LAPLACE

Contents

Laplace Transform

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1 Introduction

In mathematics the Laplace transform is an integral transform named after its discover Pierre – Simon Laplace. It takes a function of a positive real variable t (often time) to a function of a complex variable s (frequency).

The transform method provides an easy and effective means for the solution of many problems arising in engineering .

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850 – 1925), to problems in electrical engineering.

It was found that Heavisides operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.

The Laplace Transform method is a technique for solving linear differential equations with initial conditions. It is commonly used to solve electrical circuit and systems problems.

The Laplace transform is very similar to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with $t > 0$.

1.i Definition

Let f(t) be a function of t defined for all positive value of t.

Then the Laplace transforms of f (t), denoted by L { f (t)} is defined by

$$\mathbf{L\,\{f(t)\} = \int_0^{\infty} e^{-st}\, f(t)\, dt = \overline{f(s)} \quad or \quad F(s)}$$

provides that the integral exists, s is a parameter which may be a real or complex number, f(t) is called objective function defined for $t \geq 0$, $\overline{f(s)}$ or F(s) is the resultant or image function, L which transforms f(t) into f(s)is called the Laplace transformation operator.

1.ii Formulae

(1) $L\{1\} = \frac{1}{s}$ $\quad s > 0$

(2) $L\{t\} = \frac{1}{s^2}$ $\quad s > 0$

(3) $L\{t^n\} = \frac{n!}{s^{n+1}}$ or $\frac{\overline{|n+1}}{s^{n+1}}$ $\quad$ where $n = 0,1,2,3,\ldots$

(4) $L\{e^{at}\} = \frac{1}{s-a}$ $\quad s > a$

(5) $L\{e^{-at}\} = \frac{1}{s+a}$ $\quad s > 0$

(6) $L\{\sin at\} = \frac{a}{s^2 + a^2}$ $\quad s > 0$

(7) $L\{\cos at\} = \frac{s}{s^2 + a^2}$ $\quad s > 0$

(8) $L\{\sin h\, at\} = \frac{a}{s^2 - a^2}$ $\quad s > |a|$

(9) $L\{\cos h\, at\} = \frac{s}{s^2 - a^2}$ $\quad s > |a|$

(10) $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$

(11) $L\{e^{-at} \sin bt\} = \frac{b}{(s+a)^2 + b^2}$

(12) $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$

(13) $L\{e^{-at} \cos bt\} = \frac{s+a}{(s+a)^2 + b^2}$

(14) $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$

(15) $L\{e^{-at} \sinh bt = \frac{b}{(s+a)^2 - b^2}$

(16) $L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2 - b^2}$

(17) $L\{e^{-at}\cosh bt\} = \dfrac{s+a}{(s+a)^2 - b^2}$

2 Laplace transform of standard functions

1) Prove that $L\{1\} = \dfrac{1}{s}$

Proof: By definition, $L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)dt$

$$L\{1\} = \int_0^\infty e^{-st}\,(1)dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-s(\infty)} - e^{-s(0)}\right]$$

$$= -\frac{1}{s}[0-1] \qquad \{\because e^{-\infty} = 0,\ e^0 = 1$$

$\therefore \quad L\{1\} = \dfrac{1}{s}$ ... Hence proved

2) Prove that $L\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$

Proof: By definition, $L\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt$

$$= \int_0^\infty e^{(-s+a)t}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-(s-a)(\infty)} - e^{-(s-a)(0)}\right] = \frac{-1}{s-a}[0-1]$$

$\therefore L\{e^{at}\} = \dfrac{1}{s-a}$ ... Hence proved

3) Prove that $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

Proof: By definition, $L\{\sin at\} = \int_0^\infty e^{-st}.\sin at\, dt$

$$= \left\{\left[\frac{e^{-st}}{s^2+a^2}\right][-s\sin at - a\cos at]\right\}_0^\infty$$

$$\therefore \ L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$\left\{\because \int e^{-at} \sin bt \, dt = \frac{e^{-at}}{a^2 + b^2}[-a\sin bt - b\cos bt]\right.$$

OR

$$L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} = \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s - ia} - \frac{1}{s + ia}\right] = \frac{1}{2i}\left[\frac{s + ia - s + ia}{(s - ia)(s + ia)}\right]$$

$$\therefore \ L\{\sin at\} = \frac{a}{s^2 + a^2} \quad \text{... Hence proved} \quad \{\because i^2 = -1$$

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4) Prove that $\mathbf{L\{\cos at\} = \dfrac{s}{s^2 + a^2}}$

Proof: By definition, $L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$

$$= \left[\frac{e^{-st}}{s^2 + a^2}[-s\cos at + a\sin at]\right]_0^\infty$$

$$\therefore \ L\{\cos at\} = \frac{s}{s^2 + a^2}$$

OR

$$\left\{\because \int e^{-at} \sin bt \, dt = \frac{e^{-at}}{a^2 + b^2}[-a\sin bt - b\cos bt]\right.$$

$$L\{\cos at\} = L\left[\frac{e^{iat} + e^{-iat}}{2}\right]$$

$$= \frac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - ia} + \frac{1}{s + ia}\right] = \frac{1}{2}\left[\frac{s + ia + s - ia}{(s - ia)(s + ia)}\right]$$

$$\therefore L\{\cos at\} = \frac{s}{s^2 + a^2} \quad \text{... Hence proved}$$

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5) Prove that $\mathbf{L\{\sinh at\} = \dfrac{s}{s^2 - a^2}}$

Proof: $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \qquad = \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$= \frac{a}{s^2 - a^2} \qquad \text{... Hence proved}$$

6) Prove that $\mathbf{L\{\cosh at\} = \frac{s}{s^2 - a^2}}$

Proof: $L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$= \frac{s}{s^2 - a^2} \qquad \text{... Hence proved}$$

7) Prove that $\mathbf{L\{t^n\} = \frac{\sqrt{(n+1)}}{s^{n+1}} = \frac{n!}{s^{n+1}}}$

Proof: By definition $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$

Put $st = u$, $\quad t = \frac{u}{s}$

Differentiating w.r.t t, $\quad s = \frac{du}{dt}$, $\quad dt = \frac{du}{s}$

when $t \to 0$ then $u \to 0$; when $t \to \infty$ then $u \to \infty$

$$\therefore L\{t^n\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s} \qquad = \frac{1}{s^n}\frac{1}{s}\int_0^\infty e^{-u}\, u^n\, du$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-u} u^{n+1-1}\, du$$

$$\left\{\because \int_0^\infty e^{-x} x^{n-1}\, du = \overline{|n} \text{ (gamma of n)}\right.$$

$$= \frac{\overline{|(n+1)}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \qquad \text{... Hence proved}$$

3 Properties of Laplace transform

1) Linearity property:

If a, b, c be any constants and f, g, h any functions of t,

Then $L[a\,f(t) + b\,g(t) - c\,h(t)]$

$= a\,L\{f(t)\} + b\,L\{g(t) - c\,L\{h(t)\}$

2) First shifting property:

If $L\{f(t)\} = \overline{f(s)}$

Then $L\{e^{-at}\,f(t)\} = \overline{f(s+a)}$

or $L\{e^{at}\,f(t)\} = \overline{f(s-a)}$

3) Second shifting property:

If $L\{f(t)\} = \overline{f(s)}$ and

$$f(t) = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

Then $L\{f(t)\} = e^{-as}\,\overline{f(s)}$

4) Multiplication of t^n:

If $L\{f(t)\} = \overline{f(s)}$

Then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\,\overline{f(s)}$

5) Division of t:

If $L\{f(t)\} = \overline{f(s)}$

Then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_0^\infty \overline{f(s)}\,ds$

6) Change of scale property:

If $L\{f(t)\} = \overline{f(s)}$

Then $L\{f(at)\} = \dfrac{1}{a}\,\overline{f\left(\dfrac{s}{a}\right)}$

7) If $L\{f(t)\} = \overline{f(s)}$,

Then $L\left\{\displaystyle\int_0^t f(t)\,dt\right\} = \dfrac{1}{s}\,\overline{f(s)}$

8) If $L\{f(t)\} = \overline{f(s)}$,

Then $L\left\{\displaystyle\int_0^\infty e^{-at}\,f(t)\,dt\right\} = \overline{f(a)}$

9) If $L\{f(t)\} = \overline{f(s)}$

Then $L\left\{\dfrac{d}{dt}f(t)\right\} = s\,\overline{f(s)} - f(0)$

Note: $f(0) = \lim_{t\to 0} f(t)$

10) Convolution Theorem:

If $L\{f_1(t)\} = \overline{f_1(s)}$,

$L\{f_2(s)\} = \overline{f_2(s)}$

Then

$\overline{f_1(s)}\,.\,\overline{f_2(s)} =$

$$L\left\{\int_0^t f_1(u)\,.\,f_2(t-u)\,du\right\}$$

$\overline{f_1(s)}.\,\overline{f_2(s)} =$

$$L\left\{\int_0^t f_1(t-u).\,f_2(u)\,du\right\}$$

3.i Examples on Linearity Property

Example 1: Find the Laplace transform of $(t^2 + 1)^2$

Solution: Let, $L\{f(t)\} = L\{t^2 + 1)^2\}$

$f(s) = L\{t^4 + 2t^2 + 1\}$

$$= L\{t^4\} + 2L\{t^2\} + L\{1\}$$

$$= \frac{4!}{s^5} + 2\frac{2!}{s^3} + \frac{1}{s} \qquad \left\{\because L\{t^n\} = \frac{n!}{s^{n+1}}\right.$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

$$\mathbf{f(s) = \frac{24 + 4s^2 + s^4}{s^5}}$$

Example 2: Evaluate: $\mathbf{L\left\{\frac{t^2 - 3t + 2}{\sqrt{t}}\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\frac{t^2 - 3t + 2}{\sqrt{t}}\right\}$

$$\overline{f(s)} = L\left\{\frac{t^2}{t^{\frac{1}{2}}} - \frac{3t}{t^{\frac{1}{2}}} + \frac{2}{t^{\frac{1}{2}}}\right\} \qquad \left\{\because \sqrt{a} = a^{\frac{1}{2}}\right.$$

$$= L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 2t^{-\frac{1}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 2L\left\{t^{-\frac{1}{2}}\right\}$$

$$= \frac{\frac{3}{2}!}{s^{\frac{3}{2}+1}} - \frac{3\frac{1}{2}!}{s^{\frac{1}{2}+1}} + \frac{2\left(-\frac{1}{2}\right)!}{s^{-\frac{1}{2}+1}} \qquad \left\{\because L\{t^n\} = \frac{n!}{s^{n+1}}\right.$$

$$= \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{\frac{5}{2}}} - \frac{3\ \frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{2\sqrt{\pi}}{s^{\frac{1}{2}}} \qquad \left\{\because \left(\frac{-1}{2}\right)! = \sqrt{\pi}\right.$$

$$\mathbf{\overline{f(s)} = \sqrt{\frac{\pi}{s}}\left[\frac{3}{4}\frac{1}{s^2} - \frac{3}{2}\frac{1}{s} + 2\right]} \qquad \left\{\because a^{\frac{1}{2}} = \sqrt{a}\right.$$

Example 3: Find the Laplace transform of $\mathbf{(\sin 2t - \cos 2t)^2}$

Solution: Let, $L\{f(t)\} = L\{(\sin 2t - \cos 2t)^2\}$

$$= L\{\sin^2 2t + \cos^2 2t - 2\sin 2t\cos 2t\}$$

$$\{\because (a-b)^2 = a^2 + b^2 - 2ab$$

$$= L\{1 - \sin 4t\} \qquad \{\because \sin^2\theta + \cos^2\theta = 1,\ 2\sin\theta.\cos\theta = \sin 2\theta$$

$$= L\{1\} - L\{\sin 4t\}$$

$$\mathbf{\overline{f(s)} = \frac{1}{s} - \frac{4}{s^2 + 16}}$$

Example 4: Evaluate by Laplace transform of $\mathbf{\cos^2 2bt}$

Solution: Let, $L\{f(t)\} = L\{\cos^2 2\,bt\}$

$$= L\left\{\frac{1 + \cos 4bt}{2}\right\} \qquad \left\{\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right.$$

$$= \frac{1}{2}[\,L\{1\} + L\{\cos 4\,bt\}]$$

$$\overline{f(s)} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 + 16b^2}\right]$$

Example 5: **Find L.T. of $\cos t \,.\, \cos 2t \,.\, \cos 3t$**

Solution: Let, $L\{f(t)\} = L\{\cos t \,.\, \cos 2t \,.\, \cos 3t\}$

$$= L\left\{\cos t\,\frac{1}{2}[\cos 5t + \cos t\,]\right\}$$

$$\left\{\because \cos x.\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]\right.$$

$$= \frac{1}{2}L\{\cos t \,.\, \cos 5t + \cos^2 t\} \qquad \{\because \cos(-\theta) = \cos\theta$$

$$= \frac{1}{2}\,L\left\{\frac{1}{2}[\cos 6t + \cos 4t\,] + \frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{4}[L\{\cos 6t\} + L\{\cos 4t\} + L\{1\} + L\{\cos 2t\}]$$

$$\overline{f(s)} = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$$

Example 6: **Find $L\{\sin^3 t\}$**

Solution: Let, $L\{f(t)\} = L\{\sin^3 t\}$

$$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\} \qquad \left\{\because \sin^3 3\theta = \frac{3\sin\theta - \sin 3\theta}{4}\right.$$

$$= \frac{3}{4}L\{\sin t\} - \frac{1}{4}\,L\{\sin 3t\}$$

$$= \frac{3}{4}\,\frac{1}{s^2+1} - \frac{1}{4}\,\frac{3}{s^2+9} \quad = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)\,(s^2+9)}\right]$$

$$\overline{f(s)} = \frac{6}{(s^2+1)(s^2+9)}$$

Example 7: **Find $L\{\cosh 2t \,.\, \sinh 2t\}$**

Solution: Let, $L\{f(t)\} = L\{\cosh 2t.\sinh 2t\}$

$$= L\left\{\frac{e^{2t}+e^{-2t}}{2}\,\frac{e^{2t}-e^{-2t}}{2}\right\}$$

$$\left\{\because \cos h\theta = \frac{e^{\theta}+e^{-\theta}}{2}, \sin h\theta = \frac{e^{\theta}-e^{-\theta}}{2}\right.$$

$$= \frac{1}{4}L\{e^{4t} - e^{-4t}\} \qquad = \frac{1}{4}[L\{e^{4t}\} - L\{e^{-4t}\}]$$

$$\overline{\mathbf{f}(\mathbf{s})} = \frac{1}{4}\left[\frac{1}{s-4} - \frac{1}{s+4}\right]$$

Example 8: **Find** $L\{\sin^3 2t\}$

Solution: Let, $L\{f(t)\} = \{\sin^3 2t\}$

$$= L\left\{\frac{3\sin 2t - \sin 6t}{4}\right\} \qquad \left\{\because \sin^3 2\theta = \frac{3\sin 2\theta - \sin 6\theta}{4}\right.$$

$$= \frac{3}{4}\ L\{\sin 2t\} - \frac{1}{4}\ L\{\sin 6t\}$$

$$= \frac{3}{4}\ \frac{2}{s^2+4} - \frac{1}{4}\ \frac{6}{s^2+36} \qquad = \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$\overline{\mathbf{f}(\mathbf{s})} = \frac{48}{(s^2+4)(s^2+36)}$$

Example 9: **Find** $L\{\sin(\omega t + \alpha)\}$

Solution: Let, $L\{f(t)\} = L\{\sin(\omega t + \alpha)\}$

$$= L\{\sin \omega t \cos \alpha + \cos \omega t \sin \alpha\}$$

$$\{\because \sin(x+y) = \sin x.\cos y + \cos x.\sin y$$

$$= \cos\alpha\, L\{\sin \omega t\} + \sin\alpha\, [L\{\cos \omega t\}]$$

$$= \cos\alpha \frac{\omega}{s^2+\omega^2} + \sin\alpha \frac{s}{s^2+\omega^2}$$

$$\overline{\mathbf{f}(\mathbf{s})} = \frac{\omega\cos\alpha + s\sin\alpha}{s^2+\omega^2}$$

3.ii Examples on First Shifting Property

Example 10: **Find** $L\{\cos at\,.\sinh at\}$

Solution: Let, $L\,\{f(t)\} = L\{\cos at\,.\sinh at\}$

$$= L\left\{\cos at\ \frac{e^{at} - e^{-at}}{2}\right\} \qquad \left\{\because \sin h\theta = \frac{e^{\theta}-e^{-\theta}}{2}\right.$$

$$= \frac{1}{2}[L\{ e^{at.} \cos at \} - L\{ e^{-at} \cos at\}]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2 + a^2} - \frac{s+a}{(s+a)^2 + a^2} \right]$$

$$= \frac{1}{2}\left[\frac{(s-a)[(s+a)^2 + a^2] - (s+a)[(s-a)^2 + a^2]}{[(s-a)^2 + a^2][(s+a)^2 + a^2]} \right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2 + 2sa + 2a^2) - (s+a)(s^2 - 2sa + 2a^2)}{(s^2 - 2sa + 2a^2)(s^2 + 2sa + 2a^2)} \right]$$

$$= \frac{1}{2}\left[\frac{s^3 + 2as^2 + 2a^2 s - as^2 - 2a^2 s - 2a^3 - s^3 + 2as^2 - 2a^2 s - as^2 + 2a^2 s - 2a^3}{s^4 + 2as^3 + 2a^2 s^2 - 2as^3 - 4a^2 s^2 - 4a^3 s + 2a^2 s^2 + 4a^3 s + 4a^4} \right]$$

$$= \frac{1}{2}\left[\frac{2as^2 - 4a^3}{s^4 + 4a^4} \right] \qquad = \frac{1}{2} \frac{2a(s^2 - 2a^2)}{s^4 + 4a^4}$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{a(s^2 - 2a^2)}}{\mathbf{s^4 + 4a^4}}$$

Example 11: Find $\mathbf{L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}}$

Solution: Let, $L\{f(t)\} = L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$$= 2L\{e^{-3t} \cos 5t\} - 3L\{e^{-3t}.\sin 5t\}$$

$$= 2\frac{(s+3)}{(s+3)^2 + 25} - 3\frac{5}{(s+3)^2 + 25}$$

$$= \frac{2s + 6 - 15}{(s+3)^2 + 25} \qquad = \frac{2s - 9}{s^2 + 6s + 9 + 25}$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{2s - 9}}{\mathbf{s^2 + 6s + 34}}$$

Example 12: Find $\mathbf{L\{e^{2t}\cos^2 t\}}$

Solution: Let, $L\{f(t)\} = L\{e^{2t}\cos^2 t\}$

$$= L\left\{ e^{2t} \frac{1 + \cos 2t}{2} \right\} \qquad \left\{ \cos^2\theta = \frac{1 + \cos 2\theta}{2} \right.$$

$$= \frac{1}{2}[L\{ e^{2t}\} + L\{e^{2t} \cos 2t\}]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{2}}\left[\frac{\mathbf{1}}{\mathbf{s-2}} + \frac{\mathbf{s-2}}{\mathbf{(s-2)^2 + 4}} \right]$$

Example 13: **Find** $\mathbf{L\{\sqrt{t}\, e^{3t}\}}$

Solution: Let, $L\{f(t)\} = L\{e^{3t}.\sqrt{t}\}$

Now, $L\{\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\} = \dfrac{\left(\frac{1}{2}\right)!}{s^{\frac{1}{2}+1}} = \dfrac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}}$

Now, *By shifting theorem* $L\{e^{at}F(t)\} = \overline{F(s-a)}\}$

$$L\{e^{3t}.\sqrt{t}\} = \frac{1}{2}\frac{\sqrt{\pi}}{(s-3)^{\frac{3}{2}}}$$

$$\therefore \quad \overline{\mathbf{f(s)}} = \frac{\sqrt{\pi}}{2}\frac{1}{(s-3)^{\frac{3}{2}}}$$

3.iii Examples on Change of Scale Property

Example 14: **Find** $\mathbf{L\{e^{-t}.f(3t)\}}$, **if** $\mathbf{L\{f(t)\} = \frac{1}{s}e^{\frac{-1}{s}}}$

Solution: We have, $L\{f(t)\} = \dfrac{1}{s}\, e^{\frac{-1}{s}} = \overline{f(s)}$

Now, *By change of scale property* $L\{f(at)\} = \dfrac{1}{a}\overline{f\left(\frac{s}{a}\right)}$

$$L\{f(3t)\} = \frac{1}{3}\frac{1}{s/3}\, e^{-\frac{1}{s/3}} = \frac{1}{s}\, e^{-\frac{3}{s}}$$

Now, *By shifting theorem* $L\{e^{at}F(t)\} = \overline{F(s-a)}\}$

$$\mathbf{L\{e^{-t}f(3t)\} = \frac{1}{s+1}\, e^{-\frac{3}{s+1}}}$$

3.iv Examples on effect of multiplication of t^n

Example 15: **Find** $\mathbf{L\{t.\sin t\}}$

Solution: Let, $L\{f(t)\} = L\{t.\sin t\}$

Now, $L\{\sin t\} = \dfrac{1}{s^2+1} = \overline{F(s)}$ (say)

Now, *By effect of multiplication by t,*

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\{t.\sin t\} = (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-1}{(s^2+1)^2}(2s)\right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{2s}}{\mathbf{(s^2+1)^2}}$$

Example 16: **Find** $\mathbf{L\{t\,sin^2 3t\}}$

Solution: Let, $L\{f(t)\} = L\{\,t\sin^2 3t\}$

Now, $L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\} = \overline{F(s)}$ (say)

$$\left\{\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right.$$

$$= \frac{1}{2}[L\{1\} - L\{\cos 6t\}]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

Now, By effect of multiplication by t,

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\{t.\sin^2 3t\} = \frac{1}{2}(-1)\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\frac{1}{s} - \frac{d}{ds}\frac{s}{s^2+36}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(s^2+36)-s(2s)}{(s^2+36)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{-s^2+36}{(s^2+36)^2}\right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{2}}\left[\frac{\mathbf{1}}{\mathbf{s^2}} + \frac{\mathbf{-s^2+36}}{\mathbf{(s^2+36)^2}}\right]$$

Example 17: **Find** $\mathbf{L\{t\,e^{3t}\sin 2t\}}$

Solution: Let, $L\{f(t)\} = L\{t\,e^{3t}\sin 2t\}$

Now, $L\{\sin 2t\} = \frac{2}{s^2+4} = \overline{F(s)}$ (say)

Now, By effect of multiplication by t,

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\{t.\sin 2t\} = (-1)\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -2\left[\frac{-1}{(s^2+4)^2}.2s\right] \quad \left\{\because \frac{d}{dx}\frac{u}{v} = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right.$$

$$= \frac{4s}{(s^2+4)^2}$$

Now, *By shifting theorem* $L\{e^{at}F(t)\} = \overline{F(s-a)}\}$

$$L\{t\,e^{3t}\sin 2t\} = \frac{4(s-3)}{[(s-3)^2+4]^2} = \frac{\mathbf{4(s-3)}}{\mathbf{(s^2-6s+13)^2}} = \overline{\mathbf{f(s)}}$$

Example 18: **Find** $\mathbf{L\{t(2\sin 3t - 3\cos 3t)\}}$

Solution: Let, $L\{f(t)\} = L\{t(2\sin 3t - 3\cos 3t)\}$

Now, $L\{2\sin 3t - 3\cos 3t\}$

$$= 2L\{\sin 3t\} - 3L\{\cos 3t\}$$

$$= 2\frac{3}{s^2+9} - 3\frac{s}{s^2+9} \qquad = \frac{6}{s^2+9} - \frac{3s}{s^2+9}$$

$$= \frac{6-3s}{(s^2+9)} = \overline{F(s)} \quad \text{(say)}$$

Now, *By effect of multiplication by t,*

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\,\{t(2\sin 3t - 3\cos 3t)\}$$

$$= (-1)\frac{d}{ds}\frac{(6-3s)}{s^2+9}$$

$$= -\left[\frac{(s^2+9)(-3) - (6-3s)(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{-3s^2-27-12s+6s^2}{(s^2+9)^2}\right] \qquad = -\left[\frac{3s^2-12s-27}{(s^2+9)^2}\right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{3(9-s^2+4s)}}{\mathbf{(s^2+9)^2}}$$

Example 19: **Find** $\mathbf{L\{t^2\cos t\}}$

Solution: Let, $L\{f(t)\} = L\{t^2\cos t\}$

Now, $L\{\cos t\} = \frac{s}{s^2+1} = \overline{F(s)}$ (say)

Now, *By effect of multiplication by t,*

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\ \overline{F(s)}$$

$$L\{t\cos t\} = (-1)\frac{d}{ds}\frac{s}{s^2+1}$$

$$= -\left\{\frac{(s^2+1) - s(2s)}{(s^2+1)^2}\right\} \qquad \left\{\because \frac{d}{dx}\frac{u}{v} = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right.$$

$$= \frac{s^2-1}{(s^2+1)^2} = \overline{F(s)} \quad \text{(say)}$$

Now, *By again effect of multiplication of 't'*

$$L\{t^2\cos t\} = -\frac{d}{ds}\frac{s^2-1}{(s^2+1)^2}$$

$$= -\left[\frac{(s^2+1)^2(2s) - (s^2-1)2\,(s^2+1)(2s)}{[(s^2+1)^2]^2}\right]$$

$$= -\left[\frac{2s^5+4s^3+2s-4s(s^4-1)}{(s^2+1)^4}\right]$$

$$= -\left[\frac{-2s^5+4s^3+6s}{(s^2+1)^4}\right] \qquad = -\left[\frac{-2s(s^4-2s^2-3)}{(s^2+1)^4}\right]$$

$$= \frac{2s(s^2+1)(s^2-3)}{(s^2+1)^4}$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{2s(s^2-3)}}{\mathbf{(s^2+1)^3}}$$

Example 20: **Find** $\mathbf{L\{t^2\cos kt\}}$

Solution: Hint: Same as above problem no. (19)

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{2s(s^2-3k^2)}}{\mathbf{(s^2+k^2)^3}}$$

Example 21: **Find** $\mathbf{L\left\{\frac{2\sqrt{t}}{\sqrt{\pi}}\right\}}$, **if** $\mathbf{L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}}$

Solution: Given, $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}} = \overline{F(s)}$ (say)

Now, *By effect of multiplication by t,*

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\ \overline{F(s)}$$

$$L\left\{t\,\frac{1}{\sqrt{\pi t}}\right\} = (-1)\,\frac{d}{ds}\,\frac{1}{\sqrt{s}} = -\frac{d}{ds}\,\frac{1}{s^{1/2}}$$

$$L\left\{\frac{t}{\sqrt{t}}\,\frac{1}{\sqrt{\pi}}\right\} = (-1)\,\frac{d}{ds}\,s^{-\frac{1}{2}} = (-1)\left(\frac{-1}{2}\right)s^{-\frac{1}{2}-1} = \frac{1}{2}\,s^{-\frac{3}{2}}$$

$$L\left\{\frac{\sqrt{t}}{\sqrt{\pi}}\right\} = \frac{1}{2}\frac{1}{s^{3/2}}$$

$$\mathbf{L\left\{\frac{2\sqrt{t}}{\sqrt{\pi}}\right\} = \frac{1}{s^{3/2}}}$$

3.v Examples on effect of division of t

Example 22: Find the Laplace transform of $\mathbf{\dfrac{1-\cos t}{t}}$

Solution: Let, $L\{f(t)\} = L\left\{\dfrac{1-\cos t}{t}\right\}$

Now, $L\{1-\cos t\} = L\{1\} - L\{\cos t\}$

$$= \frac{1}{s} - \frac{s}{s^2+1} = \overline{F(s)} \quad \text{(say)}$$

Now, Effect of division by t, $L\left\{\dfrac{1}{t}F(t)\right\} = \displaystyle\int_s^\infty \overline{F(s)}\,ds$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \int_0^\infty \frac{1}{s}\,ds - \int_0^\infty \frac{s}{s^2+1}\,ds$$

$$= [\log s]_s^\infty - \left[\frac{1}{2}\log(s^2+1)\right]_s^\infty \quad \{\because \log\infty = \text{nothing} = 0$$

$$= (0-\log s) - \left[0 - \frac{1}{2}\log(s^2+1)\right]$$

$$= -\log s + \frac{1}{2}\log(s^2+1) \quad = \log\frac{(s^2+1)^{\frac{1}{2}}}{s}$$

$$\mathbf{\overline{f(s)} = \log\frac{\sqrt{s^2+1}}{s}}$$

Example 23: Find $\mathbf{L\left\{\dfrac{1}{t}(1-\cos at)\right\}}$

Solution: Hint: Refer problem (22) $\overline{\mathbf{f(s)}} = \mathbf{log}\ \frac{\sqrt{\mathbf{s^2+a^2}}}{\mathbf{s}}$

Example 24: **Evaluate using Laplace transform**: $\frac{\mathbf{cos\ at - cos\ bt}}{\mathbf{t}}$

Solution: Let, $L\{f(t)\} = L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Now, $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$

$$= \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} = \overline{F(s)} \quad \text{(say)}$$

Now, *Effect of division by t,* $L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\,ds$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2 + a^2) - \log(s^2 + b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log(s^2 + a^2) + \log(s^2 + b^2)\right] \quad \{\because \log\infty = 0 \text{ not defined}$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{2}}\ \mathbf{log}\frac{\mathbf{s^2 + b^2}}{\mathbf{s^2 + a^2}}$$

Example 25: **Find** $\mathbf{L}\left\{\frac{\mathbf{e^{-at} - e^{-bt}}}{\mathbf{t}}\right\}$

Solution: Let, $L\{f(t)\} = L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

Now, $L\{e^{-at} - e^{-bt}\} = L\{e^{-at}\} - L\{e^{-bt}\}$

$$= \frac{1}{s + a} - \frac{1}{s + b} = \overline{F(s)} \quad \text{(say)}$$

Now, *Effect of division by t,* $L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\,ds$

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s + a} - \frac{1}{s + b}\right) ds$$

$$= \left[\log(s + a) - \log(s + b)\right]_s^\infty$$

$$= \left[0 - \log(s + a) + \log(s + b)\right]$$

$$\overline{\mathbf{f(s)}} = \mathbf{\log\frac{(s+b)}{(s+a)}}$$

Example 26: **Find** $\mathbf{L\left\{\frac{1}{t}\left(e^{at} - e^{bt}\right)\right\}}$

Solution: Hint: same as before problems no. (25)

$$\overline{\mathbf{f(s)}} = \mathbf{\log\frac{(s-b)}{(s-a)}}$$

Example 27: **Find** $\mathbf{L\left\{\frac{\sinh t}{t}\right\}}$

Solution: Let, $\quad L\{f(t)\} = L\left\{\frac{\sinh t}{t}\right\}$

Now, $\quad L\{\sin ht\} = \frac{1}{s^2-1} = \overline{F(s)} \quad$ (say)

Now, $\quad$ *Effect of division by t,* $\quad L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\, ds$

$$L\left\{\frac{\sin ht}{t}\right\} = \int_s^\infty \frac{1}{s^2-1}\, ds$$

$$= \frac{1}{2}\left[\log\left|\frac{s-1}{s+1}\right|\right]_s^\infty \qquad \left\{\because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right.$$

$$= \frac{1}{2}\left[0 - \log\left|\frac{s-1}{s+1}\right|\right] \qquad \{\because \log\infty = 0$$

$$\overline{\mathbf{f(s)}} = \mathbf{\frac{1}{2}\log\left|\frac{s+1}{s-1}\right|} \qquad \left\{\because \log\frac{a}{b} = -\log\frac{b}{a}\right.$$

Example 28: **Find** $\mathbf{L\{t^{-1}e^{-t}\sin t\}}$

Solution: Let, $\quad L\{f(t)\} = L\{t^{-1}e^{-t}\sin t\}$

Now, $\quad L\{\sin t\} = \frac{1}{s^2+1} = \overline{F(s)} \quad$ (say)

Now, $\quad$ *Effect of division by t,* $\quad L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\, ds$

$$L\left\{\frac{1}{t}\sin t\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= [\tan^{-1}s]_s^{\infty} \qquad \left\{\because \int \frac{1}{x^2+1}dx = \tan^{-1}x + c\right.$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s \qquad \left\{\because \tan^{-1}\infty = \frac{\pi}{2}\right.$$

$$= \cot^{-1}s$$

Now, *By shifting theorem* $L\{e^{at}F(t)\} = \overline{F(s-a)}\}$

$$L\{e^{-t}t^{-1}\sin t\} = \mathbf{\cot^{-1}(s+1) = f(s)}$$

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3. vi Examples on other properties

Example 29: Find the L.T. of: $\mathbf{\frac{d}{dt}\frac{\sin t}{t}}$

Solution: Let, $L\{f(t)\} = L\left\{\frac{d}{dt}\frac{\sin t}{t}\right\}$

Now, $L\{\sin t\} = \frac{1}{s^2+1} = \overline{F(s)}$ (say)

Now, *Effect of division by t,* $L\left\{\frac{1}{t}F(t)\right\} = \int_s^{\infty}\overline{F(s)}\,ds$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty}\frac{1}{s^2+1}\,ds \qquad \left\{\because \tan^{-1}\infty = \frac{\pi}{2}\right.$$

$$L\{F_1(t)\} = [\tan^{-1}s]_s^{\infty} = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s = \overline{F_1(s)} \quad \text{(say)}$$

Now, *W.k.t.* $L\left\{\frac{d}{dt}F_1(t)\right\} = s\,\overline{F_1(s)} - F_1(0)$,

Where, $F_1(0) = \lim_{t\to 0}F_1(t)$

$$L\left\{\frac{d}{dt}\frac{\sin t}{t}\right\} = s\cot^{-1}s - f(0)$$

$$= s\cot^{-1}s - \lim_{t\to 0}\frac{\sin t}{t}$$

$$\overline{\mathbf{f(s)}} = \mathbf{s\cot^{-1}s - 1}$$

Example 30: Find $\mathbf{L\left\{\int_0^t e^t\frac{\sin t}{t}\,dt\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^t e^t \frac{\sin t}{t} dt\right\}$

Now, $L\{\sin t\} = \frac{1}{s^2+1}$

Now, Effect of division by t, $L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\, ds$

$$L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1}s]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s \quad = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

Now, By Shifting theorem, $L\{e^{at} F(t)\} = \overline{F(s-a)}$

$$L\left\{e^t . \frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

Now, W.k.t., $L\left\{\int_0^t F(t)\, dt\right\} = \frac{1}{s}\overline{F(s)}$

$$\mathbf{L\left\{\int_0^t e^t . \frac{\sin t}{t} dt\right\} = \frac{1}{s}\cot^{-1}(s-1) = \overline{f(s)}}$$

Example 31: **Find** $\mathbf{L\left\{\int_0^t x.\cosh x \; dx\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^t x.\cos hx.\, dx\right\}$

Now, $L\{\cosh x\} = \frac{s}{s^2-1}$

Now, By effect of multiplication by x,

$$L\{x^n F(x)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\{x.\cosh x\} = (-1)\frac{d}{ds}\frac{s}{s^2-1}$$

$$= -\left[\frac{(s^2-1) - s\,(2s)}{(s^2-1)^2}\right] = -\left[\frac{-s^2-1}{(s^2-1)^2}\right]$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

Now, *W.k.t.,* $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{s}\,\overline{F(s)}$

$$L\left\{\int_0^t x.\cosh x\,dx\right\} = \frac{\mathbf{(s^2+1)}}{\mathbf{s(s^2-1)^2}} = \overline{\mathbf{f(s)}}$$

Example 32: **Find** $\mathbf{L\left\{e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right\}$

Now, $L\{\sin 3t\} = \frac{3}{s^2+9}$

Now, *Effect of division by t,* $L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\,ds$

$$L\left\{\frac{\sin 3t}{t}\right\} = \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3.\frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^\infty \qquad \left\{\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\ t\right.$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{3} \qquad = \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$$

$$= \cot^{-1}\frac{s}{3}$$

Now, *W.k.t.,* $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{s}\,\overline{F(s)}$

$$L\left\{\int_s^t \frac{\sin 3t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}\frac{s}{3}$$

Now, *By Shifting theorem ,* $L\{e^{at}\,F(t)\} = \overline{F(s-a)}$

$$L\left\{e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right\} = \frac{\mathbf{1}}{\mathbf{s+4}}\,\mathbf{\cos^{-1}}\frac{\mathbf{(s+4)}}{\mathbf{3}} = \mathbf{f(s)}$$

Example 33: **Find** $\mathbf{L\left\{\cosh t \int_0^t e^x \cosh x\, dx\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\cosh t \int_o^t e^x \cos hx\, dx\right\}$

Now, $L\{\cosh x\} = \dfrac{s}{s^2-1}$

Now, By Shifting theorem, $L\{e^{at}\,F(t)\} = \overline{F(s-a)}$

$$L\{e^x \cosh x\} = \frac{s-1}{(s-1)^2-1} = \frac{(s-1)}{s^2-2s}$$

Now, W.k.t., $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{s}\,\overline{F(s)}$

$$L\left\{\int_0^t e^x \cosh x\,dx\right\} = \frac{1}{s}\,\frac{(s-1)}{(s^2-2s)} = \frac{s-1}{s^2(s-2)} = \overline{G(s)} \quad \text{(say)}$$

