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Real and Complex Matrices & Linear System of Equations

1.1 REVIEW

The student is already familier with the definition and properties of matrices. Matrix is an inevitable tool in the study of many subjects like Physics, Mechanics, Statistics, Electronic circuits and Computers. Here we will briefly review some definitions and properties of matrices.

Matrix Definition: A system of mn numbers (real or complex) arranged in the form of an ordered set of m rows, each row consisting of an ordered set of n numbers between [ ] or ( ) or || || is called a matrix of order or type $m \times n$.

Each of mn numbers constituting the $m \times n$ matrix is called an element of the matrix.

Thus we write a matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}_{m\times n} = \left[a_{ij}\right]_{m\times n}, \text{ where } 1 \le i \le m,\ 1 \le j \le n$$

In relation to a matrix, we call the numbers as scalars.

1.2 TYPES OF MATRICES

Definitions :

1. If $A = [a_{ij}]_{m \times n}$ and $m = n$, then A is called a ***Square matrix.*** A square matrix A of order $n \times n$ is sometimes called as a ***n*-rowed matrix** A or simply a square matrix of order n.

   e.g. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2nd order matrix.

2. A matrix which is not a square matrix is called a ***Rectangular matrix.***

   e.g. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2×3 matrix.

3. A matrix of order $1 \times m$ is called a ***Row matrix.***

   e.g. $[1 \quad 2 \quad 3]_{1 \times 3}$

4. A matrix of order $n \times 1$ is called a ***Column matrix.***

   e.g. $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3\times 1}$

   Row and Column matrices are also called as ***Row*** and ***Column vectors*** respectively.

5. If $A = [a_{ij}]_{n\times n}$ such that $a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, then A is called a ***Unit matrix***. It is denoted by I_n.

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. If $A = [a_{ij}]_{m \times n}$ such that $a_{ij} = 0 \ \forall \ i$ and j, then A is called a ***Zero matrix*** or a ***Null matrix***. It is denoted by O or more clearly $O_{m \times n}$.

e.g. $O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

7. Diagonal Elements of a Square matrix and Principal Diagonal :

Definition : In a matrix $A = [a_{ij}]_{n \times n}$, the elements a_{ij} of A for which $i = j$ (*i.e.* $a_{11}, a_{22}, \ldots, a_{nn}$) are called the ***diagonal elements of A***. The line along which the diagonal elements lie is called the ***principal diagonal* of *A***.

8. A square matrix all of whose elements except those in leading diagonal are zero is called ***Diagonal matrix.*** If $d_1, d_2, \ldots, d_n$ are diagonal elements of a diagonal matrix A, then A is written as $A = diag(d_1, d_2, \ldots, d_n)$.

e.g. $\text{A} = \text{diag}(3, 1, -2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

9. A diagonal matrix whose leading diagonal elements are equal is called a ***Scalar matrix***.

e.g. $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

10. Equal Matrices :

Definition: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if

(*i*) A and B are of the same type (or oder) and (*ii*) $a_{ij} = b_{ij}$ for every i and j.

Algebra of Matrices :

11. Addition of two matrices :

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ be two matrices. The matrix $C = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$, is called the sum of the matrices A and B. The sum of A and B is denoted by $A + B$.

Thus $[a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$ and $[a_{ij} + b_{ij}]_{m \times n} = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$

12. Difference of Two Matrices :

If A, B are two matrices of the same type (order) then $A + (-B)$ is taken as $A - B$.

13. Multiplication of a Matrix by a Scalar :

Let A be a matrix. The matrix obtained by multiplying every element of A by K, a scalar, is called the product of A by K and is denoted by KA or AK.

Thus if $A = [a_{ij}]_{m \times n}$, then

$KA = [Ka_{ij}]_{m \times n}$ and $[Ka_{ij}]_{m \times n} = K\,[a_{ij}]_{m \times n} = KA$.

Properties :

(*i*) $OA = O$ (null matrix), $(-1)A = -A$, called the negative of A.

(*ii*) $K_1(K_2 A) = (K_1K_2)A = K_2(K_1A)$ where K_1, K_2 are scalars.

(*iii*) $KA = O \Rightarrow A = O$ if $K \neq 0$.

(*iv*) $K_1 A = K_2 A$ and A is not a null matrix $\Rightarrow K_1 = K_2$.

14. Matrix Multiplication :

Let $A = [a_{ik}]_{m \times n}$ and $B = [b_{kj}]_{n \times p}$. Then the matrix $C = [c_{ij}]_{m \times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ is called the **product of the matrices A and B** in that order and we write $C = AB$.

In the product AB, the matrix A is called the **pre-factor** and B the **post-factor**.

If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

15. Positive Integral Powers of Square Matrices :

Let A be a square matrix. Then A^2 is defined as $A.A$. Now, by the Associative law,

$A^2 A = (AA)A = A(AA) = AA^2$ so that we write

$A^2 A = AA^2 = AAA = A^3$

Similarly we have $AA^{m-1} = A^{m-1} A = A^m$, where m is a positive integer.

Further we have $A^m A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$ where m, n are positive integers.

Note : $I^n = I$, $O^n = O$

Theorem 1: Matrix multiplication is associative.

i.e., if A, B, C are matrices, then $(AB)C = A(BC)$.

Proof: Let $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$ and $C = [c_{kl}]_{p \times q}$

Then $AB = [u_{ik}]_{m \times p}$, where $u_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ ... (1)

Also $BC = [v_{jl}]_{n \times q}$, where $v_{jl} = \sum_{k=1}^{p} b_{jk} c_{kl}$ ... (2)

Now, $A(BC)$ is an $m \times q$ matrix and $(AB)C$ is also an $m \times q$ matrix.

Let $A(BC) = [w_{il}]_{m \times q}$ where w_{il} is the $(i, j)^{\text{th}}$ element of $A(BC)$.

Then $$w_{il} = \sum_{j=1}^{n} a_{ij} v_{jl} = \sum_{j=1}^{n} \left[a_{ij} \left\{ \sum_{k=1}^{p} b_{jk} c_{kl} \right\} \right] \quad \text{[by (2)]}$$

$$= \sum_{k=1}^{p} \left[\left\{ \sum_{j=1}^{n} a_{ij} b_{jk} \right\} c_{kl} \right] \quad [\because \text{Finite summations can be interchanged}]$$

$$= \sum_{k=1}^{p} u_{ik} c_{kl} \quad \text{[From (1)]}$$

$= \text{The } (i, j)^{\text{th}}$ element of $(AB)C$

Hence, by the equality of two matrices, we have

$A(BC) = (AB)C$

Note : $(AB)C = A(BC) = ABC$

Theorem 2: Multiplication of matrices is distributive w.r.t. addition of matrices.

i.e., $A(B + C) = AB + AC$ and $(B + C) A = BA + CA$

Note : $A(B - C) = AB - AC$ and $(B - C) A = BA - CA$

Theorem 3: If A is a matrix of order $m \times n$, then $AI_n = I_n A = A$.

16. Trace of A Square Matrix :

Let $A = [a_{ij}]_{n \times n}$. Then trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$ and is denoted by '$tr\ (A)$'.

Thus $\text{tr}\,(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$

Properties : If A and B are square matrices of order n and λ is any scalar, then

(*i*) $\text{tr}\,(\lambda A) = \lambda\ \text{tr}\ A$.

(*ii*) $\text{tr}\,(A + B) = \text{tr}\ A + \text{tr}\ B$

(*iii*) $\text{tr}\,(AB) = \text{tr}\,(BA)$

17. Triangular Matrix :

A square matrix all of whose elements below the leading diagonal are zero is called an ***Upper Triangular matrix***. A square matrix all of whose elements above the leading diagonal are zero is called a ***Lower Triangular matrix.***

e.g. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ is an *upper triangular matrix*

and $\begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 & 0 \\ 2 & 1 & -8 & 5 & 0 \\ 2 & 0 & 4 & 1 & 6 \end{bmatrix}$ is a *lower triangular matrix.*

18. If A is a square matrix such that $A^2 = A$ then A is called **Idempotent.**

19. If A is a square matrix such that $A^m = O$ where m is a positive integer, then A is called **'Nilpotent'**. If m is least positive integer such that $A^m = O$ then A is called **'Nilpotent' of index *m*.**

20. If A is a square matrix such that $A^2 = I$ then A is called **Involutory.**

21. The Transpose of a Matrix :

Definition: The matrix obtained from any given matrix A, by inter changing its rows and columns is called *the* ***Transpose of A***. It is denoted by A' or A^T.

If $A = [a_{ij}]_{m \times n}$, then the transpose of A is $A' = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$

Also $(A')' = A$

Note : If A' and B' be the transposes of A and B respectively, then

(*i*) $(A')' = A$

(*ii*) $(A + B)' = A' + B'$, A and B being of the same order.

(*iii*) $(KA)' = KA'$, K is a scalar.

(*iv*) $(AB)' = B'A'$, A and B being conformable for multiplication.

Determinants :

22. Minors and Cofactors of a Square Matrix :

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. When from A the elements of ith row and jth column are deleted the determinant of $(n - 1)$ rowed matrix M_{ij} is called the **minor** of a_{ij} of A and is denoted by $|M_{ij}|$. The signed minor $(-1)^{i+j} |M_{ij}|$ is called the **cofactor** of a_{ij} and is denoted by A_{ij}.

Thus If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Note 1 : Determinant of the square matrix A can be defined as

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

or $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} = a_{12}A_{12} + a_{22}A_{2} + a_{32}A_{32}$

$$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

Therefore, in a determinant the sum of the products of the elements of any row or column with their corresponding co-factors is equal to the value of the determinant.

Note 2. If A is a square matrix of order n then $|KA| = k^n|A|$, where k is a scalar.

Note 3. If A is a square matrix of order n, then

$$|A| = |A^T|$$

Note 4. If A and B be two square matrices of the same oder, then $|AB| = |A|.|B|$

23. Adjoint of a square matrix : Let A be a square matrix of order n. The transpose of the matrix got from A by replacing the elements of A by the corresponding cofactors is called the adjoint of A and is denoted by **adj A.**

Note : For any scalar k, adj$(kA) = k^{n-1}$ adj A.

24. Singular and Non-Singular Matrices :

Definition : A square matrix A is said to be **singular** if $|A| = 0$. If $|A| \neq 0$, then A is said to be **non-singular.** Thus only non-singular matrices possess inverses.

Note : If A, B are non-singular, then AB, the product is also non-singular. Thus the product of non-singular matrices is also non-singular.

25. Inverse of a Matrix :

Let A be any square matrix, then a matrix B, if exists such that $AB = BA = I$, then B is called **inverse** of A and is denoted by A^{-1}.

Note. For AB, BA to be both defined and equal, it is necessary that A and B are both square matrices of same order. Thus a non-square matrix cannot have inverse.

26. Invertible :

A matrix is said to be **invertible,** if it posseses inverse.

Theorem 4: Every invertible matrix Possesses a unique inverse.

(or) The inverse of a matrix if it exists is unique.

Proof: Let if possible, B and C be the inverses of A. Then

$$AB = BA = I \quad \dots (1)$$

and $$AC = CA = I \quad \dots (2)$$

Now $B = BI = B(AC) = (BA)C = IC = C$

$\therefore$ $B = C$

Hence there is only one inverse of A, which is denoted by A^{-1}.

Note 1 : The inverse of A is denoted by A^{-1}.

Thus $AA^{-1} = A^{-1}A = I$

Also $A^{-1}A = AA^{-1} = I \Rightarrow (A^{-1})^{-1} = A$

$\Rightarrow$ The inverse of matrix is invertible and the inverse of the inverse of the matrix is itself.

Note 2. Since $I_n I_n = I_n$. we have $I^{-1} = I$ *i.e.*, the inverse of a unit matrix is itself.

Note 3. If A is an invertible matrix and if $A = B$ then $A^{-1} = B^{-1}$.

Theorem 5: The necessary and sufficient condition for a square matrix to possess inverse is that $|A| \neq 0$

Note : If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adj\, A)$

Theorem 6: If A is a $m \times n$ matrix and B is a $n \times p$ matrix then $(AB)' = B'A'$.

Corollary : $(ABC...Z)' = Z'Y'...C'B'A'$

Theorem 7 : If A, B are invertible matrices of the same order, then

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A')^{-1} = (A^{-1})'$

Proof: (i) We have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(IB) = B^{-1}B = I.$$

Similarly $(AB)(B^{-1}A^{-1}) = I.$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Note 1 : $(B^{-1}A^{-1})^{-1} = AB.$

Note 2 : $(A.B.......Z)^{-1} = Z^{-1}......B^{-1}A^{-1}.$

(ii) We have $A^{-1}A = AA^{-1} = I$

$\therefore$ $(A^{-1}A)' = (AA^{-1})' = I'$

$\Rightarrow$ $A'(A^{-1})' = (A^{-1})'A' = I$

$\therefore (A')^{-1} = (A^{-1})'$ by the definition of the inverse of a matrix.

27. Cramer's rule (Determinant Method):

The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$; $a_3x + b_3y + c_3z = d_3$ is given by

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$ $(\Delta \neq 0)$ where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

We notice that Δ_1, Δ_2, Δ_3 are the determinants obtained from Δ on replacing the first, second and third columns by d's respectively.

1.3 SYMMETRIC MATRIX

Definition: A square matrix $A = [a_{ij}]$ is said to be ***Symmetric*** if $a_{ij} = a_{ji}$ for every i and j.

Thus : A is a symmetric matrix $\Leftrightarrow$ $A = A'$ or $A' = A$.

1.4 SKEW-SYMMETRIC MATRIX

Definition : A square matrix $A = [a_{ij}]$ is said to be ***Skew - Symmetric*** if $a_{ij} = -a_{ji}$ for every i and j.

Thus : A is a skew –symmetric matrix $\Leftrightarrow$ $A = -A'$ or $A' = -A$.

Note : Every diagonal element of a skew-symmetric matrix is necessarily zero since $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix and $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew-symmetric matrix.

(i) A is symmetric $\Rightarrow$ KA is symmetric.

(ii) A is skew-symmetric $\Rightarrow$ KA is skew-symmetric.

1.5 ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $AA' = A'A = I$. That is $A^T = A^{-1}$.

Theorem 8: Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices in one and only way (uniquely).

OR

Show that any square matrix $A = B + C$ where B is symmetric and C is skew-symmetric matrices.

Proof: Let A be any square matrix. We can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q, \text{ say}$$

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

We have $P' = \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}(A + A')'$ $\quad [\because (KA)' = KA']$

$$= \frac{1}{2}\{A' + (A')'\} = \frac{1}{2}(A + A') = P$$

$\therefore$ P is a symmetric matrix.

Now $Q' = \left\{\frac{1}{2}(A - A')\right\}' = \frac{1}{2}(A - A')' = \frac{1}{2}\{A' - (A')'\}$

$$= \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Q$$

$\therefore$ Q is a skew-symmetric matrix.

Thus **Square matrix = Symmetric + Skew-Symmetric.**

Hence the matrix A is the sum of a symmetric matrix and a skew-symmetric matrix.

To prove that the sum is unique :

If possible, let $A = R + S$ be another such representation of A where R is a symmetric and S is a skew-symmetric matrix.

$\therefore$ $R' = R$ and $S' = -S$.

Now $A' = (R + S)' = R' + S' = R - S$ and

$$\frac{1}{2}(A + A') = \frac{1}{2}(R + S + R - S) = R$$

$$\frac{1}{2}(A - A') = \frac{1}{2}(R + S - R + S) = S$$

$\Rightarrow$ $R = P$ and $S = Q$.

Thus the representation is unique.

Theorem 9 : Prove that inverse of a non-singular symmetric matrix A is symmetric.

Proof: Since A is non-singular symmetric matrix,

$\therefore$ A^{-1} exists and $A^T = A$ ... (1)

Now we have to prove that A^{-1} is symmetric.

We have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ [by (1)]

Since $(A^{-1})^T = A^{-1}$, therefore, A^{-1} is symmetric.

Theorem 10: If A is a symmetric matrix, then prove that adj A is also symmetric.

Proof: Since A is symmetric, we have

$$A^T = A \quad \text{... (1)}$$

Now we have $(\text{adj } A)^T = \text{adj } A^T$ [$\because$ adj $A' = (\text{adj } A)'$]

$= \text{adj } A$ [by (1)]

Since $(\text{adj } A)^T = \text{adj } A$, therefore, adj A is a symmetric matrix.

Theorem 11: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Proof: Since A and B are both orthogonal matrices,

$\therefore$ $AA^T = A^TA = I$ ... (1) and $BB^T = B^TB = I$ ... (2)

Now $(AB)^T = B^TA^T$

$\therefore$ $(AB)^T(AB) = (B^TA^T)(AB)$

$= B^T(A^TA)B$

$= B^TIB$ [by (1)]

$= B^TB = I$ [by (2)]

$\therefore$ AB is orthogonal.

Similarly we can prove that BA is also orthogonal.

Theorem 12: Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof: Let A be an orthogonal matrix.

Then $A^T.A = A.A^T = I$

Consider $A^T.A = I$

Taking inverse on both sides, $(A^T.A)^{-1} = I^{-1}$

$$\Rightarrow A^{-1}(A^T)^{-1} = I$$

$$\Rightarrow A^{-1}(A^{-1})^T = I$$

$\therefore$ A^{-1} is orthogonal.

Again $A^T.A = I$

Taking transpose on both sides, $(A^T.A)^T = I^T$

$$\Rightarrow A^T.(A^T)^T = I$$

Hence A^T is orthogonal.

Note 1 : If A is orthogonal, then $|A| = \pm 1$

Since $AA^T = A^T A = I,\quad \therefore\ |A||A^T| = |I|$ ... (1)

But $|A^T| = |A|$

$\therefore (1) \Rightarrow\ |A||A| = |I|$ or $|A|^2 = |I|\ \Rightarrow |A|^2 = 1$

$\therefore\ |A| = \pm 1$

SOLVED EXAMPLES

Example 1 : Evaluate $A^2 - 3A + 9I$ where $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and I is a unit matrix.

Solution : Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now $A^2 = A.A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix} = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} \text{ and } 9I = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore A^2-3A+9I=\begin{bmatrix}-12 & -5 & 11\\ 11 & 4 & 1\\ -7 & 11 & -6\end{bmatrix}-\begin{bmatrix}3 & -6 & 9\\ 6 & 9 & -3\\ -9 & 3 & 6\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 1 & 2\\ 5 & 4 & 4\\ 2 & 8 & -3\end{bmatrix}$$

Example 2 : Express the matrix A as a sum of symmetric and skew-symmetric matrix where $A=\begin{bmatrix}3 & -2 & 6\\ 2 & 7 & -1\\ 5 & 4 & 0\end{bmatrix}$

Solution : Given $A=\begin{bmatrix}3 & -2 & 6\\ 2 & 7 & -1\\ 5 & 4 & 0\end{bmatrix}$

$$\therefore \quad A^T=\begin{bmatrix}3 & 2 & 5\\ -2 & 7 & 4\\ 6 & -1 & 0\end{bmatrix}$$

Now $A+A^T=\begin{bmatrix}3 & -2 & 6\\ 2 & 7 & -1\\ 5 & 4 & 0\end{bmatrix}+\begin{bmatrix}3 & 2 & 5\\ -2 & 7 & 4\\ 6 & -1 & 0\end{bmatrix}=\begin{bmatrix}6 & 0 & 11\\ 0 & 14 & 3\\ 11 & 3 & 0\end{bmatrix}$

Let $P=\frac{1}{2}(A+A^T)=\frac{1}{2}\begin{bmatrix}6 & 0 & 11\\ 0 & 14 & 3\\ 11 & 3 & 0\end{bmatrix}=\begin{bmatrix}3 & 0 & 11/2\\ 0 & 7 & 3/2\\ 11/2 & 3/2 & 0\end{bmatrix}$

We observe that P is symmetric.

Again $A-A^T=\begin{bmatrix}3 & -2 & 6\\ 2 & 7 & -1\\ 5 & 4 & 0\end{bmatrix}-\begin{bmatrix}3 & 2 & 5\\ -2 & 7 & 4\\ 6 & -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -4 & 1\\ 4 & 0 & -5\\ -1 & 5 & 0\end{bmatrix}$

Let $Q=\frac{1}{2}(A-A^T)=\frac{1}{2}\begin{bmatrix}0 & -4 & 1\\ 4 & 0 & -5\\ -1 & 5 & 0\end{bmatrix}=\begin{bmatrix}0 & -2 & 1/2\\ 2 & 0 & -5/2\\ -1/2 & 5/2 & 0\end{bmatrix}$

We observe that Q is skew-symmetric.

Hence $A=P+Q$ which is sum of a symmetric matrix and a skew-symmetric matrix.

Example 3 : Find the adjoint and inverse of $A=\begin{bmatrix}2 & 3 & 4\\ 4 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}$

Solution : Adjoint of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

where A_{ij} are the cofactors of the elements a_{ij}. Thus minors of a_{ij} are

$$M_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10, \quad M_{12} = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15\,; \quad M_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5, \quad M_{21} = \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = 4$$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4, \quad M_{23} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1\,; \quad M_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9, \quad M_{32} = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -14$$

and $$M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$$

Cofactors of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$

$$\therefore \text{Adjoint of } A = \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}^T = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Now $$|A| = 2(12-2) - 3(16-1) + 4(8-3)$$
$$= 20 - 45 + 20 = 40 - 45 = -5 \neq 0$$

Hence $$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{-5} \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Example 4 : Compute the adjoint and inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

Solution : det A = 1(6 – 0) – 2(4 – 2) + 0 (0 – 6) = 2 ≠ 0
$\Rightarrow$ A is non-singular $\Rightarrow$ A^{-1} exists
The matrix formed by the cofactors of elements of A be

$$B = \begin{bmatrix} (6-0) & -(4-0) & (2-0) \\ -(4-2) & (2-0) & -(1-0) \\ (0-6) & -(0-4) & (3-4) \end{bmatrix} = \begin{bmatrix} 6 & -4 & 2 \\ -2 & 2 & -1 \\ -6 & 4 & -1 \end{bmatrix}$$

$$\therefore Adj\, A = B^T = \begin{bmatrix} 6 & -2 & -6 \\ -4 & 2 & 4 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{Adj\, A}{\det A} = \frac{1}{2} \begin{bmatrix} 6 & -2 & -6 \\ -4 & 2 & 4 \\ 2 & -1 & -1 \end{bmatrix}$$

Example 5 : Find the inverse of the matrix diag$[a, b, c]$, $a \neq 0, b \neq 0, c \neq 0$.

Solution : Let $A = \text{diag}[a, b, c] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$\therefore \quad \det A = abc$

Matrix formed by the cofactors of elements of A is $\begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

$\therefore \ \text{adj } A = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

$$\therefore A^{-1} = \frac{adj\ A}{\det A} = \frac{1}{abc}\begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} = \text{diag}\left[\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right]$$

Example 6 : Define adjoint of a matrix and hence find A^{-1} by using adjoint of A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Solution : Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\det A = 1(-12 - 12) - 1(-4 - 6) + 3(-4 + 6) = -8 \neq 0$

$\Rightarrow$ A is non-singular $\Rightarrow$ A^{-1} exists.

Calculation of Cofactors :

Row 1

cofactor of $1 = (-1)^{1+1}\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -12 - 12 = -24$

cofactor of $1 = (-1)^{1+2}\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4 - 6) = 10$

cofactor of $3 = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -(-4 + 6) = 2$

Row 2

cofactor of $1 = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4 + 12) = -8$

cofactor of $3 = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -(-4 + 6) = 2$

cofactor of $-3 = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -1(-4 + 2) = 2$

Row 3

$$\text{cofactor of } -2 = (-1)^{3+1}\begin{vmatrix}1 & 3\\ 3 & -3\end{vmatrix} = 1(-3-9) = -12$$

$$\text{cofactor of } -4 = (-1)^{3+2}\begin{vmatrix}1 & 3\\ 1 & -3\end{vmatrix} = 1(-3-3) = 6$$

$$\text{cofactor of } -4 = (-1)^{3+3}\begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix} = 1(3-1) = 2$$

The matrix formed by the cofactors of elements of A is $B = \begin{bmatrix}-24 & 10 & 2\\ -8 & 2 & 2\\ -12 & 6 & 2\end{bmatrix}$

$$\therefore Adj\ A = B^T = \begin{bmatrix}-24 & -8 & -12\\ 10 & 2 & 6\\ 2 & 2 & 2\end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{\text{adj } A}{\det A} = \begin{bmatrix}3 & 1 & 3/2\\ -5/4 & -1/4 & -3/4\\ -1/4 & -1/4 & -1/4\end{bmatrix}$$

Example 7 : Find the values of 'x' such that the matrix 'A' is singular where

$$A = \begin{bmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ -2 & -4 & -(1+x)\end{bmatrix}$$

Solution : Given A is singular $\Rightarrow |A| = 0$

$$i.e., \quad \begin{vmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ -2 & -4 & -(1+x)\end{vmatrix} = 0$$

Applying $R_2 + R_3$, we get

$$\begin{vmatrix}3-x & 2 & 2\\ 0 & -x & -x\\ -2 & -4 & -1-x\end{vmatrix} = 0 \Rightarrow x\begin{vmatrix}3-x & 2 & 2\\ 0 & -1 & -1\\ -2 & -4 & -1-x\end{vmatrix} = 0$$

Applying $C_2 - C_3$, we get

$$x\begin{vmatrix}3-x & 0 & 2\\ 0 & 0 & -1\\ -2 & x-3 & -1-x\end{vmatrix} = 0 \Rightarrow x(x-3)\begin{vmatrix}3-x & 0 & 2\\ 0 & 0 & -1\\ -2 & 1 & -1-x\end{vmatrix} = 0 \quad \text{[Expand by } C_2]$$

$$\Rightarrow x(x-3)[(3-x)(-1)-0] = 0 \Rightarrow x(x-3)^2 = 0 \Rightarrow x = 0 \text{ or } x = 3.$$

Example 8 : Is the matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ orthogonal ?

Solution : Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

We have $AA^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$$

Hence the matrix A is not orthogonal.

Example 9 : Prove that $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal.

Solution : Let A be the given matrix.

$$\therefore \; A'A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}^2 = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

Hence A is orthogonal.

Example 10 : Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

(or) Determine a, b, c so that A is orthogonal where $A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$.

Solution : For orthogonal matrix, $AA^T = I$

So $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$

$$= \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$ (non - diagonal elements of I), we get

$c = \pm\sqrt{2}b, a^2 = b^2+c^2 = b^2+2b^2 = 3b^2 \Rightarrow a = \pm\sqrt{3}b$

From diagonal elements of I,

$4b^2+c^2=1 \Rightarrow 4b^2+2b^2=1 \; (\because \; c^2=2b^2) \quad \Rightarrow \quad b = \pm\frac{1}{\sqrt{6}}$

$\therefore \quad a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}$ and $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

Example 11 : Prove that the following matrix is orthogonal $\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$

We have $AA^T = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$= \begin{bmatrix} 4/9+1/9+4/9 & -4/9+2/9+2/9 & -2/9-2/9+4/9 \\ -4/9+2/9+2/9 & 4/9+4/9+1/9 & 2/9-4/9+2/9 \\ -2/9-2/9+4/9 & 2/9-4/9+2/9 & 1/9+4/9+4/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the matrix is A orthogonal.

Example 12 : Show that $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal .

Solution : Given $A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$$\therefore \quad A^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Now $A\,A^T = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

$\therefore$ A is orthogonal.

Example 13 : Solve the equations $3x + 4y + 5z = 18$, $2x - y + 8z = 13$ and $5x - 2y + 7z = 20$ by matrix inversion method.

Solution : The given equations in matrix form is $\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$ *i.e.* $AX = B$

where $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

$\det A = 3(-7+16) - 4(14-40) + 5(-4+5) = 136$

The matrix formed by the cofactors of the elements of A is

$$D=\begin{bmatrix}(-7+16) & -(14-40) & (-4+5)\\ -(28+10) & (21-25) & -(-6-20)\\ (32+5) & -(24-10) & (-3-8)\end{bmatrix}=\begin{bmatrix}9 & 26 & 1\\ -38 & -4 & 26\\ 37 & -14 & -11\end{bmatrix}$$

$$\therefore Adj\ A = D^T = \begin{bmatrix}9 & -38 & 37\\ 26 & -4 & -14\\ 1 & 26 & -11\end{bmatrix}$$

Hence $A^{-1}=\dfrac{Adj\ A}{\det A}=\dfrac{1}{136}\begin{bmatrix}9 & -38 & 37\\ 26 & -4 & -14\\ 1 & 26 & -11\end{bmatrix}$

$$\therefore X = A^{-1}B=\frac{1}{136}\begin{bmatrix}9 & -38 & 37\\ 26 & -4 & -14\\ 1 & 26 & -11\end{bmatrix}\begin{bmatrix}18\\13\\20\end{bmatrix}=\frac{1}{136}\begin{bmatrix}162-494+740\\ 468-52-280\\ 18+338-220\end{bmatrix}$$

i.e., $\begin{bmatrix}x\\y\\z\end{bmatrix}=\dfrac{1}{136}\begin{bmatrix}408\\136\\136\end{bmatrix}=\begin{bmatrix}3\\1\\1\end{bmatrix}$

$\therefore$ The solution is

$x=3, y=1, z=1$.

Example 14 : Solve the system of equations by matrix method:

$x_1+x_2+x_3=2;\ 4x_1-x_2+2x_3=-6;\ 3x_1+x_2+x_3=-18$

Solution : The given system of equations in the matrix form *i.e.*, $AX=B$ is

$$\begin{bmatrix}1 & 1 & 1\\ 4 & -1 & 2\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-6\\-18\end{bmatrix}$$

Now $\det A = 1(-3)-1(-2)+7=6\neq 0 \Rightarrow A^{-1}$ exists .

$$\therefore\quad A^{-1}=\frac{adj\ A}{\det A}=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 7 & 2 & -5\end{bmatrix}$$

We have $X=A^{-1}B$

i.e., $X=\dfrac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 7 & 2 & -5\end{bmatrix}\begin{bmatrix}2\\-6\\-18\end{bmatrix}$ or $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\dfrac{1}{6}\begin{bmatrix}-6-54\\ 4+12-36\\ 14-12+90\end{bmatrix}=\begin{bmatrix}-10\\ -20/6\\ 92/6\end{bmatrix}$

$\Rightarrow x_1=-10, x_2=-10/3, x_3=46/3$. This is the required solution.

Example 15 : Solve the equations $2x+y-z=1$, $x-y+z=2$, $5x+5y-4z=3$ by Cramer's rule.

Solution : Here $\Delta=\begin{vmatrix}2 & 1 & -1\\ 1 & -1 & 1\\ 5 & 5 & -4\end{vmatrix}=2(4-5)-1(-4-5)-1(5+5)=-3$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 5 & -4 \end{vmatrix} = 1(4-5) - 1(-8-3) - 1(10+3) = -3$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 5 & 3 & -4 \end{vmatrix} = 2(-8-3) - 1(-4-5) - 1(3-10) = -6$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 5 & 3 \end{vmatrix} = 2(-3-10) - 1(3-10) + 1(5+5) = -26 + 7 + 10 = -9$$

$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-3}{-3} = 1, \quad y = \frac{\Delta_2}{\Delta} = \frac{-6}{-3} = 2, \quad z = \frac{\Delta_3}{\Delta} = \frac{-9}{-3} = 3$

Substituting $x = 1, y = 2$ in the equation $2x + y - z = 1$, we get

$$2(1) + 2 - z = 1 \Rightarrow z = 3$$

$\therefore$ The solution is $x = 1, y = 2, z = 3$.

EXERCISE 1.1

1. Prove that $A^3 - 4A^2 - 3A + 11I = O$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

2. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and I is the unit matrix of order 3, evaluate $A^2 - 4A + 9I$.

3. Find the adjoint and inverse of a matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ show that $A^3 = A^{-1}$.

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ prove that $AA^{-1} = I$. **[JNTU 2003 (Set No. 3)]**

6. Find the inverse of the matrix $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ if $a^2 + b^2 + c^2 + d^2 = 1$.
[JNTU 2003 (Set No. 3)]

7. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ prove that $A^{-1} = A'$, where A' is the transpose of A.

8. Prove that the matrix (*i*) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (*ii*) $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal.

9. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

10. Prove that the matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

11. If A is symmetric (skew-symmetric) matrix, prove that KA is symmetric (skew-symmetric) matrix.

12. If A, B are symmetric (skew - symmetric), prove that $A + B$ is also symmetric (skew-symmetric).

13. If A and B are symmetric matrices, prove that AB is symmetric if and only if A and B commute *i.e.,* $AB = BA$.

14. If A be any matrix, prove that AA' and $A'A$ are both symmetric matrices.

15. Prove that every square real matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrices.

16. Express the matrix A as the sum of a symmetric and a skew-symmetric matrices where

(*i*) $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix}$

17. Solve the following equations by Cramer's rule:

$x + y + z = 6, \quad x - y + z = 2, \quad 2x - y + 3z = 9.$

18. Solve by matrix method the system

$x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6$

ANSWERS

2. $\begin{bmatrix} -7 & 3 & -1 \\ 3 & 1 & 5 \\ 5 & 7 & -5 \end{bmatrix}$ **3.** $\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}; -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$ **6.** $\begin{bmatrix} a - ib & -c - id \\ c - id & a + ib \end{bmatrix}$

16. (*i*) $\begin{bmatrix} 4 & 3/2 & -4 \\ 3/2 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 & 1 \\ -1/2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

17. $x = 1, y = 2, z = 3$ **18.** $x = 2, y = 1, z = 0$

1.6 COMPLEX MATRICES

Previously we have considered matrices with real elements which are called real matrices and their properties. We will now introduce matrices with complex elements which are called complex matrices and define three important types of matrices which will be used in many areas like quantum mechanics.

1. Conjugate of a matrix :

The matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called *the* ***conjugate*** *of A* and is denoted by $\overline{A}$.

Thus if $A = [a_{ij}]_{m \times n}$, then $\overline{A} = [b_{ij}]_{n \times m}$ where $b_{ij} = \overline{a}_{ij}$, the conjugate complex number of a_{ij}.

Note : If $\overline{A}$ and $\overline{B}$ be the conjugates of A and B respectively, then

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A+B)} = \overline{A} + \overline{B}$

(iii) $\overline{(KA)} = \overline{K}\,\overline{A}$, K being any complex number.

(iv) $\overline{(AB)} = \overline{A}.\overline{B}$, A and B being conformable for multiplication.

***e.g.*1.** If $A = \begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2\times 3}$, then $\overline{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2\times 3}$

***e.g.*2.** If $A = \begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

2. The transpose of the Conjugate of a Square Matrix :

If A is a square matrix and its conjugate is $\overline{A}$, then the transpose of $\overline{A}$ is $(\overline{A})^T$. It can be easily seen that $(\overline{A})^T = \overline{(A^T)}$. (*i.e.*) The transpose of the conjugate of a square matrix is same as the conjugate of its transpose.

The transposed conjugate of A is denoted by A^θ.

$\therefore\ A^\theta = (\overline{A})^T = \overline{(A^T)}$

Now $A = [a_{ij}]_{m \times n} \Rightarrow A^\theta = [b_{ij}]_{n \times m}$, where $b_{ij} = \overline{a}_{ji}$

i.e. the $(i, j)^{th}$ element of A^θ = the conjugate complex of the $(j, i)^{th}$ element of A.

e.g. If $A = \begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}$; $A^\theta = \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3\times 2}$

Note 1. $A^\theta = \overline{A}' = (\overline{A})'$ and $(A^\theta)^\theta = A$.

Note 2. If A^θ and B^θ be the transposed conjugates of A and B respectively, then

(i) $(A^\theta)^\theta = A$

(ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$

(iii) $(KA)^\theta = \overline{K}A^\theta$, where K is a complex number.

(iv) $(AB)^\theta = B^\theta A^\theta$

3. Hermitian matrix :

A square matrix A such that $A^T = \overline{A}$ or $(\overline{A})^T = A$ is called a *Hermitian matrix*.

(This can also be written as $\overline{(A^T)} = A$)

e.g. consider $A = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$. Then $\overline{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $A^T = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Since $A^T = \overline{A}$, therefore, A is Hermitian.

Note that elements of the principal diagonal of a Hermitian matrix must be real.

A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Obviously a necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

4. Skew-Hermitian matrix :

A square matrix A such that $A^T = -\overline{A}$ or $(\overline{A})^T = -A$ is called a Skew-Hermitian matrix.

(This can also be written as $\overline{(A^T)} = -A$)

Let A $= \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$.

Then $\overline{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$, $A^T = \begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix}$ and $-\overline{A} = \begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix}$

which shows $A^T = -\overline{A}$

It should be noted that elements of the leading diagonal must be all zero or all are purely imaginary.

A Skew-Hermitian matrix over the field of real numbers is nothing but a real Skew - symmetric matrix.

Obviously a necessary and sufficient condition for a matrix A to be Skew-Hermitian is that $A^\theta = -A$.

5. Unitary matrix :

A square matrix A such that $(\overline{A})^T = A^{-1}$

i.e., $(\overline{A})^T A = A(\overline{A})^T = I$ or $A^\theta A = I$ is called a unitary matrix.

e.g. $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$ is a unitary matrix.

Observation: We can observe that every real symmetric matrix is Hermitian. Similarly a real skew-symmetric matrix is Skew-Hermitian and a real orthogonal matrix is Unitary.

Thus Hermitian, Skew-Hermitian and Unitary matrices generalize symmetric, skew-symmetric and orthogonal matrices respectively.

SOLVED EXAMPLES

Example 1 : If $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is Hermitian and iA is Skew - Hermitian.

Solution : Given $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore\ \overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$. Thus $(\overline{A})^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$

Hence A is Hermitian matrix.

Let $B = i\,.\,A = \begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$. Now $\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$

$\therefore\ (\overline{B})^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$

Thus B *i.e.,* iA is a Skew-Hermitian matrix.

Example 2 : If A and B are Hermitian matrices, prove that $AB - BA$ is a Skew-Hermitian.

Solution : Given A and B are Hermitian.

$\therefore\ (\overline{A})^T = A$ and $(\overline{B})^T = B$ ..... (1)

Now $(\overline{AB - BA})^T = (\overline{AB} - \overline{BA})^T = (\overline{A}\,\overline{B} - \overline{B}\,\overline{A})^T$

$= (\overline{A}\,\overline{B})^T - (\overline{B}\,\overline{A})^T = (\overline{B})^T(\overline{A})^T - (\overline{A})^T(\overline{B})^T$

$= BA - AB$, by (1)

$= -(AB - BA)$

Thus $AB - BA$ is a Skew-Hermitian matrix.

Example 3 : Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

Solution : Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$. Then $A^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$

$\therefore\quad A^{\theta} = (\overline{A^T}) = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

Now $AA^{\theta} = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus $AA^{\theta} = \text{I}$ Hence A is a unitary matrix.

Example 4 : Show that $A = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$ is unitary if $a^2 + b^2 + c^2 + d^2 = 1$.

Solution : Given $A = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$

$\therefore \quad \overline{A} = \begin{pmatrix} a-ic & -b-id \\ b-id & a+ic \end{pmatrix}$

Hence $A^{\theta} = (\overline{A})^T = \begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

Now $AA^{\theta} = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}\begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$\therefore \quad AA^{\theta} = I$ if and only if $a^2 + b^2 + c^2 + d^2 = 1$

i.e., A is unitary if and only if $a^2 + b^2 + c^2 + d^2 = 1$.

Example 5 : If A is a Hermitian matrix, prove that iA is a Skew-Hermitian matrix.

Solution : Since A is a Hermitian matrix,

$\therefore \quad A^{\theta} = A \qquad \ldots(1)$

Now we have $(iA)^{\theta} = \bar{i}A^{\theta} \quad [\because (KA)^{\theta} = \overline{K}A^{\theta},\ K$ being a complex number$]$

$= (-i)A^{\theta} \qquad (\because \bar{i} = -i)$

$= -(iA^{\theta})$

$= -(iA) \quad$ [by (1)]

Since $(iA)^{\theta} = -(iA)$, therefore iA is a skew-Hermitian matrix.

Example 6 : If A is a Skew-Hermitian matrix, prove that iA is a Hermitian matrix.

Solution : Let A be a Skew-Hermitian matrix. Then $A^{\theta} = -A \ldots(1)$

We have $\quad (iA)^{\theta} = \bar{i}A^{\theta} = (-i)A^{\theta} = -(iA^{\theta}) = -\{i(-A)\}$ [by (1)]

$= -[-(iA)] = iA$

Since $(iA)^{\theta} = iA$, therefore, iA is a Hermitian matrix.

Example 7 : Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

Solution : Let A be any square matrix.

Now $(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A$

Since $(A + A^\theta)^\theta = A + A^\theta$, therefore, $A + A^\theta$ is a Hermitian matrix.

$\therefore\ \frac{1}{2}(A + A^\theta)$ is also a Hermitian matrix.

Now $(A - A^\theta)^\theta = A^\theta - (A^\theta)^\theta = A^\theta - A = -(A - A^\theta)$

Hence $A - A^\theta$ is a Skew-Hermitian matrix.

$\therefore\ \frac{1}{2}(A - A^\theta)$ is also a Skew-Hermitian matrix.

Now, we have $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta) = P + Q$ (say)

where P is a Hermitian matrix and Q is a Skew–Hermitian matrix.

Thus every square matrix can be expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.

To prove that the representation is unique :

Let $A = R + S$ be another such representation of A, where R is Hermitian and S is Skew-Hermitian. Then to prove that $R = P$ and $S = Q$.

Then $A^\theta = (R + S)^\theta = R^\theta + S^\theta = R - S$

($\because$ R is Hermitian and S is Skew – Hermitian)

$\therefore\ R = \frac{1}{2}(A + A^\theta) = P$ and $S = \frac{1}{2}(A - A^\theta) = Q$

Thus the representation is unique.

Example 8 : Prove that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew-symmetric.

