz = 2k^2 a^2\, \rho i \int_C \frac{dz}{(z^2 + i^2)^2} \qquad \ldots(4)$$

Again, by Cauchy's residue theorem, we have

$$\int_C \frac{dz}{(z^2 + a^2)^2} = 2\pi i \times \text{ (sum of the resides)}$$

$\therefore$ (4) becomes,
$$X - iY = -4k^2 a^2 \rho\pi \times \text{ (sum of the residues)} \qquad \ldots(5)$$

where the indicated sum of the residues is calculated at poles of $1/(z^2 + a^2)^2$ lying within the circular boundary C.

We now proceed to find the residues of $1/(z^2 + a^2)^2$. The only poles of $1/(z^2 + a^2)^2$ are at $z = \pm ia$. But only $z = ia$ lies within the boundadry C as shown in the figure. Hence we shall find residue at $z = ia$.

Now,
$$\frac{1}{z^2 + a^2} = \frac{1}{(z - ia)(z + ia)} = \frac{1}{2ia}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right)$$

$\therefore$
$$\frac{1}{(z^2 + a^2)^2} = -\frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{2}{(z - ia)(z + ia)}\right\}$$

$$= -\frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{1}{2ia}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right)\right\}$$

Hence, Reissue of $1/(z^2 + a^2)^2$ at $z = ia$ is $1/(8ia^3)$.

$\therefore$ (5) becomes
$$X - iY = -(4k^2 a^2 \rho\pi) \times (1/8ia^3) = \{(\pi\rho k^2)/2a\}i$$

$\Rightarrow$ $\quad X = 0 \quad$ and $\quad Y = -(\pi\rho k^2/2a)$,

showing that the liquid exerts a downward force on the cylinder of numerical value $(\pi\rho k^2/2a)$ per unit length.

Ex. 15. *In the two-dimensional motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2 + y^2 = a^2$ which lies in the first and fourth quadrants and the parts of y-axis which lie outside the circle. A simple source of strength m is placed at the point (f, 0) where $f > a$. Prove that the speed of the fluid at the point ($a \cos\theta$, $a \sin\theta$) of the semi-circular boundary is $(4\,a\,m\,f^2 \sin 2\theta)/(a^4 + f^4 - 2a^2 f^2 \cos 2\theta)$. Find at what point of the boundary the pressure is least ?*

Sol. Refer solution of Ex. 6. Here $f = c$. Then equation (1) gives the required value of speed of the fluid.

Second part. By Bernoulli's equation. $p + (\rho q^2)/2$ = constant. So it follows that p is least when q is maximum. Hence as explained in solution of Ex. 6 at a point $P(a \cos\theta, a \sin\theta)$, where θ is given by $\cos\theta = (a^2 + f^2)/[2(a^4 + f^4)]^{1/2}$, the pressure is least. At every other point, p is greater than that at P.

Ex. 16. *A circular cylinder is placed in a uniform stream, find the forces acting on the cylinder.*

Sol. For undisturbed motion, the complex potential is given by $w = (u - iv)z$.

Then by circle theorem, complex potential for the disturbed motion is

$$w = (u - iv)z + (u + iv)(a^2/z).$$

so that
$$dw/dz = (u - iv) - (u + iv)(a^2/z^2). \qquad \ldots(1)$$

If the pressure thrusts on the given cylinder are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then by Blasius's throrem, we have

$$X - iY = \frac{1}{2} i\rho \int_C (dw/dz)^2 \, dz \qquad \ldots(2)$$

and
$$N = \text{Real part of } -\frac{1}{2}\rho \int_C z\,(dw/dz)^2 \, dz \qquad \ldots(3)$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

From (1) and (2), we have

$$X - iY = (1/2) \times i\rho \int_C \{(u - iv) - (u + iv)(a^2/z^2)\}^2 dz = 0 \;\Rightarrow\; X = 0 \quad \text{and} \quad Y = 0$$

From (1) and (3), we have

$$N = \text{Real part of } -\frac{1}{2}\rho \int_C \left[(u - iv) - (u + iv)(a^2/z^2)\right]^2 z\,dz$$

$$= \text{Real part of } -\frac{1}{2}\rho \int_C \left[(u - iv)^2 - 2(u^2 + v^2)(a^2/z^2) + \cdots\right] z\,dz$$

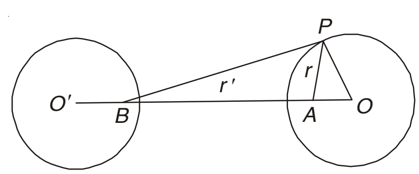
$$= \text{Real part of} - (1/2) \times \rho\{-2(u^2 + v^2)a^2\} 2\pi i = 0$$

Thus, we find that no force or couple acts on the cylinder.

Ex. 17. *Prove that for liquid circulating irrotationally in part of the fluid between two non-intersecting circles the curves of constant veloctiy are Cassini's Ovals.*

[U.P.P.C.S 1997; Rohilkhand 1994; I.A.S. 1993]

Sol. Let O and O' be the centres of the two non-intersecting circles. Let $A(a, 0)$ and $B(-a, 0)$ be the inverse points with respect to both the circles. Let P be any point on one of the given circles such that $PA = r$ and $PB = r'$.



Since A and B are inverse points of the circle with centre O, so by defintion, we have

$$OA \cdot OB = OP^2$$

Now, from similar triangles *OPA* and *OPB*, we have

$$PA/PB = OP/OB = \text{constant} \qquad \Rightarrow \qquad r/r' = \text{constant}.$$

Hence the equations of the two circles may taken as $r/r' = c_1$ and $r/r' = c_2$, where c_1 and c_2 are constants. Since these circles are two streamlines, it follows that the stream function ψ is of the form $f(r/r')$ and it being a harmonic, we take $\psi = k \log (r/r')$ because $\log r$ is the only function of r which is plane harmonic. Here k is a constant.

Now, if θ is the conjugate harmonic of r, $\phi + i\psi$ or $\psi - i\phi$ must be an analytic function of z, so that

$$\phi = -k(\theta - \theta').$$

$$\therefore \qquad w = \psi - i\phi = k\log(r/r') + ik(\theta - \theta') = k[\log r - \log r' + i\theta - i\theta']$$

$$= k[(\log r + i\theta) - (\log r' + i\theta')] = k[\log(re^{i\theta}) - \log(r'e^{i\theta'})]$$

or $\qquad w = k[\log(z-a) - \log(z+a)], \qquad$ as $\qquad re^{i\theta} = z - a \qquad$ and $\qquad r'e^{i\theta'} = z + a$

$$q = \left|\frac{dw}{dz}\right| = \left|k\left[\frac{1}{z-a} - \frac{1}{z+a}\right]\right| = \frac{2ak}{|z-a||z+a|} = \frac{2ak}{rr'}.$$

Hence the curves of equal velocity are given by q = constant or $(2ak)/rr'$ = constant or rr' = constant, which are Cassini's ovals.

EXERCISE 5 (D)

1. Show that the image system of a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle. **[Meerut 2000]**

2. Show that the force per unit length extered on a circular cylinder, radius a, due a source of strength m, at a distance c from the the axis is $(2\pi\rho m^2 a^2)/c(c^2 - a^2)^2$ **[Kanpur 2005]**

Hint. Refer Ex. 7 of Art. 5.25. Here $\mu = m$, $r = a$.

3. The boundary of a semi-infinite liquid consists of an infinite plane surmounted by the cylinder boss of semi-circular cross-section of radius a and the liquid contains a line source everywhere at a distance c from the plane and the axis of the boss, where $c = a \tan\lambda$. Show that the velocity at points on the boss is a maximum along the generators lying in the axial planes, making an angle an angle θ with the axial plane containing the line source, given by $\tan\theta = \pm\cos 2\lambda$.

4. A source is situated at the point (c, c) on the region bounded by the x-axis and the circle $x^2 + y^2 = a^2$, the source being outside the circle. Show that the fluid velocity vanishes at the points $(\pm a, 0)$ and that it will vanish at any other point on the circle provided that $2c < (2+\sqrt{2})a$.

5. If a circle be cut in half by the y-axis, forming rigid boundary and a source of strength m be on the x-axis at a distance a, equal to half the radius, from the centre, prove that the stream lines are given by

$$(16a^4 + r^4)\cos 2\theta - 17a^2r^2 = (16a^4 - r^4)\sin 2\theta \cot(\psi/m).$$

Show that stream line $\psi = m\pi/2$ leves the source in a direction perpendicular to OX and entres the sink at and angle $\pi/4$ with OX.

6. Within a rigid circular boundary of radius a there is a source of strength m at a point P distance b from the centre; at Q, R the extremities of the diameter through P are equal sinks. Find the velocity potential and stream function of two dimensional fluid motion.

7. A simple source, of strength m, is fixed at the origin O in a uniform stream of incompessible fluid moving with velocity U. Show that the velocity potential ϕ at any point P of the stream is

$(m/r) - U\,r\cos\theta$, where $OP = r$ and θ is the angle OP makes with the direction $\boldsymbol{i}$. Find the differential equation of the stream lines and show that they lie on the surfaces

$$Ur^2\sin^2\theta - 2m\cos\theta = \text{constant.}$$

OBJECTIVE QUESTIONS ON CHAPTER 5

Choose the correct alternative from the following questions

1. The image of source $+m$ with respect to a circle is a source $+m$ at the inverse point and
(*i*) a source $+m$ at the centre (*ii*) a source $+m$ at the same point
(*iii*) a sink $-m$ at the centre (*iv*) None of these. **[Kanpur 2003]**

2. The relation between ϕ and ψ is
(*i*) $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = \partial\psi/\partial x$ (*ii*) $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$
(*iii*) $\partial\phi/\partial x = -\partial\psi/\partial y$ and $\partial\phi/\partial y = \partial\psi/\partial x$ (*iv*) None of these **[Kanpur 2002]**

3. With usual notations complex potential of a doublet is
(*i*) $\mu\, e^{i\alpha}/(z-a)$ (*ii*) $\mu\, e^{-i\alpha}/(z-a)$
(*iii*) $\mu\, e^{i\alpha}/(z+a)$ (*iv*) None of these. **[Kanpur 2002]**

4. If w be the complex potential, then the magnitude of the velocity of the fluid is given by
(*i*) $|dw/d\phi|$ (*ii*) $|dw/d\psi|$ (*iii*) $|dw/dz|$ (*iv*) None of these

5. The complex potential due to a source m at $z = z'$ is
(*i*) $-m\log(z-z')$ (*ii*) $m\log(z-z')$ (*iii*) $-m\log(z+z')$ (*iv*) $m\log(z+z')$

6. How many sinks are there if the complex potential is given by $w = \log\{z-(a^2/z)\}$?
(*i*) 1 (*ii*) 2 (*iii*) 3 (*iv*) None of these

7. The family of curves given by ϕ = constant and ψ = constant intersect at
(*i*) 30° (*ii*) 45° (*iii*) 60° (*iv*) 90°

8. The velocity vector **q** is everywhere tangent to the lines in xy-plane along which
(*i*) $\phi(x, y)$ = const. (*ii*) $\psi(x, y)$ = const. (*iii*) w = const. (*iv*) None of these

9. A two-dimensional flow field is given by $\psi = xy$. Then flow is
(*i*) rotational (*ii*) irrotational (*iii*) laminar (*iv*) None of these

10. The stream function is contant along a particular stream line flow
(*i*) false statememnt (*ii*) true statement (*iii*) both of above (*iv*) None of these

11. In a conformal transformation, a source is transformed into
(*a*) an equal source (*b*) an equal sink (*c*) an equal doublet (*d*) None of these
[Agra 2005]

12. Cauchy–Riemann equations in polar form are
(*a*) $\dfrac{\partial\phi}{\partial r} = r\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\phi}{\partial\theta} = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}$ (*b*) $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r}$
(*c*) $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$, $r\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial\theta}$ (*c*) $\dfrac{\partial\phi}{\partial r} = -r\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\phi}{\partial\theta} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial r}$ **[Agra 2008]**

Answer/Hints to objective type questions

1. (*iii*). Refer result of Art. 5.21
2. (*ii*). See Eq. (3), Art 5.6
3. (*i*). See note 3, Art. 5.14
4. (*iii*). See Art. 5.8
5. (*i*). See Art. 5.13
6. (*ii*). See Ex. 1, Art. 5.15
7. (*iv*). See Art. 5.6
8. (*ii*). See Ex. 8, Art. 5.10
9. (*ii*). See Ex. 22, Art. 5.10
10. (*i*) Refer Art. 5.2
11. (*a*). See Art. 5.19B
12. (*b*). See Art. 5.7A

Miscellaneous Problems on Chapter 5

1. Show that at the points of fields of flow the equipotential surfaces cut streamlines orthogonally. **(Agra 2009)**

Hint : Use Ex. 1, page 5.6

2. Explain velocity potential and stream function and derive expressions for velocity components in terms of ϕ and ψ. Also, prove that ϕ and ψ satisfy Laplace's equation. **(Meerut 2010)**

Hint : Refer Art. 2.26 on page 2.56, Art. 5.2 on page 5.1 and Art. 5.6 on page 5.3.

3. Find complex potential of a two-dimensional source. (Meerut 2012)

Hint : Refer Art. 5.13, page 5.20

4. Write 'T' for true and 'F' for false statements :

The stream function ψ exists only in irrotation motions. **(Agra 2004, 11)**

Ans. 'F'. Refer Art.5.2

General Theory of Irrotational Motion

6.1. Connectivity. Definition.

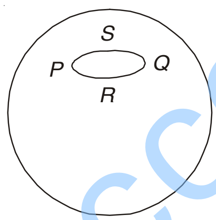
A region of space is said to be connected if a continuous curve joining any two points of the region lies entirely in the given region.

Thus the region interior to a sphere, or the region between two coaxial infinitely long cylinders are connected.

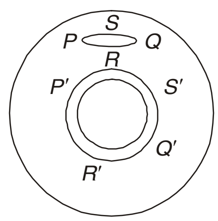
Reducible and irreducible circuits.

Definition. *A closed circuit, all of whose points lie in the region, is said to be reducible if it can be shrunk to a point of the region without passing outside of the region.*

The circuit *PRQS* in figures (*i*) and (*ii*) are reducible; the circuit *P′Q′R′S′* in figure (*ii*) is irreducible, for it cannot be made smaller than the circumference of the inner cylinder.



(*i*)



(*ii*)

Simply connected region.

Definition. *A region in which every circuit is reducible is known as simply connected.*

Thus the region interior to a sphere, the region exterior to a sphere, the region between two concentric spheres, unbounded space etc. are simply connected regions. The region between the concentric cylinders in figure (*ii*) above is not simply connected, for it contains irreducible circuits. This region can be made simply connected by inserting one boundary or barrier which may not be crossed, such as *AB* containing a generating line of each cylinder as shown in figure (*iii*).

With the insertion of this barrier each circuit in the modified region becomes reducible and hence the modified region is simply connected.

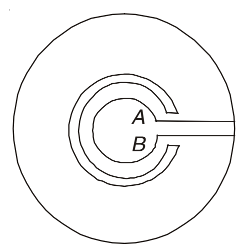


Fig. (*iii*)

Doubly connected and *n*-ply connected regions.

Definition. *A region is said to be doubly connected, if it can be made simply connected by the insertion of one barrier. Similarly, a region is said to be n-ply connected, if can be made simply connected by the insertion of n* – 1 *barriers.*

Thus the region between two coaxial infinitely long cylinders, the region exterior to an infinitely long cylinder, the region interior (or exterior) to an anchor ring etc. are doubly connected regions.

Reconcilable or irreconcilable paths and circuits.

Definition. *The paths joining two points P and Q of the region are said to be reconcilable, if either can be continuously deformed into the other without ever passing out of the region.*

Thus the paths PRQ and PSQ in figures (*i*) and (*ii*) are reconcilable and the paths $P'R'Q'$ and $P'S'Q'$ in figure (*ii*) are irreconcilable.

Note that two reconcilable paths taken together form a reducible circuit.

Two closed circuits are said to be reconcilable, if either can be continuously deformed into the other without ever passing out of the region.

Clearly reconcilable circuits are not always reducible.

6.2. Flow and circulation. **[Himachal 2003]**

If A and P be any two points in a fluid, then the value of the integral

$$\int_A^P (udx + vdy + wdz),$$

taken along any path from A to P, is called the *flow* along that path from A to P.

When a velocity potential ϕ exists, the flow from A to P is

$$= -\int_A^P \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = -\int_A^P d\phi = \phi_A - \phi_P.$$

The flow round a closed curve is known as the *circulation round* the curve. Let C be closed curve and Γ be the circulation. Also, let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then,

$$\Gamma = \int_C (udx + vdy + wdz) = \int_C \mathbf{q}\cdot d\mathbf{r},$$

where the line integral is taken round C in a counter clockwise direction and $\mathbf{q}$ is the velocity vector.

Remark. When a single-valued velocity potential exists the circulation round any closed curve is clearly zero. Again, in what follows, we shall prove that if the velocity potential is many-valued there are curves for which the circulation is zero, though it is not zero for all such paths.

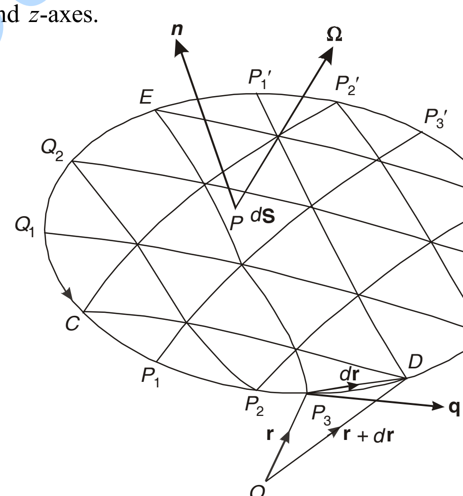
6.3. Stokes's theorem. **[Kanpur 2000, 01; Meerut 2000, 02, 09]**

Let $\mathbf{q}$ *be the velocity vector,* $\boldsymbol{\Omega}$ *the vorticity vector and S be a surface bounded by a closed curve C. Then*

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} \qquad i.e. \qquad \Gamma = \int_S \boldsymbol{\Omega}\cdot\mathbf{n}\,dS,$$

where Γ *is the circulation round C and the unit normal vector* $\mathbf{n}$ *at any point of S is drawn in the sense in which a right-handed screw would move when rotated in the sense of description of C.*

Proof. As shown in figure we observe that the given surface S can be divided up into a network of infinitesimally small triangles ΔS. Let lines be drawn from the vertices of such triangles parallel to the x, y, and z-axes.



Then we obtain a series of elementary tetrahedrons. Let $PABC$ be one of these tetrahedrons, with edges, PA, PB and PC equal to dx, dy and dz, respectively, as shown in the following figure. Let D, E, F be the mid-points of the AB, BC and CA, respectively. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

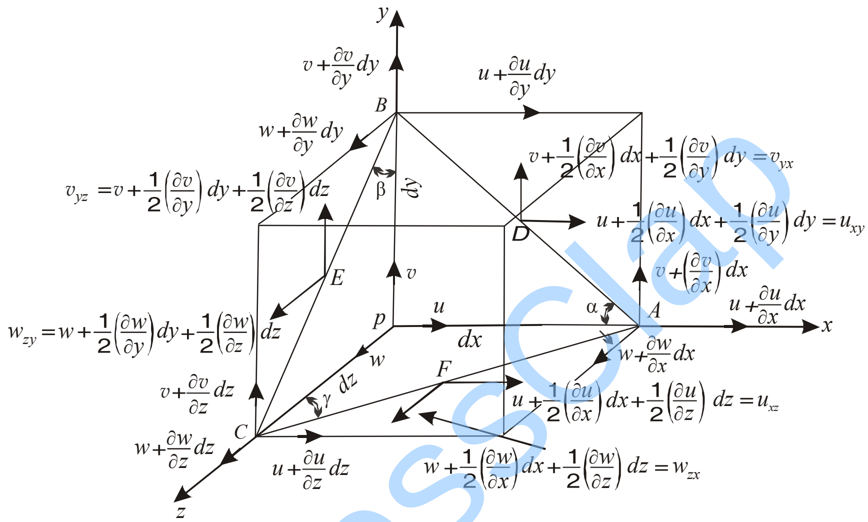
From right-angled triangle PAB, we have

$$\cos\alpha = dx / AB \quad \text{and} \quad \sin\alpha = dy / AB$$

so that
$$AB = dx/\cos\alpha = dy/\sin\alpha \qquad ...(1)$$

Similarly,
$$BC = dy/\cos\beta = dz/\sin\beta \qquad ...(2)$$

and
$$CA = dz/\cos\gamma = dx/\sin\gamma \qquad ...(3)$$



Let co-ordinates of P be (x, y, z). Also, we have

$$\text{Velocity at } P \text{ parallel to } x\text{-axis} = u = f(x, y, z), \text{ say} \qquad ..(4)$$

Then velocity at A parallel to x-axis $= f(x + dx, y, z) = f(x, y, z) + dx\,(\partial f/\partial x)$

[To first order of approximation by using Taylor's theorem]

$$= u + dx\,(\partial u/\partial x), \text{ using (4)}$$

Proceeding likewise, the velocity components at D, E and F (which contribute to the desired circulation can be calculated from those at the vertices P, A, B and B and are given by

$$u_{xy} = u + \frac{1}{2}\frac{\partial u}{\partial x}dx + \frac{1}{2}\frac{\partial u}{\partial y}dy \qquad ...(5)$$

$$u_{xz} = u + \frac{1}{2}\frac{\partial u}{\partial x}dx + \frac{1}{2}\frac{\partial u}{\partial z}dz \qquad ...(6)$$

$$v_{yx} = v + \frac{1}{2}\frac{\partial v}{\partial y}dy + \frac{1}{2}\frac{\partial v}{\partial x}dx \qquad ...(7)$$

$$v_{yz} = v + \frac{1}{2}\frac{\partial v}{\partial y}dy + \frac{1}{2}\frac{\partial v}{\partial z}dz \qquad ...(8)$$

$$w_{zx} = w + \frac{1}{2}\frac{\partial w}{\partial z}dz + \frac{1}{2}\frac{\partial w}{\partial x}dx \qquad ...(9)$$

$$w_{zy} = w + \frac{1}{2}\frac{\partial w}{\partial z}dz + \frac{1}{2}\frac{\partial w}{\partial y}dy \qquad ...(10)$$

Let the circulation be taken as positive if it rotates according to the right-handed screw rule with normal outward. Then the circulation along the sides of the triangle ABC is given by

$$d\Gamma = -u_{xy}\cos\alpha\,(AB) + v_{yx}\sin\alpha\,(AB) - v_{yz}\cos\beta\,(BC) + w_{zy}\sin\beta\,(BC) - w_{zx}\cos\gamma\,(CA) + u_{xz}\sin\gamma\,(CA)$$

$$= -u_{xy}\,dx + v_{yx}\,dy - v_{yz}\,dy + w_{zy}\,dz - w_{zx}\,dz + u_{xz}\,dx, \text{ using (1), (2) and (3)}$$

$$= (u_{xz} - u_{xy})\,dx + (v_{yx} - v_{yz})\,dy + (w_{zy} - w_{zx})\,dz$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial z}dz - \frac{\partial u}{\partial y}dy\right)dx + \frac{1}{2}\left(\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial z}dz\right)dy + \frac{1}{2}\left(\frac{\partial w}{\partial y}dy - \frac{\partial w}{\partial x}dx\right)dz, \text{ using (5) to (10)}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \qquad \text{...(11)}$$

But, $$\text{Curl}\,\mathbf{q} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and $$d\mathbf{S} = \mathbf{i}\left(\frac{1}{2}dy\,dz\right) + \mathbf{j}\left(\frac{1}{2}dz\,dx\right) + \mathbf{k}\left(\frac{1}{2}dx\,dy\right)$$

$$\therefore \quad \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \qquad \text{...(12)}$$

Using (12), (11) reduces to $$d\Gamma = \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} \qquad \text{...(13)}$$

Proceeding likewise we can obtain the circulation around all other elementary triangles, dS, of the entire surface S. We observe that the circulation along the elementary sides common to two triangles cancels and hence the remaining circulation will be that round the closed contour C. Thus, circulation round C is equal to the sum of the circulation in all elementary triangles, *i.e.*,

$$\Gamma = \int_S \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} \qquad \text{...(14)}$$

But $$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} \qquad \text{and} \qquad \mathbf{\Omega} = \text{Curl}\,\mathbf{q}$$

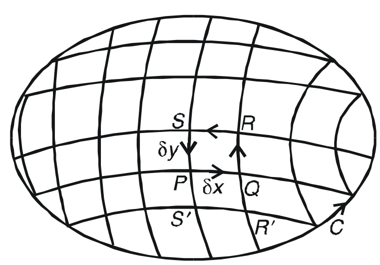
$$\therefore \quad \int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \mathbf{\Omega}\cdot d\,\mathbf{S} = \int_S \mathbf{\Omega}\cdot\mathbf{n}\,d\,\text{S} \qquad \text{...(15)}$$

6.3A. Stakes' theorem (Alternative form with proof)

The circulation Γ *round any closed curve C drawn in a fluid is equal to the surface integral of the normal component of spin* (*i.e. vorticity vector* $\mathbf{\Omega}$) *taken over any surface S, provided the surface lies wholly in the fluid, that is,*

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{q}\,dS \quad \textit{so that} \quad \Gamma = \int_S \mathbf{\Omega}\cdot\mathbf{n}\,dS,$$

where $\boldsymbol{n}$ *is the unit normal vector at any point of S.*

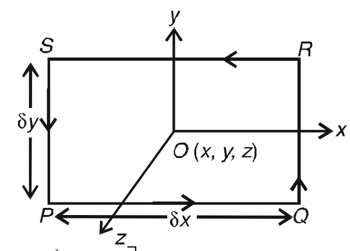


Proof. Let the given surface S be divided into small meshes by drawing a net work of lines across it as shown in the adjoining figure. Then the circulation round the edge of any finite surface is equal to the sum of the circulations, taken all in the same sense, round the boundaries of the infinitely small meshes into which the surface has been divided.

Suppose an elementary mesh be in the form of an elementary rectangular lamina $PQRS$ whose sides are δx, δy. Let the positive direction of circulation for $PQRS$ be taken from the axis of X to that of Y. Let $\boldsymbol{q}$ (u, v, w) be the velocity at the centre of intertia O (x, y, z) of the rectangle.

Now, the circulation due to the two sides QR and SP

$$= \left(\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dy - \left(\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dy = \frac{\partial v}{\partial x}dx\,dy.$$



Similarly, the circulation due to sides RS and $PQ = -(\partial u/\partial y)\,dx\,dy$.

$\therefore$ The circulation $\Gamma_{PQRS} = (\partial v/dx - \partial u/dy)\,dx\,dy$. ...(1)

It follows that the circulation round the boundary C of S

$$= \iint_S \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy\right]. \qquad ...(2)$$

While computing the L.H.S. of (1), we find that no contribution to

$$\sum_{\text{Rectangles}} \mathbf{q}\,.\,d\mathbf{r} \qquad ...(3)$$

will be made by boundary lines (such as PQ between two adjoining rectangles $PQRS$ and $PQR'S'$) because each of such a line will give equal and opposite contribution to the two rectangles adjoining it. It follows that the result of the sum (3) will be simply $\int_C \mathbf{q}\cdot d\mathbf{r}$ taken over the boundary curve C. Hence , we have

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \iint_S \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy\right]$$

or $$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{q}\,d\mathbf{S},$$

where the surface integral is taken over given surface S bounded by closed curve C and the line integral is taken once round the curve.

6.4. Kelvin's circulation theorem.

[Garhwal 2001, 03, 05; Rohilkhand 2001 Agra 2006, 08; Himanchal 2007; Garhwal 2005; Kanpur 1999; Meerut 2005, 09, 10, 12]

When the external forces are conservative and derivable from a single valued potential function and the density is a function of pressure only, the circulation in any closed circuit moving with the fluid is constant for all time.

Proof. Let C be a closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\mathbf{q}$ be the fluid velocity at any point P of the circuit and let $\mathbf{r}$ be its position vector. Then the circulation along the closed circuit C is given by

$$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} \qquad \text{or} \qquad \frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C \mathbf{q}\cdot d\mathbf{r} \qquad ...(1)$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt} = \int_C \frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) \qquad ...(2)$$

But $$\frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) = \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} + \mathbf{q}\cdot\frac{D}{Dt}d\mathbf{r} = \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} + \mathbf{q}\cdot d\mathbf{q} \qquad ...(3)$$

The Euler's equation of motion is

$$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad ...(4)$$

Let the external forces be conservative and derivable from a single valued potential function

V. Then $\mathbf{F} = -\nabla \mathbf{V}$ and hence (4) becomes

$$\frac{D\mathbf{q}}{Dt} = -\nabla \mathbf{V} - \frac{1}{\rho}\nabla p \qquad \text{...(5)}$$

$$\therefore \qquad \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} = -\nabla \mathbf{V}\cdot d\mathbf{r} - \frac{1}{\rho}\nabla p\cdot d\mathbf{r} = -d\,\mathbf{V} - \frac{dp}{\rho} \qquad \text{...(6)}$$

Also $$\mathbf{q}\cdot d\mathbf{q} = \frac{1}{2}d\,(\mathbf{q}\cdot\mathbf{q}) = \frac{1}{2}dq^2, \qquad \text{...(7)}$$

where q denotes the magnitude of the velocity vector $\boldsymbol{q}$.

Using (6) and (7), (3) reduces to $$\frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) = -d\mathbf{V} - \frac{1}{\rho}dp + \frac{1}{2}dq^2 \qquad \text{...(8)}$$

Using (8) and assuming that ρ is a single-valued function of p, (2) reduces to

$$\frac{D\Gamma}{Dt} = \left[\frac{1}{2}q^2 - \mathbf{V} - \int_C \frac{dp}{\rho}\right]_C \qquad \text{...(9)}$$

where $[\ ]_C$ denotes change in the quantity enclosed within brackets on moving once round C. Since q, V and p are single-valued functions of $\mathbf{r}$, so R.H.S. of (9) vanishes. Equation (9) gives the rate of change of flow along any closed circuit moving with the fluid. Thus, it follows that the circulation in any closed circuit moving with the fluid is constant for all time.

6.5. Permanence of irrotational motion.

[G.N.D.U. Amritser 2000, 04; Meerut 1999, 2002; Garhwal 2001, 02; Kurkshetra 1998; Rohilkhand 2002]

When the external forces are conservative and derivable from a single valued potential, and density is a function of pressure only, then the motion of an inviscid fluid, if once irrotational, remains irrotational even afterwards.

Proof. *From Stokes' theorem, the circulation is given by*

$$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{Curl}\,\mathbf{q}\cdot d\,\mathbf{S}. \qquad \text{...(1)}$$

At any time t, let the motion be irrotational so that curl $\mathbf{q} = \mathbf{0}$. Then (1) shows that $\Gamma = 0$ at that instant. Hence it follows from Kelvin's circulation theorem that $\Gamma = 0$ for all time. Hence at any subsequent time, (1) shows that

$$\int_S \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} = 0 \qquad \text{...(2)}$$

Since S is arbitrary, (2) shows that Curl $\mathbf{q} = \mathbf{0}$ at all subsequent time *i.e.* the motion remains irrotational even afterwards.