Now, $L\left\{\cosh t \int_0^t e^x \cosh x\,dx\right\} = L\left\{\frac{e^t+e^{-t}}{2}\,G(t)\right\}$

... where $G(t) = \int_0^t e^x \cosh x\,dx$

$$= \frac{1}{2}\left[L\{e^t\,G(t)\} + L\{e^{-t}\,G(t)\}\right]$$

$$= \frac{\mathbf{1}}{\mathbf{2}}\left[\overline{\mathbf{G(s-1)}} + \overline{\mathbf{G(s+1)}}\right] \qquad \text{... where } \overline{G(s)} = \frac{s-1}{s^2(s-2)}$$

Example 34: **Find** $\mathbf{L\left\{\int_0^\infty \frac{\sin t}{t}\,dt\right\}}$

Solution: Let $L\{f(t)\} = L\left\{\int_0^\infty \frac{\sin t}{t}\,dt\right\}$

Now, $L\{\sin t\} = \dfrac{1}{s^2+1}$

Now, *Effect of division by t,* $L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty \overline{F(s)}\, ds$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds = [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s \qquad \left\{\because \tan^{-1}\infty = \frac{\pi}{2}\right.$$

$$= \cot^{-1} s$$

Now, *W.k.t.,* $L\left\{\int_0^\infty e^{-at}\, F(t)\, dt\right\} = \overline{F(a)}$

$$L\left\{\int_0^\infty e^{-0t}\, \frac{\sin t}{t}\right\} = \cot^{-1}(0) \qquad \text{... put } s = 0$$

$$\therefore \quad \overline{\mathbf{f(s)}} = \frac{\boldsymbol{\pi}}{\mathbf{2}} \qquad \left\{\therefore \cot^{-1}(0) = \frac{\pi}{2}\right.$$

Example 35: **Find** $\mathbf{L\left\{\int_0^\infty e^{-2t} \sin^3 t\, dt\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^\infty e^{-2t} \sin^3 t\, dt\right\}$

Now, $L\{\sin^3 t\} = L\left\{\dfrac{3\sin t - \sin 3t}{4}\right\}$

$$= \frac{3}{4} L\{\sin t\} - \frac{1}{4} L\{\sin 3t\}$$

$$= \frac{3}{4}\,\frac{1}{s^2+1} - \frac{1}{4}\,\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

Now, *W.k.t.,* $L\left\{\int_0^\infty e^{-at}\, F(t)\, dt\right\} = \overline{F(a)}$

$$L\left\{\int_0^\infty e^{-2t}\sin^3 t\right\} = \frac{3}{4}\left[\frac{1}{2^2+1} - \frac{1}{2^2+9}\right] \qquad \{\because \text{ put } s = 2$$

$$= \frac{3}{4}\left[\frac{1}{5} - \frac{1}{13}\right] \quad = \frac{3}{4}\left[\frac{13-5}{65}\right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{6}}{\mathbf{65}}$$

Example 36: **Find** $\mathbf{L\left\{\int_0^\infty e^{-3t}\, t\, \sin t\, dt\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^\infty e^{-3t}\, t\, \sin t\, dt\right\}$

Now, $L\{\sin t\} = \dfrac{1}{s^2+1}$

Now, By effect of multiplication by t,

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{F(s)}$$

$$L\{t \sin t\} = (-1)\frac{d}{ds}\frac{1}{s^2+1}$$

$$= (-1)\left[\frac{-1}{s^2+1}\right](2s)$$

$$= \frac{2s}{(s^2+1)^2}$$

Now, W.k.t., $L\left\{\int_0^\infty e^{-at}\, F(t)\, dt\right\} = \overline{F(a)} \qquad \{\because \text{ here put } s = 3$

$$L\left\{\int_0^\infty e^{-3t}\, t\, \sin t\right\} = \frac{2(3)}{(3^2+1)^2} \quad = \frac{6}{100}$$

$$\therefore \qquad \overline{\mathbf{f(s)}} = \frac{\mathbf{3}}{\mathbf{50}}$$

Example 37: **Find** $\mathbf{L\left\{\int_0^\infty e^{-2t}\, \frac{\sin ht}{t}\, dt\right\}}$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^\infty e^{-2t}\, \frac{\sin ht}{t}\, dt\right\}$

Now, $L\{\sinh t\} = \frac{1}{s^2 - 1}$

Now, Effect of division by t, $L\left\{\frac{1}{t}F(t)\right\} = \int_s^{\infty} \overline{F(s)}\, ds$

$$L\left\{\frac{\sin ht}{t}\right\} = \int_s^{\infty} \frac{1}{s^2 - 1} ds \qquad \{\because \log \infty = 0, \text{not defined}$$

$$= \frac{1}{2(1)}\left[\log\left|\frac{s-1}{s+1}\right|\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\left|\frac{s-1}{s+1}\right|\right]$$

$$= \frac{1}{2}\log\left|\frac{s+1}{s-1}\right| \qquad \left\{\because \log\frac{a}{b} = -\log\frac{b}{a}\right.$$

Now, W.k.t., $L\left\{\int_0^{\infty} e^{-at}\, F(t)\, dt\right\} = \overline{F(a)}$

$$L\left\{\int_0^{\infty} e^{-2t}\,\frac{\sin ht}{t}\, dt\right\} = \frac{1}{2}\log\left|\frac{2+1}{2-1}\right| \qquad \{\because \text{For } s = 2$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{\log 3}$$

Example 38: Find L.T. of: $\int_0^{\infty} \frac{\cos 6t - \cos 4t}{t}\, dt$

Solution: Let, $L\{f(t)\} = L\left\{\int_0^{\infty} \frac{\cos 6t - \cos 4t}{t}\, dt\right\}$

Now, $L\{\cos 6t - \cos 4t\} = L\{\cos 6t\} - L\{\cos 4t\}$

$$= \frac{s}{s^2 + 36} - \frac{s}{s^2 + 16}$$

Now, Effect of division by t, $L\left\{\frac{1}{t}F(t)\right\} = \int_s^{\infty} \overline{F(s)}\, ds$

$$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_s^{\infty} \left(\frac{s}{s^2 + 36} - \frac{s}{s^2 + 16}\right) ds$$

$$= \frac{1}{2}[\log(s^2+36) - \log(s^2+16)]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right) \qquad \left\{\because \log\frac{a}{b} = -\log\frac{b}{a}\right.$$

Now,   *W.k.t.,* $L\left\{\int_0^{\infty} e^{-at}\ F(t)\,dt\right\} = \overline{F(a)}$

$$L\left\{\int_0^{\infty} e^{0t}\frac{\cos 6t - \cos 4t}{t}dt\right\} = \frac{1}{2}\log\left(\frac{0^2+16}{0^2+36}\right) \qquad \{\because \text{Replace } s = 0$$

$$= \log\left(\frac{16}{36}\right)^{\frac{1}{2}} = \log\sqrt{\frac{16}{36}} = \log\frac{4}{6}$$

$$\overline{\mathbf{f(s)}} = \mathbf{\log\frac{2}{3}}$$

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3. vii Examples on using definition

Example 39: **If** $f(t) = a, \quad 0 < t < b$

$= 0, \quad t > b$ Find $L\{f(t)\}$

Solution: By definition of L.T.

$$L\{f(t)\} = \int_0^{\infty} e^{-st}\ f(t)\,dt$$

$$= \int_0^{b} e^{-st}\ f(t)\,dt + \int_b^{\infty} e^{-t}\ f(t)\,dt$$

$$= \int_0^{b} e^{-st}\ a\,dt + 0$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^b = \frac{-a}{s}\left[e^{-bs} - e^0\right] = \frac{-a}{s}\left(e^{-bs} - 1\right)$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{a}}{\mathbf{s}}\left(\mathbf{1 - e^{-bs}}\right)$$

Example 40: **Find the L.T. of** $\mathbf{f(t) = t, \quad 0 < t < 4}$
$\mathbf{= 5, \quad t > 4}$

Solution: By definition of L.T.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^4 e^{-st}\ f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$$

$$= \int_0^4 e^{-st}\ t\,dt + \int_4^\infty e^{-st}\ 5\,dt$$

$$= \left[t\,\frac{e^{-st}}{-s} - \frac{e^{st}}{(-s)^2}\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= \left[\left(\frac{-4}{s}e^{-4s} - \frac{e^{-4s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] + 5\left[0 - \frac{e^{-4s}}{-s}\right]$$

$$= \frac{-4}{s}e^{-4s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + 5\,\frac{e^{-4s}}{s} \qquad \{\because e^{-\infty} = 0$$

$$= \frac{e^{-4s}}{s^2}\left[-4s - 1 + e^{4s} + 5s\right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{e^{-4s}}}{\mathbf{s^2}}\left[\mathbf{e^{4s} + s - 1}\right]$$

Example 41: **Find the L.T. of** $\mathbf{f\,(t) = 0, \quad 0 < t < 1}$
$\mathbf{= 2, \quad 1 < t < 2}$
$\mathbf{= 0, \quad t > 2}$

Solution: By definition of L.T.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\ f(t)\,dt + \int_1^2 e^{-st}\ f(t)dt + \int_2^\infty e^{-st}\ f(t)dt$$

$$= 0 + \int_1^2 e^{-st}\, 2\, dt + 0$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_1^2 \qquad = \frac{-2}{s}\left[e^{-2s} - e^{-s}\right]$$

$$\therefore \quad \overline{f(s)} = \frac{2}{s}\left(e^{-s} - e^{-2s}\right)$$

Example 42: Find the L. T. of $f(t) = 0, \quad 0 < t < 1$
$= t, \quad 1 < t < 2$
$= 0, \quad t > 2$

Solution: By definition of L. T.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= 0 + \int_1^2 e^{-st}\, t\, dt + 0$$

$$= \left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2$$

$$= \left(\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right) - \left((1)\,\frac{e^{-s}}{-s} - \frac{e^{-s(1)}}{s^2}\right)$$

$$\overline{f(s)} = -e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right) + e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

Example 43: Find the L. T. of $f(t) = (t+1), \quad 0 < t < 2$
$= 3, \quad t > 2$

Solution: By definition of L. T.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st}\ (t+1)dt\ +\ \int_2^\infty e^{-st}\ (3)dt$$

$$= \left[(t+1)\ \frac{e^{-st}}{-s} - \ \frac{e^{-st}}{s^2}\right]_0^2 + 3\ \left[\ \frac{e^{-st}}{-s}\ \right]_2^\infty$$

$$= \left[\left(3\ \frac{e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-0}}{-s} - \frac{e^{-0}}{s^2}\right)\right] + 3\left[0 - \ \frac{e^{-2s}}{-s}\ \right]$$

$$= -\ \frac{3}{s}\ e^{-2s}\ -\ \frac{e^{-2s}}{s^2}\ +\ \frac{1}{s}\ +\ \frac{1}{s^2}\ +\ \frac{3}{s}\ e^{-2s}$$

$$\overline{\mathbf{f(s)}} = -\frac{\mathbf{e^{-2s}}}{\mathbf{s^2}}\ + \frac{\mathbf{1}}{\mathbf{s}}\ + \frac{\mathbf{1}}{\mathbf{s^2}}$$

Example 44: **Find the L. T. of** $\mathbf{f\ (t) = \sin t\ \ ,\ \ 0 < \ t\ < \ \pi}$
$\mathbf{= \ 0\ \ \ \ \ ,\ \ t\ > \ \pi}$

Solution: By definition of L. T.

$$L\{f(t)\}\ =\ \int_0^\infty e^{-st}\ f(t)\ dt$$

$$= \int_0^\pi e^{-st}\ f(t)\ dt\ + \int_\pi^\infty e^{-st}\ f(t)\ dt$$

$$= \int_0^\pi e^{-st}\ \sin t\, dt\ + \int_\pi^\infty e^{-st}\ \ (0)dt$$

$$= \left[\ \frac{e^{-st}}{s^2+1}\ (-\ s\sin t - \cos t)\right]_0^\pi$$

$$\left\{\because \int e^{-at}\ \sin bt\ \ dt\ = \frac{e^{-at}}{a^2+b^2}\ [-\ a\sin bt - \ b\cos bt] + c\right.$$

$$= \left[\frac{e^{-s\pi}}{s^2\ +1}(-s\sin\pi - \cos\pi) - \frac{e^{-\infty}}{s^2+1}\ \ (-s\sin(0) - \cos(0))\right]$$

$$= \ \frac{e^{-s\pi}}{s^2\ +1}\ (1) - \ \ \frac{1}{s^2\ +1}\ (-1)\ =\ \frac{1}{s^2\ +1}[\ e^{-s\pi}\ + 1]$$

$$\overline{\mathbf{f(s)}}\ =\ \frac{\mathbf{1}}{\mathbf{s^2+1}}\ [\mathbf{e^{-\pi s}} + \mathbf{1}]\qquad \{\because \cos\pi\ =\ -1, \cos\ (0) = 1$$

Example 45: Find the L. T. of $\mathbf{f(t) = \begin{cases} (t-1)^3, & t > 1 \\ 0, & 0 < t < 1 \end{cases}}$

Solution: By definition of L. T.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} (0)\, dt + \int_1^\infty e^{-st} (t-1)^3\, dt$$

$$= 0 + \left[(t-1)^3 \frac{e^{-st}}{-s} - 3(t-1)^2 \frac{e^{-st}}{(-s)^2} + 6(t-1)\frac{e^{-st}}{(-s)^3} - \frac{6e^{-st}}{(-s)^4}\right]_1^\infty$$

$$= 0 - (-6)\frac{e^{-s}}{s^4}$$

$$\mathbf{\overline{f(s)} = \frac{6e^{-s}}{s^4}}$$

OR *Second shifting property*:

$$If\ L\{f(t)\} = \overline{f(s)}\ \ and\ \ f(t) = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

$$Then\quad L\{f(t)\} = e^{-as}\,\overline{f(s)}$$

∴ Here, $a = 1$, $f(t) = (t-1)^3$

$$L\{f(t)\} = L\{(t-1)^3\} = e^{-s} L\{(t)^3\} = e^{-s}\frac{3!}{s^{3+1}}$$

$$\mathbf{\overline{f(s)} = \frac{6e^{-s}}{s^4}}$$

4 Initial value theorem and Final value theorem

Initial value theorem:

Prove that: If $\mathbf{L[f(t)] = F(s)}$, **then** $\mathbf{\underset{t \to 0}{Lt} f(t) = \underset{s \to \infty}{Lt} s\,F(s)}$

Proof : W.k.t.

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

$$= s\,F(s) - f(0)$$

$$s\,F(s) - f(0) = L[f'(t)]$$

$$= \int_0^\infty e^{-st} f'(t)\, dt$$

$$\underset{s\to\infty}{Lt}\,[s\,F(s)-f(0)] = \underset{s\to\infty}{Lt}\int_0^\infty e^{-st}\,f'(t)\,dt$$

$$\underset{s\to\infty}{Lt}\,s\,F(s)-f(0)=0 \qquad \{\because\ e^{-\infty} = 0$$

i.e. $\underset{s\to\infty}{Lt}\,s\,F(s) = f(0) = \underset{t\to 0}{Lt}\,f(t)$

$$\textbf{Hence}\quad \underset{t\to 0}{\mathbf{Lt}}\,\mathbf{f(t)} = \underset{s\to\infty}{\mathbf{Lt}}\,\mathbf{s\,F(s)}$$

Final value theorem:

Prove that: If $L[f(t)] = F(s)$, then $\underset{t\to\infty}{Lt}\,f(t) = \underset{s\to 0}{Lt}\,s\,F(s)$

Proof : w.k.t. $L[f'(t)] = s\,L[f(t)] - f(0)$

$$s\,L[f(t)] - f(0) = L\,[f'(t)] = \int_0^\infty e^{-st}\,f'(t)\,dt$$

$$\underset{s\to 0}{Lt}\,[s\,L[f(t)] - f(0)] = \underset{s\to 0}{Lt}\int_0^\infty e^{-st}\,f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt = [f(t)]_0^\infty$$

$$\underset{s\to 0}{Lt}\,F(s) - f(0) = f\,(\infty) - f(0)$$

$$\textbf{Hence}\quad \underset{t\to\infty}{\mathbf{Lt}}\,\mathbf{f(t)} = \underset{s\to 0}{\mathbf{Lt}}\,\mathbf{s\,F(s)}$$

2.4.i

Examples on Initial value theorem and Final value theorem

Example 46:

$$\textbf{If } L[f(t)] = \frac{1}{s(s+a)},\ \textbf{find}\ \underset{t\to\infty}{Lt}\,f(t)\ \textbf{and}\ \underset{t\to 0}{Lt}\,f(t)$$

Solution: Given, $L\{f(t)\} = \dfrac{1}{s(s+a)} = F(s)$

i) Final value theorem states that,

$$\underset{t\to\infty}{Lt}\,f(t) = \underset{s\to 0}{Lt}\,s\,F(s)$$

$$= \underset{s\to 0}{Lt}\,s\,\frac{1}{s(s+a)} = \underset{s\to 0}{Lt}\,\frac{1}{s+a}$$

$$\mathop{\mathbf{Lt}}_{\mathbf{t\to\infty}} \mathbf{f(t)} = \frac{\mathbf{1}}{\mathbf{a}}$$

ii) Initial value theorem states that,

$$\mathop{Lt}_{t\to 0} f(t) = \mathop{Lt}_{s\to\infty} s\,F(s)$$

$$= \mathop{Lt}_{s\to\infty} s\,F(s) \quad = \mathop{Lt}_{s\to\infty} s\,\frac{1}{s(s+a)}$$

$$= \mathop{Lt}_{s\to\infty} \frac{1}{s+a} \quad = \frac{1}{\infty}$$

$$\mathop{\mathbf{Lt}}_{\mathbf{t\to 0}} \mathbf{f(t)} = \mathbf{0}$$

Example 47: Verify the initial and final value theorem for the function $\mathbf{f(t) = 1 + e^{-t}(\sin t + \cos t)}$

Solution: Given, $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Now, $L[f(t)] = F(s) = L\{1 + e^{-t}(\sin t + \cos t)\}$

$$= L\{1\} + L\{e^{-t}(\sin t)\} + L\{e^{-t}(\cos t)\}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

i) Initial value theorem states that, $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

$$\text{L.H.S.} = \lim_{t\to 0} f(t) = \lim_{t\to 0}[1 + e^{-t}(\sin t + \cos t)] = 1 + 1 = \mathbf{2}$$

$$\text{R.H.S.} = \lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] \quad = \lim_{s\to\infty}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] \quad = \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right]$$

$$= 1 + 1 = \mathbf{2}$$

L.H.S. = R.H.S. ... Initial value theorem verified.

ii) Final value theorem states that, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

$$\text{L.H.S.} = \lim_{t\to\infty} f(t) = \lim_{t\to\infty}[1 + e^{-t}(\sin t + \cos t)] = 1 + 0 = \mathbf{1}$$

$$\text{R.H.S.} = \lim_{s\to 0} s\,F(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] = 1 + 0 = \mathbf{1}$$

L.H.S. = R.H.S. ... Final value theorem verified.

Example 48: Verify the initial and final value theorems for $\mathbf{f(t) = 3e^{-2t}}$

Solution: Given, $f(t) = 3e^{-2t}$

$$L[f(t)] = L[3e^{-2t}] = \frac{3}{s+2} = F(s)$$

i) Initial value theorem states that, $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ s\,F(s)$

$$\text{L.H.S.} = \underset{t\to 0}{\text{Lt}}\ f(t) = \underset{t\to 0}{\text{Lt}}\ 3e^{-2t} = \mathbf{3}$$

$$\text{R.H.S.} = \underset{s\to\infty}{\text{Lt}}\ s\,F(s) = \underset{s\to\infty}{\text{Lt}}\ s\left(\frac{3}{s+2}\right)$$

$$= \underset{s\to\infty}{\text{Lt}}\ \frac{3s}{s+2} = \underset{s\to\infty}{\text{Lt}}\ \frac{3s}{s\left(1+\frac{2}{s}\right)} = \underset{s\to\infty}{\text{Lt}}\ \frac{3}{1+\left(\frac{2}{s}\right)} = \mathbf{3}$$

L.H.S. = R.H.S. ... Hence Initial value theorem verified.

ii) Final value theorem states that, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

$$\text{L.H.S.} = \underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{t\to\infty}{\text{Lt}}\ 3e^{-2t} = \mathbf{0} \qquad \{\because e^{-\infty} = 0$$

$$\text{R.H.S.} = \underset{s\to 0}{\text{Lt}}\ s\,f(s) = \underset{s\to 0}{\text{Lt}}\ s\left(\frac{3}{s+2}\right) = \mathbf{0}$$

L.H.S. = R.H.S. ... Hence Final value theorem verified.

Inverse Laplace Transform [I. L. T.]

5 Inverse Laplace Transform [I. L. T.]

5.i Definition

If $L\{f(t)\} = \overline{f(s)}$

Then f(t) is called Inverse laplace transform of $\overline{f(s)}$

i.e. $\mathbf{L^{-1}\left\{\overline{f(s)}\right\} = f(t)}$

5.ii Formulae of inverse Laplace transform

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

4) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$, $n = 1,2,3 =$ or $\frac{t^{n-1}}{\overline{|(n-1)+1}}$

5) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ or $\frac{t^n}{\overline{|n+1}}$

6) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

7) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

8) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

9) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

10) $L^{-1}\left\{\overline{f(s-a)}\right\} = e^{at}\, L^{-1}\left\{\overline{f(s)}\right\} = e^{at}\, f(t)$

11) $L^{-1}\left\{\overline{f(s-a)}\right\} = e^{at}\, L^{-1}\left\{\overline{f(s)}\right\} = e^{-at}\, f(t)$

12) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\,\frac{t^{n-1}}{(n-1)!}$

13) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\,\frac{\sin bt}{b}$

14) $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = e^{at}\,\frac{\sinh bt}{b}$

15) $L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = e^{-at}\,\frac{\sinh t}{b}$

16) $L^{-1}\left\{\frac{1}{(s+a)^2-b^2}\right\} = e^{-at}\,\frac{\sinh bt}{b}$

17) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

18) $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\,\cosh bt$

19) $L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$

20) $L^{-1}\left\{\frac{s+a}{(s+a)^2-b^2}\right\} = e^{-at}\cosh bt$

21) $L^{-1}\left\{(-1)^n\,\frac{d^n}{ds^n}\,\overline{f(s)}\right\} = t^n\,f(t)$

22) $L^{-1}\left\{\frac{\overline{f(s)}}{s}\right\} = \int_0^t f\,(t)\,dt$

23) $L^{-1}\left\{\int_s^\infty \overline{f(s)}\,ds\right\} = \frac{f(t)}{t}$

24) $L^{-1}\{\,\overline{f_1(s)}\,.\,\overline{f_2(s)}\} = \int_0^t f_1(\,t-u).f_2(u)\,du$ **OR** $\int_0^t f_1\,(u).f_2(t-u)\,du$

25) $L^{-1}\left\{s\,\overline{f(s)}\right\} = \frac{d}{dt}\,f(t),\qquad \text{if } f\,(0) = 0$

26) $L^{-1}\left\{e^{-as}\,\overline{f(s)}\right\} = \begin{cases} f(t-a)\,; & t \geq a \\ 0\,; & t < a \end{cases}$

5. iii Summary of L. T. and I. L. T. Formulae

L. T	**I. L. T.**
1) $L\,\{f(t)\} = \overline{f(s)}$	1) $L^{-1}\left\{\overline{f(s)}\right\} = f(t)$
2) $L\,\{1\} = \frac{1}{s}$	2) $L^{-1}\left\{\frac{1}{s}\right\} = 1$
3) $L\,\{t^n\} = \frac{n!}{s^{n+1}}$ or $\frac{\lvert n+1}{s^{n+1}}$	3) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

4) $L\{t\} = \frac{1}{s^2}$

5) $L\{e^{at}\} = \frac{1}{s-a}$

6) $L\{\sin at\} = \frac{a}{s^2+a^2}$

7) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

8) $L\{\cos at\} = \frac{s}{s^2+a^2}$

9) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

4) $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

5) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

6) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

7) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

8) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

9) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

6 Properties of L. T. and I. L. T.

Sr. No.	Laplace Transform (L. T.) Definition: $L\{f(t)\} = \overline{f(s)}$	Inverse Laplace Transform (I. L. T.) Definition: $L^{-1}\{\overline{f(s)}\} = f(t)$
1.	**Linearity property** $L[a f(t) - b g(t)] = a L\{f(t)\} - b L\{g(t)\}$	**Linearity property** $L^{-1}[a f(s) - b g(s)] = a L^{-1}\{\overline{f(s)}\} - b L^{-1}\{\overline{g(s)}\}$
2.	**First shifting property** $L\{e^{-at} f(t)\} = \overline{f(s+a)}$ and $L\{e^{at} f(t) = \overline{f(s-a)}$	**First shifting property** $L^{-1}\{\overline{f(s+a)}\} = e^{-at} L^{-1}\{\overline{f(s)}\}$ and $L^{-1}\{\overline{f(s-a)}\} = e^{at} L^{-1}\{\overline{f(s)}\}$
3.	**Second shifting property** If $f(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t \le a \end{cases}$ Then, $L\{f(t)\} = e^{-as}\,\overline{f(s)}$	**Second shifting property** If $L^{-1}\{e^{-as}\,\overline{f(s)}\}$ Then, $f(t) = \begin{cases} f(t-a) & ; t \ge a \\ 0 & ; t < a \end{cases}$
4.	**Multiplication of t^n** $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{f(s)}$	**Multiplication of t^n** $L^{-1}\left\{(-1)^n \frac{d^n}{ds^n}\overline{f(s)}\right\} = t^n L^{-1}\{\overline{f(s)}\}$
5.	**Division of t** $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f(s)}\,ds$	**Division of t** $L^{-1}\left\{\int_s^\infty \overline{f(s)}\,ds\right\} = \frac{f(t)}{t}$
6.	**Change of scale property** $L\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$	**Change of scale property** $L^{-1}\left\{\overline{f\left(\frac{s}{a}\right)}\right\} = a.\,f(at)$

7.	$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}\,\overline{f(s)}$	$L^{-1}\left\{\frac{1}{s}\,\overline{f(s)}\right\} = \int_0^t \left[L^{-1}\{\overline{f(s)}\}\,dt\right]$
8.	$L\left\{\int_0^\infty e^{at}\,f(t)\,dt\right\} = \overline{f(-a)},$ Replacing $s = -a$	**No Property**
9.	$L\left\{\frac{d}{dt} f(t)\right\} = s\,\overline{f(s)} - F(0)$ Where $F(0) = \lim_{t \to 0} f(t)$	$L^{-1}\{s\,\overline{f(s)}\} = \frac{d}{dt} f(t)$ If $F(0) = 0$

6. i <u>Examples on fundamental of inverse Laplace transform</u>

Example 49: Find the inverse Laplace transform of: $\mathbf{\frac{1}{(s-1)^5}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s-1)^5}\right\}$

$$= e^t\, L^{-1}\left\{\frac{1}{s^5}\right\}$$

$$= e^t\,\frac{t^4}{4!} \qquad \left\{\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}\right.$$

$$\mathbf{f(t) = \frac{e^t\, t^4}{24}} \qquad \{\because n! = n\,(n+1)\,(n-1)\ldots\ldots$$

Example 50: Find $\mathbf{L^{-1}\left\{\frac{3s+1}{(s+1)^2}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{3s+1}{(s+1)^2}\right\}$

$$= L^{-1}\left\{\frac{3(s+1)-2}{(s+1)^2}\right\} \qquad = L^{-1}\left\{\frac{3}{s+1} - \frac{2}{(s+1)^2}\right\}$$

$$= 3\,L^{-1}\left\{\frac{1}{s+1}\right\} - 2\,L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= 3\,e^{-t} - 2e^{-t}\,L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 3\,e^{-t} - 2\,e^{-t}\,t$$

$$\mathbf{f(t) = e^{-t}\,(3-2t)}$$

OR $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{3(s+1)-2}{(s+1)^2}\right\} = e^{-t} L^{-1}\left\{\frac{3s-2}{s^2}\right\}$

$$= e^{-t} L^{-1}\left\{\frac{3s}{s^2} - \frac{2}{s^2}\right\}$$

$$= e^{-t}\left[3 L^{-1}\left\{\frac{1}{s}\right\} - 2 L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$\mathbf{f(t) = e^{-t}(3 - 2t)}$$

Example 51: Find: $\mathbf{L^{-1}\left\{\frac{4s+15}{16s^2-25}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{4s+15}{16s^2-25}\right\}$

$$= 4 L^{-1}\left\{\frac{s}{16s^2-25}\right\} + 15 L^{-1}\left\{\frac{1}{16s^2-25}\right\}$$

$$= 4 L^{-1}\left\{\frac{s}{16\left(s^2-\frac{25}{16}\right)}\right\} + 15 L^{-1}\left\{\frac{1}{16\left(s^2-\frac{25}{16}\right)}\right\}$$

$$= \frac{4}{16} L^{-1}\left\{\frac{s}{s^2-\left(\frac{5}{4}\right)^2}\right\} + \frac{15}{16} L^{-1}\left\{\frac{1}{s^2-\left(\frac{5}{4}\right)^2}\right\}$$

$$= \frac{1}{4}\cosh\frac{5}{4}t + \frac{15}{16}\,\frac{1}{5/4}\sinh\frac{5}{4}t$$

$$\mathbf{f(t) = \frac{1}{4}\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t}$$

Example 52: Find $\mathbf{L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

$$= L^{-1}\left\{\frac{1}{\sqrt{2\left(s+\frac{3}{2}\right)}}\right\} = \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{\sqrt{s+\frac{3}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}\right\} = \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \frac{t^{\frac{1}{2}-1}}{\left(\frac{1}{2}-1\right)!} \qquad = \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \frac{t^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \frac{1}{t^{\frac{1}{2}}} \frac{1}{\sqrt{\pi}} \qquad \left\{\because a^{-n} = \frac{1}{a^n}\ ;\ \left(\frac{-1}{2}\right)! = \sqrt{\pi}\right.$$

$$\mathbf{f(t) = \frac{e^{-\frac{3}{2t}}}{\sqrt{2\pi t}}}$$

Example 53: Find $\mathbf{L^{-1}\left\{\frac{s+1}{s^{\frac{4}{3}}}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s+1}{s^{\frac{4}{3}}}\right\}$

$$= L^{-1}\left\{\frac{s}{s^{\frac{4}{3}}}\right\} + L^{-1}\left\{\frac{1}{s^{\frac{4}{3}}}\right\} = L^{-1}\left\{\frac{1}{s^{\frac{1}{3}}}\right\} + L^{-1}\left\{\frac{1}{s^{\frac{4}{3}}}\right\}$$

$$= \frac{t^{\frac{1}{3}-1}}{\left|\overline{\left(\frac{1}{3}-1\right)+1}\right.} + \frac{t^{\frac{4}{3}-1}}{\left|\overline{\left(\frac{4}{3}-1\right)+1}\right.} = \frac{t^{\frac{-2}{3}}}{\left|\overline{\frac{1}{3}}\right.} + \frac{t^{\frac{1}{3}}}{\left|\overline{\frac{4}{3}}\right.}$$

$$= \frac{t^{\frac{-2}{3}}}{\left|\overline{\frac{1}{3}}\right.} + \frac{t^{\frac{1}{3}}}{\frac{1}{3}\left|\overline{\frac{1}{3}}\right.} = \frac{1}{\left|\overline{\frac{1}{3}}\right.}\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right)$$

$$\mathbf{f(t) = \frac{1}{\left|\overline{\frac{1}{3}}\right.}\left(t^{-\frac{2}{3}} + 3t^{\frac{1}{3}}\right)}$$

Note: i) $n! = n(n-1)(n-2)(n-3)\ldots[n-(n-1)]$

ii) $\left|\overline{n}\right. = (n-1)\left|\overline{n-1}\right. = (n-1)(n-2)\left|\overline{n-2}\right. = \cdots$

$! =$ factorial, $\left|\overline{\ \ }\right. =$ gamma

Example 54: Find $\mathbf{L^{-1}\left\{\frac{1}{(s+4)^{\frac{3}{2}}}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s+4)^{\frac{3}{2}}}\right\}$

$$= e^{-4t} L^{-1}\left\{\frac{1}{s^{\frac{3}{2}}}\right\} \quad = e^{-4t}\frac{t^{\frac{3}{2}-1}}{\left(\frac{3}{2}-1\right)!}$$

$$= e^{-4t}\frac{t^{\frac{1}{2}}}{\frac{1}{2}!} \quad = e^{-4t}\frac{t^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}}$$

$$\mathbf{f(t) = 2e^{-4t}\sqrt{\frac{t}{\pi}}}$$

Example 55: Find $\mathbf{L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

$$= L^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\}$$

$$= L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$$

$$= e^{-4t} L^{-1}\left\{\frac{4s-4}{s^2}\right\}$$

$$= e^{-4t}\left[4\,L^{-1}\left\{\frac{1}{s}\right\} - 4\,L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$f(t) = e^{-4t}\,[4(1) - 4(t)]$$

$$\mathbf{f(t) = 4e^{-4t}(1-t)}$$

---

7 Second (2nd) Shifting Theorem

$$\mathbf{L^{-1}\left\{e^{-as}\,\overline{F(s)}\right\} = \begin{cases} F(t-a) & ; \ t \geq a \\ 0 & ; \ t < a \end{cases}}$$

7.i Examples on 2nd Shifting Theorem

Example 56: Find the inverse Laplace transform of: $\mathbf{\frac{8e^{-3s}}{s^2+4}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{8e^{-3s}}{s^2+4}\right\}$

$$= 8\,L^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\}$$

Now, $L^{-1}\left\{\overline{F(s)}\right\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$$= \frac{\sin 2t}{2} = F(t) \quad \text{(say)}$$

$W.k.t.\ 2^{nd}\ Shifting\ theorem \quad L^{-1}\left\{e^{-as}\,\overline{F(s)}\right\}$

$$= \begin{cases} F(t-a) & ;\ t \geq a \\ 0 & ;\ t < a \end{cases}$$

$$\therefore \quad L^{-1}\left\{\frac{8e^{-3s}}{s^2+4}\right\} = f(t) = \begin{cases} 8\,\dfrac{\sin 2(t-3)}{2}, & t \geq 3 \\ 0, & t < 3 \end{cases}$$

$$\mathbf{f(t)} = \begin{cases} \mathbf{4\sin 2(t-3)}, & \mathbf{t \geq 3} \\ \mathbf{0}, & \mathbf{t < 3} \end{cases}$$

Example 57: Find $\mathbf{L^{-1}\left\{\dfrac{e^{-2s}}{s^2+8s+25}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{e^{-2s}}{s^2+8s+25}\right\}$

$$= L^{-1}\left\{e^{-2s}\,\frac{1}{s^2+8s+4^2-4^2+25}\right\}$$

$$= L^{-1}\left\{e^{-2s}\,\frac{1}{(s+4)^2+9}\right\}$$

$$= L^{-1}\left\{e^{-2s}\,\frac{1}{(s+4)^2+3^2}\right\}$$

Now, $L^{-1}\left\{\overline{F(s)}\right\} = L^{-1}\left\{\dfrac{1}{(s+4)^2+3^2}\right\} = e^{-4t}\,\dfrac{\sin 3t}{3} = F(t) \quad \text{(say)}$

$W.k.t.\ 2^{nd}\ Shifting\ theorem \quad L^{-1}\left\{e^{-as}\,\overline{F(s)}\right\}$

$$= \begin{cases} F(t-a) & ;\ t \geq a \\ 0 & ;\ t < a \end{cases}$$

$$\therefore \quad \mathbf{L^{-1}\left\{e^{-2s}\,\frac{1}{s^2+8s+25}\right\} = f(t)}$$

$$= \begin{cases} \mathbf{e^{-4(t-2)}\,\dfrac{\sin 3(t-2)}{3}}, & \mathbf{t \geq 2} \\ \mathbf{0}, & \mathbf{t} < 2 \end{cases}$$

Example 58: Find $\mathbf{L^{-1}\left\{\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{e^{-as}}{(s+b)^{\frac{5}{2}}}\right\}$

Now, $L^{-1}\left\{\overline{F(s)}\right\} = L^{-1}\left\{\frac{1}{(s+b)^{\frac{5}{2}}}\right\} = e^{-bt}\, L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}$

$$= e^{-bt}\,\frac{t^{\frac{5}{2}-1}}{\left(\frac{5}{2}-1\right)!} = e^{-bt}\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = e^{-bt}\,\frac{t^{\frac{3}{2}}}{\frac{3}{4}\sqrt{\pi}}$$

$$= \frac{4}{3}\,e^{-bt}\,\frac{t^{\frac{3}{2}}}{\sqrt{\pi}} = F(t) \quad \text{(say)}$$

$W.k.t.\ 2^{nd}\ Shifting\ theorem \quad L^{-1}\left\{e^{-as}\,\overline{F(s)}\right\}$

$$= \begin{cases} F(t-a) & ;\ t \geq a \\ 0 & ;\ t < a \end{cases}$$

$$\mathbf{L^{-1}\left\{e^{-as}\frac{1}{(s+b)^{\frac{5}{2}}}\right\} = \begin{cases} \frac{4}{3}\,e^{-b(t-a)}\cdot\frac{(t-a)^{\frac{3}{2}}}{\sqrt{\pi}}; & t \geq a \\ 0 & ;\ t < a \end{cases}}$$

Example 59: **Find** $\mathbf{L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2 e^{-s}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2 e^{-s}\right\}$

Now, $L^{-1}\left\{\overline{F(s)}\right\} = L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right\} = L^{-1}\left\{\frac{1-2\sqrt{s}+s}{s^4}\right\}$

$$= L^{-1}\left\{\frac{1}{s^4}\right\} - 2\,L^{-1}\left\{\frac{\sqrt{s}}{s^4}\right\} + L^{-1}\left\{\frac{s}{s^4}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^4}\right\} - 2\,L^{-1}\left\{\frac{1}{s^{7/2}}\right\} + L^{-1}\left\{\frac{s}{s^4}\right\}$$

$$= \frac{t^3}{3!} - 2\,\frac{t^{\frac{7}{2}-1}}{\left(\frac{7}{2}-1\right)!} + \frac{t^2}{2!}$$

$$= \frac{t^3}{6} - 2\frac{t^{\frac{5}{2}}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} + \frac{t^2}{2}$$

$$= \frac{t^3}{6} - \frac{16}{15}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}} + \frac{t^2}{2} = F(t) \quad \text{(say)}$$

$W.k.t.\ 2^{nd}\ Shifting\ theorem \qquad L^{-1}\left\{e^{-as}\,\overline{F(s)}\right\}$

$$= \begin{cases} F(t-a) & ;\ t \geq a \\ 0 & ;\ t < a \end{cases}$$

$$\mathbf{L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2 e^{-s}\right\} = \begin{cases} \frac{(t-1)^3}{6} - \frac{16}{15}\frac{(t-1)^{\frac{5}{2}}}{\sqrt{\pi}} + \frac{(t-1)^2}{2} & ;\ t \geq 1 \\ 0 & ;\ t < 1 \end{cases}}$$

8 Third (3^{rd}) term

If $as^2 + bs + c$ cannot be factorised then we can use 3^{rd} term

First convert

$as^2 + bs + c$ to $a\left(s^2 + \frac{b}{a}s + \frac{c}{a}\right)$ then apply 3^{rd} term

Note: Coefficient of s must be 1, if not then make it first then apply 3^{rd} term.

$$\mathbf{3^{rd}\ term = \left(coefficient\ of\ s \times \frac{1}{2}\right)^2}$$

8.i Examples on 3^{rd} term

Example 60: **Obtain the inverse Laplace transform of** $\mathbf{\frac{s+1}{s^2+s+1}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

$$= L^{-1}\left\{\frac{s+1}{s^2+s+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= L^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{1}{2}t}\, L^{-1}\left\{\frac{s+\frac{1}{2}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{1}{2}t}\left[L^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}+\frac{1}{2}\, L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$\therefore\quad \mathbf{f(t) = e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]}$$

Example 61: **Find** $\mathbf{L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

$$= L^{-1}\left\{\frac{3s+7}{s^2-2s+1-1-3}\right\} \qquad = L^{-1}\left\{\frac{3s+7}{(s-1)^2-2^2}\right\}$$

$$= L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-2^2}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{3s+10}{s^2-2^2}\right\}$$

$$= e^t\left[3\, L^{-1}\left\{\frac{s}{s^2-2^2}\right\}+10\, L^{-1}\left\{\frac{1}{s^2-2^2}\right\}\right]$$

$$= e^t\left[3\cosh 2t+\frac{10}{2}\sinh 2t\right]$$

$$\therefore\quad \mathbf{f(t) = e^t\,(3\cosh 2t+5\sinh 2t)}$$

Note: This problem can also solve by partial fraction method by

factorising Denominator
$(s+1)(s-3)$ but answer get different and meaning is same.