Solution : Let C be a Hermitian matrix. Then $C^\theta = (C)$ ...(1)

Let us take $A = \frac{1}{2}(C + \overline{C})$ and $B = \frac{1}{2i}(C - \overline{C})$

Then obviously both A and B are real matrices.

Now we can write

$$C = \frac{1}{2}(C + \overline{C}) + i\left[\frac{1}{2i}(C - \overline{C})\right] = A + iB$$

Now we have to prove that A is symmetric and B is skew-symmetric.

$$A^T = \frac{1}{2}(C + \overline{C})^T = \frac{1}{2}\left[C^T + (\overline{C})^T\right]$$

$$= \frac{1}{2}\left[C^T + C^\theta\right] = \frac{1}{2}\left[(C^\theta)^T + C\right], \quad \text{[by (1)]}$$

$$= \frac{1}{2}\left[\left\{(\overline{C})^T\right\}^T + C\right] = \frac{1}{2}(\overline{C} + C) = A$$

$\therefore\ A$ is symmetric.

Now $B^T = \left[\frac{1}{2i}(C - \overline{C})\right]^T = \frac{1}{2i}(C - \overline{C})^T$

$$= \frac{1}{2i}\left[C^T - (\overline{C})^T\right] = \frac{1}{2i}\left[C^T - C^\theta\right]$$

$$= \frac{1}{2i}\left[(C^\theta)^T - C\right] \quad \text{[by (1)]}$$

$$= \frac{1}{2}\left[\left\{(\overline{C})^T\right\}^T - C\right] = \frac{1}{2i}(\overline{C} - C) = -\frac{1}{2i}(C - \overline{C}) = -B$$

$\therefore$ B is skew-symmetric. Hence the result.

Example 9 : Express the matrix $\begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ as the sum of Hermitian matrix and Skew-Hermitian matrix.

Solution : Let $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ ..... (1)

Then $\overline{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$

$\therefore$ $A^\theta = (\overline{A})^T = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$ ..... (2)

(1) + (2) gives, $A + A^\theta = \begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$

Let $P = \frac{1}{2}\left(A + A^\theta\right) = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$

This is a Hermitian matrix.

(1) – (2) gives, $A - A^\theta = \begin{bmatrix} 2i & 2+2i & 6-4i \\ -2+2i & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$

Let $Q = \frac{1}{2}\left(A - A^\theta\right) = \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$

which is a Skew-Hermitian matrix.

$$\therefore A = \frac{1}{2}\left(A + A^\theta\right) + \frac{1}{2}\left(A - A^\theta\right) = P + Q = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix} + \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

Example 10 : Given that $A=\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Solution : We have $I-A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}=\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

and $I+A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$

$\therefore (I+A)^{-1}=\dfrac{1}{1-(4i^2-1)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\dfrac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

Let $B=(I-A)(I+A)^{-1}$

$$=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\dfrac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\left(\overline{B}\right)^T=\dfrac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$\therefore\ B\left(\overline{B}\right)^T=\dfrac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}=\dfrac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

Thus $\left(\overline{B}\right)^T=B^{-1}$ *i.e.,* B is unitary matrix.

$\therefore\ (I-A)(I+A)^{-1}$ is a unitary matrix.

Example 11 : Show that the inverse of a unitary matrix is unitary.

Solution : Let A be a unitary matrix. Then $AA^{\theta}=I$.

i.e., $\Rightarrow (AA^{\theta})^{-1}=I^{-1}$ .... (1)

$\Rightarrow (A^{\theta})^{-1}A^{-1}=I$

$\Rightarrow (A^{-1})^{\theta}A^{-1}=I.$ Thus A^{-1} is unitary.

$\therefore B$ is unitary *i.e.,* A^{-1} is unitary.

Example 12 : Prove that the product of two unitary matrices is unitary.

Solution : Let A and B be the two unitary matrices.

$\therefore A.A^{\theta}=A^{\theta}.A=I$ and $B.B^{\theta}=B^{\theta}.B=I$ .... (1)

Consider $(AB).(AB)^{\theta}=(AB)(B^{\theta}A^{\theta})$

$=A(BB^{\theta})A^{\theta}=AIA^{\theta}$, From (1)

$=AA^{\theta}=I$, From (1)

Again consider $(AB)^{\theta}.(AB)=(B^{\theta}A^{\theta}).(AB)$

$$=B^{\theta}(A^{\theta}A)B \quad = B^{\theta}I\,B$$

$$=B^{\theta}B=I\,, \quad \text{From (1)}$$

$\therefore (AB).(AB)^{\theta} = (AB)^{\theta}.(AB)=I$

Thus AB is a unitary matrix.

Example 13 : Prove that the transpose of a unitary matrix is unitary.

Solution : Let A be a unitary matrix. Then $\overline{(A^T)}=\left(\overline{A}\right)^T = A^{-1}$

Taking transpose, we get $\left(\left(\overline{A^T}\right)\right)^T = \left(A^{-1}\right)^T = \left(A^T\right)^{-1}$

Let $A^T = B$. Now (1) becomes $\left(\overline{B}\right)^T = B^{-1}$

Thus B is unitary *i.e.,* A^T is unitary.

Example 14 : Find the Hermitian form H for $A=\begin{bmatrix}0 & i & 0\\ -i & 1 & -2i\\ 0 & 2i & 2\end{bmatrix}$ with $X=\begin{bmatrix}i\\ 1\\ -i\end{bmatrix}$

Solution : The Hermitian form of A is $H=\bar{X}^T AX$

$$=\begin{bmatrix}-i & 1 & i\end{bmatrix}\begin{bmatrix}0 & i & 0\\ -i & 1 & -2i\\ 0 & 2i & 2\end{bmatrix}\begin{bmatrix}i\\ 1\\ -i\end{bmatrix}$$

$$=\begin{bmatrix}-i & 1+1-2 & 0\end{bmatrix}\begin{bmatrix}i\\ 1\\ -i\end{bmatrix}=1 \text{ (real)}$$

Example 15 : Find the Hermitian form of $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$ with $X=\begin{bmatrix}1\\ i\end{bmatrix}$.

Solution : Given $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$ and $X=\begin{bmatrix}1\\ i\end{bmatrix}$

The Hermitian form of A is $H=\bar{X}^T AX$

$$=\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}$$

$$=\begin{bmatrix}-1 & i\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}=-1+i^2=-2, \text{ real}$$

Example 16 : Determine the skew-Hermitian form S for $A=\begin{bmatrix}2i & 3i\\ 3i & 0\end{bmatrix}$ with $X=\begin{bmatrix}4i\\ -5\end{bmatrix}$

Solution : Given $A=\begin{bmatrix}2i & 3i\\ 3i & 0\end{bmatrix}; X=\begin{bmatrix}4i\\ -5\end{bmatrix}$

We have Skew - Hermitian S for A is $=\overline{X}^T AX$

$$=\begin{bmatrix}-4i & -5\end{bmatrix}\begin{bmatrix}2i & 3i\\ 3i & 0\end{bmatrix}\begin{bmatrix}4i\\ -5\end{bmatrix}$$

$$=\begin{bmatrix}8 & -15i & 12\end{bmatrix}\begin{bmatrix}4i\\ 5\end{bmatrix}$$

$$=32i+60-60$$
$$=32i\text{, purely imaginary.}$$

Example 17 : Find the Hermitian form of $A=\begin{bmatrix}3 & 2-i\\ 2+i & 4\end{bmatrix}$ with $X=\begin{bmatrix}1+i\\ 2i\end{bmatrix}$

Solution : The Hermitian form of A is

$$H=\overline{X}^T AX$$

$$=\begin{bmatrix}1-i & -2i\end{bmatrix}\begin{bmatrix}3 & 2-i\\ 2+i & 4\end{bmatrix}\begin{bmatrix}1+i\\ 2i\end{bmatrix}$$

$$=\begin{bmatrix}(3-3i-4i-2i^2)\ (2-i-2i+i^2-8i)\end{bmatrix}\begin{bmatrix}1+i\\ 2i\end{bmatrix}$$

$$=[(5-7i)(1-11i)]\begin{bmatrix}1+i\\ 2i\end{bmatrix}$$

$$=\begin{bmatrix}(5-7i+5i-7i^2)+(2i-22i^2)\end{bmatrix}=[33]\ \text{(real)}$$

Example 18 : Find the Skew - Hermetian form for $A=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ with $X=\begin{bmatrix}1\\ i\end{bmatrix}$

Solution : We have Skew-Hermetian form

$$S=\overline{X}^T AX$$

$$=[1-i]\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}=\begin{bmatrix}i & i^2\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}$$

$$=\begin{bmatrix}i & -1\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}$$

$$[i-i]=[0]\ \text{(real)}$$

EXERCISE 1.2

1. If $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ then prove that A is a Skew - Hermitian matrix.

2. If $H = \begin{bmatrix} 2 & 3+2i & -4 \\ 3-2i & 5 & 6i \\ -4 & -6i & 3 \end{bmatrix}$ show that H is Hermitian and iH is a Skew - Hermitian matrix.

3. If $A = \begin{bmatrix} 2+3i & 1-i & 2+i \\ -2i & 4 & 2i \\ -4i & -4i & i \end{bmatrix}$ then find a Skew-Hermitian matrix.

4. Express the matrix $\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and Skew - Hermitian matrix.

ANSWERS

4. $\begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix} + \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$

ELEMENTARY TRANSFORMATIONS (OR OPERATIONS) ON A MATRIX

(*i*) Interchange of two rows : If ith row and jth row are interchanged, it is denoted by $R_i \leftrightarrow R_j$.

(*ii*) Multiplication of each element of a row with a non-zero scalar. If i^{th} row is multiplied with k then it is denoted by $R_i \leftrightarrow kR_i$.

(*iii*) Multiplying every element of a row with a non-zero scalar and adding to the corresponding elements of another row.

If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + kR_i$.

The corresponding column transformations will be denoted by writing C, instead of R, *i.e.*, by $C_i \leftrightarrow C_j$, $C_i \to C_i + KC_j$, $C_j \to C_j + KC_i$ respectively.

An elementary transformation is called a **row transformation** or a **column transformation** according as it applies to rows or columns.

Important Result :

We can prove that elementary operations on a matrix do not change its rank.

1.8 EQUIVALENCE OF MATRICES

If B is obtained from A after a finite chain of elementay transformations then B is said to be equivalent to A. Symbolically it is denoted as $B \sim A$.

Important Results :

1. If A and B are two equivalent matrices, then rank A = rank B.
2. If two matrices A and B have the same size and the same rank, then the two matrices A and B are equivalent.

1.9 SUB-MATRIX

A matrix obtained by deleting some rows or columns or both of a given matrix is called its **sub-matrix.**

e.g. Let $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$.

Then $\begin{bmatrix} 1 & 5 & 6 \\ 8 & 9 & 10 \end{bmatrix}_{2\times3}$ is a sub-matrix of A obtained by deleting third row and 4^{th} column from A.

Similarly, $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 3 & 4 & 5 & -1 \end{bmatrix}_{2\times4}$ is a sub-matrix of A obtained by deleting the second row of A and $\begin{bmatrix} 5 & 6 & 7 \\ 9 & 10 & 5 \\ 4 & 5 & -1 \end{bmatrix}$ is another sub-matrix of A.

1.10 MINOR OF A MATRIX

Let A be an $m \times n$ matrix. The determinant of a square sub-matrix of A is called a minor of the matrix. If the order of the square sub-matrix is t then its determinant is called a minor of order t.

e.g. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4\times3}$ be a matrix.

We have $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order 2.

det. $B = 2 - 3 = -1$ is a minor of order 2 and

$C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}_{3\times3}$ is a sub-matrix of order 3.

det. $C = 2(7-12) - 1(21-10) + 1(18-5) = 2(-5) - 1(11) + 1(13)$

$= -10 - 11 + 13 = -8$ is a minor of order 3.

We will now study about the rank of a matrix, which plays a vital role in relation to the application of matrices to linear equations, linear transformation, vector spaces etc.

1.11 RANK OF A MATRIX

Let A be an $m \times n$ matrix. If A is a null matrix, we define its rank to be 0 (zero).

If A is a non-zero matrix, we say that r is the rank of A if

(*i*) every $(r + 1)^{th}$ order minor of A is 0 (zero) and

(*ii*) three exists at least one rth order minor of A which is not zero.

Rank of A is denoted by $\rho(A)$.

This definition is useful to understand what rank is.

To determine the rank of A, if m, n are both greater than 4, this definition will not in general be of much use.

Note 1 : It can be noted that the rank of a non-zero matrix is the order of the highest order non-zero minor of A.

Note 2 : (1) Every matrix will have a rank.

(2) Rank of a matrix is unique.

(3) $\rho(A) \geq 1$ when A is a non-zero matrix.

(4) If A is a matrix of order $m \times n$, rank of $A = \rho(A) \leq \min(m, n)$.

(5) If $\rho(A) = r$ then every minor of A of order $r + 1$, or more is zero.

(6) Rank of the identity matrix I_n is n.

(7) If A is a matrix of order n and A is non-singular [(*i.e.*) $\det A \neq 0$], then $\rho(A) = n$.

Important Results : The following two results will help us to determine the rank of a matrix.

(*i*) The rank of a matrix is $\leq r$, if all minors of $(r + 1)^{th}$ order vanish.

(*ii*) The rank of a matrix is $\geq r$, if there is at least one minor of r^{th} order which is not equal to zero.

SOLVED EXAMPLES

Example 1 : Find the rank of the matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3\times 3}$.

Solution : We have $\det A = -1\,(18-1) - 0\,(9+5) + 6\,(3+30)$

$= -17 - 0 + 6(33) = 181 \neq 0$

Since the minor of order $3 \neq 0$,

$\therefore\ \rho(A) = 3$.

Example 2 : Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Solution : Let A be the given matrix. Then

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 1(24-25) - 2(18-20) + 3(15-16) = -1 + 4 - 3 = 0$$

$\therefore$ Rank $A \neq 3$. So it must be less than 3.

Consider a minor of order $2 = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Hence there is at least a minor of order 2 which is not zero.

$\therefore$ Rank of $A = 2$

Example 3 : Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$.

Solution : Here the matrix is of order 3×4. Its rank $\leq \min(3, 4) = 3$.

$\therefore$ Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$.

Its determinant $= 24 \neq 0$

$\therefore$ The order of the highest order non-zero minor of A is 3.

Hence the rank of the given matrix is 3.

$\therefore$ There is at least a minor of order 3 which is not equal to zero.

$\therefore$ Rank$(A) = 3$.

We find that the method of finding the rank by this method is laborious since it involves evaluation of a number of minors. The rank of a given matrix can be determined by using what are called row/column operations on a matrix.

Note : Consider the system of equations

$$\left.\begin{aligned} 3x + 2y &= 7 \\ 2x + y &= 4 \end{aligned}\right\} \quad \text{...(1)}$$

In matrix notation, this system can be written as $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

This is like $AX = B$ where $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

The matrix $[A \mid B] = \left(\begin{array}{cc|c} 3 & 2 & 7 \\ 2 & 1 & 4 \end{array}\right)$ ...(2)

is called the Augmented matrix of the system. By solving the system, we get

$$\left.\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}\right\} \quad \text{...(3)}$$

In matrix notation this is same as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The Augmented matrix of this system is $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$ ...(4)

We can notice that to solve system (1), we have to convert the Augmented matrix $[A \mid B]$ in (2) to the form (4) by a series of mathematical operations.

If we follow the usual procedure of solving system of simultaneous equations (as in (1)), we note that

(*i*) We are interchanging the positions of two equations (if necessary).

or (*ii*) We are multiplying (or dividing) an equation by a non-zero constant.

or (*iii*) We are adding k times an equation to another equation.

These three precisely correspond to

(*i*) Interchange of two rows of $[A \mid B]$.

(*ii*) Multiplication of a row of $[A \mid B]$ by a non-zero constant.

(*iii*) Adding a constant times a row to another row.

It is interesting to note that the above three operations on the rows of a matrix have a number of useful applications is (*i*) solving a system of simultaneous equations, (*ii*) determination of a rank, (*iii*) finding the inverse of *A*, (*iv*) determining whether a given set of vectors are linearly dependent or independent etc.

Hence we formalize these ideas in the form of the following definition.

1.12 ZERO ROW AND NON-ZERO ROW

If all the elements in a row of a matrix are zeros, then it is called a zero row and if there is at least one non-zero element in a row, then it is called a non-zero row.

1.13 ECHELON FORM OF A MATRIX

A matrix is said to be in Echelon form if it has the following properties.

(*i*) Zero rows, if any, are below any non-zero row.

(*ii*) The first non-zero entry in each non-zero row is equal to 1.

(*iii*) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note: The condition (*ii*) is optional.

Important Result : The number of non-zero rows in the row echelon form of *A* is the rank of *A*.

e.g. 1. $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form.

3. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is in row echelon form.

Note: The rank of the transpose of a matrix is the same as that of the original matrix.

SOLVED EXAMPLES

Example 1 : Reduce the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$ into echelon form and hence find its rank.

Solution : Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, and $R_4 \to R_4 - 6R_1$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_3$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$

Now applying $R_4 \to R_4 - R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

Finally applying $R_4 \to R_4 - R_3$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form and the number of non-zero rows is 3.

$\therefore$ Rank $(A) = \rho(A) = 3$.

Example 2 : Reduce the matrix $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form and hence find its rank.

Solution : Let A be the given matrix. Then $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we get $A \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$

Performing $R_3 \to R_3 - 5R_1$; $A \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$

Performing $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 8R_2$; $A \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$

Applying $\frac{R_3}{-4}$, we get $A \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

Applying $R_1 \to 3R_1 - 4R_3$ and $R_2 \to 3R_2 - 2R_3$, we get $A \sim \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

This is in row Echelon form. Here number of non-zero rows is 3.
Hence rank of the matrix is 3.

Example 3 : Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

Solution : Given matrix is $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

Applying $R_2 \leftrightarrow R_2 - 2R_1$, $R_3 \leftrightarrow R_3 - 3R_1$ and $R_4 \to R_4 + R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_4 \to R_4 + R_2$, we get

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in Echelon form. In the above matrix, number of non-zero rows is 2. Hence rank $(A) = 2$.

Example 4 : Reduce the matrix to Echelon form and find its rank.

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Solution : Given matrix is $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$ and $R_4 \to R_4 - R_1$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 11R_2$ and $R_4 \to R_4 + 2R_2$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_4 \to 6R_4 + R_3$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in Echelon form. Number of non-zero rows is 4.

$\therefore$ Rank of $A = \rho(A) = 4$.

Example 5 : If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ find the ranks of A, B, $A + B$, AB and BA.

Solution : (*i*) Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$\therefore\ A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix} \text{ (by } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \text{ (by } R_3 \to R_3 - R_2)$$

Here the matrix is in Echelon form. Number of non-zero rows is 2.

Hence the rank of matrix A is 2.

(*ii*) We have $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Applying $R_1 \to \dfrac{R_1}{-1}$, $R_2 \to \dfrac{R_2}{6}$ and $R_3 \to \dfrac{R_3}{5}$, we get $B \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get $B \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form and number of non-zero rows is 1.
Hence the rank of the matrix B is 1.

(iii) We have $A+B = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$

Applying $R_3 \to \dfrac{R_3}{8}$, we get $A + B \sim \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we get $A + B \sim \begin{bmatrix} 1 & 1 & 1 \\ 8 & 9 & 10 \\ 0 & -1 & -2 \end{bmatrix}$

Applying $R_2 \to R_2 - 8R_1$, we get $A + B \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$

Applying $R_3 \to R_3 + R_2$, we get $A + B \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Hence the matrix is in Echelon form and Number of non-zero rows$=2$.
$\therefore$ Rank of $A + B$ is 2

(iv) We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

We have defined the rank of null matrix is 0.

$\therefore$ Rank of AB is zero

Note : The above example indicates that $\rho(AB) \leq \rho(A)$ or $\rho(B)$

(v) Again $BA = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 7 & -10 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix}$

Applying $R_2 \to \dfrac{R_2}{6}$ and $R_3 \to \dfrac{R_3}{5}$, we get $BA \sim \begin{bmatrix} -8 & 7 & -10 \\ 8 & -7 & 10 \\ 8 & -7 & 10 \end{bmatrix}$

Now applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we get $BA \sim \begin{bmatrix} -8 & 7 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form. Since number of non-zero rows in one, rank is one.

$\therefore$ The matrix BA is of rank 1.

Important Result :

The rank of a product of two matrices cannot exceed the rank of either matrix.

Example 6 : Find rank of the marix $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Applying $R_1 \to R_3 + R_1$, we get $A \sim \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

This is in row Echelon form and number of non-zero rows is 3.

Hence rank of A is 3.

Example 7 : Define the rank of the matrix and find the rank of the following matrix

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

Solution : **Rank of a Matrix :** A number 'r' is said to be the rank of the matrix A if it possess the following two properties :

(i) There exist at least one minor of order 'r' which is non-zero.

(ii) All minors of order $(r+1)$ if they exists are zeros.

Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} \text{(Applying } R_2 \to R_2 - 2R_1\text{ ; } R_3 \to R_3 - 4R_1\text{ ; } R_4 \to R_4 - 4R_1\text{)}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(Applying } R_3 \to R_3 - R_2\text{ ; } R_4 \to R_4 - 3R_2\text{)}$$

This matrix is in Echelon form.
Rank of A = number of non - zero rows = 2

Example 8 : Determine the rank of the matrix $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{(Applying } R_2 \to R_2 + 2R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{(Applying } R_2 \to R_2 - 3R_4 \text{ and } R_3 \to R_3 - 2R_4)$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_2 \leftrightarrow R_4)$$

This is in Echelon form.
$\therefore$ Rank of A is 2 since the number of non-zero rows is 2.

Example 9 : For what value of K the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$ has rank 3.

Solution :

Let $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$

Applying $R_2 \to 4\,R_2 - R_1,\ R_3 \to 4\,R_3 - K\,R_1,\ R_4 \to 4R_4 - 9R_1$, we get

$$A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4K & 8+3K & 8-K \\ 0 & 0 & 4K+27 & 3 \end{bmatrix}$$

The given matrix is of the order 4 × 4. If its rank is 3, then we must have det A = 0

$$\Rightarrow 4\begin{vmatrix} 0 & -1 & -1 \\ 8-4K & 8+3K & 8-K \\ 0 & 4K+27 & 3 \end{vmatrix} = 0 \Rightarrow 1[(8-4K)3] - 1[(8-4\,K)\,(4K+27)] = 0$$

i.e. $(8-4K)\,(3-4K-27) = 0$

i.e. $(8-4K)\,(-24-4K) = 0$ or $4(2-K)\,(-4)\,(6+K) = 0$

$\therefore\ K = 2$ or $K = -6$

Example 10 : Find the value of k such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given rank of A = $\rho(A) = 2$ $\therefore\ |A| = 0$

$\Rightarrow 1(10k-42) - 2\,(20-21) + 3(12-3k) = 0$ or $k = 4$

Example 11 : Find the value of k it the Rank of Matrix A is 2 where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

Solution : Given matrix is $= \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix} \text{(Applying } R_3 - 3R_1, R_4 - R_1\text{)}$$

For the rank (A) to be equal to 2, we must have 3 rows is identical.

$\therefore\ k-1=-3 \Rightarrow k=-2$

Example 12 : Find the value of k such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2.

Solution : Given $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \text{(Applying } R_2 - R_1; R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & 2 \\ 0 & 0 & 2-k & 0 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

Since the rank of A is 2, there will be only 2 non-zero rows.

$\therefore$ Third row must be a zero row $\Rightarrow 2-k=0 \Rightarrow k=2$

Example 13 : Find the rank of $\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$.

Solution : Given $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

Interchanging R_1 and R_2, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_4 \to R_4 - 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_4$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

Applying $R_4 \to 9R_4 + 5R_3$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 0 & 18 & -16 \end{bmatrix}$$

This is in Echelon form.
Number of non-zero rows is 4. Thus, rank of $A = \rho(A) = 4$.

Example 14 : Find the rank of matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{(Applying } R_3 - 3R_1, R_4 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_4 - R_2, R_3 - R_2\text{)}$$

This is in Echelon form. The number of non-zero rows is 2.

$\therefore$ Rank of the matrix = 2.

Example 15 : Find the rank of $\begin{pmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{pmatrix}$

[

Solution : Let $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix} \text{(Applying } R_2 + 2R_1, R_3 + R_1, R_4 + 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \text{(Applying } \frac{R_2}{5}, \frac{R_3}{10}, \frac{R_4}{15}\text{)}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2, R_4 - R_2\text{)}$$

$\therefore$ Rank (A) = $\rho(A)$ = The number of non-zero rows in the matrix $= 2$

Example 16 : Find the rank of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{pmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 2 & 14 & 20 \\ 0 & 1 & 7 & 10 \end{bmatrix} \text{(Applying } R_2 + 2R_1; R_3 + 3R_1; R_4 - R_1 \text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 1 & 7 & 10 \\ 0 & 1 & 7 & 10 \end{bmatrix} \text{(Applying } \frac{R_3}{2} \text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2; R_4 - R_2 \text{)}$$

$\therefore$ rank (A) = ρ (A) = Number of the non-zero rows in the matrix = 2

Example 17 : Find the rank of $\begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{pmatrix}$

Solution : Given matrix is

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{pmatrix} \text{(Applying } R_2 + 2R_1, R_3 - R_1\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -8 & 4 & -32 \end{pmatrix} \text{(Applying } 4R_3\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{pmatrix} \text{(Applying } R_3 + R_2\text{)}$$

This is in Echelon form.

$\therefore\ \rho(A) =$ The number of non-zero rows in the matrix = 3.

$\therefore\ \rho(A) = 3$

Example 18 : Find the rank of $\begin{pmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{pmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix} \text{(Applying } R_2 + 2R_1; R_3 + R_1; R_4 + 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \text{(Applying } \frac{R_2}{5}; \frac{R_3}{10}; \frac{R_4}{15}\text{)}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applyinjg } R_3 - R_2; R_4 - R_2\text{)}$$

$\therefore\ \rho(A) =$ Number of non-zero rows in the matrix = 2.

Example 19 : Find the rank of $\begin{pmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{pmatrix}$

Solution : Let $A = \begin{pmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{pmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix} \text{(Applying } -R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix} \text{(Applying } R_2 - 2R_1; R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 40 & 8 & 128 \\ 0 & 8 & 4 & 31 \end{bmatrix} \text{(Applying } 8R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 40 & 8 & 128 \end{bmatrix} \text{(Applying } R_2 \leftrightarrow R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 0 & -12 & -27 \end{bmatrix} \text{(Applying } R_3 - 5R_2\text{)}$$

$\therefore$ Rank of the matrix = Number of non-zero rows in the matrix = 3.

i.e, $\rho(A) = 3$.

Example 20 : Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Solution : Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$R_3 - R_1; R_2 - 2R_1$ gives $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$R_3 + R_2$ gives $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form.

No. of non-zero rows $= 2$. Rank (A) $= 2$.

EXERCISE 1.3

Find the rank of each of the following matrices

1. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$ **2.** $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

1. 14 REDUCTION TO NORMAL FORM

There is another important method of finding rank of a matrix. We will discuss it and state some theorems without proof.

Theorem : Every $m \times n$ matrix of **rank r** can be reduced to the form I_r, $[I_r, O]$ or $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by a finite chain of elementary row or column operations, where I_r is the r-rowed unit matrix.

The above form is called **"normal form"** or "first canonical form" of a matrix.

Cor. 1 : The rank of a $m \times n$ matrix A is r if and only if it can be reduced to the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by a finite chain of elementary row and column operations.

Cor. 2: If A is an $m \times n$ matrix of rank r, there exists non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

SOLVED EXAMPLES

Example 1 : Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical form (normal) an

Solution : Given $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - C_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 8 & 5 & 0 \\ 1 & -2 & 1 & -8 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

Applying $C_2 \to \frac{C_2}{8}$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & \frac{-1}{4} & 1 & -8 \end{bmatrix}$

Applying $C_3 \to C_3 - 5C_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{4} & \frac{9}{4} & -8 \end{bmatrix}$

Applying $R_3 \to 4R_3$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 9 & -32 \end{bmatrix}$

Applying $R_3 \to R_3 + R_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$

Applying $C_3 \to \frac{C_3}{9}$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -32 \end{bmatrix}$

Applying $C_4 \to C_4 + 32C_3$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

The above matrix is in the form $[I_3 \quad O]$.

$\therefore$ Rank of A is 3.

Example 2 : Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and hence find the rank.

[JNTU 2006, 2006S, 2007S (Set No.1)]

Solution : Given $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Applying $C_1 \leftrightarrow C_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

Applying $R_2 \to \frac{R_2}{2}$, we get $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_3 \to C_3 - 2C_1$ and $C_4 \to C_4 + 2C_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_2 \to \frac{C_2}{2}$, $C_4 \to \frac{C_4}{3}$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_3 \to C_3 - C_2$, $C_4 \to C_4 - C_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$ (normal form).

Hence the rank of matrix A is 2.

Example 3 : Find the rank of the matrix $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to canonical form.

Solution : Given matrix is $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we get $A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & -6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$, and $R_4 \to R_4 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_3$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{R_2}{6}$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $C_4 \to C_4 - \frac{4}{3}C_1$, $C_4 \to 6C_4 + 10C_2$ and $C_4 \to C_4 + 6C_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $\frac{C_2}{6}$ and $R_3 \to R_4$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$ which is in canonical form.

$\therefore$ Rank of $A = 3$.

Example 4 : Reduce the matrix A to normal form and hence find its rank

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

Solution : Given $A=\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, we get $A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$

Applying $R_4 \to R_4 - 2R_3$, we get $A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_2 \to R_2 - R_3$, we get $A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$, we get $A \sim \begin{bmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_3 \to 2C_3 - 3C_1$, we get $A \sim \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_1 \to \frac{C_1}{2}$, $C_3 \to \frac{C_3}{8}$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_4 \to C_4 - 4C_1$, $C_4 \to C_4 - C_3$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is of the form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$. This is the required normal form.

Hence rank of $A = 3$.

Example 5 : By reducing the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ into normal form, find its rank.

Solution : Given $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Applying $R_2 \leftrightarrow R_1$ we get $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Applying $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$, we get

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Applying $R_4 - R_3$, we get $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 5 & 3 & 7 \end{bmatrix}$

Applying $R_4 - R_2$, we get $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

Applying $R_2 - R_{3,}$ we get $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_1 + R_2$ we get $A \sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_3 - 4R_2$, we get $A \sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $C_3 + 8C_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $\frac{R_3}{11}$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 + 7C_1$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

Applying $C_3 + 6C_2$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 + 3\,C_2$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

Applying $\frac{C_3}{3}, \frac{C_4}{2}$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 - C_3$, we get $$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is of the form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$. Hence Rank of A is 3.

Example 6 : By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Solution : Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Applying $R_2 - 2R_1, R_3 - 3R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$

Applying $\frac{R_3}{-2}$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$;

Applying $R_3 + R_2$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Applying $C_2 - 2C_1, C_3 - 3C_1, C_4 - 4C_1$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Applying $3C_3 - 2C_2, 3C_4 - 5C_2$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

Applying $\frac{C_2}{-3}, \frac{C_4}{18}$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

Applying $C_4 \leftrightarrow C_3$, we get $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is in normal form $[I_3\ O]$. Hence Rank of A is '3'.

Example 7 : Reduce A to canonical form and find its rank, if $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Applying $R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1$, we get $A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$

Applying $R_4 - R_3, 2R_1 + R_3$, we get
$$A \sim \begin{bmatrix} 2 & 0 & -2 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Applying $R_4 - R_2$, we get
$$A \sim \begin{bmatrix} 2 & 0 & -2 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 + C_1, 2C_4 - 3C_1$, we get
$$A \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 4 \\ 0 & -4 & -8 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $\frac{C_1}{2}, \frac{C_2}{-4}, \frac{C_4}{2}$ we get
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_3 + 8C_2, C_4 - 3C_2$, we get
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $\frac{C_3}{-3}, \frac{C_4}{2}$, we get
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 - C_3$, we get
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

Applying $R_2 \leftrightarrow R_3$, we get
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in the form of $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$.

$\therefore$ Rank of A is '3'.

Example 8 : Find the Rank of the matrix $A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -3\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -\end{bmatrix}$ by reducing it to the normal form.

Solution : Given matrix $A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -3\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$

Applying $R_2 \to 2R_2 - R_1, R_3 \to 2R_3 - 3R_1, R_4 \to R_4 - 3R_1$, $A \sim \begin{bmatrix}2 & 3 & -1 & -1\\ 0 & -5 & -3 & -5\\ 0 & -7 & 9 & -1\\ 0 & -6 & 3 & -4\end{bmatrix}$

Applying $C_2 \to 2C_2 - 3C_1, C_3 \to 2C_3 + C_1, C_4 \to 2C_4 + C_1$, $A \sim \begin{bmatrix}2 & 0 & 0 & 0\\ 0 & -10 & -6 & -10\\ 0 & -14 & 18 & -2\\ 0 & -12 & 6 & -8\end{bmatrix}$

Applying $R_1 \to \frac{R_1}{2}, R_2 \to \frac{R_2}{2}, R_3 \to \frac{R_3}{2}, R_4 \to \frac{R_4}{2}$, $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -5 & -3 & -5\\ 0 & -7 & 9 & -1\\ 0 & -6 & 3 & -4\end{bmatrix}$

Applying $R_3 \to 5R_3 - 7R_2; R_4 \to 5R_4 - 6R_2$, $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -5 & -3 & -5\\ 0 & 0 & 66 & 30\\ 0 & 0 & 33 & 10\end{bmatrix}$

Applying $C_2 \to \frac{C_2}{-5}; C_3 \to \frac{C_3}{-3}; C_4 \to \frac{C_4}{-5}; \frac{R_3}{-1}; \frac{R_4}{-1}$ $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 22 & 6\\ 0 & 0 & 11 & 2\end{bmatrix}$

Applying $C_3 \to C_3 - C_2; C_4 \to C_4 - C_2$, $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 22 & 6\\ 0 & 0 & 11 & 2\end{bmatrix}$

Applying $R_4 \to 2R_4 - R_3$, $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 22 & 6\\ 0 & 0 & 0 & -2\end{bmatrix}$

Applying $C_4 \to 11C_4 - 3C_3$, $A \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 22 & 0\\ 0 & 0 & 0 & -22\end{bmatrix}$

Applying $R_3 \to \frac{R_3}{22}, R_4 \to \frac{R_4}{-22}$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

$\therefore$ Rank of $A = 4$.

Example 9 : Find the rank of the matrix A by reducing it to the normal form

where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$, $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2, R_4 \to R_4 + 7R_2$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$

Applying $C_3 \to C_3 - 2C_2, C_4 \to C_4 + 5C_2$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$

Applying $R_4 \to R_4 - 6R_3$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$

Applying $C_4 \to C_4 + 2C_3$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 \end{bmatrix}$

Applying $C_4 \to \frac{C_4}{-18}$, $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$

$\therefore$ Rank of $A = 4$.

Example 10 : Find the rank of $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ **[JNTU 2008(Set No.3)]**

Solution : Let $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Operating $R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + R_1,\ R_4 \to R_4 + 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

Operating $\frac{R_2}{5}, \frac{R_3}{10}, \frac{R_4}{15}$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $\frac{C_4}{-2}$, we get

$$A \sim \begin{bmatrix} 1 & 4 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \; A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

$$\therefore \; \rho(A) = 2.$$

Hence Rank of given matrix = 2.

Example 11 : Find the rank of $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$. Then

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{ (Applying } R_1 \leftrightarrow R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{ (Applying } R_2 - R_1,\; R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{ (Applying } \frac{R_3}{2}\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (Applying } R_3 - R_2, R_4 + R_2\text{)}$$

$\therefore$ Rank (A) = ρ (A) = Number of non-zero rows in the matrix = 2

Example 12 : Find the Rank of the Matrix, by reducing it to the normal form

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Solution : Given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & k-3 \end{bmatrix} \text{(Applying } R_2 - 4R_1, R_3 - 3R_1, R_4 - R_1\text{)}$$

$$\sim \begin{bmatrix} 7 & 0 & 5 & -1 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix} \text{(Applying } 7R_1 + 2R_2, R_3 - R_2\text{)}$$

$$\sim \begin{bmatrix} 14 & 0 & 0 & 18 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix} \text{(Applying } 2R_1 + 5R_3, R_2 + 3R_3\text{)}$$

$$\sim \begin{bmatrix} 14 & 0 & 0 & 18 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix} \text{(Applying } 2R_4 + R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix} \text{(Applying } \frac{C_1}{4}, \frac{C_2}{-7}, \frac{C_3}{2}\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix} \text{(Applying } C_4 - 18C_1, C_4 - C_2, C_4 - 4C_3\text{)}$$

Let $2k-2=0 \Rightarrow k=1$. Then
The matrix will be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

$\therefore$ Rank of A = 3

Example 13 : Find the Rank of the Matrix $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to the normal form.

Solution : Given matrix is $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & -13 & -13 \\ 0 & -9 & -9 \end{bmatrix} \text{(Applying } 2R_2 - 3R_1, 2R_3 - R_1\text{)}$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \left(\text{Applying } \frac{R_2}{-13}, \frac{R_3}{-9}\right)$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_1 - 3R_2\text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_3 - 2C_1\text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_3 - C_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\text{Applying } \frac{R_1}{2}\right) \square \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

This is in normal form.

$\therefore$ Rank of the matrix = 2

Example 14 : Find the Rank of the Matrix, by reducing it to the normal form $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} (\text{Applying } R_2 - 2R_1, R_3 - 4R_1, R_4 - 4R_1)$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (\text{Applying } R_4 - 3R_2, R_3 - R_2)$$

$$\sim \begin{bmatrix} 10 & 5 & 0 & 4 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (\text{Applying } 5R_1 + 3R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(\text{Applying } \frac{C_1}{10}, \frac{C_2}{5}, \frac{C_3}{-5}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_2 - C_1, C_4 - 4C_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_4 + 7C_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_2 \leftrightarrow C_3\text{)}$$

This is of the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$ which is in normal form.

$\therefore$ Rank of A = 2

Example 15 : Find the Rank of the Matrix, by reducing it to the normal form

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & 5 \end{bmatrix} \text{(Applying } R_3 - R_1, R_2 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 10 & 11 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \text{(Applying } R_1 \to R_1 + 3R_2, R_3 \to R_3 + 2R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \text{(Applying } C_3 - 10C_1, C_4 - 11C_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \text{(Applying } 9R_2 - 2R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \left(\text{Applying } \frac{R_2}{-9}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 9 & 0 \end{bmatrix} \text{(Applying } C_3 \leftrightarrow C_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\text{Applying } C_4 - 2C_2, \frac{C_3}{9}\right)$$

$$\sim [I_3 \quad O]$$

This is in normal form
Rank of the matrix = 3

Example 16 : Find the Rank of the Matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$, by reducing it to the normal form.

Solution : Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \text{(Applying } R_2 - R_1, R_3 - 2R_1, R_4 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(Applying } R_1 - 2R_2, R_4 - R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_4 - R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_1 + R_3, R_2 - R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } C_4 - 2C_1\text{)}$$

This is of the form $\begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$ which is normal form.

$\therefore$ Rank of the matrix is 3.

Example 17 : Reduce the Matrix A to its normal form where $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

and hence find the rank.

Solution : Given matrix is $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{(Applying } R_2 - 2R_1, R_3 + R_1, R_4 - 2R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(Applying } R_1 - 2R_2, R_3 + R_2, \tfrac{R_4}{3}\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(Applying } R_1 + 2R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(Applying } C_4 - 3C_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim I_4 \text{ (Applying } R_4 \leftrightarrow R_3\text{)}$$

This is in normal form.

Rank (A) = 4.

Example 18 : Find the Rank of the Matrix, by reducing it to the normal form.

$$\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

Solution : Given matrix is $\sim \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$R_2 \leftrightarrow R_1$ gives $\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$R_2 - 2R_1; R_4 - 4R_1$ gives $\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$

$$R_2 \leftrightarrow R_3 \text{ gives } \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$$

$$R_1 + 2R_2; R_4 - R_2 \text{ gives } \sim \begin{bmatrix} 1 & 0 & -3 & 2 & 4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 9 & 9 & -4 \end{bmatrix}$$

$$\begin{matrix} 5R_1 + 3R_3; \\ 5R_2 + R_3; \\ 5R_4 - 9R_3 \end{matrix} \text{ gives } \sim \begin{bmatrix} 5 & 0 & 0 & 3 & 8 \\ 0 & 5 & 0 & 22 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 0 & -18 & 16 \end{bmatrix}$$

$$\frac{C_1}{5}, \frac{C_2}{5}, \frac{C_3}{5} \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 31 & 8 \\ 0 & 1 & 0 & 22 & 1 \\ 0 & 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & -18 & 16 \end{bmatrix}$$

$$C_4 - 31C_1; C_5 - 8C_1 \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 22 & 1 \\ 0 & 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & -18 & 16 \end{bmatrix}$$

$$C_4 - 22C_2; C_5 - C_2 \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & -4 \\ 0 & 0 & 0 & -18 & 16 \end{bmatrix}$$

$$C_4 - 7C_3; C_5 + 4C_3 \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -18 & 16 \end{bmatrix}$$

$$\frac{C_4}{-18}; \frac{C_5}{16} \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$C_5 - C_4$ gives $\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

This is of the form $[I_4 \quad O]$ which is in normal form.

Rank $(A) = 4$.

Example 19 : Find the rank of the Matrix by reducing it to the normal form.