6.6. Green's theorem.

(Agra 2012)

If ϕ, ϕ' are both single-valued and continuously differentiable scalar point functions such that $\nabla\phi$ and $\nabla\phi'$ are also continuously differentiable, then

$$\int_V (\nabla\phi\cdot\nabla\phi')\,d\,\mathbf{V} = -\int_S \phi\frac{\partial\phi'}{\partial n}d\,\mathbf{S} - \int_V \phi\nabla^2\phi'\mathbf{dV}$$

$$= -\int_S \phi'\frac{\partial\phi}{\partial n}d\,\mathbf{S} - \int_V \phi'\nabla^2\phi\,\mathbf{dV},$$

where S is closed surface bounding any simply-connected region, δn is an element of inward normal at a point on S, and V is the volume enclosed by S.

Proof. From vector calculus, we have $$\nabla\cdot(\phi\mathbf{a}) = \mathbf{a}\cdot(\nabla\phi) + \phi(\nabla\cdot\mathbf{a}), \qquad \text{...(1)}$$

where ϕ is a scalar point function and $\mathbf{a}$ is a vector point function.

Replacing **a** by $\nabla\phi'$ in (1), we get $\nabla\cdot(\phi\nabla\phi') = (\nabla\phi')\cdot(\nabla\phi) + \phi(\nabla\cdot\nabla\phi')$ ...(2)

Integrating both sides of (2) over volume V, we get

$$\int_V \nabla\cdot(\phi\nabla\phi')\,dV = \int_V (\nabla\phi')\cdot(\nabla\phi)\,dV + \int_V \phi(\nabla\cdot\nabla\phi')\,dV \qquad ...(3)$$

By Gauss divergence theorem, we have $\int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \boldsymbol{n}\cdot(\phi\nabla\phi')\,dS$,

where $\boldsymbol{n}$ is the unit vector drawn to the surface S.

or $\int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \phi(\boldsymbol{n}\cdot\nabla\phi')\,dS$ or $\int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \phi\frac{\partial\phi'}{\partial n}\,dS$ ...(4)

Again, $\nabla\cdot\nabla\phi' = \nabla^2\phi'$ and $\nabla\phi'\cdot\nabla\phi = \nabla\phi\cdot\nabla\phi'$ ...(5)

Using (4) and (5), (3) reduces to

$$-\int_S \phi\frac{\partial\phi'}{\partial n}\,dS = \int_V (\nabla\phi\cdot\nabla\phi')\,dV + \int_V \phi\nabla^2\phi'\,dV$$

or
$$\int_V (\nabla\phi\cdot\nabla\phi')\,dV = -\int_S \phi\frac{\partial\phi'}{\partial n}\,dS - \int_V \phi\nabla^2\phi'\,dV \qquad ...(6)$$

Interchanging ϕ and ϕ' in (6), we have

$$\int_V (\nabla\phi'\cdot\nabla\phi)\,dV = -\int_S \phi'\frac{\partial\phi}{\partial n}\,dS - \int_V \phi'\nabla^2\phi\,dV$$

or
$$\int_V (\nabla\phi\cdot\nabla\phi')\,dV = -\int_S \phi'\frac{\partial\phi}{\partial n}\,dS - \int_V \phi'\nabla^2\phi\,dV \qquad ...(7)$$

(6) and (7) together prove the Green's theorem

6.7. Deductions from Green's theorem.

Deduction I. Let ϕ, ϕ' be the velocity potentials of two liquid motions taking place within S. Then $\nabla^2\phi = 0 = \nabla^2\phi'$ and hence Green's theorem yields

$$\int_S \phi\frac{\partial\phi'}{\partial n}\,dS = \int_S \phi'\frac{\partial\phi}{\partial n}\,dS \quad \text{or} \quad \int_S \rho\phi\left(-\frac{\partial\phi'}{\partial n}\right)dS = \int_S \rho\phi'\left(-\frac{\partial\phi}{\partial n}\right)dS \qquad ...(1)$$

But $-\partial\phi/\partial n$ is the normal velocity inwards and $\rho\phi$ is the impulsive pressure at any point on the surface which will produce velocity potential ϕ from rest. Hence (1) shows that if there be two possible motions inside S by means of two different impulsive pressures on the boundary, then the work done by the first in acting through the displacement produced by the second must be equal to the work done by the second in acting through the displacement produced by the first.

Deduction II. Let ϕ' = constant (= k, say). Then $\nabla^2\phi' = 0 = \partial\phi'/\partial n$ everywhere. If ϕ be the velocity potential of a liquid motion within **S**, then by Green's theorem, we have

$$\int_S k\frac{\partial\phi}{\partial n}\,dS = 0 \quad \text{or} \quad \int_S \frac{\partial\phi}{\partial n}\,dS = 0. \qquad ...(2)$$

Since $\partial\phi/\partial n$ is the normal velocity outwards, $(\partial\phi/\partial n)$ dS represents the flow across dS per unit time. Then (2) shows that the total flow across S is zero, *i.e.*, the quantity of a liquid inside S remains constant.

Deduction III. Let $\phi = \phi'$ and let ϕ be the velocity potential of a liquid motion within S. Then $\nabla^2\phi = 0$ and hence Green's theorem gives

$$\int_V (\nabla\phi\cdot\nabla\phi)\,d\,V = -\int_S \phi\frac{\partial\phi}{\partial n}\,d\,S \quad\text{or}\quad \int_V \left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]d\,V = -\int_S \phi\frac{\partial\phi}{\partial n}\,d\,S \quad ...(3)$$

Let q be the velocity and ρ the density of the liquid, then (3) reduces to

$$\frac{1}{2}\rho\int_V q^2\,d\,V = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}\,d\,S \quad ...(4)$$

Clearly the L.H.S. of (4) represents the kinetic energy T of the liquid within S. Hence (4) reduces to

$$T = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}\,d\,S. \quad ...(5)$$

Now $\rho\phi$ is the impulsive pressure that would set up the motion instantaneously from rest, and $-\partial\phi/\partial n$ is the inward normal velocity at the surface. Hence (5) shows that the kinetic energy set up by impulses, in a system starting from rest, is the sum of the products of each impulse and half the velocity of its point of application. From (5), we also find that the kinetic energy of a given mass of liquid moving irrotationally in simply-connected region depends only on the motion of its boundaries.

Suppose on the boundary $\partial\phi/\partial n = 0$. Then (4) reduces to

$$\int_V q^2\,d\,V = 0 \quad ...(6)$$

Since q^2 is positive, (6) implies that $q = 0$ everywhere. Hence the liquid is at rest. Thus a cyclic irrotational motion is impossible in a liquid bounded by fixed rigid boundary.

6.8. Kinetic energy of infinite liquid. **[Meerut 2007]**

Consider an infinite mass of liquid moving irrotationally, at rest at infinity, and bounded internally by a solid surface S and externally by a large surface S'. Let ϕ be the single-valued velocity potential. Then from deduction III of Art. 6.7, the kinetic energy T of the liquid contained in the region bounded by S and S' is given by

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,d\,S - \frac{1}{2}\rho\int_{S'} \phi\frac{\partial\phi}{\partial n}\,d\,S' \quad ...(1)$$

Since there is no flow into the region across S, the equation of continuity takes the form

$$\int_S \frac{\partial\phi}{\partial n}\,d\,S + \int_{S'} \frac{\partial\phi}{\partial n}\,d\,S' = 0 \quad ...(2)$$

Multiplying (2) by $C/2$, a constant, and subtracting from (1), we get

$$T = -\frac{1}{2}\rho\int_S (\phi - C)\frac{\partial\phi}{\partial n}\,d\,S - \frac{1}{2}\rho\int_{S'} (\phi - C)\frac{\partial\phi}{\partial n}\,d\,S' = 0 \quad ...(3)$$

Since for the solid boundary S, $\int_S \frac{\partial\phi}{\partial n}\,d\,S = 0$, it follows from (2) that $\int_{S'} \rho\frac{\partial\phi}{\partial n}\,d\,S' = 0$, *i.e.*, $\int_{S'} \frac{\partial\phi}{\partial n}\,d\,S'$ is independent of S'. Let $\phi \to C$ at infinity and let the surface S' be enlarged indefinitely in all directions. Then the second integral in (3) vanishes and hence the required kinetic energy of infinite liquid is given by

$$T = -\frac{1}{2}\rho\int_S (\phi - C)\frac{\partial\phi}{\partial n}\,d\,S$$

i.e., $$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,d\,S \quad\text{as}\quad \int_S \frac{\partial\phi}{\partial n}\,d\,S = 0 \quad ...(4)$$

Remark. For the motion of liquid to exist, T must not vanish. Hence all internal boundaries must not be at rest.

6.9A. Acyclic and cyclic motions.

The motion in which the velocity potential is single-valued is called *acyclic* whereas the motion in which the velocity potential is not a single-valued is called *cyclic*.

6.9B. Some uniqueness theorems related to acyclic irrotational motion.

In what follows, we shall use the following equivalence of the expressions for the kinetic energy

$$T = \frac{1}{2}\rho\int_V q^2 d\,V = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad ...(1)$$

where the symbols have their usual meaning.

Theorem I. *There cannot be two different forms of acyclic irrational motions of a confined mass of incompressible inviscid liquid, when the boundaries have prescribed velocities.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions subject to the condition

$$\partial\phi_1/\partial n = \partial\phi_2/\partial n \text{ at each point of } S \qquad ...(2)$$

Also $$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad ...(3)$$

Let $\phi = \phi_1 - \phi_2$, then $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0.$

Hence ϕ is a solution of Laplace's equation and so it represents irrotational motion in which

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial n}(\phi_1 - \phi_2) = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0, \text{ by (2).}$$

Hence $q = 0$ by (1). But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0$ so that $\phi =$ constant or $\phi_1 - \phi_2 =$ constant.

Since the constant is of no significance, it follows that the two motions are the same.

Theorem II. *There cannot be two different forms of irrotational motion for a given confined mass of incompressible inviscid liquid whose boundaries are subject to the given impulses.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two motions subject to the conditions

$$\rho\phi_1 = \rho\phi_2 \text{ at each point } S \qquad ...(4)$$

Also, $$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad ...(5)$$

Let $\phi = \phi_1 - \phi_2$, then $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0.$

Hence ϕ is a solution of Laplace's equation and so it represents irrotational motion in which

$$\rho\phi = \rho(\phi_1 - \phi_2) = 0, \text{ by (4)}$$

Hence $q = 0$ by (1). But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0$ so that $\phi =$ constant *i.e.* $\phi_1 - \phi_2 =$ constant,

showing that the two motions are the same.

Theorem III. *Acyclic irrotational motion is impossible in a liquid bounded entirely by fixed rigid walls.*

Proof. Since at every point of the rigid boundary walls $\partial\phi/\partial n = 0$, it follows by (1) that

$$\int_V q^2 dV = 0.$$

Since q^2 cannot be negative, $q = 0$ everywhere and hence the motion will be impossible.

Theorem IV. *Acyclic irrotational motion of a liquid bounded by rigid walls will instantly cease if the boundaries are brought to rest.*

Proof. This is an immediate corollary to Theorem III.

Theorem V. *Acyclic irrotational motion is impossible in a liquid which is at rest at infinity and is bounded internally by fixed rigid walls.*

Proof. Since the liquid is at rest at infinity and there is no flow over the internal boundaries, the kinetic energy is still given by (1). Hence here $\partial\phi/\partial n = 0$ at each point of S. Hence as shown in theorem III, the motion is impossible.

Theorem VI. *The acyclic irrotational motion of a liquid at rest at infinity and bounded internally by rigid walls will instantly cease if the boundaries are brought to rest.*

Proof. This is an immediate corollary to Theorem V.

Theorem VII. *The acyclic irrotational motion of a liquid at rest at infinity, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions.Then

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad \text{...(6)}$$

Also given $\quad \partial\phi_1/\partial n = \partial\phi_2/\partial n,$ at each point of surface ...(7)

and $\quad q_1 = q_2 \quad$ at infinity ...(8)

Let $\phi = \phi_1 - \phi_2$ and $q = q_1 - q_2$ Then, we have $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ by (6). Hence ϕ must be the velocity potential of a possible motion. Furthermore,

$$\partial\phi/\partial n = \partial\phi_1/\partial n - \partial\phi_2/\partial n = 0, \quad \text{at each point of surface} \qquad \text{... (9)}$$

and $$q = q_1 - q_2 = 0 \quad \text{at infinity.} \qquad \text{... (10)}$$

From (1) and (7), we have $q = 0$. But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0$ so that ϕ = constant or $\phi_1 - \phi_2$ = constant

Since the constant is of no significance, it follows that the two motions are the same.

Theorem VIII. *If the liquid is in motion at infinity with uniform velocity, the acyclic irrotational motion, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.*

Proof. Let us superimpose on the whole system of solid and liquid a velocity equal in magnitude and opposite in direction to the velocity at infinity. The relative kinematical conditions remain unchanged and the liquid is reduced to rest at infinity. The resulting motion is then determined by theorem VII and we return to the given motion by reimposing the velocity at infinity.“

6.10. Kelvin's minimum energy theorem. **[Meerut 2003, 05, 08]**

The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Proof. Let T_1 be the kinetic energy, $\mathbf{q}_1$ the fluid velocity of the actual irrotational motion with a velocity potential ϕ Then $\quad \mathbf{q}_1 = -\nabla\phi$ ...(1)

Let T_2 be the kinetic energy, $\mathbf{q}_2$ the fluid velocity of any other possible state of motion consistent with the same normal velocity of the boundary S.

Continuity equations for the above two motions give

$$\nabla\cdot\mathbf{q}_1 = 0 \quad \text{and} \quad \nabla\cdot\mathbf{q}_2 = 0 \qquad \text{...(2)}$$

Let $\mathbf{n}$ denote the unit normal at a point of S. Then using the fact that the boundary has the same normal velocity in both motions, we have

$$\mathbf{n}\cdot\mathbf{q}_1 = \mathbf{n}\cdot\mathbf{q}_2 \qquad ...(3)$$

Now, $T_1 = \frac{1}{2}\rho\int_V q_1^2 dV = \frac{1}{2}\rho\int_V \mathbf{q}_1^2 dV$ and $T_2 = \frac{1}{2}\rho\int_V q_2^2 dV = \frac{1}{2}\rho\int_V \mathbf{q}_2^2 dV$

$$\therefore \quad T_2 - T_1 = \frac{1}{2}\rho\int_V (\mathbf{q}_2^2 - \mathbf{q}_1^2)dV = \frac{1}{2}\rho\int_V \left\{2\mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)+(\mathbf{q}_2-\mathbf{q}_1)^2\right\}dV$$

$$= \rho\int_V \mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)\,dV + \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 dV$$

$$= -\rho\int_V (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)\,dV + \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 dV, \text{ using (1)} \qquad ...(4)$$

But $\nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)] = \phi[\nabla\cdot(\mathbf{q}_2-\mathbf{q}_1)] + (\nabla\phi).(\mathbf{q}_2-\mathbf{q}_1) = (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)$, using (2)

$\therefore \int_V (\nabla\phi).(\mathbf{q}_2-\mathbf{q}_1)dV = \int_V \nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)]dV = \int_S \phi\mathbf{n}\cdot(\mathbf{q}_2-\mathbf{q}_1)\,dS$, by divergrnce Theorem

Thus, $$\int_V (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)dV = 0, \text{ Using (3)} \qquad ...(5)$$

Making use of (5), (4) reduces to

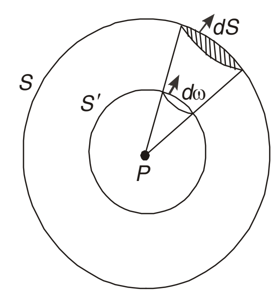
$$T_2 - T_1 = \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 dV \qquad ...(6)$$

Since R.H.S. of (6) is non-negative, we have $T_2 - T_1 \geq 0$, *i.e.*, $T_1 \leq T_2$. Hence the result.

6.11. Mean potential over spherical surface.

The mean value of ϕ *over any spherical surface, throughout whose interior* $\nabla^2\phi = 0$, *is equal to the value of* ϕ *at the centre of the sphere.*

Proof. Describe a sphere S of radius r with P as its centre. Let ϕ_P and $\overline{\phi}$ denote the value of ϕ at P and the mean value of ϕ over S. Describe another concentric sphere S' of radius unity. Then we know that a cone with vertex P which intercepts dS from S also intercepts $d\omega$ (the solid angle) from S'. We then have



$$dS/d\omega = r^2/1^2 \quad \text{so that} \quad dS = r^2 d\omega \qquad ...(1)$$

Now, $$\overline{\phi} = \frac{1}{4\pi r^2}\int_S \phi dS = \frac{1}{4\pi r^2}\int_{S'} \phi r^2 d\omega = \frac{1}{4\pi}\int_{S'} \phi\, d\omega$$

$$\therefore \quad \frac{\partial\overline{\phi}}{\partial r} = \frac{1}{4\pi}\int_{S'} \frac{\partial\phi}{\partial r} d\omega = \frac{1}{4\pi r^2}\int_S \frac{\partial\phi}{\partial r} dS \qquad ...(2)$$

Let **n** denote unit normal at any point of S. Then, we have

$$\int_S \frac{\partial\phi}{\partial r} dS = \int_S \frac{\partial\phi}{\partial n} dS = \int_V \boldsymbol{n}\cdot\nabla\phi\, dS = \int_V \nabla^2\phi\, dV, \text{ by Divergence theorem}$$

$$= 0, \quad \text{as} \quad \nabla^2\phi = 0 \text{ (given)}$$

Hence (2) reduces to $\partial\overline{\phi}/\partial r = 0$ so that $\overline{\phi}$ is independent of r. It follows that $\overline{\phi}$ has the same value over all concentric spheres with P as centre. Hence, by shrinking S to a point P, we have $\overline{\phi} = \phi_\rho$,. Hence the result.

Cor. 1. ϕ *cannot be a maximum or minimum in the interior of any region throughout which* $\nabla^2\phi = 0$.

Proof. If possible, let ϕ_P be a maximum value of ϕ at a point P. Describe a sphere S of radius r with P as its centre such that r is very small. Let $\bar{\phi}$ be the mean value of ϕ over S. Then in our case $\bar{\phi} < \phi_P$, which contradicts the above theorem. Similarly, we can show that ϕ cannot be a minimum.

Cor. 2. *In irrotational motion the maximum values of the speed must occur at the boundary.*

OR

In irrotational motion the velocity cannot be a maximum in the interior of the fluid.

Proof. Let P be a point interior to the fluid. Take P as origin and the axis of x in the direction of motion at P. Let Q be a point near to P and let q and q' be the speeds at P and Q respectively. Then we have

$$q^2 = \left(\frac{\partial \phi}{\partial x}\right)_P^2, \qquad q'^2 = \left(\frac{\partial \phi}{\partial x}\right)_Q^2 + \left(\frac{\partial \phi}{\partial y}\right)_Q^2 + \left(\frac{\partial \phi}{\partial z}\right)_Q^2$$

Now
$$\nabla^2 \left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}(\nabla^2 \phi) = 0,$$

showing that $\partial\phi/\partial x$ satisfies Laplace's equation and hence cannot be a maximum or minimum at P. It follows that there exist points such as Q very near to P such that

$$\left(\frac{\partial \phi}{\partial x}\right)_Q^2 > \left(\frac{\partial \phi}{\partial x}\right)_P^2 \qquad \text{so that} \qquad q'^2 > q^2.$$

Thus q cannot be a maximum in the interior of the fluid, and its maximum value, if any, must occur only on the boundary.

Remark. q^2 may be minimum in the interior of the fluid, for $q = 0$ at a *stagnation point.*

Cor. 3. I*n steady irrotational motion the hydrodynamical pressure has its minimum values on the boundary.*

Proof. By Bernoulli's theorem in absence of external forces, we have

$$p/\rho + q^2/2 = \text{constant}$$

Thus p is least when q^2 is greatest, and this cannot occur inside the fluid by corollary 2. Hence the minimum value of p, if any, must occur on the boundary.

Remark. The maximum value of p occurs at the stagnation points.

6.12. Mean value of velocity potential in a region with internal boundaries.

If Σ is the sold boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, then the mean value of ϕ on Σ is of the from.

$$\bar{\phi} = (M/R) + C$$

where M, C are constants, provided that the fluid extends to infinity and is at rest there.

Proof. Let the volume of fluid crossing each of the internal surfaces contained within Σ per unit time is a finite quantity $4\pi M$. Then the equation of continuity gives

$$\int_\Sigma \left(-\frac{\partial \phi}{\partial R}\right) dS = 4\pi M.$$

Let dS subtends a solid angle $d\omega$ at the centre of Σ. Then $dS = R^2 d\omega$ and hence

$$\frac{1}{4\pi}\int_\Sigma \frac{\partial \phi}{\partial R}\, d\omega = -\frac{M}{R^2} \qquad \text{or} \qquad \frac{1}{4\pi}\frac{\partial}{\partial R}\int_\Sigma \phi\, d\omega = -\frac{M}{R^2}$$

Integrating, $\dfrac{1}{4\pi}\int_{\Sigma} \phi\, d\omega = \dfrac{M}{R} + C$, C being an arbitrary constant

or $$\frac{1}{4\pi R^2}\int_{\Sigma} \phi\, dS = \frac{M}{R} + C, \quad \text{as} \quad dS = R^2\, d\omega \qquad \text{...(1)}$$

or $$\overline{\phi} = M/R + C, \qquad \text{...(2)}$$

where $\overline{\phi}$ is the mean value of ϕ on S and C is independent of R.

We now prove that C is an absolute constant. For this we must prove that C is independent of the coordinates of the centre of Σ. To prove this, let the sphere be displaced a distance δx in any direction, keeping R constant. Then (1) and (2) give

$$\frac{\partial \overline{\phi}}{\partial x}\delta x = \frac{1}{4\pi r^2}\int_{\Sigma} \frac{\partial \phi}{\partial x}\delta x\, dS = \frac{\partial C}{\partial x}\delta x \qquad \text{...(3)}$$

Since the liquid is at rest at infinity, $\partial\phi/\partial x = 0$ on Σ when $R \to \infty$. Hence for large R, (3) shows that $\partial C/\partial x = 0$. Thus we see that C is an absolute constant and the required result follows from (2).

6.13. Illustrative solved examples.

Ex. 1. (*i*) *A velocity field is given by* $\mathbf{q} = (-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2 + y^2)$. *Determine whether the flow is irrotational. Calculate the circulation round* (*a*) *a square with its corners at* (1, 0), (2, 0), (2, 1), (1, 1); (*b*) *a unit circle with centre at the origin.*

[Agra 2008; Rohilkhand 2003; 04; Meerut 2002 04, 05, 07, 08; Kanpur 2002]

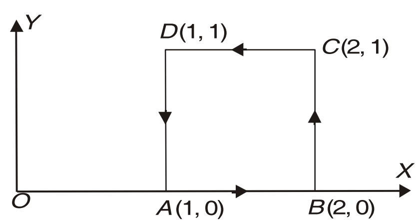
(*ii*) *Find the circulation about the square enclosed by the lines* $x = \pm 2$, $y = \pm 2$ *for the flow* $u = x + y$, $v = x^2 - y$.

Sol. Part (*i*) We have,

$$\text{Curl}\,\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/(x^2+y^2) & x/(x^2+y^2) & 0 \end{vmatrix} = \mathbf{k}\left\{\frac{(y^2-x^2)+(x^2-y^2)}{(x^2+y^2)^2}\right\} = \mathbf{0}$$

Hence curl $\mathbf{q} = \mathbf{0}$ every where except at the origin. Thus the flow is irrotational. It has a singularity at the origin where the velocity becomes infinite.

(*a*) Draw a square in the cartesian plane as follows : $A(1, 0)$, $B(2, 0)$, $C(2, 1)$, $D(1,1)$. Then circulation around the square $ABCD$ is given by



$$\Gamma = \int \mathbf{q}\cdot d\mathbf{r}$$

$$= \int_A^B \mathbf{q}\cdot d\mathbf{r} + \int_B^C \mathbf{q}\cdot d\mathbf{r} + \int_C^D \mathbf{q}\cdot d\mathbf{r} + \int_D^A \mathbf{q}\cdot d\mathbf{r}. \qquad \text{...(1)}$$

Now, $$\int_A^B \mathbf{q}\cdot d\mathbf{r} = \int_{x=1}^{x=2} [(-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2+y^2)]_{y=0}\cdot(dx\,\mathbf{i}) = \int_1^2 (x^{-1}\mathbf{j})\cdot(dx\,\mathbf{i})$$

[$\because$ along AB (*i.e.* x-axis), $y = 0$ so $dy = 0$ and hence $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} = dx\,\mathbf{i}$]

$= 0$, as $\mathbf{j}\cdot\mathbf{i} = 0$ ...(2)

Next, $$\int_C^D \mathbf{q}\cdot d\mathbf{r} = \int_{x=2}^{x=1} [(-\mathbf{i}y + \mathbf{j}x)/(x^2+y^2)]_{y=1}\cdot(dx\,\mathbf{i}) = \int_1^2 \frac{dx}{x^2+1} = \tan^{-1}2 - \tan^{-1}1 \qquad \text{...(3)}$$

Also, $\int_B^C \mathbf{q}\cdot d\mathbf{r} = \int_{y=0}^{y=1} [(-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2+y^2)]_{x=2} \cdot (dy\, \mathbf{j})$ [$\because$ along BC (*i.e.* parallel to y-axis), $x = 2$ so $dx = 0$ and hence $d\mathbf{r} = dx\, \mathbf{i} + dy\, \mathbf{j} = dy\, \mathbf{j}$]

$$= \int_0^1 \frac{2dy}{y^2+4} = 2\cdot\frac{1}{2}\left[\tan^{-1}\frac{y}{2}\right]_0^1 = \tan^{-1}\frac{1}{2} \qquad \text{...(4)}$$

and $$\int_D^A \mathbf{q}\cdot d\mathbf{r} = \int_{y=1}^{y=0} [(-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2+y^2)]_{x=1} \cdot (dy\mathbf{j}) = \int_1^0 \frac{dy}{y^2+1} = -\tan^{-1} 1 \qquad \text{...(5)}$$

Using (2), (3), (4) and (5), (1) becomes

$$\Gamma = \tan^{-1}(1/2) + \tan^{-1} 2 - \tan^{-1} 1 - \tan^{-1} 1 = \cot^{-1} 2 + \tan^{-1} 2 - \pi/4 - \pi/4$$

$$= \pi/2 - \pi/4 - \pi/4 = 0, \text{ as } \tan^{-1} 2 + \cot^{-1} 2 = \pi/2$$

Since curl $\mathbf{q} = \mathbf{0}$ everywhere inside the square path, we could have got the same result directly from Stokes's theorem.

(*b*) To obtain circulation around the unit circle with its centre at the origin, we use polar coordinates for convenience. Writing $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\mathbf{q} = \frac{-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}}{r^2} = -\frac{\sin\theta}{r}\mathbf{i} + \frac{\cos\theta}{r}\mathbf{j} \quad \text{and} \quad \mathbf{q} = u\mathbf{i} + v\mathbf{j}$$

so that $$u = -\frac{\sin\theta}{r} \quad \text{and} \quad v = \frac{\cos\theta}{r}$$

$\therefore$ $$q_r = u\cos\theta + v\sin\theta = 0 \quad \text{and} \quad q_\theta = -u\sin\theta + v\cos\theta = 1/r.$$

$\therefore$ $$\Gamma = \int \mathbf{q}\cdot d\mathbf{r} = \int_0^{2\pi} \{(1/r)\times r\}\, d\theta = 2\pi.$$

Note. The answer indicates the following facts :

1. Unlike part (a), Γ is not equal to zero, because the circle encloses the origin where a singularity exists (*i.e.*, the continuity conditions for Stokes's theorem do not hold there).

2. The circulation is independent of the radius of the circle ; in fact, it can be shown that $\Gamma = 2\pi$ for every curve enclosing the origin.

Part (*ii*) Proceed like part (*a*) of part (*i*).

Ex. 2. *Show that if* $\phi = -(ax^2 + by^2 + cz^2)/2$, $V = -(lx^2 + my^2 + cn^2)/2$ *where a, b, c; l, m, n are functions of time and a + b + c =* 0, *irrotational motion is possible with a free surface of equi-pressure if*

$$(l + a^2 + \dot{a})e^{2\int adt}, \quad (m + b^2 + \dot{b})e^{2\int bdt}, \quad (n + c^2 + \dot{c})e^{2\int cdt} \quad \textit{are constants.}$$

Sol. Given $$\phi = -(ax^2 + by^2 + cz^2)/2 \qquad \text{...(1)}$$

$$V = (lx^2 + my^2 + nz^2)/2 \qquad \text{...(2)}$$

and $$a + y + z = 0 \qquad \text{...(3)}$$

From (1), $$\partial\phi/\partial x = -ax, \quad \partial\phi/\partial y = -by, \quad \partial\phi/\partial z = -cz \qquad \text{...(4)}$$

and $$\partial^2\phi/\partial x^2 = -a, \quad \partial^2\phi/\partial y^2 = -b, \quad \partial^2\phi/\partial z^2 = -c. \qquad \text{...(5)}$$

$\therefore$ $$\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = -(a+b+c) = 0, \text{ by (3)}$$

Thus the Laplace equation $\nabla^2\phi = 0$ is satisfied and hence ϕ is velocity potential for a possible irrotational motion.

Bernoulli's equation for non-steady irrotational motion under conservative external forces with potential V is given by

$$\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V = F(t) \text{ where } F(t) \text{ is an arbitrary function of } t. \qquad ...(6)$$

Let dot denote differentiation *w.r.t.* 't'. Then (1) gives

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}\Sigma \dot{a}\, x^2 \qquad ....(7)$$

Also $$q^2 = \Sigma(\partial\phi/\partial x)^2 = \Sigma a^2 x^2 \qquad ...(8)$$

In order that a free surface of a equal pressure may exist, we have p = const. so that $Dp/Dt = 0$.

$$\therefore \quad \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = 0 \qquad ...(9)$$

Using (2), (7) and (8), (6) gives

$$\frac{p}{\rho} = -\frac{1}{2}\Sigma \dot{a}\, x^2 - \frac{1}{2}\Sigma a^2 x^2 - \frac{1}{2}\Sigma l x^2 + F(t)$$

$$\therefore \quad \frac{\partial p}{\partial t} = -\frac{1}{2}\rho[\Sigma \ddot{a}\, x^2 + 2\Sigma a\, \dot{a}\, x^2 + \Sigma \dot{l}\, x^2] + \rho F'(t)$$

and $$u\frac{\partial p}{\partial x} = \left(-\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial p}{\partial x}\right) = -\rho a x^2(\dot{a} + a^2 + l).$$

Similarly, we have

$$v\frac{\partial p}{\partial y} = -\rho b y^2(\dot{b} + b^2 + m), \qquad w\frac{\partial p}{\partial z} = -\rho c z^2(\dot{c} + c^2 + n).$$

Substituting these in (9), we obtain

$$-\frac{1}{\rho}\Sigma(\ddot{a} + 2a\,\dot{a} + \dot{l})x^2 - \rho\,\Sigma a(\dot{a} + a^2 + l)x^2 + \rho F'(t) = 0,$$

which is an identity in t and so the coefficients of x^2, y^2, z^2 and the constant must vanish separately. Thus, we obtain

$$(1/2)\times(\ddot{a} + 2a\,\dot{a} + \dot{l}) + a(\dot{a} + a^2 + l) = 0 \qquad ...(10)$$

$$(1/2)\times(\ddot{b} + 2b\,\dot{b} + \dot{m}) + b(\dot{b} + b^2 + m) = 0 \qquad ...(11)$$

$$(1/2)\times(\ddot{c} + 2c\,\dot{c} + \dot{n}) + c(\dot{c} + c^2 + n) = 0 \qquad ...(12)$$

$$F'(t) = 0 \qquad ...(13)$$

Now (13) gives $F(t)$ = const. (= C, say). Thus $F(t)$ is an absolute constant.