Example 62: **Find** $\mathbf{L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}$

$$= L^{-1}\left\{\frac{s+2}{s^2-4s+2^2-2^2+13}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\,L^{-1}\left\{\frac{s+4}{s^2+3^2}\right\}$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 4\,L^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right]$$

$$= e^{2t}\left[\cos 3t + 4\,\frac{1}{3}\sin 3t\right]$$

$$\therefore\ \mathbf{f(t) = e^{2t}\cos 3t + \frac{4}{3}\,e^{2t}\sin 3t}$$

Example 63: **Find** $\mathbf{L^{-1}\left\{\dfrac{s}{s^2+5s+16}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{s}{s^2+5s+16}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2+5s+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+16}\right\} \quad = L^{-1}\left\{\frac{s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\}$$

$$= L^{-1}\left\{\frac{\left(s+\frac{5}{2}\right)-\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\left(\frac{\sqrt{39}}{2}\right)^2}\right\}$$

$$= e^{-\frac{5}{2}t}\,L^{-1}\left\{\frac{s-\frac{5}{2}}{s^2+\left(\frac{\sqrt{39}}{2}\right)^2}\right\}$$

$$= e^{-\frac{5}{2}t}\left[L^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{39}}{2}\right)^2}\right\}-\frac{5}{2}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{39}}{2}\right)^2}\right\}\right]$$

$$= e^{-\frac{5}{2}t}\left[\cos\frac{\sqrt{39}}{2}t-\frac{5}{2}\frac{1}{\frac{\sqrt{39}}{2}}\sin\frac{\sqrt{39}}{2}t\right]$$

$$= \mathbf{e^{-\frac{5}{2}t}\left[\cos\frac{\sqrt{39}}{2}t-\frac{5}{\sqrt{39}}\sin\frac{\sqrt{39}}{2}t\right]}$$

9 Partial fraction

Types of partial fraction

1) Linear and non − repeated (distinct):

$$\frac{1}{(s+a)(s+b)(s+c)}=\frac{A}{s+a}+\frac{B}{s+b}+\frac{C}{s+c}$$

2) Linear and repeated (same):

$$\frac{1}{s^2(s+a)^2(s+b)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+a}+\frac{D}{(s+a)^2}+\frac{E}{(s+b)}+\frac{F}{(S+b)^2}$$

3) Non − linear and non − repeated (distinct):

$$\frac{1}{(s^2+a)(s^2+b)}=\frac{as+B}{s^2+a}+\frac{Cs+D}{s^2+b}$$

4) Non − linear and repeated (same):

$$\frac{1}{(s^2+a)^2(s^2+b)^2}=\frac{As+B}{s^2+a}+\frac{Cs+D}{(s^2+a)^2}+\frac{Es+F}{s^2+b}+\frac{Gs+H}{(s^2+b)^2}$$

5) Mixing

i) $$\frac{1}{s(s+a)^2(s^2+b)}=\frac{A}{s}+\frac{B}{s+a}+\frac{C}{(s+a)^2}+\frac{Ds+E}{s^2+b}$$

ii) $$\frac{1}{(s^2+a)(s^2+b)^2(s+c)}=\frac{As+B}{s^2+a}+\frac{Cs+D}{s^2+b}+\frac{Es+F}{(s^2+b)^2}+\frac{G}{s+c}$$

9.i <u>Examples on partial fraction</u>

Example 64: Find $L^{-1}\left\{\frac{s}{s^2+5s+6}\right\}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s}{s^2+5s+6}\right\}$

$$= L^{-1}\left\{\frac{s}{(s+2)(s+3)}\right\}$$

∴ By using partial fraction method,

$$\frac{s}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} \quad \ldots\ldots(1)$$

$$A = \left.\frac{s}{s+3}\right|_{s=-2} = \frac{-2}{-2+3} = \mathbf{-2}$$

$$B = \left.\frac{s}{s+2}\right|_{s=-3} = \frac{-3}{-3+2} = \mathbf{3}$$

Equation (1) becomes, $\frac{s}{(s+2)(s+3)} = \frac{-2}{s+2} + \frac{3}{s+3}$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{s}{(s+2)(s+3)}\right\} = -2L^{-1}\left\{\frac{1}{s+2}\right\} + 3\,L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\mathbf{f(t) = -2e^{-2t} + 3e^{-3t}}$$

Example 65: Find $\mathbf{L^{-1}\left\{\frac{1}{s(s+1)(s+2)(s+3)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)(s+2)(s+3)}\right\}$

∴ By partial fraction method,

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{1/6}{s} + \frac{-1/2}{s+1} + \frac{1/2}{s+2} + \frac{-1/6}{s+3}$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{1}{s(s+1)(s+2)(s+3)}\right\}$$

$$= \frac{1}{6}\,L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\,L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$f(t) = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

$$\therefore\ \mathbf{f(t) = \frac{1}{6}\left[1 - 3e^{-t} + 3e^{-2t} - e^{-3t}\right]}$$

Example 66: Find $L^{-1}\left\{\dfrac{s^2+1}{s^3+3s^2+2s}\right\}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{s^2+1}{s^3+3s^2+2s}\right\}$

$$= L^{-1}\left\{\frac{s^2+1}{s(s^2+3s+2)}\right\} = L^{-1}\left\{\frac{s^2+1}{s(s+1)(s+2)}\right\}$$

∴ By using partial fraction method,

$$\frac{s^2+1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{s^2+1}{s(s+1)(s+2)} = \frac{1/2}{s} + \frac{-2}{s+1} + \frac{5/2}{s+2}$$

Taking I. L. T. both sides

$$L^{-1}\left\{\frac{s^2+1}{s(s+1)(s+2)}\right\} = \frac{1}{2}\, L^{-1}\left\{\frac{1}{s}\right\} + 2\, L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{5}{2} L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{2}(1) - 2e^{-t} + \frac{5}{2}e^{-2t}$$

$$\therefore \quad \mathbf{f(t)} = \frac{1}{2}\left(1 - 4e^{-t} + 5e^{-2t}\right)$$

Example 67: Find $L^{-1}\left\{\dfrac{2s^2-6s+s}{s^3-6s^2+11s-6}\right\}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{2s^2-6s+s}{s^3-6s^2+11s-6}\right\}$

$$= L^{-1}\left\{\frac{2s^2-6s+s}{(s-1)(s-2)(s-3)}\right\}$$

$$\{\because\ s^3-6s^2+11s-6 = (s-1)(s-2)(s-3)$$

∴ By using partial fraction method,

$$\frac{2s^2-6s+s}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\frac{2s^2-6s+s}{(s-1)(s-2)(s-3)} = \frac{\frac{-3}{2}}{s-1} + \frac{2}{s-2} + \frac{\frac{3}{2}}{s-3}$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{2s^2-6s+s}{(s-1)(s-2)(s-3)}\right\}$$

$$= -\frac{3}{2}\ L^{-1}\left\{\frac{1}{s-1}\right\} + 2\ L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{2} L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\therefore \qquad \mathbf{f(t) = -\frac{3}{2} e^{t} + 2e^{2t} + \frac{3}{2}\ e^{3t}}$$

Example 68: **Find** $\mathbf{L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$

∴ By using partial fraction method,

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} \qquad \ldots\ldots(1)$$

Multiplying methods by $(s-1)^2(s+2)$

$$4s+5 = A\,(s-1)\,(s+2) + B(s+2) + C(s-1)^2 \quad \ldots\ldots(2)$$

Put $s = 1$ in equation (2), $9 = 3B$; $\mathbf{B = 3}$

Put $s = -2$ in equation (2), $-3 = 9C$; $\mathbf{C = \frac{-1}{3}}$

Put $s = 0, B = 3\ \&\ C = -\frac{1}{3}$ in equation (2)

$$5 = A\,(-1)(2) + 3\,(2) + \frac{-1}{3}\,(-1)^2$$

$$5 = -2A + 6 - \frac{1}{3}\ ; \quad 5 - \frac{17}{3} = 2A\ ; \quad -\frac{2}{3} = -2A\ ; \quad \mathbf{A = \frac{1}{3}}$$

Equation (1)→ $\frac{4s+5}{(s-2)^2(s+2)} = \frac{1/3}{(s-1)} + \frac{3}{(s-1)^2} + \frac{-1/3}{s+2}$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3} L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\mathbf{f(t) = \frac{1}{3}\ e^{t} + 3t.e^{t} - \frac{1}{3}\ e^{-2t}}$$

Example 69: **Find** $\mathbf{L^{-1}\left\{\frac{s+29}{(s^2+9)(s+4)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s+29}{(s^2+9)(s+4)}\right\}$

∴ By using partial fraction method,

$$\frac{s+29}{(s^2+9)(s+4)} = \frac{As+B}{s^2+9} + \frac{C}{s+4} \qquad \ldots\ldots(1)$$

$$s+29 = (As+B)(s+4) + C(s^2+9) \qquad \ldots\ldots(2)$$

Put $s = -4$ in equn(2), $25 = 25C$; $\mathbf{C = 1}$

Put $s = 0$, $C = 1$ in equation(2),

$$29 = B(4) + 1(9) \ ; \quad 29 - 9 = 4B$$

$20 = 4B$; $\mathbf{B = 5}$

Put $s = 1$, $B = 5$, $C = 1$ in equation (2),

$$30 = (A+5)(5) + (1)(10)$$

$$\frac{30-10}{5} = A+5 \ ; \quad 4 = A+5 \ ; \quad \mathbf{A = -1}$$

Equation (1) → $\dfrac{s+29}{(s^2+9)(s+4)} = \dfrac{-s+5}{s^2+9} + \dfrac{1}{s+4}$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{s+29}{(s^2+9)(s+4)}\right\}$$

$$= -L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 5L^{-1}\left\{\frac{1}{s^2+3^2}\right\} + L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\cos 3t + \frac{5\sin 3t}{3} + e^{-4t}$$

$$\mathbf{f(t) = e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t}$$

Example 70: Find $\mathbf{L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

By partial fraction,

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} \qquad \ldots\ldots(1)$$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1) \qquad \ldots\ldots(2)$$

Put $s = 1$ in equation (2), $8 = 8A$; $\mathbf{A = 1}$

Put $s = 0$ and $A = 1$ in equn (2), $3 = 5 + (-C)$; $\mathbf{C = 2}$

Put $s = -1$, $A = 1$, $C = 2$ in equn (2)

$-2 = 4 + (-B + 2)(-2)$

$-6 = -2(2 - B); \quad 3 = 2 - B; \quad 3 - 2 = -B; \quad \mathbf{B = -1}$

Equation (1) → $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{1}{s-1} + \dfrac{-s+2}{s^2+2s+5}$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{-s+2}{s^2+2s+5}\right\}$$

$$= e^t + L^{-1}\left\{\frac{-s+2}{s^2+2s+1-1+5}\right\}$$

$$= e^t + L^{-1}\left\{\frac{-(s+1)+3}{(s+1)^2+2^2}\right\}$$

$$= e^t + e^{-t} L^{-1}\left\{\frac{-s+3}{s^2+2^2}\right\}$$

$$= e^t + e^{-t}\left[(-1)L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right]$$

$$= e^t + e^{-t}\left[-\cos 2t + \frac{3\sin 2t}{2}\right]$$

$$\mathbf{f(t) = e^t - e^{-t}\cos 2t + \frac{3}{2} e^{-t}\sin 2t}$$

Example 71: **Find** $\mathbf{L^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \qquad \text{... Note}$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\mathbf{f(t) = t - \sin t}$$

Example 72: **Find** $\mathbf{L^{-1}\left\{\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\}$

By partial fraction

$$\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)} = \frac{A}{s+2} + \frac{B}{s-3} + \frac{Cs+D}{s^2+2s+5} \quad \ldots\ldots(1)$$

Multiplying both sides by $(s+2)(s-3)(s^2+2s+5)$

$$s^2 - 3 = A(s-3)(s^2+2s+5) + B(s+2)(s^2+2s+5) + (Cs+D)(s+2)(s-3) \quad \ldots\ldots(2)$$

Put $s = -2$ in equation (2), $1 = A(-5)(5)$; $\therefore \mathbf{A} = \frac{\mathbf{-1}}{\mathbf{25}}$

Put $s = 3$ in equation(2), $6 = B(5)(20)$; $\therefore \mathbf{B} = \frac{\mathbf{3}}{\mathbf{50}}$

Put $s = 0$, $A = \frac{-1}{25}$ and $B = \frac{3}{50}$ in equn (2)

$$-3 = \frac{-1}{25}(-3)(5) + \frac{3}{50}(2)(5) + (D)(2)(-3)$$

$$\frac{-21}{5} = -6D; \quad \frac{21}{5} \times \frac{1}{6} = D; \quad \therefore \mathbf{D} = \frac{\mathbf{7}}{\mathbf{10}}$$

Put $s = 1$, $A = \frac{-1}{25}$, $B = \frac{3}{50}$ and $D = \frac{7}{10}$ in equn (2)

$$-2 = \frac{-1}{25}(-2)(8) + \frac{3}{50}(3)(8) + \left(C + \frac{7}{10}\right)(3)(-2)$$

$$-2 = \frac{16}{25} + \frac{72}{50} - 6C - \frac{21}{5}$$

$$\left(-2 - \frac{16}{25} - \frac{72}{50} + \frac{21}{5}\right) = -6C; \quad \frac{3}{25} = -6C; \quad \frac{-1}{50} = C; \therefore \mathbf{C} = \frac{\mathbf{-1}}{\mathbf{50}}$$

Substituting the value of A, B, C and D in equn (1)

and Taking I. L. T. on both sides, we get

$$L^{-1}\left\{\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right\}$$

$$= \frac{-1}{25} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{3}{50} L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{50} L^{-1}\left\{\frac{s}{(s^2+2s+5)}\right\} + \frac{7}{10} L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{-1}{25} e^{-2t} + \frac{3}{50} e^{3t} - \frac{1}{50} L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+2^2}\right\} + \frac{7}{10} L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{-1}{25}e^{-2t} + \frac{3}{50}e^{3t} - \frac{1}{50}\left[\left\{\frac{s+1}{(s+1)^2+2^2}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}\right] + \frac{7}{10}e^{-t}\frac{\sin 2t}{2}$$

$$= \frac{-1}{25}e^{-2t} + \frac{3}{50}e^{3t} - \frac{1}{50}\left[e^{-t}\cos 2t - e^{-t}\frac{\sin 2t}{2}\right] + \frac{7}{20}e^{-t}\sin 2t$$

$$= \frac{-1}{50}\left[2e^{-2t} - 3e^{3t} + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t - \frac{350}{20}e^{-t}\sin 2t\right]$$

$$\mathbf{f(t) = \frac{-1}{50}\left[2e^{-2t} - 3e^{3t} + e^{-t}\cos 2t - 18\, e^{-t}\sin 2t\right]}$$

Example 73: **Find** $\mathbf{L^{-1}\left\{\frac{s+2}{s^3(s-1)^2}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{s+2}{s^3(s-1)^2}\right\}$

By partial fraction method

$$\frac{s+2}{s^3(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1} + \frac{E}{(s-1)^2} \qquad \dots\dots(1)$$

Multiplying both sides by $s^3(s-1)^2$

$$s+2 = A\,s^2(s-1)^2 + B\,s(s-1)^2 + C(s-1)^2 + Ds^3(s-1) + Es^3$$

$$s+2 = A\,s^2(s^2-2s+1) + Bs(s^2-2s+1) + C(s^2-2s+1) + Ds^3(s-1) + Es^3$$

$$s+2 = As^4 - 2As^3 + As^2 + Bs^3 - 2Bs^2 + Bs + Cs^2 - 2Cs + C + Ds^4 - Ds^3 + Es^3$$

$$s+2 = (A+D)s^4 + (-2A+B-D+E)s^3 + (A-2B+C)s^2 + (B-2C)s + C$$

Equating Coefficent on both sides.

Coefficient of s^4 → $A + D = 0$ …… (2)

Coefficient of s^3 → $-2A + B - D + E = 0$ …… (3)

Coefficient of s^2 → $A - 2B + C = 0$ …… (4)

Coefficient of s → $B - 2C = 1$ …… (5)

Constant term → $\mathbf{C = 2}$

Put $C = 2$ in equation (5), $B - 2(2) = 1$; $B = 1 + 4$; $\mathbf{B = 5}$

Put $B = 5$ & $C = 2$ in equation (4), $A - 2(5) + 2 = 0$; $\mathbf{A = 8}$

Put $A = 8$, in equation (2), $8 + D = 0$; $\mathbf{D = -8}$

Put $A = 8$, $B = 5$, $C = 2$, $D = -8$ in equation(3)

$$-2(8) + 5 - (-8) + E = 0$$

$$-3 + E = 0; \qquad \mathbf{E = 3}$$

Now, Substituting A, B, C, D, & F values in equation(1)

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{s+2}{s^3(s-1)^2}\right\} = L^{-1}\left\{\frac{8}{s} + \frac{5}{s^2} + \frac{2}{s^3} + \frac{-8}{s-1} + \frac{3}{(s-1)^2}\right\}1$$

$$= 8\,L^{-1}\left\{\frac{1}{s}\right\} + 5L^{-1}\left\{\frac{1}{s^2}\right\} + 2L^{-1}\left\{\frac{1}{s^3}\right\} - 8L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$+ 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 8(1) + 5t + 2\frac{t^2}{2!} - 8e^t + 3e^{t}.\,L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\mathbf{f(t) = 8 + 5t + t^2 - 8e^t + 3t\,e^t}$$

Example 74: **Find** $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$

Solution: Hint: By P. F. $\dfrac{1}{s^3(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3} + \dfrac{Ds+E}{s^2+1}$

Simplify by equating coffefficient method $A = -1$, $B = 0$, $C = 1$, $D = 1$, $E = 0$

$$\therefore\ \mathbf{f(t) = -1 + \frac{t^2}{2} + \cos t}$$

Example 75: **Find** $L^{-1}\left\{\dfrac{21s-33}{(s+1)(s-2)^3}\right\}$

Solution: Hint: by P. F. $\dfrac{21s-33}{(s+1)(s-2)^3}$

$$= \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

Simplify by equating coffefficient method $A = 2$, $B = -2$, $C = 4$, $D = 3$

$$\therefore\ \mathbf{f(t) = 2e^t - 2e^{2t} + 4t.e^{2t} + \frac{3}{2}t^2e^{2t}}$$

Example 76: **Find** $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\ \dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

By partial fraction

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{k_1 s + k_2}{s^2+2s+2} + \frac{k_3 s + k_4}{s^2+2s+5} \quad \ldots\ldots(1)$$

Multiplying both sides by $(s^2+2s+2)(s^2+2s+5)$

$$s^2+2s+3 = (k_1 s + k_2)(s^2+2s+5) + (k_3 s + k_4)(s^2+2s+2)$$

$$s^2+2s+3 = k_1 s^3 + 2k_1 s^2 + 5k_1 s + k_2 s^2 + 2k_2 s$$
$$+5k_2 + k_3 s^3 + 2k_3 s^2 + 2k_3 s + k_4 s^2 + 2k_4 s + 2k_4$$

$$s^2+2s+3 = (k_1+k_3)s^3 + (2k_1+k_2+2k_3+k_4)s^2$$
$$+(5k_1+2k_2+2k_3+2k_4)s + (5k_2+2k_4)$$

Equating coefficient on both sides

Coefficient of $s^3 \rightarrow k_1+k_3 = 0$ ……(2)

Coefficient of $s^2 \rightarrow 2k_1+k_2+2k_3+k_4 = 1$ ……(3)

Coefficient of $s \rightarrow 5k_1+2k_2+2k_3+2k_4 = 2$ ……(4)

Constant term $\rightarrow 5k_2+2k_4 = 3$ ……(5)

Equn (3) × 2	$4k_1 + 2k_2 + 4k_3 + 2k_4 = 2$
Equn (4)	$5k_1 + 2k_2 + 2k_3 + 2k_4 = 2$
Subtraction	$- \quad - \quad - \quad - \quad -$
	$-k_1 + 2k_3 = 0$ ……(6)

Equn (2)	$k_1 + k_3 = 0$
Equn (6)	$-k_1 + 2k_3 = 0$
Adding	$3k_3 = 0$
	$\mathbf{k_3 = 0}$

Equn (2) $\rightarrow k_1 + 0 = 0 \quad \therefore \ \mathbf{k_1 = 0}$

Put k_1 & k_3 values in equation (3)→

$$2(0) + k_2 + 2(0) + k_4 = 1$$

$$k_2 + k_4 = 1 \quad \ldots\ldots(7)$$

Now,	Eqn (7) × 2	$2k_2 + 2k_4 = 2$
	Eqn (5)	$5k_2 + 2k_4 = 3$
Subtracting		$- \quad - \quad -$
		$-3k_2 = -1$

$$\mathbf{k_2 = \frac{1}{3}}$$

Put $k_2 = \frac{1}{3}$ in equation (7)→

$$\frac{1}{3} + k_4 = 1 \; ; \quad k_4 = 1 - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}; \qquad \mathbf{k_4 = \frac{2}{3}}$$

Substituting all k_1, k_2, k_3 and k_4 values in equation (1)
and Taking I. L. T. on both sides equation(1) →

$$L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= L^{-1}\left\{\frac{0(s)+\frac{1}{3}}{s^2+2s+2} + \frac{0(s)+\frac{2}{3}}{s^2+2s+5}\right\}$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{2}{3} e^{-t} \frac{\sin 2t}{2}$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$$

$$\mathbf{f(t) = \frac{1}{3} e^{-t} (\sin t + \sin 2t)}$$

Example 77: **Find** $\mathbf{L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}}$

Solution: Hint by P. F. $\frac{s}{(s^2+1)(s^2+4)} = \frac{k_1 s + k_2}{s^2+1} + \frac{k_3 s + k_4}{s^2+4}$

Simplify by using equating coefficient method

$$k_1 = \frac{1}{3}, k_2 = 0, k_3 = \frac{-1}{3}, k_4 = 0$$

$$\mathbf{f(t) = \frac{1}{3}(\cos t - \cos 2t)}$$

Example 78: **Find** $\mathbf{L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$

We have, $\frac{s}{s^4+s^2+1} = \frac{s}{s^4+2s^2+1-s^2}$ ... Note

$$= \frac{s}{(s^2+1)^2 - s^2} \qquad \{\because \text{ use } a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{s}{(s^2+1-s)(s^2+1+s)}$$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] \qquad \text{... Note}$$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1} - \frac{1}{s^2+s+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\right]$$

... By using 3rd term

$$= \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

Taking I. L. T. on both sides , we get

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}\ L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{1}{2}t}\ \frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} - e^{-\frac{1}{2}t}\ \frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{3}}\ e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - \frac{2}{\sqrt{3}}\ e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{1}{2}\ \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left[e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right] \qquad = \frac{1}{2}\ \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left[2\ \sinh\frac{t}{2}\right]$$

$$\mathbf{f(t) = \frac{2}{\sqrt{3}}\ \sin\ \frac{\sqrt{3}}{2}t\ \sinh\frac{t}{2}}$$

Example 79: Find $\mathbf{L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{s^3}{s^4 - a^4}\right\}$

Consider, $\frac{s^3}{s^4 - a^4} = \frac{s^3}{(s^2)^2 - (a^2)^2}$

$= \frac{s^3}{(s^2 - a^2)(s^2 + a^2)}$ $\{\because$ use $a^2 - b^2 = (a - b)(a + b)$

$= \frac{s}{2}\left[\frac{1}{s^2 - a^2} + \frac{1}{s^2 + a^2}\right]$ ... Note

Taking inverse L. T. on both sides

$$L^{-1}\left\{\frac{s^3}{s^4 - a^4}\right\} = \frac{1}{2} L^{-1}\left\{\frac{s}{s^2 - a^2} + \frac{s}{s^2 + a^2}\right\}$$

$$= \frac{1}{2}\left[L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} + L^{-1}\left\{\frac{s}{s^2 + a^2}\right\}\right]$$

$$\mathbf{f(t) = \frac{1}{2}[\cosh at + \cos at]}$$

Example 80: **Find** $\mathbf{L^{-1}\left\{\frac{s + 2}{s^2(s + 3)}\right\}}$

Solution: Hint: by P. F. $\frac{s + 2}{s^2(s + 3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 3}$

$A = \frac{1}{9}, \ B = \frac{2}{3}, \ C = -\frac{1}{9}$ $\quad \mathbf{f(t) = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}}$

Example 81: **Find** $\mathbf{L^{-1}\left\{\frac{s^2}{(s^2 - a^2)^2}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{s^2}{(s^2 - a^2)^2}\right\}$

Consider, $\frac{s^2}{(s^2 - a^2)^2} = \frac{s^2}{[(s - a)(s + a)]^2} = \frac{s^2}{(s - a)^2(s + a)^2}$

By partial fraction

$$\frac{s^2}{(s - a)^2(s + a)^2} = \frac{A}{s - a} + \frac{B}{(s - a)^2} + \frac{C}{s + a} + \frac{D}{(s + a)^2} \quad \ldots\ldots(1)$$

Multiplying both sides by $(s - a)^2(s + a)^2$

$s^2 = A(s - a)(s + a)^2 + B(s + a)^2 + C(s - a)^2(s + a) + D(s - a)^2 \ldots(2)$

Put $s = a$ in equation (2)

$$a^2 = B(a+a)^2\ ;\ a^2 = 4a^2B; \qquad \mathbf{B} = \frac{\mathbf{1}}{\mathbf{4}}$$

Put $s = -a$ in equation (2)

$$(-a^2) = D(-a-a)^2;\ a^2 = 4a^2D; \qquad \mathbf{D} = \frac{\mathbf{1}}{\mathbf{4}}$$

Consider, $\dfrac{A}{s-a} + \dfrac{C}{s+a} = \dfrac{1}{2a}\left[\dfrac{1}{s-a} - \dfrac{1}{s+a}\right]$

$$\frac{A}{s-a} + \frac{C}{s+a} = \frac{\frac{1}{2a}}{s-a} + \frac{-\frac{1}{2a}}{s+a}$$

$\therefore$ Compare N^r on both sides $\mathbf{A} = \dfrac{\mathbf{1}}{\mathbf{2a}}$ & $\mathbf{C} = \dfrac{\mathbf{-1}}{\mathbf{2a}}$

$\therefore$ Substituting values of A, B, C, and D in equation(1)

and Taking inverse L. T. on both sides

$$L^{-1}\left\{\frac{s^2}{(s^2-a^2)^2}\right\} = L^{-1}\left\{\frac{\frac{1}{2a}}{s-a} + \frac{\frac{1}{4}}{(s-a)^2} + \frac{-\frac{1}{2a}}{s+a} + \frac{\frac{1}{4}}{(s+a)^2}\right\}$$

$$= \frac{1}{2a}L^{-1}\left\{\frac{1}{s-a}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s-a)^2}\right\} - \frac{1}{2a}L^{-1}\left\{\frac{1}{s+a}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+a)^2}\right\}$$

$$= \frac{1}{2a}e^{at} + \frac{1}{4}e^{at}(t) - \frac{1}{2a}e^{-at} + \frac{1}{4}e^{-at}(t)$$

$$= \frac{1}{2a}(e^{at} - e^{-at}) + \frac{t}{4}(e^{at} + e^{-at})$$

$$= \frac{1}{2a}\,2\sinh at + \frac{t}{4}\,2\cosh at$$

$$\mathbf{f(t)} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{\sinh at} + \frac{\mathbf{t}}{\mathbf{2}}\mathbf{\cosh at}$$

Example 82: **Find** $\mathbf{L^{-1}\left\{\dfrac{s^2-a^2}{(s^2+a^2)^2}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\dfrac{s^2-a^2}{(s^2+a^2)^2}\right\}$

By partial fraction

$$\frac{s^2-a^2}{(s^2+a^2)^2} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{(s^2+a^2)^2} \qquad \ldots\ldots(1)$$

Multiplying both sides by $(s^2+a^2)^2$

$$s^2 - a^2 = (As+B)(s^2+a^2) + (Cs+D)$$

$$s^2 - a^2 = As^3 + Aa^2 s + Bs^2 + Ba^2 + Cs + D$$

$$s^2 - a^2 = As^3 + Bs^2 + (Aa^2 + C)s + (Ba^2 + D)$$

Equating coefficient on both sides

Coefficient of $s^3 \rightarrow$ $\mathbf{A = 0}$

Coefficient of $s^2 \rightarrow$ $\mathbf{B = 1}$

Coefficient of $s \rightarrow$ $Aa^2 + C = 0$ ……(2)

Constant term $\rightarrow Ba^2 + D = -a^2$ ……(3)

Put $A = 0$ in equation (2) $0a^2 + C = 0$; $\mathbf{C = 0}$

Put $B = 1$ in equation (3) $(1)a^2 + D = -a^2$; $\mathbf{D = -2a^2}$

Substituting values of A, B, C and D in equn(1)

and Taking inverse L. T. on both sides

$$L^{-1}\left\{\frac{s^2 - a^2}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{\frac{0(s) + 1}{s^2 + a^2} + \frac{0(s) + (-2a^2)}{(s^2 + a^2)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} - 2a^2 L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}$$

$$= \frac{\sin at}{a} - a^2 L^{-1}\left\{\frac{1}{s}\left[\frac{\mathbf{2s}}{(\mathbf{s^2} + \mathbf{a^2})^2}\right]\right\}$$

$$= \frac{\sin at}{a} - a^2 L^{-1}\left\{\frac{1}{s}\overline{G(s)}\right\} \quad \text{... Note}$$

$$f(t) = \frac{\sin at}{a} - a^2 \int_0^t G(t)\, dt \quad \text{……(2) ... Note}$$

Where, $\overline{G(s)} = \frac{2s}{(s^2 + a^2)^2}$

Taking I. L. T. $L^{-1}\{\overline{G(s)}\} = L^{-1}\left\{\frac{2s}{(s^2 + a^2)^2}\right\}$

$$= L^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s^2 + a^2}\right)\right\}$$

$$G(t) = t\frac{\sin at}{a} \quad \left\{\because L^{-1}\left\{\frac{d}{ds} f(s)\right\} = -t f(t)\right.$$

Equn (2) becomes

$$f(t) = \frac{\sin at}{a} - a^2 \int_0^t t\frac{\sin at}{a}\, dt$$

$$= \frac{\sin at}{a} - \frac{a^2}{a}\int_0^t t\sin at\ dt$$

$$= \frac{\sin at}{a} - a\left[t.\frac{(-\cos at)}{a} - (1).\frac{(-\sin at)}{a^2}\right]_0^t$$

$$= \frac{\sin at}{a} - a\left[\frac{-t}{a}\cos at + \frac{\sin at}{a^2} - 0\right]$$

$$= \frac{\sin at}{a} + t\cos at - \frac{\sin at}{a}$$

$$\mathbf{f(t) = t\cos at}$$

Example 83: **Find** $\mathbf{L^{-1}\left\{\frac{1}{s^3 + a^3}\right\}}$

Solution: Let $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s^3 + a^3}\right\}$

Consider,

$$\frac{1}{s^3 + a^3} = \frac{1}{(s+a)(s^2 - sa + a^2)} \quad \{\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

By partial fraction

$$\frac{1}{(s+a)(s^2 - sa + a^2)} = \frac{A}{s+a} + \frac{Bs + C}{s^2 - sa + a^2} \qquad \ldots\ldots(1)$$