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & -2 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & -2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 5 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_4\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 6 & -16 & 12 \\ 0 & 1 & 2 & 3 \\ 0 & 7 & -10 & 9 \end{bmatrix} \text{(Applying } R_2 - 5R_1; R_4 - 4R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & -16 & 12 \\ 0 & 7 & -10 & 9 \end{bmatrix} \text{(Applying } R_2 \leftrightarrow R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -6 \\ 0 & 0 & -24 & -12 \end{bmatrix} \text{(Applying } R_1 + R_2; R_3 - 6R_2; R_4 - 7R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 14 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{(Applying } \frac{R_3}{-2}, \frac{R_4}{-12})$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 14 & 3 \end{bmatrix} \text{(Applying } R_3 \leftrightarrow R_4)$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \text{(Applying } 2R_1 - 5R_3, R_2 - R_3, R_4 - 7R_3)$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix} \text{(Applying } 2C_4 + 3C_1)$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix} \text{(Applying } C_4 - 4C_2)$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix} \text{(Applying } C_4 - C_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(Applying } \frac{R_1}{2}, \frac{R_3}{2}, \frac{R_4}{-8})$$

$$\sim I_4$$

$\therefore$ Rank of tthe matrix is 4.

Example 20 : Reduce the $\begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix}$ matrix, to normal form and find its rank.

[JNTU (A) June 2011, June 2013 (Set No. 2)]

Solution : Given matrix is $A = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix}$

$$\begin{matrix} R_3 - R_1 \\ R_4 - R_1 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 3 & 5 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\begin{matrix} R_3 - 3R_2 \\ R_4 - R_2 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -3 & -7 \end{bmatrix}$$

$$\begin{matrix} 7R_1 - 3R_2 \\ 7R_2 + 4R_3 \\ 7R_4 - 3R_3 \end{matrix} \text{ gives } \sim \begin{bmatrix} 7 & 0 & 0 & 65 \\ 0 & 7 & 0 & -33 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\frac{R_1}{7}, \frac{R_2}{7}, \frac{R_3}{-7}, \frac{R_4}{2} \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & \frac{65}{7} \\ 0 & 1 & 0 & -\frac{33}{7} \\ 0 & 0 & 1 & \frac{17}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} C_4 - \frac{65}{7}C_1 \\ C_4 - \frac{33}{7}C_2 \\ C_4 - \frac{17}{7}C_3 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim I_4 \text{ which is normal form.}$$

$\therefore$ Rank (A) = 4.

Example 21 : Reduce the matrix $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$ to normal form and find its rank.

Solution : Given matrix is $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$

$9R_2 - 5R_1$ gives $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 0 & -44 & 21 & -21 \\ 6 & 8 & 2 & 4 \end{bmatrix}$

$9R_3 - 6R_1$ gives $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 0 & -44 & 21 & -21 \\ 0 & 30 & 0 & 0 \end{bmatrix}$

$\frac{C_3}{3}, \frac{C_4}{3}$ gives $\begin{bmatrix} 9 & 7 & 1 & 2 \\ 0 & -44 & 7 & -7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$C_4 + C_3$ gives $\begin{bmatrix} 9 & 7 & 1 & 3 \\ 0 & -44 & 7 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_2 + 44R_3$ gives $\begin{bmatrix} 9 & 7 & 1 & 3 \\ 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\frac{R_2}{7}$ gives $\begin{bmatrix} 9 & 7 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_1 - 7R_3$ gives $\begin{bmatrix} 9 & 7 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_1 - R_2$ gives $\begin{bmatrix} 9 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\frac{C_1}{9}, \frac{C_2}{7}$ gives $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$C_2 - C_1$ gives $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ gives $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is of the form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ which is normal form.

Rank of $A = 3$.

Example 22 : Reduce the matrix to its normal form $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

Solution : Given matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

$R_2 - R_1; R_3 - 3R_1$ gives $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$

$R_3 - R_2$ gives $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 - C_1$ gives $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 - C_1; C_4 + C_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 - 2C_2; C_4 - 5C_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

This is to required normal form.

1.15 ELEMENTARY MATRIX

Definition : It is a matrix obtained from a unit matrix by a single elementary transformation.

For example, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the elementary matrices obtained from I_3 by applying the elementary operations $C_1 \leftrightarrow C_2$, $R_3 \to 2R_3$ and $R_1 \to R_1 + 2R_2$ respectively.

An important note : Consider $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \\ 8 & 1 & 1 \end{pmatrix}$

Let us interchange 1st and 3rd rows. We get $B = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 4 \end{pmatrix}$

This B is same as the matrix obtained by pre-multiplying A with the matrix E_{13} obtained from unit matrix by interchanging 1st and 3rd rows in it.

Verification : $E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$E_{13} \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 7 \\ 8 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 4 \end{pmatrix}$$

Similarly interchange of two columns of a matrix is the result of the post-multiplication by an elementary matrix obtained from unit matrix by the interchange of the corresponding columns.

Similar observations can be made regarding row/column operations.

These ideas are used in the following example.

In this connection we have the following theorem.

Theorem : Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication (post-multiplication) with corresponding elementary matrix.

SOLVED EXAMPLES

Example 1 : Obtain non-singular matrices P and Q such that PAQ is of the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ and hence obtain its rank.

Solution : We know by cor. 2 of theorem in 1.9 we can find two non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$. For this we adopt the following procedure.

We write $A = I_3 A I_3$ i.e., $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we go on applying elementary row operations and column operations on the matrix A (the left hand member of the above equation) until it is reduced to the normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$. Every row operation will also be applied to the pre-factor I_3 of the product on R.H.S. and every column operation will be applied to the post-factor I_3 of the product on the R.H.S.

Applying $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 + R_2$, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

This is of the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

We can verify that det $P = 1 \neq 0$ and det $Q = 1 \neq 0$

$\therefore$ P and Q are non-singular such that $PAQ = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$. Hence the rank of $A = 2$.

Note : P and Q obtained in the above problem are not unique.

Example 2 : If $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$, obtain non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by suitable elementary row and column operations.

Solution : We write $A = I_3 A I_4$.

$$i.e.\ \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New, we apply row operations on the prefactor I_3 in R.H.S. and column operations on postfactor I_4 in R.H.S. to get $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix} = PAQ$.

Applying $R_1 \leftrightarrow R_3$, we get $\begin{bmatrix} 1 & -4 & 11 & -19 \\ 5 & 1 & 4 & -2 \\ 3 & 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying $C_2 \to C_2 + 4C_1,\ C_3 \to C_3 - 11C_1,\ C_4 \to C_4 + 19C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 21 & -51 & 93 \\ 3 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -11 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1,\ R_3 \to R_3 - 3R_1$ and $C_2 \to \frac{C_2}{7}, \frac{C_3}{-17}, \frac{C_4}{31}$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{11}{17} & \frac{19}{31} \\ 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{3}$ and $R_3 \to \frac{R_3}{2}$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{-5}{3} \\ \frac{1}{2} & 0 & \frac{-3}{2} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{11}{17} & \frac{19}{31} \\ 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2,\ C_4 \to C_4 - C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{-5}{3} \\ \frac{1}{2} & 0 & \frac{-3}{2} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & \frac{-1}{7} & \frac{-1}{7} \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

Performing $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{-5}{3} \\ \frac{1}{2} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{17} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & \frac{-1}{7} & \frac{-1}{7} \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 4/17 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$

We can verify that P and Q are non-singular.

Since the matrix $A \sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$, therefore, the rank of A is 2.

Example 3 : Find non-singular matrices P and Q so that PAQ is of the normal form, where $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$.

Solution : We write $A = I_3 A I_4$

i.e., $$\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We will apply elementary row operations and column operations on A in L.H.S., to reduce it to normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$. Every row operation will also be applied to prefactor of the product on R.H.S. and every column operation will also be applied to the post-factor of the product on the R.H.S.

Applying $R_2 + 2R_1$ and $R_3 + R_1$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2$ gives $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\frac{1}{5}R_2$ gives $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_4 - C_3$ gives $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 - 3R_2$ gives $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_2 + 2C_1$ and $C_4 - C_1$ gives $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ gives $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Thus we have $P = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $PAQ = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

$\therefore$ Rank of given matrix is 2.

Example 4 : Find the non-singular matrices P and Q such that the normal form of A is PAQ where $A=\begin{bmatrix}1&3&6&-1\\1&4&5&1\\1&5&4&3\end{bmatrix}$. Hence find its rank.

Solution : We write $A=I_3AI_4$ i.e., $\begin{bmatrix}1&3&6&-1\\1&4&5&1\\1&5&4&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

We will apply elementary row operations and column operations on A in L.H.S. to reduce to a normal form $\begin{bmatrix}I_r&O\\O&O\end{bmatrix}$. Every row operation will also be applied to prefactor I_3 of the product on R.H.S. and every column operation will also be applied to the post-factor I_4 of the product on the R.H.S.

Applying R_2-R_1 and R_3-R_1, we get $\begin{bmatrix}1&3&6&-1\\0&1&-1&2\\0&2&-2&4\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Applying R_3-2R_2, we get $\begin{bmatrix}1&3&6&-1\\0&1&-1&2\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Applying C_2-3C_1, C_3-6C_1 and C_4+C_1, we get

$$\begin{bmatrix}1&0&0&0\\0&1&-1&2\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&-3&-6&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Applying C_3+C_2 and C_4-2C_2, we get

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Thus $\begin{bmatrix}I_2&O\\O&O\end{bmatrix}=PAQ$ where $P=\begin{bmatrix}1&0&0\\-1&1&0\\1&-2&1\end{bmatrix}$ and $Q=\begin{bmatrix}1&-3&-9&7\\0&1&1&-2\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Since $A\sim\begin{bmatrix}I_2&O\\O&O\end{bmatrix}$, therefore, rank of A is 2.

Example 5 : Find P and Q such that the normal form of $A=\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & 1\\ 3 & 1 & 1\end{bmatrix}$ is PAQ. Hence find the rank of A.

Solution : We write $A = I_3\, AI_3$, *i.e.*,

$$\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & 1\\ 3 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We will apply elementary row operations and column operations on A in L.H.S. to reduce to a normal form $\begin{bmatrix}I_r & O\\ O & O\end{bmatrix}$. Every row operation will also be applied to the prefactor of the product on R.H.S. and every column operation will also be applied to the post-factor of the product on the R.H.S.

Applying $C_2 + C_1$ and $C_3 + C_1$, we get $\begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 2\\ 3 & 4 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Applying $R_2 - R_1$ and $R_3 - 3R_1$, we get $\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 2\\ 0 & 4 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Applying $\frac{R_2}{2}$ and $\frac{R_3}{4}$, we get $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -(1/2) & 1/2 & 0\\ -(3/4) & 0 & 1/4\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Applying $R_3 - R_2$, we get $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -(1/2) & 1/2 & 0\\ -(1/4) & -(1/2) & 1/4\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Applying $C_3 - C_2$, we get $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -(1/2) & 1/2 & 0\\ -(1/4) & -(1/2) & 1/4\end{bmatrix} A \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$

Thus the L.H.S. is in the normal form $\begin{bmatrix}I_2 & O\\ O & O\end{bmatrix}$.

Hence $P_{3\times 3} = \begin{bmatrix}1 & 0 & 0\\ -(1/2) & 1/2 & 0\\ -(1/4) & -(1/2) & 1/4\end{bmatrix}$ and $Q_{3\times 3} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$

Since A $\sim \begin{bmatrix}I_2 & O\\ O & O\end{bmatrix}$, therefore, rank of A is 2.

Example 6 : If $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non-singular matrices such that PAQ is in normal form.

Solution : We write $A = I_3\ AI_4$

$$i.e, \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We will apply elementary row operations and column operations on A in L.H.S. to reduce to a normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$. Every row operation will also be applied to prefactor I_3 of the product on R.H.S. and every column operation will also be applied to the pre-factor I_4 of the product on R.H.S.

Performing $R_3 \leftrightarrow R_1$, we get $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Performing $R_2 - 2R_1$ and $R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing $R_1 + R_3$, we get $\begin{bmatrix} 1 & 0 & -4 & -8 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Performing $C_4 - 2C_3$, we get $\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Performing $C_3 + 4C_1$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Performing $5C_3 - C_2$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -1 & -24 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 20 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Performing $24C_2 - C_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -120 & 0 & 0 \\ 0 & 0 & -24 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing $\frac{R_2}{-120}$ and $\frac{R_3}{-24}$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1/120 & 1/60 \\ -1/24 & 0 & 1/12 \end{bmatrix} A \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the L.H.S. is in the normal form $[I_3 \quad O]$

Hence $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{-1}{120} & \frac{1}{60} \\ \frac{-1}{24} & 0 & \frac{1}{12} \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We can verify that P and Q are non-singular. Also rank of $A = 3$.

Example 7 : Find the non-singular matrices P and Q such that PAQ is in the normal form of the matrix and find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Solution : Since A is 3×4 matrix, we write $A = I_3\, A\, I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$ on L.H. S. and on pre-factor of A , we get

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_1,\ C_3 \to C_3 - 3C_1,\ C_4 \to C_4 + 2C_1$ on L.H.S. and on post-factor of A, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ on L.H.S and on pre-factor of A, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to (-1/6) . C_2$ on L.H.S. and on post - factor of A, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -3 & 2 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 + 5C_2, C_4 \to C_4 - 7C_2$ on L.H.S. and on post factor of A, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & -1/6 & -5/6 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$i.e. \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 1/3 & -4/3 & -1/3 \\ 0 & -1/6 & -5/6 & 7/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The matrix A is reduced to the form $\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

$\therefore$ The rank of $A = 2$.

1.16 THE INVERSE OF A MATRIX BY ELEMENTARY TRANSFORMATIONS (*Gauss-Jordan method*)

We can find the inverse of a non-singular square matrix using elementary row operations only. This method is known as Gauss-Jordan Method.

Working Rule for finding the Inverse of a Matrix :

Suppose A is a non-singular square matrix of order n. We write $A = I_n A$.

Now, we apply elementary row operations only to the matrix A and the prefactor I_n of the R.H.S. We will do this till we get an equation of the form

$$I_n = BA.$$

Then obviously B is the inverse of A.

SOLVED EXAMPLES

Example 1: Find the inverse of $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ using elementary row operations (Gauss-Jordan method).

Solution : Given $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

We write $A = I_3 A$ *i.e.,* $\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Now we will apply row operations to the matrix A (the left hand member of the above equation) until it is reduced to the form I_3. Same operations will be performed on prefactor I_3 of R.H.S.

Applying $R_1 \leftrightarrow R_3$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$

Applying $R_3 \to R_3 + 2R_1$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$

Applying $R_3 \to R_3 + 5R_2$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$

Applying $R_1 \to R_1 + 2R_2$, we get $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$

Applying $R_3 \to \frac{R_3}{8}$, we get

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{8} & \frac{6}{8} & \frac{4}{8} \\ \frac{-1}{8} & \frac{3}{8} & \frac{-2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A$$

Applying $\frac{R_2}{-1}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{8} & \frac{6}{8} & \frac{4}{8} \\ \frac{1}{8} & \frac{-3}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A = \frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix} A$$

This is of the form $I_3 = BA$ where $B = \frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

$$\therefore\ A^{-1} = B = \frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

Example 2 : Find the inverse of the matrix A using elementary operations (*i.e.*, using Gauss-Jordan method).

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Solution : We write $A = I_4 A$ *i.e.*, $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

We will apply only row operations on matrix A in L.H.S. and same row operations on I_4 in R.H.S. till we reach the result $I_4 = BA$. Then B is the inverse of A.

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$ and $R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to 2R_1 - 3R_2$, $R_3 \to 2R_3 - 11R_2$ and $R_4 \to R_4 + 2R_2$, we get

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_4$ and $R_3 \to R_3 + 6R_4$, we get

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -1 & -3 & 0 & -2 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + R_3$, we get

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -2 & -2 & 2 & 4 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

Applying $\frac{R_1}{-2}, \frac{R_2}{-2}$ and $R_3 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

This is of the form $I_4 = BA$.

$$\therefore \quad A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

Example 3 : Find the inverse of the matrix by elementary row operations

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution : We write $A = I_3 A$.

We perform elementary row operations on LHS to reduce it to I_3. We will perform the same row operations on prefactor I_3 on RHS. Then we will get an equation of the form $I_3 = BA$. Then $B = A^{-1}$. That is,

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 + 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

Applying $\frac{R_2}{2}$ and $\frac{R_3}{2}$, we get

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + R_2$, we get

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{-1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} A \quad \left(\text{Applying } R_3 \to \frac{R_3}{-2}\right)$$

Applying $R_1 \to R_1 - 6R_3$ and $R_2 \to R_2 + 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

This is of the form $I_3 = BA$ which gives

$$A^{-1}=\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Example 4 : Given $A=\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find its inverse.

Solution : We write $A = I_3 A$.

Applying row transformations, we reduce LHS into I_3.

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A$$

Applying $R_1 + R_2$, we get

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A$$

Applying $R_1 + R_3$, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A$$

Applying $\frac{R_2}{2}$ and $\frac{R_3}{3}$, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}A$$

This is of the form $I_3 = BA$

Thus $B = A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$

Example 5 : If $A = \begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{pmatrix}$, find A^{-1}

Solution : Given matrix is $A = \begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

We write $A = I_3A$

We perform elementary row operations on A in LHS and make it equivalent to I_3.

We perform the same row operations on I_3 in RHS and make the equation $I_3 = BA$.

Then B is the inverse of A.

$$\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$ gives $\sim \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$\begin{matrix} R_2 - 2R_1 \\ 2R_3 - R_1 \end{matrix}$ gives $\sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$

$R_3 - 2R_2$ gives $\sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -2 & 3 & 2 \end{bmatrix} A$

$\begin{matrix} R_1 + R_3 \\ R_2 - 3R_3 \end{matrix}$ gives $\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 7 & -11 & -6 \\ -2 & 3 & 2 \end{bmatrix} A$

$$\frac{R_1}{2}, \frac{R_2}{-1} \text{ gives } \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix} A$$

This is of the form $I_3 = BA$

$$\Rightarrow B = A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix}$$

Note 1 : To find the inverse of the matrix using column operations only we proceed as follows.

We write $A = AI_n$.

Now we go on applying column operations only to the matrix A in L.H.S. and perform the same column operations on the prefactor I_n in R.H.S. till we reach the result $I_n = AB$.

Then obviously B is the inverse of A.

Note 2 : While, in principle, A^{-1} can be calculated by the determination of the adjoint of A, the procedure is highly unpracticable in case of large n.

Even if $n=4$, we have to calculate 16 third order determinants.

For $n=5$, we have to calculate 25 determinants of 4th order *i.e.,* $25 \times 16 \times 4$ determinants of second order and perform the needed additions and subtractions.

For large n, the impracticability need not be emphasized.

The Gauss-Jordan procedure is the best to calculate the inverse of A when A is of large order. The procedure can be comfortably implemented on computer.

EXERCISE 1.4

Find the rank of the matrix by reducing it to normal form.

1. $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 5 & 6 & 10 \\ -1 & 1 & -2 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 10 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & 8 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \end{bmatrix}$ [ ]

10. Determine the non-singular matrices P and Q such that PAQ is in the normal form for A. Hence find the rank of the matrix A.

(*i*) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

(*iii*) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(*iv*) $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

11. Determine whether the matrix $A = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 2 & 4 & 2 & -3 \end{bmatrix}$ has inverse.

12. Find the inverse of the matrix

(*i*) $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ by using elementary operations.

13. Compute the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ by using elementary operations.

ANSWERS

1. 3 **2.** 3 **3.** 3 **4.** 3 **5.** 3 **6.** 2 **7.** 3 **8.** 4

9. (*i*) $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Rank $= 2$

SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Definition : An equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$ ...(1)

where $x_1, x_2, \ldots, x_n$ are unknowns and $a_1, a_2, \ldots, a_n, b$ are constants is called a linear equation in n unknowns.

1.17 DEFINITION

Consider the system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ as given below :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots(2)$$

where a_{ij}'s and $b_1, b_2, \ldots, b_m$ are constants. An ordered n-tuple $(x_1, x_2, \ldots, x_n)$ satisfying all the equations in (2) simultaneously is called a solution of the system (2).

Given a system (2), we do not known in general whether it has a solution or not. If there is at least one solution for (2), the system is said to be consistent. If (2) does not have any solution, the system is said to be inconsistent.

If the system is consistent, some questions that we may have to answer are the following

(*i*) Is the solution unique ?

(*ii*) If not how many solutions are there.

(*iii*) In the most general solution, how many independent parameters exist ?

In the following, we propose to answer these questions.

The system of equations in (2) can be conveniently written in matrix form as

$$AX = B \quad \ldots(3)$$

where $A = [a_{ij}],\ X = (x_1, x_2, \ldots, x_n)^T$ and $B = (b_1, b_2, \ldots, b_m)^T$ ...(4)

The matrix $[A \mid B]$ is called the augmented matrix of the system (2). By reducing $[A \mid B]$ to its row echelon form, all the questions related to the existence and uniqueness of the solutions of $AX = B$ can be answered satisfactorily.

1.18 DEFINITION

If $B = O$ in (3), the system is said to be ***Homogeneous.*** Otherwise the system is said to be **non-homogeneous.**

The system $AX = O$ is always consistent since $X = O$ (*i.e.*, $x_1 = 0, x_2 = 0, \ldots, x_n = 0$) is always a solution of $AX = O$.

This solution is called a ***trivial solution*** of the system.

Given $AX = O$, we try to decide whether it has a solution $X \neq O$. Such a solution, if exists, is called a ***non-trivial solution.***

Given $AX = B$, however, there is no certainity whether it has a solution (*i.e.)* whether it is *consistent.*

Let us consider few examples.

Example 1 : Consider $2x + 3y = 0$; $3x - 2y = 0$.

Here $x = 0$, $y = 0$ is a solution. We cannot find any other solution. Here $x = 0$, $y = 0$ is a unique solution.

Example 2 : Consider $2x + 3y = 0$; $4x + 6y = 0$

This is a homogeneous system of two equations in two unknowns.

$x = 0$, $y = 0$ is a solution of the system.

Further $(-3k, 2k)$ where k is any number is also a solution of the system. Taking $k = 1$, $(-3, 2)$ is a solution of the system. Hence the above homogeneous system has also non-trivial solutions. In fact, it has infinitely many non-trivial solutions.

Example 3 : Consider $2x + 3y = 5$; $3x + 2y = 5$

This system does not possess any solution. This is an inconsistent system as the two equations cannot simultaneously be satisfied by any x, y.

Example 4 : Consider $2x + 3y = 5$; $4x + 6y = 10$.

This is a system of two equations in two unknowns.

Using one of them we get $2x = 5 - 3y$ (*i.e.*) $x = \frac{5}{2} - \frac{3}{2}y$. For any $y = k$, $x = \frac{5}{2} - \frac{3}{2}k$.

Thus for any value of k, $x = \frac{5}{2} - \frac{3}{2}k$, $y = k$ is a solution of the system.

The system is consistent and has infinitely many solutions.

Note : A given system of m equations in n unknowns, may or may not have a solution whether $m < n$ or $m = n$ or $m > n$.

1.19 WORKING PROCEDURE TO SOLVE $AX = B$

Let us first consider n equations in n unknowns (*i.e.*) $m = n$.

The system will be of the form

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

....

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n.$

The above system can be written as $AX = B$ where A is an $n \times n$ matrix.

1. Solution by using A^{-1} :

If $|A| \neq 0$, we can find A^{-1}. Pre-multiplying the equation AX = B by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \ (i.e.)\ (A^{-1}A)X = A^{-1}B$$

(*i.e.*) $IX = (A^{-1}B)$ (*i.e.*) $X = A^{-1}B.$

Thus a solution is obtained. (In fact, here this is the only solution).

2. Solution by Cramer's Rule :

If $|A| \neq 0$, let us define A_i = matrix obtained by replacing the i^{th} column of A by B.

Then Cramer's rule states that, the solution is given by $x_i = \dfrac{\det A_i}{\det A}$ for $i = 1, 2, \dots, n$.

In many of the real life examples, since the number of unknowns can be large, Cramer's rule will not be useful.

Further, priorily, we do not know whether a solution for the system exists or not.

Hence we will proceed to describe a procedure known as Gauss Jordan procedure to solve the general system of m simultaneous equations in n unknowns given by $AX = B$.

SOLVED EXAMPLES

Example 1 : Write the following equations in matrix form AX = B and solve for X by finding A^{-1} : $x + y - 2z = 3,\ 2x - y + z = 0,\ 3x + y - z = 8$. **[JNTU(K)June 2009 (Set No.1)]**

Solution : Taking the matrices $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $B = \begin{pmatrix} 3 \\ 0 \\ 8 \end{pmatrix}$, we get AX = B.

Consider $A = I_3 A$. We apply row operations only and change LHS into I_3.

$R_2 - 2R_1; R_3 - 3R_1$ gives $\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 5 \\ 0 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} A$

$3R_1 + R_2, 3R_3 - 2R_2$ gives $\begin{pmatrix} 3 & 0 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & -2 & 3 \end{pmatrix} A$

$5R_1 + R_3, R_2 - R_3$ gives $\begin{pmatrix} 15 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 3 & -3 \\ -5 & -2 & 3 \end{pmatrix} A$

$\dfrac{R_1}{15}, \dfrac{R_2}{-3}, \dfrac{R_3}{5}$ gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & \frac{-2}{5} & \frac{3}{5} \end{pmatrix} A$

This is of the form $I_3 = CA$

$$\therefore A^{-1} = C = \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & \frac{-2}{5} & \frac{3}{5} \end{pmatrix}$$

Hence $\quad X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & -1 & 1 \\ -1 & \frac{-2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 5 \\ 9/5 \end{bmatrix}$$

$\therefore$ The solution is $x = \frac{8}{5}, \quad y = 5, \ z = \frac{9}{5}$

Example 2 : Solve the equations by finding the inverse of the coefficient matrix :
$x + y + z = 1, 3x + 5y + 6z = 4, 9x + 26y + 36z = 16$. **[JNTU (K)June 2009 (Set No.3)]**

Solution : This is left as an exercise to the student.

1.20 GAUSS - JORDAN PROCEDURE TO SOLVE A SYSTEM OF SIMULTANEOUS EQUATIONS IN *n* UNKNOWNS

Consider the system of m equations in n unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m.$$

(*i.e.*) $AX = B$

The augmented matrix of the above system is $[A|B] = \left(\begin{array}{c|c} a_{11} \ a_{12} \ a_{13} \dots a_{1n} & b_1 \\ a_{21} \ a_{22} \ a_{23} \dots a_{2n} & b_2 \\ \dots\dots\dots\dots & \dots \\ \dots\dots\dots\dots & \dots \\ a_{m1} \ a_{m2} \ a_{m3} \dots a_{mn} & b_m \end{array}\right)$

Step I. Using row operations only on 1st row to last row, make the (1, 1) element equal to 1. Let us not disturb the 1st row so obtained.

Fixing the first row, using this and second row to last row make all other elements of the first column equal to 0 [This is the end of stage (1)].

Step II. Now the augmented matrix looks like $[A|B] \simeq \left(\begin{array}{l|l} 1\ a'_{12}\ \ a'_{13} \ldots a'_{1n} & b'_1 \\ 0\ a'_{22}\ \ a'_{23} \ldots a'_{2n} & b'_2 \\ \ldots\ldots\ldots\ldots\ldots & \ldots \\ \ldots\ldots\ldots\ldots\ldots & \ldots \\ 0\ a'_{m2}\ a'_{m3} \ldots a'_{mn} & b'_m \end{array}\right)$

Using R_2 to R_m and row operations only, make the (2, 2) element equal to 1. Now fixing the second row, using row operations, make all other elements of 2nd column equal to zero. This does not disturb the first column obtained earlier [This is the end of stage (2)].

Step III. Now the augmented matrix looks like $[A|B] \simeq \left(\begin{array}{cccccc|c} 1 & 0 & a''_{13} & a''_{14} & \ldots & a''_{1n} & b''_1 \\ 0 & 1 & a''_{23} & a''_{24} & \ldots & a''_{2n} & b''_2 \\ 0 & 0 & a''_{33} & a''_{34} & \ldots & a''_{3n} & b''_3 \\ 0 & 0 & a''_{43} & a''_{44} & \ldots & a''_{4n} & b''_4 \\ \ldots & & & & & & \ldots \\ \ldots & & & & & & \ldots \\ 0 & 0 & a''_{m3} & a''_{m4} & \ldots & a''_{mn} & b''_m \end{array}\right)$

Continuing the procedure as in step I and step II, making use of row operations only, reduce the above $(A \mid B)$ to its row echelon form.

One of the following cases occurs. The augmented matrix becomes

$$[A \mid B] \simeq \left(\begin{array}{ccccccc|c} 1 & 0 & 0 & \ldots & a^*_{1,r+1} & \ldots & a^*_{1m} & b^*_1 \\ 0 & 1 & 0 & \ldots & a^*_{2,r+1} & \ldots & a^*_{2m} & b^*_2 \\ 0 & 0 & 1 & \ldots & a^*_{r,r+1} & \ldots & a^*_{nm} & b^*_r \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & b^*_{r+1} \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & b^*_{r+2} \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & b^*_m \end{array}\right)$$

Case (*i*) : If $b^*_{r+1} = b^*_{r+2} \ldots = b^*_m = 0$, the system of equations is consistent.

If $r = n$, the system has unique solution.

If $r < n$, the system has an infinite number of solutions. In that case

$$\begin{aligned} x_1 &= b^*_1 - a^*_{1\,r+1}\ x_{r+1} \ldots - a^*_{1n} x_n \\ x_2 &= b^*_2 - a^*_{2\,r+1}\ x_{r+1} \ldots - a^*_{2n} x_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= b^*_r - a^*_{r\,r+1}\ x_{r+1} \ldots - a^*_{2n} x_n \end{aligned}$$

Choosing $x_{n+1} = B_1,\ x_{n+2} = B_2 \ldots,\ x_n = B_{n-r}$ where $B_1, B_2, \ldots, B_{n-r}$ to be arbitrary we can write $(x_1, x_2, \ldots, x_n)$ and this solution consists of $(x - r)$ parameters (*viz.*) $B_1, B_2, \ldots, B_{n-1}$.

Case (*ii*) : If one or several of $b^*_{r+1}, b^*_{r+2}, \ldots, b^*_n$ are non-zero, the system of equations leads to an equation like 0 = non-zero quantity, which is impossible.

In this case the system does not possess any solution (*i.e.*) and the system is inconsistent.

We note that this procedure terminates [(*i.e.*) has an end] since the number of stages cannot be more than m.

The above discussion implies that in case(*i*) when the system is consistent row echelon form of A has r non-zero rows (*i.e.*) its rank = r and row echelon form of $[A \mid B]$ has also r non-zero rows.

(*i.e.*) rank of $[A \mid B]$ is also r. (*i.e.*) $\rho(A) = \rho\,[A \mid B]$

In case (*ii*) when the sysem is inconsistent $\rho(A) \neq \rho[A \mid B]$

This implies the following :

For Non-Homogeneous System :

The system $AX = B$ is consistent *i.e.* it has a solution (unique or infinite) if and only if rank of A = rank of $[A/B]$

(*i*) If rank of A = rank of $[A/B] = r <$ number of unknowns, the system is consistent, but there exists infinite number of solutions. Giving arbitrary values to $n - r$ of the unknowns, we may express the other r unknowns in terms of these.

(*ii*) If rank of A = rank of $[A/B] = r = n$ the system has unique solution.

(*iii*) If rank of $A \neq$ rank of $[A/B]$ the system is inconsistent. It has no solution.

To obtain solutions, set $(n - r)$ variables any arbitrary value and solve for the remaining unknowns.

Thus irrespective of $m < n$ or $m = n$, or $m > n$, we have decided about the existence and uniqueness of solutions of a system of linear simultaneous equations $AX = B$.

This procedure is illustrated through the following examples.

SOLVED EXAMPLES

Example 1 : Show that the equations $x + y + z = 4,\ \ 2x + 5y - 2z = 3,\ x + 7y - 7z = 5$ are not consistent.

Solution : We write the given equations in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$\text{Consider } [A|B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$$[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

We can see that rank $[A|B]=3$, since the number of non-zero rows is 3.
Applying the same row operations on A, we get from above

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the number of non-zero rows is 2 so the rank of $A=2$.

Here we have rank $(A) \neq$ rank $[A/B]$. $\therefore$ The given system is not consistent.

Example 2 : Solve the equations $x+y+z=9$; $2x+5y+7z=52$ and $2x+y-z=0$

Solution : Writing the equations in the matrix form $AX=B$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_1$ and $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -18 \end{bmatrix}$$

Applying $R_3 \to 3R_3 + R_2$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -20 \end{bmatrix}$

Writing in the form of linear equations, we get
$x + y + z = 9$ ...(1) $3y + 5z = 34$ ...(2)
and $-4z = -20 \Rightarrow z = 5$
From (2), $3y = 34 - 5z = 34 - 25 = 9 \Rightarrow y = 3$
Again from (1), $x = 9 - y - z = 9 - 3 - 5 = 1$.
$\therefore x = 1,\ y = 3,\ z = 5$ is the solution.

Note : We find $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{bmatrix}$ which is in Echelon form and number of non-zero rows is 3. Hence the rank of the matrix is 3. Again $[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$ which is in Echelon form and rank is equal to 3. Hence we have rank of A = rank of $[A|B]$ and the system of equations is consistent. Here number of unknowns is 3 = rank of A.

$\therefore$ The solution is unique.

Example 3 : Solve the system of linear equations by matrix method. $x+y+z=6, \quad 2x+3y-2z=2, \quad 5x+y+2z=13$ **[JNTU (H) June 2010 (Set No. 1)]**

Solution : Let $A=\begin{bmatrix}1&1&1\\2&3&-2\\5&1&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}6\\2\\13\end{bmatrix}$

The given equations can be written in form, AX = B

$$\Rightarrow \begin{bmatrix}1&1&1\\2&3&-2\\5&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\2\\13\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&1&1\\0&1&-4\\0&-4&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\-17\end{bmatrix}(\text{Applying } R_2-2R_1, R_3-5R_1)$$

$$\Rightarrow \begin{bmatrix}1&0&5\\0&1&-4\\0&0&-19\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\-57\end{bmatrix}(\text{Applying } R_1-R_2, R_3+4R_2)$$

$$\Rightarrow \begin{bmatrix}1&0&5\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}16\\-10\\3\end{bmatrix}\left(\text{Applying } \frac{R_3}{-19}\right)$$

$$\Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}(\text{Applying } R_1-5R_3, R_2+4R_3)$$

Hence the solution is $x=1, y=2, z=3$.

Example 4 : Discuss for what values of λ, μ the simultaneous equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (*i*) no solution (*ii*) a unique solution (*iii*) an infinite number of solutions. **[JNTU 2001, 2002S, 2004S (Set No. 1), 2005 (Set No.3)]**

Solution : The matrix form of given system of equations is

$$AX=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}=B$$

We have the augmented matrix is $[A/B]=\begin{bmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

Case I. Let $\lambda \neq 3$ then rank of A = 3 and rank of $[A/B]$ = 3, so that they have same rank. Then the system of equations is consistent. Here the number of unknowns is 3 which is same as the rank of A. The system of equations will have a unique solution. This is true for any value of μ.

Thus if $\lambda \neq 3$ and μ has any value, the given system of equations will have a unique solution.

Case II. Suppose $\lambda = 3$ and $\mu \neq 10$, then we can see that rank of A = 2 and rank of $[A/B]$ = 3. Since the ranks of A and $[A/B]$ are not equal, we say that the system of equations has no solution (inconsistent).

Case III. Let $\lambda = 3$ and $\mu = 10$. Then we have rank of A = rank of $[A/B]$ = 2.

$\therefore$ The given system of equations will be consistent.

But here the number of unknowns = 3 > rank of A.

Hence the system has infinitely many solutions.

Example 5 : Find for what values of λ the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case.

Solution : The given system can be expressed as

$$AX = B \ (i.e.) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \quad ...(1)$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda-1 \\ \lambda^2-1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda-1 \\ \lambda^2-3\lambda+2 \end{bmatrix}$$

Now the given equations will be consistent if and only if, $\lambda^2 - 3\lambda + 2 = 0$, *i.e.*, iff λ = 1 or = 2.

Case I. If λ = 1 then we have $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ so that rank of A = 2.

$[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that rank of $[A/B] = 2$ and the two ranks are equal.

Then we have the system of equations is consistent.

We can write $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Writing as linear equations, we get $x + y + z = 1, y + 3z = 0$

Let $z = k$ then $y = -3k$ and $x = 1 - y - z = 1 + 3k - k = 2k + 1$

Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k+1 \\ -3k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where k is a parameter. In this case the system will have infinite number of solutions.

Case II. $\lambda = 2$. Here $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ so that the rank of A is 2

and $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and rank $[A/B] = 2$.

Here we have rank A = rank $[A/B]$ and the system will be consistent. Here no. of unknowns = 3 > rank of A. Hence the number of solutions is infinite.

The system can be written as $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x + y + z = 1$ and $y + 3z = 1$

Take $z = c \Rightarrow y = 1 - 3c$ and $x = 1 - y - z = 1 - 1 + 3c - c = 2c$

$\therefore$ The general solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2c \\ 1-3c \\ c \end{bmatrix} = c \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Where c is a parameter. The system has infinitely many solutions.

Example 6 : Discuss for all values of λ, the system of equations

$x + y + 4z = 6$; $x + 2y - 2z = 6$; $\lambda x + y + z = 6$ with regard to consistency

Solution : The above system of equations can be written in the matrix form

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} \text{ i.e., } AX = B$$

Performing $R_2 \to R_2 - R_1, R_3 \to R_3 - \lambda R_1$, we get $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6\lambda \end{bmatrix}$

For the above system to have a unique solution we must have rank $A = 3$ = number of unknowns.

$\Rightarrow$ det. $A \neq 0 \Rightarrow (1-4\lambda)+(6-6\lambda) \neq 0 \Rightarrow \lambda \neq \frac{7}{10}$

$\therefore$ If $\lambda \neq \frac{7}{10}$ the system is consistent and the solution is unique.

If $\lambda = \frac{7}{10}$, we have $A \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & \frac{3}{10} & \frac{-18}{10} \end{bmatrix}$

Applying $R_3 \to R_3 - \frac{3}{10} R_2$, we get $A \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ so that rank of $A = 2$

Applying same operations, $[A/B] \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & \frac{18}{10} \end{bmatrix}$

Here rank $[A/B] = 3$.

$\therefore$ We have rank of $A \neq$ rank of $[A/B]$ and system of equations is inconsistent *i.e.*, no solution.

Example 7 : If $a + b + c \neq 0$, show that the system of equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ has no solution. If $a + b + c = 0$, show that it has infinitely many solutions.

Solution : Given system of equations can be expressed in the matrix equation $AX = B$

i.e. $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Performing $R_1 \to R_3$, we get $\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$

Performing $R_2 \to R_2 - R_1, R_3 \to R_3 + 2R_1$, we get $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ b-c \\ a+2c \end{bmatrix}$

Performing $R_3 \to R_3 + R_2$, we get $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c \\ b-c \\ a+b+c \end{bmatrix}$

We have rank $A = 2$ and rank $[A/B] = 2$ if $a+b+c = 0$, then rank of A = rank of $[A/B]$.

$\therefore$ The system will be consistent if $a + b + c = 0$

Number unknowns = 3> rank of A $\therefore$ Number of solutions is infinite.

If $a + b + c \neq 0$, then rank $A \neq$ rank $[A, B]$. Hence the system will be inconsistent.

Example 8 : Find the values of 'a' and 'b' for which the equations $x + y + z = 3; x + 2y + 2z = 6; x + ay + 3z = b$ have (*i*) No solution (*ii*) a unique solution (*iii*) infinite number of solutions. **[JNTU 2001]**

Solution : The system of equations can be written in the matrix form

$$AX = B, \text{ i.e.}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$$

$$R_3 - R_2 \text{ gives, } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & a-2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b-6 \end{bmatrix}$$

$$R_2 - R_1 \text{ gives, } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & a-2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ b-6 \end{bmatrix}$$

$$R_3 - R_2 \text{ gives, } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & a-3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ b-9 \end{bmatrix}$$

Case 1. When $a \neq 3$, b has any value, Rank $A = 3$ and Rank $[A/B] = 3$

No. of variables = 3; Rank A = Rank $[A/B] = 3$

So, system has unique solution.

Case 2. Suppose $a = 3, b = 9$

Then Rank $A = 2$ and Rank $[A, B] = 2$

No. of variables = 3

The system will have infinite no. of solutions with $n - r = 3 - 2 = 1$ arbitrary variable.

Case 3. $a = 3$ and $b \neq 9$

Then Rank $A = 2$ and Rank $[A/B] = 3$

Since $\rho(A) \neq \rho[A/B]$, Inconsistant

$\therefore$ The system of equations has no solution.

Example 9 : Show that the equations $x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30$ are consistent and solve them.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

Then they satisfy the equation $AX = B$.

Consider $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

$R_3 - R_1, R_2 - R_1$ gives $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$

$R_3 - 3R_2$, gives $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Above is the Echelon form of the matrix $[A/B]$. $\therefore$ Rank of $[A/B] = 2$

By the same elementary transformations, we get $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank of $A = 2$. Since rank A = rank $[A/B] = 2$, the system of equations is consistent.

Here the number of unknowns is 3. Since rank of A is less than the number of unknowns, therefore, the system of equations will have infinite no. of solutions.

We see that the given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$\Rightarrow y + 2z = 8$ and $x + y + z = 6$

Taking $z = k$ we get $y = 8 - 2k$ and $x = k - 2$ is the solution, where k is an arbitrary constant.

Example 10 : Find whether the following equations are consistent, if so solve them.
$x + y + 2z = 4;\ 2x - y + 3z = 9;\ 3x - y - z = 2$

[JNTU May 2005S, 2005 (Set No.1), (A) Dec. 2013 (Set No. 4)]

Solution : The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

i.e. AX = B

The augmented matrix $[A|B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$[A\,|\,B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$ (Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$)

$[A\,|\,B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$ (Applying $R_3 \to 3R_3 - 4R_2$)

Since Rank of A = 3 and Rank of $[A|B] = 3$ $\therefore$ Rank of A = Rank of $[A|B]$

The given system is consistent. So it has a solution.