Re writing (10), we have

$$\frac{\ddot{a} + 2\,\dot{a}\,a + \dot{l}}{\dot{a} + a^2 + l}dt + 2a\,dt = 0$$

Integrating, $$\log(\dot{a} + a^2 + l) - \log c' = -2\int a\,dt$$

or $$(\dot{a} + a^2 + l)/c' = e^{-\int 2a\,dt} \qquad \text{or} \qquad (\dot{a} + a^2 + l)e^{2\int a\,dt} = c'$$

Similarly, (11) and (12) also yields two similar expressions.

Ex. 3. *A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid satisfying the conditions for the continued existence of a velocity potential* ϕ *under conservative forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is*

$$-\rho\iint \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n}\, dS,$$

where ρ *is the density and* δn *an element of the mormal to* δS *drawn into the space considered.*

Sol. The kinetic energy T is given by $$T = -\frac{1}{2}\rho\iint \phi\frac{\partial\phi}{\partial n}\, dS$$

$$\therefore \qquad \frac{dT}{dt} = -\frac{1}{2}\rho\iint\left[\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n}dS + \phi\frac{\partial^2\phi}{\partial n\partial t}\, dS\right] \qquad \text{...(1)}$$

But $$\iint \phi\frac{\partial\phi'}{\partial n}dS = \iint \phi'\frac{\partial\phi}{\partial n}dS \qquad \text{...(2)}$$

Taking $\phi' = \partial\phi/\partial t$, (2) reduces to $$\iint \phi\frac{\partial^2\phi}{\partial n\partial t}dS = \iint \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n}dS \qquad \text{...(3)}$$

Using (3), (1) reduces to $$\frac{dT}{dt} = -\rho\iint \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial n}dS.$$

Ex. 4. *Prove that if the velocity potential at any instant be* λxyz, *the velocity at any point* $(x+\xi, y+\eta, z+\zeta)$ *relative to the fluid at the point (x, y, z) where* ξ, η, ζ *are small, is normal to the quadratic* $x\eta\zeta + y\zeta\xi + z\xi\eta$ = *contant, with centre at (x, y, z).* **[Meerut 2006]**

Sol. Let $\mathbf{q}$ = (u, v, w) and $\mathbf{q}' = (u', v', w')$ be the fluid velocities at $P(x, y, z)$ and $P'(x+\xi, y+\eta, z+\zeta)$ respectivley.

Given velocity potential = $\lambda xyz = \phi$, say Hence $u = -\partial\phi/\partial x = -\lambda yz$

Similarly, $v = -\lambda xz$ and $w = -\lambda xy$.

Again $$u' = u + \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} = -u - \lambda(\eta z + \zeta y)$$

Similarly, $v' = v - \lambda(\zeta x + \xi z)$ and $w' = w - \lambda(\xi y + \eta x)$.

$\therefore$ velocity of P' relative to P *i.e.* $(u'-u, v'-v, w'-w)$ is given by

$$\{-\lambda(\eta z + \zeta y), -\lambda(\zeta x + \xi z), -\lambda(\xi y + \eta x)\} \qquad \text{...(1)}$$

Let $$F \equiv x\eta\zeta + y\zeta\xi + z\xi\eta = \text{const.} \qquad \text{...(2)}$$

Then direction ratios of the normal at P' are $\partial F/\partial\xi$, $\partial F/\partial\eta$, $\partial F/\partial\zeta$

i.e. $y\zeta + z\eta$, $x\zeta + z\xi$, $x\eta + y\xi$. ...(3)

From (1) and (3), it follows that the velocity at P' relative to that at P is normal to the quadratic (2).

Ex. 5. *Deduce from the principle that the kinetic energy set up is a minimum that, if a mass of incompressible liquid be given at rest, completely filling a closed vessel of any shape and if any motion of the liquid be produced suddenly by giving arbitrarily prescribed normal velocities at all points of its bounding surface subject to the condition of constant volume, the motion produced is irrotational.*

Sol. Let T be the kinetic energy and let u, v, w be the components of velocity at any point.

Then, $$T = \frac{1}{2}\iint (u^2 + v^2 + w^2)\,dxdydz$$

Since T is minimum, $\delta T = 0$ and so we get

$$\iiint (u\,\delta u + v\,\delta v + w\,\delta w)\,dxdydz = 0 \qquad ...(1)$$

Since normal velocity $lu + mv + nw$ is prescribed on the bounding surface, we have

$$l\,\delta u + m\,\delta v + n\,\delta w = 0 \quad \text{on the boundary } S. \qquad ...(2)$$

Equation of continuity $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ gives

$$\frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w = 0, \quad \text{which holds at each point within the fluid.}$$

Since ϕ is finite, we have everywhere

$$\iiint \phi\left(\frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w\right)dxdydz = 0$$

or $$\iiint \left[\frac{\partial}{\partial x}(\phi\delta u) + \frac{\partial}{\partial y}(\phi\delta v) + \frac{\partial}{\partial z}(\phi\delta w)\right]dxdydz - \iiint \left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0$$

or $$-\iint \phi(l\delta u + m\delta v + n\delta w)\,dS - \iiint \left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0 \qquad ..(3)$$

Making use of (2), we have everywhere

$$\iiint \left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0 \qquad ...(4)$$

Adding (1) and (4), we have

$$\iiint \left[\left(u + \frac{\partial\phi}{\partial x}\right)\delta u + \left(v + \frac{\partial\phi}{\partial y}\right)\delta v + \left(w + \frac{\partial\phi}{\partial z}\right)\delta w\right]dxdydz = 0$$

$\therefore$ $$u + \partial\phi/\partial x = 0, \qquad v + \partial\phi/\partial y = 0, \qquad w + \partial\phi/\partial z = 0,$$

showing that u, v and w can be obtained from ϕ and hence the motion is irrotational.

Ex. 6. *Show that the theorem that under certain conditions, the motion of a frictionless fluid, if once irrotational, will always be so, is true also when each particle is acted on by a resistance varying as the velocity.*

Sol. Let P' be a point very near to P such that $PP' = \delta s$.

Let flow along PP' be Q. Then, we have $$Q = u\delta x + v\delta y + w\delta z$$

$\therefore$ $$\frac{DQ}{Dt} = \frac{D}{Dt}(u\delta x + v\delta y + w\delta z) \qquad ...(1)$$

But $$\frac{D}{Dt}(u\delta x) = \delta x\frac{Du}{Dt} + u\frac{D\delta x}{Dt} = \delta x\frac{Du}{Dt} + u\delta u. \qquad ...(2)$$

Let the components of the resistance be $(-ku, -kv, -kw)$. Then the equation of motion along x-axis is

$$\frac{Du}{Dt} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} - ku \qquad ...(3)$$

Using (3), (2) reduces to

$$\frac{D}{Dt}(u\delta x) = \delta x\left[-\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} - ku\right] + u\delta u \qquad ...(4)$$

Similarly,
$$\frac{D}{Dt}(v\delta y) = \delta y\left[-\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} - kv\right] + v\delta v \qquad ...(5)$$

and
$$\frac{D}{Dt}(w\delta z) = \delta z\left[-\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - kw\right] + w\delta w \qquad ...(6)$$

Using (4), (5) and (6), (1) reduces to

$$\frac{DQ}{Dt} = -\delta V - \frac{1}{\rho}\delta p + \frac{1}{2}\delta(u^2 + v^2 + w^2) - k(u\delta x + v\delta y + w\delta z)$$

Let Γ be the circulation along the closed curve *APA*. Then

$$\Gamma = \int_A^A (u\delta x + v\delta y + w\delta z)$$

$$\therefore \qquad \frac{D\Gamma}{Dt} = \left[-V - \int\frac{dp}{\rho} + \frac{1}{2}q^2\right]_A^A - k\int_A^A (u\delta x + v\delta y + w\delta z)$$

$= -\,k\,\Gamma$ [when p is a function of ρ and V is a single valued function]

$\therefore \qquad \Gamma = \Gamma_0 e^{-kt}$, where Γ_0 is independent of t.

Let initially the motion be irrotational so that $\xi = \eta = \zeta = 0$. Thus $\Gamma = 0$ when $t = 0$. Hence $\Gamma_0 = 0$ and so $\Gamma = 0$ is always true. Therefore by Stokes's theorem, we always have

$$\iint (l\xi + m\eta + n\zeta)\,dS = 0, \qquad \text{so that} \qquad \xi = \eta = \zeta = 0 \qquad \text{is always true.}$$

Thus if the fluid motion is once irrotational, it will be always so.

Ex. 7. *Obtain Cauchy's integral using circulation theorem.*

[**Note :** For the alternative method of getting Cauchy's integrals, refer Art. 3.12]

Sol. Let *a, b, c* be the initial co-ordinates of a particle and *x, y, z* the co-ordinates of the same particle at time *t*. Let C_0 be the initial position of the closed curve C in *yz*-plane.

From circulation theorem, we have

$$\int_{C_0} (v_0 db + w_0 dc) = \int_C (udx + vdy + wdz),$$

where u, v, w are velocity components at any time t and u_0, v_0, w_0 are initial velocity components.

Let ξ, η, ζ be the vorticity components at any time t and ξ_0, η_0, ζ_0 be the initial vorticity components.

Let *l, m, n* be direction cosines of normal to surface S which was initially in shape S_0. Then, by Stokes' theorem, we have

$$\iint_{S_0}\left(\frac{\partial w_0}{\partial b} - \frac{\partial v_0}{\partial c}\right)dS = \iint_S \left\{l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + m\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\}dS$$

or $$\iint_{S_0} \xi_0 \, db\, dc = \iint_S (l\xi + m\eta + n\zeta)\, dS$$

or $\iint_{S_0} \xi_0 \, db\, dc = \iint_S \left\{\xi \dfrac{\partial(y,z)}{\partial(b,c)} + \eta \dfrac{\partial(z,x)}{\partial(b,c)} + \zeta \dfrac{\partial(x,y)}{\partial(b,c)}\right\} db dc$, as $l dS = dy dz = \dfrac{\partial(x,y)}{\partial(b,c)} db\, dc$ etc.

$$\Rightarrow \qquad \xi_0 = \xi \frac{\partial(y,z)}{\partial(b,c)} + \eta \frac{\partial(z,x)}{\partial(b,c)} + \zeta \frac{\partial(x,y)}{\partial(b,c)}. \qquad ...(1)$$

Similarly, $$\eta_0 = \xi \frac{\partial(y,z)}{\partial(c,a)} + \eta \frac{\partial(z,x)}{\partial(c,a)} + \zeta \frac{\partial(x,y)}{\partial(c,a)}. \qquad ...(2)$$

and $$\zeta_0 = \xi \frac{\partial(y,z)}{\partial(a,b)} + \eta \frac{\partial(z,x)}{\partial(a,b)} + \zeta \frac{\partial(x,y)}{\partial(a,b)}. \qquad ...(3)$$

Multiplying (1), (2) and (3) by $\partial x/\partial a$, $\partial x/\partial b$ and $\partial x/\partial c$ respectively and then adding, we have

$$\xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c} = \xi \left\{ \frac{\partial x}{\partial a}\frac{\partial(y,z)}{\partial(b,c)} + \frac{\partial x}{\partial b}\frac{\partial(y,z)}{\partial(c,a)} + \frac{\partial x}{\partial c}\frac{\partial(y,z)}{\partial(a,b)} \right\}$$

or $$\xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c} = \xi \frac{\partial(x,y,z)}{\partial(a,b,c)}, \text{ on simplification.} \qquad ...(4)$$

In Lagrangian coordinates, the equation of continuity is

$$\rho \frac{\partial(x,y,z)}{\partial(a,b,c)} = \rho_0 \qquad \text{so that} \qquad \frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\rho_0}{\rho}. \qquad ...(5)$$

Using (5), (4) may be re-written as

$$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c}. \qquad ...(6)$$

Proceeding likewise, we also obtain

$$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \qquad ...(7)$$

and $$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \qquad ...(8)$$

Relations (6), (7) and (8) are known as *Cauchy's integrals*.

Ex. 8. *A rigid envelope is filled with homogeneous frictionless liquid; show that it is not possible, by any movements applied to the envelope, to set its contents into motion which will persist after the envelope has come to rest.*

Sol. Liquid motion is produced by movements on the boundary. Equations of motion are given by [Refer Art. 3.7]

$$u' - u = -(1/\rho)\times(\partial\tilde{w}/\partial x), \qquad ...(1)$$

$$v' - v = -(1/\rho)\times(\partial\tilde{w}/\partial y) \qquad ...(2)$$

and $$w' - w = -(1/\rho)\times(\partial\tilde{w}/\partial z), \qquad ...(3)$$

where u, v, w and u', v', w' are the velocity components at the point $P(x, y, z)$ just before and just after the impulsive action and $\tilde{w}$ is the impulsive pressure at P. Here, we have $u = v = w = 0$. Hence, (1), (2), (3) reduce to

$u' = -(1/\rho)\times(\partial\tilde{w}/\partial x)$, $\qquad v' = -(1/\rho)\times(\partial\tilde{w}/\partial y)$, $\qquad$ and $\qquad w' = -(1/\rho)\times(\partial\tilde{w}/\partial z)$.

$$\therefore \qquad u'\,dx + v'\,dy + w'dz = -\frac{1}{\rho}\left[\frac{\partial \tilde{w}}{\partial x}dx + \frac{\partial \tilde{w}}{\partial y}dy + \frac{\partial \tilde{w}}{\partial z}dz\right] = -\frac{1}{\rho}d\tilde{w} = -d\phi, \text{ say}$$

When the density ρ is constant and therefore the motion produced is irrotational. Since the pressure at any point is single valued, ϕ is single valued, *i.e.,* the motion is acyclic, then

$$\iiint q^2\,dx\,dy\,dz = -\iint \phi\frac{\partial \phi}{\partial n}dS. \qquad ...(4)$$

If $\partial\phi/\partial n = 0$ on the boundary, then (4) shows that q is zero everywhere, that is, the liquid comes to rest.

Ex. 9. *Prove that in a cylic irrotational motion of a homogeneous fluid the total momentum of the fluid contained within the sphere of any radius is equivalent to a single vector through the centre of the sphere.*

Sol. Let S and V denote surface and volume of the given sphere whose centre is O. Let ρ be the density and $\mathbf{q}$ the velocity of the fluid. Let $\mathbf{M}$ be the momentum of the fluid contained within the sphere. Then, we have

$$\mathbf{M} = \int_V \rho\mathbf{q}\,dV. \qquad ...(1)$$

Let $\mathbf{N}$ be the moment of momentum $\mathbf{M}$ about O. Then, we have

$$\mathbf{N} = \int_V \mathbf{r}\times(\rho\mathbf{q}\,dV) = \rho\int_V (\mathbf{r}\times\mathbf{q})\,dV. \qquad ...(2)$$

Since the motion is irrotational, velocity potential ϕ exists such $\mathbf{q} = -\nabla\phi$. Hence (2) becomes

$$\mathbf{N} = -\rho\int_V \mathbf{r}\times\nabla\phi\,dV. \qquad ...(3)$$

Now, $\quad \nabla\times(\phi\mathbf{r}) = \nabla\phi\times\mathbf{r} + \phi(\nabla\times\mathbf{r})$, [Refer Art. 1.6 for vector identities] ...(4)

Also, $\quad \nabla\times\mathbf{r} = \nabla\times(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$$\therefore \qquad \nabla\times\mathbf{r} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

Hence $\quad$ (4) $\Rightarrow$ $\nabla\times(\phi\mathbf{r}) = \nabla\phi\times\mathbf{r} = -\mathbf{r}\times\nabla\phi \quad \Rightarrow \quad \mathbf{r}\times\nabla\phi = -\nabla\times(\phi\mathbf{r})$. ...(5)

Using (5), (3) $\Rightarrow$ $\mathbf{N} = \rho\int_V \nabla\times(\phi\mathbf{r})\,dV$ or $\mathbf{N} = \rho\iint_S \hat{\mathbf{n}}\times(\phi\mathbf{r})\,dS = \rho\iint_S \phi(\hat{\mathbf{n}}\times\mathbf{r})\,dS,$ ...(6)

where $\hat{\mathbf{n}}$ is the inward drawn normal unit vector. Again $\hat{\mathbf{n}}$ and $\mathbf{r}$ are parallel vectors on the surface of the sphere and so $\hat{\mathbf{n}}\times\mathbf{r} = \mathbf{0}$. Then, by (6), $\mathbf{N} = \mathbf{0}$. Hence the moment of momentum $\mathbf{N}$ about O is zero and therefore $\mathbf{N}$ must pass through the centre O of the sphere.

Ex. 10. *If p denotes the pressure, V the potential of the external forces and q the velocity of* a homogeneous liquid moving irrotationally, show that *$\nabla^2 q^2$ is positive and $\nabla^2 p$ is negative provided $\nabla^2 V = 0$. Hence prove that the velocity cannot have a maximum value and the pressure cannot have a minimum value at a point in the interior of the liquid.*

Sol. Since the motion is irrotational, the velocity potential ϕ exists such that

$$\boldsymbol{q} = -\nabla\phi = -[(\partial\phi/\partial x)\mathbf{i} + (\partial\phi/\partial y)\mathbf{j} + (\partial\phi/\partial z)\mathbf{k}].$$

$$\therefore \qquad q^2 = (\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2. \qquad ...(1)$$

From vector calculus (Refer vector identities in Art. 1.6, we have

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\therefore \qquad \nabla\cdot\{\nabla(\phi\psi)\} = \nabla\cdot(\phi\,\nabla\psi) + \nabla\cdot(\psi\nabla\phi)$$

or $\quad \nabla^2(\phi\psi) = [\nabla\phi\cdot\nabla\psi + \phi\,\nabla\cdot\nabla\psi] + [\nabla\psi\cdot\nabla\phi + \psi\,\nabla\cdot\nabla\phi]$

or $\quad \nabla^2(\phi\psi) = 2\nabla\phi\cdot\nabla\psi + \psi\,\nabla^2\phi + \phi\nabla^2\psi.$ ...(2)

Replacing ϕ and ψ by $\partial\phi/\partial x$ in result (2), we have

$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2 = 2\left(\nabla\frac{\partial\phi}{\partial x}\cdot\nabla\frac{\partial\phi}{\partial x}\right) + 2\frac{\partial\phi}{\partial x}\nabla^2\left(\frac{\partial\phi}{\partial x}\right). \qquad ...(3)$$

But
$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\nabla^2\phi = 0. \qquad ...(4)$$

Also,
$$\nabla\frac{\partial\phi}{\partial x} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\frac{\partial\phi}{\partial x} = \mathbf{i}\frac{\partial^2\phi}{\partial x^2} + \mathbf{j}\frac{\partial^2\phi}{\partial y\partial x} + \mathbf{k}\frac{\partial^2\phi}{\partial z\partial x}.$$

$$\therefore \quad \nabla\frac{\partial\phi}{\partial x}\cdot\nabla\frac{\partial\phi}{\partial x} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2 + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)^2. \qquad ...(5)$$

Using (4) amd (5), (3) reduces to

$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2 = 2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2 + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)^2\right] > 0. \qquad ...(6)$$

Similarly, $\nabla^2(\partial\phi/\partial y)^2 > 0$ and $\nabla^2(\partial\phi/\partial z)^2 > 0$. ...(7)

Now, (6) and (7) $\Rightarrow$ $\nabla^2[(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2] > 0$. ...(8)

Then (1) and (9) $\Rightarrow$ $\nabla^2 q^2 > 0$, as required. ...(9)

From Bernoulli's equation, we have

$$(p/\rho) - (\partial\phi/\partial t) + q^2/2 + V = f(t).$$

$$\therefore \quad \nabla^2\left(\frac{p}{\rho}\right) - \frac{\partial}{\partial t}(\nabla^2\phi) + \frac{1}{2}\nabla^2 q^2 + \nabla^2 V = \nabla^2 f(t).$$

Now, $\nabla^2\phi = 0$ and $\nabla^2 f(t) = 0$. Also given that $\nabla^2 V = 0$.

$\therefore$ From (10), $\nabla^2 p = -\frac{1}{2}\rho\nabla^2 q^2 < 0$ as $\nabla^2 q^2 > 0$, using (9)

Second part. From Gauss' theorem, we have

$$\iint_S (l'U + m'V + n'W)dS = -\iiint_V \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)dxdydz, \qquad ...(1)$$

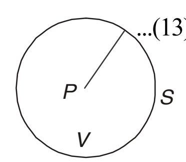
where l', m', n' are the direction cosines of the inward normal to an element δS of surface S and V is the volume enclosed by S.

Let $U = \partial\phi/\partial x$, $V = \partial\phi/\partial y$, $W = \partial\phi/\partial z$. Then (1) reduces to

$$\iint_S \left(l'\frac{\partial\phi}{\partial x} + m'\frac{\partial\phi}{\partial y} + n'\frac{\partial\phi}{\partial z}\right)dS = -\iiint_V \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)dx\,dy\,dz$$

or $\displaystyle\iint_S \frac{\partial\phi}{\partial n}dS = -\iiint_V \nabla^2\phi\,dxdydz$, where δn is an element of inward normal. ...(12)

If $\phi = q^2$, (12) $\Rightarrow$ $\displaystyle\iint_S \frac{\partial q^2}{\partial n}dS = -\iiint_V \nabla^2 q^2\,dxdydz$, ...(13)



We now apply the above result (13) to the case of a liquid contained in a small sphere (See figure) Then, since $\nabla^2 q^2 > 0$, (13) shows that

$$\iint_S \frac{\partial q^2}{\partial n}dS < 0. \qquad ...(14)$$

Since $\delta n = -\delta r$, (14) $\Rightarrow$ $\iint_S \dfrac{\partial q^2}{\partial r}\, dS > 0.$ ...(15)

If q^2 is maximun at P, then $\partial q^2/\partial r$ is negative on the surface of a small sphere surrounding P,

that is, $$\iint_S \frac{\partial q^2}{\partial r}\, dS < 0, \qquad ...(16)$$

which contradicts the results (15) and hence q^2 cannot have a maximum within the liquid. Thus q^2 can be maximum only on the boundary.

Similarly, putting $\phi = p$ in (12), we have

$$\iint_S \frac{\partial p}{\partial n}\, dS = -\iiint_V \nabla^2 p\, dx\, dy\, dz. \qquad ...(17)$$

Since $\nabla^2 p < 0$, (17) $\Rightarrow$ $\iint_S \dfrac{\partial p}{\partial n}\, dS > 0.$ ...(18)

Now proceeding as before, we find that p cannot have a minimum at a point within the liquid. Hence the pressure can be minimum only on the boundary.

Ex. 11. *Prove that irrotational acyclic motion of a liquid contained in a boundary cannot be created or destroyed by application of impulses.*

Sol. The equations of motion under inpulsive forces are

$$u' - u = I_x - (1/\rho)\,(\partial \tilde{w}/\partial x) \qquad ...(1)$$

$$v' - v = I_y - (1/\rho)\,(\partial \tilde{w}/\partial y) \qquad ...(2)$$

and $$w' - w = I_z - (1/\rho)\,(\partial \tilde{w}/\partial z), \qquad ...(3)$$

where u, v, w and u', v', w' are the velocity components at any point $P(x, y, z)$ just before and just after the impulsive action, I_x, I_y, I_z are the components of external impulsive forces per unit mass of the fluid and $\tilde{w}$ is the impulsive pressure at P.

Multiplying (1), (2) and (3) by, dx, dy and dz respectively and then adding, we have

$$(u'dx + v'dy + w'dz) - (udx + vdy + wdz)$$

$$= (I_x\, dx + I_y dy + I_z dz) - (1/\rho)\left[(\partial \tilde{w}/\partial x)dx + (\partial \tilde{w}/\partial y)dy + (\partial \tilde{w}/\partial z)dz\right]$$

$$= -dV - (1/\rho)\; d\tilde{w}, \text{ if } V \text{ is potential of the external impulses}$$

$$= -d(V + \tilde{w}/\rho), \text{ if } \rho \text{ is constant}$$

$\therefore$ $$(u'dx + v'dy + w'dz) - (udx + vdy + wdz) = -d(V + \tilde{w}/\rho). \qquad ...(4)$$

Now R.H.S. of (4) is an exact differential and hence L.H.S. of (4) must be an exact differential. Hence if $udx + vdy + wdz$ is not an exact differential, then $u'dx + v'dy + w'dz$ will also be not an exact differential, that is, when the motion is not irrotational we cannot make it irrotational by application of impulses.

Let $$udx + vdy + wdz = -\,d\phi, \qquad ...(5)$$

then, clearly $$u'dx + v'dy + w'dz = -\,d\phi', \text{ say} \qquad ...(6)$$

Then (4) becomes $$-\,d\phi' + d\phi = -\,d(V + \tilde{w}/\rho).$$

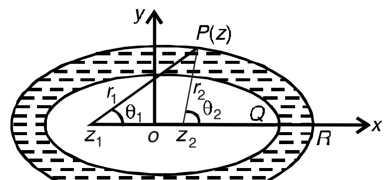
Integrating, $\; -\phi' + \phi = -(V + \tilde{w}/\rho) - C \quad$ or $\quad \phi' - \phi = V + \tilde{w}/\rho + C,$ ...(7)

where C is an arbitrary constant.

Thus, if V and $\tilde{w}$ be single valued, then from (7) we see that $\phi' - \phi$ is also single-valued. It follows that if ϕ be single valued, ϕ' must be single valued and if ϕ be many valued, ϕ' must also be many valued so that $\phi' - \phi$ is single-valued. Hence the required result follows.

Ex. 12. *Liquid of density ρ is flowing in two dimensions between the oval curves* $r_1r_2 = a^2$ *and* $r_1r_2 = b^2$ *where* r_1, r_2 *are the distances measured from two fixed points. If the motion is irrotational and quantity m per unit time crosses any line joining the bounding curves, then prove that the kinetic energy is* $(\pi\rho m^2)/\log(b/a)$.

Sol. Here we have two-dimensional irrotational motion in a region bounded by the given curves $r_1r_2 = a^2$ and $r_1r_2 = b^2$.



Let the complex potential w be

$$w = iE \log \{(z - z_1)(z - z_2)\}. \qquad ...(1)$$

But $\quad z - z_1 = r_1 e^{i\theta_1}$ and $z - z_2 = r_2 e^{i\theta_2}$

$$\therefore \quad (z - z_1)(z - z_2) = (r_1r_2)\, e^{i(\theta_1+\theta_2)} \qquad ...(2)$$

Using (2), (1) reduces to

$$\phi + i\psi = iE\,[\log (r_1r_2) + i(\theta_1 + \theta_2)].$$

$$\therefore \quad \phi = -E(\theta_1 + \theta_2) \quad \text{and} \quad \psi = E \log r_1r_2. \qquad ...(3)$$

Let the barrier be taken at $\theta_1 = \theta_2 = 0$. On the positive side $\theta_1 = \theta_2 = 0$ and hence $\phi = 0$. Again, on the negative side $\theta_1 = \theta_2 = 2\pi$ and so $\phi = 4\pi E$.

If k is the circulation, we have $k = 4\pi E$ so that $E = k/4\pi$.

$\therefore$ From (3), $\quad \phi = -(k/4\pi)(\theta_1 + \theta_2)$ and $\psi = (k/4\pi) \log r_1r_2$. ...(4)

Now, $\quad m = \psi_B - \psi_A = (k/4\pi) \log b^2 - (k/4\pi) \log a^2$, by (4)

or $\quad m = (k/4\pi) \log (b^2/a^2) = (k/4\pi) \log (b/a)^2 = (k/2\pi) \log (b/a)$

and so $\quad k = (2\pi m)/\log (b/a)$. ...(5)

Let T be required the kinetic energy. Then, we have

$$T = -\frac{1}{2}\rho k \iint \phi \frac{\partial \phi}{\partial n} dS = -\frac{1}{2}\rho km, \quad \text{as} \quad m = -\iint \phi \frac{\partial \phi}{\partial n} dS$$

$$= \frac{1}{2}\rho\, m \times \frac{2\pi m}{\log (b/a)} = \frac{\pi\rho m^2}{\log (b/a)}, \text{ using (5).}$$

Ex. 13. *Incompressible fluid of density ρ is contained between two co-axial circular cylinders, of radii a and b (a < b), and between two rigid planes perpendicular to the axis at a distance l apart. The cylinders are at rest and the fluid is circulating in irrotational motion, its velocity V at the surface of the inner cylinder. Prove that the kinetic energy is* $\pi\rho l a^2 V^2 \log (b/a)$.

Sol. For the case of irrotational two-dimensional fluid motion, we have

$$\nabla^2\psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \qquad ...(1)$$

Here ψ is function of r only. So $\quad \partial^2\psi/\partial\theta^2 = 0.$

Then (1) becomes $\quad \dfrac{d^2\psi}{dr^2} + \dfrac{1}{r}\dfrac{d\psi}{dr} = 0 \quad$ or $\quad r\dfrac{d^2\psi}{dr^2} + \dfrac{d\psi}{dr} = 0 \quad$ or $\quad d\{r(d\psi/dr)\} = 0.$

Integrating, $\quad r(d\psi/dr) = C \quad$ so that $\quad d\psi/dr = C/r.$ ...(2)

But given that $d\psi/dr = V$ when $r = a$. So (2) gives $V = C/a$ and hence $C = Va$.

Thus, $\quad$ transverse velocity $= d\psi/dr = (Va)/r.$

Again, the radial velocity is zero as $\partial\psi/\partial\theta = 0$, ψ being a function of r only.

Hence, $\quad q =$ resultant velocity $= (Va)/r.$

Let T be the required kinetic energy of the fluid. Then, we have

$$T = \int_a^b \frac{1}{2}(2\pi r\, dr\, l\rho)q^2 = \int_a^b \frac{1}{2}(2\pi r\, l\rho)\frac{V^2a^2}{r^2}\, dr = \pi\rho\, la^2V^2 \int_a^b \frac{1}{r} dr = \pi\,\rho l a^2 V^2 \log (b/a).$$

Ex. 14. *In a two-dimensional flow the velocity components are $u = Cy$, $v = 0$ (where C is a constant). Find the circulation about the circle $x^2 + y^2 - 2ay = 0$ situated in the flow.*

Sol. We know that $\Gamma = \Omega A$, ...(1)

where A = area of circular boundary of radius $a = \pi a^2$

and Ω = vorticity $= (\partial v / \partial x) - (\partial u / \partial y) = 0 - C = -C$

$\therefore$ $\Gamma = (-C) \times \pi a^2 = -C\pi a^2$ square meters per second.