Multiplying both sides by $(s+a)\ (s^2 - sa + a^2)$

$$1 = A(s^2 - sa + a^2) + (Bs + C)(s + a) \qquad \ldots\ldots(2)$$

Put $s = -a$ in equn(2)→ $1 = A(a^2 + a^2 + a^2)$; $\mathbf{A = \frac{1}{3a^2}}$

Put $s = 0$ and $A = \frac{1}{3a^2}$ in equn(2)→$1 = \frac{1}{3a^2}(a^2) + C(a)$;

$$(a)C = 1 - \frac{1}{3}\ ;\ \mathbf{C = \frac{2}{3a}}$$

Now, from equn(2)

$$1 = As^2 - Aas + Aa^2 + Bs^2 + Bas + Cs + Ca$$

$$1 = (A+B)\ s^2 + (-Aa + Ba + C)s + (Aa^2 + Ca)$$

Equating coefficient on both sides

Coefficient of s^2 → $A + B = 0$; $B = -A$; $\mathbf{B = \frac{-1}{3a^2}}$

Substituting values of A, B, and C in equation (1) & Taking inverse L. T. on both sides,

$$L^{-1}\left\{\frac{1}{s^3+a^3}\right\} = L^{-1}\left\{\frac{\frac{1}{3a^2}}{s+a} + \frac{\frac{-1}{3a^2}+\frac{2}{3a}}{s^2-as+a^2}\right\}$$

$$= \frac{1}{3a^2}L^{-1}\left\{\frac{1}{s+a}\right\} - \frac{1}{3a^2}L^{-1}\left\{\frac{s}{s^2-as+a^2}\right\} + \frac{2}{3a}L^{-1}\left\{\frac{1}{s^2-as+a^2}\right\}$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}L^{-1}\left\{\frac{s}{s^2-as+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+a^2}\right\}$$

$$+ \frac{2}{3a}L^{-1}\left\{\frac{1}{s^2-as+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+a^2}\right\}$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}L^{-1}\left\{\frac{s}{\left(s-\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right\} + \frac{2}{3a}L^{-1}\left\{\frac{1}{\left(s-\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right\}$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}L^{-1}\left\{\frac{\left(s-\frac{a}{2}\right)+\frac{a}{2}}{\left(s-\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$+ \frac{2}{3a}L^{-1}\left\{\frac{1}{\left(s-\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}e^{\frac{a}{2}t}L^{-1}\left\{\frac{s+\frac{a}{2}}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\} + \frac{2}{3a}e^{\frac{a}{2}t}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}e^{\frac{a}{2}t}\left[L^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\} + \frac{a}{2}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}\right]$$

$$+ \frac{2}{3a}e^{\frac{a}{2}t}\frac{\sin\sqrt{\frac{3}{2}}at}{\frac{\sqrt{3}}{2}a}$$

$$= \frac{1}{3a^2}\ e^{-at} - \frac{1}{3a^2}\ e^{\frac{a}{2}t}\left[\cos\frac{\sqrt{3}}{2}\ at + \frac{a}{2}\frac{\sin\frac{\sqrt{3}}{2}at}{\frac{\sqrt{3}}{2}a}\right] + \frac{2}{3a}e^{\frac{a}{2}t}\frac{2}{\sqrt{3}a}\sin\frac{\sqrt{3}}{2}at$$

$$= \frac{1}{3a^2}e^{-at} - \frac{1}{3a^2}e^{\frac{a}{2}t}\cos\frac{\sqrt{3}}{2}at - \frac{e^{\frac{a}{2}t}}{3\sqrt{3}a^2}\ \sin\frac{\sqrt{3}}{2}\ at$$

$$+ \frac{4}{3}\ \frac{e^{\frac{a}{2}t}}{\sqrt{3}a^2}\sin\frac{\sqrt{3}}{2}\ at$$

$$\mathbf{f(t) = \frac{e^{-at}}{3a^2} - \frac{e^{\frac{a}{2}t}}{3a^2}\cos\frac{\sqrt{3}}{2}at + \frac{e^{\frac{a}{2}t}}{\sqrt{3}a^2}\sin\ \frac{\sqrt{3}}{2}at}$$

Example 84: **Find** $\mathbf{L^{-1}\left\{\frac{s+1}{(s^2+2s\ +2)^2}\right\}}$

Solution: Let, $\quad L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s+1}{(s^2+2s\ +2)^2}\right\}$

Consider, $\quad \frac{d}{ds}\left\{\frac{1}{s^2+2s+2}\right\} = \frac{-(2s+2)}{(s^2+2s+2)^2} = \frac{-2(s+1)}{(s^2+2s\ +2)^2}$

$$\therefore\ \frac{s+1}{(s^2+2s\ +1)^2} = \frac{-1}{2}\ \frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right)$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{s+1}{(s^2+2s\ +1)^2}\right\} = \frac{-1}{2}\ L^{-1}\left\{\frac{d}{ds}\left\{\frac{1}{s^2+2s+2}\right\}\right\}$$

Now, $\quad W.k.t,\ L^{-1}\left\{(-1)^n\frac{d^n}{ds^n}\ \overline{F(s)}\right\} = t^n L^{-1}\{\overline{F(s)}\} = t^n F(t),$

$$Here\ n = 1$$

$$L^{-1}\left\{\frac{s+1}{(s^2+2s\ +1)^2}\right\} = \frac{-1}{2}\ (-t)\ F(t) = \frac{1}{2}t\,F(t) \qquad \ldots\ldots(1)$$

Where $\ F(t)\ = L^{-1}\{\overline{F(s)}\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$

$$= L^{-1}\left\{\frac{1}{s^2+2s+1^2-1^2+2}\right\}$$

$$F(t)\ = L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} = e^{-t}\sin t$$

Equation (1) → $\quad \mathbf{f(t) = \frac{1}{2}t\,e^{-t}\sin t}$

Example 85: Find $\mathbf{L^{-1}\left\{\dfrac{3s+1}{(s+1)^4}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\dfrac{3s+1}{(s+1)^4}\right\}$

$$= L^{-1}\left\{\frac{3(s+1)-2}{(s+1)^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{3s-2}{s^4}\right\} \qquad = e^{-t}L^{-1}\left\{\frac{3s}{s^4}\ \frac{-2}{s^4}\right\}$$

$$= e^{-t}\left[3L^{-1}\left\{\frac{1}{s^3}\right\} - 2L^{-1}\left\{\frac{1}{s^3}\right\}\right]$$

$$= e^{-t}\left[3\frac{t^2}{2!} - 2\frac{t^3}{3!}\right] \qquad = \frac{3}{2}e^{-t}t^2 - \frac{2}{6}e^{-t}t^3$$

$$\mathbf{f(t) = e^{-t}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)}$$

Example 86: Find $\mathbf{L^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\}}$

Solution: Let $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\}$

Consider $\dfrac{s}{s^4+4a^4} = \dfrac{s}{(s^2)^2+(2a^2)^2}$

$$= \frac{s}{(s^2)^2 + \mathbf{2(s^2)(2a^2)} + (2a^2)^2 - \mathbf{2(s^2)(2a^2)}} \qquad \ldots \text{Note}$$

$$= \frac{s}{(s^2+2a^2)^2 - 4a^2s^2}$$

$$= \frac{s}{(s^2+2a^2)^2 - (2as)^2}$$

$$= \frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)} \qquad \{\because\ a^2-b^2 = (a-b)(a+b)$$

By partial fraction

$$\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}$$

$$= \frac{k_1s+k_2}{s^2+2a^2-2as} + \frac{k_3s+k_4}{s^2+2a^2+2as} \qquad \ldots\ldots(1)$$

Multiplying both sides by $(s^2+2a^2-2as)(s^2+2a^2+2as)$

$$s = (k_1s+k_2)(s^2+2a^2+2as) + (k_3s+k_4)(s^2+2a^2-2as)$$

$$s = k_1s^3 + 2a^2k_1s + 2ak_1s^2 + k_2s^2 + 2a^2k_2 + 2ak_2s + k_3s^3 + 2a^2k_3s - 2ak_3s^2 + k_4s^2 + 2a^2k_4 - 2ak_4s$$

$$s = (k_1+k_3)s^3+(2ak_1+k_2-2ak_3+k_4)s^2$$
$$+(2a^2k_1 + 2ak_2 + 2a^2k_3 - 2ak_4)s + (2a^2k_2 + 2a^2k_4)$$

Equating coefficient on both sides

Coefficient of $s^3 \rightarrow k_1 + k_3 = 0$ ... ... (2)

Coefficient of $s^2 \rightarrow 2ak_1 + k_2 - 2ak_3 + k_4 = 0$ ... ... (3)

Coefficient of $s \rightarrow 2a^2k_1 + 2ak_2 + 2a^2k_3 - 2ak_4 = 1$ ... ... (4)

Constant term $\rightarrow 2a^2k_2 + 2a^2k_4 = 0$

$$k_2 + k_4 = 0 \quad \text{... ... (5)}$$

Equation (3) $\rightarrow 2ak_1 - 2ak_3 = 0$

i.e. $k_1 - k_3 = 0$ ... ... (6)

Now, Adding equn(2) and (6)$\rightarrow 2k_1 = 0$; $\mathbf{k_1 = 0}$; $\mathbf{k_3 = 0}$

Putting $k_1 = 0$; $k_3 = 0$ in equn(3) $\rightarrow k_2 + k_4 = 0$ ... ... (7)

Putting $k_1 = 0$; $k_3 = 0$ in equn(4) $\rightarrow 2ak_2 - 2ak_4 = 1$ ... ... (8)

Now,

Equation (7) × 2a $\quad 2ak_2 + 2ak_4 = 0$

Equation (8) $\quad 2ak_2 - 2ak_4 = 1$

Adding $\quad 4ak_2 = 1$

$$\mathbf{k_2 = \frac{1}{4a}}$$

Equation(7) $\rightarrow \dfrac{1}{4a} + k_4 = 0$; $\mathbf{k_4 = \dfrac{-1}{4a}}$

Substituting values of k_1, k_2, k_3 and k_4 in equaion(1)

$$L^{-1}\left\{\frac{s}{s^4 + a^4}\right\} = L^{-1}\left\{\frac{0(s) + \frac{1}{4a}}{s^2 + 2a^2 - 2as} + \frac{0(s) + \left(-\frac{1}{4a}\right)}{s^2+2a^2 + 2as}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2 - 2as + 2a^2}\right\} - \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2 + 2as + 2a^2}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2 - 2as + a^2 - a^2 + 2a^2}\right\}$$
$$- \frac{1}{4a}L^{-1}\left\{\frac{1}{s^2 + 2as + a^2 - a^2 + 2a^2}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{(s-a)^2 + a^2}\right\} - \frac{1}{4a}L^{-1}\left\{\frac{1}{(s+a)^2 + a^2}\right\}$$

$$= \frac{1}{4a}e^{at}\frac{\sin at}{a} - \frac{1}{4a}e^{-at}\frac{\sin at}{a}$$

$$= \frac{1}{4a^2}\sin at\,(e^{at} - e^{-at}) \quad = \frac{1}{4a^2}\sin at\;2\sinh at$$

$$\mathbf{f(t) = \frac{1}{2a^2}\sin at\;\sinh at}$$

10 Logarithmic function

Steps: 1)First simplifying $\overline{f(s)}$ by using rules of logarithm

2) First differentiate $\overline{f(s)}$ w.r.t. 's' i.e. $\frac{d}{ds}\overline{f(s)}$

3) Taking I.L.T. on bothsides ; $L^{-1}\left\{\frac{d}{ds}\overline{f(s)}\right\} = -t\,f(t)$

and proceed.

10.i Examples on Logarithmic function

Example 87: **Evaluate by using inverse Laplace transform of**

$$\mathbf{\log\left(\frac{s+b}{s+a}\right)}$$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\log\left(\frac{s+b}{s+a}\right)\right\}$

Consider, $\overline{f(s)} = \log\left(\frac{s+b}{s+a}\right) = \log(s+b) - \log(s+a)$

Differentiating w.r.t.'s' on both sides

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}\,[\log(s+b) - \log(s+a)]$$

$$= \frac{d}{ds}\log(s+b) - \frac{d}{ds}\log(s+a)$$

$$\frac{d}{ds}\overline{f(s)} = \frac{1}{s+b} - \frac{1}{s+a}$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds}\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s+b} - \frac{1}{s+a}\right\}$$

$$-t\;f(t) = L^{-1}\left\{\frac{1}{s+b}\right\} - L^{-1}\left\{\frac{1}{s+a}\right\}$$

$$-t\;f(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-bt} - e^{-at}}{-t}$$

$$\mathbf{f(t) = \frac{e^{-at} - e^{-bt}}{t}}$$

Example 88: **Find** $\mathbf{L^{-1}\left\{\log\left(\frac{1-s}{1+s}\right)\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\log\left(\frac{1-s}{1+s}\right)\right\}$

Consider, $\overline{f(s)} = \log\left(\frac{1-s}{1+s}\right) = \log(1-s) - \log(1+s)$

Differentiating w.r.t.'s' on both sides

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}\left[\log(1-s) - \log(1+s)\right]$$

$$= \frac{d}{ds}\log(1-s) - \frac{d}{ds}\log(1+s)$$

$$= \frac{1}{1-s}(-1) - \frac{1}{1+s}(1)$$

$$\frac{d}{ds}\overline{f(s)} = \frac{-1}{1-s} - \frac{1}{1+s}$$

i.e. $\frac{d}{ds}\overline{f(s)} = \frac{1}{s-1} - \frac{1}{s+1}$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds}\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\}$$

$$-t\,f(t) = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$-t\,f(t) = e^{t} - e^{-t}$$

$$f(t) = \frac{e^{t} - e^{-t}}{-t} = \frac{2\sinh t}{-t}$$

$$\mathbf{f(t) = \frac{-2\sinh t}{t}}$$

Example 89: **Find** $\mathbf{L^{-1}\left\{\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right\}}$

Solution: Hint: Solve by as like above problem ,

Ans. $\mathbf{f(t) = \frac{-\sinh t}{t}}$

Example 90: **Find** $\mathbf{L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Consider, $\overline{f(s)} = \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

$$\overline{f(s)} = \frac{1}{2}[\log(s^2+a^2) - \log(a^2+b^2)]$$

Differentiating w.r.t. 's'

$$\frac{d}{ds}f(s) = \frac{d}{ds}\frac{1}{2}[\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= \frac{1}{2}\left[\frac{d}{ds}\log(s^2+a^2) - \frac{d}{ds}\log(s^2+b^2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+a^2}2s - \frac{1}{s^2+b^2}2s\right]$$

$$\frac{d}{ds}f(s) = \frac{2}{2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds}f(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$$

$$-t\,f(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - L^{-1}\left\{\frac{s}{s^2+b^2}\right\}$$

$$f(t) = \frac{\cos at - \cos bt}{-t}$$

$$\mathbf{f(t) = \frac{\cos bt - \cos at}{t}}$$

Example 91: **Find** $\mathbf{L^{-1}\left\{\log\left(1+\frac{a^2}{s^2}\right)\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\log\left(1+\frac{a^2}{s^2}\right)\right\}$

Consider, $\overline{f(s)} = \log\left(\frac{s^2+a^2}{s^2}\right)$

$$\overline{f(s)} = \log(s^2+a^2) - \log s^2$$

$$\overline{f(s)} = \log(s^2+a^2) - 2\log s$$

Differentiating w.r.t. 's'

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}[\log(s^2+a^2) - 2\log s]$$

$$= \frac{d}{ds}\log(s^2+a^2) - 2\frac{d}{ds}\log s$$

$$\frac{d}{ds}\overline{f(s)} = \frac{1}{s^2+a^2}2s - 2\frac{1}{s}$$

Taking inverse L. T. on both sides

$$L^{-1}\left\{\frac{d}{ds}\overline{f(s)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - 2\ L^{-1}\left\{\frac{1}{s}\right\}$$

$$-t\,f(t) = 2\cos at - 2\ (1)$$

$$f(t) = \frac{2\cos at - 2}{-t}$$

$$\mathbf{f(t) = 2\ \frac{(1-\cos at\,)}{t}}$$

Example 92: **Find** $\mathbf{L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2 - a^2}{s^2}\right)\right\}}$

Ans. $\mathbf{f(t) = \frac{1-\cosh at}{t}}$

Example 93: **Find** $\mathbf{L^{-1}\left\{\log\frac{s^2+1}{s(s+1)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\log\frac{s^2+1}{s(s+1)}\right\}$

Consider, $\overline{f(s)} = \log\frac{s^2+1}{s(s+1)} \quad = \log(s^2+1) - \log s(s+1)$

$$\overline{f(s)} = \log(s^2+1) - [\log s + \log(s+1)]$$

$$\overline{f(s)} = \log(s^2+1) - \log s - \log(s+1)$$

Differentiating w.r.t. 's' on both sides

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}[\log(s^2+1) - \log s - \log(s+1)]$$

$$= \frac{d}{ds}\log(s^2+1) - \frac{d}{ds}\log s - \frac{d}{ds}\log(s+1)$$

$$\frac{d}{ds}f(s) = \frac{1}{s^2+1}\ 2s - \frac{1}{s} - \frac{1}{s+1}$$

Taking inverse L. T. on both sides,

$$L^{-1}\left\{\frac{d}{ds}f(s)\right\} = 2\ L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$-t\ f(t) = 2\cos t - 1 - e^{-t}$$

$$\mathbf{f(t) = \frac{1}{t}(1 + e^{-t} - 2\cos t)}$$

Example 94: **Find** $\mathbf{L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}$

Consider, $\overline{F(s)} = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$

$$\overline{F(s)} = \log(s^2+1) - \log s^2$$

Differentiating w.r.t. 's' on both sides

$$\frac{d}{ds}\overline{F(s)} = \frac{d}{ds}\log(s^2+1) - \frac{d}{ds}\log s^2$$

$$= \frac{1(2s)}{s^2+1} - \frac{1}{s^2}2s$$

$$\frac{d}{ds}\overline{F(s)} = \frac{2s}{s^2+1} - \frac{2}{s}$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds}\overline{F(s)}\right\} = 2L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{1}{s}\right\}$$

$$-t\ \overline{F(t)} = 2\cos t - 2\ (1)$$

$$\overline{F(t)} = \frac{2(\cos t - 1)}{-t}$$

$$\overline{F(t)} = \frac{2(1-\cos t)}{t}$$

Now, W.k.t., $L^{-1}\left\{\frac{1}{s}\overline{F(s)}\right\} = \int_0^t \overline{F(t)}\ dt$

$$\mathbf{L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2(1-\cos t)}{t}\ dt}$$

Example 95: **Find** $\mathbf{L^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}$

Consider, $\overline{f(s)} = \cot^{-1}\left(\frac{s}{2}\right)$

Differentiating w.r.t. 's' on both sides

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right) \qquad \left\{\because \frac{d}{ds}\cot^{-1}x = \frac{-1}{x^2+1}\right.$$

$$\frac{d}{ds}\overline{f(s)} = \frac{-1}{\left(\frac{s}{2}\right)^2+1}\left(\frac{1}{2}\right) = \frac{-1}{\frac{s^2}{4}+1}\left(\frac{1}{2}\right) = \frac{-4}{s^2+4}\left(\frac{1}{2}\right)$$

$$\frac{d}{ds}\overline{f(s)} = \frac{-2}{s^2+4}$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds}\overline{f(s)}\right\} = -2\,L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$-t\ f(t) = -2\frac{\sin 2t}{2}$$

$$-t\ f(t) = -\sin 2t$$

$$f(t) = \frac{-\sin 2t}{-t}$$

$$\mathbf{f(t) = \frac{\sin 2t}{t}}$$

Example 96: Find $\mathbf{L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

Consider, $\overline{f(s)} = \tan^{-1}\left(\frac{2}{s^2}\right)$

Differentiating w.r.t. 's' on both sides

$$\frac{d}{ds}\overline{f(s)} = \frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)$$

$$\frac{d}{ds}\overline{f(s)} = \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\cdot\frac{d}{ds}\left(\frac{2}{s^2}\right) \qquad \left\{\because \frac{d}{ds}\tan^{-1}x = \frac{1}{1+x^2}\right.$$

$$= \frac{1}{1+\frac{4}{s^4}}\cdot 2\,(-2)\,s^{-3} \qquad \left\{\because \frac{d}{ds}\frac{1}{x^2} = \frac{d}{ds}x^{-2}\right.$$

$$= \frac{-4s^{-3}}{\frac{s^4+4}{s^4}} \qquad \left\{\because \frac{d}{ds}x^n = nx^{n-1}\right.$$

$$= \frac{s^4}{s^3}\left(\frac{-4}{s^4+4}\right)$$

$$= \frac{-4s}{s^4+4} = \frac{-4s}{s^4+4s^2+4-4s^2}$$

$$= \frac{-4s}{(s^2+2)^2-(2s)^2}$$

$$= \frac{-4s}{(s^2+2-2s)(s^2+2+2s)} \qquad \{\because a^2-b^2 = (a-b)(a+b)$$

$$= \frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2} \qquad \text{... Note}$$

$$\frac{d}{ds} f(s) = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1} \qquad \{\because \text{ 3rd term}$$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{d}{ds} f(s)\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2+1^2}\right\}$$

$$-t\ f(t) = e^{-t}\sin t - e^{t}\sin t$$

$$f(t) = \frac{\sin t\,(e^{-t}-e^{t})}{-t} = \frac{\sin t\,(e^{t}-e^{-t})}{t} = \frac{\sin t\ 2.\sinh t}{t}$$

$$\mathbf{f(t) = \frac{2\sin t.\sinh t}{t}}$$

Example 97: **If s is sufficiently large show using series expansion of** $\tan^{-1}\left(\frac{a}{s}\right)$ **that** $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\} = \frac{\sin at}{t}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$

w.k.t. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

Replace $x = \frac{a}{s}$

$$\tan^{-1}\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{\left(\frac{a}{s}\right)^3}{3} + \frac{\left(\frac{a}{s}\right)^5}{5} - \frac{\left(\frac{a}{s}\right)^7}{7} + \cdots$$

$$\tan^{-1}\left(\frac{a}{s}\right) = \frac{a}{s} - \frac{a^3}{3}\frac{1}{s^3} + \frac{a^5}{5}\frac{1}{s^5} - \frac{a^7}{7}\frac{1}{s^7} + \cdots$$

Taking Inverse L.T. on both sides

$$L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$$

$$= a\,L^{-1}\left\{\frac{1}{s}\right\} - \frac{a^3}{3}L^{-1}\left\{\frac{1}{s^3}\right\} + \frac{a^5}{5}L^{-1}\left\{\frac{1}{s^5}\right\} - \frac{a^7}{7}L^{-1}\left\{\frac{1}{s^7}\right\} + \cdots$$

$$= a(1) - \frac{a^3}{3}\frac{t^2}{2!} + \frac{a^5}{5}\frac{t^4}{4!} - \frac{a^7}{7}\frac{t^6}{6!} + \cdots$$

$$= \frac{1}{t}\left[at - \frac{(at)^3}{3!} + \frac{(at)^5}{5!} - \frac{(at)^7}{7!} + \cdots\right] \qquad \ldots \text{Note}$$

$$\mathbf{\overline{f(t)} = \frac{1}{t}\sin at} \qquad \left\{\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ here } x = at\right.$$

Example 98: Show that

$$\mathbf{L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2}}$$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\}$

Consider, w.r.t. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Replace $x = \frac{1}{s}$, $\cos\left(\frac{1}{s}\right) = 1 - \frac{\left(\frac{1}{s}\right)^2}{2!} + \frac{\left(\frac{1}{s}\right)^4}{4!} - \frac{\left(\frac{1}{s}\right)^6}{6!}$

$$= 1 - \frac{1}{2!}\frac{1}{s^2} + \frac{1}{4!}\frac{1}{s^4} - \frac{1}{6!}\frac{1}{s^6}$$

Multiplying both sides by $\frac{1}{s}$, $\frac{1}{s}\cos\frac{1}{s} = \frac{1}{s} - \frac{1}{2!}\frac{1}{s^3} + \frac{1}{4!}\frac{1}{s^5} - \frac{1}{6!}\frac{1}{s^7}$

Taking inverse L.T. on both sides

$$L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2!}L^{-1}\left\{\frac{1}{s^3}\right\} + \frac{1}{4!}L^{-1}\left\{\frac{1}{s^5}\right\} - \frac{1}{6!}L^{-1}\frac{1}{s^7}$$

$$\overline{f(t)} = 1 - \frac{1}{2!}\frac{t^2}{2!} + \frac{1}{4!}\frac{t^4}{4!} - \frac{1}{6!}\frac{t^6}{6!}$$

$$\text{L.H.S.} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2}$$

L.H.S. = R.H.S. ... **Hence proved**

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11 Convolution Theorem

$$\mathbf{L^{-1}\left\{\overline{f(s)}\right\} = f(t) = \int_0^t f_1(t-u).f_2(u)\,du \quad OR}$$

$$\mathbf{L^{-1}\left\{\overline{f(s)}\right\} = f(t) = \int_0^t f_1(u).f_2(t-u)\,du}$$

Steps:

1) Given $\overline{f(s)}$, split it into two parts $\overline{f_1(s)}$ and $\overline{f_2(s)}$ (say).

2) Taking inverse Laplace transform of both $\overline{f_1(s)}$ & $\overline{f_2(s)}$ we get $f_1(t)$ & $f_2(t)$ respectively.

3) Any one function replace $t = u$ and other replace $t = t - u$ and put in formula and proceed.

11.i Examples on convolution theorem

Example 99: **Using convolution theorem find** $\mathbf{L^{-1}\left\{\frac{a}{s(s-a)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{a}{s(s-a)}\right\}$

Consider, $\overline{f(s)} = \frac{a}{s(s-a)} = \frac{a}{s}.\frac{1}{s-a}$

Let $\overline{f_1(s)} = \frac{a}{s}$, $L^{-1}\{\overline{f_1(s)}\} = L^{-1}\left\{\frac{a}{s}\right\} = a = f_1(t)$

Let $\overline{f_2(s)} = \frac{1}{s-a}$, $L^{-1}\{\overline{f_2(s)}\} = L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} = f_2(t)$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}.\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t a\,e^{au}\,du \quad = a\int_0^t e^{au}\,du$$

$$= a\left[\frac{e^{au}}{a}\right]_0^t \quad = [e^{au}]_0^t \quad = e^{at} - e^{a(0)} \quad = e^{at} - e^0$$

$$\mathbf{f(t) = e^{at} - 1}$$

Example 100: **Using C.T. find** $\mathbf{L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

$$\overline{f(s)} = \frac{1}{s(s^2+a^2)} = \frac{1}{s}.\frac{1}{s^2+a^2}$$

Let $\overline{f_1(s)} = \frac{1}{s}$; $\quad L^{-1}\{\overline{f_1(s)}\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = f_1(t)$

$$\overline{f_2(s)} = \frac{1}{s^2+a^2} \ ; \ L^{-1}\{\overline{f_2(s)}\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = f_2(t)$$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}.\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t 1.\frac{\sin au}{a}\,du \qquad = \frac{1}{a}\int_0^t \sin au\,du$$

$$= \frac{1}{a}\left[\frac{-\cos au}{a}\right]_0^t \qquad = \frac{-1}{a^2}\,[\cos au]_0^t$$

$$= \frac{-1}{a^2}\,[\cos at - \cos a\,(0)] \qquad = \frac{-1}{a^2}\,[\cos at - 1]$$

$$\mathbf{f(t) = \frac{1}{a^2}\,(1 - \cos at)}$$

Example 101: Using C.T. find inverse of $\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{(s-2)(s+2)^2}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$

$$\overline{f(s)} = \frac{1}{(s-2)(s+2)^2} = \frac{1}{s-2}\,\frac{1}{(s+2)^2}$$

Let, $\overline{f_1(s)} = \frac{1}{s-2}$; $\quad L^{-1}\{\overline{f_1(s)}\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = f_1(t)$

$$\overline{f_2(s)} = \frac{1}{(s+2)^2} \ ; \ L^{-1}\{\overline{f_2(s)}\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = t.e^{-2t}$$
$$= f_2(t)$$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}.\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t e^{2(t-u)} . e^{-2u} . u \; du$$

$$= \int_0^t e^{2t} . e^{-2u} . e^{-2u} . u \, du \quad = \int_0^t e^{2t} e^{-4u} \, u \, du$$

$$= e^{2t} \int_0^t u \, e^{-4u} \, du \quad = e^{2t} \left[u \frac{e^{-4u}}{-4} - \frac{e^{-4u}}{(-4)(-4)} \right]_0^t$$

$$= e^{2t} \left[t \, \frac{e^{-4t}}{-4} - \frac{1}{16} \, e^{-4t} - \left(0 - \frac{e^{4(0)}}{16} \right) \right]$$

$$= e^{2t} \left[\frac{-1}{4} t \, e^{-4t} - \frac{1}{16} \, e^{-4t} + \frac{1}{16} \, e^{0} \right]$$

$$= \frac{1}{16} \left(-4t \, e^{-2t} - e^{-2t} + e^{2t} \right)$$

$$\mathbf{f(t) = \frac{1}{16} \left(e^{2t} - e^{-2t} - 4te^{-2t} \right)}$$

Example 102: **Using C. T. find** $\mathbf{L^{-1}\left\{ \frac{1}{(s+1)(s^2+1)} \right\}}$

Solution: Let, $L^{-1}\left\{ \overline{f(s)} \right\} = L^{-1}\left\{ \frac{1}{(s+1)(s^2+1)} \right\}$

$$\overline{f(s)} = \frac{1}{s+1} . \frac{1}{s^2+1}$$

Let $\overline{f_1(s)} = \frac{1}{s+1}$; $f_1(t) = e^{-t}$

$\overline{f_2(s)} = \frac{1}{s^2+1}$; $f_2(t) = \sin t$

By convolution theorem

$$f(t) = L^{-1}\left\{ \overline{f(s)} \right\} = L^{-1}\left\{ \overline{f_1(s)} . \overline{f_2(s)} \right\} = \int_0^t f_1(t-u) . f_2(u) \, du$$

$$= \int_0^t e^{-(t-u)} \sin u \, du = \int_0^t e^{-t} e^{u} \sin u \; du$$

$$= e^{-t} \int_0^t e^u \sin u \; du \qquad \left\{\because \int e^{ax} \sin bx \, dx\right.$$

$$= \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] + c$$

$$= e^{-t} \left[\frac{e^u}{1^2 + 1^2} [(1) \sin u - (1) \cos u]\right]_0^t$$

$$= e^{-t} \left[\frac{e^t}{2} [\sin t - \cos t] - \frac{e^0}{2} (\sin 0 - \cos 0)\right]$$

$$= e^{-t} \left[\frac{e^t}{2} (\sin t - \cos t) - \frac{1}{2} (-1)\right] \qquad \{\because \sin 0 = 0, \quad \cos 0 = 1$$

$$= \frac{1}{2} (\sin t - \cos t) + \frac{1}{2} e^{-t}$$

$$\mathbf{f(t) = \frac{1}{2} (\sin t - \cos t + e^{-t})}$$

Example 103: Using C.T. find $\mathbf{L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

$$\overline{f(s)} = \frac{s^2}{(s^2 + 4)^2} = \frac{s}{s^2 + 4} \cdot \frac{s}{s^2 + 4}$$

Let $\overline{f_1(s)} = \frac{s}{s^2 + 4}$; $f_1(t) = \cos 2t$

$\overline{f_2(s)} = \frac{s}{s^2 + 4}$; $f_2(t) = \cos 2t$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)} \,.\, \overline{f_2(s)}\} = \int_0^t f_1(t - u).f_2(u) \, du$$

$$= \int_0^t \cos 2(t - u) \,.\, \cos 2u \; du$$

$$= \int_0^t \cos(2t - 2u). \cos 2u \, du$$

$$= \int_0^t \frac{1}{2}\,[\cos(2t - 2u + 2u) + \cos(2t - 2u - 2u)]du$$

$$\left\{\because \cos A . \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\,,\right.$$

Here $A = 2t - 2u$ & $B = 2u$

$$= \frac{1}{2}\int_0^t [\cos(2t) + \cos(2t - 4u)]\,du$$

$$= \frac{1}{2}\left[\cos 2t \int_0^t 1du + \int_0^t \cos(2t - 4u)\,du\right]$$

$$= \frac{1}{2}\left[\cos 2t\ [u]_0^t + \left[\frac{\sin(2t - 4u)}{-4}\right]_0^t\right]$$

$$= \frac{1}{2}\left\{\cos 2t\ (t - 0) - \frac{1}{4}\,[\sin(2t - 4t) - \sin(2t - 4(0)]\right\}$$

$$= \frac{1}{2}\left[t \cos 2t - \frac{1}{4}\,(\sin(-2t) - \sin 2t)\right]$$

$$= \frac{1}{2} t . \cos 2t - \frac{1}{8}(-2 \sin 2t\,) \qquad \{\because \sin(-\theta) = -\sin\theta$$

$$\mathbf{f(t) = \frac{1}{2} t . \cos 2t + \frac{1}{4} \sin 2t}$$

Example 104: **Using C.T. find** $\mathbf{L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\}$

$$\overline{f(s)} = \frac{1}{(s^2 + a^2)^2} . \frac{1}{(s^2 + a^2)}$$

Let $\overline{f_1(s)} = \frac{1}{s^2 + a^2}$; $f_1(t) = \frac{\sin at}{a}$

$\overline{f_2(s)} = \frac{1}{s^2 + a^2}$; $f_2(t) = \frac{\sin at}{a}$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)} . \overline{f_2(s)}\} = \int_0^t f_1(t - u) . f_2(u)\,du$$

$$= \int_0^t \frac{\sin a(t-u)}{a} . \frac{\sin au}{a}\, du$$

$$= \frac{1}{a^2} \int_0^t \sin(at - au) \sin au\, du$$

$$\left\{ \sin A . \sin B = -\frac{1}{2} [\cos(A+B) - \cos(A-B)] \right.$$

$$= \frac{-1}{2a^2} \int_0^t [\cos at - \cos(at - 2au)]\, du$$

$$= \frac{-1}{2a^2} \left[\cos at \int 1\, du - \int_0^t \cos(at - 2au)\, du \right]$$

$$= -\frac{1}{2a^2} \left\{ \cos at\, [u]_0^t - \left[\frac{\sin(at - 2au)}{-2a} \right]_0^t \right\}$$

$$= \frac{-1}{2a^2} \left\{ \cos at\ (t-0) + \frac{1}{2a} \left[\sin(at - 2at) - \sin\big(at - 2a(0)\big) \right] \right\}$$

$$= \frac{-1}{2a^2} \left\{ t \cos at + \frac{1}{2a} (-\sin at - \sin at) \right\} \qquad \{ \because \sin(-\theta) = -\sin\theta$$

$$= \frac{-1}{2a^2} \left\{ t \cos at + \frac{1}{2a} (-2 \sin at) \right]$$

$$\mathbf{f(t) = \frac{1}{2a^3} [\sin at - at \cos at]}$$

Example 105: **Using C.T. find** $\mathbf{L^{-1} \left\{ \frac{(s+2)^2}{(s^2 + 4s + 8)^2} \right\}}$

Solution: Let, $L^{-1} \left\{ \overline{f(s)} \right\} = L^{-1} \left\{ \frac{(s+2)^2}{(s^2 + 4s + 8)^2} \right\}$

$$\overline{f(s)} = \frac{(s+2)^2}{(s^2 + 4s + 8)^2} = \frac{s+2}{(s^2 + 4s + 8)} . \frac{s+2}{(s^2 + 4s + 8)}$$

Let, $\overline{f_1(s)} = \frac{s+2}{s^2 + 4s + 8} = \frac{(s+2)}{s^2 + 4s + 2^2 - 2^2 + 8}$

$$= \frac{(s+2)}{(s+2)^2 + 2^2} = \overline{f_2(s)} \qquad \{ \because \text{By using } 3^{rd} \text{ term}$$

$$f_1(t) = e^{-2t} \cos 2t = f_2(t)$$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}\,.\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t e^{-2(t-u)}\cos 2(t-u)\,.\,e^{-2u}.\cos 2u\;du$$

$$= \int_0^t e^{-2t}.e^{2u}\cos(2t-2u)\;.\,e^{-2u}\cos 2u\;du$$

$$= e^{-2t}\int_0^t \cos(2t-2u).\cos 2u\;du \qquad \{\because e^{2u}.e^{-2u} = e^0 = 1$$

$$= e^{-2t}\int_0^t \frac{1}{2}[\cos(2t-2u+2u)+\cos(2t-2u-2u)]\,du$$

$$\left\{\because \cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]\right.$$

$$= \frac{e^{-2t}}{2}\int_0^t [\cos 2t + \cos(2t-4u)]\,du$$

$$= \frac{e^{-2t}}{2}\left[\cos 2t\int_0^t 1\,du + \int_0^t \cos(2t-4u)\,du\right]$$

$$= \frac{e^{-2t}}{2}\left[\cos 2t\,[u]_0^t + \left[\frac{\sin(2t-4u)}{-4}\right]_0^t\right]$$

$$= \frac{e^{-2t}}{2}\left[\cos 2t\,[t-0] - \frac{1}{4}\,[\sin(2t-4t)-\sin\big(2t-4(0)\big)\,]\right]$$

$$= \frac{e^{-2t}}{2}\left[t\cos 2t - \frac{1}{4}[\sin(-2t)-\sin 2t]\right]$$

$$\mathbf{f(t)} = \frac{\mathbf{e^{-2t}}}{\mathbf{2}}\left[\mathbf{t\cos 2t} + \frac{\mathbf{\sin 2t}}{\mathbf{2}}\right] \qquad \{\because \sin(-\theta) = \sin\theta$$

Example 106: **Use C.T. find** $\mathbf{L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

$$\overline{f(s)} = \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2}$$

Let $\overline{f_1(s)} = \overline{f_2(s)} = \dfrac{s}{s^2 + a^2}$

$$L^{-1}\left\{\overline{f_1(s)}\right\} = L^{-1}\left\{\overline{f_2(s)}\right\} = L^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos at = f_1(t) = f_2(t)$$

$$f_1(t) = f_2(t) = \cos at$$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}.\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t \cos a(t-u).\cos au\,du \qquad = \int_0^t \cos(at - au).\cos au\,du$$

$$= \int_0^t \frac{1}{2}[\cos(at - au + au) + \cos(at - au - au)]du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(at - 2au)]du$$

$$\left\{\because \cos A.\cos B = \frac{1}{2}\,[\cos(A+B) + \cos(A-B)]\right.$$

$$= \frac{1}{2}\left[\cos at\int_0^t 1du + \int_0^t \cos(at - 2au)\,du\right]$$

$$= \frac{1}{2}\left[\cos at\,[u]_0^t + \left[\frac{\sin(at - 2au)}{-2a}\right]_0^t\right]$$

$$= \frac{1}{2}\left[\cos at\,(t - 0) - \frac{1}{2a}\,\left[\sin(at - 2at) - \sin\big(at - 2a(0)\big)\right]\right]$$

$$= \frac{1}{2}\left[t\cos at - \frac{1}{2a}\,(\sin(-at) - \sin at)\right]$$

$$= \frac{1}{2}\left[t\cos at - \frac{1}{2a}\,(-2\sin at)\right]$$

$$\mathbf{f(t) = \frac{1}{2a}\,[at\cos at + \sin at]}$$

Example 107: Use C.T. find $\mathbf{L^{-1}\left\{\dfrac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right\}}$