Since Rank of A = Rank of $[A/B]$ = number of unknowns,

$\therefore$ The given system has a unique solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$\Rightarrow$ $x + y + 2z = 4$ .... (1) $-3y - z = 1$ .... (2) $-17z = -34$ or $z = 2$ .... (3)

Substituting $z = 2$ in (2) $\Rightarrow -3y - 2 = 1 \Rightarrow -3y = 3 \Rightarrow y = -1$

Substituting $y = -1, z = 2$ in (1), we get

$x - 1 + 4 = 4 \Rightarrow x = 1$

$\therefore$ $x = 1, y = -1, z = 2$ is the solution.

Example 11 : Find whether the following system of equations are consistent. If so solve them.

$x + 2y + 2z = 2;\ 3x - 2y - z = 5;\ 2x - 5y + 3z = -4;\ x + 4y + 6z = 0$

[JNTU May 2005S, Sep. 2008, (H) June 2010 (Set No. 4), (A) Dec. 2013 (Set No. 4)]

Solution : The given equations can be written in the matrix form as AX = B

$$(i.e.)\ \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

The Augmented matrix $[A, B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1; R_4 \to R_4 - R_1$, we get $[A, B] \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

Applying $R_3 \to 8R_3 - 9R_2$ and $R_4 \to 4R_4 + R_2$, we get

$$[A, B] \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left(\text{Applying } \frac{R_3}{55}, \frac{R_4}{9} \text{ and } R_4 \to R_4 - R_3\right)$

Since Rank of A = 3 and Rank of $[A, B] = 3$, we have Rank of A = Rank of $[A, B]$.

$\therefore$ The given system is consistent and it has solution.

Since Rank of A = Rank of $[A, B]$ = number of unknowns

$\therefore$ The given system has a unique solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x + 2y + 2z = 2$ .... (1), $\quad -8y - 7z = -1$ .... (2) and $z = -1$

Put $z = -1$ in (2) $\quad \Rightarrow -8y + 7 = -1 \Rightarrow -8y = -8 \Rightarrow y = 1$

Put $y = 1, z = -1$ in (1) $\Rightarrow x + 2 - 2 = 2 \Rightarrow x = 2$

$\therefore x = 2, y = 1, z = -1$ is the solution.

Example 12 : Find the value of λ for which the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ will have infinite number of solutions and solve them with that λ value. **[JNTU May 2005S]**

Solution : The given system of equations can be written in the matrix form as AX = B

(*i.e.*) $\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$

The Augmented matrix $[A, B] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$

Applying $R_2 \to 3R_2 - R_1, R_3 \to R_3 - 2R_1$, $[A, B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda - 8 & -9 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, $[A, B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda + 5 & 0 \end{bmatrix}$

If $\lambda = -5$, Rank of A = 2 and Rank of $[A, B] = 2$

Number of unknowns = 3

$\therefore$ Rank of A = Rank of $[A, B] \neq$ number of unknowns

Hence when $\lambda = -5$, the given system is consistent and it has an infinite number of solutions.

If $\lambda = -5$ the given system becomes $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$

$3x - y + 4z = 3$ ..... (1) and $7y - 13z = -9$ ..... (2)

From (2), Let $z = k$ · Then

$7y - 13k = -9 \Rightarrow 7y = 13k - 9 \Rightarrow y = (13k - 9)/7$

Substituting the value of y in (1) , we get

$$3x - \frac{1}{7}(13k - 9) + 4k = 3 \Rightarrow 3x = \frac{13}{7}k - 4k + 3 - \frac{9}{7}$$

$$3x = -\frac{15}{7}k + \frac{12}{7} \Rightarrow x = \frac{1}{7}(-5k + 4)$$

$\therefore$ $x = \frac{1}{7}(-5k+4), y = \frac{1}{7}(13k-9), z = k$ is the solution.

Example 13 : Find whether the following set of equations are consistent if so, solve them.

$x_1 + x_2 + x_3 + x_4 = 0$
$x_1 + x_2 + x_3 - x_4 = 4$
$x_1 + x_2 - x_3 + x_4 = -4$
$x_1 - x_2 + x_3 + x_4 = 2$ **[JNTU M2005]**

Solution : Given system of equations is

$x_1 + x_2 + x_3 + x_4 = 0$
$x_1 + x_2 + x_3 - x_4 = 4$
$x_1 + x_2 - x_3 + x_4 = -4$
$x_1 - x_2 + x_3 + x_4 = 2$

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$

Then the system of equations is of the form $AX = B$.

Consider $[A \mid B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$

Applying $R_4 \to R_4 - R_1$, $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & -2 & 0 & -4 \\ 0 & -2 & 0 & 0 & 2 \end{bmatrix}$$

This is in Echelon form. Number of non-zero rows is 4.

$\therefore$ Rank $A = 4$ and rank $[A \mid B] = 4$.

Hence the given system of equations is consistent.

Writing in the equation form, we get

$$-2x_2 = 2 \Rightarrow x_2 = -1$$
$$-2x_3 = -4 \Rightarrow x_3 = 2.$$
$$-2x_4 = 4 \Rightarrow x_4 = -2$$

and $x_1 + x_2 + x_3 + x_4 = 0$

Substituting the values of x_2, x_3, x_4 in the last equation, we get

$$x_1 - 1 + 2 - 2 = 0 \Rightarrow x_1 = 1.$$

Thus $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2$ is the only solution.

Example 14 : Prove that the following set of equations are consistent and solve them.

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

[JNTU May 2006, April 2007, Aug. 2007, 2008, (H) June 2009, (K) 2009S, (K) May 2010 (Set No. 1)]

Solution : The given system of equations can be written in the matrix form as follows:

$$AX = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix} = B$$

The Augmented matrix of the given equations is

$$[A|B] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix} \text{[Applying } R_1 \leftrightarrow R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \text{[Applying } R_2 - 3R_1 \text{ and } R_4 - 2R_1\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \left[\text{Applying } \frac{R_2}{-3}\right]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix} \text{[Applying } R_3 - 10R_2 \text{ and } R_4 + 7R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix} \left[\text{Applying } \frac{3}{29} R_2\right]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left[\text{Applying } R_4 + \frac{17}{3} R_3\right]$$

Thus the matrix [A|B] has been reduced to Echelon form.

$\therefore$ Rank [A|B] = no. of non-zero rows = 3

By the same row operations, we have

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank (A) = 3

Since Rank (A) = Rank [A|B] = 3, therefore the given equations are consistent.

Also rank (A) = 3 = no. of unknowns.

Hence the given equations have unique solution.

The given equations are equivalent to the equations

$$x + 2y = 4;\ y - \frac{2}{3}z = \frac{11}{3};\ z = -4$$

On solving these equations, we get

$$x = 2, y = 1, z = -4.$$

Example 15 : Test for consistency and hence solve the system :

$x + y + z = 6,\ x - y + 2z = 5,\ 3x + y + z = 8,\ 2x - 2y + 3z = 7$ **[JNTU 2008S (Set No.3)]**

Solution : Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Then the system of equations can be written as $AX = B$

The Augmented matrix of the given equations is

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix} \text{(Applying } R_2 - R_1, R_3 - 3R_1, R_4 - 2R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_4 - 2R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

Thus the matrix [A/B] has been reduced to Echelon form.

$\therefore$ Rank [A/B] = no. of non-zero rows = 3

By applying same operations, we have Rank (A) = 3.

Since Rank (A) = Rank [A/B], therefore the given equations are consistent.

Hence the given equations have a unique solution.

From above Echelon form, we get

$-3z = -9 \Rightarrow z = 3$

$-2y + z = 1 \Rightarrow -2y = -y \Rightarrow y = 2$

$x + y + z = 6 \Rightarrow x = 6y - 3 = 6 - 2 - 3 = 1$

$\therefore$ The solution is $x = 1, y = 2, z = 3$.

Example 16 : Show that the equations $x - 4y + 7z = 14$, $3x + 8y - 2z = 13$, $7x - 8y + 26z = 5$ are not consistent. **[JNTU 2008S (Set No.3)]**

Solution : Consider $A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}, B = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Then the given system of equations can be written as AX = B.

The Augmented matrix of the given equations is

$$[A/B] = \begin{bmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{bmatrix}$$

Operating $R_2 - 3R_1$ and $R_3 - 7R_1$, we get

$$\begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{bmatrix}$$

Operating $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{bmatrix}$$

Number of non-zero rows is 3. $\therefore$ Rank $(A, B) = 3$.

We can observe that number of non-zero rows is 2. $\therefore$ Rank (A) = 2.

Since Rank (A) $\neq$ Rank [A/B], the system is inconsistent.

Example 17 : Test for the consistency of $x+y+z=1, x-y+2z=1, x-y+2z=5, 2x-2y+3z=1, 3x+y+z=2$. **[JNTU 2008S (Set No.4), (A) Nov. 2010 (Set No. 3)]**

Solution : Given system can be written as AX = B

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 5 \\ 2 & -2 & 3 & 1 \end{bmatrix}$$

Applying $R_2 - R_1, R_3 - R_1, R_4 - 2R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 4 \\ 0 & -4 & 1 & -1 \end{bmatrix}$$

Applying $R_3 - R_2, R_4 - R_2$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -2 & 0 & -1 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

We can see that number of non-zero rows = 4.

$\therefore$ Rank [A,B] = 4.

But $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Applying Same operations)

$\therefore$ Rank (A) = 3.

Since Rank (A) $\neq$ Rank [A,B], therefore the system is not consistent.

Example 18 : Solve the system of equations $x+2y+3z=1,\ 2x+3y+8z=2,\ x+y+z=3$.

[JNTU(H) June 2009 (Set No.2), (A) Nov. 2010 (Set No. 2)]

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Then system can be written as $AX = B$

Consider $[A,B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix} \text{(Applying } R_2 - 2R_1, R_3 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

This is in Echelon form. Number of non-zero rows is 3. $\therefore \rho[A,B] = 3$.

Now $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$ and $B \sim \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Number non-zero rows is 3.

$\therefore$ rank $(A) = \rho(A) = 3$

$\therefore\ \rho[A,B] = \rho(A) = 3$

$\therefore$ The above system has unique solution.

Now solve $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\Rightarrow x + 2y + 3z = 1$

$-y + 2z = 0$

$-4z = 2$

$\Rightarrow -2z = 1 \Rightarrow z = -1/2$

$\Rightarrow -y + 2\left(\frac{-1}{2}\right) = 0 \Rightarrow -y - 1 = 0$

$\Rightarrow -y = 1 \Rightarrow y = -1$

Now $x + 2y + 3z = 1$

$\Rightarrow\ x + 2(-1) + 3\left(\frac{-1}{2}\right) = 1$

$\Rightarrow x - 2 - \frac{3}{2} = 1 \Rightarrow x = \frac{9}{2}$.

Thus $X = \begin{bmatrix} \frac{9}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$ is the unique solution.

Example 19 : Solve the system of equations $x + y + z = 6,\ x - y + 2z = 5,\ 3x + y + z - 8$

[JNTU (H) June 2009 (Set No.3)]

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 6 \\ 5 \\ -8 \end{bmatrix}$

Then the given system of equations are of the form $AX = B$

Consider $[A,B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & -8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -26 \end{bmatrix} \quad (\text{Applying } R_2 - R_1 \text{ and } R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -25 \end{bmatrix} \text{ (Applying } R_3 - R_2)$$

No. of non-zero rows is 3.

$\therefore$ The rank of $[A, B] = \rho[A,B] = 3$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Also we can easily see that rank $(A) = \rho(A) = 3$

$\therefore$ $\rho[A,B] = \rho(A) = n = 3$

$\therefore$ The given system has unique solution.

$$B \sim \begin{bmatrix} 6 \\ -1 \\ -25 \end{bmatrix}$$

Consider $AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -25 \end{bmatrix}$$

$$\Rightarrow -3z = -25 \Rightarrow z = \frac{25}{3}, -2y + z = -1 \text{ and } x + y + z = 6$$

On solving these equations, we get

$$y = \frac{-1-z}{-2} = \frac{1}{2}(1+z) = \frac{1}{2}\left(1 + \frac{25}{3}\right) = \frac{14}{3} \text{ and } x = 6 - (y+z) = 6 - \left(\frac{14}{3} + \frac{25}{3}\right) = 6 - 13 = -7$$

$\therefore$ The solution is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 14/3 \\ 25/3 \end{bmatrix}$$

Example 20 : Show that the system of equations $x + 2y + z = 3, 2x + 3y + 2z = 5,$ $3x - 5y + 5z = 2,\ 3x + 9y - z = 4$ are consistent and solve them. **[JNTU(K) June 2009 (Set No.1)]**

Solution : Take $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; B = \begin{pmatrix} 3 \\ 5 \\ 2 \\ 4 \end{pmatrix}$

The Angmented matrix of the given system of equations is

$$[A, B] = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{pmatrix} \text{(Applying } R_2 - 2R_1, R_3 - 3R_1, R_4 - 3R_1\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 0 & -4 & -8 \end{pmatrix} \text{(Applying } R_4 + 3R_2\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix} \text{(Applying } \frac{R_4}{-4}\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \text{(Applying } R_3 - 11R_2\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \text{(Applying } \frac{R_3}{2}\text{)}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{(Applying } R_4 - R_3\text{)}$$

This is in Echelon form. Number of non-zero rows = 3.

$\therefore$ Rank $[A, B] = \rho[A, B] = 3$.

$$\text{Similarly } A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is in Echelon form. Number of non-zero rows is 3.

$\therefore$ Rank A = ρ (A) = 3.

ρ[A, B] = ρ(A) $\Rightarrow$ The system is consistent.

From $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}$

we get $x + 2y + z = 3$

$-y = -1$

$z = 2$

$\Rightarrow y = 1;\ z = 2$

$\Rightarrow x = -1$

$\therefore$ X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is the solution.

Example 21 : Find whether the following system of equations are consistent. If so solve them. $x + y + 2z = 9,\ x - 2y + 2z = 3,\ 2x - y + z = 3,\ 3x - y + z = 4$

[JNTU(H) June 2010 (Set No. 3)]

Solution : The given system of equations is non-homogeneous and can be written in the matrix form AX = B.

i.e., $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 3 \\ 4 \end{bmatrix}$

Augmented matrix is [A, B]

$$= \begin{bmatrix} 1 & 1 & 2 & 9 \\ 1 & -1 & 2 & 3 \\ 2 & -1 & 1 & 3 \\ 3 & -1 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & -2 & 0 & -6 \\ 0 & -3 & -5 & -15 \\ 0 & -4 & -5 & -23 \end{bmatrix} \text{(Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & -3 & -5 & -15 \\ 0 & -4 & -5 & -23 \end{bmatrix} \left(\text{Applying } R_2 \to \frac{R_2}{-2}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & -5 & -11 \end{bmatrix} (\text{Applying } R_1 \to R_1 - R_2, R_3 \to R_3 + 3R_2, R_4 \to R_4 + 4R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -5 \end{bmatrix} (\text{Applying } R_4 \to R_4 - R_3)$$

This is in Echelon form. We observe that rank A = 3 and rank [A, B] = 4.
rank (A) $\neq$ rank [A, B]
Hence the given system of equations is not consistent.

Example 22 : Find the values of p and q so that the equations $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+pz=q$ have

i. No solution *ii.* Unique solution

iii. An infinite number of solutions. **[JNTU(H) Dec. 2010]**

Solution : Given equations are $2x+3y+5z=9, 7x+3y+2z=8, \ 2x+3y+pz=q$

The equations can be written in the matrix form $AX = B$

i.e., $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ q \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 0 & -15 & -31 \\ 0 & 0 & p-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -47 \\ q-9 \end{bmatrix} \quad (\text{Applying } 2R_2 - 7R_1, R_3 - R_1)$$

Applying same operations,

$$[A,\ B] \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & p-5 & q-9 \end{bmatrix}$$

We have $\det A = \begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{vmatrix}$

$$= 2(3p-6) - 3(7p-4) + 5(21-6)$$

$$= 6p - 12 - 21p + 12 + 75 \quad = -15p + 75$$

$\det A = 0 \Rightarrow p = 5$

Case I : When $p = 5, q \neq 9$

The rank (A) = 2 and rank [A, B] = 3

The system will be inconsistent. The system will not have any solution.

Case II : When $p \neq 5$, $\det A \neq 0$

The system will have unique solution.

Case III : $p = 5, q = 9$

rank A = 2 and rank [A, B] = 2

Number of variables = 3

The system will be consistent and will have infinite number of solutions.

Example 23 : Find whether the following system of equations are consistent. If so solve them.

$$x + 2y - z = 3, 3x - y + 2z = -1, 2x - 2y + 3z = 2, x - y + z = -1$$

[JNTU(H) June 2011 (Set No. 3)]

Solution : Given system of equations is

$$x + 2y - z = 3, 3x - y + 2z = -1, 2x - 2y + 3z = 2, \quad x - y + z = -1$$

The given system can be written in matrix form $AX = B$ *i.e.*

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ -4 \\ -4 \end{bmatrix} \text{(Applying } R_2 - 3R_1, R_3 - 2R_1, R_4 - R_1\text{)}$$

$$\Rightarrow \begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \\ 32 \\ 2 \end{bmatrix} \text{(Applying } 7R_1 + 2R_2, 7R_3 - 6R_2, 7R_4 - 3R_2\text{)}$$

$$\Rightarrow \begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ \frac{32}{5} \\ -2 \end{bmatrix} \text{(Applying } \frac{R_3}{5}, \frac{R_4}{-1}\text{)}$$

$$\Rightarrow \begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ \frac{32}{5} \\ -\frac{42}{5} \end{bmatrix} \text{(Applying } R_4 - R_3\text{)}$$

Number of non-zero rows in A = 3. $\therefore$ Rank (A) = 3.

No. of non-zero rows in [A, B] = 4. Rank [A, B] = 4

Rank (A) $\neq$ Rank [A, B]

The given system of equations is inconsistent.

No solution exists.

Example 24 : Determine whether the following equations will have a solution, if so solve them.

$x_1 + 2x_2 + x_3 = 2, 3x_1 + x_2 - 2x_3 = 1; 4x_1 - 3x_2 - x_3 = 3, 2x_1 + 4x_2 + 2x_3 = 4$.

[JNTU(H) June 2011 (Set No. 4)]

Solution : Given equations are

$x_1 + 2x_2 + x_3 = 2; 3x_1 + x_2 - 2x_3 = 1; 4x_1 - 3x_2 - x_3 = 3;\ 2x_1 + 4x_2 + 2x_3 = 4$.

Writing the equation in the matrix form $AX = B$, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & -11 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -5 \\ 0 \end{bmatrix} \text{(Applying } R_2 - 3R_1, R_3 - 4R_1, R_4 - 2R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -11 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 6 \end{bmatrix} \text{(Applying } \frac{R_2}{-5}\text{)}$$

$$\sim\begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 6\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\1\\6\\0\end{bmatrix} \text{(Applying } R_1-2R_2, R_3+11R_2\text{)}$$

We can observe that A is in Echelon form. Number of non-zero rows = 3.

Rank (A) = Rank [A,B] = 3 = no. of variables.

The system is consistent and solution is unique.

$6x_3 = 6 \Rightarrow x_3 = 1$

$x_2 + x_3 = 1 \Rightarrow x_2 = 0$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3 = 1$

$\therefore$ The solution is $x_1 = 1, x_2 = 0, x_3 = 1$

Example 25 : Find whether the following system of equations are consistent. If so solve them. $2x - y + z = 5, 3x + y - 2z = -2, x - 3y - z = 2$ **[JNTU(H) June 2012 (Set No. 2)]**

Solution : Given of equations are $2x - y + z = 5; 3x + y - 2z = -2; x - 3y - z = 2$

Taking $A=\begin{bmatrix}2 & -1 & 1\\3 & 1 & -2\\1 & -3 & -1\end{bmatrix}; X=\begin{bmatrix}x\\y\\z\end{bmatrix}; B=\begin{bmatrix}5\\-2\\2\end{bmatrix}$, we get $AX = B$

Consider Augmented matrix, $[A,B]=\begin{bmatrix}2 & -1 & 1 & 5\\3 & 1 & -2 & -2\\1 & -3 & -1 & 2\end{bmatrix}$

Taking $2R_2 - 3R_1; 2R_3 - R_1$, we get $\sim\begin{bmatrix}2 & -1 & 1 & 5\\0 & 5 & -7 & -19\\0 & -5 & -3 & -1\end{bmatrix}$

Taking $5R_1 + R_2; R_3 + R_2$, we get $\sim\begin{bmatrix}10 & 0 & -2 & 6\\0 & 5 & -7 & -19\\0 & 0 & -10 & -20\end{bmatrix}$

No. of non-zero rows $= 3$

Rank A = Rank $[A,B] = 3$.

No. of variables $= 3$

$\therefore$ The given system is consistent and unique.

From the matrix , $-10z = -20 \Rightarrow z = 2$

$5y - 7z = -19 \Rightarrow 5y = -5 \Rightarrow y = -1$

$10x - 2z = 6 \Rightarrow 10x = 10 \Rightarrow x - 1$

$\therefore$ $x = 1, y = -1, z = 2$ is the unique solution.

Example 26 : Find the values of 'a' and 'b' for which the equations, $x+y+z=3, x+2y+2z=6, \quad x+9y+az=b$ have

i) No solution

ii) A unique solution

iii) Infinite number of solutions. **[JNTU (H) June 2012]**

Solution : Given equations are $x+y+z=3, x+2y+3z=6, x+9y+az=b$

Taking $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 9 & a\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 6\\ b\end{bmatrix}$

The given equations can be written as $AX=B$

Augmented matrix is $[A, B]=\begin{bmatrix}1 & 1 & 1 & 3\\ 1 & 2 & 3 & 6\\ 1 & 9 & a & b\end{bmatrix}$

$R_2-R_1;$
R_3-R_1 gives $\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 1 & 2 & 3\\ 0 & 8 & a-1 & b-3\end{bmatrix}$

$R_1-R_2;$
R_3-8R_2 gives $\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 3\\ 0 & 0 & a-17 & b-27\end{bmatrix}$

Case I : Let $a=17, b=27$.

Then rank of A $=$ rank of $[A,B]=2$.

No. of variables $=3$

$\therefore$ The system is consistent and there will be infinite number of solutions.

Case II : Let $a=17, b\neq 27$.

Rank of A $=2$ and Rank of $[A,B]=3$

$\therefore$ The system is inconsistant. (no solution)

Case III : $a\neq 17, b\neq 27$

Rank A $=$ Rank $[A,B]=3=$ no. of variables

$\therefore$ The system will be consistent and there will be unique solution.

Example 27 : Solve the system of linear equations by matrix method.

$x+y+z=6, \quad 2x+3y-2z=2, \quad 5x+y+2z=13$ **[JNTU (H) June 2010 (Set No. 1)]**

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 2 \\ 13 \end{bmatrix}$

The given equations can be written in form, AX = B

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & -4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ -17 \end{bmatrix} (\text{Applying } R_2 - 2R_1, R_3 - 5R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ -57 \end{bmatrix} (\text{Applying } R_1 - R_2, R_3 + 4R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -10 \\ 3 \end{bmatrix} \left(\text{Applying } \frac{R_3}{-19}\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} (\text{Applying } R_1 - 5R_3, R_2 + 4R_3)$$

Hence the solution is $x = 1, y = 2, z = 3$.

Example 28 : Solve the system $2x - y + 4z = 12; 3x + 2y + z = 10; x + y + z = 6$; if it is consistent. **[JNTU (A) June 2011, June 2013 (Set No. 2)]**

Solution : Given system of equations is $2x - y + 4z = 12; 3x + 2y + z = 10; x + y + z = 6$

Writing in the form of matrix equation $AX = B$

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 6 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \text{ gives } \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \end{bmatrix}$$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_2 \end{matrix} \text{ gives } \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_1+R_2 \\ R_3-3R_2 \end{matrix} \text{ gives } \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ 24 \end{bmatrix}$$

Rank of A = 3. With the same operations rank [A,B] = 3
The system of equations is consistent. Also no. of variables = 3 .
Hence the solution is unique.

$8z = 24 \Rightarrow z = 3$

$x - z = -2 \Rightarrow x = -2 + z = 1$

$-y - 2z = -8$

$-y = -8 + 2z = -2 \Rightarrow y = 2$

$\therefore\ x = 1, y = 2, z = 3$ is the unique solution.

Example 29 : If consistent, solve the system of equations, $x + y + z + t = 4; x - z + 2t = 2;$ $y + z - 3t = -1; x + 2y - z + t = 3$. **[JNTU (A) June 2011 (Set No. 3)]**

Solution : Given system of equations is

$x + y + z + t = 4; x - z + 2t = 2; y + z - 3t = -1;\ x + 2y - z + t = 3$

Taking $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}; B = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 3 \end{bmatrix}$

We get the system as $AX = B$

Consider augumented matrix, $[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & 2 & -1 & 1 & 3 \end{bmatrix}$

$$\begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} R_3 - R_1 \\ R_4 - R_1 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 2 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{matrix} R_3 - R_2 \\ R_4 - 2R_2 \end{matrix} \text{ gives } \sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 5 & 3 \end{bmatrix}$$

$R_4 + 2R_3$ gives $\sim \begin{bmatrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 9 & 9 \end{bmatrix}$

This is in Echelon form. Number of non-zero rows is 4.

$\therefore$ Rank (A,B) = 4. Also rank of (A) = 4.

The given system is consistent. Also number of variables = 4.

The system will have a unique solution.

From the matrix $9t = 9 \Rightarrow t = 1$

$z + 2t = 3 \Rightarrow z = 1$

$y + z - 3t = -1 \Rightarrow y = 1$

$x - z + 2t = 2 \Rightarrow x = 1$

$\therefore$ $x = 1, y = 1, z = 1, t = 1$ is the unique solution.

Example 30 : Test for consistency and if consistent solve the system,

$5x + 3y + 7t = 4;\ 3x + 26y + 2t = 9;\ 3x + 26y + 2t = 9$ **[JNTU (A) June 2011 (Set No. 4)]**

Solution : Given system of equations is

$5x + 3y + 7t = 4; 3x + 26y + 2t = 9; 7x + 2y + 10t = 5$

Let $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

Then given system is of the form $AX = B$

The Augmented matrix is

$$[A \mid B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix} \text{(Applying } 5R_2 - 3R_1, 5R_3 - 7R_1\text{)}$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix} \text{(Applying } \frac{R_2}{11}\text{)}$$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 + R_2\text{)}$$

Number of non-zero rows = 2

Rank A = 2 = Rank (A | B)

$\therefore$ The given system is consistent.

Number of variables = 3

Number of solutions is infinite.

From the matrix

$5x+3y+7z=4$

$11y-z=3$

Let $z=k$. Then

$11y=3+k \Rightarrow y=\frac{3+k}{11}$

$5x=4-3y-7z=4-3\left(\frac{3+k}{11}\right)-7k$

$=\frac{44-9-3k-77k}{11}=\frac{53-80k}{11} \Rightarrow x=\frac{53-80k}{55}$

This will give infinite number of solutions.

Example 31 : Solve the equations $2x+3y+5z=9; 7x+3y-2z=8; 2x+3y+\lambda z=u$

[JNTU (H) May 2012 (Set No. 2)]

Solution : Given simultaneous equations are

$$2x+3y+5z=9; 7x+3y-2z=8; 2x+3y+\lambda z=u$$

Taking $A=\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & -2\\ 2 & 3 & \lambda\end{bmatrix}; X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}9\\ 8\\ u\end{bmatrix}$

We can write the system of equations as $AX=B$

Augumented matrix is $[A,B]=\begin{bmatrix}2 & 3 & 5 & 9\\ 7 & 3 & -2 & 8\\ 2 & 3 & \lambda & u\end{bmatrix}$

$2R_2-7R_1; R_3-R_1$ gives $\begin{bmatrix}2 & 3 & 5 & 9\\ 0 & -15 & -39 & -47\\ 0 & 0 & \lambda-5 & u-9\end{bmatrix}$

Case I : Let $\lambda=5, u=9$. Then number of non - zero rows $=2$.

Rank of the matrix $[A,B]=2$ = rank of A

The system of equation is consistant.

No. of variable $=3$.

There will be infinite number of solutions.

Case II : Let $\lambda = 5, u \neq 9$. We will have rank $[A, B] = 3$ and rank (A) $= 2$.
The system will be inconsistant. There will be no solution.

Case III : If $\lambda \neq 5, u \neq 9$ then rank$(A, B) =$ rank $(A) = 3 =$ no. of variables.
Then there will be a unique solution.

Example 32 : Find the values of λ and u so that the system of equations $2x+3y+5z=9$; $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ has (*i*) unique solution (*ii*) No solution (*iii*) infinite no. of solutions. **[JNTU (H) May 2013]**

Solution : Given equations can be written as $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Consider the augumented matrix $[A, B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

$2R_2 + 7R_1; R_3 - R_1$ gives $\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$

Case (*i*) : $\lambda \neq 5$ and $\mu \neq 9$. Rank (A) = Rank (A, B) $= 3 =$ no. of variables.
The system will have unique solution.

Case (*ii*) : When $\lambda = 5$ and $\mu \neq 9$. Rank (A) $= 2$ and Rank (A, B) $= 3$
The system is not consistant. There will be no solution.

Case (*iii*) : When $\lambda = 5$ and $\mu = 9$.
Rank (A) = Rank (A, B) $= 2$ and no. of variables = 3.
$\therefore$ The system is consistant. It will have infinite no. of solutions.

EXERCISE 1. 5

1. Solve the system $x - 4y + 7z = 8$; $3x + 8y - 2z = 6$; $7x - 8y + 26z = 31$.
2. Show that the equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve them. **[JNTU 2003S (Set No. 3)]**
3. Solve completely the equations $x + y + z = 3$, $3x - 5y + 2z = 8$, $5x - 3y + 4z = 14$.
4. Solve the equations $\lambda x + 2y - 2z = 1$; $4x + 2\lambda y - z = 2$; $6x + 6y + \lambda z = 3$ for all values of λ.
5. Solve $x - y + 2z + t - 2 = 0$; $3x + 2y + t - 1 = 0$; $4x + y + 2z + 2t - 3 = 0$
6. Solve $5x + 3y + 7z = 4$; $3x + 26y + 2z = 9$; $7x + 2y + 10z = 5$.
7. Solve $x - 2y - 5z = -9$; $3x - y + 2z = 5$; $2x + 3y - z = 3$; $4x - 5y + z = -3$.
8. Solve $x + 2y + z = 14$; $3x + 4y + z = 11$; $2x + 3y + z = 11$.

9. Solve $x-y+2z=4$; $3x+y+4z=6$; $x+y+z=1$.

10. Solve $x+y+z=6$; $x-y+2z=5$; $2x-2y+3z=7$. **[JNTU2000, Sup. 2008 (Set No. 3)]**

11. Test for consistency and solve

(*i*) $2x + 3y + 7z = 5$; $3x + y - 3z = 12$; $2x + 19y - 47z = 32$ **[JNTU 2003S (Set No. 4)]**

(*ii*) $u+2v+2w=1; 2u+v+w=2, 3u+2u+2w=3, v+w=0$ **[JNTU (A) June 2013 (Set No. 3)]**

(*iii*) $2x+y+5z=4; 3x-2y+2z=2; 5x-8y-4z=1$ **[JNTU (A) June 2013 (Set No. 1)]**

12. Investigate for what values of λ and μ the simultaneous equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have

(*i*) no solution (*ii*) a unique solution (*iii*) an infinite number of solutions.

13. Find the values of a and b for which the equations $x + ay + z = 3$, $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent. When will these equations have a unique solution. **[JNTU 2004S (Set No. 1)]**

14. Show that the system $x+2y-5z=-9, 3x-y+2z=5, 2x+3y-z=3, 4x-5y+z=-3$ is consistent and solve it. **[JNTU (K) June 2009 (Set No.3)]**

ANSWERS

1. Inconsistent **2.** $x=-1, y=4, z=4$ **3.** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23/8 \\ 1/8 \\ 0 \end{bmatrix} + k \begin{bmatrix} -7/8 \\ -1/8 \\ 1 \end{bmatrix}$

4. When $\lambda \neq 2$ unique solution exists, when $\lambda = 2$, $x = \frac{1}{2} - k$, $y = k$, $z = 0$

5. $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} \frac{-4}{5} \\ \frac{6}{5} \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{-3}{5} \\ \frac{2}{5} \\ 0 \\ 1 \end{bmatrix}$ **6.** $x = \frac{1}{11}(7-16k), y = \frac{1}{11}(3+k), z = k$

7. $x = \frac{1}{2}; y = \frac{3}{2}; z = \frac{5}{2}$ **8.** Inconsistent **9.** $x = \frac{5-3k}{2}; y = \frac{k-3}{2}; z = k$

10. $x = 1, y = 2, z = 3$ **11.** (*i*) No solution **12.** (*i*) $\lambda = 5, \mu \neq 9$ (*ii*) $\lambda \neq 5$
(*iii*) $\lambda = 5, \mu = 9$ **13.** $a = -1, b = 6$; unique solution if $a \neq -1$

1.21 VECTORS

Definition : An ordered n-tuple of numbers is called an ***n*-vector.** The n numbers which are called **components** of the vector may be written in a horizontal or in a vertical line, and thus a vector will appear either as a row or column matrix. A vector over real numbers is called a **Real Vector** and the vector over complex numbers is called a **Complex Vector**.

e.g. $[2\ 3\ 0\ 1]$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are two vectors.

The n-vector space :

The set of all n-vectors of a field F, to be denoted by $V_n(F)$, is called the n-vector space over F. The elements of the field F are known as scalars relatively to the vectors.

The multiple kX of an n-vector X by a scalar k is the n-vector whose components are the products by k of the components of X.

The sum of two n-vectors X_1, X_2 is the n-vector whose components are the sums of the corresponding components of X_1 and X_2.

e.g. (1) Suppose $X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Then $2X = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$

(2) Suppose $X_1 = \begin{bmatrix} 5 \\ -6 \\ 7 \\ 8 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$. Then $X_1 + X_2 = \begin{bmatrix} 8 \\ -2 \\ 9 \\ 9 \end{bmatrix}$

These two operations on ***n*-vectors** are only special cases of the same operations on matrices, the laws corresponding to these operations are same as those in the case of matrices. It is also clear that if X_1 and X_2 are two n-vectors over a field F then $k_1X_1 + k_2X_2$ is also a vector over F where k_1 and k_2 are elements of F.

The important concepts of linearly dependent and linearly independent set of vectors will now be introduced.

1.22 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

1. Linearly dependent set of vectors : A set $\{X_1, X_2, \ldots X_n\}$ of r vectors is said to be a linearly dependent set, if there exist r scalars $k_1, k_2, \ldots k_r$, not all zero, such that $k_1X_1 + k_2X_2 + \ldots + k_rX_r = O$, where, O, denotes the n vector with components all zero.

2. Linearly independent set of vectors : A set $\{X_1, X_2 \ldots X_r\}$ of r vectors is said to be linearly independent set, if the set, is not linearly dependent, *i.e.,* if $k_1X_1 + k_2X_2 + \ldots + k_rX_r = O$, where, O, denotes the n vector with components all zero.

$\Rightarrow k_1 = 0, \quad k_2 = 0, \quad \ldots, \quad k_r = 0$

3. A vector as a linear combination of a set of vectors : A vector X which can be expressed in the form $X = k_1X_1 + k_2X_2 + \ldots + k_rX_r$ is said to be a linear combination of the set of vectors $X_1, X_2, \ldots, X_r$.

Here $k_1, k_2, \ldots, k_r$ are any numbers.

Important Results :

(*i*) If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the rest of the members.

(*ii*) If a set of members is linearly independent then no member of the set can be expressed as a linear combination of the rest of the members.

1.23 CONSISTENCY OF SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Consider a system of m homogeneous linear equations in n unknowns, namely

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0 \\ a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots (1)$$

We write

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ \vdots & \\ a_{m1} & a_{m2}\ldots a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \quad \ldots (2)$$

Then (1) can be written as $AX = O$ which is the matrix equation.

Here A is called the *coefficient matrix.* It is clear that

$x_1 = 0 = x_2 = x_3 = \ldots. = x_n$ is a solution of (1) *i.e.,* $X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of (*2*).

This is called **trivial solution** of $AX = O$.

This $AX = O$ is always **consistent** (*i.e.*) it has a solution.

The ***trivial solution*** is called the **zero solution**.

A zero solution is not linearly independent (*i.e.),* it is linearly dependent.

Theorem : The number of linearly independent solutions of the linear system $AX = O$ is $n–r$, r being the rank of the matrix $A_{m\times n}$ and n being the number of variables.

Note :

1. If A is a non-singular matrix. (*i.e.,* det $A \neq 0$) then the linear system $AX = O$ has only the zero solution.
2. The system $AX = O$ possesses a non-zero solution if and only if A is a non-singular matrix.

Working Rule for Finding the Solutions of the Equation $AX = O$:

Let rank of $A = r$ and rank of $[A/B] = r_1$.

Since all b's are zero, $r = r_1$, then

I.(*i*) If $r = n$(number of variables) $\Rightarrow$ the system of equations have only trivial solution (*i.e.*, zero solution).

(*ii*) If $r < n \Rightarrow$ the system of equations have an infinite number of non-trivial solutions, we shall have $n - r$ linearly independent solutions.

To obtain infinite solutions, set $(n - r)$ variables any arbitrary value and solve for the remaining unknowns.

If the number of equations is less than the number of unknowns, it has a non-trivial solution. The number of solutions of the equation $AX = O$ will be infinite.

II. If the number of equations is less than number of variables, the solution is always other than a trivial solution.

III. If the number of equations = number of variables, the necessary and sufficient condition for solutions other than a trivial solution is that the determinant of the coefficient matrix is zero.

SOLVED EXAMPLES

Example 1 : Solve completely the system of equations

$$x + y - 2z + 3w = 0;\ x - 2y + z - w = 0$$

$$4x + y - 5z + 8w = 0;\ 5x - 7y + 2z - w = 0$$

Solution : The given system of equations in matrix form is

$$AX = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$

Applying $R_2 - R_1$, $R_3 - 4R_1$ and $R_4 - 5R_1$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$

Applying $R_4 \to \dfrac{R_4}{4}$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$

Applying $R_3 - R_2$ and $R_4 - R_2$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in the Echelon form. We have

rank A = the number of non-zero rows in this Echelon form = 2

Since rank $A(= 2)$ is less than the number of unknowns (= 4), therefore, the given system has infinite number of non-trivial solutions.

$\therefore$ Number of independent solutions = 4 – 2 = 2.

Now, we shall assign arbitrary values to 2 variables and the remaining 2 variables shall be found in terms of these. The given system of equations is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the equations $x + y - 2z + 3w = 0,\ -3y + 3z - 4w = 0$

Taking $z = \lambda$ and $w = \mu$, we see that $x = \lambda - \frac{5}{3}\mu,\ y = \lambda - \frac{4}{3}\mu,\ z = \lambda,\ w = \mu$ constitutes the general solution of the given system.

Example 2 : Determine the values of λ for which the following set of equations may possess non-trivial solution:

$3x_1 + x_2 - \lambda x_3 = 0,\ 4x_1 - 2x_2 - 3x_3 = 0,\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

For each permissible value of λ, determine the general solution.

Solution : The given system of equations is equivalent to the matrix equation,

$$AX = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

The given system possess non-trivial solution, if rank of $A <$ number of unknowns.

i.e., Rank of $A < 3$

$$i.e., \quad \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$

$\Rightarrow \quad -4\lambda^2 - 32\lambda + 36 = 0 \quad \Rightarrow \quad \lambda^2 + 8\lambda - 9 = 0$

$\Rightarrow \quad (\lambda + 9)(\lambda - 1) = 0 \quad \therefore \lambda = -9 \quad \text{or} \quad \lambda = 1$

Case I. For $\lambda = -9$, the given system reduces to

$$3x_1 + x_2 + 9x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$-18x_1 + 4x_2 - 9x_3 = 0$$

Now rank of $A = 2 < 3$ (number of variables) $\left(\because \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -10 \neq 0\right)$

$\therefore$ System has infinite number of solutions.

$\therefore$ Number of independent solutions $= 3 - 2 = 1$.

Let $x_1 = 2K$ and from the first two equations, we get

$$x_2 + 9x_3 = -6K \text{ and } -2x_2 - 3x_3 = -8K$$

On solving $x_2 = 6K$ and $x_3 = \frac{-4}{3}K$, we get

$x_1 = 2K$, $x_2 = 6K$ and $x_3 = \frac{-4}{3}K$ as the general solution of the given system.

Case II. For $\lambda = 1$, the given system reduces to

$$3x_1 + x_2 - x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 + x_3 = 0$$

Now rank of $A = 2 < 3$ (number of variables)

Hence the system has infinite number of solutions.

$\therefore$ Number of independent solutions = 3 – 2 = 1.

Let $x_1 = K$ and from the first two equations, we get

$$x_2 - x_3 = 3K \text{ and } -2x_2 - 3x_3 = -4K$$

On solving, $x_2 = -K$ and $x_3 = 2K$, where K is a constant

$\therefore$ $x_1 = K$, $x_2 = -K$ and $x_3 = 2K$ is the general solution of the given system.

Example 3 : Solve the system of equations
$x + y - 3z + 2w = 0;\ 2x - y + 2z - 3w = 0; 3x - 2y + z - 4w = 0; -4x + y - 3z + w = 0.$

Solution : The given homogeneous system of linear equations can be expressed as

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(1)}$$

i.e., $AX = O$

Consider $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$

We will reduce this matrix to Echelon form using elementary row operations and find its rank.

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 + 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Performing $\frac{R_3}{-5}$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix}$

Performing $R_4 \to R_4 - 5R_3$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 9 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & -1 \end{bmatrix}$

Performing $R_3 \to 3R_3 + R_2$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -5 & -1 \end{bmatrix}$

Performing $R_4 \to 3R_4 + 5R_3$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

This is in Echelon form. The number of non-zero rows = 4

$\therefore$ Rank $A = 4$. Here number of variables = $n = 4$

We have number of non-zero solutions is $n-r = 4 - 4 = 0$

$\therefore$ There are no non-zero solutions.

$\therefore$ $x = y = z = w = 0$ is the only solution.