EXERCISES

1. Show that the kinetic energy of a volume V of liquid of constant density ρ that is moving irrotationally with velocity potential ϕ is $-\dfrac{1}{2}\rho \displaystyle\int_S \phi \frac{\partial \phi}{\partial n} dS$,

where S denotes the surface of V and n the normal into the liquid. **[Meerut 2007]**

[**Hint.** Refer deduction **III** of Art 6.7]

2. Prove that the circulation in any closed path moving with the fluid is constatnt for all time, provided that the fluid is barotropic and the external forces are conservative. Deduce the theorem of the permanence of irrorational motion.

3. Prove that acyclic irrotatinal motion produces in an infinite liquid bounded intermally and externally by given velocities on the boundary may not cease when the boundaries are brought to rest.

4. State and prove uniqueness, theorem. **[Kanpur 2001, G.N.D.U. 1998]**

[**Solution : Statment of uniqueness theorem.** *There cannot be two diffeent forms of irrotational motion for a given confined mass of liquid when boundaries have prescribed velocities or are subject to given impulses*.

Proof. Give proofs of theorem I and theorem II given in Art. 6.9 B.

5. State and prove Kelvin's circulation theorem. Also prove that the irrotational motion is permanent. **[Agra 2008; Meerut 1999; 2002]**

[**Hint.** Refer Art. 6.4 and Art 6.5]

6. State and prove Kelvin's theorem of constancy of circulation. **[Meerut 2001, 02]**

[**Hint.** Refer Art 6.4]

7. State and prove Stokes' theorem for circulation. **[Agra 2005]**

OBJECTIVE QUESTIONS ON CHAPTER 6

Multiple choice questions

Choose the correct alternative from the following questions

1. Let C be a closed curve and Γ be the circulation, then

(*i*) $\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r}$ (*ii*) $\Gamma = \int_C \mathbf{q} \times d\mathbf{r}$ (*iii*) $\Gamma = \int_C |\mathbf{q}|\, dr$ (*iv*) None of these

2. In usual notations, Stoke's theorem is

(*i*) $\int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \times d\mathbf{S}$ (*ii*) $\int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \cdot dS$

(*iii*) $\int_C \mathbf{q} \times d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \cdot dS$ (*iv*) $\int_C \mathbf{q} \times d\mathbf{r} = \int_C \text{curl}\, \mathbf{q} \times dS$

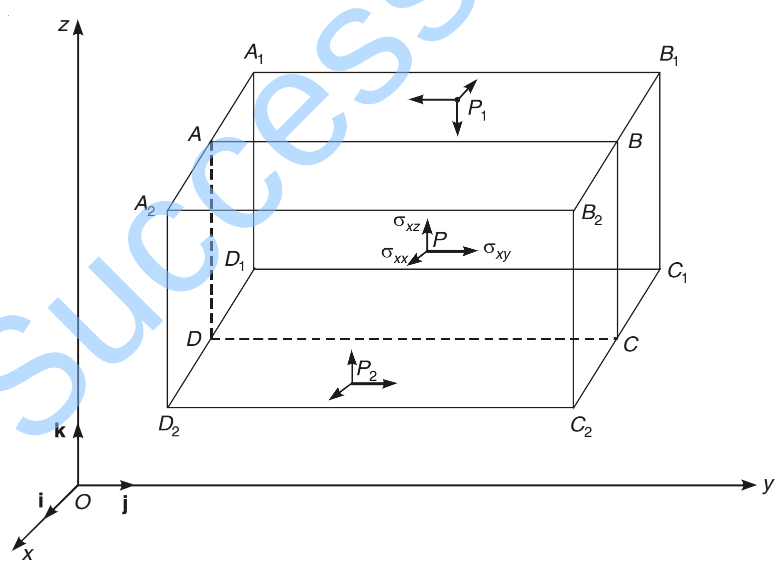
3. The motion in which the velocity potential is single–valued is called

(*i*) Laminar (*ii*) Turbulent (*iii*) Cyclic (*iv*) Acyclic **(Agra 2011)**

The Navier-Stokes Equations And The Energy Equation

14.1. The Navier-Stokes equations of motion of a viscous fluid. (Agra 2005, 07, 08, 09, 10, Garhwal 2005; Himanchal 2000, 01, 02, 03, 09; Kanpur 2004, 09; Meerut 2000, 01, 08, 09, 10, 12)

With $P(x, y, z)$ as centre and edges of lengths δx, δy, δz parallel to fixed coordinate axes, construct an elementary rectangular parallelepiped as shown in the figure. We consider the motion of above mentioned parallelepiped of viscous fluid. We suppose that the element is moving with the fluid and mass $\rho\,\delta x\delta y\delta z$ of the fluid remains constant. Let coordinates of points P_1 and P_2 be $(x-\delta x/2, y, z)$ and $(x+\delta x/2, y, z)$ respectively.



At P, the force components parallel to OX, OY, OZ on the rectangular surface $ABCD$ of area $\delta y\delta z$ through P and having $\mathbf{i}$ as unit normal are

$$[\sigma_{xx}\delta y\delta z, \qquad \sigma_{xy}\delta y\delta z, \qquad \sigma_{xz}\delta y\delta z].$$

At P_2, since $\mathbf{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_2B_2C_2D_2$ (parallel to $ABCD$) of area $\delta y\delta z$ are

$$\left[\left(\sigma_{xx}+\frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\delta z, \qquad \left(\sigma_{xy}+\frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\delta z, \qquad \left(\sigma_{xz}+\frac{\delta x}{2}\frac{\partial\sigma_{xz}}{\partial x}\right)\delta y\delta z\right] \quad ...(1)$$

At P_1, since $-\mathbf{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_1B_1C_1D_1$ (parallel to $ABCD$) of area $\delta y \delta z$ are.

$$\left[-\left(\sigma_{xx} - \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y \delta z, \qquad -\left(\sigma_{xy} - \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y \delta z, \qquad -\left(\sigma_{xz} - \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y \delta z\right] \quad ...(2)$$

Hence the forces on the parallel planes $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ passing through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial \sigma_{xx}}{\partial x}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{xy}}{\partial x}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{xz}}{\partial x}\delta x \delta y \delta z\right] \quad ...(3)$$

together with couples whose moments*. (to the third order of smallness) are

$$-\sigma_{xz}\delta x \delta y \delta z \text{ about } OY \qquad \text{and} \qquad \sigma_{xy}\delta x \delta y \delta z \text{ about } OZ. \quad ...(4)$$

Similarly, the forces on the parallel planes perpendicular to the y-axis are equivalent to a single force at P with components

$$\left[\frac{\partial \sigma_{yx}}{\partial y}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{yy}}{\partial y}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{yz}}{\partial y}\delta x \delta y \delta z\right] \quad ...(5)$$

together with couples whose moments are

$$-\sigma_{yx}\delta x \delta y \delta z \text{ about } OZ \qquad \text{and} \qquad \sigma_{yz}\delta x \delta y \delta z \text{ about } OX. \quad ...(6)$$

Again, the forces on the parallel planes perpendicular to the z-axis are equivalent to a single force at P with components

$$\left[\frac{\partial \sigma_{zx}}{\partial z}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{zy}}{\partial z}\delta x \delta y \delta z, \qquad \frac{\partial \sigma_{zz}}{\partial z}\delta x \delta y \delta z\right] \quad ...(7)$$

together with couples whose moments are

$$-\sigma_{zy}\delta x \delta y \delta z \text{ about } OX \qquad \text{and} \qquad \sigma_{zx}\delta x \delta y \delta z \text{ about } OY. \quad ...(8)$$

Thus, the surface forces on all the six faces of the rectangular parallelepiped ($A_1B_1C_1D_1$, $A_2B_2C_2D_2$) are equivalent to a single force at P having components

$$\left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x \delta y \delta z, \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right)\delta x \delta y \delta z, \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x \delta y \delta z\right] \quad ...(9)$$

together with a vector couple having components

$$\left[(\sigma_{yz} - \sigma_{zy})\,\delta x \delta y \delta z, \qquad (\sigma_{zx} - \sigma_{xz})\,\delta x \delta y \delta z, \qquad (\sigma_{xy} - \sigma_{yx})\,\delta x \delta y \delta z\right]. \quad ...(10)$$

Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_x\,\mathbf{k}$ ...(11)

be the velocity of the fluid at $P(x, y, z)$ at any time t and external body force at P per unit mass respectively.

Clearly the total body force on the elementary rectangular parallelepiped has components

---

* **Convention of sign of a couple.** If a couple in the plane XOY causes rotation from OX towards OY, then it shall be represented by a positive length along OZ. Similarly, a couple in the plane YOZ which would cause rotation from OY towards OZ will be represented by a positive length along OX and a couple in the plane ZOX causing rotation from OZ towards OX will be represented by a positive length along OY.

$$(B_x\, \rho\, \delta x \delta y \delta z, \qquad B_y\, \rho\, \delta x \delta y \delta z, \qquad B_z\, \rho\, \delta x \delta y \delta z).$$

Taking account of surface forces and body forces, we find that the total force component in the **i**-direction on the element of fluid under consideration is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x \delta y \delta z + B_x\, \rho\, \delta x \delta y \delta z.$$

Since the mass $\rho\, \delta x \delta y \delta z$ of the element is treated to be constant, the equation of motion of the element in the **i**-direction (*i.e.* OX) is

$$(\rho\, \delta x \delta y \delta z)\frac{Du}{Dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x \delta y \delta z + B_x\, \rho\, \delta x \delta y \delta z$$

or
$$\rho \frac{Du}{Dt} = \rho\, B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}.$$

Thus by cyclic permutation we obtain three equations of motion in the **i**, **j**, **k** directions (*i.e.* OX, OY, OZ) :

$$\rho \frac{Du}{Dt} = \rho\, B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \qquad ...(12a)$$

$$\rho \frac{Dv}{Dt} = \rho\, B_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \qquad ...(12b)$$

$$\rho \frac{Dw}{Dt} = \rho\, B_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \qquad ...(12c)$$

The constitutive equations for a Newtonian (viscous) compressible fluid are given by [refer equation (32*a*) to (32*f*) in Art. 13.14]

$$\left.\begin{aligned}
\sigma_{xx} &= 2\mu(\partial u/\partial x) - (2\mu/3)\times \nabla\cdot \mathbf{q} - p\\
\sigma_{yy} &= 2\mu(\partial v/\partial y) - (2\mu/3)\times \nabla\cdot \mathbf{q} - p\\
\sigma_{zz} &= 2\mu(\partial w/\partial z) - (2\mu/3)\times \nabla\cdot \mathbf{q} - p\\
\sigma_{xy} &= \sigma_{yx} = \mu\left(\partial u/\partial y + \partial v/\partial x\right)\\
\sigma_{yz} &= \sigma_{zy} = \mu\left(\partial v/\partial z + \partial w/\partial y\right)\\
\sigma_{zx} &= \sigma_{xz} = \mu\left(\partial w/\partial x + \partial u/\partial z\right)
\end{aligned}\right\} \qquad ...(13)$$

Using (13), equations (12*a*) to (12*c*) may be expressed in terms of the velocity derivatives as follows :

$$\rho \frac{Du}{Dt} = \rho\, B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot \mathbf{q})\right\}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad ...(14a)$$

$$\rho \frac{Dv}{Dt} = \rho\, B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla\cdot \mathbf{q})\right\}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad ...(14b)$$

$$\rho \frac{Dw}{Dt} = \rho\, B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla\cdot \mathbf{q})\right\}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad ...(14c)$$

The above three equations are called the *Navier-Stokes equations of motion for a viscous compressible fluid in cartesian coordinates.*

Particular Case: Incompressible viscous fluid flow. **(Himanchal 2007)**

The above system of equation (14*a*), (14*b*) and (14*c*) become further simplified in the case of incompressible fluids (ρ = constant) even if the temperature in not constant. First, as already shown in Art.2.8 we have $\nabla \cdot \mathbf{q} = 0.$ Secondly, since temperature variation are, generally, speaking, small in this case, the viscosity may be taken to be constant. Writing the acceteration terms in full, the equations of motion for incompressible flow are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho\, B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad ...(14a')$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho\, B_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad ...(14b')$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho\, B_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad ...(14c')$$

Deduction of equations of motion for some particular cases :

In what follows, we shall use the following result :

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad ...(15)$$

(*i*) Viscous compressible fluid with constant viscosity (Vector form) (Meerut 2000)

Let the coefficient of viscosity, μ, be constant. Then equations (14*a*) to (14*c*) may be expressed in vector form

$$\rho\left[\partial\mathbf{q}/dt + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = \rho\mathbf{B} - \nabla p + \mu\nabla^2\mathbf{q} + (\mu/3)\times\nabla(\nabla\cdot\mathbf{q}) \qquad ...(16)$$

Now, $\quad \mathbf{q}\times(\nabla\times\mathbf{q}) = \nabla(\mathbf{q}\cdot\mathbf{q}) - (\mathbf{q}\cdot\nabla)\mathbf{q} \quad$ or $\quad \mathbf{q}\times(\nabla\times\mathbf{q}) = \nabla(\mathbf{q}^2/2) - (\mathbf{q}\cdot\nabla)\mathbf{q}$

$$\therefore \qquad (\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(\mathbf{q}^2/2) - \mathbf{q}\times(\nabla\times\mathbf{q}) \qquad ...(i)$$

Again, $\quad \nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - (\nabla\cdot\nabla)\mathbf{q} \quad$ or $\quad \nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}$

$$\therefore \qquad \nabla^2\mathbf{q} = \nabla(\nabla\cdot\mathbf{q}) = -\nabla\times(\nabla\times\mathbf{q}) \qquad ...(ii)$$

Using (*i*) and (*ii*), (16) may be re-written as

$$\rho\left[\frac{\partial\mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times(\nabla\times\mathbf{q})\right] = \rho\mathbf{B} - \nabla p + \mu[\nabla(\nabla\cdot\mathbf{q}) - \nabla\times(\nabla\times\mathbf{q})] + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{q})$$

or

$$\rho\left[\frac{\partial\mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times(\nabla\times\mathbf{q})\right] = \rho\mathbf{B} - \nabla p + \frac{4}{3}\mu\nabla(\nabla\cdot\mathbf{q}) - \mu\nabla\times(\nabla\times\mathbf{q}). \qquad ...(16)'$$

(*ii*) Viscous incompressible fluid with constant viscosity. Let ρ and μ be constants for the given incompressible fluid. Further, for such a fluid $\nabla\cdot\mathbf{q} = 0.$ If $\nu = \mu/\rho$ be the kinematic viscosity, then (16) reduces to

$$\partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\nabla^2\mathbf{q}. \qquad ...(17)$$

(*iii*) Non-viscous incompressible fluid. For such fluid $\mu = 0$ and hence (16) further reduces to

$$\partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p, \qquad ...(18)$$

which is the well known Euler's equation* Note that (18) is valid for both incompressible flows and compressible flows. For incompressible flows ρ is constant while for compressible flows, ρ is usually a function of both pressure and temperature.

* It was obtained independently in Art. 3.1 of chapter 3. Refer equation (7) of that article.

For viscous incompressible fluid, $\nabla \cdot \mathbf{q} = 0$ and so (16)' reduces to

$$\rho\left[\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = \rho\mathbf{B} - \nabla p - \mu\nabla\times(\nabla\times\mathbf{q}). \qquad ...(18a)$$

where we have also used relation (*i*).

Comparing the above equation (18*a*) with (18), we find that for incompressible flow the equation of motion differs from Euler's equation of motion for non-viscous flow by the term $-\mu\nabla\times(\nabla\times\mathbf{q})$. This term, due to viscosity, increases the complexity by increasing the order of the differential equation of the motion. Hence an additional boundary condition is required. This is provided by the condition that there must be no slip between a viscous fluid and its boundary. It follows that we cannot arrive at the solution of the corresponding non-viscous flow problem by solving (18*a*) and then letting $\mu \to 0$.

(*iv*) **Plane two-dimensional flow of incompressible viscous fluid.**

Here we have $w = 0$ and $\partial/\partial z = 0$. Then (14*a*) to (14*c*) reduce to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho\, B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad ...(19a)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho\, B_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad ...(19a)$$

and
$$0 = \rho\, B_z. \qquad ...(19c)$$

14.2. The energy equation-conservation of energy. [Himanchal 1999, 2000, 03, 06]

Consider motion of a viscous compressible (Newtonian) fluid. We propose to consider conservation of energy on the basis of the first law of thermodynamics. According to this law the total energy added to the system (both by heat and by work done on the fluid) increases the internal energy per unit mass of the fluid. Let Q be the heat added per unit mass of fluid through conduction and E be internal energy per unit mass of fluid. Then the rate of work done W by the normal and shearing stresses on a unit volume of the fluid is given by

$$W = \sigma_{xx} \in_{xx} + \sigma_{yy} \in_{yy} + \sigma_{zz} \in_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx} \qquad ...(1)$$

Then the first law of thermodynamics (in terms of variation of energy) may be re-written as

$$\rho(dQ/dt) + W = \rho(dE/dt). \qquad ...(2)$$

The relations between the stresses and the rates of strain (constitutive equations) are given by [see (32*a*) to (32 *f*) in Art. 13.14]

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu \in_{xx} - (2/3)\times\mu\nabla\cdot\mathbf{q} - p \\ \sigma_{yy} &= 2\mu \in_{yy} - (2/3)\times\mu\nabla\cdot\mathbf{q} - p \\ \sigma_{zz} &= 2\mu \in_{zz} - (2/3)\times\mu\nabla\cdot\mathbf{q} - p \end{aligned}\right\} \qquad ...(3a)$$

and
$$\sigma_{xy} = \mu\,\gamma_{xy}, \qquad \sigma_{yz} = \mu\,\gamma_{yz}, \qquad \sigma_{zx} = \mu\,\gamma_{zx} \qquad ...(3b)$$

Also
$$\in_{xx} = \partial u/\partial x, \qquad \in_{yy} = \partial v/\partial y, \qquad \in_{zz} = \partial w/\partial z \qquad ...(4a)$$

and
$$\left.\begin{aligned} \gamma_{xy} &= \partial u/\partial y + \partial v/\partial x \\ \gamma_{yz} &= \partial v/\partial z + \partial w/\partial y \\ \gamma_{zx} &= \partial w/\partial x + \partial u/\partial z \end{aligned}\right\} \qquad ...(4b)$$

Using results (3*a*) and (3*b*), (1) reduces to

$W = \{2\mu \in_{xx} - (2/3)\times\mu\nabla\cdot\mathbf{q} - p\} \in_{xx}$ + two similar terms + $\mu\gamma_{xy}\cdot\gamma_{xy}$ + two similar terms

$= 2\mu(\in_{xx}^2 + \in_{yy}^2 + \in_{zz}^2) - (2/3)\times\mu(\nabla\cdot\mathbf{q})(\in_{xx} + \in_{yy} + \in_{zz}) - p(\in_{xx} + \in_{yy} + \in_{zz}) + \mu(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)$

$= -p(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + \mu[2\{\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\}$

$-(2/3)\times(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)^2 + (\partial u/\partial y + \partial v/\partial x)^2 + (\partial v/\partial z + \partial w/\partial y)^2 + (\partial w/\partial x + \partial u/\partial z)^2\}$

[Using 4 (*a*) and 4 (*b*) and the fact that $\nabla\cdot\mathbf{q} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\cdot(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$]

Thus, $$W = -p\nabla\cdot\mathbf{q} + \mathbf{\Phi}, \qquad ...(5)$$

where $\mathbf{\Phi}$ denotes the *dissipation function* and it represents the time rate at which energy is being dissipated per unit volume through the action of viscosity. Hence we have

$$\mathbf{\Phi} = \mu[\,2\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\} - (2/3)\times(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)^2$$
$$+(\partial u/\partial y + \partial v/\partial x)^2 + (\partial v/\partial z + \partial w/\partial y)^2 + (\partial w/\partial x + \partial u/\partial z)^2\,] \qquad ...\ (6)$$

Using (5), (2) reduces to $$\rho(dQ/dt) + \mathbf{\Phi} = \rho(dE/dt) + p\nabla\cdot\mathbf{q} \qquad ...(7)$$

The equation of continuity, for compressible (viscous) fluid is given by [Refer Equation (5) in Art. 2.7]

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{q} = 0 \quad \text{so that} \quad \frac{p}{\rho}\nabla\cdot\mathbf{q} = -\frac{p}{\rho^2}\frac{D\rho}{Dt} \qquad ...(8)$$

Now, $$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{D\rho}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}, \quad \text{so that} \quad -\frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{Dp}{Dt} \qquad ...(9)$$

From (8) and (9), we have

$$\frac{p}{\rho}\nabla\cdot\mathbf{q} = \frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{Dp}{Dt} \quad \text{or} \quad p\nabla\cdot\mathbf{q} = \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{Dp}{Dt} \qquad ...(10)$$

Using (10), (7) reduces to

$$\rho\frac{dQ}{dt} + \mathbf{\Phi} = \rho\frac{dE}{dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{Dp}{Dt} = \rho\frac{D}{Dt}\left(E + \frac{p}{\rho}\right) - \frac{Dp}{Dt} = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}, \qquad ...(11)$$

where $h = E + p/\rho$ is the *enthalpy** of the fluid per unit mass.

We now evaluate Q. According to the *Fourier's heat-conduction law, heat* flux f crossing an area (*i.e.*, quantity of heat per unit time) is proportional to the temperature gradient along the surface. Hence, $$f = -k(\partial T/\partial n),$$

where k is the thermal conductivity of the fluid, and the negative sign signifies that the direction of the flux is opposite to that of the temperature gradient.

Refer figure of Art 2.9, Chapter 2. Let there be a fluid particle at $P(x, y, z)$. Let T and ρ be the temperature and density of the fluid at P. Construct a small parallelepiped with edges of length parallel to their respective coordinate axes, having P at one of the angular points as shown in the figure just referred. Then we have

The heat flow through the face $PQRS$ per unit time $= -k(\partial T/\partial x)\,\delta y\delta z = f(x, y, z)$, ...(12)

* It is also known as the total heat content (heat introduced into the system).

$\therefore$ The heat flow through the opposite face $P'Q'R'S'$ per unit time

$$= f(x+\delta x, y, z) = f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots, \text{ by Taylor's theorem} \qquad ...(13)$$

Hence the net gain in energy per unit time within the fluid element in the x-direction (due to flow through faces $PQRS$ and $P'Q'R'S'$) from (12) and (13) is

$$= f(x,y,z) - \left[f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots \right] = -\delta x \frac{\partial}{\partial x} f(x,y,z)$$

[to the first order of approximation]

$$= -\delta x \frac{\partial}{\partial x}\left(-k \delta y \delta z \frac{\partial T}{\partial x} \right) = \delta x \delta y \delta z \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right)$$

Similarly, the net gains in energy per unit time within the fluid element in y- and z-directions are given by $\delta x \delta y \delta z \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right)$ and $\delta x \delta y \delta z \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right)$ respectively.

Hence the total quantity of heat introduced in the fluid element during time δt is

$$\delta t \delta x \delta y \delta z \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) \right].$$

Hence the rate of heat added by conduction per unit volume is given by

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) \qquad ...(14a)$$

i.e. $$\nabla \cdot (k \nabla T) \qquad ...(14b)$$

Thus, $$\rho(dQ/dt) = \nabla \cdot (k \nabla T) \qquad ...(15)$$

Using (15) and assuming that there is no direct heating from chemical reaction and radiation heating, the required energy equation from (11) is given by

$$\nabla \cdot (k \nabla T) + \mathbf{\Phi} = \rho(Dh/Dt) - (Dp/Dt). \qquad ...(16)$$

In cartesian coordinates the energy equation for viscous compressible fluid reduces to

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) + \mathbf{\Phi} = \rho \frac{D(C_p T)}{Dt} - \frac{Dp}{Dt}, \qquad ...(17)$$

where $k = c_p\ T$ and c_p is specific heat at constant pressure. Using the kinetic theory of gases together with experiments, μ and k are found to be functions of the temperature only for gases having ordinary densities.

Energy equation for special cases :

(*i*) **Viscous incompressible fluid.** **[Himanchal 2003, 09]**

When the fluid is taken as incompressible viscous fluid, then k = constant and μ = constant. Further-more, the equation of continuity for such a fluid is given by

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \qquad ...(18)$$

Hence the dissipation function $\mathbf{\Phi}'$ for the present problem is given by [on using (6)]

$$\mathbf{\Phi}' = \mu \left[2\left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] \qquad ...(19)$$

If c_v be the specific heat at constant volume, then $c_p = c_v = c$ for an incompressible fluid. Here c is the specific heat of the fluid. With the above mentioned discussion, the energy equation

(17) assumes the form

$$k(\partial^2 T/dx^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2) + \mathbf{\Phi}' = \rho c(DT/Dt) - (Dp/Dt) \qquad ...(20)$$

(*ii*) **Non-viscous fluid.** Since $\mu = 0$ for such fluids, $\mathbf{\Phi} = 0$ by (6). Then (17) yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho\frac{D(C_p T)}{Dt} - \frac{Dp}{Dt} \qquad ...(21)$$

(*iii*) **Non-viscous incompressible fluid.** As before k = constant and $\mu = 0$ (hence $\mathbf{\Phi} = 0$). Also $c_p = c_v = c$. Hence the energy equation (21) assumes the following form

$$k(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2) = \rho c\,(DT/Dt) - (Dp/Dt) \qquad ...(22)$$

14.3. Equation of state for perfect fluid.

The equation of state of a substance is a relation between its pressure, temperature and specific volume. There exist an equation of state corresponding to a given homogeneous substance, solid, liquid or gas. The relationship may be expresses as

$$f(\rho, p, T) = 0, \qquad ...(1)$$

which is known as the *equation of state*. The exact nature of the function f is, in general, very complicated and varies from fluid to fluid. However, for a perfect gas or an ideal gas the equation of state is given by

$$p = \rho R T \qquad ...(2)$$

or

$$p = (c_p - c_v)\rho T, \qquad ...(3)$$

where R is called the *gas constant* and c_p and c_v are specific heats at constant pressure and volume respectively. Relation (2) is also known as *Boyle's law*.

14.4. Diffusion of vorticity. [Agra 2005, 06; Kolkata 2006; Himanchal 2001]

The Navier-stokes equation for viscous incompressible fluid is given by

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times\mathbf{\Omega} = \mathbf{B} - \nabla\int\frac{dp}{\rho} + \nu\nabla^2\mathbf{q} \qquad ...(1)$$

where $\mathbf{\Omega}$ = vorticity vector = $\nabla\times\mathbf{q}$. ...(2)

Let the body forces be conservative so that $\nabla\times\mathbf{B} = \mathbf{0}$. ...(3)

On taking the curl of both sides of (1) and using (3), we obtain

$$\nabla\times(\partial\mathbf{q}/\partial t) - \nabla\times(\mathbf{q}\times\mathbf{\Omega}) = \nu\nabla\times(\nabla^2\mathbf{q})$$

or

$$(\partial/\partial t)\,(\nabla\times\mathbf{q}) - [(\mathbf{\Omega}\cdot\nabla)\,\mathbf{q} - (\mathbf{q}\cdot\nabla)\,\mathbf{\Omega}] = \nu\nabla^2(\nabla\times\mathbf{q})$$

or

$$\partial\mathbf{\Omega}/\partial t + (\mathbf{q}\cdot\nabla)\,\mathbf{\Omega} = (\mathbf{\Omega}\cdot\nabla)\,\mathbf{q} + \nu\,(\nabla^2\,\mathbf{\Omega}), \text{ using (2)}$$

or

$$D\mathbf{\Omega}/Dt = (\mathbf{\Omega}\cdot\nabla)\,\mathbf{q} + \nu\nabla^2\mathbf{\Omega}, \qquad ...(4)$$

which is known as *vorticity transport equation*.

The first term on *R.H.S* of (4) represents the rate at which $\mathbf{\Omega}$ varies for a given particle when the vortex lines move with the fluid, strengths of the vortices remaining constant. Since this term is negligible for slow motion, approximate form of (4) is

$$D\mathbf{\Omega}/Dt = \nu\nabla^2\mathbf{\Omega}. \qquad ...(5)$$

In the special case of two-dimensional flow, with reference to fixes axes, we have

$$\mathbf{q} = u(x, y)\,\mathbf{i} + v(x, y)\,\mathbf{j}$$

Then $\Omega = \nabla\times\mathbf{q} = (\partial v/\partial x - \partial u/\partial y)\,\mathbf{k}$ and $(\mathbf{\Omega}\cdot\nabla)\,\mathbf{q} = (\partial v/dx - \partial u/\partial y)(d\mathbf{q}/\partial z) = \mathbf{0}$, showing that (4) reduces to (5) for a two dimensional case.

It follows that for slow three-dimensional motion, or for two-dimensional motion, (3) describes the manner in which vorticity is transmitted throughout a viscous fluid.

Remark Equation (5) is of the same form as the equation of heat conduction in a liquid. Hence vorticity diffuses through a liquid in almost the same way as heat does. By analogy it follows that vorticity cannot be generated within the interior of a viscous fluid. In fact it is transmitted from the boundaries into the fluid. As an example, a sailing ship will generate vortices in its wake, arising from the hull which is a moving boundary. As time passes, the disturbance is soon damped out as the vortices diffuse through the water.

14.5. Equations for vorticity and circulation. To prove that $d\Gamma/dt = \nu\nabla^2\Gamma$

(Himanchal 2003, 09; Meerut 1998)

The equations of motion for viscous fluid are given by

$$\left.\begin{aligned} \frac{Du}{Dt} &= -\frac{\partial Q}{\partial x} + \frac{1}{3}\nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 u \\ \frac{Dv}{Dt} &= -\frac{\partial Q}{\partial y} + \frac{1}{3}\nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 v \\ \frac{Dw}{Dt} &= -\frac{\partial Q}{\partial z} + \frac{1}{3}\nu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\nu\nabla^2 w \end{aligned}\right\} \qquad ...(1)$$

where $$Q = V + \int \frac{dp}{\rho}.$$

The above equations (1) can also be re-written as

$$Du/Dt - 2(v\xi - w\eta) = -(\partial\chi/\partial x) + \nu\,\nabla^2 u \qquad ...(2)$$

$$Dv/Dt - 2(w\xi - u\zeta) = -(\partial\chi/\partial y) + \nu\,\nabla^2 v \qquad ...(3)$$

$$Dw/Dt - 2(u\eta - v\xi) = -(\partial\chi/\partial z) + \nu\,\nabla^2 w \qquad ...(4)$$

where $$\chi = p/\rho + q^2/2 + V.$$

Differentiating (3) and (4) partially *w.r.t.*'z', and 'y' respectively and subtracting, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) - 2\left[\frac{\partial}{\partial y}(u\eta - v\xi) - \frac{\partial}{\partial z}(w\xi - u\zeta)\right] = \nu\nabla^2\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

or $$\frac{\partial\xi}{\partial t} - \left[\eta\frac{\partial u}{\partial y} + u\frac{\partial\eta}{\partial y} - \xi\frac{\partial v}{\partial y} - v\frac{\partial\xi}{\partial y} - \xi\frac{\partial w}{\partial z} - w\frac{\partial\xi}{\partial z} + \zeta\frac{\partial u}{\partial z} + u\frac{\partial\zeta}{\partial z}\right] = \nu\nabla^2\xi$$

or $$\frac{\partial\xi}{\partial t} + \xi\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} - u\left(\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right) + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} = \nu\nabla^2\xi$$

or $$\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} - \xi\frac{\partial u}{\partial x} - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} = \nu\nabla^2\xi$$

or $$\frac{D\xi}{Dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\,\nabla^2\xi \qquad ...(5)$$

Similarly, $$\frac{D\eta}{Dt} = \xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z} + \nu\,\nabla^2\eta \qquad ...(6)$$

$$\frac{D\zeta}{Dt} = \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z} + \nu\,\nabla^2\zeta \qquad ...(7)$$

The first three terms on the R.H.S. in equations (5), (6) and (7) represent the rates at which ξ, η, ζ vary for a given particle, when the vortex lines move with the fluid and their strengths remain constant. When the motion is very slow, these terms can be neglected and the remaining terms give the variations of vorticity. Since the resulting equations are the same in form as standard equation of conduction of heat, hence as in conduction of heat, we can say that vortex-motion cannot originate in the interior of a viscous liquid but must be diffused inwards from the boundary.