Solution: Let, $L^{-1}\{\overline{f(s)}\} = L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$\overline{f(s)} = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}$$

Let $\overline{f_1(s)} = \frac{s}{s^2+a^2}$; $f_1(t) = \cos at$

$\overline{f_2(s)} = \frac{s}{s^2+b^2}$; $f_2(t) = \cos bt$

By convolution theorem

$$f(t) = L^{-1}\{\overline{f(s)}\} = L^{-1}\{\overline{f_1(s)}\,.\,\overline{f_2(s)}\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t \cos a(t-u).\cos bu\,du$$

$$= \int_0^t \cos(at-au).\cos bu\,du$$

$$= \int_0^t \frac{1}{2}[\cos(at-au+bu)+\cos(at-au-bu)]\,du$$

$$\left\{\because \cos A.\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]\right.$$

$$= \frac{1}{2}\left[\int_0^t \cos(at-au+bu)du + \int_0^t \cos(at-au-bu)du\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin(at-au+bu)}{(-a+b)}\right]_0^t + \left[\frac{\sin(at-au-bu)}{(-a-b)}\right]_0^t\right]$$

$$= \frac{1}{2}\left\{\frac{\sin(bt)}{-(a-b)} - \frac{\sin(at)}{-(a-b)} + \left[\frac{\sin(-bt)}{-(a+b)} - \frac{\sin(at)}{-(a+b)}\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin bt}{-(a-b)} + \frac{\sin at}{(a-b)} + \frac{\sin bt}{(a+b)} + \frac{\sin at}{a+b}\right\}$$

$$\{\because \sin(-\theta) = -\sin\theta$$

$$= \frac{1}{2}\left\{\left(\frac{-1}{a-b}+\frac{1}{a+b}\right)\sin bt + \left(\frac{1}{a-b}+\frac{1}{a+b}\right)\sin at\right\}$$

$$= \frac{1}{2}\left\{\frac{-a-b+a-b}{(a-b)(a+b)}\ \sin bt + \frac{a+b+a-b}{(a-b)(a+b)}\ \sin at\right\}$$

$$= \frac{1}{2}\left\{\frac{-2b}{a^2-b^2}\ \sin bt + \frac{2a}{a^2-b^2} \sin at\right\}$$

$$= \frac{1}{2}\left\{\frac{2a \sin at - 2b \sin bt}{a^2-b^2}\right\}$$

$$\mathbf{f(t) = \frac{a \sin at - b\sin bt}{a^2 - b^2}}$$

Example 108: **Using C.T. find** $\mathbf{L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$

$$\overline{f(s)} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+9}$$

Let , $\overline{f_1(s)} = \frac{1}{s^2+1}$; $f_1(t) = \sin t$

$\overline{f_2(s)} = \frac{1}{s^2+3^2}$; $f_2(t) = \frac{\sin 3t}{3}$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\overline{f_1(s)}\,.\overline{f_2(s)}\right\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t \sin(t-u)\frac{\sin 3u\,du}{3}$$

$$\left\{\because \sin A - \sin B = \frac{-1}{2}[\cos(A+B) - \cos(A-B)]\right.$$

$$= \int_0^t \frac{1}{3}\left(\frac{-1}{2}\right)[\cos(t-u+3u) - \cos(t-u-3u)]\,du$$

$$= \frac{-1}{6}\int_0^t [\cos(t+2u) - \cos(t-4u)]du$$

$$= \frac{-1}{6}\left[\int_0^t \cos(t+2u)\,du - \int_0^t \cos(t-4u)\,du\right]$$

$$= \frac{-1}{6}\left[\left[\frac{\sin(t+2u)}{2}\right]_0^t - \left[\frac{\sin(t-4u)}{-4}\right]_0^t\right]$$

$$= \frac{-1}{6}\left[\frac{1}{2}(\sin(3t)-\sin t)+\frac{1}{4}(\sin(-3t)-\sin t)\right]$$

$$= \frac{-1}{6}\left[\frac{1}{2}\sin 3t - \frac{1}{2}\sin t - \frac{1}{4}\sin 3t - \frac{1}{4}\sin t\right]$$

$$= \frac{-1}{6}\left[\left(\frac{1}{2}-\frac{1}{4}\right)\sin 3t + \left(\frac{-1}{2}-\frac{1}{4}\right)\sin t\right]$$

$$= \frac{-1}{6}\left[\frac{1}{4}\sin 3t - \frac{3}{4}\sin t\right] \quad = \frac{-1}{24}\sin 3t + \frac{1}{8}\sin t$$

$$\mathbf{f(t) = \frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]}$$

Example 109: Using C.T. find $\mathbf{L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\}}$

Solution: Let, $L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\}$

$$\overline{f(s)} = \frac{s}{(s^2+1)(s^2+4)(s^2+9)}$$

$$= \frac{1}{(s^2+1)(s^2+9)}\cdot\frac{s}{s^2+4}$$

Let $\overline{f_1(s)} = \frac{1}{(s^2+1)(s^2+9)}$

Consider, $s^2 = p$

By partial fraction

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{(p+1)(p+9)} = \frac{A}{(p+1)} + \frac{B}{(p+9)} \qquad \dots(1)$$

$$\frac{1}{(p+1)(p+9)} = \frac{1}{(p+1)(8)} + \frac{1}{(p+9)(-8)} = \frac{\frac{1}{8}}{(s^2+1)} + \frac{\frac{-1}{8}}{(s^2+9)}$$

Equation (1) →

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{8}\sin t - \frac{1}{8}\frac{\sin 3t}{3}$$

$$f_1(t) = \frac{1}{8}\sin t - \frac{1}{24}\sin 3t$$

Now, $\overline{f_2(s)} = \dfrac{s}{s^2+4}$; $f_2(t) = \cos 2t$

By convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\overline{f_1(s)}\,.\overline{f_2(s)}\right\} = \int_0^t f_1(t-u).f_2(u)\,du$$

$$= \int_0^t \left(\frac{1}{8}\sin u - \frac{1}{24}\sin 3u\right)\cos 2(t-u)\,du$$

$$= \frac{1}{8}\int_0^t \left[\sin u\,.\cos(2t-2u) - \frac{1}{3}\sin 3u\,.\cos(2t-2u)\right]du$$

$$\left\{\because \sin A\,.\cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]\right.$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}[\sin(2t-u)+\sin(3u-2t)] - \frac{1}{6}[\sin(u+2t)+\sin(5u-2t)]\right]du$$

$$= \frac{1}{16}\left[\left[\frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3}\right]_0^t\right] - \frac{1}{48}\left[\left[-\cos(u+2t) + \frac{\cos(5u-2t)}{5}\right]_0^t\right]$$

$$= \frac{1}{16}\left[\cos t + \frac{\cos t}{3} - \cos 2t - \frac{\cos 2t}{3}\right] - \frac{1}{48}\left[-\cos 3t + \frac{\cos 3t}{5} + \cos 2t - \frac{\cos 2t}{5}\right]$$

$$= \frac{1}{16}\left(1+\frac{1}{3}\right)\cos t + \left[\frac{1}{16}\left(-1-\frac{1}{3}\right) - \frac{1}{48}\left(-1+\frac{1}{5}\right)\right]\cos 2t - \frac{1}{48}\left(1-\frac{1}{5}\right)\cos 3t$$

$$\mathbf{f(t) = \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t}$$

Example 110: If

$$L\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}} \text{ show that } \int_0^t J_0(x)\, J_0(t-x)\, dx = \sin t$$

Solution: By Convolution theorem

$$f(t) = L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\{\overline{f_1(s)}\,.\overline{f_2(s)}\} = \int_0^t f_1(x).\, f_2(t-x)\, dx \quad \ldots\ldots (1)$$

Let $f_1(x) = J_0(x) = f_2(x)$

Given, $L\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}}$

$\therefore\ \overline{f_1(s)} = \overline{f_2(s)} = \frac{1}{\sqrt{s^2+1}}$

$$\therefore L^{-1}\left\{\overline{f_1(s)}.\ \overline{f_2(s)}\right\} = L^{-1}\left\{\frac{1}{\sqrt{s^2+1}}.\ \frac{1}{\sqrt{s^2+1}}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$\therefore$ Equation (1) $\rightarrow$ $\int_0^t J_0(x)\,.\, J_0(t-x)\, dx = \sin t$ ... **Hence proved.**

12 L. T. of Periodic Function

If $f(t+T) = f(t)$

$\therefore$ f (t) is periodic function of period T

Ex. $\sin(t+2\pi) = \sin t$

$\therefore\ T = \text{Period} = 2\pi$

$f(t+rT) = f(t),\ r = 0,1,2,3.$ i.e. $f(t+T) = f(t+2T)$

$= f(t+3T) = \ldots = f(t)$

Definition: $$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st}\, f(t)\, dt$$

12.i Examples on L. T. of Periodic Function

Example 111: Find the L. T. of the following periodic functions

$$f(t) = \frac{Kt}{T},\ 0 < t < T \quad \text{and} \quad f(t) = f(t+T)$$

Solution: Given, $f(t) = \frac{Kt}{T}$, $0 < t < T$

and $f(t) = f(t+T)$

$\therefore$ f(t) is periodic function of period T = T

$\therefore$ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{Kt}{T}\, dt$$

$$= \frac{K}{T} \frac{1}{(1-e^{-sT})} \int_0^T t.e^{-st}\, dt$$

$$= \frac{K}{T} \frac{1}{(1-e^{-sT})} \left[t\frac{e^{-st}}{-s} - (1)\frac{(1)e^{-st}}{(-s)^2} \right]_0^T$$

$$= \frac{K}{T} \frac{1}{(1-e^{-sT})} \left[\left(T\frac{e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} \right) - \left(0 - \frac{e^0}{s^2} \right) \right]$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{K}}{\mathbf{T}} \frac{\mathbf{1}}{(\mathbf{1-e^{-sT}})} \left[\frac{\mathbf{-T\,e^{-sT}}}{\mathbf{s}} - \frac{\mathbf{e^{-sT}}}{\mathbf{s^2}} + \frac{\mathbf{1}}{\mathbf{s^2}} \right]$$

Example 112: Find the L. T. of f(t) if

$$\mathbf{f(t) = a \sin pt,\quad 0 < t < \frac{\pi}{p}}$$

$$\mathbf{= 0,\qquad \frac{\pi}{p} < t < \frac{2\pi}{p}}$$

and $\mathbf{f(t) = f\left(t + \frac{2\pi}{p}\right)}$

Solution: Given, $f(t) = a\sin pt \quad ; \quad 0 < t < \frac{\pi}{p}$

$$= 0 \qquad ; \quad \frac{\pi}{p} < t < \frac{2\pi}{p}$$

and $f(t) = f\left(t + \frac{2\pi}{p}\right)$ ... periodic function

$\therefore$ f(t) is periodic function of period $\frac{2\pi}{p}$ $\quad \therefore T = \frac{2\pi}{p}$

$\therefore$ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi}{p}s}} \left[\int_0^{\frac{\pi}{p}} e^{-st}\ a \sin pt\,dt + \int_{\frac{\pi}{p}}^{\frac{2\pi}{p}} e^{-st}\,(0)dt \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi}{p}s}} \left[a \int_0^{\frac{\pi}{p}} e^{-st} \sin pt\ dt\ + 0 \right]$$

$$= \frac{a}{1-e^{\frac{-2\pi}{p}s}} \left[\frac{e^{-st}}{s^2+p^2} (-s \sin pt - p \cos pt) \right]_0^{\frac{\pi}{p}}$$

$$\left\{\because \int e^{-ax} \sin bx\ dx = \frac{e^{-ax}}{a^2+b^2}(-a \sin bx - b \cos bx) + c\right.$$

$$= \frac{a}{1-e^{\frac{-2\pi}{p}s}} \left[\frac{e^{-\frac{\pi}{p}s}}{s^2+p^2} \left[-s \sin p\left(\frac{\pi}{p}\right) - p \cos p\left(\frac{\pi}{p}\right)\right] - \frac{e^0}{s^2+p^2} [0 - p \cos p\,(0)] \right]$$

$$\{\because \sin 0 = 0,\ \sin\pi = 0,\ \cos 0 = 0,\ \cos\pi = -1$$

$$= \frac{a}{1-\left(e^{-\frac{\pi s}{p}}\right)^2} \left[\frac{e^{-\frac{\pi}{p}s}}{s^2+p^2}(p) - \frac{1}{s^2+p^2}\ (-p) \right]$$

$$= \frac{a}{1-\left(e^{-\frac{\pi s}{p}}\right)^2} \left[\frac{p}{s^2+p^2} \left(e^{-\frac{\pi}{p}s} + 1\right) \right]$$

$$= \frac{ap}{(s^2+p^2)\left(1-e^{-\frac{\pi s}{p}}\right)\left(1+e^{-\frac{\pi s}{p}}\right)} \left(e^{-\frac{\pi}{p}s} + 1\right)$$

$$\overline{\mathbf{f}}(\mathbf{s}) = \frac{\mathbf{ap}}{(\mathbf{s^2+p^2})\left(\mathbf{1-e^{-\frac{\pi s}{p}}}\right)}$$

Example 113: Find the Laplace transform of the function

$$\mathbf{f(t) = \sin\omega t\quad,\quad 0 < t < \frac{\pi}{\omega}}$$

$$= \mathbf{0} \qquad , \quad \frac{\boldsymbol{\pi}}{\boldsymbol{\omega}} < t < \frac{\mathbf{2\pi}}{\boldsymbol{\omega}}$$

Solution: Here, $f(t)$ is a periodic function with period $T = \frac{2\pi}{\omega}$

∴ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{\frac{-2\pi}{\omega}s}} \left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\, (0) dt \right]$$

$$\left\{ \because \text{ w.k.t.} \int e^{-at}.\sin bt = \frac{e^{-at}}{a^2+b^2}(-a \sin bt - b \cos bt) + c \right.$$

$$= \frac{1}{1-e^{\frac{-2\pi}{\omega}s}} \left\{ \left[\frac{e^{-st}}{s^2+\omega^2} [-s \sin \omega t - \omega \cos \omega t] \right]_0^{\frac{\pi}{\omega}} \right\}$$

$$= \frac{1}{1-e^{\frac{-2\pi}{\omega}s}} \left[\frac{e^{-\frac{\pi}{\omega}t}}{s^2+\omega^2} \left[-s \sin\omega \frac{\pi}{\omega} - \omega \cos \omega \frac{\pi}{\omega} \right] \right.$$

$$\left. - \frac{e^0}{s^2+\omega^2} [-s \sin \omega(0) - \omega \cos \omega(0)] \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi}{\omega}s}} \left[\frac{e^{-\frac{\pi}{\omega}t}}{s^2+\omega^2} \left(-\omega\,(-1)\right) - \frac{1}{s^2+\omega^2} \left(-\omega(1)\right) \right]$$

$$= \frac{\omega}{\left[1^2 - \left(e^{-\frac{\pi}{\omega t}}\right)^2\right](s^2+\omega^2)} \left[e^{-\frac{\pi}{\omega t}} + 1 \right]$$

$$\{ \because \cos 0 = 1, \cos \pi = -1, \sin 0 = 0, \sin \pi = 0$$

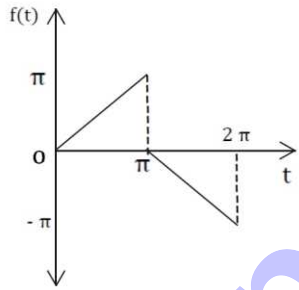
$$= \frac{\omega}{\left(1 - e^{-\frac{\pi}{\omega t}}\right)\left(1 + e^{-\frac{\pi}{\omega t}}\right)(s^2+\omega^2)} \left[e^{-\frac{\pi}{\omega t}} + 1 \right]$$

$$\therefore \quad \overline{\mathbf{f(s)}} = \frac{\boldsymbol{\omega}}{\left(\mathbf{1} - \mathbf{e}^{-\frac{\boldsymbol{\pi}}{\boldsymbol{\omega}\mathbf{T}}}\right)(\mathbf{s^2} + \boldsymbol{\omega}^{\mathbf{2}})}$$

Example 114: Draw the graph of the periodic function

$$\mathbf{f(t) = \begin{cases} t & , \ 0 < t < \pi \\ \pi - t & , \ \pi < t < 2\pi \end{cases}}$$

and find its Laplace transform.



Solution: Given,

$$f(t) = \begin{cases} t & , \ 0 < t < \pi \\ \pi - t & , \ \pi < t < 2\pi \end{cases}$$

Note: $f(t) = t$ is a straight line passing through centre.

Here f(t) is a periodic function of period $T = 2\pi$ and its graph is in fig.

∴ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \left[t \frac{e^{-st}}{-s} - \frac{(1) e^{-st}}{(-s)^2} \right]_0^{\pi} + \left[(\pi - t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{(-s)^2} \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \left[\frac{\pi e^{-s\pi}}{-s} - \frac{e^{-s\pi}}{s^2} - \left(0 - \frac{e^0}{s^2} \right) \right] \right.$$

$$+ \left[\frac{(\pi - 2\pi) e^{-s2\pi}}{-s} + \frac{e^{-s2\pi}}{s^2} \right.$$

$$\left. \left. - \left(\frac{(\pi - \pi) e^{-s\pi}}{-s} + \frac{e^{-s\pi}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{-\pi}{s} e^{-s\pi} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} + \frac{\pi}{s} e^{-2\pi s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-s\pi}}{s^2} \right\}$$

$$\mathbf{\overline{f(s)} = \frac{1}{1 - e^{-2\pi s}} \left\{ \frac{\pi}{s} \left(e^{-2\pi s} - e^{-\pi s} \right) + \frac{1}{s^2} \left(1 + e^{-2\pi s} - 2e^{-\pi s} \right) \right\}}$$

Example 115: If $\mathbf{f(t) = \frac{t}{a} \quad , \quad 0 < t < a}$

$$\mathbf{= \frac{1}{a} (2a - t), \quad a < t < 2a}$$

and $\mathbf{f(t) = f(t + 2a)}$ **Find L {f (t)}**

Solution: Given, $f(t) = f(t+2a)$

$\therefore$ f (t) is a periodic function of period $T = 2a$

$\therefore$ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a e^{-st} \frac{t}{a}\, dt + \int_a^{2a} e^{-st} \frac{1}{a} (2a-t) dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \frac{1}{a} \int_0^a e^{-st} .t\, dt + \frac{1}{a} \int_a^{2a} e^{-st} (2a-t) dt \right\}$$

$$= \frac{1}{(1-e^{-2as})} \frac{1}{a} \left\{ \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{(-s)(-s)} \right]_0^a + \left[(2a-t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{(-s)(-s)} \right]_a^{2a} \right\}$$

$$= \frac{1}{a(1-e^{-2as})} \left\{ \frac{-a}{s} e^{-as} - \frac{1}{s^2} e^{-as} - \left(0 - \frac{1e^0}{s^2} \right) + 0 + \frac{1}{s^2} e^{-2as} - \left((2a-a) \frac{e^{-as}}{-s} + \frac{1}{s^2} e^{-as} \right) \right\}$$

$$= \frac{1}{a(1-e^{-2as})} \left\{ \frac{-a}{s} e^{-as} - \frac{1}{s^2} e^{-as} + \frac{1}{s^2} + \frac{1}{s^2} e^{-2as} + \frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as} \right\}$$

$$= \frac{1}{a\,(1-e^{-2as})} \left\{ \frac{1}{s^2} (1 - 2e^{-as} + e^{-2as}) \right\}$$

$$= \frac{1}{a\,[1^2 - (e^{-as})^2]} \frac{1}{s^2} (1-e^{-as})^2$$

$$= \frac{1}{as^2 (1-e^{-as})(1+e^{-as})} (1-e^{-as})^2 \qquad \{\because (a-b)^2 = a^2 - 2ab + b^2$$

$$= \frac{1}{as^2} \frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{as^2}\frac{1-\frac{e^{-\frac{as}{2}}}{e^{\frac{as}{2}}}}{1+\frac{e^{-\frac{as}{2}}}{e^{\frac{as}{2}}}} \quad \left[\text{Note: } e^{-as} = e^{-\frac{as}{2}-\frac{as}{2}} = e^{-\frac{as}{2}}.e^{-\frac{as}{2}} = \frac{e^{-\frac{as}{2}}}{e^{\frac{as}{2}}}\right]$$

$$= \frac{1}{as^2}\frac{\left(e^{\frac{as}{2}} - e^{-\frac{as}{2}}\right)/e^{\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)/e^{\frac{as}{2}}}$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{as^2}}\ \mathbf{tanh}\left(\frac{\mathbf{as}}{\mathbf{2}}\right) \qquad \left\{\because \tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right.$$

Example 116: **Find the Laplace transform of the square wave function of a period is defined as**

$$\mathbf{f(t) = 1 \quad ; \quad 0 < t < \frac{a}{2}}$$

$$\mathbf{= -1 \quad ; \quad \frac{a}{2} < t < a}$$

Solution: Given, f (t) is a periodic function of period T = a

∴ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}\ (1)dt + \int_{\frac{a}{2}}^{a} e^{-st}\,(-1)dt\right]$$

$$= \frac{1}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{a}\right\}$$

$$= \frac{-1}{s(1-e^{-as})}\left\{[e^{-st}]_0^{\frac{a}{2}} - [e^{-st}]_{\frac{a}{2}}^{a}\right\}$$

$$= \frac{-1}{s(1-e^{-as})}\left\{\left(e^{-\frac{as}{2}t} - e^0\right) - \left(e^{-as} - e^{-\frac{as}{2}t}\right)\right\}$$

$$= \frac{-1}{s(1-e^{-as})}\left[e^{-\frac{as}{2}t} - 1 - e^{-as} + e^{-\frac{as}{2}t}\right]$$

$$= \frac{-1}{s(1-e^{-as})}\left[2e^{-\frac{as}{2}t} - 1 - e^{-as}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1 - 2e^{-\frac{as}{2}t} + e^{-as}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left(1 - e^{-\frac{as}{2}}\right)^2$$

$$= \frac{1}{s\left[1-\left(e^{-\frac{as}{2}}\right)^2\right]}\left(1 - e^{-\frac{as}{2}}\right)^2$$

$$= \frac{1}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1 - e^{-\frac{as}{2}}\right)^2$$

$$= \frac{1}{s}\,\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}$$

$$= \frac{1}{s}\,\frac{e^{\frac{as}{4}}-e^{-\frac{as}{4}}}{e^{\frac{as}{4}}+e^{-\frac{as}{4}}} \qquad \text{... Multiplying Nr and Dr by } e^{\frac{as}{4}}$$

$$\mathbf{\overline{f(s)} = \frac{1}{s}\tanh\left(\frac{as}{4}\right)} \qquad \left\{\because \tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right.$$

Example 117: **If $\mathbf{f(t) = t^2,\ 0 < t < 2,\ f(t) = f(t+2)}$.**

Find $\mathbf{L\{f(t)\}}$

Solution: Given, $f(t) = f(t+2)$

$\therefore$ f(t) is a periodic function of period $T = 2$

$\therefore$ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\,t^2\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - (2t)\frac{e^{-st}}{(-s)(-s)} + 2\frac{e^{-st}}{(-s)(-s)(-s)}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left\{(2)^2\frac{e^{-2s}}{-s} - 2(2)\frac{e^{-2s}}{s^2} + 2\frac{e^{-2s}}{-s^3} - \left(0 - 0 + \frac{e^0}{-s^3}\right)\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{\frac{-4e^{-2s}}{s} - \frac{4}{s^2}e^{-2s} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right\}$$

$$= \frac{1}{(1-e^{-2s})}\left[\left(\frac{-2e^{-2s}}{s^3}\right)(2s^2+2s+1)+\frac{2}{s^3}\right]$$

$$\overline{\mathbf{f}(\mathbf{s})} = \frac{\mathbf{2-2e^{-2s}(2s^2+2s+1)}}{\mathbf{s^3(1-e^{-2s})}}$$

Example 118: **If** $\mathbf{f(t) = t\ ,\ 0 < t < 1}$

$\mathbf{= 0\ ,\ 1 < t < 2}$

$\mathbf{f(t) = f(t+2)}$ **Find** $\mathbf{L\{f(t)\}}$

Solution: Given, $f(t+2) = f(t)$

∴ $f(t)$ is a periodic function of period $T = 2$

∴ By definition of periodic function,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-s(2)}}\left[\int_0^1 e^{-st}\ t\,dt + \int_1^2 e^{-st}\ (0)dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[t\frac{e^{-st}}{-s} - 1\frac{e^{-st}}{(-s)^2}\right]_0^1$$

$$= \frac{1}{1-e^{-2s}}\left[(1)\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - \left(0 - \frac{e^0}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s}}\frac{e^{-s}}{s^2}(-s-1+e^s)$$

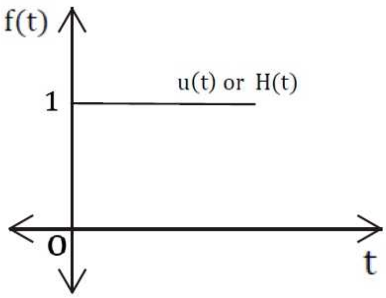
$$\overline{\mathbf{f}(\mathbf{s})} = \frac{\mathbf{e^{-s}(e^s-s-1)}}{\mathbf{s^2(1-e^{-2s})}}$$

13 Unit Step Function or Heavisides Unit Step Function

At times, we come across such fractions of which the inverse transform cannot be determind from the formulas so far derived.

In order cover such cases, we introduce the unit step function (or Heavisides unit function).

Definition: Unit step function is a cover which has the value zero at all points to the left of the origin and is unity at all the points on the right of origin.



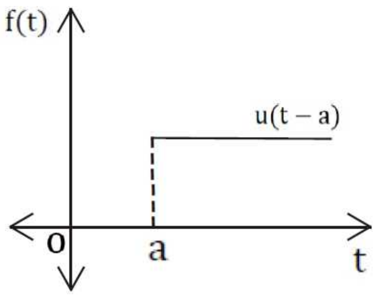
It is denoted as H(t) or U(t)

$$u(t) = 0\,,\ t < 0$$
$$= 1\,,\ t \geq 0$$

$$\mathbf{L\{u(t)\} = \frac{1}{s}}$$

Displaced unit step function:

Definition: It represents the curve u(t) which is displaced to the right through a distance 'a' along the direction of t − axis denoted by u(t − a).



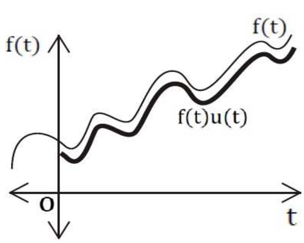
$$u(t-a) = 0\,,\quad t < a$$
$$= 1\,,\quad t \geq a$$

$$\mathbf{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

<u>Application of unit step and displaced unit step function</u>:

1) Representation of $f(t)$ for $t \geq 0$:

When the function f(t) is multiplied by u(t). Then f(t) u(t) will represents the part of f(t) on the right of the origin, the part of the left of the origin being cut off.

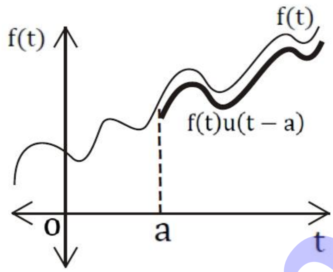


$$\therefore\ f(t)\,u(t) = 0,\quad t < 0$$
$$= f(t),\ t > 0$$

$$\mathbf{L\{f(t).u(t)\} = L\{f(t)\} = \overline{f(s)}}$$

2) Representation of $f(t)$ for $t \geq a$:

When the function $f(t)$ is multiplied by $u(t-a)$ will represent the part of $f(t)$ on the right of $t = a$. The part of the left of $t = a$ being cut off.



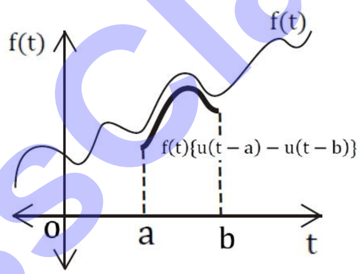
$\therefore\ f(t)u(t-a) = 0, \quad t < a$

$\qquad\qquad\qquad = f(t), \quad t \geq a$

$$\mathbf{L\{f(t)u(t-a)\} = e^{-as}\, L\{f(t+a)\}}$$

3) Representation of $f(t)$ for $a \leq t \leq b$:

When the function $f(t)$ is multiplied by $u(t-a) - u(t-b)$, Then $f(t)\{u(t-a) - u(t-b)\}$ will represent the part of $f(t)$ in $a \leq t \leq b$ the part on the left of $t = a$ and on the right of $t = b$ being cut off.



$$\therefore\ f(t)\{u(t-a) - u(t-b)\} = \begin{cases} 0, & t < a \\ f(t), & a \leq t \leq b \\ 0, & t > b \end{cases}$$

$\mathbf{L\{f(t-a)u(t-a)\} = e^{-as}\, L\{f(t)\}}$ ... **called 2^{nd} shifting property.**

Note:

1) $L\{u(t)\} = \dfrac{1}{s}$

2) $L\{u(t-a)\} = \dfrac{1}{s}\ e^{-as}$

3) $L\{f(t-a)u(t-a)\} = e^{-as}\, L\{f(t)\} = e^{-as}\, \overline{f}(s)$

4) $L\{f(t).u(t-a)\} = e^{-as}\, L\{f(t+a)\}$

5) If $f(t-a) = F(t)$, then $f(t) = F(t+a)$

6) When $a = 0$, $L\{f(t).u(t)\} = L\{f(t)\} = \overline{f(s)}$

7) $L^{-1}\left\{\overline{f(s)}\right\} = f(t)\, u(t)$

8) $L^{-1}\left\{e^{-as}\, \overline{f(s)}\right\} = f(t-a)\, u(t-a)$

13.i Examples on L.T. of Unit Step Function

Example 119: Find L.T. of $(t-1)^2\, u(t-1)$

Solution: We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

Here $a = 1$

$$\therefore L\{f(t-1)u(t-1)\} = e^{-s}\,L\{f(t)\} \quad \ldots\ldots(1)$$

Now,

$$f(t-1) = (t-1)^2$$

$$\therefore f(t) = [(t+1)-1]^2 = t^2$$

$$\therefore L\{f(t)\} = L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

Equation (1) → $\mathbf{L\{(t-1)^2\, u\,(t-1)\} = \frac{2}{s^3}\, e^{-s}}$

Example 120: Find L.T. of $\sin t\;u(t-\pi)$

Solution: We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

Here $a = \pi$

$$\therefore L\{f(t-\pi)u(t-\pi)\} = e^{-\pi s}L\{f(t)\} \quad \ldots\ldots(1)$$

Now,

$$f(t-\pi) = (\sin t)$$

$$\therefore f(t) = \sin(t+\pi) = -\sin t$$

$$\therefore L\{f(t)\} = \overline{f(s)} = L\{-\sin t\} = -L\{\sin t\} = \frac{-1}{s^2+1}$$

Equation (1) → $\mathbf{L\{\sin t\, u(t-\pi)\} = \frac{-e^{-\pi s}}{s^2+1}}$

Example 121: Find L.T. of $e^{-3t}\,u(t-2)$

Solution: We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

Here, $a = 2$

$$\therefore L\{f\,(t-2)u\,(t-2)\} = e^{-2s}\,L\{f(t)\} \quad \ldots\ldots(1)$$

Now,

$$f(t-2) = e^{-3t}$$

$$\therefore f(t) = e^{-3(t+2)} = e^{-3t}e^{-6}$$

$$\therefore L\{f(t)\} = e^{-6}L\{e^{-3t}\} = e^{-6}\frac{1}{s+3}$$

Equation(1) → $\mathbf{L\{e^{-3t}\,u(t-2)\} = e^{-2s}\frac{e^{-6}}{s+3} = \frac{1}{s+3}\,e^{-(2s+6)}}$

Example 122: Find L. T. of $\mathbf{(1+2t-3t^2+4t^3)\,u(t-2)}$

Solution: We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

If $f(t-a) = F(t)$, then $f(t) = F(t+a)$

Here $a = 2$

$$\therefore L\{f(t-2)\,u(t-2)\} = e^{-2s}\,L\{f(t)\} \quad \ldots\ldots(1)$$

Now,

$$f(t-2) = 1+2t-3t^2+4t^3$$

$$\therefore f(t) = 1+2(t+2)-3(t+2)^2+4(t+2)^3$$

$$= 1+2t+4-3(t^2+4t+4)+4(t^3+6t^2+12t+8)$$

$$= 1+2t+4-3t^2-12t-12+4t^3+24t^2+48t+32$$

$$f(t) = 4t^3+21t^2+38t+25$$

$$\therefore L\{f(t)\} = 4\,L\{t^3\}+21\,L\{t^2\}+38\,L\{t\}+25\,L\{1\}$$

$$= 4\frac{3!}{s^{3+1}}+21\frac{2!}{s^{2+1}}+38\frac{1}{s^2}+25\frac{1}{s}$$

$$= \frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}$$

∴ Equation(1)→ $\mathbf{L\{f(t-2)\,u(t-2)\} = e^{-2s}\left(\frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}\right)}$

Example 123: Obtain the Laplace transform of
$\mathbf{e^{-t}[1-u(t-2)]}$

Solution:

$$L\{e^{-t}(1-u(t-2))\}$$

$$= L\{e^{-t}-e^{-t}u(t-2)\}$$

$$= L\{e^{-t}\} - L\{e^{-t}u(t-2)\}$$

$$\{\because L\{f(t-a)\,u(t-a)\} = e^{-as}L\{f(t)\}$$

$$= \frac{1}{s+1} - e^{-2s}\, L\left\{e^{-(t+2)}\right\} \qquad \begin{cases} \because\ f(t-a) = e^{-t} \\ \therefore\ f(t) = e^{-(t+a)} \end{cases}$$

$$= \frac{1}{s+1} - e^{-2s} e^{-2}\, L\{e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s-2}\frac{1}{s+1} \quad = \frac{1-e^{-2(s+1)}}{s+1}$$

$$\mathbf{L\{e^{-t}[1-u(t-2)]\} = \frac{1-e^{-2(s+1)}}{s+1}}$$

Example 124: **Using L.T. evaluate**

$$\int_0^\infty \mathbf{e^{-t}\left(1+2t-t^2+t^3\right) H(t-1)\, dt}$$

Solution: We have,

$L\{f(t-a)\,u(t-a)\} = e^{-as}\, L\{f(t)\}$ and

If $f(t-a) = F(t)$, then $f(t) = F(t+a)$

Now, $L\{(1+2t-t^2+t^3)\, H(t-1)\}$

Here, $a = 1$

$\therefore L\{(1+2t-t^2+t^3)\, H(t-1)\}$

$$= e^{-s}\, L\,\{1+2(t+1)-(t+1)^2+(t+1)^3\}$$

$$= e^{-s}\, L\{1+2t+2-(t^2+2t+1)+(t^3+3t^2+3t+1)\}$$

$$= e^{-s}\, L\{(1+2-1+1)+(2t-2t+3t)+(-t^2+3t^2)+t^3\}$$

$$= e^{-s} L\{3+3t+2t^2+t^3\}$$

$$= e^{-s}[3L\{1\}+3L\{t\}+2L\{t^2\}+L\{t^3\}]$$

$$= e^{-s}\left[3\frac{1}{s}+3\frac{1}{s^2}+2\frac{(2!)}{s^3}+\frac{3!}{s^4}\right]$$

$$= e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

Now, $L\left\{\int_0^\infty e^{-t}(1+2t+t^2+t^3)H(t-1)\, dt\right\}$

$$= e^{-1}\left(\frac{3}{1}+\frac{3}{1^2}+\frac{4}{1^3}+\frac{6}{1^4}\right) = e^{-1}\,(16)$$

$$\left\{\because \int_0^\infty e^{-at}\, f(t) = f(a)\,, \because \text{put } s = 1\right.$$

$$\therefore \quad \mathbf{L\left\{\int_0^\infty e^{-t}(1+2t+t^2+t^3)H(t-1)\,dt\right\} = \frac{16}{e}}$$

Example 125: **Express the following in terms of unit step function and hence find their Laplace transform.**

$$\mathbf{If \quad f(t) = (t-a)^4, \quad t > a}$$
$$\mathbf{= 0 \quad , \quad 0 < 1 < a, \quad find\ L\{f(t)\}}$$

Solution: If can be express in unit step form as

$f(t) = (t-a)^4\, u(t-a)$

We have,

$L\{f(t-a)\,u(t-a)\} = e^{-as}\, L\{f(t)\}$ and

If $f(t-a) = F(t)$, then $f(t) = F(t+a)$

Here $a = a$

$\therefore L\{(t-a)^4\, u(t-a)\} = e^{-as}\, L\{f(t)\}$ ...... (1)

Now, $f(t-a) = (t-a)^4$

$\therefore \quad f(t) = (t+a-a)^4 = t^4$

$\therefore \quad L\{f(t)\} = L\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$

Equation (1) → $\mathbf{L\{f(t-a)\,u(t-a)\} = \frac{4!\,e^{-as}}{s^5} = \frac{24}{s^5}e^{-as}}$

Example 126: $\mathbf{f(t) = e^{-t}, \quad 0 \le t \le 3}$

$\mathbf{= 0 \quad , \quad t > 3}$

Solution: It can be expressed in unit step form as

$f(t) = e^{-t}\{u(t-0) - u(t-3)\}$

$= e^{-t}u(t) - e^{-t}\, u(t-3)$

$= e^{-t}u(t) - e^{-(t-3+3)}u(t-3)$

$f(t) = e^{-t}u(t) - e^{-3}.e^{-(t-3)}u(t-3)$

Taking L. T. on both sides

$L\{f(t)\} = L\{e^{-t}u(t)\} - e^{-3}L\{e^{-(t-3)}u(t-3)\}$

$= \frac{1}{s+1} - e^{-3}e^{-3s}\frac{1}{s+1}$ ... L {f(t − a) u(t − a)}

$= e^{-as}\,\overline{f(s)}$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1 - e^{-3(s+1)}}}{\mathbf{s + 1}}$$

Example 127: **If** $\mathbf{f(t) = e^t \cos t \quad , \quad 0 < t < \pi}$
$\mathbf{= e^t \sin t \quad , \quad t > \pi}$ **find** $\mathbf{L\{f(t)\}}$

Solution:

<u>The first part</u>

Let $f_1(t) = e^t \cos t\,, \ 0 < t < \pi$ *written as*

$$f_1(t) = e^t \cos t\{u(t) - u(t-\pi)\} \qquad \ldots\ldots(i)$$

<u>The second part</u>

Let $f_2(t) = e^t \sin t\,, \ t > \pi$ written as

$$f_2(t) = e^t \sin t\{u(t - \pi)\} \qquad \ldots\ldots(ii)$$

From equation (i) & (ii)

$$\therefore f(t) = f_1(t) + f_2(t) = e^t.\cos t\{u(t) - u(t-\pi)\} + e^t \sin t\{u(t-\pi)\}$$
$$= e^t \cos t\, u(t) - e^t \cos t\, u(t-\pi) + e^t \sin t\ u(t-\pi)$$
$$= e^t \cos t\, u(t) - e^{(t-\pi)+\pi} \cos[(t-\pi)+\pi]u(t-\pi)$$
$$+e^{(t-\pi)+\pi} \sin[(t-\pi)+\pi]\, u(t-\pi)$$
$$= e^t.\cos t\ u(t) - e^{\pi} e^{t-\pi}\big(-\cos(t-\pi)\big)\, u(t-\pi)$$
$$+ e^{\pi} e^{(t-\pi)}[-\sin(t-\pi)]u(t-\pi)$$
$$f(t) = e^t.\cos t\ u(t) + e^{\pi} e^{(t-\pi)} \cos(t-\pi)\ u(t-\pi)$$
$$- e^{\pi} e^{(t-\pi)} \sin(t-\pi)\, u(t-\pi)$$

Taking L.T. on both sides

$$L\{f(t)\} = L\{e^t.\cos t\ u(t)\} + e^{\pi} L\big\{e^{(t-\pi)} \cos(t-\pi)\, u(t-\pi)\big\}$$
$$-e^{\pi}\ L\{e^{(t-\pi)}\sin(t-\pi)u(t-\pi)\}$$

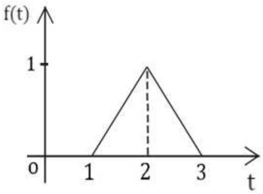
We have,

$$L\{f(t-a)\, u(t-a)\} = e^{-as}\ L\{f(t)\} \ \text{ and}$$

If $f(t-a) = F(t)$, then $f(t) = F(t+a)$

$$\therefore \mathbf{L\{f(t)\}} = \frac{\mathbf{s-1}}{\mathbf{(s-1)^2+1}} + \mathbf{e^{\pi} e^{-\pi s}} \frac{\mathbf{(s-1)}}{\mathbf{(s-1)^2+1}}$$
$$- \mathbf{e^{\pi} e^{-\pi s}} \frac{\mathbf{1}}{\mathbf{(s-1)^2+1^2}}$$

Example 128: Express the function shown in fig. in terms of unit step function and find its Laplace transform.