Example 4 : Solve $x_1 + 2x_3 - 2x_4 = 0;\ 2x_1 - x_2 - x_4 = 0;\ \ x_1 + 2x_3 - x_4 = 0;$
$4x_1 - x_2 + 3x_3 - x_4 = 0.$

Solution : Writing the given system of equations in matrix form $AX = O$ ...(1), we have

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will perform elementary row operations on A on L.H.S. We will perform same row operations on O in right side But the matrix O will remain unaltered because of these row operations.

Applying $R_2 \to R_2 \to 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 4R_1$, we get

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_2$, we get $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \leftrightarrow R_4$, we get $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Here we can write system of equations as

$$\left.\begin{aligned} x_1 + 2x_3 - 2x_4 &= 0 \\ -x_2 - 4x_3 + 3x_4 &= 0 \\ -x_3 + 4x_4 &= 0 \\ x_4 &= 0 \end{aligned}\right\} \quad \ldots(2)$$

Solving (2), we get $x_4 = 0,\ x_3 = 0,\ x_2 = 0$ and $x_1 = 0$.

Note : In the above form of the matrix A, number of non-zero rows is equal to 4. Its rank is 4. Number of variables is equal to 4. The number of non-zero solutions = $n - r = 0$. Hence $x_1 = 0 = x_2 = x_3 = x_4$ is the only solution.

Example 5 : Solve the system of equations

$x+3y+2z=0,\ 2x - y+3z = 0,\ \ 3x-5y+4z = 0,\ x+17y+4z = 0.$

Solution : Given system of equations can be written as $AX = O$ ... (1) where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}_{4\times 3},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1},\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 1}$$

Performing $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing $R_2 \to \dfrac{R_2}{-1}, R_3 \to \dfrac{R_3}{-2}$ and $R_4 \to \dfrac{R_4}{2}$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Performing $R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$, we get $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ... (2)

We observe that in the L.H.S. of the matrix, the number of non-zero rows is 2 so that the rank of the matrix is 2. We have the number of unknowns is 3.

$\therefore$ Number of non-zero solutions is 3–2 = 1.

We will find the non-zero solution in the following way.

Writing from the matrix equation (2), we get

$$x + 3y + 2z = 0$$
$$7y + z = 0$$

Let $z = k$ so that $y = \frac{-k}{7}$. From this, we get

$$x = -3y - 2z = \frac{3k}{7} - 2k = \frac{3k - 14k}{7} = \frac{-11k}{7}$$

$\therefore$ Solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-11k}{7} \\ \frac{-k}{7} \\ k \end{bmatrix} = k \begin{bmatrix} \frac{-11}{7} \\ \frac{-1}{7} \\ 1 \end{bmatrix}$ for all real values of k.

Example 6 : Solve the system of equations
$x + 2y + (2+k)z = 0;\ 2x + (2+k)y + 4z = 0;\ 7x + 13y + (18+k)z = 0$ for all values of k.

Solution : The given system can be expressed as

$$\begin{bmatrix} 1 & 2 & 2+k \\ 2 & 2+k & 4 \\ 7 & 13 & 18+k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ (i.e.)\ AX = O \qquad \ldots (1)$$

where $A = \begin{bmatrix} 1 & 2 & 2+k \\ 2 & 2+k & 4 \\ 7 & 13 & 18+k \end{bmatrix}$

If the given system of equations is to possess non-trivial solution,

Rank of A < number of variables *i.e.*, Rank of $A < 3$

$i.e.$, det $A = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 2+k \\ 2 & 2+k & 4 \\ 7 & 13 & 18+k \end{vmatrix} = 0$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 7R_1$, we get $\begin{vmatrix} 1 & 2 & 2+k \\ 0 & k-2 & -2k \\ 0 & -1 & 4-6k \end{vmatrix} = 0$ [Expand by C_1]

$\Rightarrow (k-2)\,(4-6k) - 2k = 0 \ \Rightarrow 3k^2 - 7k + 4 = 0 \Rightarrow\ k = 1$ (or) $k = \frac{4}{3}$

Now three cases arise.

Case I. When $k \neq 1$ and $k \neq \frac{4}{3}$

In this case det $A \neq 0$. Then rank $A = 3$ and number of variables is 3.

$\therefore$ The number of independent solutions is $n - r = 3 - 3 = 0$

$\therefore$ $x = 0 = y = z$ is the only solution.

Case II. If $k = 1$, the equation (1) reduces to $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We find that number of non-zero rows is 2 in the matrix equivalent to A.

Hence Rank of $A = 2$ and number of variables = 3.

$\therefore$ The number of non-zero solutions is $3-2 = 1$.

The given system of equations is now equivalent to

$$x + 2y + 3z = 0, -y - 2z = 0$$

Let $z = k$. Then $y = -2k$ and $x = -2y - 3z = 4k - 3k = k$

Thus $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, k is a parameter.

This is the general solution of the given system.

Case III. If $k = \frac{4}{3}$, the equation (1) reduces to $\begin{bmatrix} 1 & 2 & \frac{10}{3} \\ 0 & \frac{-2}{3} & \frac{-8}{3} \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - \frac{3}{2}R_2$, we get

$$\begin{bmatrix} 1 & 2 & \frac{10}{3} \\ 0 & \frac{-2}{3} & \frac{-8}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here number of non-zero rows is 2. Hence rank of $A = 2$. The number of variables is 3.

$\therefore$ The number of independent solutions = $3-2 = 1$

The given system is now equivalent to

$x + 2y + \frac{10}{3}z = 0$ and $\frac{-2}{3}y - \frac{-8}{3}z = 0$

Let $z = k$. Then $y = -4k$ and $x = \frac{14}{3}k$

$\therefore$ Solution is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} \frac{14}{3} \\ -4 \\ 1 \end{bmatrix}$, where k is a parameter.

which is the general solution of the given system.

Example 7 : Solve the equations $x+y-z+t=0;\ x-y+2z-t=0;\ 3x+y+t=0.$

Solution : Given system of equations can be written as $AX = O$ ... (1)

where

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}_{3\times 4},\quad X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}_{4\times 1},\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

We write $AX = O$ and apply elementary row operations to reduce the matrix A in the L.H.S. to Echelon form. We will perform the same row operations on matrix in R.H.S. But it will not alter since it is a null matrix.

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$ we get $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ... (2)

We find that in the above equivalent matrix, number of non-zero rows is 2, so that the rank of A is 2. We have the number of variables is 4.

$\therefore$ Number of non-zero solutions = 4–2 = 2. We will find them in the following manner.

From (2), we have

$$x+y-z+t=0$$
$$-2y+3z-2t=0$$

Let $t = k_1$ and $z = k_2$. Then $2y = 3z - 2t = 3k_2 - 2k_1 \Rightarrow y = \frac{3}{2}k_2 - k_1$

Again $x = -y+z-t = -\frac{3}{2}k_2 + k_1 + k_2 - k_1 = \frac{-1}{2}k_2$

$\therefore$ Solution is given by $X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}k_2 \\ \frac{3}{2}k_2 - k_1 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

The independent solutions are $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

Any solution of system (1) is a linear combination of these two solutions.

$\therefore$ The number of solutions is infinite.

Example 8 : Solve the system $\lambda x + y + z = 0;\ x + \lambda y + z = 0;\ x + y + \lambda z = 0$ if the system has non-zero solution only.

Solution : Writing the above equations in matrix form, we get $AX = O$ ... (1), where

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the system has non-zero solution then we must have rank $A < 3$.

$(i.e.)$ det. $A = 0 \Rightarrow \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$, we get $\begin{vmatrix} \lambda & 1 & 1 \\ 1-\lambda & \lambda-1 & 0 \\ 1-\lambda & 0 & \lambda-1 \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(1-\lambda)\begin{vmatrix} \lambda & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0$ [Taking $(1-\lambda)$ common from R_2 and R_3]

$\Rightarrow (1-\lambda)^2\left[\lambda(1-0) - 1(-1-0) + 1(0+1)\right] = 0$

$\Rightarrow (1-\lambda)^2\ (2+\lambda) = 0 \Rightarrow \lambda = 1, 1$ or $\lambda = -2$

Case I. Let $\lambda = 1$. Then $AX = O$ becomes $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_1,\ R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Writing in linear equations, we get $x + y + z = 0$

Let $z = k_2$ and $y = k_1$. Then $x = -k_1 - k_2$

$\therefore$ The solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Linearly independent solutions are $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Case II. Let $\lambda = -2$. Then we can write $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_2$, we get $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$, we get $\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 + R_2$, we get $\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Writing in linear equations, we get $x - 2y + z = 0$ and $-3y + 3z = 0 \Rightarrow y = z$

Let $y = z = k$. Then $x = 2y - z = 2k - k = k$.

$\therefore\ X = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the only non-zero solution.

Example 9 : Show that the only real number λ for which the system

$x + 2y + 3z = \lambda x;\ \ 3x + y + 2z = \lambda y\ ;\ 2x + 3y + z = \lambda z$

has non-zero solution is 6 and solve them, when $\lambda = 6$.

[JNTU 2005-May, 2006, 2006S (Set No. 1), Sep. 2008 (Set No.1)]

Solution : Given system can be expressed as $AX = O$ where

$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here number of variables $= n = 3$.

The given system of equations possess a non-zero (non-trivial) solution, if

Rank of $A <$ number of unknowns *i.e.*, Rank of $A < 3$

For this we must have det. $A = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get $(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)\left[(-2-\lambda)(-1-\lambda)+1\right]=0$

$\Rightarrow (6-\lambda)(\lambda^2+3\lambda+3)=0 \Rightarrow \lambda=6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 + 3R_1, R_3 \to 5R_3 + 2R_1 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x+2y+3z=0$ and $-19y+19z=0 \Rightarrow y=z$

Since Rank of A < number of unknowns (Rank of $A=2$, number of unknowns = 3), therefore, the given system has infinite number of non-trivial solutions.

Let $z=k \Rightarrow y=k$ and $-5x+2k+3k=0 \Rightarrow x=k$

$\therefore\ x=k, y=k, z=k$ is the solution.

Example 10 : Solve :

$2x+3ky+(3k+4)z=0;\ x+(k+4)y+(4k+2)z=0;\ x+2(k+1)y+(3k+4)z=0$

Solution : The given system can be written as $AX = O$, where

$$A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The given system of equations possess a non-zero solution, if

Rank of A < number of variables *i.e.*, Rank of $A < 3$

For this, we must have

det. $A = 0$ *i.e.*, $\begin{vmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0$

Performing $R_1 \leftrightarrow R_3$, we get $\begin{vmatrix} 1 & 2k+2 & 3k+4 \\ 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \end{vmatrix} = 0$

Performing $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we get $\begin{vmatrix} 1 & 2k+2 & 3k+4 \\ 0 & -k+2 & k-2 \\ 0 & -k-4 & -3k-4 \end{vmatrix} = 0$

$$\Rightarrow (k-2)\begin{vmatrix} 1 & 2k+2 & 3k+4 \\ 0 & -1 & 1 \\ 0 & -k-4 & -3k-4 \end{vmatrix} = 0 \qquad [\text{Expand by } C_1]$$

$$\Rightarrow (k-2)\ [1(3k+4+k+4)] = 0 \Rightarrow (k-2)\ (4k+8) = 0$$

$$\Rightarrow 4(k-2)\ (k+2) = 0 \quad \therefore k = 2 \ \text{(or)}\ k = -2$$

Now three cases arise.

Case I. Suppose $k \neq \pm 2$. Then det. $A \neq 0 \Rightarrow$ rank of $A = 3$

We have number variables is 3.

$\therefore$ The number of solutions is $n-r = 3 - 3 = 0$

$\therefore$ $x = 0 = y = z$ is the only solution. (Trivial solution)

Case II. If $k = 2$ then $AX = O$ becomes $\begin{bmatrix} 2 & 6 & 10 \\ 1 & 6 & 10 \\ 1 & 6 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Performing $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 2 & 6 & 10 \\ 1 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Performing $R_2 \to R_2 - R_1$, we get $\begin{bmatrix} 2 & 6 & 10 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The equivalent matrix in L.H.S. is having 2 non-zero rows.

$\therefore$ Its rank = 2

Since number of variables is 3, we have number of independent solutions = 3–2 = 1. We will try to find it.

From the above matrix equation, we have

$$2x + 6y + 10z = 0$$

$$-x = 0 \Rightarrow x = 0$$

Let $z = k$. Then $6y = -10k \Rightarrow y = \dfrac{-10k}{6} = \dfrac{-5k}{3}$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-5k}{3} \\ k \end{bmatrix} = k\begin{bmatrix} 0 \\ \frac{-5}{3} \\ 1 \end{bmatrix}$$ where k is a parameter.

which is the general solution of the given system.

Case III. If $k = -2$, then the system of equations can be written as

$$\begin{bmatrix} 2 & -6 & -2 \\ 1 & 2 & -6 \\ 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing $R_1 \to R_1/2$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 1 & 2 & -6 \\ 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing $R_2 \to R_2 - R_1$ and $R_3 - R_1$, we get $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 5 & -5 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Performing $R_2 \to \frac{R_2}{5}$, we get $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Performing $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Above matrix is in Echelon form and the number of non-zero rows is 2.

$\therefore$ Rank of $A = 2$. Since the number of variables is 3, we have the number of non-zero solutions = 3 – 2 = 1.

From the above matrix equation, we get

$$x - 3y - z = 0$$

$$y - z = 0$$

Let $z = k$ so that $y = k$

$\therefore \; x = 3y + z = 4k$

The infinite solutions are $X = \begin{bmatrix} 4k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Example 11 : Solve the system of equations

$x + 3y - 2z = 0;\; 2x - y + 4z = 0;\; x - 11y + 14z = 0.$

[JNTU 2002, (K) May 2010 (Set No. 2), (A) Nov. 2010 (Set No. 1), Dec. 2013 (Set No. 2)]

Solution : Writing $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then the given system can be written as $AX = O$

Now $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

Applying $R_3 - R_1$ and $R_2 - 2R_1$, we get $A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$

Applying $R_3 - 2R_2$, we get $A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

Thus the matrix is in Echelon form. Number of non-zero rows is 2.

$\therefore$ The rank of matrix is 2.

Since number of variables is 3, this will have 3–2 = 1 non-zero solution.

The corresponding equations are, $x + 3y - 2z = 0$ and $-7y + 8z = 0$

Let $z = k$. Then $y = \frac{8}{7}k$ and $x = -3y + 2z = \frac{-24}{7}k + 2k = \frac{-10k}{7}$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-10}{7}k \\ \frac{8}{7}k \\ k \end{bmatrix} = \frac{k}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix}$$

$$i.e.\ X = \frac{k}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix}.$$

which is the general solution of the given system.

Example 12 : Find the values of λ for which the equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent and find the ratio of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greater of these values. **[JNTU 2004S (Set No. 4)]**

Solution : Given system of equations can be written in the matrix form as

$$AX = \begin{bmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

The given equations will be consistent, if $|A| = 0$

[Since for non-trivial solution of homogeneous system, rank of A< no. of unknowns.]

$$i.e., \quad \text{if} \begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

Applying $R_2 - R_1$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$

Applying $C_3 + C_2$, we get $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$ [Expand by R_2]

or if $(\lambda - 3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda-1) \end{vmatrix} = 0$ or if $6\lambda(\lambda-3)^2 = 0$

or if $\lambda = 0$ or 3

(*i*) When $\lambda = 0$, the equations become

$$-x + y = 0 \quad \ldots (1)$$
$$-x - 2y + 3z = 0 \quad \ldots (2)$$
$$2x + y - 3z = 0 \quad \ldots (3)$$

Solving (1), (2) and (3), we get

$$x = y = z$$

(*ii*) When $\lambda = 3$, equations become identical (each is $2x + 10y + 6z = 0$)

Example 13 : Determine whether the following equations will have a non-trivial solution if so solve them. $4x + 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$, $2x + y + w = 0$.

[JNTU May 2006, 2006S (Set No.2)]

Solution : Given equations will have a non-trivial solution since the number of equations is less than the number of unknowns.

In other words the system will have a non-trivial solution if and only if the rank r of the coefficient matrix is less than n the number of unknowns.

Given equations can be written in the matrix form as follows :

$$AX = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$

or $$AX = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} z \\ y \\ x \\ w \end{bmatrix} = O \text{ (Interchanging the variables } x \text{ and } z)$$

We shall now reduce the coefficient matrix A to the Echelon form by applying only elementary row operations.

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \text{ (Applying } R_2 - 4R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \left(\text{Applying } \frac{R_2}{-5}\right)$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

Thus the matrix A has been reduced to Echelon form.

$\therefore$ Rank (A) = Number of non-zero rows = 2 < 4 (unknowns)

Hence the given system will have 4 – 2 i.e., 2 linearly independent solutions.

Now the given system of equations is equivalent to

$$AX = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ w \end{bmatrix} = O$$

i.e., $z + 2y + 4x + 3w = 0$

and $y + 2x + w = 0$

Choose $x = k_1$ and $w = k_2$. Then solving these two equations, we get

$y = -2x - w = -2k_1 - k_2$ and $z = -4x - 2y - 3w = -4k_1 - 2(-2k_1 - k_2) - 3k_2 = -k_2$.

$\therefore$ The solution is

$x = k_1, y = -2k_1 - k_2, z = -k_2$ and $w = k_2$, where k_1 and k_2 are arbitrary constants.

Example 14 : Solve the system of equations $x + y + w = 0, y + z = 0, x + y + z + w = 0, x + y + 2z = 0$. **[JNTU 2008, (H) June 2009 (Set No.1)]**

Solution : The equations can be written in matrix form Ax = O.

$$\text{where} \quad A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \; O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Consider } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix} \text{(Applying } R_3 \to R_3 - R_1, R_4 \to R_4 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{(Applying } R_4 - 2R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{(Applying } R_1 + R_4\text{)}$$

$\therefore$ Rank (A) = 4 and Number of variables = 4

Therefore, there is no non-zero solution.

Hence $x = y = z = w = 0$ is the only solution.

Example 15 : Solve the system $2x - y + 3z = 0, 3x + 2y + z = 0$ and $x - 4y + 5z = 0$.

[JNTU 2008S (Set No.1)]

Solution : The Given system can be written as AX = O

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \leftrightarrow R_1$, we get

$$A \sim \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

Applying $R_2 - 3R_1$ and $R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{bmatrix}$$

Applying $2R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $\frac{R_2}{14}$, we get

$$A \sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the number of non-zero rows is 2, we have rank (A) = 2

Here number of variables = 3

The system will have $n - r = 3 - 2 = 1$ non-zero solution.

From the matrix, we have $y - z = 0 \Rightarrow y = z$

Let $y = z = k$. Then

$x - 4y + 5z = 0 \Rightarrow x = 4y - 5z = 4k - 5k = -k$

$\therefore$ The solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Example 16 : Find all the solutions of the system of equations :

$x + 2y - z = 0, 2x + y + z = 0, x - 4y + 5z = 0$ **[JNTU 2008S (Set No.1)]**

Solution : Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix}$. Then

$\det A = 1(5+4) - 2(10-1) - 1(-8-1)$

$= 9 - 18 + 9 = 0$

$\therefore$ The rank of A is < 3.

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus rank (A) = 2. $\therefore$ Number of non-zero solutions = $n - r = 3 - 2 = 1$.

From the above matrix, $-3y + 3z = 0 \Rightarrow y = z$

Let $z = k$. Then $y = k$.

$x + 2y - z = 0 \Rightarrow x = z - 2y = k - 2k = -k$

$\therefore$ The solutions are given by $X = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Example 17 : Solve completely the system of equations :

$x + 3y - 2z = 0,\ 2x - y + 4z = 0,\ x - 11y + 14z = 0$. **[JNTU (A)June 2009 (Set No.1)]**

Solution : Taking $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we get

$AX = O$ is the matrix form of equations.

Consider $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix} \text{(Applying } R_2 - 2R_1,\ R_3 - R_1\text{)}$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -7 & 8 \end{pmatrix} \text{(Applying } \frac{R_3}{2}\text{)}$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}. \text{(Applying } R_3 - R_2\text{)}$$

This is in Echelon form.

$\therefore\ \rho(A) = 2 = r$.

n = number of unknowns = 3.

$\therefore\ r < n$

$\therefore$ Number of linearly independent solutions $= n - r = 3 - 2 = 1$

$\therefore$ The above system has infinite solutions.

Now solve $AX = O$.

i.e., $\begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad x + 3y - 2z = 0 \qquad \ldots (1)$

$-7y + 8z = 0 \Rightarrow y = \frac{8}{7}z. \qquad \ldots (2)$

From (1) and (2), we have

$x + \frac{24}{7}z - 2z = 0 \Rightarrow x + \frac{10z}{7} = 0 \quad \Rightarrow x = -\frac{10}{7}z$

Taking $z = c$, we get $x = -\frac{10}{7}c$; $\quad y = \frac{8}{7}c$

$$\therefore\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{10}{7}c \\ \frac{8}{7}c \\ c \end{pmatrix} = \frac{c}{7}\begin{pmatrix} -10 \\ 8 \\ 7 \end{pmatrix}$$

This gives the solution of the system of equations.

Example 18 : Find all the non-trivial solutions of $2x - y + 3z = 0,\ \ 3x + 2y + z = 0,$ $x - 4y + 5z = 0$.

Solution : Given equations form homogeneous system and can be written in the matrix form as

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see that, det $A \neq 0$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(By } R_2 \to 2R_2 - 3R_1; R_3 \to 2R_3 - R_1\text{)}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(By } R_3 \to R_3 + R_2\text{)}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(By } R_2 \to \frac{R_2}{7}\text{)}$$

The coefficient matrix is in Echelon form. Number of non-zero rows is 2. Thus its rank is 2. It will have 3–2 = 1, non-trivial solution.

We can have the equations,

$$2x - y + 3z = 0, \quad y - z = 0 \Rightarrow y = z = k \text{ (say)}$$

$$\therefore \quad 2x = y - 3z = k = 3k = -2k$$

Thus $X = \begin{bmatrix} -2k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is the solution for the above system of equations.

Example 19 : Solve completely the equations $3x + 4y - z - 6w = 0; 2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0;\ x + 3y + 13z + 3w = 0$ **[JNTU (H) May 2012 (Set No. 3)]**

Solution : Given equations $3x + 4y - z - 6w = 0; 2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0; x + 3y + 13z + 3w = 0$

Taking, $A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}; O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

the given system becomes $AX = O$.

Consider $A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

we will reduce this to Echelon form using elementary operations,

$R_4 \leftrightarrow R_1$ gives $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}$

$R_2 - 2R_1$;
$R_3 - 2R_1$; gives
$R_4 - 3R_1$
$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -10 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$

$R_4 - R_3$ gives $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\frac{R_2}{-3}, \frac{R_3}{-5}$ gives $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 - R_2$ gives $\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Number of non-zero rows $= 2$
$\therefore$ Rank (A) $= 2$ and no. of variables $= 4$
$\therefore$ There will be $4 - 2 = 2$ non trivial solutions.

$R_1 - R_2$ gives $\begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $z = k_1$ and $w = k_2$ then $y = -8k_1 - 3k_2$ ... (1)

$x = -2 - 5z = -2(-8k_1 - 3k_2) - 5k_1 = 11k_1 + 6k_2$

The solution is given by

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11k_1 + 6k_2 \\ -8k_1 - 3k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ gives solution completely.

Example 20 : Solve completely the system of equations

$2x - 2y + 5z + 3w = 0; 4x - y + z + w = 0, 3x - 2y + 3z + 4w = 0; \; x - 3y + 7z + 6w = 0$

[JNTU (H) May 2011 (Set No. 3), Nov. 2010 (Set No. 3)]

Sol. Taking $A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as $AX = O$

Consider $A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}$

We find rank (A) using elementary row transformations.

$R_1 \leftrightarrow R_4$ gives $\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{bmatrix}$

$R_2 - 4R_1$; $R_3 - 3R_1$; $R_4 - 2R_1$ gives $\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$

$R_2 - R_3$ gives $\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$

$4R_3 - 7R_4$ gives $\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & -9 & 7 \\ 0 & 4 & -9 & -9 \end{bmatrix}$

$R_4 - R_2$ gives $\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form. No. of non-zero rows $= 3$.

$\therefore$ Rank (A) $= 3$

No. of variables $= 4$

$\therefore$ No. of independent solutions is $4 - 3 = 1$

$x - 3y + 7z + 6w = 0$

$4y - 9z - 9w = 0$

$-9z + 7w = 0$

Taking $w = c$; we get $z = \frac{7}{9}c; y = 4c; x = (5/9)c$ we get as the solution.

Example 21 : Solve completely the system of equations:

$x + 2y + 3z = 0$; $3x + 4y + 4z = 0; 7x + 10y + 12z = 0$ **[JNTU (A) Jan 2014 (Set No. 2)]**

Solution : Taking $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as $AX = O$

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$R_2 - 3R_1; R_3 - 7R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$

$R_3 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

This is in Echelon form. No. of non-zero rows is 3.

$\therefore$ The rank of $A = 3$

$\therefore$ No. of variables $= 3$

$\therefore$ No. of non-zero solutions $= 3 - 3 = 0$

$\therefore$ $x = 0, y = 0, z = 0$ is the only solution.

EXERCISE 1.6

1. Solve completely the system of equations:

(*i*) $x + 2y + 3z = 0$; $3x + 4y + 4z = 0$; $7x + 10y + 12z = 0$.

(*ii*) $4x + 2y + z + 3w = 0$; $6x + 3y + 4z + 7w = 0$; $2x + y + w = 0$.

[JNTU 2004S, Sep. 2008 (Set No.2)]

(*iii*) $3x + 4y - z - 6w = 0$; $2x + 3y + 2z - 3w = 0$; $2x + y - 14z - 9w = 0$;

$x + 3y + 13z + 3w = 0$. **[JNTU 2002 (Set No. 3)]**

(*iv*) $2x - 2y + 5z + 3w = 0$; $4x - y + z + w = 0$; $3x - 2y + 3z + 4w = 0$;
$x - 3y + 7z + 6w = 0$.

2. Show that the system of equations

$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$

can possess a non-trivial solution only if $\lambda = 1$, $\lambda = -3$.

Obtain the general solution in each case. **[JNTU 2003S (Set No. 1)]**

ANSWERS

1. (*i*) $x = 0, y = 0, z = 0$ (*ii*) $x = 1, y = -21 - m, z = -m, w = m$

(*iii*) $x = 11k_1 + 6k_2$, $y = -8k_1 - 3k_2$, $z = k_1$, $w = k_2$

(*iv*) $x = \frac{5}{9}k$, $y = 4k$, $z = \frac{7}{9}k$, $w = k$

2. $\lambda = 1$, $x_1 = 2t - s$, $x_2 = t$, $x_3 = s$; $\lambda = -3$, $x_1 = t$, $x_2 = -2t$, $x_3 = t$

1.24 SYSTEM OF LINEAR EQUATIONS - TRIANGULAR SYSTEMS

Consider the system of n linear algebraic equations in n unknowns,

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n \end{aligned}\right\} \quad \dots\dots (1)$$

Take $A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & .. & .. & a_{nn} \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ .. \\ b_n \end{bmatrix}$

Then (1) in matrix form is $AX = B$

We consider two particular cases

Case (i) : Suppose the coefficient matix A is such that all the elements above the leading diagonal are zero. That is, A is a lower triangular matrix of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & .. & .. & 0 \\ a_{21} & a_{22} & 0 & 0 & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & an_3 & .. & .. & .. & a_{nn} \end{bmatrix}$$

In this case the system (1) will be of the form

$$\left.\begin{aligned} &a_{11}x_1 = b_1 \\ &a_{21}x_1 + a_{22}x_2 = b_2 \\ &a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n \end{aligned}\right\} \quad \dots\dots (2)$$

This type of system is called **lower triangular system.**

From the system of equations (2), we get

$$x_1 = \frac{b_1}{a_{11}}, x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}} = \frac{1}{a_{22}}\left[b_2 - \frac{a_{21}}{a_{11}}b_1\right] \text{ and so on.}$$

The values $x_1, x_2, \dots\dots, x_n$ given above constitute the exact solution of (2).

This method of constructing the exact solution, when the system is lower triangular is called **method of forward substitution.**

Case (ii) : Suppose the coefficient matrix A is such that all the elements below the leading diagonal are all zero, *i.e.* when A is an upper triangular matrix of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & .. & .. & a_{1n} \\ 0 & a_{22} & .. & .. & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. & a_{nn} \end{bmatrix}$$

In this case the system (1) will be of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned}\right\} \quad \ldots\ldots (3)$$

A system of the above type is called an **upper triangular system.**

From (3), we easily get

$$x_n = \frac{b_n}{a_{nn}};$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}\, x_n}{a_{n-1,n-1}} = \frac{1}{a_{n-1,n-1}}\left[b_{n-1} - \frac{a_{n-1,n}}{a_{nn}} b_n\right] \text{ and so on.}$$

The values of $x_n, x_{n-1}, \ldots\ldots, x_1$ as given above form an exact solution of (1).

This method of constructing the exact solution when the system is upper triangular is called **method of backward substitution.**

1.25 SOLUTION OF LINEAR SYSTEMS - DIRECT METHODS

The solution of a linear system of equations can be found out by numerical methods known as **direct method** and **iterative methods.** We will discuss the Gauss elimination method and its modification. Matrix inversion method and Gauss - Jordan Method are already discussed in the previous chapters.

1. Gaussian Elimination Method

This method of solving a system of n linear equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution . We discuss the method here for $n = 3$. The method is analagous for $n > 3$.

Consider the system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \ldots\ldots (1)$$

The augmented matrix of this system is

$$[A,B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \quad \text{.... (2)}$$

Performing $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$, we get

$$[A,B]\sim\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & \alpha_{32} & \alpha_{33} & \beta_3 \end{bmatrix} \quad \text{.... (3)}$$

where $\alpha_{22} = a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$; $\alpha_{23} = a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right)$

$\alpha_{32} = a_{32} - \left(\frac{a_{31}}{a_{11}}\right)a_{12}$; $\alpha_{33} = a_{33} - \left(\frac{a_{31}}{a_{11}}\right)a_{13}$

$\beta_2 = b_2 - \left(\frac{a_{21}}{a_{11}}\right)b_1$; $\beta_3 = b_3 - \left(\frac{a_{31}}{a_{11}}\right)b_1$

Here we assume $a_{11} \neq 0$

We call $\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}$ as **multipliers for the first stage**. a_{11} is called **first pivot.**

Now applying $R_3 \to R_3 - \frac{\alpha_{32}}{\alpha_{22}}(R_2)$, we get

$$[A,B]\sim\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & 0 & \gamma_{33} & \Delta_3 \end{bmatrix} \quad \text{.... (4)}$$

where $\gamma_{33} = \alpha_{33} - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\alpha_{23}$; $\Delta_3 = \beta_3 - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\beta_2$

We have assumed $\alpha_{22} \neq 0$.

Here the **multiplier** is $-\frac{\alpha_{32}}{\alpha_{22}}$ and **new pivot** is α_{22}.

The augmented matrix (4) corresponds to an upper triangular system which can be solved by backward substitution. The solution obtained is exact.

Note : If one of the elements $a_{11}, \alpha_{22}, \alpha_{33}$ are zero the method is modified by rearranging the rows, so that the pivot is non- zero.

This procedure is called **partial pivoting**. If this is impossible then the matrix is singular and the system has no solution.

SOLVED EXAMPLES

Example 1 : Solve the equations

$2x_1 + x_2 + x_3 = 10;\ 3x_1 + 2x_2 + 3x_3 = 18;\ x_1 + 4x_2 + 9x_3 = 16$

using Gauss-Elimination method.

Solution : The Augmented matrix of the given system is $[A\,|\,B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$

Performing $R_2 \to 2R_2 - 3R_1$ and $R_3 \to 2R_3 - R_1$, we get

$$[A\,|\,B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_2$, we get $[A\,|\,B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix}$

This Augmented matrix corresponds to the following upper triangular system.

$2x_1 + x_2 + x_3 = 10;\ x_2 + 3x_3 = 6;\ -4x_3 = -20$

$\Rightarrow\ x_3 = 5, x_2 = -9, x_1 = 7$ by backward substitution.

$\therefore$ The solution is $x_1 = 7, x_2 = -9, x_3 = 5$

Example 2 : Solve the system of equations :

$3x + y - z = 3;\ 2x - 8y + z = -5;\ x - 2y + 9z = 8$ using Gauss elimination method.

Solution : The Augmented matrix is $[A\,|\,B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$

Performing $R_2 \to R_2 - \frac{2}{3}R_1;\ R_3 \to R_3 - \frac{1}{3}R_1$, we get

$$[A, B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & -7/3 & 28/3 & 7 \end{bmatrix}$$

Performing $R_3 \to R_3 - \frac{7}{26}R_2, [A|B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & 0 & 693/78 & 231/26 \end{bmatrix}$

From this, we get

$3x + y - z = 3$; $\frac{-26}{3}y + \frac{5}{3}z = -7$, $\frac{693}{78}z = \frac{231}{26}$

$\Rightarrow z = 1, y = 1$ and $x = 1$.

$\therefore$ The solution is $x = 1, y = 1, z = 1$.

Example 3 : Solve the equations $x + y + z = 6; 3x + 3y + 4z = 20; 2x + y + 3z = 13$ using partial pivoting Gaussian elimination method.

Solution : The augmented matrix of the system is $[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$

Performing the operations $R_2 \to R_2 - 3R_1$ and $R_3 \to R_2 - 2R_1$, we get

$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$

Using the operation $R_2 \leftrightarrow R_3$, we get $[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

This corresponds to the upper triangular system $x + y + z = 6; -y + z = 1; z = 2$

$\therefore$ By backward substitution, we get $x = 3, y = 1, z = 2$.

Example 4 : Solve the system of equations $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$. **[JNTU 2008, (K) Nov.2009S (Set No.3)]**

Solution : The given system of equation can be written in the matrix form as AX = B.

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

The Augumented matrix of the given system is

$$[A,B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

Operating $R_1 \leftrightarrow R_3$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

Operating $R_3 \to 7R_3 - 5R_2$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

This augmented matrix corresponds to the following upper triangular system.

$$x + 2y + z = 4 \quad \text{... (1)}$$
$$-7y - 3z = -11 \quad \text{... (2)}$$
$$8z = -8$$

By back substitution, we have

$$z = -1 \quad \text{... (3)}$$

Substituting equation (3) in equation (2), we get

$$-7y = -11-3 \Rightarrow y = 2$$

Now from (1), we have

$$x + 4 - 1 = 4 \Rightarrow x = 1.$$

∴ The solution is $x = 1, y = 2, z = -1$.

Example 5 : Solve the system of equations $x + 2y + 3z = 1$, $2x + 3y + 8z = 2$, $x + y + z = 3$.

[JNTU 2008 (Set No.4)]

Solution : The given non-homogeneous linear system of equations are

$x + 2y + 3z = 1$; $2x + 3y + 8z = 2$; $x + y + z = 3$.

These can be written in matrix form as $AX = B$

$$\text{where} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Applying $R_2 - 2R_1$, $R_3 - R_1$, we get

$$[A,B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix}$$

Applying $R_3 - R_2$, we get

$$[A, B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

Rank of [A, B] = 3. Using the same operations, rank (A) = 3.

$\therefore$ The system of equations is consistent.

From the above Augmented matrix,

$$-4z = 2 \Rightarrow z = \frac{-1}{2}$$

$$-y + 2z = 0 \Rightarrow -y - 1 = 0 \Rightarrow y = -1$$

and $x + 2y + 3z = 1 \Rightarrow x - 2 - \frac{3}{2} = 1 \Rightarrow x = \frac{9}{2}$

$\therefore$ The solution is $x = \frac{9}{2}$, $y = -1$, $z = -\frac{1}{2}$

Example 6 : Solve the equations

$$3x + y + 2z = 3;\ 2x - 3y - z = -3;\ x + 2y + z = 4$$

Using Gauss elimination method **[JNTU 2008S (Set No.4)]**

Solution : The given system can be written as AX = B

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} \text{(Applying } R_2 - 2R_1, R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} \text{(Applying } 7R_3 - 5R_2\text{)}$$

This corresponds to upper triangular system.

This gives $8z = -8 \Rightarrow z = -1$

$7y + 3z = 11 \Rightarrow 7y = 11 - 3z = 14 \Rightarrow y = 2$

$x + 2y + z = 4 \Rightarrow x = 4 - 2y - z = 4 - 4 + 1 = 1$

$\therefore$ The solution is $x = 1, y = 2, z = -1$

Example 7: Express the following system in matrix form and solve by Gauss Elimination method.

$2x_1 + x_2 + 2x_3 + x_4 = 6;\qquad 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1;\qquad 2x_1 + 2x_2 - x_3 + x_4 = 10$

[JNTU 2008, 2008S, (H), (A) 2009, (K) 2009S, (K)May 2010 (Set No.1)]

Solution : The Augmented matrix of the given equations is

$$[A,B] = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_2 \to \frac{R_2}{6}$, we get

$$[A,B] \sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_1 \to R_2$ gives,

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, and $R_4 \to R_4 - 2R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{bmatrix}$$

Performing $R_3 \to 3\ R_3 - 7\ R_2$, $R_4 \to 3R_4 - 4R_2$, we obtain

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & -9 & 3 & 18 \end{bmatrix}$$

Performing $R_4 \to R_4 - 3\ R_3$, we obtain

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 0 & 39 & 117 \end{bmatrix}$$

This corresponds to the upper triangular system

$$x_1 - x_2 + x_3 + 2x_4 = 6 \quad \ldots (1)$$
$$3x_2 - 3x_4 = -6 \quad \ldots (2)$$
$$-3x_3 - 12x_4 = -33 \quad \ldots (3)$$
$$39x_4 = 117 \quad \ldots (4)$$

By back substitution,

(4) $\Rightarrow$ $x_4 = 3$

Substituting the value of x_4 in (3) and (2), we get

$x_3 = -1$ and $x_2 = 1$

Now substituting the values of x_2, x_3 and x_4 in (1), we get

$x_1 - 1 - 1 + 6 = 6 \Rightarrow x_1 = 2$

$\therefore$ The solution is $x_1 = 2,\ x_2 = 1, x_3 = -1, x_4 = 3$

2. Gauss - Jordan Method

[Not included in Syllabus]

This method is a modification of the Gauss's elimination method. In this method the unknowns are eliminated so that the system is in diagonal form. This can be done with or without using pivoting. The method is illustrated in the examples given below.

SOLVED EXAMPLES

Example 1 : Using Gauss - Jordan method, solve the system

$2x + y + z = 10; 3x + 2y + 3z = 18; x + 4y + 9z = 16$.

Solution : The Augmented matrix of the system is $[A|B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$

Using the operations $R_2 \to R_2 - \frac{3}{2}R_1$ and $R_3 \to R_3 - \frac{1}{2}(R_1)$, we get

$$[A|B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

Using the operation $R_3 \to R_3 - 7R_2$, we get $[A|B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix}$

Using the operations $R_1 \to R_1 + \frac{1}{2}R_3$, $R_2 \to R_2 + \frac{3}{4}R_3$ and $R_1 \to R_1 - 2R_2$, we get

$$[A|B] \sim \begin{bmatrix} 2 & 0 & 0 & 14 \\ 0 & 1/2 & 0 & -9/2 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

We see that A is reduced to diagonal form.

We get $2x = 14 \Rightarrow x = 7$; $\frac{1}{2}y = \frac{-9}{2} \Rightarrow y = -9$; $-2z = -10 \Rightarrow z = 5$

Thus $x = 7, y = -9, z = 5$ is the solution.

Example 2 : Solve the equations $10x + y + z = 12$; $2x + 10y + z = 13$ and $x + y + 5z = 7$ by Gauss - Jordan method.

Solution : The Augmented matrix is $[A, B] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$

Performing $R_1 \to R_1 - 9R_3$, $[A, B] \sim \begin{bmatrix} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$;

Performing $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$, $[A, B] \sim \begin{bmatrix} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{bmatrix}$;

Performing $R_2 \to -R_2 + 3R_3, [A, B] \sim \begin{bmatrix} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{bmatrix}$

Performing $R_1 \to R_1 + 8R_2, [A, B] \sim \begin{bmatrix} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{bmatrix}$;

Performing $R_3 / (-473), R_1 \to R_1 - 420R_3, R_2 \to R_2 - 58R_3, [A, B] \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\therefore$ The solution is $x = y = z = 1$.

Example 3 : Solve the equations

$10x_1 + x_2 + x_3 = 12$; $x_1 + 10x_2 - x_3 = 10$ and $x_1 - 2x_2 + 10x_3 = 9$ by Gauss - Jordan method.

Solution : The matrix form of the given system is $AX = B$ where

$$A = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; B = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

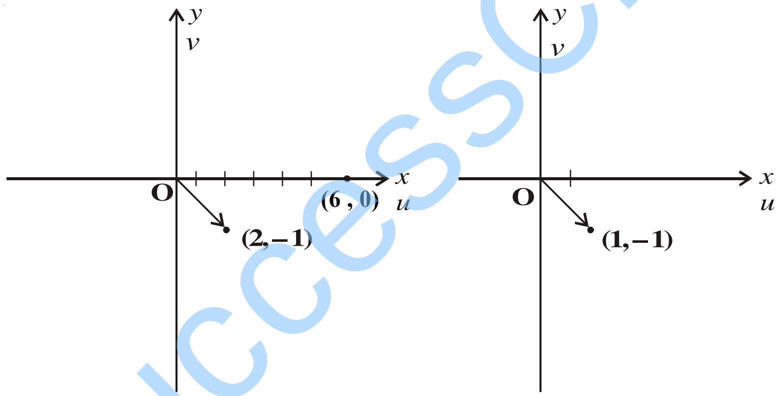
EIGEN VALUES AND EIGEN VECTORS

2.1 INTRODUCTION

Consider the transformation

$$\left.\begin{aligned} u &= 5x + 4y \\ v &= x + 2y \end{aligned}\right\} \qquad \ldots (1)$$

This transformation transforms a vector (x, y) to the vector (u, v) where u, v are obtained by using the equations in (1). For example $(x, y) = (2, -1)$ is transformed to $(u, v) = (6, 0)$.