Let Γ be the circulation round a closed circuit moving with the fluid. Then, we have

$$\Gamma = \int_C (udx + vdy + wdz)$$

$$\therefore \qquad \frac{D\Gamma}{Dt} = \int_C \left(\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz\right) + \int_C (u\,du + v\,dv + w\,dw) \qquad ...(8)$$

Since circulation is taken round a closed circuit, the second integral on R.H.S. in (8) is zero

$\therefore$ Hence,
$$\frac{D\Gamma}{Dt} = \int_C \left(\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz\right) \qquad ...(9)$$

Now, from equation (1),
$$D\mathbf{q}/Dt = -\nabla\left(p/\rho + V\right) + \nu\,\nabla^2\mathbf{q} \qquad ...(10)$$

From (9) and (10), we get

$$\frac{D\Gamma}{Dt} = \int_C \left\{-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+V\right)dx - \frac{\partial}{\partial y}\left(\frac{p}{\rho}+V\right)dy - \frac{\partial}{\partial z}\left(\frac{p}{\rho}+V\right)dz + \nu(\nabla^2 u\,dx + \nabla^2 v\,dy + \nabla^2 w\,dz)\right]$$

or
$$\frac{D\Gamma}{Dt} = -\int_C d\left(\frac{p}{\rho}+V\right) + \nu\int_C \nabla^2(udx + vdy + wdz)$$

$$D\Gamma/Dt = \nu\,\nabla^2\Gamma. \qquad [\because \text{ the first integral is zero for a closed circuit}]$$

14.6A. Dissipation of energy. definition.

Dissipation of energy is that energy which is dissipated in a viscous liquid in motion on account of the internal friction.

Determination of the rate of dissipation of energy of a fluid due to viscosity.

[Agra 2007; Meerut 2005]

Suppose we follow a particle of viscous incompressible fluid of fixed mass $\rho\,\delta V$ and moving with velocity $\mathbf{q}$ at any time t. Then its kinetic energy is $(1/2)\times(\rho\delta V)\mathbf{q}^2$. Hence the rate of gain of kinetic energy at time t as we follow the particle is given by

$$\frac{D}{Dt}\left(\frac{1}{2}\rho\,\delta V\,\mathbf{q}^2\right) = \rho\,\delta V\,\mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}.$$

Let the total volume be V and S be the total surface enclosing the volume V.

Hence the total rate of gain of kinetic energy dT/dt. (say), of the total volume V, is given by

$$\frac{dT}{dt} = \int_V \rho\,\mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}dV = \rho\int_V \mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}dV. \qquad ...(1)$$

The Navier-Stokes equation for viscous incompressible fluid is given by [refer equation

(17) in Art. 14.1) $$\frac{D\mathbf{q}}{Dt} = \mathbf{B} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q},$$

where **B** is the body force. Using this relation, (1) reduces to

$$\frac{dT}{dt} = \rho\int_V \mathbf{q}\cdot\left[\mathbf{B} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q}\right]dV$$

or $$\frac{dT}{dt} = \int_V \mathbf{q}\cdot(\rho\mathbf{B})\,dV - \int_S p\,\mathbf{q}\cdot\mathbf{n}\,dS + \rho\int_V (\mathbf{q}\cdot\frac{\mu}{\rho}\nabla^2\mathbf{q})\,dV \qquad ...(2)$$

[On using Gauss divergence theorem in the second term]

The first term in R.H.S. of (2) represents the rate at which the external force **B** is doing work throughout the mass of the liquid while the second term on R.H.S. of (2) represents the rate at which the pressure is doing work on the boundary. It follows that for an ideal fluid $(\mu = 0)$, the rate of increase of kinetic energy equals the rate at which work is done by the body forces and pressures at the boundary. Hence, if D is the rate of dissipation of energy due to viscosity, then by virtue of (2), we have

$$D = -\mu\int_V (\mathbf{q}\cdot\nabla^2\mathbf{q})\,dV. \qquad ...(3)$$

Let Ω denote the vorticity vector. Then $$\mathbf{\Omega} = \nabla\times\mathbf{q} \qquad ...(4)$$

Now, consider the vector identity $$\nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}. \qquad ...(5)$$

Since the fluid is incompressible, we have $$\nabla\cdot\mathbf{q} = 0 \qquad ...(6)$$

Using (4) and (6), (5) reduces to

$$\nabla\times\mathbf{\Omega} = -\nabla^2\mathbf{q} \quad \text{so that} \quad \nabla^2\mathbf{q} = -\nabla\times\mathbf{\Omega} \qquad ...(7)$$

From (7), $$\mathbf{q}\cdot\nabla^2\mathbf{q} = -\mathbf{q}\cdot(\nabla\times\mathbf{\Omega}) \qquad ...(8)$$

Now, $$\nabla\cdot(\mathbf{q}\times\mathbf{\Omega}) = \mathbf{\Omega}\cdot(\nabla\times\mathbf{q}) - \mathbf{q}\cdot(\nabla\times\mathbf{\Omega}) = \mathbf{\Omega}\cdot\mathbf{\Omega} - \mathbf{q}\cdot(\nabla\times\mathbf{\Omega}), \quad \text{by (4)}$$

$$\Rightarrow \nabla\cdot(\mathbf{q}\times\mathbf{\Omega}) = \mathbf{\Omega}^2 - \mathbf{q}\cdot(\nabla\times\mathbf{\Omega}) \quad \text{or} \quad -\mathbf{q}\cdot(\nabla\times\mathbf{\Omega}) = \nabla\cdot(\mathbf{q}\times\mathbf{\Omega}) - \mathbf{\Omega}^2. \qquad ...(9)$$

From (8) and (9), we have $$\mathbf{q}\cdot\nabla^2\mathbf{q} = \nabla\cdot(\mathbf{q}\times\mathbf{\Omega}) - \mathbf{\Omega}^2. \qquad ...(10)$$

Using (10), (3) reduces to

$$D = -\mu\int_V [\nabla\cdot(\mathbf{q}\cdot\mathbf{\Omega}) - \mathbf{\Omega}^2]\,dV \quad \text{or} \quad D = \mu\int_V \mathbf{\Omega}^2 dV - \mu\int_V \nabla\cdot(\mathbf{q}\cdot\mathbf{\Omega})\,dV$$

$$\therefore \quad D = \mu\int_V \mathbf{\Omega}^2 dV - \mu\int_S \mathbf{n}\cdot(\mathbf{q}\cdot\mathbf{\Omega})\,dS, \text{ by Gauss divergence theorem} \qquad ...(11)$$

In case the boundary is at rest and there is no slip between fluid and boundary so that **q** = **0** on S, then (11) reduces to

$$D = \mu\int_V \mathbf{\Omega}^2 dV \quad \text{or} \quad D = 4\mu\int_V (\xi^2 + \eta^2 + \zeta^2)\,dxdydz,$$

where ξ, η, ζ are components of vorticity vector $\mathbf{\Omega}$, *i.e.*,

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

14.6 B. Dissipation of energy (cartesian form).

[Agra 1999; 2006; Allahabad 2002; Garhwal 2000; Kanpur 1997; 1998; Kolkata 2000]

The kinetic energy T at time t of a portion of fluid bounded by S is given by

$$T = \frac{1}{2}\iiint \rho(u^2 + v^2 + w^2)\, dx\, dy\, dz \qquad ...(1)$$

Hence, differentiating following the motion of the same portion, we have

$$\frac{DT}{Dt} = \iiint \rho\left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right) dx\, dy\, dz \qquad ...(2)$$

Navier-Stokes equations of motion (refer Art. 14.1) are given by

$$\left.\begin{aligned} \rho\frac{Du}{Dt} &= \rho B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho\frac{Dv}{Dt} &= \rho B_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \rho\frac{Dw}{Dt} &= \rho B_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{aligned}\right\} \qquad ...(3)$$

Using (3), (2) reduces to

$$\frac{DT}{Dt} = \iiint \rho(uB_x + vB_y + wB_z)\, dxdydz$$

$$+\iiint\left[u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + v\left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right) + w\left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\right] dxdydz \qquad ...(4)$$

The first term on R.H.S. of (4) represents the rate at which the external forces are doing work throughout the mass of the fluid. The second term on R.H.S. of (4) may be re-written as

$$\iiint\left[\frac{\partial}{\partial x}(u\,\sigma_{xx}) + \frac{\partial}{\partial y}(u\,\sigma_{xy}) + \frac{\partial}{\partial z}(u\,\sigma_{xz}) + \ldots - \left\{\frac{\partial u}{\partial x}\sigma_{xx} + \frac{\partial u}{\partial y}\sigma_{xy} + \frac{\partial u}{\partial z}\sigma_{xz} + \ldots\right\}\right] dxdydz$$

$$= -\iint [u(l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz}) + v(l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz}) + w(l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz})] dS$$

$$-\iiint\left[\frac{\partial u}{\partial x}\sigma_{xx} + \frac{\partial v}{\partial y}\sigma_{yy} + \frac{\partial w}{\partial z}\sigma_{zz} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sigma_{xy} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\sigma_{yz} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\sigma_{zx}\right] dxdydz, \qquad ...(5)$$

where l, m, n are the direction cosines of the inward drawn normal to dS.

Now, we know that (refer Art. 13.5)

$$\left.\begin{aligned} \sigma_{nx} &= l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz} \\ \sigma_{ny} &= l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz} \\ \sigma_{nz} &= l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz} \end{aligned}\right\} \qquad ...(6)$$

Using (6), the first integral in (5) may be re-written as

$$-\iint (u\sigma_{nx} + v\sigma_{ny} + w\sigma_{nz})\, dS, \qquad ...(7)$$

where the suffix n indicates a normal to dS, and this integral represents the rate at which the kinetic energy is being increased by the action of the stresses on the boundary of the fluid.

Using the constitutive equations for an incompressible viscous fluid $(\nabla \cdot \mathbf{q} = 0)$, we have (refer Art. 13.12 and Art 13.14).

$$\left.\begin{aligned}\sigma_{xx} &= -p + 2\mu \in_{xx} = -p + 2\mu(\partial u/\partial x)\\ \sigma_{yy} &= -p + 2\mu \in_{yy} = -p + 2\mu(\partial v/\partial y)\\ \sigma_{zz} &= -p + 2\mu \in_{zz} = -p + 2\mu(\partial w/\partial z)\\ \sigma_{xy} &= 2\mu \in_{xy} = \partial u/\partial y + \partial v/\partial x\\ \sigma_{yz} &= 2\mu \in_{yz} = \partial v/\partial z + \partial w/\partial y\\ \sigma_{zx} &= 2\mu \in_{zx} = \partial w/\partial x + \partial u/\partial z\end{aligned}\right\} \quad ...(8)$$

Again, the equation of continuity $\nabla \cdot \mathbf{q} = 0$ may be written as

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad ...(9)$$

Using (8), the second term on R.H.S. of (5)

$$= \iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) - \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right] dxdydz$$

$$= -\iiint \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\} dxdydz, \text{ by (9)}$$

$$= -\iiint \mathbf{\Phi}\, dxdydz, \quad ...(10)$$

where $$\mathbf{\Phi} = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}, \quad ...(11)$$

$\mathbf{\Phi}$ is called the *dissipation function.*

$\therefore$ The rate of dissipation of energy $= \iiint \mathbf{\Phi}\, dxdydz$

$$= \mu \iiint\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\} dxdydz, \text{ by (11)}$$

$$= \mu \iiint\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 - 2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right\} dxdydz, \text{ using (9)},$$

$$= \mu \iiint\left\{\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2 + 4\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial v}{\partial y}\right) + 4\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right) + 4\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right) dxdydz \quad ...(12)$$

Now, $\iiint \left(\frac{\partial w}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial z} \frac{\partial v}{\partial y} \right) dxdydz$

$$= \iiint \left\{ \frac{\partial}{\partial y} \left(w \frac{\partial v}{\partial z} \right) - w \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial}{\partial z} \left(w \frac{\partial v}{\partial y} \right) + w \frac{\partial^2 v}{\partial y \partial z} \right\} dxdydz = - \iint \left(mw \frac{\partial v}{\partial z} - nw \frac{\partial v}{\partial y} \right) dS$$

If $w = 0$ on the boundary, then $$\iiint \left(\frac{\partial w}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial z} \frac{\partial v}{\partial y} \right) dxdydz = 0 \qquad ...(13)$$

Similarly, $$\iiint \left(\frac{\partial u}{\partial z} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial w}{\partial z} \right) dxdydz = 0 \qquad ...(14)$$

and $$\iiint \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} \right) dxdydz = 0 \qquad ...(15)$$

If , $\xi, \eta, \zeta,$ are the components of the vorticity vector, then we know that

$$\xi = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \qquad \eta = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \qquad \zeta = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \qquad ...(16)$$

Using (13), (14), (15) and (16), (12) reduces to

The rate of dissipation of energy $= 4\mu \iiint (\xi^2 + \eta^2 + \zeta^2)\, dxdydz.$ ...(17)

This is the rate of dissipation of energy for a liquid filling a closed vessel.

Remark As an application of the Art. 14.6. B, consider the following example.

Illustrative solved example. *Prove that for a liquid filing up a vessel in the form of a surface of revolution which is rotating about its axis (z-axis) with the angular velocity* ω *the rate of dissipation of energy has an additional term*

$$2\mu\omega \iiint (l\, Du + m\, Dv)\, dS, \qquad \text{where} \qquad D = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right)$$

and l, m, n are the direction cosines of the inward drawn normal at the element dS of the surface of the vessel. **[Meerut 2004]**

Sol. First do the whole Art. 14.6.B. Next, we have, here

$$u = -\omega y, \qquad v = \omega x, \qquad w = 0. \qquad ...(18)$$

Hence as in the above article 12.7 B, the integral on **L.H.S.** of (15) will not vanish. So we have an additional term which will be calculated as follows.

The additional term

$$= 4\mu \iiint \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) dxdydz = 4\mu \iiint \left\{ \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} \right) \right\} dxdydz$$

$$= -4\mu \iint \left(lv \frac{\partial u}{\partial y} - mv \frac{\partial u}{\partial x} \right) dS = -4\mu \iint v \left(l \frac{\partial u}{\partial y} + m \frac{\partial v}{\partial y} \right) dS$$

[$\because$ For two-dimensional case, equation of continuity is $\partial u / \partial x + \partial v / \partial y = 0$]

$$= -4\mu\omega \iint x \left(l \frac{\partial u}{\partial y} + m \frac{\partial v}{\partial y} \right) dS, \text{ using (18)} \qquad ...(19)$$

Similarly, the same expression

$$= 4\mu \iiint \left\{ \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) \right\} dxdydz = -4\mu \iint \left(mu\frac{\partial v}{\partial x} - lu\frac{\partial v}{\partial y} \right) dS$$

$$= -4\mu \iint u\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x} \right) dS \qquad [\because \ \partial u/\partial x + \partial v/\partial y = 0 \text{ as before }]$$

$$= 4\mu\omega \iint y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x} \right) dS \text{, using (18)} \qquad \text{...(20)}$$

Taking the mean of the above two expressions (19) and (20), the additional term

$$= 2\mu\omega \iint \left\{ y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x} \right) - x\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y} \right) \right\} dS$$

$$= 2\mu\omega \iint \left\{ l\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right)u + m\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right)v \right\} dS$$

$$= 2\mu\,\omega \iint (l\,Du + m\,Dv)\,dS, \text{ since } D = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \text{ (given)}$$

14.7 Illustrative solved examples.

The reader is advised to study carefully and remember all results of articles 14.10, 14.11 and 14.12. These results can be used directly while solving problems.

Ex. 1. *Show that for an incompressible steady flow with constant viscosity, the velocity components* $u(y) = y\dfrac{U}{h} + \dfrac{h^2}{2\mu}\left(-\dfrac{dp}{dx}\right)\dfrac{y}{h}\left(1 - \dfrac{y}{h}\right)$, $v = w = 0$ *satisfy the equation of motion, when the body force is neglected. h, U, dp/dx are constants and* $p = p(x)$. **(Meerut 2007, 11)**

Sol. Given $\qquad u(y) = (yU/h) + (h^2/2\mu)(-dp/dx)(y/h)(1 - y/h) \qquad$ ...(1)

$v = 0 \qquad$ and $\qquad w = 0 \qquad$ ...(2)

The equation of motion for viscous incompressible fluid is given by

$$\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q} \qquad \text{...(3)}$$

Here $\qquad \partial \mathbf{q}/\partial t = \mathbf{0}$, the motion being steady. $\qquad$ ...(4)

and $\qquad \mathbf{B} = \mathbf{0}$, as the body force is neglected. $\qquad$ ...(5)

Since $\quad v = w = 0, \qquad$ we have $\qquad \mathbf{q} = \mathbf{i}u$

$\therefore \qquad \nabla^2\mathbf{q} = \mathbf{i}\nabla^2 u \qquad$ ...(6)

Given that $p = p(x)$ so that $\qquad \nabla p = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z} \right) p = \mathbf{i}\dfrac{dp}{dx} \qquad$ ...(7)

Also $\qquad \mathbf{q}\cdot\nabla = (\mathbf{i}u)\cdot\left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z} \right) = u\dfrac{\partial}{\partial x}$

$\therefore \qquad (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \left(u\dfrac{\partial}{\partial x} \right)(\mathbf{i}u) = \mathbf{i}u\dfrac{\partial u}{\partial x} = 0$, as $u = u(y)$, given $\qquad$ ...(8)

Substituting (4), (5), (6), (7) and (8) into (3), we have

$$0 = -\frac{1}{p}\frac{dp}{dx} + \nu\nabla^2 u \quad \text{or} \quad \frac{1}{p}\frac{dp}{dx} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}. \qquad \text{...(9)}$$

$$\left[\because \nu = \mu/\rho \text{ and } \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\right]$$

Now from (1), $$\frac{du}{dy} = \frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}\left(1 - \frac{2y}{h}\right),$$ as dp/dx is given to be constant

$\therefore$ $$\frac{d^2u}{dy^2} = 0 - \frac{h}{2\mu}\frac{dp}{dx}\left(-\frac{2}{h}\right) = \frac{1}{\mu}\frac{dp}{dx},$$

which is the same as (9). This proves that the equation of motion is satisfied.

Ex. 2. *Consider an inviscid, incompressible, steady flow with negligible body force whose velocity components are* $q_r = U(1 - R^3/r^3)\cos\theta$, $q_\theta = -U(1 + R^3/2r^3)\sin\theta$, $q_\phi = 0$ *in spherical coordinates where R is a constant. Is the equation of motion satisfied.*

Sol. Here $$q_r = U(1 - R^3/r^3)\cos\theta \qquad \ldots(1)$$

$$q_\theta = -U(1 + 2R^3/r^3)\sin\theta \qquad \ldots(2)$$

and $$q_\phi = 0 \qquad \ldots(3)$$

Equations of motion for non-viscous fluid in absence of body forces are given by [Refer equations (11a) to (11c) and equation (2) of Art. 14.12 of this chapter]

$$\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(4)$$

$$\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} - \frac{q_\phi^2 \cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(5)$$

$$\frac{Dq_\phi}{Dt} + \frac{q_\phi q_r}{r} + \frac{q_\theta q_\phi \cot\theta}{r} = -\frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial \phi} \qquad \ldots(6)$$

Here $$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \qquad \ldots(7)$$

For steady flow, $\partial/\partial t = 0$, Also, here $q_\phi = 0$. Hence (4), (5) and (6) may be rewritten as

$$q_r\frac{\partial q_r}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(8)$$

$$q_r\frac{\partial q_\theta}{\partial r} + \frac{q_\theta}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r q_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(9)$$

$$0 = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} \qquad \ldots(10)$$

Equation (10) shows that p is function of r and θ alone. From (1) and (2), we have

$$\frac{\partial q_r}{\partial r} = \frac{3UR^3}{r^4}\cos\theta \quad \text{and} \quad \frac{\partial q_r}{\partial \theta} = -U\left(1 - \frac{R^3}{r^3}\right)\sin\theta \qquad \ldots(11)$$

$$\frac{\partial q_\theta}{\partial r} = \frac{3UR^3}{2r^4}\cos\theta \quad \text{and} \quad \frac{\partial q_\theta}{\partial \theta} = -U\left(1 - \frac{R^3}{2r^3}\right)\cos\theta \qquad \ldots(12)$$

Using (1), (2) and (11), (8), reduces to

$$\frac{3U^2R^3}{r^4}\left(1 - \frac{R^3}{r^3}\right)\cos^2\theta + \frac{U^2}{r}\left(1 - \frac{R^3}{r^3}\right)\left(1 + \frac{R^3}{2r^3}\right)\sin^2\theta - \frac{U^2}{r}\left(1 + \frac{R^3}{2r^3}\right)^2\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta+\frac{U^2}{r}\sin^2\theta\left[\left(1-\frac{R^3}{r^3}\right)\left(1+\frac{R^3}{2r^3}\right)-\left(1+\frac{R^3}{2r^3}\right)^2\right]=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta-\frac{3U^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(13)$$

Using (1), (2) and (12) in (9) and proceeding as above, the reader can verify that

$$\frac{3U^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta+\frac{3U^3R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(14)$$

Differentiating (13) partially with respect to θ, we get

$$-\frac{6U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta-\frac{3U^2R^3}{r^4}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$\frac{9U^2R^3}{r^4}\left(\frac{R^3}{2r^3}-1\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta} \qquad ...(15)$$

Finally, differentiating (14) partially with respect to r, we get

$$\frac{3}{2}U^2R^3\left(-\frac{3}{r^4}+\frac{6R^3}{r^7}\right)\sin\theta\cos\theta+\frac{3}{2}U^2R^3\left(-\frac{3}{r^4}-\frac{6R^3}{2r^7}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$-\frac{9}{2}\frac{U^2R^3}{r^4}\left(1-\frac{2R^3}{r^3}\right)\sin\theta\cos\theta-\frac{9}{2}\frac{U^2R^3}{r^4}\left(1+\frac{R^3}{r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$\frac{9U^2R^3}{r^4}\left(\frac{R^3}{2r^3}-1\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta} \qquad ...(16)$$

Since (15) and (16) are identical, it follows that the equations of motion are satisfied.

Ex. 3 (*a*) *Define circulation. Show that the time rate of change of circulation in a closed circuit, drawn in a viscous incompressible fluid under the action of conservative forces, moving with the fluid depends only on the kinematic viscosity and the space rate of change of vorticity components at the the contour. Hence state and prove Kelvin's circulation theorem.*

[Himanchal 1998; 2003]

(*b*) *Derive the time rate of change of circulation of a closed curve drawn in a viscous incmpressible fluid, moving with the fluid.* **[Himanchal 2002, 05, 07]**

Sol. (*a*) Let C be a closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\boldsymbol{q}$ be the fluid velocity at any point P of circuit and let $\boldsymbol{r}$ be its position vector. Then the circulation along the closed C is given by

$$\Gamma=\oint_C \mathbf{q}\cdot d\boldsymbol{r} \qquad \text{so that} \qquad \frac{D\Gamma}{Dt}=\frac{D}{Dt}\oint_C \mathbf{q}\cdot d\boldsymbol{r} \qquad ...(1)$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt}=\oint_C\frac{D}{Dt}(\mathbf{q}\cdot d\boldsymbol{r}) \qquad ...(2)$$

But $$\frac{D}{Dt}(\mathbf{q}\cdot d\boldsymbol{r})=\frac{D\mathbf{q}}{Dt}\cdot d\boldsymbol{r}+\mathbf{q}\cdot\frac{D}{Dt}d\boldsymbol{r}=\frac{D\mathbf{q}}{Dt}\cdot d\boldsymbol{r}+\mathbf{q}\cdot d\mathbf{q} \qquad ...(3)$$

Equation of motion for viscous incompressible fluid with constant viscosity in vector form (refer equation (17) of Art. 14.1) is given by

$$D\mathbf{q}/Dt = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q} \qquad ...(4)$$

Let the external forces be conservative and derivable from a single valued potential V. Then $\mathbf{B} = -\nabla V$ and hence (4) reduces to

$$D\mathbf{q}/Dt = -\nabla V - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q} \qquad ...(5)$$

$\therefore$ $$(D\mathbf{q}/Dt)\cdot d\mathbf{r} = -\nabla V\cdot d\mathbf{r} - (1/\rho)\times\nabla p\cdot d\mathbf{r} + \nu\,\nabla^2\mathbf{q}\cdot d\mathbf{r} \qquad ...(6)$$

If $\mathbf{\Omega}$ is the vorticity vector, then with help of a vector identity, we obtain

$$\text{curl }\mathbf{\Omega} = \nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}$$

or $$\text{curl }\mathbf{\Omega} = -\nabla^2\mathbf{q}, \qquad \text{as } \nabla\cdot\mathbf{q} = 0 \text{ for incompressible fluid}$$

Hence (6) can be re-written as

$$(D\mathbf{q}/Dt)\cdot d\mathbf{r} = -dV - (1/\rho)\,dp - \nu\,(\text{curl }\mathbf{\Omega})\cdot d\mathbf{r} \qquad ...(7)$$

Also, we have $$\mathbf{q}\cdot d\mathbf{q} = (1/2)\times d\,(\mathbf{q}\cdot\mathbf{q}) = (1/2)\times d\mathbf{q}^2 \qquad ...(8)$$

Using (7) and (8), (3) reduces to

$$\frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) = -dV - \frac{1}{\rho}dp - \nu\,(\text{curl }\mathbf{\Omega})\cdot d\mathbf{r} + \frac{1}{2}d\mathbf{q}^2 \qquad ...(9)$$

Using (9) and assuming that ρ is a single valued function of p only, (2) reduces to

$$\frac{D\Gamma}{Dt} = \oint_C \left\{\left(\frac{1}{2}d\mathbf{q}^2 - dV - \frac{1}{\rho}dp\right) - \nu\,(\text{curl }\mathbf{\Omega})\cdot d\mathbf{r}\right\}$$

or $$\frac{D\Gamma}{Dt} = \left[\frac{1}{2}\mathbf{q}^2 - V - \oint_C \frac{1}{\rho}dp\right]_c - \nu\oint_C (\text{curl }\Omega)\cdot d\mathbf{r}, \qquad ...(10)$$

where the symbol $[\ ]_C$ denotes change in the quantity enclosed within brackets on moving once round C. Since $\boldsymbol{q}$, V and p are single-valued functions of $\boldsymbol{r}$, it follows that the first term on the R.H.S. of (10) vanishes.

Then, (10) reduces to $$\frac{D\Gamma}{Dt} = -\nu\oint_C (\text{curl }\Omega)\cdot d\mathbf{r}, \qquad ...(11)$$

showing that the rate of change of circulation in a closed circuit, drawn in a viscous incompressible fluid, moving with the fluid depends only on the kinematic vicosity ν and on the space rate of change of the vorticity components at the contour.

As a particular case, let $\nu = 0$, *i.e.*, let the fluid be inviscid. Then (11), reduces to $D\Gamma/Dt = 0$, which is well known Kelvin's circulation theorem, namely, *the circulation round any closed circuit moving with the fluid does not change with the time, provided the fluid is inviscid, the field of force is conservative and density is a single valued function of pressure only.*

(*b*) Refer part (*a*). Omit the particular case given at the end.

Ex. 4. *Write Navier-Stokes equations in cartesian co-ordinates. Simplify the equations when*

(*a*) *Fluid is incompressible and dynamic vicosity is constant*

(*b*) *The fluid is incompressible and viscous effects are neligible.*

[Andhra 2002, 03, 06; Kanpur 2003, Meerut 1996]

Sol. (*a*) For incompressible fluid, $\nabla \cdot \mathbf{q} = 0$ *i.e.*, $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0.$
Also, given that μ = dynamic viscosity = constant.
Re-writing equation (14*a*) of Art 14.1, we have

$$\rho \frac{Du}{Dt} = \rho B_x - \frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z} + \mu \frac{\partial^2 u}{\partial z^2}$$

or
$$\rho \frac{Du}{Dt} = \rho B_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

Since $D/Dt = \partial/\partial t + u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z)$, and $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, the above equation reduces to

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho B_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Similarly, equations (14b) and (14c) of Art. 14.1 yield

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho B_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

and
$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho B_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

(*b*) For incompressible fluid, $\nabla \cdot \mathbf{q} = 0.$ Also, if viscous effects are negligible, then setting $\mu = 0$ in equations of part (a), the required equations are

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho B_x - \frac{\partial p}{\partial x},$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho B_y - \frac{\partial p}{\partial y}$$

and
$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho B_z - \frac{\partial p}{\partial z}$$

Ex. 5. *Consider the case of simple Couette* (*see Art. 16.3A of chapter* 16) *flow with the velocity and temperature distributions as follows* :

$u = Uy/h,$ $v = 0,$ p = constant ...(1)

$$\frac{T - T_\omega}{T_\infty - T_\omega} = \frac{y}{h} + \frac{\mu U^2}{2k(T_\infty - T_\omega)} \left(\frac{y}{h} \right) \left(1 - \frac{y}{h} \right), \qquad ...(2)$$

where T_ω *and* T *are temperatures (constant in value) of stationary and moving plates, respectively, and* μ, *h and k are constants. Verify that* (1) *and* (2) *are the solutions of the energy equation for steady viscous incompressible fluid.* **[Garhwal 1996, 98, Meerut 1998]**

Sol. The energy equation for a two-dimensional viscous incompressible fluid is [Refer equation (20) in Art. 14.3]

$$k(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) + \mathbf{\Phi}' = \rho C(DT/Dt) - (Dp/Dt) \qquad ...(3)$$

where
$$\mathbf{\Phi}' = \mu[2\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2\} + (\partial u/\partial y + \partial v/\partial x)^2] \qquad ...(4)$$

Given that u and T are functions of y alone, v = 0 and p = constant. Also for steady motion, $\partial/\partial t = 0.$ Hence

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0 \qquad \text{and} \qquad \mathbf{\Phi}' = \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

Hence (3) reduces $\qquad k(d^2T/dy^2) + \mu(d^2u/dy^2) = 0$ ...(5)

From (1), $\qquad \partial u/dy = U/h$ ...(6)

From (2), $$\frac{dT}{dy} = (T_\infty - T_\omega)\frac{1}{h} + \frac{\mu U^2}{2kh}\left(1 - \frac{2y}{h}\right)$$

and hence $$\frac{d^2T}{dy^2} = 0 + \frac{\mu U^2}{2kh}\left(-\frac{2}{h}\right) = -\frac{\mu U^2}{kh^2}. \qquad ...(7)$$

Using (6) and (7) in (5), we have

$$k\left(-\frac{\mu U^2}{kh^2}\right) + \mu\frac{U^2}{h^2} = 0 \qquad i.e. \qquad 0 = 0,$$

showing that (1) and (2) satisfy the energy equation (3) for steady flow.