Solution: Form given fig. the function is in unit step function as:

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

$$\therefore f(t) = (t-1)\{u(t-1)-u(t-2)\} + (3-t)\{u(t-2)-u(t-3)\}$$

$$= (t-1)\,u(t-1) - (t-1)\,u(t-2) + (3-t)\,u(t-2) - (3-t)\,u(t-3)$$

$$= (t-1)\,u(t-1) - t\,u(t-2) + u(t-2) + 3u(t-2) - t\,u(t-2) + (t-3)u(t-3)$$

$$f(t) = (t-1)\,u(t-1) - (t-1-3+t)\,u(t-2) + (t-3)\,u(t-3)$$

$$f(t) = (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

Now,

$$L\{f(t)\} = L\{(t-1)\,u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)\}$$

$$= L\{(t-1)\,u(t-1)\} - 2\,L\{(t-2)\,u(t-2)\} + L\{(t-3)\,u(t-3)\}$$

$$= e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$$

$$= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2} = \frac{e^{-s}(1 - 2e^{-s} + e^{-2s})}{s^2}$$

$$\overline{\mathbf{f}}(\mathbf{s}) = \frac{\mathbf{e^{-s}(1 - e^{-s})^2}}{\mathbf{s^2}}$$

Example 129: Express in terms of unit step function and hence find L. T.,

$$\mathbf{f(t)} = \begin{cases} \mathbf{t^2}, & \mathbf{0 < t < 1} \\ \mathbf{4t}, & \mathbf{t > 1} \end{cases} \qquad \textbf{Find } \mathbf{L\{f(t)\}}$$

Solution: Given, $f(t) = \begin{cases} t^2, & 0 < t < 1 \\ 4t, & t > 1 \end{cases}$

The given f(t) in terms of unit step function as:

$$f(t) = t^2u(t) + (4t - t^2)\,u(t-1)$$

Taking L.T. on both sides,

$$L\{f(t)\} = L\{t^2u(t)\} + L\{(4t - t^2)u(t-1)\}$$

Now, $L\{f(t)\} = L\{f_1(t)\} + e^{-s}L\{f_2(t)\}$ ……(1)

We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

$$f_1(t) = t^2 \;\&\; f_2(t-a) = f_2(t-1) = (4t - t^2) \qquad \{\because a = 1$$

$$\therefore f_2(t) = 4(t+1) - (t+1)^2 = 4t + 4 - t^2 - 2t - 1$$

$$f_2(t) = 2t + 3 - t^2$$

$\therefore$ Equation (1) → $L\{f(t)\} = L\{t^2\} + e^{-s}L\{2t + 3 - t^2\}$

$$\mathbf{L\{f(t)\} = \frac{2}{s^3} + e^{-s}\left(\frac{2}{s^2} + \frac{3}{s} - \frac{2}{s^3}\right)}$$

Example 130: Express in terms of Heavisides unit step function and hence find L.T.,

$$\mathbf{f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}}$$

Solution: Given, $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

The given function is in unit step function written as:

$$f(t) = \cot u(t) + (\cos 2t - \cos t)\,u(t-\pi)$$
$$+ (\cos 3t - \cos 2t)\,u(t - 2\pi)$$

Taking L.T. on both sides.

$$L\{f(t)\} = L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)u(t-\pi)\}$$
$$+ L\{(\cos 3t - \cos 2t)u(t - 2\pi)\}$$

$$L\{f(t)\} = L\{f_1(t)\} + L\{f_2(t)\} + L\{f_3(t)\} \qquad \text{……(1)}$$

We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$

$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

i) $L\{f_1(t)\} = L\{\cos t\; u(t)\}$, Here $f(t) = \cos t$

$$L\{\cos t\; u(t)\} = L\{\cos t\} = \frac{s}{s^2+1} \qquad \therefore\; \mathbf{L\{f_1(t)\} = \frac{s}{s^2+1}}$$

ii) $L\{f_2(t)\} = L\{(\cos 2t - \cos t)\, u(t-\pi)\}$

Here $f(t-\pi) = \cos 2t - \cos t$

$$\therefore \quad f(t) = \cos 2(t+\pi) - \cos(t+\pi)$$
$$= \cos(2t+2\pi) - \cos(t+\pi)$$
$$= \cos(2\pi+2t) - \cos(\pi+t)$$
$$= \cos 2t - (-\cos t)$$
$$\{\because \cos(2\pi+\theta) = \cos\theta,\ \cos(\pi+\theta) = -\cos\theta$$
$$f(t) = \cos 2t + \cos t$$

$$\therefore L\{(\cos 2t - \cos t)\, u(t-\pi)\} = e^{-\pi s} L\{f(t)\} = e^{-\pi s} L\{\cos 2t + \cos t\}$$
$$= e^{-\pi s}[L\{\cos 2t\} + L\{\cos t\}]$$
$$\therefore \mathbf{L\{f_2(t)\} = e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]}$$

iii) $L\{f_3(t)\} = L\{(\cos 3t - \cos 2t)u(t-2\pi)\}$

Here $f(t-2\pi) = \cos 3t - \cos 2t$

$$\therefore \quad f(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$$
$$= \cos(3t+6\pi) - \cos(2t+4\pi)$$
$$= \cos(6\pi+3t) - \cos(4\pi+2t)$$
$$f(t) = \cos 3t - \cos 2t \qquad \{\because \cos(2n\pi+\theta) = \cos\theta$$

$$\therefore L\{f(t-2\pi)\, u(t-2\pi)\} = e^{-2\pi s} L\{f(t)\} = e^{-2\pi s} L\{\cos 3t - \cos 2t\}\}$$
$$= e^{-2\pi s}[L\{\cos 3t\} + L\{\cos 2t\}]$$
$$\therefore \mathbf{L\{f_3(t)\} = e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]}$$

$\therefore$ Equation (1) →

$$\mathbf{L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]}$$

Example 131: **Express in terms of unit step function and hence find L.T.,**
$$\mathbf{f(t) = \begin{cases} \sin t &, \ 0 \le t < \pi \\ \sin 2t &, \ \pi \le t < 2\pi \\ \sin 3t &, \ t \ge 2\pi \end{cases}}$$

Solution: Given,
$$f(t) = \begin{cases} \sin t &, \ 0 \le t < \pi \\ \sin 2t &, \ \pi \le t < 2\pi \\ \sin 3t &, \ t \ge 2\pi \end{cases}$$

Given $f(t)$ is in unit step function as follows:

$$f(t) = \sin t\,[\,u\,(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)]$$
$$+ \sin 3t\,[u(t-2\pi)]$$
$$= \sin t\;u(t) - \sin t\;u(t-\pi) + \sin 2t\;u(t-\pi)$$
$$- \sin 2t\,u(t-2\pi) + \sin 3t\,u(t-2\pi))$$
$$f(t) = \sin t\,u(t) + (\sin 2t - \sin t)\,u(t-\pi)$$
$$+ (\sin 3t - \sin 2t)\,u(t-2\pi)$$

We have,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\} \text{ and}$$
$$\text{If } f(t-a) = F(t), \text{then } f(t) = F(t+a)$$

Taking L. T. on both sides

$$L\{f(t) = L\{\sin t\;u(t)\} + L\,\{(\sin 2t - \sin t)u(t-\pi)\}$$
$$+ L\,\{\sin 3t - \sin 2t)u(t-2\pi)\}$$
$$= L\{\sin t\} + e^{-\pi s}L\,\{\sin 2(t+\pi) - \sin(t+\pi)\}$$
$$+ e^{-2\pi s}L\{\sin 3(t+2\pi) \qquad - \sin 2(t+2\pi)\}$$
$$= L\{\sin t\} + e^{-\pi s}L\{\sin(2t+2\pi) - \sin(t+\pi)\}$$
$$+ e^{-2\pi s}L\{\sin(3t+6\pi) - \sin(2t+4\pi)\}$$
$$= L\{\sin t\} + e^{-\pi s}L\{\sin 2t + \sin t\} + e^{-2\pi s}L\{\sin 3t - \sin 2t\}$$
$$\{\text{ Note: } \sin(\pi+\theta) = -\sin\theta,$$
$$\sin(2n\pi+\theta) = \sin\theta$$

$$\therefore \mathbf{L\{f(t)\}} = \frac{\mathbf{1}}{\mathbf{s^2+1}} + \mathbf{e^{-\pi s}}\left(\frac{\mathbf{2}}{\mathbf{s^2+4}} + \frac{\mathbf{1}}{\mathbf{s^2+1}}\right)$$
$$+ \mathbf{e^{-2\pi s}}\left(\frac{\mathbf{3}}{\mathbf{s^2+9}} - \frac{\mathbf{2}}{\mathbf{s^2+4}}\right)$$

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13. ii Examples on I. L. T. of Unit Step Function

Example 132: Find the inverse Laplace transform by using unit step function $\dfrac{\mathbf{e^{-s}}}{\mathbf{(s+1)^2}}$

Solution: We have, $L^{-1}\left\{e^{-as}\,\overline{f(s)}\right\} = f(t-a)\,u(t-a)$

Here $a = 1$

$$\therefore\ L^{-1}\left\{e^{-s}\,\overline{f(s)}\right\} = f\,(t-1)\,u(t-1) \qquad \ldots\ldots(1)$$

Where, $\overline{f(s)} = \dfrac{1}{(s+1)^2}$

$$\therefore L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\, t = f(t)$$

$\therefore\ f(t-1) = e^{-(t-1)}(t-1)$ ... Replace $t = t-1$

Eqution (1) → $L^{-1}\left\{e^{-s}\,\overline{f(s)}\right\} = e^{-(t-1)}(t-1)\, u(t-1)$

$$\therefore \quad \mathbf{L^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} = (t-1)e^{-(t-1)}\, u(t-1)}$$

Example 133: Find the inverse Laplace transform by using unit step function. $\dfrac{\mathbf{e^{-\pi s}}}{\mathbf{s^2+4}}$

Solution: We have, $L^{-1}\left\{e^{-as}\,\overline{f(s)}\right\} = f(t-a)\, u(t-a)$

Here $a = \pi$

$\therefore L^{-1}\left\{e^{-\pi s}\,\overline{f(s)}\right\} = f(t-\pi)\, u(t-\pi)$ ... ... (1)

Where, $\overline{f(s)} = \dfrac{1}{s^2+4}$

$$\therefore L^{-1}\left\{\overline{f(s)}\right\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{\sin 2t}{2} = f(t)$$

$$\therefore f(t-\pi) = \frac{1}{2}\sin 2(t-\pi) = \frac{1}{2}\sin(2t-2\pi) = \frac{1}{2}\sin[-(2\pi-2t)]$$

$$= \frac{-1}{2}\sin(2\pi-2t) = \frac{-1}{2}(-\sin 2t) \quad \begin{cases} \because \sin(-\theta) = -\sin\theta \\ \sin(2\pi-\theta) = -\sin\theta \end{cases}$$

$$f(t-\pi) = \frac{1}{2}\sin 2t$$

$\therefore$ Equation(1) → $\mathbf{L^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+4}\right\} = \dfrac{1}{2}\sin 2t\ \ u(t-\pi)}$

Example 134: Find the inverse Laplace transform by using unit step function. $\dfrac{\mathbf{s\, e^{-\frac{s}{2}} + \pi\, e^{-s}}}{\mathbf{s^2+\pi^2}}$

Solution: We have,

$$L^{-1}\left\{\frac{s\, e^{-\frac{s}{2}} + \pi\, e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{\frac{s\, e^{-\frac{s}{2}}}{s^2+\pi^2}\right\} + L^{-1}\left\{\frac{\pi\, e^{-s}}{s^2+\pi^2}\right\} \quad \dots\dots (1)$$

Now, $L^{-1}\left\{e^{-as}\,\overline{f(s)}\right\} = f(t-a)\, u(t-a)$

i) For $\dfrac{s\,e^{-\frac{s}{2}}}{s^2+\pi^2}$

Here $a = \dfrac{1}{2}$

$$L^{-1}\left\{e^{-\frac{s}{2}}\,\overline{f(s)}\right\} = f\left(t-\frac{1}{2}\right)u\left(t-\frac{1}{2}\right) \qquad \ldots\ldots(2)$$

Where, $\overline{f(s)} = \dfrac{s}{s^2+\pi^2}$

$\therefore L^{-1}\left\{\overline{f(s)}\right\} = f(t) = \cos\pi t$

$$\therefore f\left(t-\frac{1}{2}\right) = \cos\pi\left(t-\frac{1}{2}\right) \quad = \cos\left(\pi t-\frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2}-\frac{\pi}{t}\right) \qquad \{\because \cos(-\theta) = \cos\theta$$

$$f\left(t-\frac{1}{2}\right) = \sin\frac{\pi}{t} \qquad \left\{\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right.$$

Equation (2) → $\mathbf{L^{-1}\left\{e^{-\frac{s}{2}}.\dfrac{s}{s^2+\pi^2}\right\} = \sin\dfrac{\pi}{t}\,u\left(t-\dfrac{1}{2}\right)}$ $\ldots\ldots(3)$

ii) For $\dfrac{\pi\,e^{-s}}{s^2+\pi^2}$

Here $a = 1$

$$L^{-1}\left\{e^{-s}\,\overline{f(s)}\right\} = f(t-1)\,u(t-1) \qquad \ldots\ldots(4)$$

Where, $\overline{f(s)} = \dfrac{\pi}{s^2+\pi^2}$

$$L^{-1}\left\{\overline{f(s)}\right\} = f(t) = \sin\pi t$$

$\therefore\ f(t-1) = \sin\pi(t-1) \quad = \sin(\pi t-\pi)$

$$= -\sin(\pi-\pi t)$$

$$f(t-1) = -\sin\pi t \qquad \{\because \sin(-\theta) = -\sin\theta, \qquad \sin(\pi-\theta) = \sin\theta$$

Equation (4) → $L^{-1}\left\{e^{-s}.\dfrac{\pi}{s^2+\pi^2}\right\} = -\sin\pi t\ u(t-1)$ $\ldots\ldots(5)$

From equation (3) & (5) equation (1) →

$$L^{-1}\left\{\frac{se^{-\frac{s}{2}}+\pi e^{-s}}{s^2+\pi^2}\right\} = \sin\pi t\ u\left(t-\frac{1}{2}\right) + [-\sin\pi t\ u(t-1)]$$

$$\mathbf{f(t) = \sin\pi t\left[u\left(t-\frac{1}{2}\right)-u(t-1)\right]}$$

Example 135: Find the inverse Laplace transform of $\frac{e^{-s}-3e^{-3s}}{s^2}$ by using unit step function

Solution: We ahve,

$$L^{-1}\left\{\frac{e^{-s}-3e^{-3s}}{s^2}\right\} = L^{-1}\left\{e^{-s}\ \frac{1}{s^2}\right\} - 3\ L^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} \quad \ldots\ldots(1)$$

By Second shifting property:

$$L^{-1}\{e^{-as}\ \overline{f(s)}\} = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

Now, $$L^{-1}\left\{e^{-s}\ \frac{1}{s^2}\right\} = \begin{cases} (t-1), & t > 1 \\ 0, & t < 1 \end{cases}$$

In unit step function it can be written as, $= (t-1)u(t-1)$

Now, $$L^{-1}\left\{e^{-3s}\ \frac{1}{s^2}\right\} = \begin{cases} (t-3), & t > 3 \\ 0, & t < 3 \end{cases}$$

In unit step function it can be written as, $= (t-3)\ u(t-3)$

$\therefore$ Equation (1) → $$\mathbf{L^{-1}\left\{\frac{e^{-3}-3e^{-3s}}{s^2}\right\}}$$

$$= \mathbf{(t-1)\ u(t-1) - 3(t-3)\ u(t-3)}$$

Example 136: Find inverse L.T. of $\frac{s\ e^{-as}}{s^2-\omega^2}$, a > 0 by using unit step function

Solution: We have $L^{-1}\left\{e^{-as}\ \frac{s}{s^2-\omega^2}\right\}$

By Second shifting property:

$$L^{-1}\{e^{-as}\ \overline{f(s)}\} = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

$$L^{-1}\left\{e^{-as}\ \frac{s}{s^2-\omega^2}\right\} = \begin{cases} \cosh\omega(t-a), & t > a \\ 0 & t < 0 \end{cases}$$

In unit step function it can be written as, $\mathbf{f(t)}$

$$= \mathbf{\cosh\omega(t-a)\ u(t-a)}$$

Example 137: Find inverse L.T. of $\frac{e^{-cs}}{s^2(s+a)}$, c > 0 by using unit step function

Solution: $$L^{-1}\left\{e^{-cs}\ \frac{1}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\ \overline{f(s)}\right\}$$

Where $\overline{f(s)} = \dfrac{1}{s^2(s+a)}$

By partial fraction, $\dfrac{1}{s^2(s+a)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+a}$ …… (1)

Multiplying both sides by $s^2(s+a)$

$$1 = As\,(s+a) + B(s+a) + Cs^2 \quad \text{…… (2)}$$

Put $s = 0$ in equation (2), $1 = B\,(0+a)$; $\mathbf{B = \dfrac{1}{a}}$

Put $s = -a$ in equation (2), $1 = C\,(-a)^2$; $\mathbf{C = \dfrac{1}{a^2}}$

Put $s = a$, $B = \dfrac{1}{a}$ & $C = \dfrac{1}{a^2}$ in equation (2)

$$1 = A\,(a)\,(a+a) + \frac{1}{a}(a+a) + \frac{1}{a^2}\,(a)^2$$

$$1 = 2a^2A + 2 + 1$$

$$1 - 3 = 2a^2\,A; \qquad \frac{-2}{2a^2} = A; \qquad \mathbf{A = \frac{-1}{a^2}}$$

∴ Equation (1) → $\dfrac{1}{s^2(s+a)} = \dfrac{-1}{a^2}\dfrac{1}{s} + \dfrac{1}{a}\dfrac{1}{s^2} + \dfrac{1}{a^2}\dfrac{1}{(s+a)}$

Taking I. L. T. on both sides.

$$L^{-1}\left\{\frac{1}{s^2(s+a)}\right\} = \frac{-1}{a^2}\,L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{a}\,L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{a^2}L^{-1}\left\{\frac{1}{s+a}\right\}$$

$$L^{-1}\{\overline{f(s)}\} = \frac{-1}{a^2}\,(1) + \frac{t}{a} + \frac{1}{a^2}\,e^{-at} = f(t)$$

By Second shifting property:

$$L^{-1}\{e^{-as}\,\overline{f(s)}\} = \begin{cases} f(t-a), & t > a \\ 0, & t \le a \end{cases}$$

Now, $L^{-1}\left\{e^{-cs}\dfrac{1}{s^2(s+a)}\right\} = \begin{cases} \dfrac{-1}{a^2} + \dfrac{1}{a}\,(t-c) + \dfrac{1}{a^2}\,e^{-a(t-c)}, & t > c \\ 0, & t \le c \end{cases}$

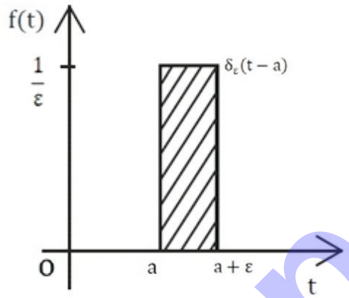
In unit step function it can be written as,

$$\left[\frac{-1}{a^2} + \frac{1}{a}\,(t-c) + \frac{1}{a^2}\,e^{-a(t-c)}\right]\,u(t-c)$$

$$\mathbf{L^{-1}\left\{e^{-cs}\frac{1}{s^2(s+a)}\right\} = \frac{1}{a^2}\left[\,a(t-c) - 1 + e^{-a(t-c)}\right]u(t-c)}$$

14 Unit Impulse function Or Dirac delta function

The idea of a very large force acting for a very short time is of frequent occurrence in mechanism. To deal with such and similar ideas, we introduce the unit impulse function.
(also called dirac delta function)



Thus unit Impulse function is considered as the limiting form of the function

$$\delta\varepsilon(t-a) = \frac{1}{\varepsilon}, \quad a \le t \le a+\varepsilon$$
$$= 0, \quad \text{Otherwise}$$

As $\varepsilon \to 0$. It is clear from fig. That as $\varepsilon \to 0$, the height of the strip increase indefinitely and the width decrease in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \infty \quad \text{for } t = a; \quad \delta(t-a) = 0$$

Such that

$$\int_0^\infty \boldsymbol{\delta(t-a)\,dt = 1} \quad (\mathbf{a \ge 0}) \qquad \text{... for } t \ne a$$

<u>Laplace Transform of unit impulse function</u>:

If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta\varepsilon(t-a)\,dt = \int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}\,dt$$
$$= (a+\varepsilon-a)\,f(n).\frac{1}{\varepsilon}$$
$$= f(n) \quad \begin{cases} \text{where } a < n < a+\varepsilon \\ \text{By mean value then for integral} \end{cases}$$

As $\varepsilon \to 0$, we get $$\int_0^\infty \mathbf{f(t)\,\delta(t-a)\,dt = f(a)}$$

Note:

1) $L\{\delta(t-a)\} = e^{-as}$

2) $L\{\delta f(t)\} = 1$

3) $L^{-1}\{1\} = \delta(t)$

4) $L\{f(t)\,\delta(t-a)\} = e^{-as}\,f(a)$

5) $L\{f(t)\,\delta(t)\} = f(0)$

14.i Examples on L.T. of Unit Impulse Function

Example 138: Evaluate $\displaystyle\int_0^\infty \sin 2t\;\delta\left(t-\frac{\pi}{4}\right)dt$

Solution: We know that, $\displaystyle\int_0^\infty f(t).\,\delta(t-a)\,dt = f(a)$

Here, $a = \frac{\pi}{4}$, $f(t) = \sin 2t$

$$\therefore \int_0^\infty \sin 2t\,\delta\left(t-\frac{\pi}{4}\right)dt = \sin\left(2\,\frac{\pi}{4}\right) = \mathbf{1}$$

Example 139: $\mathbf{L\left\{\frac{1}{t}\;\delta(t-a)\right\}}$

Solution: W.k.t. $L\{\delta(t-a)\} = e^{-as}$

$$\therefore L\left\{\frac{1}{t}\;\delta(t-a)\right\} = \int_s^\infty L\{\delta(t-a)\}\,ds$$

$$= \int_s^\infty e^{-as} = \left[\frac{e^{-as}}{-a}\right]_s^\infty$$

$$= \frac{-1}{a}\left[e^{-\infty} - e^{-as}\right]$$

$$= \frac{-1}{a}\left[0 - e^{-as}\right] \qquad \{\because e^{-\infty} = 0$$

$$\overline{\mathbf{f(s)}} = \frac{\mathbf{1}}{\mathbf{a}}\,\mathbf{e^{-as}}$$

Application to Differential equation

15 Solution of linear Differential equation.

Laplace transforms of linear differential equation are as

1. $L\left\{\frac{d^3y}{dx^3}\right\} = s^3\,\overline{y(s)} - s^2\,y(0) - s\,y'(0) - y''(0)$
2. $L\left\{\frac{d^2y}{dx^2}\right\} = s^2\,\overline{y(s)} - s\,y(0) - y'(0)$
3. $L\left\{\frac{dy}{dx}\right\} = s\,\overline{y(s)} - y(0)$
4. $L\{y\} = \overline{y(s)}$

15.i Examples on solution of linear Differential equation

Example 140: Solve the differential equation by using Laplace transform $\boldsymbol{\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0}$

given $\boldsymbol{y(0) = y'(0) = 0}$ **and** $\boldsymbol{y''(0) = 6}$

Solution: Given D.E. is

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Taking L.T. on both sides

$$L\left\{\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} - \frac{dy}{dx} - 2y\right\} = L\{0\}$$

$$L\left\{\frac{d^3y}{dx^3}\right\} + 2L\left\{\frac{d^2y}{dx^2}\right\} - L\left\{\frac{dy}{dx}\right\} - 2L\{y\} = 0$$

$$\left[s^3\,\overline{y(s)} - s^2\,y(0) - s\,y'(0) - y''(0)\right] + 2\left[s^2\,\overline{y(s)} - s\,y(0) - \right.$$

y′0− s ys− y0−2ys=0

$$(s^3 + 2s^2 - s - 2)\,\overline{y(s)} - 6 = 0$$

$$(s^3 + 2s^2 - s - 2)\,\overline{y(s)} = 6$$

$$\overline{y(s)} = \frac{6}{(s^3 + 2s^2 - s - 2)}$$

$$\overline{y(s)} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$\therefore \quad \overline{y(s)} = \frac{6}{(s-1)6} + \frac{6}{(s+1)(-2)} + \frac{6}{(s+2)(3)}$$

$$\overline{y(s)} = \frac{1}{s-1} + \frac{-3}{s+1} + \frac{2}{s+2}$$

Taking I. L. T. on both sides , we get

$$L^{-1}\left\{\overline{y(s)}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} - 3\,L^{-1}\left\{\frac{1}{s+1}\right\} + 2\,L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\mathbf{y(t) = e^{t} - 3e^{-t} + 2e^{-2t}}$$

Example 141: Use transform method to solve

$$\mathbf{\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2, \ \frac{dx}{dt} = -1 \text{ at } t = 0}$$

Solution: Given D. E. is $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \quad \ldots\ldots(1)$

$$x = 2, \quad t = 0 \rightarrow \quad x(t) = x(0) = 2$$

$$\frac{dx}{dt} = -1, \quad t = 0 \rightarrow \quad x'(t) = x'(0) = -1$$

Now, Taking L. T. on both sides of equation(1)

$$L\left\{\frac{d^2x}{dt^2}\right\} - 2L\left\{\frac{dx}{dt}\right\} + L\{x\} = L\{e^t\}$$

$$\therefore \ s^2\,\overline{x(s)} - s\,x(0) - x'(0) - 2\left[\left(s\,\overline{x(s)} - x(0)\right)\right] + \overline{x(s)} = \frac{1}{s-1}$$

$$\therefore \ (s^2 - 2s + 1)\,\overline{x(s)} - 2s - (-1) + 2(2) = \frac{1}{s-1}$$

$$\therefore \ (s^2 - 2s + 1)\,\overline{x(s)} = \frac{1}{s-1} + 2s - 5$$

$$= \frac{1 + (s-1)(2s-5)}{s-1} \qquad = \frac{1 + 2s^2 - 5s - 2s + 5}{s-1}$$

$$(s-1)^2\,\overline{x(s)} = \frac{2s^2 - 7s + 6}{s-1}$$

$$\overline{x(s)} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

By partial fraction

$$\frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \qquad \ldots\ldots(1)$$

Multiplying both sides by $(s-1)^3$

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C$$

$$2s^2 - 7s + 6 = A(s^2 - 2s + 1) + Bs - B + C$$
$$2s^2 - 7s + 6 = As^2 + (-2A + B)s + (A - B + C)$$

Equating coefficient on both sides

Coefficient of $s^2 \rightarrow$ $\mathbf{A = 2}$

Coefficient of $s \rightarrow$ $-2A + B = -7$; $-2(2) + B = -7$; $\mathbf{B = -3}$

Constant terms $\rightarrow A - B + C = 6$; $2 - (-3) + C = 6$; $\mathbf{C = 1}$

$\therefore$ Eqution(1) becomes by Taking I. L. T. on both sides

$$L^{-1}\left\{\frac{2s^2 - 7s + 6}{(s-1)^3}\right\} = 2\,L^{-1}\left\{\frac{1}{s-1}\right\} - 3\,L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$L^{-1}\{\overline{x(s)}\} = 2e^t - 3te^t + \frac{t^2}{2!}\,e^t$$

$$\therefore \quad \mathbf{x(t) = e^t\left[2 - 3t + \frac{t^2}{2}\right]}$$

Example 142: **Solve $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ by using Laplace transform**

Given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

Solution: Given D. E. can be written as:

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2\,e^t$$

$$y(0) = 1,\ \ y'(0) = 0,\ \ y''(0) = -2$$

Taking L. T. on both sides

$$L\left\{\frac{d^3y}{dt^3}\right\} - 3\,L\left\{\frac{d^2y}{dt^2}\right\} + 3\,L\left\{\frac{dy}{dt}\right\} - L\{y\} = L\{t^2e^t\}$$

$$\therefore \left[s^3\,\overline{y(s)} - s^2y(0) - s\,y'(0) - y''(0)\right] - 3\left[s^2\overline{y(s)} - s\,y(0) - y'(0)\right]$$
$$+ 3\left[s\,\overline{y(s)} - y(0)\right] - \overline{y(s)} = \frac{2}{(s-1)^3}$$

$$\therefore (s^3 - 3s^2 + 3s - 1)\,\overline{y(s)} - s^2(1) - (-2) + 3\,s(1) - 3(1) = \frac{2}{(s-1)^3}$$

$$\therefore (s-1)^3\,\overline{y(s)} - s^2 + 2 + 3s - 3 = \frac{2}{(s-1)^3}$$

$$\therefore (s-1)^3\,\overline{y(s)} = \frac{2}{(s-1)^3} + (s^2 - 3s + 1)$$

$$\therefore \ \mathbf{\overline{y(s)}} = \frac{\mathbf{2}}{\mathbf{(s-1)^6}} + \frac{\mathbf{s^2 - 3s + 1}}{\mathbf{(s-1)^3}} \qquad \dots\dots(1)$$

Take $\overline{y_1(s)} = \dfrac{s^2 - 3s + 1}{(s-1)^3}$

By partial fraction

$$\frac{s^2 - 3s + 1}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \qquad \dots\dots(2)$$

Multiplying both sides by $(s-1)^3$

$$s^2 - 3s + 1 = A(s-1)^2 + B(s-1) + C$$

$$s^2 - 3s + 1 = A(s^2 - 2s + 1) + Bs - B + C$$

$$s^2 - 3s + 1 = As^2 - 2As + A + Bs - B + C$$

$$s^2 - 3s + 1 = As^2 + (-2A + B)s + (A - B + C)$$

Equating coefficient on both sides,

Coefficient of $s^2 \to \mathbf{A = 1}$

Coefficient of $s \to -2A + B = -3$; $\quad -2(1) + B = -3$; $\quad \mathbf{B = -1}$

Constant term $\to A - B + C = 1$; $\quad 1 - (-1) + C = 1$; $\quad \mathbf{C = -1}$

$$\therefore \text{Equation (2)} \to \frac{s^2 - 3s + 1}{(s-1)^3} = \frac{1}{s-1} + \frac{-1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\text{Equation (1)} \to \quad y(s) = \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\overline{y(s)}\right\} = 2\,L^{-1}\left\{\frac{1}{(s-1)^6}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$y(t) = 2\,\frac{t^5}{5!}\,e^t + e^t - te^t - \frac{t^2}{2!}\,e^t$$

$$\mathbf{y(t) = e^t\left(\frac{1}{60}\,t^5 + 1 - t - \frac{1}{2}\,t^2\right)}$$

Example 143: Solve the D. E. by using Laplace transform

$$\mathbf{\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4t + e^{3t}, \ \text{with} \ y(0) = 1, \ y'(0) = -1.}$$

Solution: Given D. E. is $\frac{d^2y}{dt^2} - \frac{3dy}{dt} + 2y = 4t + e^{3t}$,

$y(0) = 1, \; y'(0) = -1$

Taking L. T. on both sides, we get

$$L\left\{\frac{d^2y}{dt^2}\right\} - 3\,L\left\{\frac{dy}{dt}\right\} + 2\{y\} = 4\,L\{t\} + L\{e^{3t}\}$$

$$\left[s^2\,\overline{y(s)} - s\,y(0) - y'(0)\right] - 3\left[s\,\overline{y(s)} - y(0)\right] + 2\,\overline{y(s)}$$

$$= 4\,\frac{1}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,\overline{y(s)} - s(1) - (-1) + 3(1) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,\overline{y(s)} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,\overline{y(s)} - (s-4) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,\overline{y(s)} - (s-4) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,\overline{y(s)} = \frac{4}{s^2} + \frac{1}{s-3} + (s-4)$$

$$\overline{y(s)} = \frac{4}{s^2(s^2-3s+2)} + \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-4}{s^2-3s+2}$$

$$\overline{y(s)} = \frac{4(s-3) + s^2 + (s-4)(s^2)(s-3)}{s^2(s-3)(s^2-3s+2)}$$

i. e. $\overline{y(s)} = \frac{4(s-3) + s^2 + (s-4)(s^2)(s-3)}{s^2(s-3)(s-1)(s-2)}$

$\{\because s^2 - 3s + 2 = (s-1)(s-2)$

By partial fraction

$$\overline{y(s)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1} + \frac{E}{s-2} \qquad \ldots\ldots(1)$$

Multiplying both sides by $s^2(s-1)(s-2)(s-3)$

$$4(s-3) + s^2 + (s-4)s^2(s-3) = As(s-1)(s-2)(s-3)$$
$$+B(s-1)(s-2)(s-3) + C(s^2)(s-1)(s-2)$$
$$+D(s^2)(s-2)(s-3) + E(s^2)(s-1)(s-3) \qquad \ldots\ldots(2)$$

Now,

$$B = \left.\frac{4(s-3)+s^2+(s-4)s^2(s-3)}{(s-1)(s-2)(s-3)}\right|_{s=0} = \frac{-12}{-6} \quad ; \quad \mathbf{B = 2}$$

$$C = \left.\frac{4(s-3)+s^2+(s-4)s^2(s-3)}{s^2(s-1)(s-2)}\right|_{s=3} = \frac{9}{18} \quad ; \quad \mathbf{C = \frac{1}{2}}$$

$$D = \left.\frac{4(s-3)+s^2+(s-4)(s^2)(s-3)}{s^2(s-2)(s-3)}\right|_{s\,=\,1} = \frac{-1}{2} \quad ; \quad \mathbf{D = \frac{-1}{2}}$$

$$E = \left.\frac{4(s-3)+s^2+(s-4)s^2(s-3)}{s^2(s-1)(s-3)}\right|_{s=2} = \frac{8}{-4} \quad ; \quad \mathbf{E = -2}$$

Now,

Put $s = 4,\ \ B = 2,\ \ C = \frac{1}{2},\ \ D = \frac{-1}{2}, E = -2$ in eqation (2)

$$4(1) + 16 = A(4)(3)(2)(1) + 2(3)(2)(1) + \frac{1}{2}(16)(3)(2) + \left(\frac{-1}{2}\right)(16)(2)(1) + (-2)(16)(3)(1)$$

$$20 = 24A + 12 + 48 - 16 - 96 ; \quad 20 = 24A - 52 ; \quad 72 = 24A ; \quad \mathbf{A = 3}$$

∴ Equation (1) becomes by taking I. L. T. on both sides

$$L^{-1}\left\{\overline{y(s)}\right\} = 3\,L^{-1}\left\{\frac{1}{s}\right\} + 2\,L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{2}\,L^{-1}\left\{\frac{1}{s-1}\right\} - 2\,L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(t) = 3 + 2t + \frac{1}{2}\,e^{3t} - \frac{1}{2}\,e^{t} - 2e^{2t}$$

i. e. $\mathbf{y\,(t) = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}\,e^{3t}}$

Example 144: **Solve the D. E. by using L. T.**

$$\mathbf{y''(t) - y'(t) - 2y(t) = 20\sin 2t,}$$

with $\mathbf{y(0) = -1,\ \ y'(0) = 2}$.