We notice that the vector (6, 0) is not along (2, –1). In general the vector (u, v) will not be along (x, y). However, it is possible that there exist some vectors (x, y) such that the transformed vector (u, v) and the original vector (x, y) will have the same direction. For example take $(x, y) = (1, -1)$. This after using the transformation (1), will be transformed to $(u, v) = (1, -1)$ which has the same direction of (x, y) (by chance here we had (u, v) identical with (x, y) !]. Notice that $(x, y) = (4, 1)$ using (1) will be transformed to (24, 6) which is along (4, 1) itself since (24, 6) = 6 (4, 1). Hence we observe that while the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

in general changes the magnitude and direction of 'many' vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ after the transformation, there exist some vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ whose direction will be same as that of the transformed vector $\begin{pmatrix} u \\ v \end{pmatrix}$.

Let us now proceed to a more general situation.

Consider a vector $X = (x_1, x_2, \cdots, x_n)^T$
in the n dimensional space, where $x_1, x_2, \cdots, x_n$ are real.

If λ is a scalar, we say that $\lambda X = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)^T$ has direction along that of X. Consider the transformation

$$\begin{aligned} y_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ y_2 &= a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\ &------------------ \\ &------------------ \\ y_n &= a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n \end{aligned}$$

In matrix notation this is same as $Y = AX$ ... (2)

$$\left.\begin{aligned} \text{where} \quad & Y = (y_1, y_2, \cdots, y_n)^T \\ & X = (x_1, x_2, \cdots, x_n)^T \\ \text{and} \quad & A = [a_{ij}],\ 1 \le i, j \le n \end{aligned}\right\} \quad \ldots (3)$$

The equation (3) in general transforms every n dimensional vector X into another n dimensional vector Y whose direction can be different from that of X. We note that, there may exist vector X such that after the use of equation (3), $Y = AX$ will be precisely some λX where λ is a scalar which means that X and $Y = \lambda X$ are along the same direction. Such vectors are called ***characteristic vectors*** or ***eigen vectors*** of the transformation. Since the transformation is defined by the $n \times n$ matrix A, we refer to such vectors as characteristic vectors or eigen vectors of the matrix A itself.

The eigen values and eigen vectors of a matrix that we discuss in this chapter have lots of applications in various engineering disciplines.

2.2 DEFINITION

Let $A = [a_{ij}]$ be an $n \times n$ matrix. A *non–zero* vector X is said to be a characteristic vector of A if there exists a scalar λ such that $AX = \lambda X$.

If $AX = \lambda X$, $(X \neq O)$ we say that X is eigen vector or characteristic vector of A corresponding to the eigen value or characteristic value λ of A.

. Then

Example 1 : Take $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$AX = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot X$$

We say that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value 1 of A.

We also note that if $X = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, then

$$AX = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 6 \cdot X$$

We also say that $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value 6 of A.

But $X = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is not an eigen vector of A, since $AX = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 26 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ for any scalar λ.

e.g., Consider $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$; $X = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

We have $AX = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 30 \\ -30 \\ 15 \end{pmatrix} = 15 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 15 \cdot X$

Hence $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value 15 of A.

Consider $X = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

We have $AX = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = OX$

Hence $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value 0 of A.

Note: We notice that an eigen value of a square matrix A can be 0. But a zero vector cannot be an eigen vector of A.

2.3 TO FIND THE EIGEN VECTORS OF A MATRIX

Let $A = [a_{ij}]$ be a $n \times n$ matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition, $AX = \lambda X$

i.e., $AX = \lambda IX$

i.e., $AX - \lambda IX = O$

i.e., $(A - \lambda I) X = O$

Note that this is a homogeneous system of n equations in n unknowns.

This will have a non-zero solution X, if and only if

$$|A - \lambda I| = 0$$

$(A - \lambda I)$ is called characteristic matrix of A. Also $|A - \lambda I|$ is a polynomial in λ of degree n and is called the characteristic polynomial of A. Also $|A - \lambda I| = 0$ is called the **characteristic equation** of A. This will be a polynomial equation in λ of degree n.

Solving this equation, we get the roots $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_n$ of the characteristic equation. These are the characteristic roots or eigen values of the matrix.

Corresponding to each one of these n eigen values, we can find the characteristic vector X. Consider the homogeneous system

$$(A - \lambda_i I)X_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The non-zero solution X_i of this system is the eigen vector of A corresponding to the eigen value λ_i.

SUMMARY

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be the given matrix.

Its characteristic matrix is $A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{pmatrix}$

On the other hand, $|A - \lambda I|$ by expansion is a polynomial $\phi(\lambda)$ of degree n. This is called the characteristic polynomial of A.

The characteristic equation of A is

$$\phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be its roots. These are the characteristic roots of A. These are also referred to as **eigen values** or **latent roots** or **proper values** of A.

Consider each one of these eigen values say λ. The eigen vector X corresponding to the eigen value λ is obtained by solving the homogeneous system

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and determining the non-trivial solution.

We shall illustrate this procedure through examples.

SOLVED EXAMPLES

Example 1 : Find the characteristic roots of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution : Given matrix is $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e.,} \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1[2-3+\lambda] = 0$

$\Rightarrow$ $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, on simplification

$\Rightarrow$ $(\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$

$\Rightarrow$ $(\lambda-1)(\lambda-1)(\lambda-5) = 0 \quad \therefore \; \lambda = 1, 1, 5$

Hence the characteristic roots of A are 1, 1, 5.

Example 2 : Find the eigen values and the corresponding eigen vectors of $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

[JNTU (K) Nov. 2009S (Set No.3)]

Solution : Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Its characteristic matrix = $A - \lambda I = \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}$

Characteristic equation of A is

$$|A - \lambda I| = 0 \quad (i.e.) \quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad \dots (1)$$

i.e., $(5-\lambda)(2-\lambda) - 4 = 0$

On simplification, we get $\lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow$ $(\lambda-1)(\lambda-6) = 0$ ... (2)

The roots of the equation are $\lambda = 1, 6$. Hence the eigen values of the matrix A are 1,6.

Consider the system $\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ... (3)

To get the eigen vector X corresponding to each eigen value λ we have to solve the above system.

Eigen vector corresponding to $\lambda = 1$.

Put $\lambda = 1$ in the system (3), we get $\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The system of equations is

$$4x_1 + 4x_2 = 0$$
$$x_1 + x_2 = 0$$

This implies that $x_2 = -x_1$. Taking $x_1 = \alpha$, we get $x_2 = -\alpha$.

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $\alpha \neq 0$ is a scalar.

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigen vector of A corresponding to the eigen value $\lambda = 1$.

Eigen vector corresponding to the eigen value $\lambda = 6$.

Put $\lambda = 6$ in (3). We get $\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore -x_1 + 4x_2 = 0$ and $x_1 - 4x_2 = 0$

This implies that $x_1 = 4x_2$. Taking $x_2 = \alpha$, we get $x_1 = 4\alpha$

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is eigen vector of A corresponding to the eigen value $\lambda = 6$.

Example 3 : Find the eigen values and the corresponding eigen vectors of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

Solution : If λ is an eigen value of A and X is the corresponding eigen vector, then by definition

$$(A - \lambda I)X = O \ (i.e.) \begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots(1)$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \quad [\text{Exand by } C_1]$$

$\Rightarrow (1-\lambda)(2-\lambda)(-2-\lambda) = 0 \ \therefore \lambda = 1, 2, -2.$

Important Observation : For upper triangular, lower triangular and diagonal matrices, the eigen values are given by the diagonal elements.

Eigen vector of A corresponding to $\lambda = 1$

Put $\lambda = 1$ in (1). We get $\begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$(i.e.) \quad \left.\begin{aligned} 2x_2 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ -3x_3 &= 0 \end{aligned}\right\} \quad \ldots (2)$$

This implies that $x_3 = 0$ and hence $x_2 = 0$. Note that we cannot find x_1 from these equations. As x_1 is not present in any of these equations, it follows that x_1 can be arbitrary.

Hence $x_1 = \alpha, \ x_2 = 0, \ x_3 = 0$

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ where $\alpha \neq 0$.

Hence $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is the eigen vector of A corresponding to $\lambda = 1$.

[Here in this case one may wrongly conclude that $x_1 = 0$ in which case $\begin{pmatrix}0\\0\\0\end{pmatrix}$ will be the solution. Note that $\begin{pmatrix}0\\0\\0\end{pmatrix}$ cannot be eigen vector].

Eigen vector of A corresponding to $\lambda = 2$

Put $\lambda = 2$ in (1). We get $\begin{pmatrix}-1 & 2 & -1\\ 0 & 0 & 2\\ 0 & 0 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Hence the equations are

$$\left.\begin{aligned}-x_1 + 2x_2 - x_3 &= 0\\ 2x_3 &= 0\\ -4x_3 &= 0\end{aligned}\right\} \quad \text{..... (3)}$$

This implies that $x_3 = 0$. $\therefore\ -x_1 + 2x_2 = 0$ or $x_1 = 2x_2$.

Let $x_2 = \beta$ Then $x_1 = 2\beta$

$\therefore\ x_1 = 2\beta,\quad x_2 = \beta,\quad x_3 = 0.$

Hence $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2\beta\\ \beta\\ 0\end{pmatrix} = \beta\begin{pmatrix}2\\1\\0\end{pmatrix}$ is eigen vector of A corresponding to $\lambda = 2$.

Eigen vector of A corresponding to $\lambda = -2$.

Put $\lambda = -2$ in (1). We get $\begin{pmatrix}3 & 2 & -1\\ 0 & 4 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\text{Hence}\quad \left.\begin{aligned}3x_1 + 2x_2 - x_3 &= 0\\ 4x_2 + 2x_3 &= 0\\ 0 &= 0.\end{aligned}\right\} \quad \text{... (4)}$$

$\Rightarrow\ 2x_3 = -4x_2 \quad \therefore\ x_3 = -2x_2$ and $3x_1 + 2x_2 - x_3 = 0$

$\Rightarrow\ 3x_1 + 2x_2 + 2x_2 = 0 \Rightarrow 3x_1 + 4x_2 = 0 \Rightarrow 3x_1 = -4x_2$

$\therefore\ x_1 = \dfrac{-4x_2}{3};\ x_2 = x_2;$ and $x_3 = -2x_2$

Taking $x_2 = \delta,\quad x_1 = -\dfrac{4\delta}{3}, x_2 = \delta,\quad x_3 = -2\delta.$

Hence $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-4\delta/3\\ \delta\\ -2\delta\end{pmatrix} = \delta\begin{pmatrix}-4/3\\ 1\\ -2\end{pmatrix}$ is eigen vector of A corresponding to $\lambda = -2$.

Thus the eigen values of A are 1, 2, -2 and the corresponding eigen vectors are respectively $\alpha(1, 0, 0)^T$; $\beta(2, 1, 0)^T$; $\delta(-4/3, 1, -2)^T$.

An observation :

The sum of the eigen values is 1 + 2 – 2 = 1. This is same as trace of A = sum of the principal diagonal elements of A.

Further the product of the eigen values is equal to the determinant of the matrix *i.e.,* $|A| = (1)(2)(-2) = -4$.

Example 4 : Find the eigen values and the corresponding eigen vectors of

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Solution : If X is an eigen vector of A corresponding to the eigen value λ of A, we have $(A - \lambda I)X = O$

$$\text{i.e.,} \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots(1)$$

The characteristic equation of A is

$$|A - \lambda I| = 0 \ (i.e.) \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

Expanding by R_1, we get

$$(-2-\lambda)\left[(1-\lambda)(-\lambda) - 12\right] - 2\left[-2\lambda - 6\right] - 3\left[-4 + 1 - \lambda\right] = 0.$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2 - \lambda - 12\right] + 4(\lambda + 3) + 3(\lambda + 3) = 0$$

$$\Rightarrow -(\lambda + 2)(\lambda - 4)(\lambda + 3) + 7(\lambda + 3) = 0$$

$$\Rightarrow (\lambda + 3)\left[-(\lambda + 2)(\lambda - 4) + 7\right] = 0$$

$$\Rightarrow -(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$$

$\therefore$ The eigen values of A are – 3, – 3, 5.

Eigen vector of A corresponding to $\lambda = -3$

Put $\lambda = -3$ in (1). We get $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The Augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right)$

Performing $R_2 - 2R_1, R_3 + R_1$, we get $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Hence we have

$$x_1 + 2x_2 - 3x_3 = 0$$

$$0 = 0 \quad \Rightarrow \quad x_1 = -2x_2 + 3x_3$$
$$0 = 0$$

Thus taking $x_2 = \alpha$ and $x_3 = \beta$, we get $x_1 = -2\alpha + 3\beta$; $x_2 = \alpha$; $x_3 = \beta$

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = -3$.

(Here we are getting $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ as eigen vectors of A corresponding to $\lambda = -3$.

A linear combination of these two vectors is also an eigen vector of A corresponding to $\lambda = -3$).

Eigen vector corresponding to $\lambda = 5$

Putting $\lambda = 5$ in (1), we get $\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider the Augmented matrix of the system

$$\left(\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right)$$

Performing $R_1 \leftrightarrow R_3$ and $R_1 \to (-)R_1$, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right)$$

Performing $R_2 - 2R_1$, $R_3 + 7R_1$, we get $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right)$

Performing $\frac{R_2}{-8}$, we get $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right)$

Performing $R_1 - 2R_2$, $R_3 - 16R_2$, we get $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

This implies that

$$x_1 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$0 = 0$$

Taking $x_3 = \alpha_1$, we get $x_1 = -\alpha_1$; $x_2 = -2\alpha_1$.

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha_1 \\ -2\alpha_1 \\ \alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

Hence the eigen vector of A corresponding to $\lambda = 5$ is $(-1, -2, 1)^T$.

Thus the eigen values of A are -3, -3 and 5.

$\therefore$ The eigen vector corresponding to $\lambda = -3$ is $\alpha(-2, 1, 0)^T + \beta(3, 0, 1)^T$ and

The eigen vector corresponding to $\lambda = 5$ is $\alpha_1(-1, -2, 1)^T$.

An observation :

Here again sum of the eigen values of A is $-3-3+5=-1$ and this is same as trace of A.
The product of the eigen values is $(-3)(-3)5 = 45$ and this is same as the determinant of A.

Example 5 : Find the Eigen values and Eigen vectors of the following matrix :

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$

[JNTU 2001, 2006 (Set No.4)]

Solution : Let $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow (5-\lambda)[(6-\lambda)(7-\lambda)-4] + 2[-2(7-\lambda)] = 0$

$\Rightarrow (5-\lambda)[42 - 13\lambda + \lambda^2 - 4] + 4\lambda - 28 = 0$ or $(5-\lambda)(\lambda^2 - 13\lambda + 38) + 4\lambda - 28 = 0$

$\Rightarrow 5\lambda^2 - 65\lambda^2 + 190 - \lambda^3 + 13\lambda^2 - 38\lambda + 4\lambda - 28 = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0 \Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$\Rightarrow (\lambda-3)(\lambda-6)(\lambda-9) = 0$

$\therefore$ Eigen values are 3, 6, 9.

Eigen vector corresponding to '3'

$$(A-3I)X = O \Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_1 + R_2$, we get $\begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $\left(\frac{1}{2}\right)R_3 - R_1$, we get $\begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{aligned} x_2 + 2x_3 &= 0 \\ -2x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

Let $x_3 = k$. Then

$x_2 = -2k$ and $-2x_1 - 6k + 2k = 0 \Rightarrow -2x_1 - 4k = 0 \Rightarrow x_1 = -2k$

$$\therefore \text{ Eigen vector is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to '6'

$$(A-6I)\,X = O \Rightarrow \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 - 2R_1$, we get $\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 - \frac{R_2}{2}$, we get $\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad -x_1 - 2x_2 = 0$$
$$4x_2 + 2x_3 = 0$$

Let $x_3 = k$. Then

$$4x_2 = -2k \Rightarrow x_2 = -k/2 \qquad \therefore\ x_1 = k$$

Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -k/2 \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Eigen vector corresponding to '9'

$$(A-9I)\,X = O \quad \Rightarrow \begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $2R_2 - R_1$, we get $\begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 + \frac{R_2}{2}$, we get $\begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad -4x_1 - 2x_2 = 0$ and $-4x_2 + 4x_3 = 0$

Let $x_3 = k$. Then $x_2 = k$ and $x_1 = -k/2$

$\therefore$ Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Example 6 : Find the characteristic roots of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and the corresponding eigen vectors . **[JNTU May 2005S, (A) Nov. 2010, (H) May 2012]**

Solution : The characteristic equation of A is $|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$

$\Rightarrow (6-\lambda)[(3-\lambda)^2-1]+2[-2(3-\lambda)+2]+2[2-2(3-\lambda)]=0$

$\Rightarrow (6-\lambda)[9+\lambda^2-6\lambda-1]+2[-6+2\lambda+2]+2[2-6+2\lambda]=0$

$\Rightarrow (6-\lambda)[\lambda^2-6\lambda+8]+2[2\lambda-4]+2[2\lambda-4]=0$

$\Rightarrow \lambda^3-12\lambda^2+36\lambda-32=0 \Rightarrow (\lambda-2)(\lambda^2-10\lambda+16)=0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8)=0 \Rightarrow \lambda=2,2,8$

The eigen values of A are 2, 2, 8.

The eigen vector of A corresponding to $\lambda = 2$.

$(A-\lambda I)X=O \Rightarrow (A-2I)X=O$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2+R_1, R_3 \to 2R_3-R_1$, we get $\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1-2x_2+2x_3=0 \Rightarrow 2x_1-x_2+x_3=0$

Let $x_2=k_1, x_3=k_2$. Then

$2x_1-k_1+k_2=0 \Rightarrow 2x_1=k_1-k_2 \Rightarrow x_1=\frac{k_1}{2}-\frac{k_2}{2}$

$\therefore X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{k_1}{2}-\frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}=k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}+k_2\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda=2$ where k_1 and k_2 are arbitrary constants (both are not equal to zero simultaneously).

The eigen vector of A corresponding to $\lambda = 8$

$$(A-8I)X=O \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2-R_1, R_3 \to R_3+R_1 \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0$ and $-3x_2 - 3x_3 = 0 \Rightarrow x_2 + x_3 = 0$

Put $x_3 = k$. Then $x_2 + k = 0 \Rightarrow x_2 = -k$ and $x_1 - k - k = 0 \Rightarrow x_1 - 2k = 0 \Rightarrow x_1 = 2k$

$\therefore$ X $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} k$ is the eigen of vector of A corresponding to $\lambda = 8$ where k is any non - zero arbitrary constant.

Example 7 : Find the eigen values and the corresponding eigen vectors of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[JNTU May 2006, (H) June 2011 (Set No. 1)]

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e., $\begin{vmatrix} -\lambda & 0 & 1 \\ \lambda & -\lambda & 1 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0$ (Applying $C_1 - C_2$ and $C_2 - C_3$)

i.e., $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0$ (Applying $R_2 \to R_2 + R_1$)

i.e., $-\lambda [-\lambda (1-\lambda) - 2\lambda] = 0$ [Expanding by C_1]

i.e., $\lambda^2 (1-\lambda + 2) = 0$

or $\lambda^2 (3-\lambda) = 0$

$\therefore$ $\lambda = 0, 0, 3$

To find the eigen vectors for the corresponding eigen values *i.e.*, 0, 0, 3 we will consider the matrix equation

$(A - \lambda I)\,X = O$

i.e., $$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad .... (1)$$

Eigen vector of A corresponding to $\lambda = 0$

By putting $\lambda = 0$, the matrix equation (1) will become

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $x_1 + x_2 + x_3 = 0$

Choose $x_2 = \alpha$ and $x_3 = \beta$. Then $x_1 = -(\alpha + \beta)$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the eigen vectors of A corresponding to eigen value $\lambda = 0$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A linear combination of these two vectors is also an eigen vector of A corresponding to $\lambda = 0$.

Eigen vector of *A* corresponding to $\lambda = 3$

Putting $\lambda = 3$ in (1), we get

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $-2x_1 + x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 - 2x_3 = 0$

On solving the first two equations by cross-multiplication, we get

$x_1 = k,\ x_2 = k,\ x_3 = k.$

Hence the eigen vector of A corresponding to eigen value $\lambda = 1$ is $\begin{bmatrix} k \\ k \\ k \end{bmatrix}$

By putting $k = 1$, we get the simplest eigen vector as $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus the eigen values and eigen vectors of A are 0,0,3 and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example 8 : Find the eigen values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. **[JNTU May 2006 (Set No. 3), 2008, (A) May 2012 (Set No.3)]**

Solution : Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

On expanding, we get

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$ *i.e.* $\lambda(\lambda^2 - 18\lambda + 45) = 0$

or $\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore$ $\lambda = 0, 3, 15$

To find eigen vectors for the corresponding eigen values 0, 3 and 15, we will consider the matrix equation

$(A - \lambda I)X = 0$

i.e., $\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .... (1)

Eigen vector corresponding to eigen value $\lambda = 0$

Putting $\lambda = 0$ in (1), we obtain

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

Solving the first two equations by cross-multiplication, we get

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

i.e., $\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$ or $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k$

$\therefore$ $x_1 = k, x_2 = 2k, x_3 = 2k$

Choosing $k = 1$, we get

$x_1 = 1, x_2 = x_3 = 2$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 0$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Eigen vector corresponding to $\lambda = 3$

Putting $\lambda = 3$ in (1), we get

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 0.x_3 = 0$

Solving any of the above two equations, we get

$x_1 = -2,\ x_2 = -1,\ x_3 = 2$ (taking $k = 1$)

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 3$ is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 15$

Putting $\lambda = 15$ in (1), we get

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0,\ -6x_1 - 8x_2 - 4x_3 = 0,\ 2x_1 - 4x_2 - 12x_3 = 0.$

Solving any two of the above equations by cross multiplication, we get

$x_1 = 2,\ x_2 = -2,\ x_3 = 1.$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Hence the eigen values of A are 0, 3, 15 and the corresponding eigen vectors of A are

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Example 9 : Verify that the sum of eigen values is equal to the trace of 'A' for the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vectors.

[JNTU May 2007 (Set No.4)]

Solution : Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[-3+\lambda+1]+1[1-5+\lambda]=0$

$\Rightarrow (3-\lambda)^2 (5-\lambda)+[-3+\lambda-2+\lambda]+(\lambda-4)=0$

$\Rightarrow (3-\lambda)^2 (5-\lambda)+(3\lambda-9)=0 \Rightarrow (3-\lambda)^2 (5-\lambda)+3(\lambda-3)=0$

$\Rightarrow (\lambda-3)\ [(5-\lambda)(\lambda-3)+3]=0 \Rightarrow (\lambda-3)(-\lambda^2+8\lambda-15+3)=0$

$\Rightarrow (\lambda-3)(-\lambda^2+8\lambda-12)=0 \Rightarrow (\lambda-3)(\lambda-2)(\lambda-6)=0$

$\therefore\ \lambda = 2,3,6$

Sum of Eigen values $= 3+6+2=11$

i.e. Trace of A = 3 + 6 + 2 = 11

Thus sum of eigen values = trace (A), is verified.

Eigen Vectors corresponding to $\lambda = 3$

Consider $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$ gives $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 + R_2$ gives $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We can write $-y+z=0 \Rightarrow y=z$

and $-x+y=0 \Rightarrow x=y$

Taking $x=y=z=c$, we get

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ which is the required eigen vector corresponding to $\lambda = 3$.

Similarly we can find other eigen vectors.

Example 10 : Find the Eigen values and the corresponding Eigen vectors of the matrix

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

[JNTU Sep. 2008, (H) Dec. 2011 (Set No. 2)]

Solution : Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of 'A' is $|A - \lambda I| = 0$.

i.e., $\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(2-\lambda)\,[(5-\lambda)\,(3-\lambda)] - 2\,(2(3-\lambda)) = 0$

$\Rightarrow$ $(2-\lambda)\,(15 - 5\lambda - 3\lambda + \lambda^2) - 4\,(3-\lambda) = 0$

$\Rightarrow$ $30 - 10\lambda - 6\lambda + 2\lambda^2 - 15\lambda + 5\lambda^2 + 3\lambda^2 - \lambda^3 - 12 + 4\lambda = 0$

$\Rightarrow$ $-\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$

$\Rightarrow$ $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

We observe that $\lambda = 1$ is a root.

Dividing with $(\lambda - 1)$, we get

$(\lambda - 1)\;(\lambda^2 - 9\lambda + 18) = 0$

i.e., $(\lambda - 1)\;(\lambda^2 - 6\lambda - 3\lambda + 18) = 0$

i.e., $(\lambda - 1)\,(\lambda\,(\lambda - 6) - 3\,(\lambda - 6)) = 0$

or $(\lambda - 1)\,(\lambda - 3)\,(\lambda - 6) = 0$

$\therefore$ $\lambda = 1, 3, 6$

$\therefore$ The Eigen values of 'A' are 1, 3, 6.

Case 1: The Eigen vector corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 2x_2 = 0$ $\cdots$ (1)

and $2x_3 = 0 \Rightarrow x_3 = 0$

Let $x_2 = k$. Then

(1) $\Rightarrow x_1 + 2k = 0 \Rightarrow x_1 = -2k$

$\therefore$ The Eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Case 2 : The Eigen vector corresponding to $\lambda = 3$ is given by

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + 2x_2 = 0 \Rightarrow 2x_2 = x_1$

and $\quad 2x_1 + 2x_2 = 0 \Rightarrow x_2 = 0$

and $\quad -3x_1 = 0 \Rightarrow x_1 = 0$

Let $\quad x_3 = k.$

The Eigen vector corresponding to $\lambda = 3$ is $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Case 3 : The Eigen vector corresponding to $\lambda = 6$ is given by

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; 2x_1 - x_2 = 0$ and $-3x_3 = 0$

$\therefore \quad 2x_1 = x_2$

Let $x_1 = k$. Then $\; x_2 = 2k$

$\therefore$ The eigen vector corresponding to $\lambda = 6$ is $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

The Eigen vectors corresponding to $\lambda = 1, 3, 6$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Example 11 : Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

[JNTU 2008S, (K) May 2010 (Set No.1)]

Solution : Let $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$. Then

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix}$$

The characteristic equation is given by $|A - \lambda I| = 0 \Rightarrow (1-\lambda)\,[(4-\lambda)\,(-3-\lambda) + 12] = 0$

$\Rightarrow\ (1-\lambda)(-12-4\lambda+3\lambda+\lambda^2+12)=0\ \Rightarrow (1-\lambda)(\lambda^2-\lambda)=0\ \Rightarrow\ (1-\lambda)\lambda(\lambda-1)=0$

$\lambda=1,1,0$ are the eigen values.

Eigen vector corresponding to $\lambda=1$

Let $(A-\lambda I)X=O\ \Rightarrow (A-I)X=O$

$$\Rightarrow \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying R_3-R_1 and $2R_2+R_1$, we get $\begin{bmatrix} 0 & -6 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -6y-4z=0 \Rightarrow 3y=-2z$

Let $z=k$. Then $y=-\frac{2}{3}k$. Let $x=k_1$

$$X=\begin{bmatrix} k_1 \\ -\frac{2}{3}k \\ k \end{bmatrix}=k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+\frac{1}{3}k\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$X_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2=\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda=1$.

Eigen vector corresponding to $\lambda=0$.

Solving $(A-\lambda I)X=O \Rightarrow AX=O$

$$i.e., \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x-6y-4z=0; 4y+2z=0; -6y-3z=0$

The second and third equations are same *i.e.*, $2y+z=0$

$\therefore$ Solving $x-6y-4z=0$ and $2y+z=0$, we have

$$\frac{x}{-6+8}=\frac{y}{0-1}=\frac{z}{2+0} \text{ or } \frac{x}{2}=\frac{y}{-1}=\frac{z}{2}=k \text{ (say)}$$

i.e., $x=2k, y=-k, z=2k$. Taking $k=1$ we have $x=2, y=-1, z=2$

$\therefore$ The eigen vector corresponding to $\lambda=0$ is $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Example 12 : Determine the characteristic roots and the corresponding characteristic vectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **[JNTU(H) June 2009 (Set No.2)]**

Solution : Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)]=0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0 \Rightarrow -\lambda(\lambda^2-18\lambda+45)=0$$

$$\Rightarrow -\lambda[(x-15)(\lambda-3)]=0 \Rightarrow \lambda = 0, 15, 3$$

Case (*i*) : Let $\lambda = 0$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to the root $\lambda = 0$.

Solving $(A - \lambda I)X = O$

$\Rightarrow (A - 0I)X = O$ or $AX = O$.

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$ gives

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + 3R_1;\ R_3 - 4R_1$ gives

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$ gives

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 - 4x_2 + 3x_3 = 0$

$-5x_2 + 5x_3 = 0 \Rightarrow x_2 = x_3$

$\therefore\ 2x_1 - 4x_3 + 3x_3 = 0 \Rightarrow 2x_1 - x_3 = 0$

$\Rightarrow 2x_1 = x_3 \Rightarrow x_1 = \frac{x_3}{2}$

Put $x_3 = c_1 \Rightarrow x_1 = \frac{c_1}{2}; x_2 = c_1$

$$\therefore\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{c_1}{2}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is characteristic vector corresponding to characteristic root $\lambda = 0$

Case (*ii*) : Let $\lambda = 15$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to the root $\lambda = 15$.

Solving $(A - \lambda I)X = O \Rightarrow (A - 15I)\,X = 0$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$ gives

$$\begin{bmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{1}{2}R_1; -\frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ -7 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 3R_1; R_3 + 7R_1$ gives

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 10 & 20 \\ 0 & -20 & -40 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$ gives

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 10 & 20 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; x_1 - 2x_2 - 6x_3 = 0$

$10x_2 + 20x_3 = 0 \Rightarrow x_2 = -2x_3$

$x_1 + 4x_3 - 6x_3 = 0 \Rightarrow x_1 = 2x_3$

Let $x_3 = c_2$

$\Rightarrow x_1 = 2c_2; x_2 = -2c_2$

$$\therefore \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2c_2 \\ -2c_2 \\ c_2 \end{bmatrix} = c_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is characteristic vector corresponding to characteristic root $\lambda = 15$.

Case (*iii*) : Let $\lambda = 3$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to the root $\lambda = 3$.

Solving $(A - 3I)X = O$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ gives

$$\begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{1}{2}R_1; \frac{1}{2}R_2$ gives

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -2 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + 3R_1; R_3 - 5R_1$ gives

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_2$ gives

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - 2x_2 = 0$

$$-4x_2 - 2x_3 = 0 \Rightarrow -2x_2 - x_3 = 0 \Rightarrow -2x_2 = x_3 \Rightarrow x_2 = -\frac{1}{2}x_3$$

$$\therefore \quad x_1 + 2\left(\frac{1}{2}\right)x_3 = 0 \Rightarrow x_1 = -x_3$$

Let $x_3 = c_3$. Then

$x_1 = -c_3$ and $x_2 = -\frac{1}{2}c_3$

$$\therefore \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c_3 \\ -\frac{1}{2}c_3 \\ c_3 \end{bmatrix} = \frac{c_3}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is characteristic vector corresponding to characteristic root $\lambda = 3$.

Example 13 : Find the eigen values and eigen vectors of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

[JNTU (A)June 2009 (Set No.2)]

Solution : Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$2R_3$ gives

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 4 & -2 & 6-2\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_3$, we get

$$\begin{vmatrix} 2-\lambda & 0 & -4+2\lambda \\ -2 & 3-\lambda & -1 \\ 4 & -2 & 6-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 1 & 0 & -2 \\ -2 & 3-\lambda & -1 \\ 4 & -2 & 6-2\lambda \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + 2C_1(2-\lambda)$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 3-\lambda & -5 \\ 4 & -2 & 14-2\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\,[(3-\lambda)\,(14-2\lambda)-10] = 0$

$\Rightarrow (2-\lambda)\,[(42-6\lambda-14x+2\lambda^2-10)] = 0$

$\Rightarrow (2-\lambda)\,[(2\lambda^2-20\lambda+32)] = 0$

$\Rightarrow (2-\lambda)\,(\lambda-8)\,(\lambda-2) = 0$

$\Rightarrow \lambda = 2, 2, 8$

Case (*i*) : Let $\lambda = 2$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to the root $\lambda = 2$.

Then $(A - \lambda I)X = O \Rightarrow (A-2I)X = O$

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$ gives

$$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 + R_1$ and $R_3 - 2R_1$ gives

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 + x_3 = 0$$

Let $x_2 = c_1$ and $x_3 = c_2$. Then $x_1 = \frac{x_2}{2} - \frac{x_3}{2} = \frac{c_1}{2} - \frac{c_2}{2}$

$$\therefore\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} - \frac{c_2}{3} \\ c_1 \\ c_2 \end{bmatrix} = \frac{c_1}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{c_2}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ are the eigen vectors corresponding to the eigen value $\lambda = 2$.

Case (*ii*) : Let $\lambda = 8$

Solving $(A - \lambda I)X = O \Rightarrow (A - 8I)X = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 - R_1, R_3 + R_1$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{-3}$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_1 \to \frac{R_1}{-2}$, we get

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 - x_3 = 0;$

$x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

and $x_1 - x_3 - x_3 = 0 \Rightarrow x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$

Put $x_3 = c_2 \Rightarrow x_1 = 2c_2$, $x_2 = -c_2$

$$\therefore \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2c_2 \\ -c_2 \\ c_2 \end{bmatrix} = c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 8$.

Example 14 : Find the eigen values and the corresponding eigen vectors of

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

[JNTU (H) June 2010 (Set No.4)]

Solution : Given matrix is $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$ [Expand by R_1]

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda) - 2] - 1[2 - 2(2-\lambda)] = 0$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4)-1[2\lambda-2]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4)+2(\lambda-1)=0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4-2]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+2)=0$$

$$\Rightarrow \lambda=1,\ \lambda=\frac{5\pm\sqrt{25-8}}{2}=\frac{5\pm\sqrt{17}}{2}$$

$\therefore \quad \lambda=1,\dfrac{5+\sqrt{17}}{2},\dfrac{5-\sqrt{17}}{2}$ are the characteristic (eigen) values.

Eigen Vector Corresponding to eigen value $\lambda = 1$

Let X_1 be the corresponding eigen vector. Then

$(A-I)X_1=O$

$$\Rightarrow \begin{bmatrix}0&0&-1\\1&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_3-2R_2 \text{ gives} \begin{bmatrix}0&0&-1\\1&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_1=0, x_1+x_2+x_3=0.$$

Let $x_2=k_1, x_3=-x_2=-k_1$. Then

$$X_1=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\k_1\\-k_1\end{bmatrix}=k_1\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix}0\\1\\-1\end{bmatrix}$.

Example 15 : Find the eigen values and the corresponding eigen vectors of

$$\begin{bmatrix}3&2&2\\1&2&2\\-1&-1&0\end{bmatrix}$$

[JNTU (H) Jan. 2012 (Set No. 1)]

Solution : Given matrix is $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)[(2-\lambda)(-\lambda)+2] - 2[-\lambda+2] + 2[-1+(2-\lambda)] = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 2\lambda + 2) + 2\lambda - 4 + 2 - 2\lambda = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 2\lambda + 2) - 2 = 0$

$\Rightarrow 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda - 2 = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$ or $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\therefore\ \lambda = 2, 2, 1$ are the roots.

Eigen vector corresponding to $\lambda = 2$

The eigen vectors of A corresponding to λ are given by

$(A - \lambda I)X = O$ *i.e.*, $(A - 2I)X = O$

$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 + R_1$ gives $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

From the matrix, $x_2 = 0$ and $x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$

Taking $x_3 = k \Rightarrow x_1 = -2k$

The eigen vector corresponding to $\lambda = 2$ is

$\begin{bmatrix} -2k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$; $X_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Eigen vector corresponding to $\lambda = 1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \; x_1 + x_2 + x_3 = 0$

and $x_1 + x_2 + 2x_3 = 0 \Rightarrow x_3 = 0$. Take $x_2 = k \Rightarrow x_1 = -k$

$\therefore \; X_2 = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = -k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is eigen vector corresponding to $\lambda = 1$.

Example 16 : Find the eigen values are Eigen vectors of $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

[JNTU (H) May 2012, (A) Nov. 2012 (Set No. 2)]

Solution : Characteristic equation is $(A - \lambda I) = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow (2-\lambda)(5-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$

$\Rightarrow \lambda = 6,1$ are the eigen values.

Eigen vector corresponding to $\lambda = 6$

Let $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-6 & 1 \\ 4 & 5-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$ gives $\begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + x_2 = 0$

Let $x_1 = k \Rightarrow x_2 = 4k$

$\therefore \; X = \begin{bmatrix} k \\ 4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$

Eigen value corresponding to $\lambda = 1$

$$\begin{bmatrix} 2-1 & 1 \\ 4 & 5-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_1 = k \Rightarrow x_2 = -k$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Example 17 : Find the sum and product of the eigen value of $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

[JNTU (H) May 2012, (A) May 2012 (Set No. 2)]

Solution : Sum of the eigen values = trace of the matrix $= 2+4+2 = 8$

Product of the eigen values

$$A = \text{determinant of } A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 2(8-0) - 1(6-2) - 1(0-4) = 16 - 4 + 4 = 16$$

2.4 PROPERTIES OF EIGEN VALUES

Theorem 1 :

The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

i.e., if A is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its n eigen values, then

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = Tr(A) \quad \text{and} \quad \lambda_1.\lambda_2.\lambda_3 \ldots \lambda_n = \det(A)$$

[JNTU 2003S, 2004S, 2006, (A) May 2012 (Set No. 4)]

Proof: Characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding this, we get

$(a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda) - a_{12}$(a polynomial of degree $n-2$)

$+ a_{13}$(a polynomial of degree $n-2$) $+ \cdots = 0$

i.e., $(-1)^n(\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn})$ + *a* polynomial of degree $(n-2) = 0$

i.e., $(-1)^n[\lambda^n - (a_{11} + a_{22} \ldots + a_{nn})\lambda^{n-1}$ + *a* polynomial of degree $(n-2)]$

+ *a* polynomial of degree $(n-2)$ in $\lambda = 0$

$\therefore (-1)^n \lambda^n + (-1)^{n+1}(\text{Trace } A)\lambda^{n-1}$ + *a* polynomial of degree $(n-2)$ in $\lambda = 0$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of this equation,

$$\text{sum of the roots} = -\frac{(-1)^{n+1} Tr(A)}{(-1)^n} = Tr(A).$$

Further $|A - \lambda I| = (-1)^n\lambda^n + \ldots + a_0$

Put $\lambda = 0$. Then $|A| = a_0$

$$(-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} \cdots + a_0 = 0.$$

$$\Rightarrow \text{Product of the roots} = \frac{(-1)^n a_0}{(-1)^n} = a_0 = |A| = \det A$$

Hence the result.

Theorem 2:

If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value of A^n corresponding to the eigen vector X. **[JNTU 2000, 2003 (Set No. 4)]**

(or) Prove that if $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values of A then $\lambda_1^k, \lambda_2^k, \ldots\ldots\ldots, \lambda_n^k$ are the eigen values of A^k. **[JNTU (A) Dec. 2011]**

Proof: Since λ is an eigen value of A corresponding to the eigen vector X, we have

$$AX = \lambda X. \qquad \ldots(1)$$

Premultiply (1) by A, $A(AX) = A(\lambda X)$

(*i.e.*) $(AA)X = \lambda(AX)$ (*i.e.*) $A^2X = \lambda \cdot \lambda X = \lambda^2 X$ [using (1)].

Hence λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorem is true to $n = \mathbf{2}$. Let the result be true for $n = k$.

Then $A^kX = \lambda^kX$

Premultiplying this by A and using $AX = \lambda X$, we get $A^{k+1}X = \lambda^{k+1}X$

which implies that λ^{k+1} is eigen value of A^{k+1} with X itself as the corresponding eigen vector. Hence by the principle of mathematical induction, the theorem is true for all positive integers n.

Theorem 3 :

Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the latent roots of A, then A^3 has latent roots $\lambda_1^3, \lambda_2^3, \ldots, \lambda_n^3$.

[JNTU 2000, 2003 (Set No. 4), 2004S (Set No. 4), Sep 2008 (Set No. 1)]

Proof: Put $n = 3$ in the above theorem.

Theorem 4 :

A square matrix A and its transpose A^T have the same eigen values.

[JNTU 2002 (Set No. 4), 2003S (Set No. 2), 2004S (Set No. 4)]

Proof: We have $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$\therefore\ |(A - \lambda I)^T| = |A^T - \lambda I|$ or $|A - \lambda I| = |A^T - \lambda I|$ $\quad (\because\ |A^T| = |A|)$

$\therefore\ |A - \lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

i.e. λ is an eigen value of A if and only if λ is an eigen value of A^T.

Thus the eigen values of A and A^T are same.

Theorem 5 :

If A and B are n rowed square matrices and if A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values. **[JNTU 2003, 2005S, (A) May 2012 (Set No. 1)]**

Proof: Given A is invertible $\Rightarrow$ A^{-1} exists.

We know that if A and P are the square matrices of order n such that P is non-singular, then A and $P^{-1}AP$ have the same eigen values.

Taking $A = BA^{-1}$ and $P = A$, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen values

i.e., BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

i.e., BA^{-1} and $(A^{-1}B)I$ have the same eigen values

or BA^{-1} and $A^{-1}B$ have the same eigen values.

Theorem 6 :

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of a matrix A, then $k\lambda_1, k\lambda_2, ..., k\lambda_n$ are the eigen values of the matrix kA, where k is a non-zero scalar.

[JNTU 2002 (Set No. 1), Sep 2008 (Set No.1)]

Proof: Let A be a square matrix of order n.

Then $|kA - \lambda kI| = |k(A - \lambda I)| = k^n|A - \lambda I|$ $\quad (\because\ |kA| = k^n|A|)$

Since $k \neq 0$, therefore $|kA - \lambda kI| = 0$ if and only if $|A - \lambda I| = 0$

i.e., $k\lambda$ is an eigen value of kA if and only if λ is an eigen value of A.