Ex. 6. *Consider a two-dimensional viscous incompressible steady flow with velocity components* $\qquad q_r = q_\theta = 0, \qquad q_z = (1/4\mu)\times(dp/dz)\, r^2 + A\log r + B$ ...(1)

and $\qquad p = p(z),$ ...(2)

where A, B, and μ *are constants and* $0 \le r_0 \le r$. *Is the equation of motion with negligible body force satisfied ?* **[Kolkota 2001]**

Sol. The equations of motion for viscous incompressible flow in absence of body forces are [refer equations (2), (10), (11*a*), (11b) and (11*c*) of Art. 14.11 of this chapter]

$$\rho\left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right) \qquad ...(3)$$

$$\rho\left(\frac{Dq_\theta}{Dt} - \frac{q_r q_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r}\right) \qquad ...(4)$$

$$\rho\frac{Dq_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 q_z \qquad ...(5)$$

with $$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial r} \qquad ...(6)$$

and $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \qquad ...(7)$$

For a two dimensional flow, $\qquad \partial/\partial\theta = 0, \qquad \partial/\partial z = 0$

and for steady motion $\partial/\partial t = 0$. With $q_r = q_\theta = 0$ and $p = p(z)$, (3) and (4) are identically equal to zero. Furthermore, (5) reduces to

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2q_z}{dr^2} + \frac{1}{r}\frac{dq_z}{\partial r}\right) \qquad ...(8)$$

From (1), $$\frac{dq_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)r + \frac{A}{r} \qquad \text{...(9)}$$

and $$\frac{d^2 q_z}{dr^2} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right) - \frac{A}{r^2} \qquad \text{...(10)}$$

Using (9) and (10) in (8), we get

$$0 = -\frac{dp}{dz} + \mu\left[\frac{1}{2\mu}\frac{dp}{dz} - \frac{A}{r^2} + \frac{1}{r}\left(\frac{r}{2\mu}\frac{dp}{dz} + \frac{A}{r}\right)\right], \qquad i.e. \qquad 0 = 0.$$

Thus we find that the equations of motion are satisfied.

14.8. Vorticity equation or vorticity transport equation.

Theorem. *Show that the vorticity vector* $\boldsymbol{\Omega}$ *of an incompressible viscous fluid moving under no external forces satisfies the differential equation*

$D\boldsymbol{\Omega}/Dt = (\boldsymbol{\Omega}\cdot\nabla)q + \nu\nabla^2\boldsymbol{\Omega}$ where ν is the kinematic coefficient of viscosity.

[Agra 2000, 05, 06; Kolkata 2006; Himanchal 2000, 02, 03, 09; Meerut 2011]

Proof. Navier-Stokes equation for incompressible viscous fluid with constant viscosity (refer equation (17) in Art. 14.1) is

$$\partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\nabla^2\mathbf{q}. \qquad \text{...(1)}$$

Let the forces be conservative. Then there exists a force potential V such that $\mathbf{B} = -\nabla V$.

Again, by vector calculus $$\nabla\mathbf{q}^2 = \nabla(\mathbf{q}\cdot\mathbf{q}) = 2[(\mathbf{q}\cdot\nabla)\mathbf{q} + \mathbf{q}\times\text{curl }\mathbf{q}.]$$

or $$(\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - \mathbf{q}\times\text{curl }\mathbf{q}$$

or $$(\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - 2\mathbf{q}\times\boldsymbol{\Omega} \qquad \left[\text{Taking } \boldsymbol{\Omega} = (1/2)\times\text{curl }\mathbf{q}\right]$$

Then (1) reduces to $$\partial\mathbf{q}/dt + \nabla(q^2/2) - 2\mathbf{q}\times\Omega = -\nabla V - (1/\rho)\times\nabla p + \nu\nabla^2\mathbf{q}$$

or $$\partial\mathbf{q}/\partial t - 2\mathbf{q}\times\Omega = -\nabla(V + p/\rho + q^2/2) + \nu\nabla^2\mathbf{q}$$

Taking curl of both sides and using the results curl grad $\equiv 0$ and curl $(\partial\mathbf{q}/\partial t) = \partial(\text{curl }\mathbf{q})/\partial t$ $= 2(\partial\boldsymbol{\Omega}/\partial t)$ and curl $\nabla^2\mathbf{q} \equiv \nabla^2\text{curl }\mathbf{q} = 2\nabla^2\boldsymbol{\Omega}$, we obtain

$$\partial\boldsymbol{\Omega}/\partial t - \text{curl }(\mathbf{q}\times\boldsymbol{\Omega}) = \nu\nabla^2\boldsymbol{\Omega}$$

or $$\partial\boldsymbol{\Omega}/\partial t - [\mathbf{q}\,div\,\boldsymbol{\Omega} - \boldsymbol{\Omega}\,div\,\mathbf{q} + (\boldsymbol{\Omega}\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\boldsymbol{\Omega}] = \nu\nabla^2\boldsymbol{\Omega}$$

or $$\partial\boldsymbol{\Omega}/\partial t + (\mathbf{q}\cdot\nabla)\boldsymbol{\Omega} = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{q} + \nu\nabla^2\boldsymbol{\Omega}$$

[$\because$ Equation of continuity is $div\ \mathbf{q} = 0$ Also $div\ \boldsymbol{\Omega} = div$ curl $\mathbf{q} = 0$]

or $$D\boldsymbol{\Omega}/Dt = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{q} + \nu\nabla^2\boldsymbol{\Omega}, \qquad \text{...(2)}$$

which is known as *vorticity equation or vorticity transport equation.*

Remark. Let $\boldsymbol{\Omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$, $\mathbf{q} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$. Then the above vector equation (2) in Cartesian form reduces to

$$\left.\begin{aligned} \frac{D\xi}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u + \nu\nabla^2\xi \\ \frac{D\eta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v + \nu\nabla^2\eta \\ \frac{D\zeta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w + \nu\nabla^2\zeta \end{aligned}\right\} \qquad \text{...(3)}$$

where $\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, $\zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

14.9. Diffusion of a vortex filament.

Let there be a vortex filament of strength k along the axis of z in an infinite liquid. The motion will be in circles about the z-axis, the vorticity at distance r from the axis being a function of r only. We have, therefore $w = 0$, $\xi = \eta = 0$, ...(1)

and u, v are independent of z.

We know that (refer remark of Art. 14.8)

$$\left.\begin{aligned}\frac{D\xi}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u + \nu\nabla^2\xi \\ \frac{D\eta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v + \nu\nabla^2\eta \\ \frac{D\zeta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w + \nu\nabla^2\zeta\end{aligned}\right\} \qquad ...(2)$$

Using (1), (2) reduces to $D\zeta / Dt = \nu\,\nabla^2\zeta$ ...(3)

Let u, v be of the form $u = -(y/r)\times f(r)$, $v = (x/r)\times f(r)$, ...(4)

where $r^2 = x^2 + y^2$ ...(5)

$$\therefore \quad \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \frac{\partial\zeta}{\partial t} - \frac{y\,f(r)}{r}\left(\frac{x}{r}\frac{\partial\zeta}{\partial r}\right) + \frac{x\,f(r)}{r}\left(\frac{y}{r}\frac{\partial\zeta}{\partial y}\right), \text{ by (4)}$$

Thus, $D\zeta / Dt = \partial\zeta / \partial t$ ...(6)

Also, $\nabla^2\zeta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\zeta$ ...(7)

Using (6) and (7), (3) reduces to $\frac{\partial\zeta}{\partial t} = \nu\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r}\right)$ ...(8)

To solve (8), we assume that $\zeta = (1/t)\times f(\chi)$, ...(9)

where $\chi = r / 2\sqrt{\nu t}$ ...(10)

Then, $\frac{\partial\zeta}{\partial t} = -\frac{1}{t^2}f(\chi) - \frac{r}{2t^2\sqrt{\nu t}}\frac{df}{d\chi}$ and $\frac{\partial\zeta}{\partial r} = \frac{1}{2\sqrt{\nu t}}\frac{df}{d\chi}$

Using the above relations, (8) reduces to

$$\chi\frac{d^2 f}{d\chi^2} + \frac{df}{d\chi}(1 + 2\chi^2) + 4\chi f = 0 \quad \text{or} \quad \frac{d}{d\chi}\left(\chi\frac{df}{d\chi}\right) + \frac{d}{d\chi}(2f\chi^2) = 0.$$

Integrating, $\chi(df/\partial\chi) + 2\chi^2 f = C$, where C is constant of integration.

Hence, $df / d\chi + 2\chi f = C/\chi$. ...(11)

When $\chi = 0$ *i.e.*, $r = 0$, f and f' are both infinite, therefore, we have $C = 0$ and (11) reduces to

$$df / d\chi + 2\chi f = 0 \quad \text{or} \quad (1/f)df + 2\chi d\chi = 0$$

Integrating, $\log f + \chi^2 = \log A$, where A is constant of integration ...(12)

From (12), $$f = Ae^{-\chi^2}$$ ...(13)

From (9), (10) and (13), we have $\zeta = (1/t) \times Ae^{-\chi^2} = (1/t) \times Ae^{-(r^2/4\nu t)}$ ...(14)

By Stokes's theorem, circulation Γ round a circle of radius r is given by

$$\Gamma = \int_0^r 2\zeta\, dS \text{ over the circle } = \int_0^r 2\zeta \cdot 2\pi r\, dr = \frac{4\pi A}{t}\int_0^r re^{-(r^2/4\nu t)}\, dr, \quad \text{by (14)}$$

$\therefore$ $$\Gamma = 8\pi A\nu\,(1 - e^{-r^2/4\nu t}).$$ ...(15)

Let $\Gamma \to \Gamma_1$ as $t \to 0$. So (15) gives

$\Gamma_1 = 8\pi A\nu$ so that $A = \Gamma_1 / 8\pi\nu$

$\therefore$ From (15), $$\Gamma = \Gamma_1(1 - e^{-r^2/4\nu t})$$ ...(16)

and from (14), $$\zeta = (\Gamma_1 / 8\pi\nu\, t) \times e^{-r^2/4\nu t}$$ ...(17)

Also, if v be the velocity, then $\Gamma = 2\pi r v$

$\therefore$ from (16), $\Gamma_1(1 - e^{-r^2/4\nu t}) = 2\pi r v$ or $v = (\Gamma_1/2\pi r) \times (1 - e^{-r^2/4\nu t})$. ...(18)

For small values of r, (18) reduces to

$$v = \frac{\Gamma_1}{2\pi r}\left[1 - \left\{1 - \left(-\frac{r^2}{4\nu t}\right) + \frac{1}{2!}\left(-\frac{r^2}{4\nu t}\right)^2 - \dots\right\}\right] = \frac{\Gamma_1}{2\pi}\left\{\frac{r}{4\nu t} - \frac{r^2}{2 \times (4\nu t)^2} - \dots\right\}$$ ...(19)

From (19), we see that as $r \to 0$, then $v \to 0$.

Again on the axis $r = 0$, so from (17), $\zeta = \zeta_0 = \Gamma_1 / (8\pi\nu t)$. ...(20)

Hence for very small values of r, from (19) $v = \dfrac{\Gamma_1}{2\pi} \cdot \dfrac{r}{4\nu t} = \dfrac{\Gamma_1 r}{8\pi\nu t}$. ...(21)

From (20) and (21), $v = \zeta_0\, r$.

From (18), it follows that as t increases from 0 to ∞, v decreases from $\Gamma_1 / 2\pi r$ to 0.

14.10. Summary of basic equations governing the flow of viscous fluid in certesian *co-ordinates* (*x*, *y*, *z*) :

Case I For flow of viscous compressible fluid

Equation of continuity : [Refer equation (8) of Art. 2.9]

$$\partial\rho / \partial t + \partial(\rho u) / \partial x + \partial(\rho v) / \partial y + \partial(\rho w) / \partial z = 0$$ ...(1)

The Navier-Stokes equation: [Refer equations (14*a*), (14*b*), (14*c*), Art 14.1]

$$\rho\{\partial u / \partial t + u(\partial u / \partial x) + v(\partial u / \partial y) + w(\partial u / \partial z)\} = \rho B_x - \partial p / \partial x$$

$$+ \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\right)\right\}$$ ...(2*a*)

$$\rho\{\partial v/\partial t + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z)\} = \rho B_y - \partial p/\partial y$$

$$+\frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} \quad \text{...(2b)}$$

$$\rho\{\partial w/\partial t + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z)\} = \rho B_z - \partial p/\partial z$$

$$+\frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} \quad \text{...(2c)}$$

Energy equation : [Refer equation (17), Art 14.2]

$$\rho\left[\frac{\partial(C_pT)}{\partial t} + u\frac{\partial(C_pT)}{\partial x} + v\frac{\partial(C_pT)}{\partial y} + w\frac{\partial(C_pT)}{\partial z}\right] = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

$$+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\Bigg\} + \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right.$$

$$\left. - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \quad \text{...(3)}$$

Equation of state: $p = \rho RT$ ...(4)

Thus, we have six equations. But for a the flow of compressible fluids, the coefficient of viscosity μ and the coefficient of thermal conductivity k are not constants but depend on temperature. Therefore, we have eight unknowns (u, v, w, p, ρ, T, μ and k) instead of six and we require two additional equations to solve a problem of flow of viscous compressible fluid. Let these two additional equations, in general froms, be given by

$$\mu = \mu(T) \quad \text{and} \quad k = k(T) \quad \text{...(5)}$$

For air, the variation of viscosity μ with absolute temperature T is given by the following

Sutherland's formula, $$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/2}\frac{T_\infty + S_1}{T + S_1}, \text{ approximately} \quad \text{...(6)}$$

where μ_∞ denotes the viscosity at a reference temperature T_∞ and S_1 is a constant. For air, $S_1 = 110°$ K.

The above formula (6) is quite complicated. From either the simple kinetic theory of gases, of empirical data, the coefficient of viscosity μ may, be expressed quite accurately as a power of

the absolute temperature, $$\mu/\mu_\infty = (T/T_\infty)^m, \; 0.5 \le m \le 1 \quad \text{...(7)}$$

For air at ordinary temperature, we take $m = 0.76$. As the temperature increases, m decreases, towards 0.5

It has been shown that at high temperatures the relation (6) can be well approximated by (7), when $0.5 \le m \le 0.75$ and at low temperature the appropriate value of m is 1.

Now, non dimensional number P_r (See Art. 15.7) is defined as

$$P_r = \text{Prandial number} = (\mu C_p)/k \quad \text{...(8)}$$

It has been shown that P_r is constant for air even at large temperature differences. Since C_p is also nearly constant for a wide range of temperatures around ordinary temperatures, it follows from (8) that the coefficient of heat conductivity k is directly proportional to μ. Therefore, the dependence of the coefficient of heat conductivity k on temperature is of similar nature as that of viscosity.

The components of stress at any point (*x*, *y*, *z*).

$\sigma_{xx} = 2\mu(\partial u/\partial x) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$ $\quad\sigma_{yy} = 2\mu(\partial v/\partial y) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$

$\sigma_{zz} = 2\mu(\partial w/\partial z) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$ $\quad\sigma_{xy} = \sigma_{yx} = \mu(\partial u/\partial y + \partial v/\partial x),$

$\sigma_{yz} = \sigma_{zy} = \mu(\partial v/\partial z + \partial w/\partial y),$ $\quad\sigma_{zx} = \sigma_{xz} = \mu(\partial w/\partial x + \partial u/\partial z)$

The components of the heat-flux vector

$Q_x = -k(\partial T/\partial x),\qquad Q_y = -k(\partial T/\partial y),\qquad Q_z = -k(\partial T/\partial z)$

Case II For flow of viscous incompressible fluid

While dealing with incompressible fluid flow, we suppose that fluid properties such as density ρ, coefficient of viscosity μ and coefficient of heat conductivity k are nearly constant. Accordingly, the number of unknown quantities reduce to five (u, v, w, p and T), which are obtained with the help of the following fundamental equations

Equation of continuity: $\quad \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad$ ...(1)'

The Narier-Stokes equations:

$$\rho\{\partial u/\partial t + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)\}$$
$$= \rho B_x - \partial p/\partial x + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad ...(2a)'$$

$$\rho\{\partial v/\partial t + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z)\}$$
$$= \rho B_y - \partial p/\partial y + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad ...(2b)'$$

$$\rho\{\partial w/\partial t + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z)\}$$
$$= \rho B_z - \partial p/\partial z + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad ...(2c)'$$

The energy equation

$$C_p T\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$+\mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \quad ...(3)'$$

The components of stress at any point (*x*, *y*, *z*)

$\sigma_{xx} = 2\mu(\partial u/\partial x) - p,\qquad \sigma_{yy} = 2\mu(\partial v/\partial y) - p,\qquad \sigma_{zz} = 2\mu(\partial w/\partial z) - p$

$\sigma_{xy} = \sigma_{yx} = \mu(\partial u/\partial y + \partial v/\partial x),\quad \sigma_{yz} = \sigma_{zy} = \mu(\partial v/\partial z + \partial w/\partial y),\quad \sigma_{zx} = \sigma_{xz} = \mu(\partial w/\partial x + \partial u/\partial z)$

The components of the heat flux vector

$Q_x = -k(\partial T/\partial x),\qquad Q_y = -k(\partial T/\partial y),\qquad Q_z = -k(\partial T/\partial z)$

Main difference between the compressible fluid flow and incompressible fluid flow

Observing carefully the above fundamental equations of compressible fluid flow and incompressible fluid flow, we see that, in compressible fluid flow, the equations of motion and energy are coupled whereas in an incompressible fluid flow, with constant fluid properties ρ, μ, k, the equations of motion and energy are uncoupled. Accordingly, while dealing with flow of incompressible fluid flow, the equation of continuity and equations of motion are first solved for u, v, w and finally the equation of energy is solved for the temperature.

Remark The above fundamental equations are solved subject to given initial and boundary conditions. The boundary condition are those required by geometrical considerations, together with the no-slip condition which states that on a wall the tangential component of relative velocity must be zero. To solve energy equation some conditions must be imposed on the temperature on the boundary and will be provided by the given problem.

14.11. Summary of basic equations governing the flow of viscous fluid in cylindrical co-ordinates (r, θ, z).

Equation of continuity (Refer Art. 2.10)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r\, q_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta)+\frac{\partial}{\partial z}(\rho q_z)=0 \qquad ...(1)$$

(*a*) **Cylinrical coordinate system.** The Navier – stokes equations of motion of viscous compressible fluids in cylindrical coordinates (r, θ, z) are given by

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} \qquad ...(1a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta+\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r} \qquad ...(1b)$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z+\frac{\partial \sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{zr}}{r} \qquad ...(1c)$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z}. \qquad ...(2)$$

Constitutive equations are given by

$$\left.\begin{aligned}\sigma_{rr}&=-p+2\mu\in_{rr}-(2/3)\times\mu\nabla\cdot\mathbf{q}\\ \sigma_{\theta\theta}&=-p+2\mu\in_{\theta\theta}-(2/3)\times\mu\nabla\cdot\mathbf{q}\\ \sigma_{zz}&=-p+2\mu\in_{zz}-(2/3)\times\mu\nabla\cdot\mathbf{q}\end{aligned}\right\} \qquad ...(3)$$

$$\sigma_{r\theta}=\mu\gamma_{r\theta}, \qquad \sigma_{\theta z}=\mu\gamma_{\theta z}, \qquad \sigma_{zr}=\mu\gamma_{zr} \qquad ...(4)$$

Again the components of the rates of strain are given by

$$\in_{rr}=\frac{\partial q_r}{\partial r}, \qquad \in_{\theta\theta}=\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}, \qquad \in_{zz}=\frac{\partial q_z}{\partial z} \qquad ...(5)$$

$$\left.\begin{aligned}\gamma_{r\theta}&=\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\\ \gamma_{\theta z}&=\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\\ \gamma_{zr}&=\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\end{aligned}\right\} \qquad ...(6)$$

Using (3), (4), (5) and (6), the equations of motion (1*a*) to (1*c*) may be re-written as follows:

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[\mu\left(2\frac{\partial q_r}{\partial r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)\right]+\frac{2\mu}{r}\left(\frac{\partial q_r}{\partial r}-\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}-\frac{q_r}{r}\right) \qquad ...(7a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{2}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{2q_\theta}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)\right]+\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right), \quad \text{...(7b)}$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial q_z}{\partial z}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)\right]+\frac{\mu}{r}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right) \quad \text{...(7c)}$$

where
$$\nabla\cdot\mathbf{q}=\frac{\partial q_r}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}+\frac{q_r}{r}. \quad \text{...(8)}$$

For some particular flows, equations (7*a*) to (7*c*) take the following forms:

(*i*) **Viscous compressible fluid with constant viscosity.**

Let μ be constant. Then (7*a*) to (7*c*) reduce as follows.

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 q_r-\frac{q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}+\frac{1}{3}\frac{\partial}{\partial r}(\nabla\cdot\mathbf{q})\right], \quad \text{...(9a)}$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial \theta}-\frac{q_\theta}{r^2}+\frac{1}{3r}\frac{\partial}{\partial \theta}(\nabla\cdot\mathbf{q})\right] \quad \text{...(9b)}$$

and
$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\mu\left[\nabla^2 q_z+\frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{q})\right], \quad \text{...(9c)}$$

where
$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial}{\partial z^2}. \quad \text{...(10)}$$

(*ii*) **Viscous incompressible flow.** Let ρ and μ be both constant. Also $\nabla\cdot\mathbf{q}=0$ for incompressible fuids. Then (9*a*) to (9*c*) become

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 q_r-\frac{q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right), \quad \text{...(11a)}$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial \theta}-\frac{q_\theta}{r}\right) \quad \text{...(11b)}$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\mu\nabla^2 q_z \quad \text{...(11c)}$$

(*iii*) **Non viscous fluid.** With $\mu=0$, equations (9*a*) to (9*c*) reduce to

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r} \quad \text{...(12a)}$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta} \quad \text{...(12b)}$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z} \quad \text{...(12c)}$$

(*iv*) **Axi-symmetric flow of incompressible fluids** $(\partial/\partial\theta=0)$:

$$\rho\left\{\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}+q_z\frac{\partial q_r}{\partial z}+\frac{q_\theta^2}{r}\right\}=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rq_r)\right\}+\frac{\partial^2 q_r}{\partial z^2}\right] \quad \text{...(13a)}$$

$$\rho\left\{\frac{\partial q_\theta}{\partial t}+q_r\frac{\partial q_\theta}{\partial r}+q_z\frac{\partial q_\theta}{\partial z}+\frac{q_r q_\theta}{r}\right\}=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rq_\theta)\right\}+\frac{\partial^2 q_\theta}{\partial z^2}\right] \quad ...(13b)$$

$$\rho\left\{\frac{\partial q_z}{\partial t}+q_r\frac{\partial q_z}{\partial r}+q_z\frac{\partial q_z}{\partial z}\right\}=\rho B_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)+\frac{\partial^2 q_z}{\partial z^2}\right] \quad ...(13c)$$

Energy eqution:

(i) **For viscous compressible fluid:** Equation of energy of a viscous compressible fluid in cylinderical polar coordinates (r,θ,z) is given by

$$\rho\frac{D}{Dt}(C_pT)=\frac{Dp}{Dt}+\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\mathbf{\Phi}, \quad ...(14)$$

where $$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z},$$

$$\mathbf{\Phi}=2\mu\left[\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)^2+\left(\frac{\partial q_z}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)^2-\frac{1}{3}(\nabla\cdot\mathbf{q})^2\right] \quad ...(15)$$

and $$\nabla\cdot\mathbf{q}=\frac{\partial q_r}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}+\frac{q_r}{r}. \quad ...(16)$$

(ii) **For viscous incompressible fluid:** Equation of energy of a viscous incompressible fluid for which k, ρ and μ are constants in cylindrical polar coordinates (r,θ,z) is given by

$$\rho C_v\frac{DT}{Dt}=k\nabla^2T+\Phi, \quad \text{where} \quad \frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z} \quad ...(17)$$

and $$\Phi=2\mu\left[\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)^2+\left(\frac{\partial q_z}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)^2\right]. \quad ...(18)$$

Equation of state : $$p=\rho RT \quad ...(19)$$

The components of stress at any point (r, θ, z)

(i) **For compressible viscous fluid**

$$\sigma_{rr}=2\mu(\partial q_r/\partial r)-(2/3)\times\mu\nabla\cdot\mathbf{q}-p \quad ...(20a)$$

$$\sigma_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)-\frac{2}{3}\mu\nabla\cdot\mathbf{q}-p \quad ...(20b)$$

$$\sigma_{zz}=2\mu(\partial q_z/\partial z)-(2/3)\times\mu\nabla\cdot\mathbf{q}-p \quad ...(20c)$$

$$\sigma_{r\theta}=\sigma_{\theta r}=\mu\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)=\mu\left\{r\frac{d}{dr}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\} \quad ...(20d)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = \mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}\right), \qquad \sigma_{zr} = \sigma_{rz} = \mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right) \qquad ...(20e)$$

(*ii*) **For incompressible visocus fluid**

$$\sigma_{rr} = 2\mu\left(\frac{\partial q_r}{\partial r}\right), \qquad \sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right), \qquad \sigma_{zz} = 2\mu\frac{\partial q_z}{\partial z} \qquad ...(21a)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left(\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right) = \mu\left\{r\frac{d}{dr}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\} \qquad ...(21b)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = \mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial r}\right), \qquad \sigma_{zr} = \sigma_{rz} = \mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right) \qquad ...(21c)$$

The components of heat-flux vector are

$$Q_r = -k\,(\partial T/\partial r), \qquad Q_\theta = -(k/r)\times(\partial T/\partial \theta), \qquad Q_z = -k\,(\partial T/\partial z) \quad ...(22)$$

14.12. Summary of basic equations geverning the flow of viscous fluid in spherical coordinates (r, θ, ϕ).

Equation of continutiy (Refer Art. 2.11)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho \sin\theta\, q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho\, q_\phi) = 0 \qquad ...(1)$$

(*b*) **Spherical coordinate system.** The Navier–Stokes equations of motion of viscous compressible fluids in spherical coordinates (r, θ, ϕ) are given by

$$\rho\left(\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r}\right) = \rho B_r + \frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{rr})}{\partial r} + \frac{\partial(\sigma_{r\theta}\sin\theta)}{\partial \theta} + \frac{\partial \sigma_{\phi r}}{\partial \theta}\right] - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r} \quad ...(1a)$$

$$\rho\left(\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} - \frac{q_\phi^2 \cot\theta}{r}\right) = \rho B_\theta + \frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{r\theta})}{\partial r} + \frac{\partial(\sigma_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{\partial \sigma_{\theta\phi}}{\partial \phi}\right]$$

$$+ \frac{\sigma_{r\theta}}{r} - \frac{\sigma_{\phi\phi}\cot\theta}{r} \qquad ...(1b)$$

$$\rho\left(\frac{Dq_\phi}{Dt} + \frac{q_\phi q_r}{r} - \frac{q_\phi q_\theta \cot\theta}{r}\right) = \rho B_z + \frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{\phi r})}{\partial r} + \frac{\partial(\sigma_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi}\right]$$

$$+ \frac{\partial_{\phi r}}{r} - \frac{\sigma_{\theta\phi}\cot\theta}{r} \qquad ...(1c)$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}. \qquad ...(2)$$

The constitutive equations are given by

$$\left.\begin{aligned} \sigma_{rr} &= -p + 2\mu \in_{rr} - (2\mu/3)\times\nabla\cdot\mathbf{q} \\ \sigma_{\theta\theta} &= -p + 2\mu \in_{\theta\theta} - (2\mu/3)\times\nabla\cdot\mathbf{q} \\ \sigma_{\phi\phi} &= -p + 2\mu \in_{\phi\phi} - (2\mu/3)\times\nabla\cdot\mathbf{q} \end{aligned}\right\} \qquad ...(3)$$

$$\sigma_{r\theta} = \mu\,\gamma_{r\theta}, \qquad \sigma_{\theta\phi} = \mu\,\gamma_{\theta\phi}, \qquad \sigma_{\phi r} = \mu\,\gamma_{\phi r} \qquad ...(4)$$

Again, the components of the rates of strain are given by

$$\left.\begin{aligned}&\epsilon_{rr}=\frac{\partial q_r}{\partial r}, \qquad \epsilon_{\theta\theta}=\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}, \qquad \epsilon_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}+\frac{q_r}{r}+\frac{q_\theta\cot\theta}{r}\\ &\gamma_{r\theta}=r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta},\ \gamma_{\theta\phi}=\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\theta}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi},\ \gamma_{\phi r}=\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\end{aligned}\right\} \quad ...(5)$$

Using (3), (4) and (5), the equations of motion (1*a*) to (1*b*) and (1*c*) may be re-written as:

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[\mu\left(2\frac{\partial q_r}{\partial r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\}\right]$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}\right]$$

$$+\frac{\mu}{r}\left[4\frac{\partial q_r}{\partial r}-\frac{2}{r}\frac{\partial q_\theta}{\partial\theta}-\frac{4q_r}{r}-\frac{2}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}-\frac{2q_\theta\cot\theta}{r}+r\cot\theta\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{\cot\theta}{r}\frac{\partial q_r}{\partial\theta}\right] \quad ...(6a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{2}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{2q_r}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}\right]+\frac{\partial}{\partial r}\left[\mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\}\right]$$

$$+\frac{\mu}{r}\left[2\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}-\frac{q_\theta\cot\theta}{r}\right)\cot\theta+3\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)\right\}\right] \quad ...(6b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\mu\left(\frac{2}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}+\frac{2q_r}{r}+\frac{2q_\theta\cot\theta}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{\partial}{\partial r}\left[\mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}\right]$$

$$+\frac{\mu}{r}\left[3\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)\right\}+2\cot\theta\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}\right] \quad ...(6c)$$

where
$$\nabla\cdot\mathbf{q}=\frac{1}{r}\frac{\partial}{\partial r}(r^2 q_r)+\frac{1}{r\sin\theta}\frac{\partial(q_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}. \quad ...(7)$$

For some particular flows, equations (6*a*), (6*b*) and (6*c*) take the following forms:

(*i*) **Viscous compressible fluid with constant viscosity i.e. with μ = constant.**