Solution: Given D. E. is

$$y''(t) - y'(t) - 2y(t) = 20\sin 2t\,,\ \ y(0) = -1,\ \ y'(0) = 2$$

Taking L. T. on both sides.

$$L\{y''(t)\} - L\{y'(t) - 2L\{y(t)\} = 20\ L\{\sin 2t\}$$

$$s^2\,\overline{y(s)} - s\,y(0) - y'(0) - \left\{s\,\overline{y(s)} - y(0)\right\} - 2\,\overline{y(s)} = 20\,\frac{2}{s^2+4}$$

$$(s^2 - s - 2)\,\overline{y(s)} - s(-1) - 2 + (-1) = \frac{40}{s^2+4}$$

$$(s^2 - s - 2)\,\overline{y(s)} + s - 3 = \frac{40}{s^2+4}$$

$$(s^2 - s - 2)\,\overline{y(s)} = \frac{40}{s^2+4} + 3 - s$$

$$\overline{y(s)} = \frac{40}{(s^2+4)(s^2-s-2)} + \frac{3-s}{s^2-s-2}$$

$$\overline{y(s)} = \frac{40}{(s^2+4)(s+1)(s-2)} + \frac{3-s}{(s+1)(s-2)}$$

$$\overline{y(s)} = \frac{40+(3-s)(s^2+4)}{(s^2+4)(s+1)(s-2)}$$

By partial fraction method

$$\overline{y(s)} = \frac{40+(3-s)(s^2+4)}{(s^2+4)(s+1)(s-2)} = \frac{As+B}{s^2+4} + \frac{C}{s+1} + \frac{D}{s-2} \quad \ldots\ldots(1)$$

Multiplying both sides by $(s^2+4)(s+1)(s-2)$

$$40+(3-s)(s^2+4) = (As+B)(s+1)(s-2) + C(s^2+4)(s-2) + D(s^2+4)(s+1) \quad \ldots\ldots(2)$$

Now, $C = \left.\dfrac{40+(3-s)(s^2+4)}{(s^2+4)(s-2)}\right|_{s=-1} = \dfrac{60}{-15}$; $\mathbf{C = -4}$

$D = \left.\dfrac{40+(3-s)(s^2+4)}{(s^2+4)(s+1)}\right|_{s=2} = \dfrac{48}{24}$; $\mathbf{D = 2}$

Now, Put $s = 0$, $C = -4$, $D = 2$ in equn(2)

$$40 + (3)(4) = B(1)(-2) + (-4)(4)(-2) + 2(4)(1)$$

$$52 = -2B + 32 + 8\,; \quad 52 - 40 = -2B\,; \quad 12 = -2B$$

$$\frac{12}{-2} = B\,; \quad \mathbf{B = -6}$$

Now, Put $s = 3$, $B = -6$, $C = -4$, $D = 2$, in equn(2)

$$40 + 0 = [A(3) + (-6)](4)(1) + (-4)(13)(1) + 2(13)(4)$$

$$40 - 28 = 12A\,; \quad 12 = 12A \quad \therefore \mathbf{A = 1}$$

Now, Equation (1) → and taking I.L.T. on both sides

$$L^{-1}\left\{\overline{y(s)}\right\} = L^{-1}\left\{\frac{(1)s + (-6)}{s^2 + 4}\right\} + (-4)L^{-1}\left\{\frac{1}{s+1}\right\} + 2L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(t) = L^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} - 6\,L^{-1}\left\{\frac{1}{s^2 + 2^2}\right\} - 4\,L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$+\ 2\,L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(t) = \cos 2t - 6\,\frac{\sin 2t}{2} - 4e^{-t} + 2e^{2t}$$

i. e. $$\mathbf{y(t) = \cos 2t - 3\sin 2t - 4e^{-t} + 2e^{2t}}$$

Example 145: **Solve** $\mathbf{(D^2 + 2D + 5)y = e^{-t}\sin t,\ \ y(0) = 0,}$

$$\mathbf{y'(0) = 1,\quad D = \frac{d}{dt}}\ \textbf{using L. T.}$$

Solution: Given D. E. can be written as:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$$

Taking L. T. on both sides.

$$L\left\{\frac{d^2y}{dt^2}\right\} + 2L\left\{\frac{dy}{dt}\right\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$\therefore\ s^2\,\overline{y(s)} - s\,y(0) - y'(0) + 2\left[s\ \overline{y(s)} - y(0)\right] + 5\,\overline{y(s)}$$

$$= \frac{1}{(s+1)^2 + 1}$$

$$\therefore (s^2 + 2s + 5)\,\overline{y(s)} - 1 = \frac{1}{s^2 + 2s + 1 + 1}$$

$$(s^2 + 2s + 5)\,\overline{y(s)} = \frac{1}{s^2 + 2s + 2} + 1$$

$$\therefore\ \overline{y(s)} = \frac{1}{(s^2 + 2s + 2)(s^2 - 2s + 5)} + \frac{1}{s^2 + 2s + 5}$$

$$\therefore\ \overline{y(s)} = \overline{y_1(s)} + \frac{1}{s^2 + 2s + 5} \qquad \ldots\ldots(1)$$

Take, $\overline{y_1(s)}$ By using partial fraction method

$$\overline{y_1(s)} = \frac{1}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

$$= \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{s^2 + 2s + 5} \qquad \ldots\ldots(2)$$

Multiplying both sides by $(s^2 + 2s + 2)(s^2 + 2s + 5)$

$$1 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$$

$$1 = As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$$

$$1 = (A + C)s^3 + (2A + B + 2C + D)s^2 + (5A + 2B + 2C + 2D)s + (5B + 2D)$$

Equating coefficient on both sides

Coeffient of s^3 → $A + C = 0$ ……(3)

Coeffient of s^2 → $2A + B + 2C + D = 0$ ……(4)

Coeffient of s → $5A + 2B + 2C + 2D = 0$ ……(5)

Constant term → $5B + 2D = 1$ ……(6)

Equation (4) × 2 $4A + 2B + 4C + 2D = 0$

Equation (5) $5A + 2B + 2C + 2D = 0$

Substrating $- \quad - \quad - \quad -$

$$-A + 2C = 0 \quad \text{……(7)}$$

Equation (3) $A + C = 0$

Equation (7) $-A + 2C = 0$

Adding $3C = 0$ $\quad \therefore \mathbf{C = 0}$

Equation (3) → $\mathbf{A = 0}$

Equation (5) → $2B + 2D = 0$ ……(8)

Now

Equation (6) $5B + 2D = 1$

Equation (8) $2B + 2D = 0$

Subtrating $- \quad - \quad -$

$3B = 1$ $\quad \mathbf{B = \frac{1}{3}}$

Equation (6) → $5\left(\frac{1}{3}\right) + 2D = 1, \quad 2D = 1 - \frac{5}{3} = \frac{3-5}{3};$

$2D = -\frac{2}{3}; \quad \mathbf{D = \frac{-1}{3}}$

Equation (2) → $\overline{y_1(s)} = \dfrac{0(s) + \frac{1}{3}}{s^2 + 2s + 2} + \dfrac{0(s) + \left(-\frac{1}{3}\right)}{s^2 + 2s + 5}$

Equation(1) → $\overline{y(s)}$

$$= \frac{1}{3}\frac{1}{s^2 + 2s + 2} - \frac{1}{3}\frac{1}{s^2 + 2s + 5} + \frac{1}{s^2 + 2s + 5}$$

$$\overline{y(s)} = \frac{1}{3}\,\frac{1}{s^2+2s+2} + \frac{2}{3}\,\frac{1}{s^2+2s+5}$$

$$\overline{y(s)} = \frac{1}{3}\,\frac{1}{(s+1)^2+1^2} + \frac{2}{3}\,\frac{1}{(s+1)^2+2^2}$$

Taking I. L. T. on both sides,

$$L^{-1}\left\{\overline{y(s)}\right\} = \frac{1}{3}\,L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\,L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$y(t) = \frac{1}{3}\,e^{-t}\sin t + \frac{2}{3}\,e^{-t}\,\frac{\sin 2t}{2}$$

$$\therefore\quad \mathbf{y\,(t) = \frac{1}{3}\,e^{-t}(\sin t + \sin 2t)}$$

Example 146: **Solve** $\mathbf{(D^2+n^2)x = a\sin(nt+\propto)}$ **, given** $\mathbf{x(0) = x'(0) = 0}$ **using L. T.**

Solution: Given D. E. can be written as : $D^2x + n^2x$

$$= a\sin(nt+\propto)$$

Taking L. T. on both sides.

$$L\{D^2x\} + n^2\,L\{x\} = a\,L\{\sin(nt+\propto)\}$$

$$s^2\,\overline{x(s)} - s\,x(0) - x'(0) + n^2\,\overline{x(s)}$$

$$= a\,L\{\sin nt\cos\propto + \cos nt\sin\propto\}$$

$$\therefore (s^2+n^2)\,\overline{x(s)} = a\,[\cos\propto L\{\sin nt\} + \sin\propto\,L\{\cos nt\}]$$

$$= a\cos\propto\frac{n}{s^2+n^2} + a\sin\propto\frac{s}{s^2+n^2}$$

$$\therefore\ \overline{x(s)} = a\cos\propto\frac{n}{(s^2+n^2)^2} + a\sin\propto\frac{s}{(s^2+n^2)^2}$$

Taking I. L. T. on both sides,

$$L^{-1}\left\{\overline{x(s)}\right\} = an\cos\propto\,L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\propto\,L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}$$

$\therefore$ Solve it by Convolution theorem

$$L^{-1}\left\{\overline{x(s)}\right\} = an\cos\propto\int_0^t \frac{\sin n(t-u)}{n}\frac{\sin n(u)}{n}\,du$$

$$+\ a\sin\propto\int_0^t \cos n(t-u)\,\frac{\sin nu}{n}\,du$$

$$x(t) = an \cos \propto \frac{1}{n^2}\int_0^t \sin(nt - nu)\sin(nu)\,du$$

$$+ a \sin \propto \frac{1}{n}\int_0^t \cos(nt - nu)\sin(nu)\,du$$

$$\begin{cases} \because \sin A . \sin B = \dfrac{-1}{2}[\cos(A+B) - \cos(A-B)]; \\ \cos A . \sin B = \dfrac{1}{2}[\sin(A+B) - \sin(A-B)] \end{cases}$$

$$\therefore\ x(t) = \frac{a}{n}\cos \propto \int_0^t \frac{-1}{2}[\cos(nt - nu + nu) - \cos(nt - nu - nu)]\,du$$

$$+ \frac{a}{n}\sin \propto \int_0^t \frac{1}{2}[\sin(nt - nu + nu) - \sin(nt - nu - nu)]\,du$$

$$= \frac{-a}{2n}\cos \propto \int_0^t [\cos nt - \cos(nt - 2nu)]du +$$

$$\frac{a}{2n}\sin \propto \int_0^t [\sin nt - \sin(nt - 2nu)]du$$

$$= \frac{-a}{2n}\cos \propto \left\{\left[\cos nt(u) - \frac{\sin(nt - 2nu)}{-2n}\right]_0^t\right\}$$

$$+ \frac{a}{2n}\sin \propto \left\{\left[\sin nt(u) - \frac{(-\cos(nt - 2nu))}{-2n}\right]_0^t\right\}$$

$$= \frac{-a}{2n}\cos \propto \left\{t\cos nt + \frac{1}{2n}\sin(nt - 2nt) - \frac{\sin nt}{2n}\right\}$$

$$+ \frac{a}{2n}\sin \propto \left\{t\sin nt - \frac{1}{2n}\cos(nt - 2nt) + \frac{\cos nt}{2n}\right\}$$

$$= \frac{-a}{2n}\cos \propto \left[t\cos nt - \frac{1}{2n}\sin nt - \frac{\sin nt}{2n}\right]$$

$$+ \frac{a}{2n}\sin \propto \left[t\sin nt - \frac{1}{2n}\cos nt + \frac{\cos nt}{2n}\right]$$

$$= \frac{-a}{2n}\cos \propto \left[t\cos nt - \frac{1}{n}\sin nt\right] + \frac{a}{2n}\sin \propto (t\sin nt)$$

$$= \frac{a\cos\propto}{2n^2}(\sin nt - nt\cos nt) + \frac{a\sin\propto}{2n}\, t\sin nt$$

Example 147: **Solve** : $\frac{d^2x}{dt^2} + 9x = \cos 2t$, **if** $x(0) = 1$,

$$x\left(\frac{\pi}{2}\right) = -1$$

Solution: Given, $\frac{d^2x}{dt^2} + 9x = \cos 2t$ ... ... (1)

$x(0) = 1,\ x\left(\frac{\pi}{2}\right) = -1$

But $x'(0)$ is not given. Let $x'(0) = a$

Now, Taking L. T. both sides of equn(1)

$$L\left\{\frac{d^2x}{dt^2}\right\} + 9\,L\{x\} = L\{\cos 2t\}$$

$$s^2\,\overline{x(s)} - s\,x(0) - x'(0) + 9\,\overline{x(s)} = \frac{s}{s^2+4}$$

$$\therefore\ (s^2+9)\overline{x(s)} = \frac{s}{s^2+4} + s + a$$

$$\therefore\ \overline{x(s)} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9} \qquad \text{... ... (2)}$$

Take $\overline{x_1(s)} = \frac{s}{(s^2+4)(s^2+9)}$

By partial fraction,

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9} \qquad \text{... ... (3)}$$

Multiplying both sides by $(s^2+4)(s^2+9)$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$s = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$s = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D)$$

Equating coeffient on both sides.

Coeffient of $s^3 \rightarrow A + C = 0$ ... ... (4)

Coeffient of $s^2 \rightarrow B + D = 0$ ... ... (5)

Coeffient of $s \rightarrow 9A + 4C = 1$ ... ... (6)

Coeffient term $\rightarrow 9B + 4D = 0$ ... ... (7)

Now,

Equation (4) × 4 $\quad 4A + 4C = 0$

Equation (6) $\quad 9A + 4C = 1$

Subtrating $\quad - \quad - \quad -$

$$-5A = -1 \qquad \mathbf{A = \frac{1}{5}}$$

Now, equation (4) → $\mathbf{C = \frac{-1}{5}}$

Equation (5) × 4 $\quad 4B + 4D = 0$

Equation (7) $\quad 9B + 4D = 0$

Subtrating $\quad - \quad - \quad -$

$$-5B = 0 \qquad \mathbf{B = 0}$$

Equation (5) → $\mathbf{D = 0}$

Equation (3) → $$\frac{s}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}s + 0}{s^2+4} + \frac{-\frac{1}{5}s + 0}{s^2+9}$$

Equation (2) →

$$\overline{x(s)} = \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{a}{s^2+9}$$

$$\overline{x(s)} = \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{a}{s^2+9}$$

Taking I. L. T. on both sides

$$L^{-1}\left\{\overline{x(s)}\right\} = \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{4}{5}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + a\,L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + a\,\frac{\sin 3t}{3}$$

$$\mathbf{x(t) = \frac{1}{5}\left(\cos 2t + 4\cos 3t + \frac{5}{3}a\sin 3t\right)} \qquad \ldots\ldots(8)$$

Now, given $x\left(\frac{\pi}{2}\right) = -1$, at $t = \frac{\pi}{2}$

Equation (8) → $$x\left(\frac{\pi}{2}\right) = \frac{1}{5}\left(\cos 2\frac{\pi}{2} + 4\cos 3\frac{\pi}{2} + \frac{5}{3}a\sin 3\frac{\pi}{2}\right)$$

$$-1 = \frac{1}{5}\left(-1 + 4(0) + \frac{5}{3}a(-1)\right)$$

$$\left\{\because \sin\frac{\pi}{2} = 1, \sin\frac{3\pi}{2} = -1\ \cos\pi = -1, \cos\frac{3\pi}{2} = 0\right.$$

$$-1 = \frac{-1}{5} - \frac{a}{3}; \quad \frac{a}{3} = -\frac{1}{5} + 1; \quad \frac{a}{3} = \frac{4}{5}; \quad a = \frac{4}{5}.3; \quad \mathbf{a = \frac{12}{5}}$$

$\therefore$ Equation (8) → $x(t) = \frac{1}{5}\left(\cos 2t + 4\cos 3t + \frac{5}{3}.\frac{12}{5}.\sin 3t\right)$

$$\mathbf{x(t) = \frac{1}{5}\ (\cos 2t + 4\cos 3t + 4\sin 3t)}$$

Example 148: Solve $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given $y(0) = 1$ using L.T.

Solution: Given, $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$

Taking L.T. on both sides.

$$\therefore \quad L\left\{\frac{dy}{dt} + 2y + \int_0^t y\,dt\right\} = L\{\sin t\}$$

$$L\left\{\frac{dy}{dt}\right\} + 2\,L\{y\} + L\left\{\int_0^t y\,dt\right\} = \frac{1}{s^2+1}$$

$$s\,\overline{y(s)} - y(0) + 2\overline{y(s)} + \frac{1}{s}\,\overline{y(s)} = \frac{1}{s^2+1}$$

$$\left(s + 2 + \frac{1}{s}\right)\overline{y(s)} = \frac{1}{s^2+1} + 1$$

$$\frac{s^2+2s+1}{s}\,\overline{y(s)} = \frac{1}{s^2+1} + 1$$

$$\frac{(s+1)^2}{s}\,\overline{y(s)} = \frac{1}{s^2+1} + 1$$

$$\overline{y(s)} = \frac{s}{(s^2+1)(s+1)^2} + \frac{s}{(s+1)^2} = \overline{y_1(s)} + \frac{s}{(s+1)^2} \quad \ldots\ldots(1)$$

Take by partial fraction

$$\therefore \overline{y_1(s)} = \frac{s}{(s^2+1)(s+1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \quad \ldots\ldots(2)$$

Multiplying both sides by $(s^2+1)(s+1)^2$

$$s = (As+B)(s+1)^2 + C(s^2+1)(s+1) + D(s^2+1)$$

$$s = (As+B)(s^2+2s+1) + C(s^3+s^2+s+1) + Ds^2 + D$$

$$s = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Cs + C + Ds^2 + D$$

$$s = (A + C)s^3 + (2A + B + C + D)s^2 + (A + 2B + C)s + (B + C + D)$$

Equating coefficient on both sides.

Coefficient of s^3 → $A + C = 0$ ...... (3)

Coefficient of s^2 → $2A + B + C + D = 0$ ...... (4)

Coefficient of s → $A + 2B + C = 1$ ...... (5)

Constant term → $B + C + D = 0$ ...... (6)

Now,

Equation (3) → $A + C = 0$ put in equation (5)

Equation (5) → $2B = 1$, $\mathbf{B = \frac{1}{2}}$

Equation (6) → $B + C + D = 0$ put in equation (4)

Equation (4) → $2A = 0$, $\mathbf{A = 0}$

Put $A = 0$ in equation (3), $\mathbf{C = 0}$

Put $B = \frac{1}{2}$, $C = 0$, in equn (6) $\frac{1}{2} + 0 + D = 0$; $\mathbf{D = -\frac{1}{2}}$

∴ Equation(2) → $$\frac{s}{(s^2+1)(s+1)^2} = \frac{0(s) + \frac{1}{2}}{s^2+1} + \frac{0}{s+1} + \frac{\frac{-1}{2}}{(s+1)^2}$$

Equation(1) → $$\overline{y(s)} = \frac{\frac{1}{2}}{s^2+1} - \frac{\frac{1}{2}}{(s+1)^2} + \frac{s}{(s+1)^2}$$

$$\overline{y(s)} = \frac{1}{2}\frac{1}{s^2+1} + \frac{s - \frac{1}{2}}{(s+1)^2}$$

Now, Taking I.L.T. on both sides

$$L^{-1}\left\{\overline{y(s)}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{s - \frac{1}{2}}{(s+1)^2}\right\}$$

$$y(t) = \frac{1}{2}\sin t + L^{-1}\left\{\frac{(s+1) - \frac{3}{2}}{(s+1)^2}\right\}$$

$$= \frac{1}{2}\sin t + e^{-t}L^{-1}\left\{\frac{s - \frac{3}{2}}{s^2}\right\}$$

$$= \frac{1}{2}\sin t + e^{-t}\left[L^{-1}\left\{\frac{s}{s^2} - \frac{\frac{3}{2}}{s^2}\right\}\right]$$

$$= \frac{1}{2}\sin t \ + e^{-t}\left[L^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\ L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{2}\sin t \ + e^{-t}\left[1 - \frac{3}{2}\ t\right]$$

$$\mathbf{y(t)} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{\sin t} \ + \mathbf{e^{-t}} - \frac{\mathbf{3}}{\mathbf{2}}\ \mathbf{t.e^{-t}}$$

Example 149: **Solve** $\mathbf{t\,y'' + 2y' + t\,y = \cos t}$ **given that** $\mathbf{y(0) = 1,\ \ y'(0) = 0}$.

Solution: Given, $t\,y'' + 2y' + t\,y = \cos t$

Taking L. T. on both sides,

$$L\{t\,y''\} + 2\,L\{y'\} + L\{t\,y\} = L\{\cos t\}$$

$$\left[\text{W.k.t. } L\{t^n\ f(t)\} = (-1)^n \frac{d^n}{ds^n}\ \overline{f(s)}\right]$$

$$-\frac{d}{ds}\left[s^2\overline{y(s)} - \ s\,y(0) - y'(0)\right] + 2\left[s\,\overline{y(s)} - y(0)\right] + \left(-\frac{d}{ds}\ \overline{y(s)}\right) = \frac{s}{s^2+1}$$

$$-\frac{d}{ds}\ s^2\ \overline{y(s)} + \frac{d}{ds}s + \frac{d}{ds}0 + 2s\,\overline{y(s)} - 2 - \frac{d}{ds}\overline{y(s)} = \frac{s}{s^2+1}$$

$$-\left[s^2\frac{d}{ds}\overline{y(s)} + \overline{y(s)}\ 2s\right] + 1 + 0 + 2s\,\overline{y(s)} - 2 - \frac{dy}{ds}\overline{y(s)} = \frac{s}{s^2+1}$$

$$-s^2\frac{d}{ds}\overline{y(s)} \ - 2s\ \overline{y(s)} + 2s\,\overline{y(s)} - \frac{d}{ds}\overline{y(s)} = \frac{s}{s^2+1} \ + 1$$

$$-\left(s^2+1\right)\frac{d}{ds}\overline{y(s)} = \frac{s}{s^2+1} + 1$$

$$\therefore\ \frac{d}{ds}\overline{y(s)} = \frac{-s}{(s^2+1)^2} - \frac{1}{s^2+1}$$

Taking I. L. T. on both sides

$$\therefore\ L^{-1}\left\{\frac{d}{ds}\overline{y(s)}\right\} = L^{-1}\left\{\frac{-s}{(s^2+1)^2} \ - \frac{1}{s^2+1}\right\}$$

$$= -\,L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$-t\,y(t) = -\frac{1}{2(1)}\,t\sin t - \sin t \quad \left\{\because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t.\sin at\right.$$

$$\therefore\ y(t) = \frac{1}{2}\sin t + \frac{1}{t}\sin t$$

$$\text{Or }\ \mathbf{y(t) = \frac{1}{2}\sin t\left(1+\frac{2}{t}\right)}$$

Example 150: Solve Using L.T. $\mathbf{y'' + 4y = f(t),\ \ y(0) = 0,}$
$\mathbf{y'(0) = 1}$

Where, $\mathbf{f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}}$

Solution: Given, $y'' + 4y = f(t)$ ……(1)

Where, $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

It can be express in unit step functions:

$$f(t) = L[u(t) - u(t-1)]$$

Equation(1) → $y'' + 4y = u(t) - u(t-1)$

Taking L.T. on both sides,

$$L\{y''\} + 4L\{y\} = L\{u(t)\} - L\{u(t-u)\}$$

Given, $y(0) = 0,\ \ y'(0) = 1$

$$s^2\bar{y} - s\,y(0) - y'(0) + 4\bar{y} = \frac{1}{s} - e^{-s}\frac{1}{s}$$

$$s^2\bar{y} - 1 + 4\,\bar{y} = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$(s^2+4)\,\bar{y} = \frac{1}{s} - \frac{e^{-s}}{s} + 1$$

$$\bar{y} = \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)} + \frac{1}{s^2+4} = y_1 - y_2 + \frac{1}{s^2+4} \quad \ldots\ldots(2)$$

Now, By partial fraction

$$\text{Let }\ y_1 = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad \ldots\ldots(3)$$

Multiplying both sides by $s(s^2+4)$

$$1 = A(s^2+4) + (Bs+C)(s)$$

$$1 = As^2 + 4A + Bs^2 + Cs$$

$$1 = (A+B)\,s^2 + Cs + 4A$$

Equating coefficient on both sides,

Coefficient of $s^2 \rightarrow A + B = 0$

Coefficient of $s \rightarrow \mathbf{C = 0}$

Constant term $\rightarrow 4A = 1\ ;\ \mathbf{A = \frac{1}{4}}$, $\quad \therefore \mathbf{B = \frac{-1}{4}}$

Equation (3) $\rightarrow$ $$y_1 = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2 + 4} \qquad \ldots\ldots(4)$$

Now, By partial fraction,

Let $$y_2 = \frac{e^{-s}}{s(s^2+4)} = e^{-s}\frac{1}{s(s^2+4)} = e^{-s}\, y_1$$

$$= e^{-s}\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s + 0}{s^2+4}\right]$$

$$y_2 = \frac{\frac{1}{4}e^{-s}}{s} + \frac{-\frac{1}{4}se^{-s}}{s^2+4} \qquad \ldots\ldots(5)$$

Now, Using equation (4) & (5) equation (2) becomes $\rightarrow$

Equation (2) $\rightarrow$ $$\bar{y} = \frac{\frac{1}{4}}{s} + \frac{\frac{-1}{4}s}{s^2+4} - \left(\frac{\frac{1}{4}e^{-s}}{s} + \frac{-\frac{1}{4}se^{-s}}{s^2+4}\right) + \frac{1}{s^2+4}$$

Taking I. L. T. on both sides,

$$L^{-1}\{\bar{y}\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{1}{4}L^{-1}\left\{\frac{e^{-s}}{s}\right\} + \frac{1}{4}L^{-1}\left\{\frac{se^{-s}}{s^2+2^2}\right\}$$
$$+ L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}\,u(t-1) + \frac{1}{4}\cos 2(t-1)\ u(t-1)$$
$$+ \frac{\sin 2t}{2}$$

$$\therefore\ \mathbf{y(t) = \frac{1}{4} - \frac{\cos 2t}{4} + \frac{\sin 2t}{2} - \frac{u(t-1)}{4} + \frac{\cos 2(t-1)u(t-1)}{4}}$$

16 Simultaneous L. D. E. with constant coefficient by L. T.

Let x and y be dependend variable occurring in the simultaneous system and 't' independent variable.

Steps:

1) Given the D. E. Ex. $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$

Or Ex. $D^2x + Dy + ay = 0$ and $D^2y + Dx + by = 0$

2) Take L. T. of given D. E.

3) Using initial conditions, we get two simultaneous equation in $\bar{x}$ & $\bar{y}$

4) Solve these equation simultaneously we get $\bar{x}$ & $\bar{y}$

5) Take I. L. T. of $\bar{x}$ & $\bar{y}$ we get solution $x(t)$ & $y(t)$

16.i Examples on Simultaneous L. D. E. with constant coefficient by L. T

Example 151: Solve $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$ using L. T. being given $x(0) = 8$, $y(0) = 3$

Solution: Given, $\frac{dx}{dt} = 2x - 3y$; $\frac{dy}{dt} = y - 2x$

Taking L. T. on both sides,

$$L\left\{\frac{dx}{dt}\right\} = 2\,L\{x\} - 3L\{y\}$$

$$L\left\{\frac{dy}{dt}\right\} = L\{y\} - 2L\{x\}$$

$$s\,\bar{x} - x(0) = 2\bar{x} - 3\bar{y} \qquad\qquad s\,\bar{y} - y(0) = \bar{y} - 2\bar{x}$$

$$s\,\bar{x} - 8 = 2\bar{x} - 3\bar{y} \qquad\qquad s\,\bar{y} - 3 = \bar{y} - 2\bar{x}$$

$$(s-2)\bar{x} + 3\bar{y} = 8 \quad \ldots\ldots(1) \qquad 2\bar{x} + (s-1)\bar{y} = 3 \quad \ldots\ldots(2)$$

Now,

Equation (1) × (s − 1) $\quad (s-1)(s-2)\,\bar{x} + 3(s-1)\,\bar{y} = 8(s-1)$

Equation (2) × 3 $\quad 6\,\bar{x} + 3(s-1)\,\bar{y} = 9$

Subtrating $\quad - \quad - \quad\quad -$

$$[(s-1)(s-2) - 6]\,\bar{x} = 8s - 8 - 9$$

$$\therefore (s^2 - 3s + 2 - 6)\,\bar{x} = 8s - 17$$

$$\therefore \bar{x} = \frac{8s-17}{s^2 - 3s - 4} = \frac{8s-17}{(s+1)(s-4)}$$

$\therefore$ By partial fraction, $\bar{x} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$

Taking inverse L. T. on both sides.

$$L^{-1}(\bar{x}) = 5L^{-1}\left\{\frac{1}{s+1}\right\} + 3\,L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$\mathbf{x(t) = x = 5e^{-t} + 3e^{4t}}$$

Now,

$$\begin{array}{lr} \text{Equation}(1)\times 2 & 2(s-2)\,\bar{x} + 6\bar{y} = 16 \\ \text{Equation}(2)\times(s-2) & 2(s-2)\,\bar{x} + (s-1)(s-2)\bar{y} = (s-2)3 \\ \text{Subtrating} & - \quad\quad - \quad\quad\quad - \\ \hline & [\,6-(s-1)(s-2)]\,\bar{y} = 16-3(s-2) \end{array}$$

$$[\,6-(s^2-3s+2)]\,\bar{y} = 16-3s+6$$

$$(6-s^2+3s-2)\,\bar{y} = 22-3s$$

$$(-s^2+3s+4)\,\bar{y} = 22-3s$$

$$(s^2-3s-4)\,\bar{y} = -22+3s$$

$$\bar{y} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)}$$

By partial fraction, $\bar{y} = \dfrac{3s-22}{(s+1)(s-4)} = \dfrac{5}{s+1} + \dfrac{(-2)}{s-4}$

Taking Inverse L. T. on both sides.

$$L^{-1}\{\bar{y}\} = 5\,L^{-1}\left\{\frac{1}{s+1}\right\} - 2\,L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$\mathbf{y(t) = y = 5e^{-t} - 2e^{4t}}$$

$$\therefore\ \mathbf{x = 5e^{-t} + 3e^{4t}\ \text{and}\ y = 5e^{-t} - 2e^{4t}}$$

Example 152: Solve the simultaneous equations $\mathbf{\dfrac{dx}{dt} + 5x - 2y = t,,\ \ \dfrac{dy}{dt} + 2x + y = 0}$ **being given** $\mathbf{x = y = 0}$ **when** $\mathbf{t = 0}$

Solution: Given, $\dfrac{dy}{dt} + 2x + y = 0,\qquad \dfrac{dx}{dt} + 5x - 2y = t$

Taking L. T. on both sides

$$L\left\{\frac{dy}{dt} + 2x + y\right\} = L\{0\} \qquad L\left\{\frac{dx}{dt} + 5x - 2y\right\} = L\{t\}$$

$$L\left\{\frac{dy}{dt}\right\} + 2L\{x\} + L\{y\} = 0 \qquad L\left\{\frac{dx}{dt}\right\} + 5L\{x\} - 2L\{y\} = \frac{1}{s^2}$$

$$s\,\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \qquad s\,\bar{x} - x(0) + 5\bar{x} - 2\bar{y} = \frac{1}{s^2}$$

Given, $x = 0,\ y = 0$ when $t = 0$; $\therefore\ x(0) = 0,\quad y(0) = 0$

$$(s+1)\bar{y} + 2\bar{x} = 0 \quad \ldots\ldots(1)$$

$$-2\bar{y} + (s+5)\bar{x} = \frac{1}{s^2} \quad \ldots\ldots(2)$$

Equation (1) × 2 $\quad 2(s+1)\bar{y} + 4\bar{x} = 0$

Equation (2) × (s + 1) $\quad -2(s+1)\bar{y} + (s+1)(s+5)\bar{x} = \dfrac{s+1}{s^2}$

Adding

$$[4+(s+1)(s+5)]\bar{x} = \frac{s+1}{s^2}$$

$$(4+s^2+6s+5)\bar{x} = \frac{s+1}{s^2}$$

$$(s^2+6s+9)\bar{x} = \frac{s+1}{s^2}$$

$$\therefore \bar{x} = \frac{s+1}{s^2(s^2+6s+9)} = \frac{s+1}{s^2(s+3)^2}$$

By partial fraction,

$$\bar{x} = \frac{s+1}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2} \qquad \ldots\ldots(i)$$

Multiplying both sides by $s^2(s+3)^2$

$$s+1 = As(s+3)^2 + B(s+3)^2 + Cs^2(s+3) + Ds^2$$

$$s+1 = As(s^2+6s+9) + B(s^2+6s+9) + Cs^3 + 3Cs^2 + Ds^2$$

$$s+1 = As^3 + 6As^2 + 9As + Bs^2 + 6Bs + 9B + Cs^3 + 3Cs^2 + Ds^2$$

$$s+1 = (A+C)s^3 + (6A+B+3C+D)s^2 + (9A+6B)s + 9B$$

Equating coefficient on both sides,

Coefficient of s^3 → $A + C = 0$ $\quad\ldots\ldots(3)$

Coefficient of s^2 → $6A + B + 3C + D = 0$ $\quad\ldots\ldots(4)$

Coefficient of s → $9A + 6B = 1$ $\quad\ldots\ldots(5)$

Constant term → $9B = 1$ $\quad\ldots\ldots(6)$

$$\mathbf{B = \frac{1}{9}}$$

Equation (5) → $9A + 6.\frac{1}{9} = 1$; $9A = 1 - \frac{2}{3} = \frac{1}{3}$; $\quad 9A = \frac{1}{3}$;

$$\mathbf{A = \frac{1}{27}}$$

Equation (3) → $\mathbf{C = \dfrac{-1}{27}}$

Equation (4) → $6.\frac{1}{27} + \frac{1}{9} + 3\left(\frac{-1}{27}\right) + D = 0$; $\quad \frac{2}{9} + \frac{1}{9} - \frac{1}{9} + D$