Thus $k\lambda_1, k\lambda_2, ... ,k\lambda_n$ are the eigen values of the matrix kA if $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of the matrix A.

Theorem 7 :

If λ is an eigen value of the matrix A then $\lambda + k$ is an eigen value of the matrix $A+kI$.

Proof: Let λ be an eigen value of A and X the corresponding eigen vector.

Then, by definition $\quad AX = \lambda X \quad \ldots (1)$

Now $(A + KI)\,X = AX + kIX = \lambda X + kX = (\lambda + k)X$ [by (1)] $\quad \ldots (2)$

From (2), we see that the scalar $\lambda + k$ is an eigen value of the matrix $A + KI$ and X is a corresponding eigen vector.

Theorem 8 :

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A, then $\lambda_1 - k, \lambda_2 - k, ..., \lambda_n - k$ are the eigen values of the matrix $(A - KI)$, where k is a non-zero scalar.

[JNTU 2005S (Set No. 3), Sep 2008, (H) June 2011 (Set No.1)]

Proof: Since $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A,

$\therefore$ The characteristic polynomial of A is

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda) \qquad \ldots (1)$$

Thus the characteristic polynomial of $A - kI$ is

$$(A - kI - \lambda I)X = |A - (k + \lambda)I|$$
$$= [\lambda_1 - (\lambda + k)].[\lambda_2 - (\lambda + k)]...[\lambda_n - (\lambda + k)], \text{ from } (1)$$
$$= [(\lambda_1 - k) - \lambda][(\lambda_2 - k) - \lambda]...[(\lambda_n - k) - \lambda]$$

This shows that the eigen values of $A - kI$ are $\lambda_1 - k, \lambda_2 - k, ..., \lambda_n - k$.

Theorem 9 :

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$.

Proof: First we will find the eigen values of the matrix $A - \lambda I$. **[JNTU 2003]**

Since $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A,

$\therefore$ The characteristic polynomial of A is $|A - kI| = (\lambda_1 - k)(\lambda_2 - k) \ldots (\lambda_n - k) \quad \ldots (1)$

where k is a scalar.

The characteristic polynomial of the matrix $(A - \lambda I)$ is

$$|A - \lambda I - kI| = |A - (\lambda + k)I|$$
$$= [\lambda_1 - (\lambda + k)][\lambda_2 - (\lambda + k)] \ldots [\lambda_n - (\lambda + k)], \quad \text{from } (1)$$
$$= [(\lambda_1 - \lambda) - k][(\lambda_2 - \lambda) - k] \ldots [(\lambda_n - \lambda) - k]$$

which shows that the eigen values of $A - \lambda I$ are $\lambda_1 - \lambda, \lambda_2 - \lambda, ..., \lambda_n - \lambda$.

We know that if the eigen values of A are $\lambda_1, \lambda_2, ..., \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$.

Thus the eigen values of $(A - \lambda I)^2$ are $(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, ..., (\lambda_n - \lambda)^2$.

Theorem 10 :

If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector X itself.

[JNTU 2003, 2003S, 2005 (Set No. 3)]

(OR)

Prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

[JNTU 2004S (Set No. 3)]

Proof: Since A is non-singular and product of the eigen values is equal to $|A|$, it follows that none of the eigen values of A is 0.

$\therefore$ If λ is an eigen value of the non-singular matrix A and X is the corresponding eigen vector, $\lambda \neq 0$ and $AX = \lambda X$. Premultiplying this with A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$$

$$\therefore X = \lambda A^{-1}X \qquad \Rightarrow A^{-1}X = \lambda^{-1}X \qquad (\because \lambda \neq 0)$$

Hence by definition it follows that λ^{-1} is an eigen value of A^{-1} and X is the corresponding eigen vector.

Theorem 11 :

If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A. **[JNTU 2003, 2005 (Set No. 3), (H) May 2012]**

Proof: Since λ is an eigen value of a non-singular matrix, therefore, $\lambda \neq 0$.

Also λ is an eigen value of A implies that there exists a non-zero vector X such that

$$AX = \lambda X \qquad \dots (1)$$

$$\Rightarrow (\text{adj } A)AX = (\text{adj } A)(\lambda X) \Rightarrow [(\text{adj } A)A]\, X = \lambda(\text{adj } A)X$$

$$\Rightarrow |A|IX = \lambda\, (\text{adj } A)X \qquad [\because (\text{adj } A)A = |A|I]$$

$$\Rightarrow \frac{|A|}{\lambda}X = (\text{adj } A)X \quad \text{or} \quad (\text{adj } A)X = \frac{|A|}{\lambda}X$$

Since X is a non-zero vector, therefore, from the relation (1) it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

Theorem 12 :

If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value.

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}. Since A is an orthogonal matrix, therefore,

$$A^{-1} = A'$$

$$\therefore \frac{1}{\lambda} \text{ is an eigen value of } A'.$$

But the matrices A and A' have the same eigen values, since the determinants $|A - \lambda I|$ and $|A' - \lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorem 13 :

If λ is an eigen value of A, then prove that the eigen value of
$B = a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$. **[JNTU 2003S, (A) May 2012 (Set No. 4)]**

Proof: If X be the eigen vector corresponding to the eigen value λ, then

$$AX = \lambda X \qquad \dots (1)$$

Premultiply by A on both sides, $A\,(AX) = A(\lambda X)$

$$\Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda\,(\lambda X) = \lambda^2 X$$

This shows that λ^2 is an eigen value of A^2.

We have $B = a_0 A^2 + a_1 A + a_2 I$

$$\therefore \quad BX = (a_0 A^2 + a_1 A + a_2 I)X = a_0 A^2 X + a_1 AX + a_2 X$$
$$= a_0 \lambda^2 X + a_1 \lambda X + a_2 X = (a_0 \lambda^2 + a_1 \lambda + a_2)X$$

This shows that $a_0\lambda^2 + a_1\lambda + a_2$ is an eigen value of B and the corresponding eigen vector of B is X.

Theorem 14 :

Suppose that A and P be square matrices of order n such that P is non-singular. Then A and $P^{-1}AP$ have the same eigen values. **[JNTU 2002, Sep 2008 (Set No.4)]**

Proof: Consider the characteristic equation of $P^{-1}AP$. It is

$$|(P^{-1}AP) - \lambda I| = |P^{-1}AP - \lambda P^{-1}IP| \quad (\because I = P^{-1}IP)$$
$$= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P|$$
$$= |A - \lambda I|, \text{ since } |P^{-1}||P| = 1$$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same.

Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary :

If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values. **[JNTU 2002]**

Proof: Notice that $AB = IAB = (B^{-1}B)(AB) \qquad = B^{-1}(BA)(B)$

Using the above theorem BA and $B^{-1}(BA)B$ have the same eigen values. *i.e.,* BA and AB have the same eigen values.

Theorem 15 :

The eigen values of a triangular matrix are just the diagonal elements of the matrix.

[JNTU 2002 (Set No. 2), 2003 (Set No. 2)]

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0$

i.e. $(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$.

$\therefore \ \lambda = a_{11}, a_{22}, \dots, a_{nn}$.

Hence the eigen values of A are $a_{11}, a_{22}, \dots, a_{nn}$ which are just the diagonal elements of A.

Note : Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorem 16 :

The eigen values of a real symmetric matrix are always real (or real numbers).

[JNTU (A) June 2010 (Set No.3)]

Proof: Let λ be an eigen value of a real symmetric matrix A and let X be the corresponding eigen vector

Then $AX = \lambda X$ ...(1)

Take the conjugate $\overline{A}\overline{X} = \overline{\lambda}\overline{X}$

Taking the tranpose $\overline{X}^T(\overline{A})^T = \overline{\lambda}\overline{X}^T$; since $\overline{A} = A$ and $A^T = A$, we have

$\overline{X}^T A = \overline{\lambda}\,\overline{X}^T$

Post multiply by X, we get $\overline{X}^T AX = \overline{\lambda}\,\overline{X}^T X$ ...(2)

Premultiply (1) with $\overline{X}^T$, we get $\overline{X}^T AX = \lambda \overline{X}^T X$ ...(3)

(2) – (3) gives $(\lambda - \overline{\lambda})\ \overline{X}^T X = 0$; But $\overline{X}^T X \neq 0$

$\therefore\ \lambda = \overline{\lambda} \Rightarrow \lambda$ is real

Hence the result follows.

Theorem 17 :

For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a real symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors. Let $\lambda_1 \neq \lambda_2$. We want to show that X_1 is orthogonal to X_2 (*i.e.*) $X_1^T X_2 = 0$.

Since X_1, X_2 are eigen vectors of A corresponding to the eigen values λ_1, λ_2, we have

$AX_1 = \lambda_1 X_1$ ...(1) $AX_2 = \lambda_2 X_2$ ...(2)

Premultiply (1) by X_2^T

$$X_2^T AX_1 = \lambda_1 X_2^T X_1$$

Taking transpose, we have $X_1^T A^T (X_2^T)^T = \lambda_1 X_1^T (X_2^T)^T$

(*i.e.*) $X_1^T AX_2 = \lambda_1 X_1^T X_2$ ... (3)

Premultiplying (2) by X_1^T, we get $X_1^T AX_2 = \lambda_2 X_1^T X_2$ ... (4)

Hence from (3) and (4), we get $(\lambda_1 - \lambda_2)\ X_1^T X_2 = 0$

$\Rightarrow X_1^T X_2 = 0$, since $\lambda_1 \neq \lambda_2$

$\therefore X_1$ is orthogonal to X_2.

Hence the theorem.

Note : (*i*) If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

(*ii*) If λ is an eigen value of A and $f(A)$ is any polynomial in A, then the eigen value of $f(A)$ is $f(\lambda)$.

Theorem 18 :

Prove that the two eigen vectors corresponding to the two different eigen values are linearly independent. **[JNTU May 2006, (H) Dec. 2011 (Set No. 1)]**

Proof: Let A be a square matrix. Let X_1 and X_2 be the two eigen vectors of A corresponding to two distinct eigen values λ_1 and λ_2. Then

$$AX_1 = \lambda_1 X_1 \text{ and } AX_2 = \lambda_2 X_2 \quad \text{... (1)}$$

Now we shall prove that the eigen vectors X_1 and X_2 are Linearly independent.

Let us assume that the X_1 and X_2 are Linearly dependent.

Then for two scalars k_1 and k_2 not both zeros such that $k_1X_1 + k_2X_2 = O$ ... (2)

Multiplying both sides of (2) by A, we get

$$A\,(k_1X_1 + k_2X_2) = A\,(O) = O$$

$$\Rightarrow k_1(AX_1) + k_2(AX_2) = O$$

$$\Rightarrow k_1(\lambda_1X_1) + k_2(\lambda_2X_2) = O \quad \text{... (3)} \quad \text{[using (1)]}$$

(3) – λ_2 (2) gives

$$k_1\,(\lambda_1 - \lambda_2)\,X_1 = O$$

$$\Rightarrow k_1 = 0 \;\; [\because \lambda_1 \neq \lambda_2 \text{ and } X_1 \neq O]$$

$$\Rightarrow k_2 = 0$$

But this contradicts our assumption that k_1, k_2 are not zeros. Hence our assumption that X_1 and X_2 are linearly dependent is wrong. Hence the statement is true.

Algebraic and Geometric Multiplicity of a Characteristic root:

Def: Suppose A is n×n matrix. If λ_1 is a characterstic root of order t of the characterstic equation of A, then t is called the **algebraic multiplicity** of λ_1.

Def: If s is the number of linearly indipendent characterstic vectors corresponding to the characterstic vector λ_1, then s is called the **geometric multiplicity** of λ_1.

Note : The geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity. i.e., $s \leq t$.

SOLVED EXAMPLES

Example 1 : Find the eigen values and eigen vectors of the matrix A and its inverse, where $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ **[JNTU 2000]**

Solution : Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of 'A' is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)[(2-\lambda)(3-\lambda)] = 0 \Rightarrow \lambda = 1, 2, 3$

$\therefore$ Characteristic roots of A are 1, 2, 3.

To find characteristic vector of '1'.

For $\lambda = 1$, the eigen vector of A is given by

$$(A - I)X = O \quad \Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x_2 + 4x_3 = 0,\ x_2 + 5x_3 = 0$ and $2x_3 = 0$

This implies that $x_3 = 0$ and hence $x_2 = 0$. Notice that we cannot find x_1 from these equations. As x_1 is not present in any of these equations, it follows that x_1 can be arbitrary.

Hence $x_1 = \alpha,\ x_2 = 0,\ x_3 = 0$.

$\therefore X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ where $\alpha \neq 0$ is the eigen vector corresponding to $\lambda = 1$.

To find characteristic vector of '2'

For $\lambda = 2$, the eigen vector of A is given by

$$(A - 2I)X = O \quad \Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0,\ 5x_2 = 0$ and $x_3 = 0$

We take $x_1 = k$

$\therefore X = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Hence the characteristic vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

To find characteristic vector of '3'

$$(A - 3I)X = O \Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} -2x_1 + 3x_2 + 4x_3 &= 0 \\ -x_2 + 5x_3 &= 0 \end{aligned}$$

Let $x_3 = k$. Then $x_2 = 5k$ and $-2x_1 + 15k + 4k = 0$

$\Rightarrow 2x_1 = 19k \Rightarrow x_1 = \frac{19}{2}k$

$\therefore X = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix}$ or $X = \frac{k}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$

Hence Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ *i.e.,* $1, \frac{1}{2}, \frac{1}{3}$ (Refer Theorem 2.10) and eigen vectors of A^{-1} are same as eigen vectors of the matrix A.

Example 2 : Determine the eigen values and eigen vectors of

$B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$ **[JNTU 2004S (Set No. 4)]**

Solution : We have $A^2 = A.A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}$

$$\therefore B = 2A^2 - \frac{1}{2}A + 3I = \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

Characteristic equation of B is $|B - \lambda I| = 0$

$$i.e. \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 105\lambda - 2376 = 0 \Rightarrow (\lambda - 33)(\lambda - 72) = 0 \Rightarrow \lambda = 33$ or 72

$\therefore$ Eigen values of B are 33 and 72.

For $\lambda = 33$, the eigen vector of B is given by $(B - 33I)X = O$

$$i.e. \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} i.e.\ x_1 = x_2 \text{ or } \frac{x_1}{1} = \frac{x_2}{1}$$

$\therefore$ The eigen vector for $\lambda = 33$ is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 72$, the eigen vector of B is given by $(B - 72I)X = O$

$$i.e. \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 39x_1 - 78x_2 = 0$ or $x_1 = 2x_2$

$\therefore \quad \frac{x_1}{2} = \frac{x_2}{1}$

$\therefore$ The eigen vector for $\lambda = 72$ is $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Hence the eigen vectors of B are $(1, 1)^T, (2, 1)^T$.

Example 3 : For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

[JNTU 2003]

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \text{ [Expand by } C_1]$$

i.e., $(1-\lambda)[(3-\lambda)(-2-\lambda)-0]=0$

i.e., $(1-\lambda)(3-\lambda)(2+\lambda)=0$ or $\lambda=1,3,-2$

$\therefore$ Eigen values of A are 1, 3, – 2.

We know that if λ is an eigen value of A and $f(A)$ is a polynomial in A, then the eigen value of $f(A)$ is $f(\lambda)$.

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$.

Then the eigen values of $f(A)$ are $f(1)$, $f(3)$ and $f(-2)$.

$\therefore f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 3 + 5 - 6 - 2 = 4$

[$\because$ eigen values of I are 1, 1, 1]

$f(3) = 3(3)^3 + 5(3)^2 - 6(3) + 2(1) = 81 + 45 - 18 + 2 = 110$

$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2(1) = -24 + 20 + 12 + 2 = 10$

Thus the eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4, 110, 10.

Example 4 : Verify that the geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity given the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution : Proceeding as in Solved Example 3, we obtain the characteristic roots are –3, –3, 5. Here –3 is a multiple root of order 2.

Hence the algebraic multiplicity of the characteristic root '–3' is 2.

The characterstic roots corresponding to $\lambda = -3$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Thus the geometric multiplicity of $\lambda = -3$ is 2.

Here we notice that geometric multiplicity = algebraic multiplicity

2.5 EIGEN VALUES OF HERMITIAN, SKEW-HERMITIAN AND UNITARY MATRICES

The matrices under consideration have the property that their eigen values can be generalized. We give the properties in the following theorems.

Theorem 1: The Eigen values of a Hermitian matrix are all real.

[JNTU 2005, (A) June 2010, 2011, Dec. 2011, May 2012 (Set No. 4), Dec. 2013 (Set No. 2)]

Proof: Let A be a Hermitian matrix. If X be the eigen vector corresponding to the eigen value λ of A, then

$$AX = \lambda X \qquad \ldots(1)$$

Premultiplying both sides of (1) by X^θ, we get

$$X^\theta AX = \lambda X^\theta X \qquad \ldots(2)$$

Taking conjugate transpose of both sides of (2), we get

$$(X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$$

i.e. $X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$ $[\because (ABC)^\theta = C^\theta B^\theta A^\theta$ and $(KA)^\theta = \bar{K}A^\theta]$

or $X^\theta A^\theta X = \bar{\lambda} X^\theta X$ $[\because (X^\theta)^\theta = X, (A^\theta)^\theta = A]$ $\ldots(3)$

From (2) and (3), we have

$$\lambda X^\theta X = \bar{\lambda} X^\theta X \quad i.e., \; (\lambda - \bar{\lambda}) X^\theta X = 0$$

$\Rightarrow \quad \lambda - \bar{\lambda} = 0 \qquad (\because X^\theta X \neq O$, as X being non-zero vector)

$\therefore \quad \lambda = \bar{\lambda}$.

Hence λ is real.

Thus the eigen values of a Hermitian matrix are all real.

Corollary 1. The eigen values of a real symmetric matrix are all real.

[JNTU 2002S, 2004S(Set No. 4), (A) JUNE 2011 (Set No. 4)]

If the elements of a Hermitian matrix A are all real, then A is a real symmetric matrix. Thus a real symmetric matrix is Hermitian and therefore the result follows.

Corollary 2. The eigen values of a Skew-Hermitian matrix are either purely imaginary or zero. **[JNTU 2002, (A) June 2010 (Set No.4)]**

Let A be the Skew-Hermitian matrix. If X be the eigen vector corresponding to the eigen value λ of A, then

$$AX = \lambda X \text{ or } (iA)X = (i\lambda)X$$

From this it follows that $i\lambda$ is an eigen value of iA which is Hermitian (since A is Skew-Hermitian $\therefore A^\theta = -A$.

Now $(iA)^\theta = \bar{i}A^\theta = iA^\theta = -i(-A) = iA$. Hence iA is Hermitian).

Hence $i\lambda$ is real. Therefore either λ must be zero or purely imaginary.

Theorem 2: The eigen values of a Skew-Hermitian matrix are purely imaginary or zero. **[JNTU 2002, 2002S]**

Proof: Let A be a Skew-Hermitian matrix whose eigen value is λ with the corresponding eigen vector X. Then, we have

$$(A - \lambda)X = 0$$

$$\Rightarrow AX = \lambda X \qquad \ldots(1)$$

$$\therefore \bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow (\bar{A}\bar{X})^T = \bar{\lambda}\bar{X}^T$$

$\Rightarrow \overline{X}^T \overline{A}^T = \overline{\lambda}\overline{X}^T$ ...(2)

Since A is Skew-Hermitian, we have

$\overline{A}^T = -A$

$\therefore \overline{X}^T(-A) = \overline{\lambda}(\overline{X})^T$

$\therefore$ We have $(\overline{X}^T A)\, X = \overline{\lambda}(\overline{X})^T\, \overline{X}$ ...(3)

But from (1), we have

$$\overline{X}^T AX = (\overline{X})^T \lambda \overline{X}$$
$$= \lambda\, (\overline{X})^T\, \overline{X} \quad ...(4)$$

Since $\overline{X}^T X \neq 0$, (2) and (4) gives $-\overline{\lambda} = \lambda \Rightarrow \lambda + \overline{\lambda} = 0$

For this we must have real $\lambda = 0$ *i.e.* λ is purely imaginary or $\lambda = 0$.

Cor. "The eigen values of a Skew-Symmetric matrix are *purely imaginary or zero*". If the elements of Skew Hermitian matrix are all real then it is a Skew - Symmetric matrix.

Theorem 3: The Eigen values of an unitary matrix have absolute value 1.

[JNTU 2002 S, 2003 (Set No. 1)]

Proof: Let A be a square unitary matrix whose eigen value is λ with corresponding eigen vector X.

Then we have $AX = \lambda X$ ...(1)

$\Rightarrow \quad \overline{A}\overline{X} = \overline{\lambda}\overline{X} \Rightarrow \overline{X}^T \overline{A}^T = \overline{\lambda}\overline{X}^T$ ...(2)

Since A is unitary, we have $(\overline{A})^T A = I$ ...(3)

(1) and (2) gives, $\overline{X}^T \overline{A}^T \cdot AX = \lambda\overline{\lambda}\, \overline{X}^T X$

(*i.e.*) $\overline{X}^T X = \lambda\overline{\lambda}\, \overline{X}^T X$, by (3) $\Rightarrow \overline{X}^T X(1 - \lambda\overline{\lambda}) = O$

Since $\overline{X}^T\, X \neq O$, we must have $1 - \lambda\overline{\lambda} = 0 \Rightarrow \lambda\overline{\lambda} = 1$

Since $|\lambda| = |\overline{\lambda}|$ we must have $|\lambda| = 1$

Cor 1. "The characteristic root of an orthogonal matrix is of unit modulus".

If the elements of Unitary matrix are all real then it is an Orthogonal matrix.

[JNTU (A) Dec. 2013 (Set No. 1)]

Cor 2. The only *eigen values* of unitary matrix (and orthogonal matrix) *can be* + 1 or – 1.

Theorem 4 : Prove that transpose of a unitary matrix is unitary.

Proof : Let A be a unitary matrix.

Then $A \cdot A^\theta = A^\theta \cdot A = I$

where A^θ is the transposed conjugate of A.

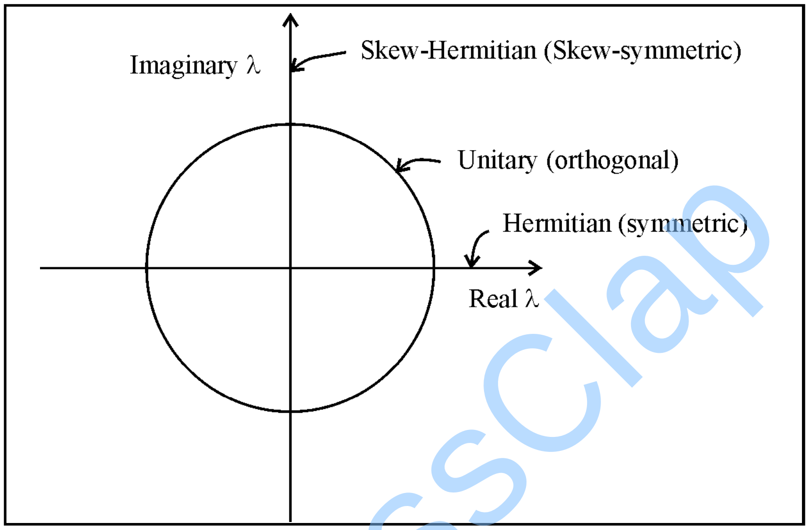
$\therefore (AA^\theta)^T = (A^\theta A)^T = (I)^T$

$\Rightarrow \quad (A^\theta)^T A^T = A^T (A^\theta)^T = I$

$\Rightarrow \quad (A^T)^\theta \cdot A^T = A^T \cdot (A^T)^\theta = I$

Hence A^T is a unitary matrix.

Location of Eigen values of Complex Matrices



Location of the eigen values of Hermitian, Skew-Hermitian and Unitary matrices in the complex λ-plane.

SOLVED EXAMPLES

Example 1 : Find the eigen values of the following matrices :

(*i*) $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ **[JNTU M-2006]** (*ii*) $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ (*iii*) $C = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

Solution : (*i*) Given matrix is $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (4-\lambda)(7-\lambda) - (1-9\lambda^2) = 0$

$\Rightarrow \quad 28 - 11\lambda + \lambda^2 - 10 = 0$

$\Rightarrow \quad \lambda^2 - 11\lambda + 18 = 0 \Rightarrow (\lambda-9)(\lambda-2) = 0 \Rightarrow \lambda = 9$ or $\lambda = 2$

Also $\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} = A^T$

$\therefore$ A is Hermitian.

This verifies that the eigen values of a Hermitian matrix are real.

(*ii*) We have $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ so $\overline{B} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$

and $B^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$ so that $\overline{B} = -B^T$. Thus B is a Skew-Hermitian matrix.

The characteristic equation of B is $|B - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$ and the roots are $4i, -2i$.

This verifies that the roots of Skew-Hermitian matrices will be purely imaginary or zero.

(*iii*) $C = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$. Now $\overline{C} = \begin{bmatrix} \frac{-1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{1}{2}\sqrt{3} & \frac{-1}{2}i \end{bmatrix}$ and $\overline{C}^T = \begin{bmatrix} \frac{-1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{1}{2}\sqrt{3} & \frac{-1}{2}i \end{bmatrix}$

We can see that $(\overline{C})^T C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus C is a Unitary matrix.

The characteristic equation of C is $|C - \lambda I| = 0 \Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i-\lambda \end{vmatrix} = 0$

which gives $\lambda = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$ as eigen values.

We also find $\left|\frac{\pm\sqrt{3}}{2} + \frac{1}{2}i\right|^2 = 1$.

Thus the characteristic roots of unitary matrix have absolute value 1.

Example 2 : Find the eigen values and eigen vectors of the Hermitian matrix

$\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$. **[JNTU 2006S (Set No.4)]**

Solution : Let $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

Its characteristic equation is given by $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$ *i.e.,* $(2-\lambda)^2 - \left[3^2 - (4i)^2\right] = 0$

i.e., $4 - 4\lambda + \lambda^2 - 9 - 16 = 0$

i.e., $\lambda^2 - 4\lambda - 21 = 0$ or $(\lambda+3)(\lambda-7) = 0$ or $\lambda + 3 = 0, \lambda - 7 = 0$

Thus the Eigen values of A are $\lambda = -3, 7$ which are real.

If x_1, x_2 be the components of an eigen vector corresponding to the eigen value λ,

we have $[A - \lambda I]X = \begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda = -3$, eigen vectors are given by $\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $5x_1 + (3+4i)\,x_2 = 0$ or $x_1 = -\left(\dfrac{3+4i}{5}\right)x_2$ or $\dfrac{x_1}{-3-4i} = \dfrac{x_2}{5}$

Eigen vector is $\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$

Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -3$.
For $\lambda = 7$, eigen vectors are given by

$\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *i.e.,* $-5x_1 + (3+4i)\,x_2 = 0$ or $\dfrac{x_1}{3+4i} = \dfrac{x_2}{5}$

$\therefore$ Eigen vector is $\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$.

Hence the eigen vectors of A are $\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$.

Example 3 : Prove that $\dfrac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$ is a unitary matrix. Find its eigen values.

Solution : Let $A = \dfrac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$. Then $\bar{A} = \dfrac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$ and $(\bar{A})^T = \dfrac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

Consider $A\,.\,(\bar{A})^T = \dfrac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix} . \dfrac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}\,i - \sqrt{3}\,i \\ -\sqrt{3}\,i + \sqrt{3}\,i & 3+1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus $(\bar{A})^T = A^{-1}$. *i.e.,* A is a unitary matrix.

The characteristic equation of A is $|\,A - \lambda I\,| = 0$

i.e., $\begin{vmatrix} \dfrac{i}{2} - \lambda & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{i}{2} - \lambda \end{vmatrix} = 0$ *i.e.,* $\left(\dfrac{i}{2} - \lambda\right)^2 - \left(\dfrac{\sqrt{3}}{2}\right)^2 = 0$

i.e., $-\dfrac{1}{4} - i\lambda + \lambda^2 - \dfrac{3}{4} = 0$ or $\lambda^2 - i\lambda - 1 = 0$ $\therefore \lambda = \dfrac{i \pm \sqrt{i^2+4}}{2} = \dfrac{i \pm \sqrt{3}}{2}$

Thus the eigen values of A are $\dfrac{\sqrt{3}+i}{2}$ and $\dfrac{-\sqrt{3}+i}{2}$.

Example 4 : Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary and determine the eigen values and eigen vectors. **[JNTU (A) Dec. 2013 (Set No. 2)]**

Solution : Given $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow A^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

$\therefore \; A^\theta = (\bar{A}^T) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Now $AA^\theta = \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1+2 & (1+i)-(1+i) \\ (1-i)-(1-i) & 2+1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

Thus A is an unitary marix.

Example 5 : Prove that the determinant of a unitary matrix is of unit modulus.

Solution : Let A be an unitary matrix.

Then $AA^\theta = I \Rightarrow |AA^\theta| = |I|$

$\Rightarrow |A||A^\theta| = 1 \;\; \Rightarrow |A||\bar{A}^T| = 1$

$\Rightarrow |A||\bar{A}| = 1 \;\; \Rightarrow |A|^2 = 1$

$\Rightarrow |A|$ is of unit modulus.

Hence if A is unitary then $|A|$ is of unit modulus.

Example 6 : Show that the matrix $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is Skew - Hermitian and hence find eigen values and eigen vectors.

Solution : Let $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Then $\bar{A} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

Now $(\bar{A})^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -A$

Hence A is Skew - Hermitian.

The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - i^2 = 0$$

$\therefore\ \lambda = \pm i$ are the eigen values.

Case I : $\lambda = i$

The eigen vector corresponding to $\lambda = i$ is given by

$(A - iI)X = O$

$$\Rightarrow \left\{ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \right\} X = O$$

$$\Rightarrow \begin{bmatrix} -i & i \\ i & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$ gives $\begin{bmatrix} -i & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow\ i(x_2 - x_1) = 0 \Rightarrow x_1 = x_2$$

$$\therefore\ X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = i$

Case II : $\lambda = -i$

The eigen vector corresponding to $\lambda = -i$ is given by $(A + iI)X = O$

$$\Rightarrow \begin{bmatrix} i & i \\ i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow\ x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\therefore\ X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -i$.

Example 7 : Find the Eigen vectors of the Hermitian matrix $A = \begin{bmatrix} a & b+ic \\ b-ic & k \end{bmatrix}$

[JNTU (A) May 2013]

Solution : $A = \begin{bmatrix} a & b+ic \\ b-ic & k \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b+ic \\ b-ic & k-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(k-\lambda) - (b+ic)(b-ic) = 0$$

$$\Rightarrow (a-\lambda)\,(k-\lambda) = b^2 + c^2$$

$$\Rightarrow \lambda^2 - (a+k)\lambda + (ak - b^2 - c^2) = 0$$

$$\therefore \lambda = \frac{(a+k) \pm \sqrt{(a+k)^2 - 4(ak - b^2 - c^2)}}{2}$$

$$= \frac{(a+k) \pm \sqrt{a^2 + k^2 + 2ak - 4ak + 4(b^2 + c^2)}}{2}$$

$$= \frac{(a+k) \pm \sqrt{(a-k)^2 + 4(b^2 + c^2)}}{2} \text{ are the eigen values.}$$

The corresponding eigen vectors are $\begin{bmatrix} \dfrac{-b^2 + c^2}{(a-\lambda)(b-\lambda)} & 1 \end{bmatrix}^T$.

Example 8 : Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a Skew-Hermitian matrix and also Unitary.

Find eigen values and the corresponding eigen vectors of A.

[JNTU May, Sep. 2006 (Set No. 4)]

Solution : Given $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$. Now $\overline{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$$\therefore \quad (\overline{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = -A$$

Hence A is Skew-Hermitian matrix.

Now we prove that A is Unitary. For this, we have to show that $A(\overline{A})^T = (\overline{A})^T A = I$.

Now $$A(\overline{A})^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also $\quad (\bar{A})^T A = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore \quad A(\bar{A})^T = (\bar{A})^T A = I.$

Hence A is unitary matrix.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0 \qquad$ [Expand by R_1]

i.e., $(i-\lambda)(\lambda^2+1) = 0$ or $\lambda^3 - i\lambda^2 + \lambda - i = 0$ or $(\lambda + i)(\lambda - i)^2 = 0$

$\therefore \quad \lambda = -i, i, i.$

To find the eigen vectors for the corresponding eigen values, we will consider the matrix equation

$(A - \lambda I)X = O$

i.e. $\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{.... (1)}$

Eigen vector corresponding to $\lambda = -i$

Putting $\lambda = -i$ in (1), we get

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2ix_1 = 0,\ x_2 + x_3 = 0 \ \Rightarrow\ x_1 = 0,\ x_2 = -x_3 \quad \text{or} \quad x_1 = 0,\ \frac{x_2}{1} = \frac{x_3}{-1}$

$\therefore$ Eigen vector corresponding to $\lambda = -i$ is $X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = i$.

Putting $\lambda = i$ in (1), we get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -ix_2 + ix_3 = 0,\ ix_2 - ix_3 = 0 \ \Rightarrow\ x_2 = x_3.$

Choose $x_1 = c_1$, where c_1 is arbitrary. Then we have two Linearly independent eigen vectors (with $x_1 = 0, x_2 = 1$ and $x_1 = 1, x_2 = 0$).

$\therefore$ Eigen vectors corresponding to $\lambda = i$ are $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Example 9 : Prove that the modulus of each latent root (eigen value) of a Unitary matrix is unity.

Solution : Let A be a unitary matrix. Then $A^\theta A = I$ ..... (1)

Let λ be an eigen value of A. Then $AX = \lambda X$ ..... (2)

Taking conjugate transpose of both sides of (1), we get

$(AX)^\theta = \bar{\lambda} X^\theta$ or $X^\theta A^\theta = \bar{\lambda} X^\theta$ ..... (3)

From (2) and (3), we have $(X^\theta A^\theta)(AX) = \lambda\bar{\lambda} XX^\theta$

or $X^\theta (A^\theta A) X = \lambda\bar{\lambda} XX^\theta$ or $X^\theta I X = \lambda\bar{\lambda} XX^\theta$ [by (1)]

or $X^\theta X = \lambda\bar{\lambda} XX^\theta$ or $X^\theta X(\lambda\bar{\lambda} - 1) = 0$ ..... (4)

Since $X^\theta X \neq 0$, (4) gives

$\lambda\bar{\lambda} - 1 = 0$ or $\lambda\bar{\lambda} = 1$ or $|\lambda|^2 = 1 \Rightarrow |\lambda| = 1$.

Example 10 : Verify that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ has eigen values with unit modulus.

Solution : Given $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$\therefore \bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

If $(\bar{A})^T (A) = I$ then the matrix A is unitary matrix.

Now $(\bar{A})^T (A) = \frac{1}{4}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} (1-i^2)+(1-i^2) & (1-i^2)+(1+i)^2 \\ (1+i)^2+(1-i)^2 & (1-i^2)+(1-i^2) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus A is a unitary matrix.

Hence the Eigen values of the matrix A have unit modulus.

Example 11 : Find the eigen vectors of the Skew - Hermiton matrix $A = \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$

[JNTU (A) Nov. 2012 (Set No. 2), (H) May 2012 (Set No. 2)]

Solution : The characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2i-\lambda & 3i \\ 3i & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2i-\lambda)(-\lambda) - 9i^2 = 0$$

$$\Rightarrow -2i\lambda + \lambda^2 + 9 = 0 \Rightarrow \lambda = \frac{2i \pm \sqrt{4i^2 - 36}}{2} = \frac{2i \pm 2\sqrt{10}\, i}{2} = i \pm \sqrt{10}\, i$$

$\therefore$ Eigen values are $(1+\sqrt{10})\, i$ and $(1-\sqrt{10})\, i$

Eigen vector corresponding to $(1+\sqrt{10})\, i$

Let $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix} - \begin{bmatrix} (1+\sqrt{10})i & 0 \\ 0 & (1+\sqrt{10})i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\sqrt{10})i & 3i \\ 3i & -(1+\sqrt{10})\, i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-\sqrt{10})i\, x_1 + 3ix_2 = 0$$

$$\Rightarrow (1-\sqrt{10})i\, x_1 = -3ix_2$$

$$\frac{x_1}{-3i} = \frac{x_2}{(1-\sqrt{10})\, i}$$

Eigen vector is $= \begin{bmatrix} -3i \\ (1-\sqrt{10})i \end{bmatrix}$

Eigen vector corresponding to $(1-\sqrt{10})i$

$$\begin{bmatrix} 2i-(1-\sqrt{10})i & 3i \\ 3i & -1+\sqrt{10}\, i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1+\sqrt{10})ix_1 + 3ix_2 = 0$$

$$\frac{x_1}{-3i} = \frac{x_2}{(1+\sqrt{10})i}$$

$\therefore$ Eigen vector is, $\begin{bmatrix} -3i \\ (1+\sqrt{10})\, i \end{bmatrix}$

2.6 DIAGONALIZATION OF A MATRIX

Diagonalization of a Matrix : A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Also the matrix P is then said to diagonalize A or transform A to diagonal form.

Similarity of Matrix: Let A and B be square matrices of order n. Then B is said to be similar to A if there exists a non-singular matrix P of order n such that $B = P^{-1}AP$. It is denoted by $A \sim B$.

The transformation $Y = PX$ is called similarity transformation.

Thus a matrix is said to be diagonalizable if it is similar to a diagonal matrix. We now prove some important theorems relating to the diagonalization.

Theorem 1: An n × n matrix is diagonalizable if and only if it possesses n linearly independent eigen vectors.

Proof: Let A be diagonalizable. Then A is similar to a diagonal matrix $D = \text{dia}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Therefore there exists an invertible matrix $P = [X_1, X_2, \ldots, X_n]$ such that

$$P^{-1}AP = D$$

i.e., $AP = PD$

i.e., $A[X_1, X_2, X_3, \ldots X_n] = [X_1, X_2, \ldots, X_n]\ \text{dia}[\lambda_1, \lambda_2, \ldots, \lambda_n]$

i.e., $[AX_1, AX_2, \ldots AX_n] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3, \ldots, \lambda_4 X_4]$

i.e., $AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n.$

So $X_1, X_2, \ldots, X_n$ are eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respecitvely. Since the matrix P is non-singular its column vectors $X_1, X_2, \ldots, X_n$ are linearly Independent. Therefore A possesses n linearly indipendent eigen vectors.

Conversely given that $X_1, X_2, \ldots, X_n$ be eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively and these eigen vectors are linearly independent.

Define $P = (X_1, X_2, \ldots, X_n)$

Since the n columns of P are linearly independent, $|P| \neq 0$.

Hence P^{-1} exists.

Consider

$$AP = A[X_1\ X_2\ \ldots\ X_n] = [AX_1\ AX_2\ \ldots\ AX_n]$$
$$= [\lambda_1 X_1\ \lambda_2 X_2 \ldots\ \lambda_n X_n]$$
$$= [X_1\ X_2\ \ldots\ X_n]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & \ldots & \ldots & \lambda_n \end{bmatrix}$$
$$= PD \text{ where } D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

$\therefore\quad AP = PD$

$\Rightarrow\quad P^{-1}(AP) = P^{-1}(PD)$

$\Rightarrow\quad P^{-1}AP = (P^{-1}P)D = ID = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Hence the result follows.

Note : **1.** If $X_1, X_2, ..., X_n$ are not linearly independent, this result is not true.

2. If the eigen values of A are all distinct, then it has n linearly independent eigen vectors and so it is diagonalizable.

3. The reduction of matrix A to the diagonal form by a non-singular matrix P using the transformation $P^{-1}AP$ is called the similarity transformation.

4. An interesting special case can be considered here. Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1, \lambda_2, ..., \lambda_n$. Then the corresponding eigen vectors $X_1, X_2, ..., X_n$ are pairwise orthogonal.

Hence if $P = (e_1, e_2, ..., e_n)$, where $e_1 = \frac{X_1}{\|X_1\|}$ $e_2 = \frac{X_2}{\|X_2\|}, ..., e_n = \frac{X_n}{\|X_n\|}$

then P will be an orthogonal matrix.

$i.e.$ $P^T P = PP^T = I.$ Hence $P^{-1} = P^T$

$\therefore \quad P^{-1}AP = D \Rightarrow P^T AP = D$

2.7 MODAL AND SPECTRAL MATRICES

Def : The matrix P in the above result which diagonalise the square matrix A is called the **modal matrix** of A and the resulting diagonal matrix D is known as **Spectral matrix.**

Note: The diagonal elements of D are the eigen values of A and they occur in the same order as is the order of their corresponding eigen vectors in the column vectors of P.

Theorem 2: If the eigen values of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

2.8 CALCULATION OF POWERS OF A MATRIX

We can obtain the powers of a matrix by using diagonalisation.

Let A be the square matrix. Then a non-singular matrix P can be found such that

$D = P^{-1}AP$

$\therefore \; D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P \quad (\because PP^{-1} = I)$

Similarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$ ..... (1)

To obtain A^n, pre-multiply (1) by P and post-multiply by P^{-1}.

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n(PP^{-1}) = A^n$

$$\text{Hence } A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 & ... & 0 \\ 0 & \lambda_2^n & 0 & ... & 0 \\ 0 & 0 & \lambda_3^n & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_n^n \end{bmatrix} P^{-1}$$

SOLVED EXAMPLES

Example 1 : Determine the modal matrix P for $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ and hence diagonalize A.

Solution : The characteristic equation of A is

$$|A - \lambda I| = 0 \quad (i.e.) \quad \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

On expansion and for factorization we note that this is $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$.

Thus the eigen values of A are $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$ (say).

The corresponding characteristic vectors X are

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{(Workout ?)}$$

Notice that $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct.

Hence X_1, X_2, X_3 are linearly independent.

Further A is a real symmetric matrix. Hence X_1, X_2, X_3 are pairwise orthogonal.

The matrix $P = (X_1 \ \ X_2 \ \ X_3) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ is modal matrix of A.

Also $P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.