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 q_r-\frac{2q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}-\frac{2q_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial\phi}\right.$$

$$+\frac{1}{3}\left\{\frac{\partial^2 q_r}{\partial r^2}+\frac{2}{r}\left(\frac{\partial q_r}{\partial r}-\frac{q_r}{r}\right)+\frac{1}{r}\frac{\partial^2 q_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\left(\frac{\partial q_\theta}{\partial\theta}+q_\theta\cot\theta\right)+\frac{\cot\theta}{r}\frac{\partial q_\theta}{\partial r}+\frac{1}{r\sin\theta}\left(\frac{\partial^2 q_\phi}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial q_\phi}{\partial\phi}\right)\right\}\Bigg] \quad ...(8a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r^2}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\Bigg[\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}$$

$$+\frac{1}{3}\left\{\frac{1}{r^2}\frac{\partial^2 q_\theta}{\partial\theta^2}+\frac{1}{r}\left(\frac{\partial^2 q_r}{\partial\theta\partial r}+\frac{2}{r}\frac{\partial q_r}{\partial\theta}+\frac{\cot\theta}{r}\frac{\partial q_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 q_\phi}{\partial\theta\partial\phi}-\cot\theta\frac{\partial q_\phi}{\partial\phi}-q_\theta\right)\right\}\Bigg] \quad ...(8b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\mu\Bigg[\nabla^2 q_\phi-\frac{q_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi}$$

$$+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\phi}+\frac{1}{3r\sin\theta}\left\{\frac{1}{r\sin\theta}\frac{\partial^2 q_\phi}{\partial\phi^2}+\frac{\partial^2 q_r}{\partial\phi\partial r}+\frac{2}{r}\frac{\partial q_r}{\partial\phi}+\frac{1}{r}\frac{\partial^2 q_\theta}{\partial\phi\partial\theta}+\frac{\cos\theta}{r\sin\theta}\frac{\partial q_\theta}{\partial\varphi}\right\}\Bigg] \quad ...(8c)$$

(*ii*) **Viscous incompressible fluid.** For such fluids, ρ = constant, μ = constant and $\nabla\cdot\mathbf{q}=0$. Hence (8*a*) to (8*c*) become

$$\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}=B_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 q_r-\frac{2q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}-\frac{2q_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial\phi}\right) \quad ...(9a)$$

$$\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r^2}=B_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left(\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}\right) \quad ...(9b)$$

$$\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}=B_\phi-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$+\nu\left(\nabla^2 q_\phi-\frac{q_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\varphi}\right) \quad ...(9c)$$

where $$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2} \quad ...(10)$$

(*iii*) **Non-viscous fluid (with $\mu = 0$).** Then (8*a*) to (8*c*) become

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r} \quad ...(11a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta} \quad ...(11b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \quad ...(11c)$$

Energy Equation

(*i*) **For viscous compressible fluid:** Equation of energy of viscous compressible fluid in spherical polar coordinates (r,θ,ϕ) is

$$\rho \frac{D}{Dt}(C_p T) = \frac{Dp}{Dt} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \mathbf{\Phi}, \quad \text{...(12)}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}, \quad \text{...(13)}$$

$$\mathbf{\Phi} = \mu\left[\left\{2\left(\frac{\partial q_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi} + \frac{q_r}{r} + \frac{q_\theta \cot\theta}{r}\right)^2\right\} + \left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\}^2\right.$$

$$\left. + \left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi}\right\}^2 + \left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}^2\right] - \frac{2}{3}\mu(\nabla\cdot\mathbf{q})^2 \quad \text{...(14)}$$

and
$$\nabla\cdot\mathbf{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{\sin\theta}\frac{\partial}{\partial r}(q_\theta \sin\theta)\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}. \quad \text{...(15)}$$

(*ii*) **For viscous incompressible fluid:** Equation of energy of a viscous incompressible fluid in spherical polar coordinates (r, θ, z) is given by

$$\rho C_v \frac{DT}{Dt} = k\nabla^2 T + \Phi, \quad \text{...(16)}$$

where
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}, \quad \text{...(17)}$$

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \quad \text{...(18)}$$

and
$$\Phi \equiv \mu\left[2\left\{\left(\frac{\partial q_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}\right)^2\right\}\right.$$

$$\left. + \left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\}^2 + \left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi}\right\}^2 + \left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}^2\right]. \quad \text{...(19)}$$

Equation of state : $p = \rho R T$ ...(20)

The components of stress at any point (r, θ, ϕ)

(*i*) **For compressible viscous fluid**

$$\sigma_{rr} = 2\mu\left(\frac{\partial q_r}{\partial r}\right) - \frac{2}{3}\mu\nabla\cdot\mathbf{q}, \quad \text{...(21}a)$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial_\theta} + \frac{q_r}{r}\right) - \frac{2}{3}\mu\nabla\cdot\mathbf{q} \quad \text{...(21}b)$$

$$\sigma_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}\right) - \frac{2}{3}\mu\nabla\cdot\mathbf{q} \quad \text{...(21}c)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\} = \mu\left(\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right) \quad \text{... (21}d)$$

...(21*d*)

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}, \qquad ...(21e)$$

$$\sigma_{\phi r} = \sigma_{r\phi} = \mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\} \qquad ...(21f)$$

(*ii*) **For incompressible viscous fluid**

$$\sigma_{rr} = 2\mu\frac{\partial q_r}{\partial r}, \qquad \sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{q_r}{r}\right), \qquad ...(22a)$$

$$\sigma_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}\right) \qquad ...(22b)$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\} \qquad ...(22c)$$

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\} \qquad ...(22d)$$

$$\sigma_{\phi r} = \sigma_{r\phi} = \mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\} \qquad ..(22e)$$

The components of heat-flux vector are

$$Q_r = -k\frac{\partial T}{\partial r}, \qquad Q_\theta = -\frac{k}{r}\frac{\partial T}{\partial\theta}, \qquad Q_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial\phi} \qquad ...(23)$$

EXERCISES

1. Find an expression for the rate of dissipation of energy of a liquid due to viscosity. Discuss the motion of a viscous liquid for which there is no dissipation of a viscosity.

Prove that for a liquid filling a closed vessel which is at rest, the rate of dissipation of energy due to viscosity is $\mu\iiint(\text{curl }\mathbf{q})^2\, dxdydz$,

where μ is the coefficient of viscosity and $\mathbf{q}$ the velocity vector.

2. Show that the velocity field $u(y) = -(h^2/8\mu)(dp/dx)\{1-4(y/h)^2\}, v = w = 0,$

satisfies the equation of motion for the two-dimensional steady flow a viscous incompressible fluid with constant viscosity and constant pressure gradient.

3. The velocity components $q_r(r,\theta) = -U(1-a^2/r^2)\cos\theta, \quad q_\theta(r,\theta) = U(1+a^2/r^2)\sin\theta$

satisfy the equation of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. U and a are constants.

4. Derive Navier Stokes equations of motion for viscous compressible fluid and also deduce the equation for viscous compressible fluid with constant viscosity.

[Himanchal 2001; 03, 10; Meerut 2000, 01, 02]

5. Derive the hydrodymical equations of motion of viscous and incompressible fluid in cartesian form as obtained by Navier and Stokes. **[Garhwal 2005]**

6. Starting from the Navier-Stokes equation for the motion of an incompressible fluid moving under conservative forces, prove that the vorticity $\mathbf{\Omega}$ satisfies the differential equation $D\mathbf{\Omega}/dt = (\mathbf{\Omega}\cdot\nabla)\mathbf{q} + \nu\nabla^2\mathbf{\Omega}$, ν being coefficient of kinetic viscosity. **[Himanchal 2000, 02]**

7. Define the principle of energy conservation. Derive energy equation for a compressible fluid and deduce it for incompressible fluids. **[Himanchal 1999, 99, 2000, 01]**

8. Derive vorticity transport equation and show that vorticity cannot originate within the interior of a viscous fluid but must be diffused from the boundary into the fluid.

Hint : Refer Art. 14.4 and its remark. **[Himanchal 1998, 2001]**

9. Derive the equation of energy for an incompressible fluid motion with constant fluid properties. **[Himanchal 2003, 09]**

10. Sate the constitutive equations for an isotropic Newtonian fluid and use it to derive the Navier-Stokes equations of motion for a viscous compressible fluid. **[Himanchal 1999]**

11. Find an expression for the rate of dissipation of energy of a liquid due to viscosity. **[Kanpur 2005]**

12. Derive Navier – Stokes equations of motion of a viscous fluid **(Himanchal 2009; Meerut 2008)**

13. Derive Navier – Stokes equations of motion of an incompressible fluid. **(Himanchal 2007)**

14. Write a short note on viscosity in a viscous incompressible fluid motion" **(Himanchal 2007)**

15. Derive the equation of energy with constant viscosity and heat conductivity of fluid. **(Himanchal 2007)**

16. Define the law of conservation of energy. Derive equation of energy and deduce it for flow of a viscous incompressible fluid. **(Himanchal 2006)**

17. Find the Navier – Stokes equations of motion for the flow of incompressible viscous fluid in cartesian coordinates. State the Green' theorem. **(Agra 2005, 06)**

18. For a non-viscous incompressible fluid the Navier – Stokes equations of motion are

(*a*) $\frac{\partial \mathbf{q}}{\partial t} - (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} + \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{q}$ (*b*) $\frac{\partial \mathbf{q}}{\partial t} - (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{q}$

(*c*) $\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} + \frac{1}{\rho}\nabla p + \nu^2 \nabla^2 \mathbf{q}$ (*d*) None of these **[Agra 2003, 05]**

Hint: Ans. (*d*). See equation (17), Art. 14.1.

19. Rate of dissipation of energy when there is no slip of the boundary is:

(*a*) $\mu \iiint (\xi^2 + \eta^2 + \zeta^2) dV$ (*b*) $2\mu \iiint (\xi^2 + \eta^2 + \zeta^2) dV$

(*c*) $3\mu \iiint (\xi^2 + \eta^2 + \zeta^2) dV$ (*d*) $4\mu \iiint (\xi^2 + \eta^2 + \zeta^2) dV$ **(Agra 2006)**

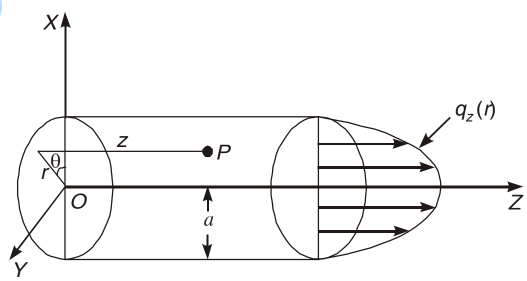
Hint: Ans. (*d*). Refer equation (17), Art. 14.6*B*.

20. Obtain Navier–Stokes equations of motion for viscous fluid in cartesian coordinates. **[Agra 2009; 10]**

16.4*A*. Flow through a circular pipe-The Hagen-Poiseuille flow.

[Himanchal 2000, 01, 02, 03, 07; 09, 10; Meerut 2003, 10, 12; Garhwal 1999; Kanpur 2002; Kurukshetra 1999]

Consider the laminar steady flow, without body forces of an incompressible fluid through an infinite circular pipe of radius a with axial symmetry as shown in the following figure.



For the present problem, we consider all basic equations in cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe. Clearly, the radial and tangential velocity components are zero, *i.e.* $q_r = q_\theta = 0$. Due to axial symmetry of flow, q_z will be independent of θ. Further, the equation of continuity for steady flow, namely, [Refer Art. 2.10, chapter 2]

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0 \quad \text{reduces to} \quad \frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence q_z is function of r alone, *i.e.* $q_z = q_z(r)$

Thus, for the given problem, $q_r = 0,\quad q_\theta = 0\quad$ and $\quad q_z = q_z(r)$ ...(1)

For the present steady axi-symmetric flow of incompressible fluid with velocity components (1), the equations of motion [refer 11 (*a*) to 11 (*c*) in Art. 14.11 of chapter 14] in cylindrical coordinates reduce to

$$0 = -(\partial p/\partial r) \qquad ...(2)$$

$$0 = -(1/r) \times (\partial p/\partial \theta) \qquad ...(3)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)\right] \qquad ...(4)$$

(2) and (3) show that p is independent of r and θ. Thus p is function of z alone. Further q_z is function of r alone by (1). Hence (4) may be re-written as

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)\right]=\frac{dp}{dz} \qquad \ldots(5)$$

Differentiating both sides of (5) *w.r.t.* `z', we find

$$0=\frac{d^2p}{dz^2} \qquad \text{or} \qquad \frac{d}{dz}\left(\frac{dp}{dz}\right)=0$$

so that $\qquad dp/dz$ = const. = P (say). ...(6)

We may take $\qquad P = (p_2 - p_1)/l,$ ...(7)

where p_1, p_2 denote the values of p, at the ends of a length l of the circular pipe. In what follows, we now write $q_z = u$. Then, using (6), (5) reduces to

$$\frac{d}{dz}\left(r\frac{du}{dr}\right)=\frac{\Pr}{\mu} \qquad \ldots(8)$$

Integrating (8), $\qquad r\dfrac{du}{dr}=\dfrac{Pr^2}{2\mu}+A \qquad$ or $\qquad \dfrac{du}{dr}=\dfrac{Pr}{2\mu}+\dfrac{A}{r}$ ...(9)

Integrating (9), $\qquad u=(Pr^2/4\mu)+A\log r+B,$ ...(10)

where the constants A and B are to be found by using the boundary conditions. Now u must be finite on the axis of the tube (where $r = 0$). So we must take $A = 0$ in (10) because otherwise u would become infinite when $r = 0$. Thus (10) reduces to

$$u=(Pr^2/4\mu)+B. \qquad \ldots(11)$$

Since the circular boundary of the tube is at rest, the no-slip condition at the wall gives rise to the following boundary condition

$$u = 0 \qquad \text{at} \qquad r = a. \qquad \ldots(12)$$

Using (12), (11) gives $B = -(Pa^2/4\mu)$. Hence (11) becomes

$$u=-(Pa^2/4\mu)\times\{1-(r/a)^2\}, \qquad \ldots(13)$$

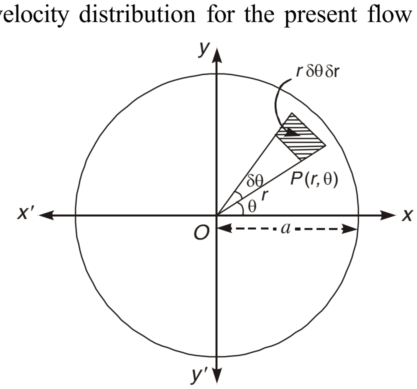
which has the form of a paraboloid of revolution as shown in the figure on page 16.9.

(*i*) **To determine the maximum and average velocities.**

From (13), it follows that the maximum velocity $u_{\max}$ can be obtained by putting $r = 0$ in it. Thus maximum velocity occurs on the axis of the pipe and is given by

$$u_{\max}=-(Pa^2/4\mu) \qquad \ldots(14)$$

where $P < 0$. The average velocity distribution for the present flow is given by



$$u_a = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a urdrd\theta \;= -\frac{P}{4\pi\mu}\int_0^{2\pi}\int_0^a r(1-r^2/a^2)drd\theta, \text{ by (13)}$$

$$= -\frac{P}{4\pi\mu}\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4a^2}\right]_0^a d\theta = -\frac{P}{4\pi\mu}\times\frac{a^2}{4}\int_0^{2\pi} d\theta = -\frac{Pa^2}{16\pi\mu}\times 2\pi$$

Thus, $$u_a = -(Pa^2/8\mu) = (1/2)\times u_{\max}, \text{ by (14)} \qquad ...(15)$$

The volumetric flow per unit time over any section is given by

$$Q = \pi a^2 \times u_a = -(\pi a^4 P)/8\mu. \qquad ...(16)$$

(*ii*) **To determine shearing stress, skin friction and the coefficient of friction.**

Using (13), the shearing stress distribution for the present flow is given by

$$\sigma_{rz} = -\mu\left(\frac{du}{dr}\right) \quad \text{or} \quad \sigma_{rz} = -\mu\left(\frac{Pa^2}{4\pi}\right)\frac{2r}{a^2} = -\frac{rP}{2}. \qquad ...(17)$$

Then the skin friction (*i.e.* shearing stress at the wall $r = a$) is given by

$$[\sigma_{rz}]_{r=a} = -\frac{aP}{2} = 4\mu\frac{u_a}{a}, \text{ using (14) and (15)} \qquad ...(18)$$

∴ Drag per unit length of the tube $= 2\pi a\times[\sigma_{rz}]_{r=a} = 2\pi a\times(-aP/2) = -\pi a^2 P.$

The (local) coefficient of friction C_f is given by

$$C_f = \frac{[\sigma_{rz}]_{r=a}}{(1/2)\times\rho u_a^2} = \frac{(4\mu u_a)/a}{(1/2)\times\rho u_a^2} = 16\times\frac{\pi}{2a\rho u_a}. \qquad ...(19)$$

If Re = $(2a\rho u_a)/\mu$ = Reynold's number, then (19) reduces to

$$C_f = 16/\text{Re}, \qquad ...(20)$$

showing that skin friction can be obtained from the knowledge of Re. The above formula is used to determine energy losses in pipe flows.

16.4*B*. Laminar steady flow between two coaxial circular cylinders

[Agra 2008; I.A.S. 2001; Meerut 2003, 04, 09; Garhwal 1993; G.N.D.U. Amritsar 2003]

For the present problem, we consider all basic equations in cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe. Clearly, the radial and tangential velocity components are zero *i.e.* $q_r = q_\theta = 0$. Due to axial symmetry of flow, q_z will be independent of θ. Further the equation of continuity for steady flow, namely,

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0 \quad \text{reduces to} \quad \frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence q_z is function of r alone, *i.e.* $q_z = q_z(r)$.

Thus, $$q_r = 0, \qquad q_\theta = 0 \qquad \text{and} \qquad q_z = q_z(r). \qquad ...(1)$$

For the present steady axi-symmetrical flow of incompressible fluid with velocity components (1), the equations of motion in cylinderical coordinates reduce to

$$0 = -\partial p/\partial r \qquad ...(2)$$

$$0 = -(1/r)\times(\partial p/\partial\theta) \qquad ...(3)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)\right] \qquad ...(4)$$

(2) and (3) show that p is independent of r and θ. Thus p is function of z alone. Further q_z is function of r alone by (1). Hence (4) may be re-written as

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)\right]=\frac{dp}{dz}. \qquad \ldots(5)$$

Differentiating both sides of (5) *w.r.t.* `z', we find

$$0=\frac{d^2p}{dz^2} \qquad \text{or} \qquad \frac{d}{dz}\left(\frac{dp}{dz}\right)=0$$

so that $\qquad dp/dz$ = const. = P, say. $\qquad \ldots(6)$

We may take $\qquad P=(p_2-p_1)/l, \qquad \ldots(7)$

where p_1, p_2 denote the values of p at the ends of a length l of the tube. In what follows, we now write $q_z=u$. Then using (6), (5) reduces to

$$\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{\Pr}{\mu} \qquad \ldots(8)$$

Integrating (8), $\quad r\dfrac{du}{dr}=\dfrac{\Pr^2}{2\mu}+A \qquad$ or $\qquad \dfrac{du}{dr}=\dfrac{Pr}{2\mu}+\dfrac{A}{r} \qquad \ldots(9)$

Integrating (9), $\qquad u=(\Pr^2/4\mu)+A\log r+B, \qquad \ldots(10)$

where A and B are arbitrary constants of integration.

Suppose there are two coaxial circular cylinders of radii a and b ($b>a$) through which laminar steady flow without body forces of an incompressible fluid takes place along the axial direction as shown in the adjoining figure. Since the circular boundaries of both the tubes are at rest, the no-slip conditions at their walls give rise to the following boundary conditions.

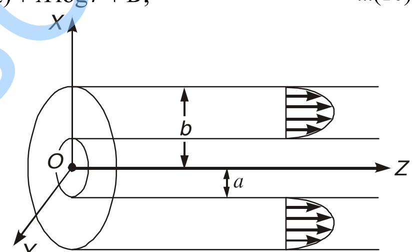


Fig (*i*)

$u=0$ at $r=a$; and $u=0$ at $r=b$. $\qquad \ldots(11)$

Using (11), (10) gives

$$0=(Pa^2/4\mu)+A\log a+B \qquad \text{and} \qquad 0=(Pb^2/4\mu)+A\log b+B$$

Solving these, $\quad A=-\dfrac{P}{4\mu}\times\dfrac{b^2-a^2}{\log(b/a)}, \quad$ and $\quad B=-\dfrac{Pa^2}{4\mu}+\dfrac{P}{4\mu}\times\dfrac{b^2-a^2}{\log(b/a)}\log a.$

Substituting these values in (1), $\quad u=-\dfrac{P}{4\mu}\left[a^2-r^2+(b^2-a^2)\dfrac{\log(r/a)}{\log(b/a)}\right] \qquad \ldots(12)$

(*i*) **To determine volumetric rate of flow Q and average velocity. [Himanchal 2003]**

The flux of the fluid (*i.e.* volumetric flow per unit time over any section of the annulus) Q is given by

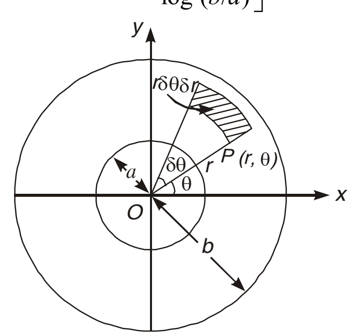


Fig. 2

$$Q = \int_0^{2\pi}\int_a^b u\, r\, dr\, d\theta \;= -\frac{P}{4\mu}\int_0^{2\pi}\int_a^b r\left[a^2 - r^2 + (b^2 - a^2)\frac{\log (r/a)}{\log (b/a)}\right] dr\, d\theta$$

$$= -\frac{P}{4\mu}[\theta]_0^{2\pi} \times \int_a^b r\left[a^2 - r^2 + (b^2 - a^2)\frac{\log (r/a)}{\log (b/a)}\right] dr$$

$$= -\frac{\pi P}{2\mu}\int_a^b \left[r\,(a^2 - r^2) + \frac{b^2 - a^2}{\log (b/a)} \times r \log \frac{r}{a}\right] dr$$

$$= -\frac{\pi P}{2\mu}\left[\left\{\frac{a^2 r^2}{2} - \frac{r^4}{4}\right\}_a^b + \frac{b^2 - a^2}{\log(b/a)}\left\{\left(\frac{r^2}{2}\log\frac{r}{a}\right)_a^b - \int_a^b \frac{r^2}{2} \times \frac{1}{(r/a)} \times \frac{1}{a}\, dr\right\}\right]$$

$$= -\frac{\pi P}{2\pi}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{2}(b^4 - a^4) + \frac{b^2 - a^2}{\log (b/a)}\left\{\frac{b^2}{2}\log\frac{b}{a} - \frac{1}{4}\left[r^2\right]_a^b\right\}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{4}(b^4 - a^4) + \frac{b^2 - a^2}{\log (b/a)}\left\{\frac{b^2}{2}\log\frac{b}{a} - \frac{1}{4}(b^2 - a^2)\right\}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{4}(b^4 - a^4) + \frac{b^2}{2}(b^2 - a^2) - \frac{(b^2 - a^2)^2}{4 \log (b/a)}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{1}{2}(b^2 - a^2)(b^2 + a^2) - \frac{1}{4}(b^4 - a^4) - \frac{(b^2 - a^2)^2}{4 \log(b/a)}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{1}{2}(b^4 - a^4) - \frac{1}{4}(b^4 - a^4) - \frac{(b^4 - a^2)^2}{4 \log (b/a)}\right] = -\frac{\pi P}{8\mu}\left[(b^4 - a^4) - \frac{(b^2 - a^2)^2}{\log (b/a)}\right] \quad \ldots(13)$$

The average velocity u_a in the annuals is given by

$$u_a = \frac{Q}{\pi (b^2 - a^2)} = -\frac{P}{8\mu}\left[b^2 + a^2 - \frac{b^2 - a^2}{\log (b/a)}\right], \text{ using (13)} \quad .(13)'$$

(*ii*) **To determine the stress and the skin frictions (*i.e.*, the shearing stress at the walls of the inner and outer cylinders).** **[Meerut 2009]**

Using (3), the shearing stress distribution is given by

$$\sigma_{rz} = \mu\frac{du}{dr} = -\frac{P}{4}\left[-2r + \frac{b^2 - a^2}{\log (b/a)} \times \frac{1}{r}\right] = -\frac{P}{4}\left[\frac{b^2 - a^2}{r \log (b/a)} - 2r\right] \quad \ldots(14)$$

Hence the skin frictions at the inner and outer cylinder are respectively given by

$$(\sigma_{rz})_{r=a} = -\frac{P}{4}\left[\frac{b^2 - a^2}{a \log (b/a)} - 2a\right] \quad \ldots(15)$$

and

$$(\sigma_{rz})_{r=b} = -\frac{P}{4}\left[\frac{b^2 - a^2}{b \log (b/a)} - 2b\right]. \quad \ldots(16)$$

From (15) and (16), it follows that skin frictions at both walls are positive ; however, the velocity gradient at the wall of the outer cylinder is negative as shown in the figure (i).

16.5. Laminar steady flow of incompressible viscous fluid in tubes of cross-section other than circular.

In usual practice the pipes of different shapes are employed in order to transport a given fluid. Accordingly, we now study steady flows of viscous incompressible fluids through infinite pipes of various cross-sections.

In such cases we take the only component of velocity, different from zero, to be the velocity parallel to the axis of the tube.

Taking z-axis along the axis of the tube, we take $u = v = 0$ and hence the equation of continuity gives $\partial w/\partial z = 0$ so that

$$w = w\,(x, y), \quad \ldots (i)$$

i.e. w is a function of x and y only. The equations of motion are

$$0 = -\,\partial p/\partial x, \quad \ldots(ii)$$

$$0 = -\,\partial p/\partial y, \quad \ldots(iii)$$

$$0 = -\partial p/\partial z + \mu(\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2) \quad \ldots(iv)$$

From (*ii*) and (*iii*), we see that p is independent of x and y. Hence (*iv*) reduces to

$$\mu(\partial^2 w/\partial x^2 + \partial w/\partial y^2) = dp/dz \quad \ldots (iv)'$$

Differentiating both sides of (*iv*) *w.r.t.* `z', we get

$$0 = \frac{d}{dz}\left(\frac{dp}{dz}\right), \quad \text{giving} \quad \frac{dp}{dz} = \text{constant} = -P, \text{ say}$$

$\therefore$ (*iv*)′ reduces to $\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = -P/\mu$, ...(*v*)

with the boundary condition $w = 0$ on the surface of the tube.

Thus the problem reduces to solving Poission's equation (*v*) with the boundary condition $w = 0$ on the surface of the tube. Direct solution of (*v*) is not easy. So to simplify the solution we convert (*v*) into a Laplace equation by the transformation :

$$w = w_1 - (P/4\mu) \times (x^2 + y^2), \quad \ldots(vi)$$

then w_1 satisfies the equation $\partial^2 w_1/\partial x^2 + \partial^2 w_1/\partial y^2 = 0$ ...(*vii*)

with the boundary condition $w_1 = (P/4\mu) \times (x^2 + y^2)$ on the surface of the tube.

Thus in order to solve the problem for a particular boundary, we take

$$w = w_1 + B - (P/4\mu) \times (x^2 + y^2), \quad \ldots(viii)$$

where B is a constant, w_1 is a suitable solution of the two-dimensional Laplace's equation and apply the condition that $w = 0$ on the surface of the tube, then B is found out.

To illustrate the whole procedure, we shall take the cross-section of the tubes as ellipse, equilateral triangle and rectangle.

Case I. Tube having elliptic cross-section.

[Meerut 2012, Kurukshetra 2000 Himanchal 1998, 2000]

Let the cross section of the tube to be an ellipse $x^2/a^2 + y^2/b^2 = 1.$ ...(1)

Let $w = A\,(x^2 - y^2) + B - (P/4\mu) \times (x^2 + y^2).$ ...(2)

On the boundary of the pipe $w = 0$. Hence the boundary is given by

$$0 = A(x^2 - y^2) + B - (P/4\mu) \times (x^2 + y^2)$$

or $$\frac{1}{B}\left(\frac{P}{4\mu} - A\right)x^2 + \frac{1}{B}\left(\frac{P}{4\mu} + A\right)y^2 = 1 \qquad ...(3)$$

(3) must now be identical to (1) and hence, we have

$$\frac{1}{B}\left(\frac{P}{4\mu} - A\right) = \frac{1}{a^2} \quad \text{and} \quad \frac{1}{B}\left(\frac{P}{4\mu} + A\right) = \frac{1}{b^2}. \qquad ...(4)$$

Solving (4), $$A = \frac{P}{4\mu} \times \frac{a^2 - b^2}{a^2 + b^2} \quad \text{and} \quad B = \frac{P}{4\mu} \times \frac{a^2 b^2}{a^2 + b^2} \qquad ...(5)$$

Putting these values of A and B, (2) gives

$$w = \frac{P}{4\mu}\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) + \frac{P}{4\mu}\frac{a^2 b^2}{a^2 + b^2} - \frac{P}{4\mu}(x^2 + y^2) = \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \qquad ...(6)$$

Now the flux Q (*i.e.* the volume discharged through the tube per unit time) can be obtained by double integration over the elliptic section (1) and is given by

$$Q = \iint w\, dxdy = \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\iint (1 - x^2/a^2 - y^2/b^2)\, dxdy$$

$$= \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left[\iint dxdy - \frac{1}{a^2}\iint x^2 dxdy - \frac{1}{b^2}\iint y^2\, dxdy\right]$$

$$= \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left[\pi ab - \frac{1}{a^2} \times \pi ab \times \frac{a^2}{4} - \frac{1}{b^2} \times \pi ab \times \frac{b^2}{4}\right],$$ the second and third integrals being moments of inertia,and Ist integral = area of elliptic cross-section = πab]

Thus, $$Q = (\pi P a^3 b^3)/4\mu(a^2 + b^2)$$

Case II Tube having equilateral triangular cross-section.

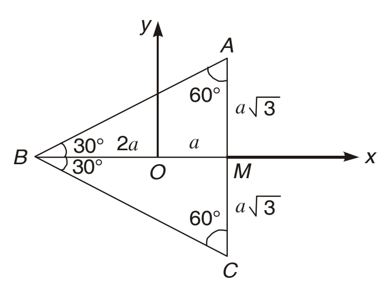
[Kurukshetra 1999, Meerut 2000, Himachal 1998; Kanphur 1998, 2000]

Let $$w = A(x^3 - 3xy^2) + B - (P/4\mu) \times (x^2 + y^2). \qquad ..(1)$$

On the boundary of the pipe $w = 0$. Hence the boundary is given by

$$A(x^3 - 3xy^2) + B - (P/4\mu) \times (x^2 + y^2) = 0 \qquad ...(2)$$

If $x = a$ be a part of the boundary, then



$$\text{A}(a^3 - 3ay^2) + B - (P/4\mu) \times (a^2 + y^2) = 0$$

or $$Aa^3 + B - (P/4\mu) \times a^2 - y^2(3aA + P/4\mu) = 0$$

so that $$Aa^3 + B - Pa^2/4\mu = 0 \quad \text{and} \quad -(3aA + P/4\mu) = 0$$

Solving these, $$A = -\frac{P}{12a\mu}, \quad B = \frac{Pa^2}{3\mu}. \qquad ...(3)$$

Putting these values of A and B in (2), the boundary is given by

$$-\frac{P}{12a\mu}(x^3 - 3xy^2) + \frac{Pa^2}{3\mu} - \frac{P}{4\mu}(x^2 + y^2) = 0$$

or $x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3 = 0$ or $(x-a)(x+2a-y\sqrt{3})(x+2a+y\sqrt{3}) = 0$

i.e. the boundary consists of $x = a$, $y = (x+2a)/\sqrt{3}$, $y = -(x+2a)/\sqrt{3}$

which represent sides AC, BA and BC of an equilateral triangle ABC as shown in the figure. Here $BM = 3a$. Origin of the coordinate system is taken at centre of the triangle, x-axis along BM and y-axis is parallel to AC. Side $AC = 2\,AM = 2 \times BM \tan 30^\circ = 2 \times 3a \times (1/\sqrt{3}) = 2\sqrt{3}a$ Putting values of A and B in (1), we get

$$w = -(P/12a\mu)\times(x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3). \qquad ...(4)$$

If Q be the flux of the fluid over an area of equilateral triangular cross-section, we have

$$Q = \iint w\,dxdy = -\frac{P}{12a\mu}\int_{x=-2a}^{a}\int_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}}(x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3)\,dxdy$$

$$= -\frac{P}{12a\mu}\int_{-2a}^{a}\left\{\left(x^3 + 3ax^2 - 4a^3\right)[y]_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}} - 3(x-a)\left[\frac{1}{3}y^3\right]_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}}\right\}dx$$

$$= -\frac{P}{6\sqrt{3}a\mu}\int_{-2a}^{a}\left(\frac{2}{3}x^4 + \frac{10}{3}ax^3 + 4a^2x^2 - \frac{8}{3}a^3x - \frac{10a^4}{3}\right)dx = \frac{27}{20\sqrt{3}}\times\frac{Pa^4}{\mu}$$

$$\text{Average flow} = \frac{\text{Flux}}{\text{Area}} = \frac{27}{20\sqrt{3}}\times\frac{Pa^4}{\mu}\times\frac{1}{(1/2)\times(3a)\times(2a\sqrt{3})} = \frac{3}{20}\times\frac{Pa^2}{\mu}.$$

Case III Tube having rectangular cross-section.