$= 0$; $\quad \mathbf{D = \dfrac{-2}{9}}$

Now, equation (i) → $\bar{x} = \dfrac{\frac{1}{27}}{s} + \dfrac{\frac{1}{9}}{s^2} + \dfrac{-\frac{1}{27}}{s+3} + \dfrac{-\frac{2}{9}}{(s+3)^2}$

Taking inverse L. T. on both sides,

$$L^{-1}\{\bar{x}\} = x = \frac{1}{27}L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{9}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{27}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9}L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$\mathbf{x = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}}$$

Now,

Substituting the value of $\bar{x}$ in equation (2)

$$-2\bar{y} + (s+5)\frac{(s+1)}{s^2(s+3)^2} = \frac{1}{s^2}$$

$$-2\bar{y} = \frac{1}{s^2} - \frac{(s+1)(s+5)}{s^2(s+3)^2} = \frac{(s+3)^2 - (s+1)(s+5)}{s^2(s+3)^2}$$

$$= \frac{s^2+6s+9-s^2-6s-5}{-2s^2(s+3)^2}$$

$$\bar{y} = \frac{4}{-2s^2(s+3)^2} = \frac{-2}{s^2(s+3)^2}$$

By partial fraction,

$$\bar{y} = \frac{-2}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2} \qquad \dots\dots(ii)$$

Multiplying both sides by $s^2(s+3)^2$

$$-2 = As(s+3)^2 + B(s+3)^2 + Cs^2(s+3) + Ds^2$$

$$-2 = As(s^2+6s+9) + B(s^2+6s+9) + Cs^3 + 3Cs^2 + Ds^2$$

$$-2 = As^3 + 6As^2 + 9As + Bs^2 + 6Bs + 9B + Cs^3 + 3Cs^2 + Ds^2$$

$$-2 = (A+C)s^3 + (6A+B+3C+D)s^2 + (9A+6B)s + 9B$$

Equating coefficient on both sides,

Coefficient of s^3 → $A + C = 0$ …… (7)

Coefficient of s^2 → $6A + B + 3C + D = 0$ …… (8)

Coefficient of s → $9A + 6B = 0$ …… (9)

Constant term → $9B = -2$ …… (10)

$$\mathbf{B = -\frac{2}{9}}$$

Equation (9) → $9A + 6\left(\frac{-2}{9}\right) = 0;\ \ 9A - \frac{4}{3} = 0;\ \ 9A = \frac{4}{3}\ ;\ \ \mathbf{A = \frac{4}{27}}$

Equation (7) → $\mathbf{C = -\frac{4}{27}}$

Equation (8) → $6.\frac{4}{27} + \left(\frac{-2}{9}\right) + 3\left(\frac{-4}{27}\right) + D = 0;$

$$\frac{8}{9} - \frac{2}{9} - \frac{4}{9} + D = 0;\ \ \mathbf{D = \frac{-2}{9}}$$

Now, Equation (ii) → $\bar{y} = \frac{\frac{4}{27}}{s} + \frac{-\frac{2}{9}}{s^2} + \frac{-\frac{4}{27}}{s+3} + \frac{-\frac{2}{9}}{(s+3)^2}$

Taking inverse L. T. on both sides,

$$L^{-1}\{\bar{y}\} = \frac{4}{27}L^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{9}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{4}{27}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9}L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$\mathbf{y = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}t\,e^{-3t}}$$

$$\therefore\ \mathbf{x = \frac{1}{27} + \frac{t}{9} - \frac{e^{-3t}}{27} - \frac{2te^{-3t}}{9}\ \ \text{and}\ \ y = \frac{4}{27} - \frac{2t}{9} - \frac{4e^{-3t}}{27} - \frac{2te^{-3t}}{9}}$$

Example 153: Solve using L. T. $\frac{dx}{dt} - y = e^t,\ \ \frac{dy}{dt} + x = \sin t$ with $y(0) = 0,\ \ x(0) = 1$

Solution: Given, $\frac{dx}{dt} - y = e^t\ ;\ \ \frac{dy}{dt} + x = \sin t$

Taking L. T. on both sides,

$L\left\{\frac{dx}{dt}\right\} - L\{y\} = L\{e^t\}$ $\qquad$ $L\left\{\frac{dy}{dt}\right\} + L\{x\} = L\{\sin t\}$

$s\,\bar{x} - x(0) - \bar{y} = \frac{1}{s-1}$ $\qquad$ $s\,\bar{y} - y(0) + \bar{x} = \frac{1}{s^2+1}$

$s\,\bar{x} - \bar{y} = \frac{1}{s-1} + 1$ $\qquad$ $s\,\bar{y} + \bar{x} = \frac{1}{s^2+1}$

$s\,\bar{x} - \bar{y} = \frac{1+s-1}{s-1}$ $\qquad$ $\bar{x} + s\,\bar{y} = \frac{1}{s^2+1}$

$s\,\bar{x} - \bar{y} = \frac{s}{s-1}$ ......(1) $\qquad$ $\bar{x} + s\,\bar{y} = \frac{1}{s^2+1}$ ......(2)

Now,

$$\text{Equation (1)} \times s \qquad s^2\,\bar{x} - s\,\bar{y} = \frac{s^2}{s-1}$$

$$\text{Equation (2)} \qquad \bar{x} + s\,\bar{y} = \frac{1}{s^2+1}$$

Adding

$$(s^2+1)\,\bar{x} = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$\therefore \quad \bar{x} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} \qquad \ldots\ldots(3)$$

Let $\bar{x}_1 = \dfrac{s^2}{(s-1)(s^2+1)}$

By partial fraction,

$$\bar{x}_1 = \frac{s^2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \qquad \ldots\ldots(4)$$

Multiplying both sides by $(s-1)(s^2+1)$

$$s^2 = A(s^2+1) + (Bs+C)(s-1)$$

$$s^2 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$s^2 = (A+B)s^2 + (-B+C)s + (A-C)$$

Equating coefficient on both sides.

Coefficient of $s^2 \rightarrow \quad A + B = 1 \qquad \ldots\ldots(5)$

Coefficient of $s \rightarrow \quad -B + C = 0 \qquad \ldots\ldots(6)$

Constant terms $\rightarrow \quad A - C = 0 \qquad \ldots\ldots(7)$

Equation (5) $\rightarrow \quad B + C = 1 \qquad \ldots\ldots(8)$

Adding equation (6) & (8) $-B + C + B + C = 0 + 1;$

$$2C = 1\,; \quad \mathbf{C} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2}}\,; \quad \mathbf{B} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$\text{Equation (4)} \rightarrow \quad \bar{x}_1 = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1}$$

$$\text{Equation (3)} \rightarrow \quad \bar{x} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} + \frac{1}{(s^2+1)^2}$$

Taking inverse L. T. on both sides,

$$L^{-1}\bar{x} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$x = \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}(\sin t - t\cos t)$$

$$\left[\text{w.k.t.} L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)\right.$$

$$x = \frac{1}{2}[e^t + \cos t + \sin t + \sin t - t\cos t]$$

$$\mathbf{x = \frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t]}$$

Now, Substitute $\bar{x}$ from equation(3) to equation(2)

Equation (2) → $s\,\bar{y} = \frac{1}{s^2+1} - \frac{s^2}{(s-1)(s^2+1)} - \frac{1}{(s^2+1)^2}$

$$\bar{y} = \frac{1}{s(s^2+1)} - \frac{s}{(s-1)(s^2+1)} - \frac{1}{s(s^2+1)^2}$$

$$= \frac{s^2+1-1}{s(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

$$\bar{y} = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)} \quad \ldots\ldots(9)$$

Take, $\bar{y}_1 = \frac{s}{(s-1)(s^2+1)}$

By partial fraction, $\bar{y}_1 = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \quad \ldots\ldots(10)$

Multiplying both sides by $(s-1)(s^2+1)$

$$s = A(s^2+1) + (Bs+C)(s-1)$$

$$s = As^2 + A + Bs^2 - Bs + Cs - C$$

$$s = (A+B)s^2 + (-B+C)s + (A-C)$$

Equating coefficient on both sides,

Coefficient of s^2 → $A + B = 0$ ……(11)

Coefficient of s → $-B + C = 1$ ……(12)

Constant term → $A - C = 0$ ……(13)

Equation (11) → $C + B = 0$ ……(14)

Adding equation (12) and (14) $-B + C + C + B = 1 + 0$

$$2C = 1, \quad \mathbf{C = \frac{1}{2}}; \quad \mathbf{A = \frac{1}{2}}; \quad \mathbf{B = -\frac{1}{2}}$$

Equation (10) → $\bar{y}_1 = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}$

Equation (9) → $\bar{y} = \dfrac{s}{(s^2+1)^2} - \left[\dfrac{\frac{1}{2}}{s-1} + \dfrac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}\right]$

Taking inverse L. T. on both sides,

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$y = \frac{1}{2}\ t\ \sin t - \frac{1}{2}\ e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

$$\mathbf{y = \frac{1}{2}\ [t\ sin t - e^t + cos\ t - sin\ t]}$$

$$\therefore\ \mathbf{x = \frac{1}{2}[e^t + cos\ t + 2\ sin\ t - t\ cos\ t]} \quad \textbf{and}$$

$$\mathbf{y = \frac{1}{2}[t\ sin t - e^t + cos\ t - sin\ t]}$$

Example 154: **Solve using L. T.** $\mathbf{(D^2 - 3)x - 4y = 0}$, $\mathbf{x + (D^2 + 1)y = 0}$, **With** $\mathbf{x(0) = y(0) = y'(0) = 0,\ x'(0) = 2}$

Solution: Given,

$$(D^2 - 3)x - 4y = 0 \quad \ldots\ldots(1)$$

$$x + (D^2 + 1)y = 0 \quad \ldots\ldots(2)$$

Now, Taking L. T. on both sides of equation (1) & (2)

Equation (1) → $L\{D^2x - 3x - 4y\} = L\{0\}$

$$[s^2\,\bar{x} - s\,x(0) - x'(0)] - [3\bar{x} - x(0)] - 4\bar{y} = 0$$

$$s^2\,\bar{x} - 2 - 3\bar{x} - 4\bar{y} = 0 \ ; \quad (s^2 - 3)\bar{x} - 4\bar{y} = 2 \quad \ldots\ldots(3)$$

Equation (2) → $L\{x + D^2y + y\} = L\{0\}$

$$\bar{x} + [s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = 0$$

$$\bar{x} + s^2\bar{y} + \bar{y} = 0 \ ; \quad \bar{x} + (s^2+1)\bar{y} = 0 \quad \ldots\ldots(4)$$

Now,

Equation (3) × (s^2+1) $\quad (s^2+1)(s^2-3)\bar{x} - 4(s^2+1)\bar{y} = 2(s^2+1)$

Equation (4) × 4 $\quad 4\,\bar{x} + 4(s^2+1)\bar{y} = 0$

Adding

$$[(s^2+1)(s^2-3) + 4]\bar{x} = 2(s^2+1)$$

$$(s^4 - 2s^2 - 3 + 4)\bar{x} = 2(s^2+1)$$

$$(s^4 - 2s^2 + 1)\bar{x} = 2(s^2+1)$$

$$\bar{x} = \frac{2(s^2+1)}{(s^4-2s^2+1)} = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{2(s^2+1)}{[(s-1)(s+1)]^2}$$

$$\therefore\ \bar{x} = \frac{2(s^2+1)}{(s-1)^2(s+1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2} \qquad \ldots\{\text{Note}$$

Taking inverse L. T. on both sides,

$$L^{-1}\{\bar{x}\} = L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$x = te^t + te^{-t} = t(e^t + e^{-t})$$

$$\mathbf{x = 2\,t\cosh t} \qquad \{\because e^t + e^{-t} = 2\cos ht$$

Now,

Equation (3)	$(s^2-3)\bar{x} - 4\bar{y}$	$= 2$
Equation (4) $\times (s^2-3)$	$(s^2-3)\bar{x} + (s^2+1)(s^2-3)\bar{y}$	$= 0$
Substracting	$- \quad -$	$-$
	$[-4-(s^2+1)(s^2-3)]\bar{y}$	$= 2$

$$\therefore [-4-(s^4-2s^2-3)]\,\bar{y} = 2$$

$$(-4-s^4+2s^2+3)\,\bar{y} = 2$$

$$(-s^4+2s^2-1)\,\bar{y} = 2$$

$$(s^4-2s^2+1)\,\bar{y} = -2$$

$$(s^2-1)^2\bar{y} = -2$$

$$\bar{y} = \frac{-2}{(s^2-1)^2} = \frac{-2}{[(s-1)(s+1)]^2}$$

By partial fraction

$$\bar{y} = \frac{-2}{(s-1)^2(s+1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \qquad \ldots(5)$$

Multiplying both sides by $(s-1)^2(s+1)^2$

$$-2 = A(s-1)(s+1)^2 + B(s+1)^2 + C(s-1)^2(s+1) + D(s-1)^2 \qquad \ldots\ldots(6)$$

Put $s = 1$ in equation(6), $-2 = B(1+1)^2$; $-2 = 4B$; $\mathbf{B = \frac{-1}{2}}$

Put $s = -1$ in equation(6), $-2 = D(-1-1)^2$; $-2 = 4D$;

$$\mathbf{D = \frac{-1}{2}}$$

Put $s = 0$ and put $B = D = \frac{-1}{2}$ in equation(6)

$$-2 = A(-1)(1)^2 + \left(\frac{-1}{2}\right)(1)^2 + C(-1)^2(1) + \left(\frac{-1}{2}\right)(-1)^2$$

$$-2 = -A - \frac{1}{2} + C - \frac{1}{2}\,; \quad -1 = -A + C\,;$$

$$-A + C = -1 \qquad \ldots\ldots(7)$$

Put $s = 2$; $B = D = \dfrac{-1}{2}$ in equation (6)

$$-2 = A(1)(3)^2 + \left(\frac{-1}{2}\right)(3)^2 + C(1)^2(3) + \left(\frac{-1}{2}\right)(1)^2$$

$$-2 = 9A - \frac{9}{2} + 3C - \frac{1}{2}$$

$$-2 = 9A + 3C - 5\,; \quad 9A + 3C = 3\,;$$

$$3A + C = 1 \quad \ldots\ldots(8)$$

Now,

Equation (7) $\quad -A + C = -1$

Equation (8) $\quad 3A + C = 1$

Subtracting $\quad - \quad - \quad -$

$$-4A = -2$$

$$A = \frac{2}{4} = \frac{1}{2}\,; \quad \mathbf{A} = \frac{\mathbf{1}}{\mathbf{2}}$$

Equation (8) → $3.\dfrac{1}{2} + C = 1\,;\ C = 1 - \dfrac{3}{2} = \dfrac{-1}{2}$; $\quad \mathbf{C} = \dfrac{\mathbf{-1}}{\mathbf{2}}$

∴ Equation (5) → $\bar{y} = \dfrac{\frac{1}{2}}{s-1} + \dfrac{-\frac{1}{2}}{(s-1)^2} + \dfrac{-\frac{1}{2}}{s+1} + \dfrac{-\frac{1}{2}}{(s+1)^2}$

Taking inverse L. T. on both sides

$$L^{-1}\{\bar{y}\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$y = \frac{1}{2} e^{t} - \frac{1}{2} te^{t} - \frac{1}{2} e^{-t} - \frac{1}{2} te^{-t}$$

$$= \frac{1}{2}\left[e^{t} - te^{t} - e^{-t} - te^{-t}\right] \qquad = \frac{1}{2}\left[e^{t} - e^{-t} - t(e^{t} + e^{-t})\right]$$

$$= \frac{1}{2}\left[2 \sin h\, t - t\, 2\cos h\, t\right]$$

$$\mathbf{y = \sin h\, t - t \cos h\, t}$$

$$\therefore\ \mathbf{x = 2t \cos h\, t \quad and \quad y = \sin h\, t - t \cos h\, t}$$

Exercise

1: Find the Laplace transforms of

1) $L\{f(t)\} = \begin{cases} 4, & 0 \le t < 1 \\ 3, & t > 1 \end{cases}$

2) $f(x) = \begin{cases} \sin(x - \frac{\pi}{3}), & x > \pi/3 \\ 0, & x < \pi/3 \end{cases}$

3) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t - 1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$

4) If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t}f(2t)]$

5) $L\left\{\frac{(\sin t \sin 5t)}{t}\right\}$

6) $L\left\{\frac{e^{at} - \cos bt}{t}\right\}$

7) $L\left\{2^t + \frac{\cos 2t - \cos 3t}{t} + t.\sin t\right\}$

8) $L\left\{\int_0^\infty \frac{e^{-\sqrt{2}t} \sin h\, t \sin t}{t} dt\right\}$

9) $L\left\{\int_0^\infty te^{-t} \sin^4 t\, dt\right\}$

10) Prove that:

(i) $L\left\{\int_0^\infty \frac{e^{-2t} \sinh t}{t} dt\right\} = \frac{1}{2}\log 3$

(ii) $L\left\{\int_0^\infty \frac{e^{-t} \sin^2 t}{t} dt\right\} = \frac{1}{4}\log 5$

2: Find the Inverse Laplace transforms of

1) $L^{-1}\left\{\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}\right\}$

2) $L^{-1}\left\{\frac{s}{(2s-1)(3s-1)}\right\}$

3) $L^{-1}\left\{\frac{s^2 - 10s + 13}{(s-7)(s^2-5s+6)}\right\}$

4) $L^{-1}\left\{\log\left[\frac{s+1}{(s+2)(s+3)}\right]\right\}$

5) $L^{-1}\left\{\log \frac{s^2+1}{(s-10)^2}\right\}$

6) $L^{-1}\{\cot^{-1}(s)\}$

3: Using Convolution theorem evaluate

1) $L^{-1}\left\{\frac{1}{(s^2+4s+13)^2}\right\}$

2) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

3) $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

4) $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

5) $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$

4: Solve the following Differential equations by Laplace transform method

1) $(D^2 - 1)x = a\cosh t,\ \ x(0) = x'(0) = 0$

2) $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3,\ \ y(0) = 5$

3) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t, \quad y = \dfrac{dy}{dt} = 0$ when $t = 0$

4) $y'' + 2y' + 5y = 5y = 5(t-2), \qquad y(0) = 0, \ y'(0) = 0$

5) $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2e^{2t},$

when $y = 1, \dfrac{dy}{dt} = 0, \dfrac{d^2y}{dt^2} = -2$ at $t = 0$

5: Solve the following Simultaneous equations by using Laplace transforms

1) $\dfrac{dx}{dt} + y = \sin t, \dfrac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$

2) $3\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x = 1. \dfrac{dx}{dt} + 4\dfrac{dy}{dt} + 3y = 0;$ given $x = 0, y = 0$ when $t = 0$

3) $(D-2)x - (D+1)y = 6e^{3t}; \ (2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3, \ y = 0$, when $t = 0$

Answers

1: 1. $\dfrac{4}{s} - \dfrac{e^{-s}}{s}$ 2. $\dfrac{e^{-\frac{\pi s}{3}}}{s^2+1}$

3. $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3(2+3s+3s^2)} + \dfrac{e^{-3s}}{s^2(5s-1)}$

4. $e^{-\frac{2\pi s}{3}} \dfrac{s}{(s-1)(s^2-2s+5)}$ 5. $\dfrac{1}{2}\log\{(s^2+36)(s^2+16)\}$

6. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{(s-a)^2}\right)$ 7. $\dfrac{1}{s-\log 2} + \dfrac{2s}{(s^2+1)^2} + \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$

8. $\dfrac{\pi}{8}$ 9. $\dfrac{8(s+1)}{s(s^2+2s+17)}$

2: 1. $\dfrac{1}{2}\left(\dfrac{\cos 5t}{2} - \dfrac{\sin 5t}{2}\right) - 4\cosh 3t + 6\sinh 3t$

2. $3e^{\frac{t}{2}} + 2e^{\frac{t}{3}}$ 3. $2e^{3t} - \dfrac{3}{5}e^{2t} - \dfrac{2}{5}e^{7t}$

4. $e^{-t} - e^{-2t} - e^{-3t}$ 5. $\dfrac{2}{t}(e^t - \cos t)$ 6. $\dfrac{\sin t}{t}$

3: 1. $\frac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)$ 2. $\frac{e^{-bt} - e^{-at}}{a-b}$ 3. $\frac{t^2}{2} + \cos t - 1$

4. $t(e^{-t}+1) + 2(e^{-t}-1)$ 5. $\frac{1}{16(e^{2t} - e^{-2t} - 4te^{-2t})}$

4: 1. $X = \frac{at}{2}\sinh t$ 2. $Y = 4e^{2t}(1+t) - 7e^{t}$

3. $y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$

4. $y = \frac{-12}{5} + \frac{12}{5}e^{-t}\cos 2t + \frac{7}{10}e^{-t}\sin 2t$

5. $y = e^{2t}(x^2 - 6x + 12) - e^{t}(15x^2 + 7x + 11)$

5: 1. $x = e^{t} + e^{-t}, y = e^{-t} - e^{t} + \sin t$ 3. $x = 2 + \frac{t^2}{2}, y = -1 - \frac{t^2}{2}$

2. $x = \frac{1}{10}(5 - 2e^{-t} - 3e^{6t/11}), y = \frac{1}{5}(e^{-t} - e^{6t/11})$

3. $x = e^{6}(1+2t) + 2e^{3t}$, $y = \sinh t + \cosh t - e^{-3t} - te^{t}$

APPENDIX

USEFUL FORMULAE

I. ALGEBRA

[A] Product Formula: Fundamental Identities

1. $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$
2. $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$
3. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad = a^3 + b^3 + 3ab(a+b)$
4. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad = a^3 - b^3 - 3ab\,(a-b)$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
6. $(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$
7. $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$
8. $(a-b-c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$
9. $(a+b+c)^3 = a^3 + b^3 + c^3 + 3ab(a+b) + 3ac(c+a) + 3bc(b+c) + 6abc$
10. $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
11. $(x+a)(x+b) = x^2 + (a+b)x + ab$
12. $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ac)x + abc$

[B] Binomial Formula:

Note : ${}^nC_0 = {}^nC_n = 1, \; {}^nC_1 = n, \; 1! = 1, \; 0! = 1$

$$(a+b)^n = a^n + na^{n-1}\,b + \frac{n(n-1)}{2!}\,a^{n-2}\,b^2 + \frac{n(n-1)(n-2)}{3!}\,a^{n-3}\,b^3 + \ldots + b^n$$

[C] Laws of Indices:

[I] **Powers**: Bases (positive real numbers) a, b and powers(rational numbers): n, m.

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a \times b)^m = a^m \times b^m$
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
5. $(a^m)^n = a^{m.n}$
6. $a^o = 1$
7. $a^1 = a$

8. $a^{\infty} = \infty$

9. $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$

10. $a^{-m} = \frac{1}{a^m}$

[II] Roots: Bases a, b and Power (rational numbers): n, m , where $a, b \geq 0$

1. $\sqrt{a} = a^{\frac{1}{2}}$
2. $\sqrt[n]{a} = a^{\frac{1}{n}}$
3. $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$
4. $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
6. $\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{a.b}$

[D] Rules of logarithm:

Positive numbers: x, y, a, c, k and Natural number : n

1. $\log_a (x^n) = n \log_a x$
2. $\log_a (xy) = \log_a x + \log_a y$
3. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
4. $\log_a a = 1$
5. $a^{\log_a^x} = x$
6. $\log_a 1 = 0$
7. $\log_a 0 = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}$

Common logarithm to base 10:

$\log_{10} x = \log x$; $\log x = \frac{1}{\ln 10} \ln x = 0.43429 \ln x$

Natural logarithm to base e: $\ln x = \frac{1}{\log e} \log x = 2.30258 \log x$

II. SERIES

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2) $a^x = e^{x \log a} = 1 + x\log a + \frac{(x \log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \cdots$

3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

5) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

6) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

7) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

8) $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

III. TRIGONOMETRIC FORMULAE

[A] Relationship among trigonometric functions

1) $\frac{1}{\sin A} = \text{cosec } A$

2) $\frac{1}{\text{cosec } A} = \sin A$

3) $\frac{1}{\cos A} = \sec A$

4) $\frac{1}{\sec A} = \cos A$

5) $\frac{1}{\tan A} = \cot A$

6) $\frac{1}{\cot A} = \tan A$

7) $\frac{\sin A}{\cos A} = \tan A$

8) $\frac{\cos A}{\sin A} = \cot A$

[B] The circular function formulae by Euler's method

1. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

2. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

3. $\tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$

4. $\cot x = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$

5. $\sec x = \frac{2}{e^{ix} + e^{-ix}}$

6. $\text{cosec } x = \frac{2i}{e^{ix} - e^{-ix}}$

[C] Trigonometric Ratios of Allied Angles:

(**1**) $\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

(**2**) $\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

(**3**) $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

(**4**) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

(**5**) $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

(**6**) $\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

(**7**) $\sin(2\pi + \theta) = \sin\theta$

$\cos(2\pi + \theta) = \cos\theta$

[D] Addition and Subtraction Or Sum Difference Formulae:

1) $\sin(x+y) = \sin x.\cos y + \cos x.\sin y$

2) $\sin(x-y) = \sin x.\cos y - \cos x.\sin y$

3) $\cos(x+y) = \cos x.\cos y - \sin x.\sin y$

4) $\cos(x-y) = \cos x.\cos y + \sin x.\sin y$

5) $\cot(x+y) = \dfrac{\cot x.\cot y - 1}{\cot y + \cot x}$

6) $\cot(x-y) = \dfrac{\cot x.\cot y + 1}{\cot y - \cot x}$

7) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x.\tan y}$

8) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x.\tan y}$

[E] Fundamental OR Pythagoras Identities:

1) $\sin^2 x + \cos^2 x = 1$ 3) $1 + \cot^2 x$

2) $1 + \tan^2 x = \sec^2 x$ $= \text{cosec}^2 x$

[F] Multiple and Sub – Multiple Angle Formulae:

1) $\sin 2x = 2\sin x.\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

2) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

3) $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

$= 1 - 2\sin^2 x = 2\cos^2 x - 1$

4) $\sin 3x = 3\sin x - 4\sin^3 x$ i.e. $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

5) $\cos 3x = 4\cos^3 x - 3\cos x$ i.e. $\cos^3 x = \dfrac{3\cos x + \cos 3x}{4}$

6) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ i.e. $\tan^3 x$

$= 3\tan x - \tan 3x(1 - 3\tan^2 x)$

[G] Factorization OR Sum − To − Product Formulas:

1) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

2) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

3) $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

4) $\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{\mathbf{B}-\mathbf{A}}{\mathbf{2}}\right)$

[H] Defactorization OR Product − To − Sum Formulas:

1) $\sin x.\cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$

2) $\cos x.\sin y = \frac{1}{2}[\sin(x+y)-\sin(x-y)]$

3) $\cos x.\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$

4) $\sin x.\sin y = \frac{-1}{2}[\cos(x+y)-\cos(x-y)]$

[I] Useful Results:

1) $1+\sin x = \left[\cos\frac{x}{2}+\sin\frac{x}{2}\right]^2$

2) $1-\sin x = \left[\cos\frac{x}{2}-\sin\frac{x}{2}\right]^2$

3) $1+\cos x = 2\cos^2\frac{x}{2}$ i.e. $\cos^2 x = \frac{1+\cos 2x}{2}$

4) $1-\cos x = 2\sin^2\frac{x}{2}$ i.e. $\sin^2 x = \frac{1-\cos 2x}{2}$

5) $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

[J] Properties of Inverse Trigonometric Function:

Property 1

$\sin^{-1}(\sin x) = x$

$\cos^{-1}(\cos x) = x$

Property 2

$\sin(\sin^{-1} x) = x$

$\cos(\cos^{-1} x) = x$

Property 3

$\cot^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

$\operatorname{cosec}^{-1}\left(\frac{a}{b}\right) = \sin^{-1}\left(\frac{b}{a}\right)$

$\sec^{-1}\left(\frac{a}{b}\right) = \cos^{-1}\left(\frac{b}{a}\right)$

Property 4

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

Property 5

1) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$ if $xy < 1$

2) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$ if $xy > -1$

Property 6

1) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$

2) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$

3) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$

4) $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

5) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$

6) $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$

IV. HYPERBOLIC FORMULAE

[A] The hyperbolic function formulae by Euler's method

1. $\sinh x = \frac{e^x - e^{-x}}{2}$ 2. $\cosh x = \frac{e^x + e^{-x}}{2}$

[B] Addition and Subtraction Or Sum − Difference Formulae:

1. $\sinh(x+y) = \sinh x . \cosh y + \cosh x . \sinh y$
2. $\sinh(x-y) = \sinh x . \cosh y - \cosh x . \sinh y$
3. $\cosh(x+y) = \cosh x . \cosh y + \sinh x . \sinh y$
4. $\cosh(x-y) = \cosh x . \cosh y - \sinh x . \sinh y$

[C] Fundamental OR Pythagoras Identities:

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \operatorname{sech}^2 x$
3. $\coth^2 x - 1 = \operatorname{cosech}^2 x$

[D] Multiple and Sub − Multiple Angle Formulae:

1. $\sinh 2x = 2\sinh x.\cosh x$
2. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
3. $\tanh 2x = \dfrac{2\tan hx}{1 + \tanh^2 x}$
4. $\sinh 3x = 3\sinh x + 4\sinh^3 x$ i.e. $\sinh^3 x = \dfrac{\sinh 3x - 3\sinh x}{4}$
5. $\cosh 3x = 4\cosh^3 x - 3\cosh x$ i.e. $\cosh^3 x = \dfrac{\cosh 3x + 3\cosh x}{4}$

[E] Defactorization OR Product − To − Sum Formulas:

1. $\sinh x.\cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)]$
2. $\cosh x.\sinh y = \frac{1}{2}[\sinh(x+y) - \sinh(x-y)]$
3. $\cosh x.\cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)]$
4. $\sinh x.\sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$

[F] Factorization OR Sum − To − Product Formulas:

1. $\sinh A + \sinh B = 2\sinh\left(\frac{A+B}{2}\right).\cosh\left(\frac{A-B}{2}\right)$
2. $\sinh A - \sinh B = 2\cosh\left(\frac{A+B}{2}\right).\sinh\left(\frac{A-B}{2}\right)$
3. $\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right).\cosh\left(\frac{A-B}{2}\right)$
4. $\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right).\sinh\left(\frac{A-B}{2}\right)$

[G] Useful Results:

1. $1 + \sinh x = \left[\cosh\frac{x}{2} + \sinh\frac{x}{2}\right]^2$
2. $1 - \sinh x = \left[\cosh\frac{x}{2} - \sinh\frac{x}{2}\right]^2$
3. $1 + \cosh x = 2\cosh^2\dfrac{x}{2}$ i.e. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
4. $\cosh x - 1 = 2\sin^2\dfrac{x}{2}$ i.e. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
5. $\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

[H] Relations between Hyperbolic Funtions

1. $\sinh(-x) = -\sinh x$
2. $\cosh(-x) = \cosh x$
3. $\tanh(-x) = -\tanh x$
4. $\coth(-x) = -\coth x$
5. $\text{sech}(-x) = \text{sech}\, x$
6. $\text{cosech}(-x) = -\text{cosech}\, x$

[I] Relationship between hyperbolic and trigonometric functions

1. $\sin(ix) = i\sinh x$
2. $\cos(ix) = \cosh x$
3. $\tan(ix) = i\tanh x$
4. $\cot(ix) = -i\coth x$
5. $\sec(ix) = \text{sech}\, x$
6. $\text{cosec}(ix) = -i\,\text{cosech}\, x$
7. $\sinh(ix) = i\sin x$
8. $\cosh(ix) = \cos x$
9. $\tanh(ix) = i\tan x$
10. $\text{cosec}(ix) = -i\,\text{cosech}\, x$
11. $\text{sech}(ix) = \sec x$
12. $\cot(ix) = -i\cot x$

V. DERIVATIVES AND INTEGRATION

[A] Product Rule

1) $\frac{d}{dx}(u.v) = u\frac{d}{dx}v + v\frac{d}{dx}u$

2) $\frac{d}{dx}(u.v.w) = u.v\frac{d}{dx}w + u.w\frac{d}{dx}v + v.w\frac{d}{dx}u$

[B] Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$$

[C] Chain Rule

$$\frac{du}{dx} = \frac{du}{dv}.\frac{dv}{dx}$$

[D] Parametric Function

$x = f(t),\ \ y = f(t),$ where t is a parameter. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

[E] Composite function

$$\frac{d}{dx}\sin(\log x) = \cos(\log x)\frac{d}{dx}(\log x) = \cos(\log x).\frac{1}{x}$$

[F] Integration by Parts or LIATE rule

L = Logarithmic function **Ex.** Log x, $\log(x^2 + 1)$ etc.

I = Inverse trigonometric function **Ex.** $\sin^{-1}x, \tan^{-1}x$ etc.

A = Algebraic function **Ex.** $x^2 + 1, x + 1, x^a$ etc

T = Trigonometric function **Ex.** sin x, cos x, cos sec x etc.

E = Exponential function **Ex.** $e^x, a^x, 5^x$ etc.

$$\int u.v\ dx = u\int v\ dx - \int\left[\frac{d}{dx}u\int v\ dx\right]dx$$

$$\int_a^b u.v.dx = \left[u\int v\ dx\right]_a^b - \int_a^b\left[\frac{d}{dx}u\int v\ dx\right]dx$$

[G] Useful Result:

1. $\int e^x\,[f(x) + f'(x)dx = e^x.f(x) + c$
2. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$
3. $\int [f(x)]^n\, f'(x) = \frac{f(x)^{n+1}}{n+1} + c$
4. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$
5. $\int e^{f(x)}\, f'(x)\,dx = e^{f(x)} + c$
6. $\int a^{f(x)}.f'(x)\,dx = \frac{a^{f(x)}}{\log a} + c$

[H] CALCULUS FORMULAE [Circular Functions]

Sr. No.	DERIVATIVES	INTEGRATION (ANTI – DERIVATIVES)
1.	$\frac{d}{dx}k = 0$	$\int k\ dx = k.x + c$
2.	$\frac{d}{dx}x = 1$	$\int 1\ dx = x + c$
3.	$\frac{d}{dx}(x)^n = nx^{n-1}$	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$

4. $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$ $\quad\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$

5. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ $\quad\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$

6. $\frac{d}{dx}(\log x) = \frac{1}{x}$ $\quad\int \frac{1}{x}\,dx = \log x + c$

7. $\frac{d}{dx}(e^x) = e^x$ $\quad\int e^x\,dx = e^x + c$

8. $\frac{d}{dx}(a^x) = a^x(\log a)$ $\quad\int a^x\,dx = \frac{a^x}{\log a} + c$

9. $\frac{d}{dx}(\sin x) = \cos x$ $\quad\int \cos x\,dx = \sin x + c$

10. $\frac{d}{dx}(\cos x) = -\sin x$ $\quad\int \sin x\,dx = -\cos x + c$

11. $\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad\int \sec^2 x\,dx = \tan x + c$

12. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\quad\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

13. $\frac{d}{dx}(\sec x) = \sec x.\tan x$ $\quad\int \sec x.\tan x\,dx = \sec x + c$

14. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$ $\quad\int \operatorname{cosec} x.\cot x\,dx = -\operatorname{cosec} x + c$

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

16. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$

17. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

18. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$ $\quad\int \frac{-1}{1+x^2}\,dx = \cot^{-1} x + c$

19. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\quad\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

20. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$ $\int \frac{-1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + c$

21. $\int \tan x\, dx = \log(\sec x) + c$

22. $\int \cot x\, dx = \log(\sin x) + c$

23. $\int \sec x\, dx = \log(\sec x + \tan x) + c$

24. $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + c$

25. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

27. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

28. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

29. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a + x}{a - x}\right| + c$

30. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + c$

31. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

32. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$

33. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2 + x^2}\right) + c$

34. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$

35. $\int e^{-ax} \sin bx\, dx = \frac{e^{-ax}}{a^2 + b^2}(-a\sin bx - b\cos bx) + c$

36. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

37. $\int e^{-ax}\cos bx\,dx = \dfrac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx) + c$

[I] Properties of Definite Integrals

Property 1 : $\int_a^b f(x)dx = \int_a^b f(t)dt$

Property 2 : $\int_a^b f(x)dx = -\int_b^a f(x)dx$ …… $\int_a^a f(x)dx = 0$

Property 3 : $\int_a^b f(x)\,dx = \int_c^b f(x)\,dx + \int_c^b f(x)\,dx$

Property 4 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Property 5 : $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Property 6 : $\int_0^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx$

Property 7:

(i) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx,$ if f is an even function,

i.e. if $f(-x) = f(x)$

(ii) $\int_{-a}^a f(x)dx = 0,$ if f is an odd function,

i.e. if $f(-x) = -f(x)$