Consider the flow through a rectangular pipe whose corss-section is bounded by the lines $x = \pm a$ and $y = \pm b$.

Take $$w = w_1 - (P/4\mu)\times(x^2 + y^2), \qquad ...(1)$$

where w_1 is a plane harmonic.

Now $w = 0$, on the boundary $x = \pm a$, $y = \pm b$. ...(2)

Boundary conditions (2) and (1) show that on the boundary $x = \pm a, y = \pm b.$ we must have

$$w_1 = (P/4\mu)\times(x^2 + y^2)$$

Take again, $$w_1 = w_2 + (P/4\mu)\times(x^2 - y^2) + K, \qquad ...(3)$$

where since $(x^2 - y^2)$ and w_1 are plane harmonic functions, w_2 is also plane harmonic such that on the boundary

$$\frac{P}{4\mu}(x^2 + y^2) = w_2 + \frac{P}{4\mu}(x^2 - y^2) + K, \quad i.e., \quad w_2 = \frac{P}{2\mu}y^2 - K = \frac{P}{2\mu}(y^2 - b^2), \qquad ...(4)$$

where $$K = Pb^2/2\mu. \qquad ...(5)$$

$\therefore$ From (1) and (3), we have

$$w = w_2 + \frac{P}{4\mu}(x^2 - y^2) + K - \frac{P}{4\mu}(x^2 + y^2) = w_2 + \frac{P}{4\mu}(x^2 - y^2) + \frac{Pb^2}{2\mu} - \frac{P}{4\mu}(x^2 + y^2) \text{ by (5)}$$

$$\therefore \quad w = w_2 + (P/2\mu)\times(b^2 - y^2), \qquad ...(6)$$

where w_2 is a plane harmonic such that on the boundary $w = 0$ so that

$$w_2 = (P/2\mu)\times(y^2 - b^2), \quad \text{when} \quad x = \pm a, \quad y = \pm b. \qquad ...(7)$$

Since w_2 is a plane harmonic, it must satisfy the Laplace's equation

$$\partial^2 w_2/\partial x^2 + \partial^2 w_2/\partial y^2 = 0. \qquad ...(8)$$

Let $$w_2 = X(x)\,Y(y). \qquad ...(9)$$

$\therefore$ From (8), $\dfrac{1}{X}\dfrac{d^2X}{dx^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = 0$ or $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}$ ...(10)

L.H.S. of (10) is function of x alone whereas *R.H.S.* of (10) is function of y alone. So (10) is valid only if each side is a constant, say λ^2. So (10) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2, \quad -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \;\Rightarrow\; \frac{d^2X}{dx^2} - \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + \lambda^2 Y = 0. \qquad ...(11)$$

Solving (11), we get
$$\left.\begin{aligned} X &= A\cosh\lambda x + B\sinh\lambda x \\ Y &= C\cos\lambda y + D\sin\lambda y. \end{aligned}\right\} \qquad ...(12)$$

$\therefore$ A solution of (8) is given by

$$w_2 = \Sigma\,(A\cosh\lambda x + B\sinh\lambda x)\,(C\cos\lambda y + D\sin\lambda y) \qquad ...(13)$$

Since $w_2 = (P/2\mu)\times(y^2 - b^2)$ when $x = \pm a$ and also when $y = \pm b$, the terms containing sinh λx and sin λy must be taken zero in (13), so we have

$$w_2 = \Sigma E_\lambda \cosh\lambda x \cos\lambda y, \text{ where } E_\lambda\,(= AC) \text{ are new arbitrary constants} \qquad ...(14)$$

When $y = \pm b$, $w_2 = 0$ from (7). H[illegible] $\cos\lambda b = 0$,

giving $\lambda b = \{(2m+1)\pi\}/2$, m being an integer so that $\lambda = \{(2m+1)\pi\}/2b$

$\therefore$ From (14), $$w_2 = \sum_{m=0}^{\infty} E_{2m+1}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}. \qquad ...(15)$$

Using boundary condition (7), namely, when $x = \pm a$, $w_2 = (P/2\mu)\times(y^2 - b^2)$, (15) gives

$$\frac{P}{2\mu}(y^2 - b^2) = \sum_{m=0}^{\infty} E_{2m+1}\cosh\left\{(2m+1)\frac{\pi a}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\} \qquad ...(16)$$

Multiplying both sides of (16) by $\cos\left\{(2m+1)\dfrac{\pi y}{2b}\right\}$ and integrating between the limits $-b$ and b and noting that

$$\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}\cos\left\{(2n+1)\frac{\pi y}{2b}\right\}dy = \begin{cases} 0, & n \neq m \\ b, & n = m \end{cases}$$

and $$\int_{-b}^{b}(y^2 - b^2)\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= \left[(y^2 - b^2)\times\frac{2b}{(2m+1)\pi}\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b} - \int_{-b}^{b} 2y\times\frac{2b}{(2m+1)\pi}\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= -\frac{4b}{(2m+1)\pi}\int_{-b}^{b} y\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= -\frac{4b}{(2m+1)\pi}\left[\left[y\times\frac{-2b}{(2m+1)\pi}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b} - \int_{-b}^{b}\frac{-2b}{(2m+1)\pi}\cos\left\{(2m+1\frac{\pi y}{2b}\right\}dy\right]$$

$$= -\frac{8b^2}{(2m+1)^2\pi^2}\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy = -\frac{8b^2}{(2m+1)^2\pi^2}\times\frac{2b}{(2m+1)\pi}\left[\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b}$$

$$= -\frac{16b^3}{(2m+1)^3\pi^3}\times 2\sin\left\{(2m+1)\frac{\pi}{2}\right\} = -\frac{32b^3(-1)^m}{(2m+1)^3\pi^3},$$

we obtain
$$\frac{P}{2\mu}\times\frac{-32b^3(-1)^m}{(2m+1)^3\pi^3} = b\,E_{2m+1}\cosh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

giving
$$E_{2m+1} = -\frac{P}{\mu}\times\frac{16b^2}{(2m+1)^3\pi^3}\times\frac{(-1)^m}{\cosh\{(2m+1)\pi a/2b\}}$$

$\therefore$ From (15),
$$w_2 = -\sum_{m=0}^{\infty}\frac{P}{\mu}\times\frac{16b^2(-1)^m}{(2m+1)^3\pi^3}\times\frac{\cosh\{(2m+1)\pi x/2b\}}{\cosh\{(2m+1)\pi a/2b\}}\cos\left\{2m+1)\frac{\pi y}{2b}\right\}$$

Putting this value of ω_2 in (6), we have

$$w = \frac{P}{2\mu}(b^2-y^2) - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}\frac{\cosh\{(2m+1)\pi x/2b\}}{\cosh\{(2m+1)\pi a/2b\}}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\} \quad ...(17)$$

Flux Q of the fluid over an area of rectangular cross-section, is given by

$$Q = \int_{y=-b}^{b}\int_{x=-a}^{a} w\,dxdy = \frac{P}{2\mu}\int_{-b}^{b}\int_{-a}^{a}(b^2-y^2)\,dxdy - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}\times\int_{-b}^{b}\int_{-a}^{a}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dxdy$$

$$= \frac{P}{2\mu}\int_{-b}^{b}(b^2-y^2)\,dy\times\int_{-a}^{a}dx - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}\times\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy\times\int_{-a}^{a}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}dx$$

$$= \frac{P}{2\mu}\left(2b^3-\frac{2b^3}{3}\right)\times 2a - \frac{p}{\mu}\times\frac{16b^2}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}$$

$$\times\frac{2b}{(2m+1)\pi}\left[\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b}\times\frac{2b}{(2m+1)\pi}\left[\sinh\left\{(2m+1)\frac{\pi x}{2b}\right\}\right]_{-a}^{a}$$

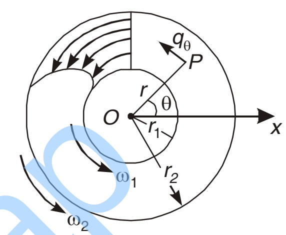
$$= \frac{4Pab^3}{3\mu} - \frac{64Pb^4}{\mu\pi^5} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^5 \cosh\{(2m+1)\pi a/2b\}} \times 2\,(-1)^m \times 2\sinh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

$$= \frac{4Pab^3}{3\mu} - \frac{256Pb^4}{\mu\pi^5} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^5} \tanh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

16.6. Laminar flow between two concentric rotating cylinders- couette flow.

[Meerut 2011, 12; Himachal 2001, 06, 07, 10]

Consider two infinitely long, concentric circular cylinders of radii r_1 and r_2 rotating with constant angular velocities ω_1 and ω_2. Let there be viscous incompressible fluid in the annular space. Then the cylinders induce a steady, axi-symmetric, tangential motion in the fluid. Let z-axis be taken along the axis of the cylinders. Since the motion is only tangential, we have $q_r = 0$, $q_z = 0$. Then the continuity equation in cylindrical coordinates reduces to $\partial q_\theta/\partial\theta = 0$, so that q_θ depends on r and z only.



Again, the cylinder being very long, the flow will not depend on z. Hence $q_\theta = q_\theta(r)$. Hence the equations of motion (refer 13 (a) to 13 (c), Art. 14.11 of chapter 14) in cylindrical coordinates for the present problem reduce to

$$\rho\,(q_\theta^2/r) = \partial p/\partial r \qquad ...(1)$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{\partial^2 q_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r^2}\right] \quad \text{or} \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left\{\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right)\right\} \qquad ...(2)$$

$$0 = -(\partial p/\partial z) \qquad ...(3)$$

Equation (3) shows that p is independent of z. Since q_θ is function of r only and the flow is axially symmetric, it follows that p must be either a function of r or a constant. Hence we take $\partial p/\partial\theta = 0$ in (2) and so it may be re-written as

$$\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = 0. \qquad ...(4)$$

Integrating (4), $\quad dq_\theta/dr + q_\theta/r = 2A$, A being an arbitrary constant.

or $\quad \dfrac{1}{r}\dfrac{d}{dr}(rq_\theta) = 2A \quad$ or $\quad \dfrac{d}{dr}(rq_\theta) = 2Ar.$

Integrating it, $\quad rq_\theta = Ar^2 + B \quad$ or $\quad q_\theta = Ar + B/r, \qquad ...(5)$

where A and B are constants of integration to be determined. These constants are obtained from the boundary conditions :

$$\left.\begin{aligned} q_\theta &= r_1\omega_1 \quad at \quad r = r_1 \\ q_\theta &= r_2\omega_2 \quad at \quad r = r_2 \end{aligned}\right\} \qquad ...(6)$$

Using (6), (5) yields $\quad r_1\omega_1 = Ar_1 + B/r_1 \quad$ and $\quad r_2\omega_2 = Ar_2 + B/r_2 \qquad ...(7)$

On solving (7), $\quad A = \dfrac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}, \quad$ and $\quad B = -\dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1).$

Substituting these values into (5), we obtain

$$q_\theta = \frac{1}{r_2^2 - r_1^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)\, r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right] \qquad ...(8)$$

As explained earlier, equation (1) may be written as

$$\frac{dp}{dr} = \frac{\rho}{r} q_\theta^2 \qquad \text{or} \qquad \frac{dp}{dr} = \frac{\rho}{r\,(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2) r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right]^2$$

or $$\frac{dp}{dr} = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 r - \frac{2 r_1^2 r_2^2}{r}(\omega_2 r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1) + \frac{r_1^4 r_2^4}{r^3}(\omega_2 - \omega_1)^2\right] \qquad ...(9)$$

Integration of (9) gives

$$p = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \frac{r^2}{2} - 2 r_1^2 r_2^2 (\omega r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1) \log r - \frac{r_1^4 r_2^4}{2r^2}.(\omega_2 - \omega_1)^2\right] + C, \qquad ...(10)$$

where C is constant of integration to be determined. Suppose that $p = p_1$ at $r = r_1$ Then (10) gives

$$p = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \frac{r_1^2}{2} - 2 r_1^2 r_2^2 (\omega_2 r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1) \log r_1 - \frac{r_1^4 r_2^4}{2r_1^2}.(\omega_2 - \omega_1)^2 + C\right] \qquad ...(11)$$

Subtracting (11) from (10) and re-writing the resulting equation, we have

$$p = p_1 + \frac{\rho}{(r_2^2 - r_1^2)^2}\Bigg[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \left(\frac{r^2 - r_1^2}{2}\right) - 2 r_1^2 r_2^2 (\omega_2 r_2^2 - \omega_1^2 r_1^2)(\omega_2 - \omega_1) \log \frac{r}{r_1}$$
$$- (r_1^4 r_2^4 / 2) \times (\omega_2 - \omega_1)^2 (1/r^2 - 1/r_1^2) \Bigg] \qquad ...(12)$$

The shearing stress for the present problem is given by*

$$\sigma_{r\theta} = \mu (dq_\theta / dr - q_\theta / r) \qquad ...(13)$$

Substituting the value of q_θ given by (8) into (13), we have

$$\sigma_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}\, \frac{r_1^2 r_2^2}{r^2}(\omega_2 - \omega_1) \qquad ...(14)$$

Hence the shearing stress at the walls of the outer and inner cylinders are given by

$$(\sigma_{r\theta})_{r = r_2} = \frac{2\mu r_1^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1) \qquad ...(15)$$

and $$(\sigma_{r\theta})_{r = r_1} = \frac{2\mu r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1). \qquad ...(16)$$

Deduction. Let the inner cylinder be at rest, *i.e.*, $\omega_1 = 0$. Then, writing $q_\theta = r\omega$, (8) gives

$$\omega = \frac{\omega_2 r_2^2 (r^2 - r_1^2)}{r^2 (r_2^2 - r_1^2)} = \frac{\omega_2 r_2^2}{r_2^2 - r_1^2}\left(1 - \frac{r_1^2}{r^2}\right) \qquad ...(17)$$

where ω is the angular velocity of the fluid at any point $P(r, \theta)$.

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* Refer results (4) and (6) of Art. 14.11 in chapter 14

For the present problem there would be only tangential stress $\sigma_{r\theta}$ given by

$$\sigma_{r\theta} = \mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right) = \mu\left(\omega + r\frac{d\omega}{dr} - \omega\right) = \mu r\frac{d\omega}{dr}$$

$\therefore$ Moment of $\sigma_{r\theta}$ about the common axis of the cylinders

$$= (\sigma_{r\theta} \times 2\pi r) \times r \;= 2\pi\mu r^3 \frac{d\omega}{dr} = 4\pi\omega_2\mu\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}, \text{ using (7)}$$

16.7 Illustrative solved examples

Ex. 1. *Incompressible liquid is flowing steadily through a circular pipe. Prove that the mean pressure is constant over the cross section and that the rate of flow is* $\pi a^4 (p_1 - p_2)/8\mu l$, *where* p_1 *and* p_2 *are the pressures over sections at distance* l *apart.* **[Himanchal 1999; Agra 2000, 06; 09; 11, Meerut 2004; Nagpur 2003, 06; Mumbai 2005; Patna 2003]**

Sol. Refer Art. 16.4 A, We have $\qquad P = (p_2 - p_1)/l = -(p_1 - p_2)/l \qquad$ ...(1)

and so $\qquad Q = -\dfrac{\pi a^4 P}{8\mu} = \dfrac{\pi a^4 (p_1 - p_2)}{8\mu l}$, using (1)

Ex. 2. *The space between two co-axial cylinders of radii a and b is filled with viscous fluid, and the cylinders are made to rotate with angular velocities* ω_1, ω_2. *Prove that in steady motion the angular velocity of the fluid is given by*

$$\omega = \{a^2 (b^2 - r^2)\omega_1 + b^2 (r^2 - a^2)\omega_2\}/r^2(b^2 - a^2)$$

[Agra 1998; Meerut 2001; Himanchal 1999]

Sol. Setting $r_1 = a$ and $r_2 = b$ in equation (8) of Art. 16.6, we have

$$q_\theta = \frac{1}{b^2 - a^2}\left[(\omega_2 b^2 - \omega_1 a^2) r - \frac{a^2 b^2}{r}(\omega_2 - \omega_1)\right]$$

or $$\omega r = [(\omega_2 b^2 - \omega_1 a^2) r^2 - a^2 b^2 (\omega_2 - \omega_1)]/r(b^2 - a^2)$$

or $$\omega = \{a^2 (b^2 - r^2)\omega_1 + b^2 (r^2 - a^2)\omega_2\}/r^2(b^2 - a^2)$$

Ex. 3. *A viscous liquid flows steadily parallel to the axis in the annular space between two co-axial cylinders of radii a, na (n > 1). Show that the rate of discharge is*

$$\frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right],$$ *where P is the pressure gradient.* **(Meerut 2007)**

Sol. From equation (13) of Art. 16.4*B*, we have

$$Q = -\frac{\pi P}{8\mu}\left[b^4 - a^4 - \frac{(b^2 - a^2)^2}{\log(b/a)}\right] \qquad ...(i)$$

Here $b = an$. Also replacing P by $-P$ in (*i*) for the present problem, we have

$$Q = \frac{\pi P}{8\mu}\left[a^4 (n^4 - 1) - \frac{a^4 (n^2 - 1)^2}{\log n}\right] = \frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right].$$

Ex. 4. (*a*) *Determine the maximum value of the velocity profile in the annular space between two coaxial cylinders.*

(*b*) If a = 50 *mm*, b = 75 *mm*, and the volumetric flow of water, Q = 0.006 m^3/s, calculate (*i*) the pressure drop (*ii*) the maximum value of u and (*iii*) the shearing stress at the wall of both cylinders. Assume that μ = 1.01 *g/ms*. **[Garwhal 1995, 97]**

Sol. Part (a). Refer Art. 16.4*B*. We have

$$u = -\frac{P}{4\mu}\left[a^2 - r^2 + (b^2 - a^2)\frac{\log(r/a)}{\log(b/a)}\right]. \qquad ...(1)$$

From (1),
$$\frac{du}{dr} = -\frac{P}{4\mu}\left[-2r + \frac{b^2 - a^2}{\log(b/a)}\frac{1}{r}\right]. \qquad ...(2)$$

For the maximum value of u, we must have $du/dr = 0$, and so

$$-2r + \frac{b^2 - a^2}{\log(b/a)}\frac{1}{r} = 0 \quad \text{so that} \quad r = \pm\left[\frac{b^2 - a^2}{2\log(b/a)}\right]^{1/2}, \qquad ...(3)$$

which gives the values of r for which u will be maximum. Putting this value of r in (1), the required maximum velocity is given by

$$u_{\max} = -\frac{P}{4\mu}\left[a^2 - \frac{b^2 - a^2}{2\log(b/a)} + \frac{b^2 - a^2}{\log(b/a)}\log\frac{1}{a}\left\{\frac{b^2 - a^2}{2\log(b/a)}\right\}^{1/2}\right]$$

$$= -\frac{Pa^2}{4\mu}\left[1 - \frac{n^2 - 1}{2\log n}\left\{1 - \log\frac{n^2 - 1}{2\log n}\right\}\right], \qquad ...(4)$$

where $n = b/a = 75/50 = 1.5$.

Part (*b*). (*i*). From Art 16.4 B, we have

$$Q = -\frac{\pi P}{8\mu}\left[(b^4 - a^4) - \frac{(b^2 - a^2)^2}{\log(b/a)}\right] \quad \text{or} \quad Q = -\frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right].$$

$$0.006 = -\frac{3.14 \times (0.05)^4 P}{8 \times (1.01 \times 10^{-3})}\left[(1.5)^4 - 1 - \frac{\{(1.5)^2 - 1\}^2}{\log 1.5}\right]$$

or $\quad 0.006 = -\ 0.000486P \quad$ so that $\quad P = dp/dz = -\ 12.3\ N/m^3$

(*ii*) The mamimum velocity is given by (4) as follows

$$u_{\max} = \frac{12.3 \times (0.05)^2}{4 \times (1.01 \times 10^{-3})}\left[1 - \frac{(1.5)^2 - 1}{2\log 1.5}\left\{1 - \log\frac{(1.5)^2 - 1}{2\log 1.5}\right\}\right] = 0.96\, m/s.$$

(*iii*) The shearing stress at the walls of the two cylinders are given by [Refer equations (15) and (16) in Art. 16.4*B*]

$$(\sigma_{rz})_{r=a} = -\frac{P}{4}\left[\frac{b^2 - a^2}{a\log(b/a)} - 2a\right] = -\frac{Pa}{4}\left[\frac{n^2 - 1}{\log n} - 2\right] = \frac{12.3 \times 0.05}{4}\left[\frac{(1.5)^2 - 1}{\log 1.5} - 2\right] = 0.184\, N/m^2.$$

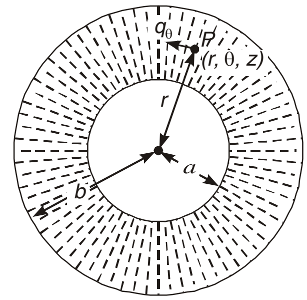
$$(\sigma_{rz})_{r=b} = -\frac{P}{4}\left[\frac{b^2 - a^2}{b\log(b/a)} - 2b\right] = -\frac{Pb}{4}\left[\frac{1 - (1/n)^2}{\log n} - 2\right] = \frac{12.3 \times 0.05}{4}\left[\frac{1 - (1/1.5)^2}{\log 1.5} - 2\right] = 0.145\, N/m^2$$

Ex. 5. *A liquid occupying the space between two co-axial circular cylinders is acted upon by a force c/r per unit mass, where r is the distance from the axis, the lines of force being circles around the axis. Prove that in the steady motion the velocity at any point is given by the formula*

$$\frac{c}{2\nu}\left\{\frac{b^2}{r}\frac{r^2 - a^2}{b^2 - a^2}\log\frac{b}{a} - r\log\left(\frac{r}{a}\right)\right\},$$

where a, b are the two radii and ν is the coefficient of kinematic viscosity.

[Agra, 2000, 02, 06; Kanpur 2002, Kolkata 2006, Rajasthan 2001]



Sol. Consider two infinitely, long, concentric circular cylinders of radii a and b ($b > a$). Let there be viscous incompressible fluid in the annular space. Since the lines of force are circles around the axis of the cylinders, this will produce steady, axi-symmetric, tangential motion in the fluid. Let z-axis be taken along the axis of the cylinders. Since the motion is only tangential, we have

$$q_r = 0 \quad \text{and} \quad q_z = 0 \qquad \text{...(1)}$$

Hence the continuity equation in cylindrical coordinates reduces to

$$\partial q_\theta / \partial \theta = 0,$$

showing that q_θ depends on r and z only. Furthermore, the cylinders being very long, the flow will not depend on z. Hence, we suppose that

$$q_\theta = q_\theta(r) = r\omega, \qquad \text{...(2)}$$

where ω is the angular velocity of the liquid at any point $P(r, \theta, z)$.

Here, body force $\mathbf{B} = \mathbf{B}(\mathbf{B}_r, \mathbf{B}_\theta, \mathbf{B}_z) = \mathbf{B}(0, c/r, 0)$. Hence $\mathbf{B}_\theta = c/r$ ...(3)

The Navier-Stokes's equation in θ-direction for axi-symmetric flow of incompressible fluid is given by (Refer equation 13 (b) in Art. 14.11, chapter 14)

$$\rho\left(\frac{\partial q_\theta}{\partial t} + q_r\frac{\partial q_\theta}{\partial r} + q_z\frac{\partial q_\theta}{\partial z} + \frac{q_r q_\theta}{r}\right) = \rho B_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\times\frac{\partial}{\partial r}(rq_\theta)\right\} + \frac{\partial^2 q_\theta}{\partial z^2}\right] \qquad \text{...(4)}$$

For the present steady $(\partial/\partial t = 0)$ and axi-symmetric ($\partial/\partial\theta = 0$) flow, using (1), (2) and (3), (4) reduces to

$$0 = \frac{\rho c}{r} + \mu\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(r^2\omega)\right\} \quad \text{or} \quad \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(r^2\omega)\right\} = -\frac{\rho c}{\mu r} = -\frac{c}{r\nu}, \quad \text{as} \quad \nu = \frac{\mu}{\rho}$$

or $$\frac{d}{dr}\left\{\frac{1}{r}\left(2r\omega + r^2\frac{d\omega}{dr}\right)\right\} = -\frac{c}{r\nu} \quad \text{or} \quad \frac{d}{dr}\left(2\omega + r\frac{d\omega}{dr}\right) = -\frac{c}{r\nu}$$

or $$2(d\omega/dr) + \{d\omega/dr + r(d^2\omega/dr^2)\} = -(c/r\nu)$$

or $$r\frac{d^2\omega}{dr^2} + 3\frac{d\omega}{dr} = -\frac{c}{r\nu} \quad \text{or} \quad r^3\frac{d^2\omega}{dr^2} + 3r^2\frac{d\omega}{dr} = -\frac{cr}{\nu}$$

or $$\frac{d}{dr}\left(r^3\frac{d\omega}{dr}\right) = -\frac{cr}{\nu}.$$

Integrating, $$r^3\frac{d\omega}{dr} = -\frac{cr^2}{2\nu} + A \quad \text{or} \quad \frac{d\omega}{dr} = -\frac{c}{2r\nu} + \frac{A}{r^3}$$

or $$d\omega = \left(-\frac{c}{2r\nu} + \frac{A}{r^3}\right)dr.$$

Integrating, $$\omega = -\frac{c}{2\nu}\log r - \frac{A}{2r^2} + B, \qquad \text{...(5)}$$

where A and B are constants of integration to be determined. To determine A and B, we use the boundary conditions :

$$\omega = 0 \quad \text{at} \quad r = 0 \qquad \text{...(6A)}$$
$$\omega = 0 \quad \text{at} \quad r = b \qquad \text{...(6B)}$$

Using (6*A*), (5) gives $$0 = -\frac{c}{2\nu}\log a - \frac{A}{2a^2} + B \qquad ...(7A)$$

Using (6B), (5) gives $$0 = -\frac{c}{2\nu}\log b - \frac{A}{2b^2} + B \qquad ...(7B)$$

Subtracting (7*A*) from (5), $$\omega = -\frac{A}{2}\left(\frac{1}{r^2} - \frac{1}{a^2}\right) - \frac{c}{2\nu}\log\left(\frac{r}{a}\right) \qquad ...(8)$$

Subtracting (7*B*) from (7*A*), we have

$$0 = -\frac{A}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{c}{2\nu}\log\left(\frac{b}{a}\right) \qquad \text{so that} \qquad A = \frac{ca^2b^2}{\nu(b^2 - a^2)}\log\left(\frac{b}{a}\right)$$

Substituting this value of *A* in (8), we have

$$\omega = -\frac{ca^2b^2\log(b/a)}{2\nu(b^2 - a^2)}\left(\frac{1}{r^2} - \frac{1}{a^2}\right) - \frac{c}{2\nu}\log\left(\frac{r}{a}\right) \quad \text{or} \quad \frac{q_\theta}{r} = \frac{c}{2\nu}\frac{r^2 - a^2}{b^2 - a^2}\frac{b^2}{r^2}\log\frac{b}{a} - \frac{c}{2\nu}\log\frac{r}{a}, \text{ using (2)}$$

$$\therefore \qquad q_\theta = \frac{c}{2\nu}\left\{\frac{b^2}{r}\frac{r^2 - a^2}{b^2 - a^2}\log\frac{b}{a} - r\log\left(\frac{r}{a}\right)\right\}.$$

Ex. 6. *Oil is filled between two concentric rotating cylinders with radii 5 in. and (11/2) in. Assume that* $\mu = 0.005$ *lbf-sec/ft*^{3}. *The inner cylinder rotates at a speed of 5 rpm, while the outer cylinder is at rest. Calculate the stress at the wall of the inner cylinder.*

Sol. Refer Art 16.6 Here, $r_2 = \frac{11}{2} \times \frac{1}{12} = \frac{11}{24}\, ft,$ $\qquad r_1 = 5 \times \frac{1}{12} = \frac{5}{12}\, ft,$

$\omega_1 = 5/60 = (1/12)\, rps,$ $\qquad \omega_2 = 0$ and $\qquad \mu = 0.005\ lbf - \sec/ft^2$

Shear stress at the wall of the inner cylinder $= (\sigma_{r\theta})_{r = r_1} = \dfrac{2\mu r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1)$

$$\therefore \qquad (\sigma_{r\theta})_{r = 5/12} = \frac{2 \times 0.005 \times (11/24)^2}{(11/24)^2 - (5/12)^2} \times \left(0 - \frac{1}{12}\right) = -\ 0.0048\ lbf/ft^2$$

Ex. 7. *In the case of steady flow of compressible liquid flowing steadily through a circular pipe of radius a, show that the mass which crosses any section per unit time is*

$$\pi a^4 (p_1 - p_2)(\rho_1 + \rho_2)/16\mu l,$$

where ρ_1 *and* ρ_2 *are the densities at two sections at distance l apart. It is assumed that the temperature is constant, and the velocity gradient in the direction of the axis may be neglected in comparison with its gradient in the direction of a radius.* **[Agra 2008, Meerut 2006]**

Sol. Using cylindrical coordinates with *z*-axis along the axis of the pipe, we have $q_r = 0 = q_\theta$.

The continuity equation reduces to $$\frac{\partial}{\partial z}(\rho q_z) = 0. \qquad ...(1)$$

Since temperature is constant, equation of state is given by $\qquad p = k\rho. \qquad ...(2)$

Using (2), (1) may be written as $$\frac{\partial}{\partial z}(p q_z) = 0. \qquad ...(3)$$

For present steady motion in absence of body forces, the Navier-Stokes equations reduce to

$$0 = -(\partial p/\partial r), \qquad ...(4)$$