However, since X_1, X_2, X_3 are pairwise orthogonal, take

$$P = \left(\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}, \frac{X_3}{\|X_3\|} \right) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Then $P^{-1} = P^T$ [$\because$ A is symmetric]

$\therefore$ Diagonalised matrix =

$$P^{-1}AP = P^T AP$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \\ 0 & -3/\sqrt{3} & 12/\sqrt{6} \\ -2/\sqrt{2} & 3/\sqrt{3} & 6/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Note : If A is non-singular matrix and its eigen values are distrinct then the matrix P is found by grouping the eigen vectors of A into square matrix and the diagonal matrix has the eigen values of A as its elements.

Example 2 : Determine the modal matrix P of A $= \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Solution : The characteristic equation of A is $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0$

Thus the eigen values are $\lambda = 5$, $\lambda = -3$, and $\lambda = -3$.

When $\lambda = 5$, we have $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly for the eigen value $\lambda = -3$ we can have two linearly independent eigen vectors $X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Observe that X_1, X_2 and X_3 are not orthogonal. But they are linearly independent. Consider $P = (X_1 \quad X_2 \quad X_3)$

$$\therefore \quad P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \text{Modal matrix of A}$$

Now $\det P = 1(-1) - 2(2) + 3(0-1) = -1 - 4 - 3 = -8$

$$\therefore \quad P^{-1} = \frac{1}{\det P}(\text{adj } P) = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$P^{-1}A = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} = \frac{-1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$$

and $P^{-1}AP = \frac{-1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag}(5, -3, -3)$

Hence $P^{-1}AP$ is a diagonal matrix.

Example 3 : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find (*a*) A^8 (*b*) A^4

Solution : The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

which gives $(1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\therefore$ The characteristic values are $\lambda = 1, \lambda = 2, \lambda = 3$.

Characteristic vector corresponding to $\lambda = 1$

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ -4 & 4 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (i.e.) \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y + z = 0, y + z = 0$ and $-4x + 4y + 2z = 0$

Take $z = k$. We have $y = -k$. Then $4x = 4y + 2z = -4k + 2k = -2k$

Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{k}{2} \\ -k \\ k \end{bmatrix} = \frac{-k}{2}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

Characteristic vector corresponding to $\lambda = 2$

We have $\begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e,. $$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x + y + z = 0,\ z = 0$ and $-4x + 4y + z = 0$

i.e., $-x + y = 0, z = 0$

Let $x = k$. Then $y = k$. Now $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the characteristic vector corresponding to $\lambda = 2$.

Characteristic vector corresponding to $\lambda = 3$

$$\begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we have $-2x + y + z = 0$; $-y + z = 0$ and $-4x + 4y = 0$

Let $y = k$. Then $z = k$ and $x = k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector of A corresponding to $\lambda = 3$ is $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Consider $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

We have $|P| = -1$ and $P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

We have $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diag}\,(1, 2, 3) = D$ (say)

(*a*) $D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$

We have $A^8 = PD^8P^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}.$$

(*b*) $D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

Now $A^4 = PD^4\, P^{-1}$.

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+64-162 & 1-48+162 & 0-16+81 \\ -2+64-162 & 2-48+162 & 0-16+81 \\ 2+0-162 & -2-0+162 & 0-0+81 \end{bmatrix} = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$$

Example 4 : (*a*) Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4. Find the eigen values and eigen vectors of A.

(*b*) Determine the eigen values of A^{-1}.

(*c*) Diagonalize $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^8.

Solution : (*a*) Characteristic equation of A is given by $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$ [Expand by R_1]

i.e., $(1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right]-0-1\left[2-2(2-\lambda)\right]=0$

i.e., $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ or $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\therefore\ \lambda = 1, \lambda = 2$ or $\lambda = 3$

Thus the eigen values of A are 1, 2 and 3.

If x_1, x_2, x_3 be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A-\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{..... (1)}$$

For $\lambda = 1$, eigen vectors are given by $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $0 . x_1 + 0 . x_2 - x_3 = 0$ and $x_1 + x_2 + x_3 = 0$

i.e. $x_3 = 0$ and $x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = 0, \ x_1 = -x_2$

$\therefore \ x_1 = 1, x_2 = -1, x_3 = 0$

Eigen vector is $[1 \ -1 \ 0]^T$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 1$.

For $\lambda = 2$, eigen vectors are given by

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 0 . x_2 + x_3 = 0$ and $2x_1 + 2x_2 + x_3 = 0$

Solving, $\dfrac{x_1}{0-2} = \dfrac{-x_2}{1-2} = \dfrac{x_3}{2-0}$ or $\dfrac{x_1}{-2} = \dfrac{x_2}{1} = \dfrac{x_3}{2}$ or $x_1 = -2, x_2 = 1, x_3 = 2$

Eigen vector is $[-2 \ 1 \ 2]^T$.

Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 2$

For $\lambda = 3$, eigen vectors are given by

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{[Putting } \lambda = 3 \text{ in (1)]}$$

i.e., $-2x_1 + 0 . x_2 - x_3 = 0$ and $x_1 - x_2 + x_3 = 0$

Solving, $\dfrac{x_1}{0-1} = \dfrac{-x_2}{-2+1} = \dfrac{x_3}{2-0}$ or $\dfrac{x_1}{-1} = \dfrac{x_2}{1} = \dfrac{x_3}{2}$ or $x_1 = -1, x_2 = 1, x_3 = 2$

Eigen vector is $[-1 \ 1 \ 2]^T$.

Writing the three eigen vectors of the matrix A as the three columns, the required transformation matrix is

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The matrix P is called **Modal matrix** of A.

$$\therefore \quad P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

Now $P^{-1}AP = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$ (say)

Hence $A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$

(b) The eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$ (Refer Th. 10)

(c) The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) - 6(3-\lambda) = 0$

$\Rightarrow (3-\lambda)\left[(1-\lambda)(2-\lambda) - 6\right] = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 3\lambda - 4) = 0$

$\Rightarrow (3-\lambda)(\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 3, 4$ are the eigen values of A

The eigen vector corresponding to $\lambda = 3$

$(A - 3I)\,X = O$

$$\Rightarrow \begin{bmatrix} 1-3 & 6 & 1 \\ 1 & 2-3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the matrix, $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

Taking $x_1 = x_2 = k$

$x_3 = 2x_1 - 6x_2 = 2k - 6k = -4k$

$$\therefore \ X = \begin{bmatrix} k \\ k \\ -4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \ X_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \text{ is the eigen vector corresponding to } \lambda = 3$$

Eigen vectorcorresponding to $\lambda = 4$

$(A - 4I)\,X = O$

$$\begin{bmatrix} 1-4 & 6 & 1 \\ 1 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get, $x_3 = 0$, $x_1 - 2x_2 = 0$

$\Rightarrow x_1 = 2x_2$ taking $x_2 = k$ we get $x_1 = 2k$

$$\therefore \ X_2 = \begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ is the eigen vector corresponding to } \lambda = 4$$

Eigen vector corresponding to $\lambda = -1$

$(A + I)\,X = O$

$$\begin{bmatrix} 1+1 & 6 & 1 \\ 1 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_3 = 0 \Rightarrow x_3 = 0$

$x_1 + 3x_2 = 0 \Rightarrow x_1 = -3x_2$ taking $x_2 = k, x_1 = -3k$

$$\therefore \ X = \begin{bmatrix} -3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore\ X_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

Consider $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$

$\det P = 1(0) - 2(4) - 3(4) = -8 - 12 = -20 \neq 0$

$$P^{-1} = \frac{-1}{20}\begin{bmatrix} 0 & 0 & -5 \\ -4 & 12 & 4 \\ 4 & 8 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{-1}{20}\begin{bmatrix} 0 & 0 & -5 \\ -4 & 12 & 4 \\ 4 & 8 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D = \text{diag}(3, 4, -1)$$

$D^8 = \begin{bmatrix} 6561 & 0 & 0 \\ 0 & 65536 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can calculate $A^8 = PD^8P^{-1}$

Example 5 : Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ [ ]

Solution : Given $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow\ (8 - \lambda)[(-3 - \lambda)(1-\lambda) - 8] + 8[4(1 - \lambda) + 6] - 2[-16 - 3(-3 - \lambda)] = 0$

$\Rightarrow\ -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore\ \lambda = 3, 1, 2.$

$\therefore$ Diagonal matrix is $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

To find the eigen vector for $\lambda = 3$.

$$\begin{vmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 5x_1 - 8x_2 - 2x_3 &= 0 \\ 4x_1 - 6x_2 - 2x_3 &= 0 \\ 3x_1 - 4x_2 - 2x_3 &= 0 \end{aligned}$$

For solving let $x_2 = x_3 = k_1$. Then $x_1 = 2x_2 = 2k_1$ $\therefore X_1 = \begin{bmatrix} 2k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 3$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

To find the eigen vector when $\lambda = 1$

$$\begin{bmatrix} 8-1 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 7x_1 - 8x_2 - 2x_3 = 0 \quad &...(1) \\ 4x_1 - 4x_2 - 2x_3 = 0 \quad &...(2) \\ 3x_1 = 4x_2 \quad &...(3) \end{aligned}\right\}$$

For solving, let $x_2 = k_2$. Then
$x_1 = (4/3)k_2$, $x_2 = k_2$, $x_3 = (2/3)k_2$.

$\therefore X_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ Eigen vector corresponding to $\lambda = 1$ is

To find the eigen vector when $\lambda = 2$

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 - \frac{1}{2}R_1$ and $R_2 - R_1$, we get

$$\begin{bmatrix} 6 & -8 & -2 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For solving, let $x_2 = k_3$. Then

$x_1 = (3/2)k_3,\ x_2 = k_3,\ x_3 = (1/2)k_3.$

$\therefore$ Eigen vector corresponding to $\lambda = 2$ is $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Thus $P = [X_1\ X_2\ X_3] = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Hence $P^{-1}AP = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Example 6 : Diagonalise the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 0 \\ 1 & 2-\lambda & 3-\lambda \\ -1 & -1 & -\lambda-1 \end{vmatrix} = 0 \text{ (Applying } C_3 + C_2)$$

$\Rightarrow (-1-\lambda)[(2-\lambda)(-\lambda-1)+1(3-\lambda)]-2[(-\lambda-1)+1(3-\lambda)] = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$ (on simplification)

$\Rightarrow (\lambda-1)(\lambda^2-5) = 0$

$\Rightarrow \lambda = 1;\ \lambda^2 = 5$

$\Rightarrow \lambda = 1;\ \pm\sqrt{5}$

$\therefore$ A is diagonalisible.

[Constructing matrix $P = [e_1\ e_2\ e_3]$ is left as an exercise to the student]

$\therefore$ The required diagonalised matrix is $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$

Example 7 : Show that the martix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonaizable. Also find the diagonal form and a diagonaizing matrix P.

Solution : The characteristic equation of A is

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix} = 0$$

Applying $R_2 - R_1$ and $R_3 - R_1$, we get

$$(1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 + \lambda)(1 + \lambda)(3 - \lambda) = 0$$

∴ Characteristic Roots (*i.e.*, eigen values) of A are –1, –1, 3.

The eigen vectors X_1 of A corresponding to the eigen vlaue –1 are given by

$$(A + I) X_1 = 0$$

i.e., $$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 \to 2R_1$, we get

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the rank of the coefficient matrix is one, it will have (3 – 1) = 2 linearly independent solution.

From the above matrix equation, we have $-2x_1 + x_2 + x_3 = 0$

Clearly $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are two linearly independent solutions of this equation.

So X_1 and X_2 are two linearly independent eigen vectors of A corresponding to the eigen value –1. Thus the geometric multiplicity of the eigen value –1 is equal to its algebraic multiplicity.

Consider the eigen vectors of A corresponding to the eigen value 3 are given by $(A - 3I)\lambda = 0$

i.e., $$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$, we get

$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $\frac{R_1}{4} \frac{R_2}{4}$, we get

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix is 2.

$\therefore$ It will have (3 – 2) = 1 independent solutions.

From the matrix, we can write the equations as

$-3x_1 + x_2 + x_3 = 0$, $x_1 - x_2 = 0$

Let $x_1 = k \Rightarrow x_2 = k$. From this we get $x_3 = 2k$

Thus the eigen vector corresponding to eigen vlaue 3 is given by $X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

The geometric multiplicity of eigen value 3 is one and its algebraic multiplicity is also 1. Thus each eigen value of A has its gemetric multiplicity equal to its algebraic multiplicity.

$\therefore$ A is similar to a diagonal matrix i.e., it can be diagonalized.

$$\text{Let } P = [X_1, X_2, X_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

The columns of P are linearly independent eigen vectors of A corresponding to the eigen values –1, –1, 3. P will transform A into the diagonal form D by the relation

$$P^{-1}AP = D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 8 : Show that the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized.

Solution : Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation is $\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(2-\lambda)[(2-\lambda)(1-\lambda)] = 0$

$\Rightarrow$ $\lambda = 2, 2, 1$

$\therefore$ 2, 2, 1 are the characteristic values of A

The characteristic vector corresponding to $\lambda = 2$ is given by (A – 2I) X = O

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$ gives

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is 2. So, these equations will have 3 – 2 = 1 independent solution. Thus the geometric multiplication of the eigen value 2 is 1 and algebraic multiplicity is 2. Since the algebraic multiplicity is not equal to geometric multiplicity, A is not similar to a digonal matrix. Thus, the matrix cannot be diagonalized.

2.9 NILPOTENT MATRIX

A non-zero matrix A is said to be **nilpotent**, if for some positive n, $A^n = O$.

Theorem 1 : A non-zero matrix is nilpotent if and only if all its eigen values are equal to zero.

Theorem 2 : A non-zero nilpotent matrix cannot be similar to a diagonal matrix *i.e.*, it cannot be diagonalised.

Example 9 : Prove that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

Solution : Given $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Thus A is nilpotent and hence can not diagonalised.

(or) The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0, 0 \text{ are the characteristic values.}$$

For $\lambda = 0$ the characteristic vector is given by $(A - \lambda I)X = 0 \Rightarrow AX = O$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0, \text{ Let } x_1 = k$$

Then the characteristic vector is $\begin{bmatrix} k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The given matrix has only one linearly independent characteristic vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to repeated characteristic value 0.

∴ The matrix is not diagonaizable.

2.10 ORTHOGONAL REDUCTION OF REAL SYMMETRIC MATRICES

Definition : Let A and B be two square matrices of order n. Then B is said to be orthogonally similar to A, if there exists an orthogonal matrix P such that $B = P^{-1}AP$.

If A and B are orthogonally similar then, they are similar.

Theorem 1 : Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

Theorem 2 : A real symmetric matrix of order n has n mutully orthogonal real eigen vectors.

Theorem 3 : Any two eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

Theorem 4 : If λ occurs exactly p times as an eigen value of real symmetric matrix A, then A has p, but not more than p mutually orthogonal real eigen vectors corresponding to λ.

Working Rule for Orthogonal Reduction of Real Symmetric Matrix :

Suppose A is a real symmetric matrix.

(*i*) Find the eigen values of A.

(*ii*) If the all the eigen values of A are different then they are all orthogonal.

(*iii*) If λ is an eigen value of A having p is its algebraic multiplicity, we shall be able to find an orthonormal set of p eigen vectors of A corresponding to their eigen values.

(*iv*) This process is repeated for each eigen vlaue of A.

(*v*) Since the eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are mutually orthogonal, then n eigen vectors found in this manner constitute an ortho normal set.

(*vi*) The matrix P having as its columns the members of the ortho normal set obtained above, is orthogonal and is such that $P^{-1}AP$ is a diagonal matrix.

We will illustrate the above with few examples

SOLVED EXAMPLES

Example 1 : Find an orthogonal matrix that will diagonaize the real symmetric matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Also find the resulting diagonal matrix.

Solution : The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$

Applying $C_1 + C_3 - 2C_2$, we get

$$\begin{vmatrix} -\lambda & 2 & 3 \\ 2\lambda & 4-\lambda & 6 \\ -\lambda & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda \begin{vmatrix} -1 & 2 & 3 \\ 2 & 4-\lambda & 6 \\ -1 & 6 & 9-\lambda \end{vmatrix} = 0$$

Applying $R_2 + 2R_1$ and $R_3 - R_1$, we get

$$\lambda \begin{vmatrix} -1 & 2 & 3 \\ 0 & 8-\lambda & 12 \\ 0 & 4 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda(\lambda^2 - 14\lambda) = 0$

$\Rightarrow \quad \lambda^2(\lambda - 14) = 0$

$\therefore$ The eigen values of A are 0, 0, 14.

The eigen vector corresponding to the eigen value $\lambda = 14$ is given by

$(A - 14I)X = O$

$$\Rightarrow \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, we get the equations

$-13x_1 + 2x_2 + 3x_3 = 0$ and $2x_1 - 10x_2 + 6x_3 = 0$

Solving, we get $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$

$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 14$

To find the eigen vector corresponding to $\lambda = 0$.

We write $AX = O$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$

We can easily see that $x_1 = 0, x_2 = 3, x_3 = -2$ is a solution.

Then $X_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$ is an eigen vector corresponding to the eigen value $\lambda = 0$. Let the eigen vector orthogonal to this is $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then we have

$x + 2y + 3z = 0$ (X_3 satisfies the equation)

$0x + 3y - 2z = 0$ ($\because X_2, X_3$ are orthogonal)

From this, we get $X_3 = \begin{bmatrix} -13 \\ 2 \\ 3 \end{bmatrix}$ is another eigen vector corresponding to $\lambda = 0$

Normalizing these vectors,

For $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \|X_1\| = \sqrt{1+4+9} = \sqrt{14}$, $\frac{X_1}{\|X_1\|} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$

Similarly $\frac{X_2}{\|X_2\|} = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{13}} \\ \frac{-2}{\sqrt{182}} \end{bmatrix}$, $\frac{X_3}{\|X_3\|} = \begin{bmatrix} \frac{-13}{\sqrt{182}} \\ \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \end{bmatrix}$

Consder = $P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{14}} & 0 & \frac{-13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{182}} \\ \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{182}} \end{bmatrix}$$

We can verify that

$$P^{-1}AP = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{diag}(14, \quad 0, \quad 0)$$

Example 2 : Diagonalize the matrix where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, by orthogonal reduction.

Solution : The characteristic equation is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0 \quad \Rightarrow (1-\lambda)(\lambda^2 - 6\lambda + 8) = 0 \quad \Rightarrow (1-\lambda)(\lambda-4)(\lambda-2) = 0$

$\therefore$ The eigen values of A are 1, 2, 4.

We can find out the eigen vectors corresponding to the above eigen values as

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ and } \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

We notice that these vectors are pairwise orthogonal.

Normalizing these vectors, we get $\frac{X_1}{\|X_1\|} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \frac{X_2}{\|X_2\|} = \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}, \frac{X_3}{\|X_3\|} = \begin{bmatrix}0\\\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

Consider $P = \begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

Then $P^T = P^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

We can easily verify that $PAP^{-1} = D$ where $D = \text{diag}(1, 2, 4) = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 4\end{bmatrix}$

Thus A is reduced to diagonal form by orthogonal reduction.

Example 3 : Find the diagonal matrix orthogonally similar to the following real symmetric matrix. Also obtain the transforming matrix. $A = \begin{bmatrix}7 & 4 & -4\\4 & -8 & -1\\-4 & -1 & -8\end{bmatrix}$

[JNTU (H) Mar. 2012 (Set No. 4)]

Solution : The characteristic equation of A is

$$\begin{vmatrix}7-\lambda & 4 & -4\\4 & -8-\lambda & -1\\-4 & -1 & -8-\lambda\end{vmatrix} = 0$$

$C_3 + C_2$ gives $\begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & -9-\lambda \\ -4 & -1 & -9-\lambda \end{vmatrix} = 0$

$(-9-\lambda)\begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & 1 \\ -4 & -1 & 1 \end{vmatrix} = 0$

$(-9-\lambda)\left[(7-\lambda)\{-8-\lambda+1\} - 4\{4+4\}\right] = 0$

$(-9-\lambda)\left[(7-\lambda)(-\lambda-7) - 32\right] = 0$

$(-9-\lambda)(\lambda^2 - 49 - 32) = 0$

$(-9-\lambda)(\lambda^2 - 81) = 0$

$\lambda = 9, -9, -9$ are the eigen values.

Eigen vector corresponding to $\lambda = 9$

Let $(A - \lambda I)X = O$

$\Rightarrow \begin{bmatrix} 7-9 & 4 & -4 \\ 4 & -8-9 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 + R_2$ gives, $\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ 0 & -18 & -18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

we have, $-18x_2 - 18x_3 = 0 \Rightarrow x_3 = -x_2$

$-2x_1 + 4x_2 - 4x_3 = 0 \Rightarrow -2x_1 + 4x_2 + 4x_2 = 0$

$\Rightarrow -2x_1 + 8x_2 = 0 \Rightarrow x_1 = 4x_2$

take $x_2 = k$ then $x_1 = 4k$ and $x_3 = -k$

Then $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4k \\ k \\ -k \end{bmatrix} = k\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 9$

Eigen vector corresponding to $\lambda = -9$

Let $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 7+9 & 4 & -4 \\ 4 & -8+9 & -1 \\ -4 & -1 & -8+9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we have, $4x_1 + x_2 - x_3 = 0$ take $x_2 = k_1, x_3 = k_2$ then $4x_1 = x_3 - x_2 = k_2 - k_1$

$$\therefore \text{ X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_2 - k_1}{4} \\ k_1 \\ k_2 \end{bmatrix} = \frac{k_1}{4} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + \frac{k_2}{4} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$\therefore \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ are two mutually orthogonal vectors corresponding to $\lambda = -9$

Normalizing, we get

$$\text{P} = \begin{bmatrix} \frac{4}{\sqrt{18}} & -\frac{1}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{-1}{\sqrt{18}} & 0 & \frac{4}{\sqrt{17}} \end{bmatrix}$$ is the required orthogonal matrix that will diagonalise A.

Thus $\text{P}^{-1}\text{AP} = \text{P}^{\text{T}}\text{AP} = \text{diag }(9, -9, -9)$

Example 4 : Determine the diagonal matrix orthogonally similar to the following symmetric matrix $\text{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution : The characteristic equation is, $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$\text{C}_1 + \text{C}_2 + \text{C}_3$ gives, $\begin{vmatrix} 3-\lambda & -1 & 1 \\ 3-\lambda & 5-\lambda & -1 \\ 3-\lambda & -1 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (3-\lambda)\begin{vmatrix} 1 & -1 & 1 \\ 1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$R_2 - R_1, R_3 - R_1$

$$(3-\lambda)\begin{vmatrix} 1 & -1 & 1 \\ 0 & 6-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(6-\lambda)(2-\lambda) = 0$$

$\therefore\ \lambda = 2, \lambda = 3, \lambda = 6$ are the eigen values which are all distinct.

Eigen vector corresponding to $\lambda = 2$

Let $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$ gives, $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore\ x_2 = 0$ and $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$ take $x_1 = k \Rightarrow x_3 = -k$

$\therefore$ Eigen vector is, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Eigen vector corresponding to $\lambda = 3$

$(A - 3I)\ X = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We have $-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$ and $-x_1 + 2x_2 - x_3 = 0 \Rightarrow x_1 = 2x_2 - x_3 = x_3$
taking $x_3 = k$ we get $x_1 = x_2 = k$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Eigen vector corresponding to $\lambda = 6$

$$(A - 6I)X = 0 \Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + R_2$ gives, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_2 - 4x_3 = 0 \Rightarrow x_2 = -2x_3$ and $-x_1 - x_2 - x_3 = 0 \Rightarrow x_1 = -x_2 - x_3 = 2x_3 - x_3 = x_3$

taking $x_3 = k$ we get $x_2 = -2k$ and $x_1 = k$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore\ X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$

Take $P = \left[\frac{X_1}{\sqrt{2}}, \frac{X_2}{\sqrt{3}}, \frac{X_3}{\sqrt{6}}\right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

Since P is orthogonal $P^T = P^{-1}$ and $P^{-1}AP = P^TAP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \text{diag}\,(2,3,6)$

Example 5 : Determine the diagonal matrix orthogonally similar to the following symmetric matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution : The characteristic equation of A is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $R_3 + R_2$, we get

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 0 & 2-\lambda & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$C_3 - C_2$ gives, $(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 4 \\ -2 & 3-\lambda & \lambda-4 \\ 0 & 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)(-\lambda+4)+4(-2)\right] = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 10\lambda + 24 - 8) = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 10\lambda + 16) = 0$

$\Rightarrow (2-\lambda)(\lambda-8)(\lambda-2) = 0$

$\therefore\ \lambda = 8, 2, 2$ are the characteristc values.

Eigen vector corresponding to $\lambda = 8$

$(A - 8I)X = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$ gives, $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

From matrix, we get

$-x_1 - x_2 + x_3 = 0$ and $-x_2 - x_3 = 0$. Taking $x_3 = k$, we get $x_2 = -k$

and $x_1 = -x_2 + x_3 = 2k$.

Then $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Thus $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 8$.

Eigen vector corresponding to $\lambda = 2$.

This is of algebric multiplicity 2.

So we will find two mutually orthogonal eigen vectors corresponding to $\lambda = 2$.

$(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - x_2 + x_3 = 0$

Taking $x_3 = k_2$ and $x_2 = k_1$ we get $2x_1 = k_1 - k_2$

$$X = \begin{bmatrix} \frac{k_1 - k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = \frac{k_1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{k_2}{2} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ are eigen vectors corresponding to $\lambda = 2$.

These two vectors are not orthogonal.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be another eigen vector corresponding to $\lambda = 2$ and orthogonal to X_2.

$2x - y + z = 0; x + 2y = 0$.

Then, $x = -2, y = 1, z = 5$ is a solution.

Then take $X_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are and $X_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ are the required eigen vectors.

Normalizing we get,

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix} \text{ and } P^{-1} = P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

We can verify that, $P^{-1}AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{diag}(8,2,2)$

Example 6 : Diagonalize the matrix by an orthogonal transformation $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$. Also find the matrix of the transformation.

Solution : The characteristic equation is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix}$

$\Rightarrow (2-\lambda)[(6-\lambda)(2-\lambda)-0]+1[0-4(6-\lambda)]=0$

$\Rightarrow (6-\lambda)[(2-\lambda)^2-4]=0$

$\Rightarrow (6-\lambda)(\lambda^2-4\lambda+4-4)=0$

$\Rightarrow (6-\lambda)\lambda(\lambda-4)=0$

$\therefore$ The eigen values are $\lambda = 0, \lambda = 4, \lambda = 6$

Since the eigen values are all different diagonal matrix $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Eigen values corresponding to $\lambda = 0$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - 2R_1 \begin{bmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_3 = 0$, $6x_2 = 0$

$\Rightarrow x_2 = 0; x_3 = -2x_1$.

Take, $x_1 = k \Rightarrow x_3 = -2k$

Eigen vector is $\begin{bmatrix} k \\ 0 \\ -2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 4$

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 2-4 & 0 & 1 \\ 0 & 6-4 & 0 \\ 4 & 0 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0 \Rightarrow x_2 = 0$

$\Rightarrow \ -2x_1 + x_3 = 0 \Rightarrow x_3 = 2x_1$

Take, $x_1 = k \Rightarrow x_3 = 2k$

The eigen vector is $\begin{bmatrix} k \\ 0 \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\therefore$ Eigen vector corresponding to $\lambda = 4$ is $X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 6$

$$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 2-6 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 + R_3$ gives $\begin{bmatrix} -4 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_3 = 0 \Rightarrow x_3 = 0$

$\Rightarrow -4x_1 + x_3 = 0 \Rightarrow x_1 = 0$

Take $x_2 = k$

The eigen vector is $\begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Diagonalision matrix $= P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 2 & 0 \end{bmatrix}$ such that $P^T AP = D$

EXERCISE 2.1

1. Determine the eigen values and the corresponding eigen vectors of the following matrices:

(*i*) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (*ii*) (*a*) $\begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$ (*b*) $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ (*v*) $\begin{bmatrix} 6 & 3 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **[JNTU 2003, 2005S (Set No. 3)]**

(*vi*) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (*vii*) $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **[JNTU 2003S(Set No. 2)]**

(*viii*) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ **[JNTU 2003S, 2004S, (K) Nov. 2009S(Set No. 4)]** (*ix*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(*x*) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **[JNTU 1995, 2005S (Set No. 4)]** (*xi*) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$ **[JNTU 2008S (Set 3)]**

(*xii*) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}$ **[JNTU 2008S, (K) May 2010 (Set No. 2)]**

2. Show that the eigen values of a triangular matrix are its diagonal elements. **[JNTU 2002]**

3. Find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

4. Diagonalize the matrix (*i*) $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ **[JNTU (A) June 2013 (Set No. 1)]**

(*iii*) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$ **[JNTU (A) June 2013 (Set No. 2)]**

(*iv*) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ **[JNTU (A) June 2013 (Set No. 3)]**

(*v*) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ **[JNTU (A) June 2013 (Set No. 4)]**

5. Diagonalize the matrix (*i*) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

[JNTU 2004, (H) June 2009 (Set No. 1)]

6. Diagonalize the matrix A where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ **[JNTU 2003S (Set No. 4),M2005]**

7. Determine the eigen values of A^{-1} where $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

[JNTU 2003S (Set No. 3), 2004S (Set No. 3)]

8. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ find A^4.

9. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find A^6.

10. For the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, determine the matrix P such that $P^{-1}AP$ is a diagonal matrix.

11. Indicate whether the following matrices are Hermitian, Skew-Hermitian or Unitary and find their eigen values and eigen vectors.

(*i*) $\begin{bmatrix} 4 & i \\ -i & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ (*iii*) $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ (*v*) $\begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$

12. Show that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. Find the eigen values and eigen vectors.

13. Verify that the matrix $A=\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ has eigen values with unit modulus.
[Hint : Refer Th.3**]**

ANSWERS

1.(*i*) Characteristic values 0, 5; Characteristic roots $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(*ii*) (*a*) Characteristic values 3, – 1; Characteristic roots $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ (*b*) 4, 6; $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(*iii*) $\lambda = 1, \lambda = 2, \lambda = 3;$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

(*iv*) $\lambda = 1, \lambda = 2, \lambda = 3;$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

(*v*) 2, 2, 8; $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(*vi*) 1, 2, 3; $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(*vii*) 1, 2, – 2 (*viii*) 3, 2, 5; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ (*ix*) –2, 3, 6; $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(*x*) 5, –3, –3; $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

3. $625\,I$ **4.** diag $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **5. (*i*)** $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. Cannot diagonalize. **7.** $1, \frac{1}{2}, \frac{1}{3}$

8. $\begin{bmatrix} 171 & 371 & 193 \\ 325 & 873 & 371 \\ 144 & 325 & 171 \end{bmatrix}$ **9.** $\begin{bmatrix} 1366 & -1365 & 1365 \\ -1365 & 1366 & -1365 \\ 1365 & -1365 & 1366 \end{bmatrix}$ **10.** $\begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$

11. (*i*) Skew Hermitian; Eigen values : $i, -i$; Eigen vectors : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(*ii*) Hermitian; Eigen values : $3+\sqrt{2}, 3-\sqrt{2}$; Eigen vectors : $[-i,\ 1-\sqrt{2}]^T, [-i,\ 1+\sqrt{2}]^T$

(*iii*) Unitary; Eigen values : $1, -1$; Eigen vectors : $[1,\ i-i\sqrt{2}]^T, [1,\ i+i\sqrt{2}]^T$

(*iv*) Hermitian; Eigen values : 9, 2

(*v*) Skew Hermitian; Eigen values : $4i, -2i$

(*vi*) Skew Hermitian; Eigen values : $1 \pm \sqrt{10}\, i$

CAYLEY- HAMILTON THEOREM

We now give some difinitions and proceed to prove Cayley - Hamilton Theorem.

2.11 DEFINITIONS

1. Matrix Polynomial :

An expression of the form $F(x) = A_0 + A_1x + A_2x^2 + \cdots + A_mx^m, A_m \neq 0,$ where $A_0, A_1, A_2, \cdots, A_m$ are matrices each of order $n \times n$ over a field F, is called a matrix polynomial of degree m.

The symbol x is called indeterminate and will be assumed that it is commutative with every matrix coefficient.

The matrices themselves are matrix polynomials of zero degree.

2. Equality of Matrix Polynomials :

Two matrix polynomials are equal if and only if the coefficients of like powers of x are the same.

3. Addition and Multiplication of Polynomials :

Let $G(x) = A_0 + A_1x + A_2x^2 + \cdots + A_mx^m$ and

$H(x) = B_0 + B_1x + B_2x^2 + \cdots + B_kx^k.$

We define : If $m > k$, then

$G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1)x + \ldots + (A_k + B_k)x^k + A_{k+1}x^{k+1} + \ldots + A_mx^m.$

Similarly, we have $G(x) + H(x)$ when $m = k$ and $m < k$.

Also $G(x) \cdot H(x) = A_0B_0 + (A_0B_1 + A_1B_0)x$
$+ (A_0B_2 + A_1B_1 + A_2B_0)x^2 + \ldots + A_mB_kx^{k+m}.$

Note that the degree of the product of two matric polynomials is less than or equal to the sum of their degree.

Theorem : Every square matrix, whose elements are polynomials in x, can be expressed as a matric polynomials in x of degree m, where m is the highest power of x having by any element of the matrix.

We illustrate the theorem by an example.

Consider the matrix $A = \begin{bmatrix} 2x & 3x^2+4 & x-x^2+x^3 \\ x^3-5 & 0 & 4+7x^2 \\ 6 & 8x^2 & -3x^2+2 \end{bmatrix}$

We write

$$A = \begin{bmatrix} 0+2x+0x^2+0x^3 & 4+0x+3x^2+0x^3 & 0+1x-1x^2+1x^3 \\ -5+0x+0x^2+x^3 & 0+0x+0x^2+0x^3 & 4+0x+7x^2+0x^3 \\ 6+0x+0x^2+0x^3 & 0+0x+8x^2+0x^3 & 2+0x-3x^2+0x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 0 \\ -5 & 0 & 4 \\ 6 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 7 \\ 0 & 8 & -3 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= A_0 + A_1x + A_2x^2 + A_3x^3$$

2.12 THE CAYLEY-HAMILTON THEOREM

Theorem: Every square matrix satisfies its own characteristic equation.

[JNTU 2002S]

Proof: Let A be an n-rowed square matrix. Then

$|A - \lambda I| = 0$ is the characteristic equation of A.

Let $|A - \lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-1} + a_3\lambda^{n-2} + \ldots + a_n]$

Since all the elements of $A - \lambda I$ are at most of first degree in λ, all the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $(n - 1)$ or less and hence adj.$(A - \lambda I)$ can be written as a matrix polynomials in λ.

Let adj. $(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda^1 + B_{n-1}$, where $B_0, B_1, \ldots, B_{n-1}$ are n-rowed matrices.

Now $(A - \lambda I)$ adj$(A - \lambda I)$

$$= |A - \lambda I| I_n (A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1})$$
$$= (-1)^n[\lambda^n + a_1\lambda^{n-1} + \ldots + a_n]I.$$

Comparing coefficients of like powers of λ, we obtain

$$-B_0 = (-1)^n I,$$
$$AB_0 - B_1 = (-1)^n a_1 I,$$
$$AB_1 - B_2 = (-1)^n a_2 I$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AB_{n-1} = (-1)^n a_n I.$$

Premultiplying the above equations successively by $A^n, A^{n-1}, \ldots, I$ and adding, we obtain

$$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \cdots + (-1)^n a_n I$$

$$\Rightarrow (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I] = 0$$

Which implies that A satisfies its characteristic equation.

Applications of Cayley - Hamilton Theorem :

The important applications of Cayley - Hamilton theorem are

1. To find the inverse of a matrix.
2. To find higher powers of the matrix.

Remark **:** Determination of A^{-1} using Cayley-Hamilton theorem :

A satisfies its characteristic equation

$$i.e. (-1)^n[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots. + a_n I] = O$$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots. + a_n I = O$$

$$\Rightarrow A^{-1}[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots. + a_n I] = O \quad \text{(Multiplying by } A^{-1} \text{ on both sides)}$$

if A is non-singular, then we have

$$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} - \ldots - a_{n-1} I$$

$$\Rightarrow \quad A^{-1} = \left(\frac{-1}{a_n}\right)[A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I]$$

Thus by Cayley - Hamilton theorem, we can find out the inverse of a matrix (if it exists *i.e.*, $a_n \neq 0$) by computing the linear combination of higher powers of A.

SOLVED EXAMPLES

Example 1 : If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Cayley- Hamilton theorem. Hence find A^{-1}.

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \quad \begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$i.e.$ $(2-\lambda)[-6-3\lambda+2\lambda+\lambda^2] - 5[-2-\lambda] - 1(3-6+2\lambda) = 0$

$i.e.$ $(2-\lambda)(-6-\lambda+\lambda^2) + 10 + 5\lambda + 3 - 2\lambda = 0$

$i.e.$ $-12 - 2\lambda + 2\lambda^2 + 6\lambda + \lambda^2 - \lambda^3 + 13 + 3\lambda = 0$

$i.e.$ $-\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$

$i.e.$ $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$

To Verify Cayley-Hamilton theorem, we have to show that

$$A^3 - 3A^2 - 7A - I = O.$$

That is, to prove that A satisfies $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$, the characteristic equation of A.

$$\text{Now} \quad A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

$$\text{Now} \quad A^3 - 3A^2 - 7A - I = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix}$$

$$+ \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ This verifies Cayley – Hamilton theorem.

To find A^{-1}

$$A^3 - 3A^2 - 7A - I = 0 \quad \Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = 0$$

$$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0 \quad \Rightarrow A^{-1} = A^2 - 3A - 7I$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ 15 & -9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

Check : $AA^{-1} = I$

Example 2 : Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley-Hamilton theorem.

Solution : Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e. \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$i.e., \ (1-\lambda)[2-\lambda-2\lambda+\lambda^2-1]+[0-2]=0$$

$$i.e. \ (1-\lambda))(1-3\lambda+\lambda^2)-2=0$$

or $\quad \lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0$

By Cayley-Hamilton theorem, A satisfies its characteristic equation.

$$\therefore A^3 - 4A^2 + 4A + I = O$$

$$\Rightarrow A^{-1}(A^3 - 4A^2 + 4A + I) = O \quad (\because |A| = -1 \neq 0)$$

$$\Rightarrow A^2 - 4A + 4I + A^{-1} = O$$

$$\Rightarrow A^{-1} = -A^2 + 4A - 4I \qquad \ldots (1)$$

Now $A^2 = A.A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \text{[by (1)]}$$

$$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

***Check* :** $AA^{-1} = I$

Example 3: State Cayley-Hamilton theorem and use it to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}.$$ **[JNTU 2001]**

Solution : Cayley-Hamilton theorem : Every square matrix satisfies its own characteristic equation.

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\,[(1+\lambda)^2 - 4] - 2[-2(1+\lambda) - 12] + 3[2 + 3(1+\lambda)] = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

By Cayley-Hamilton theorem , we have

$A^3 + A^2 - 18A - 40I = O$

Multiplying with A^{-1} on both sides, we get

$A^2 + A - 18I = 40A^{-1} \quad \Rightarrow \quad A^{-1} = \frac{1}{40}\left[A^2 + A - 18I\right]$

We have $A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Example 4 : Using Cayley-Hamilton theorem find the inverse and A^4 of the matrix

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}.$$ **[JNTU 2002]**

Solution : Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A - \lambda I| = 0$

$$\text{i.e.} \quad \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

Performing $R_1 - R_3$ and $R_2 + R_3$, we get $\begin{vmatrix} 1-\lambda & 0 & \lambda-1 \\ 0 & 1-\lambda & 1-\lambda \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$

$$\Rightarrow \quad (1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2[(-1-\lambda-2) - 1(-6)] = 0$

$\Rightarrow (1-\lambda)^2[3-\lambda] = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, we must have

$$A^3 - 5A^2 + 7A - 3I = O \qquad \ldots(1)$$

We have $A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$

To find A^{-1}, multiply with A^{-1}

$$A^{-1}[A^3 - 5A^2 + 7A - 3I] = O$$

$\Rightarrow \quad A^2 - 5A + 7I - 3A^{-1} = O$

$\Rightarrow \quad 3A^{-1} = A^2 - 5A + 7I$

$$\Rightarrow \quad A^{-1} = \frac{1}{3}[A^2 - 5A + 7I] \qquad \ldots(2)$$

Now $A^2 - 5A + 7I = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ [by (2)]

Multiplying (1) with A, we have

$$A^4 - 5A^3 + 7A^2 - 3A = O$$

$\Rightarrow A^4 = 5A^3 - 7A^2 + 3A$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Example 5 : Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$.

Verify Cayley-Hamilton theorem and hence find A^{-1}. **[JNTU 2002]**

Solution : Given $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$.

Characteristic polynomial is det $(A - \lambda I)$

Now det $(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix}$

Applying $R_3 \to R_3 + R_2$, we get $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 0 & 4-\lambda & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 0 & 1 & 1 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$, we get

$$|A - \lambda I| = (4-\lambda)\begin{vmatrix} 3-\lambda & 1 & 0 \\ -1 & 5-\lambda & -6+\lambda \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (4-\lambda)[(3-\lambda)(6-\lambda) - 1(0)] = (4-\lambda)[18 - 3\lambda - 6\lambda + \lambda^2]$$

$$= (4-\lambda)[\lambda^2 - 9\lambda + 18] = -\lambda^3 + 13\lambda^2 - 54\lambda + 72$$

Since A satisfies the characteristic equation, we must have

$$-A^3 + 13A^2 - 54A + 72I = O$$

We have $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$ $\therefore$ $A^2 = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix}$

Substituting the values of A, A^2, A^3 in $-A^3 + 13A^2 - 54A + 72I$, we get

$-A^3 + 13A^2 - 54A + 72I$

$$= -\begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix} + 13\begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - 54\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} + 72\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore -A^3 + 13A^2 - 54A + 72I = O$

Hence Cayley-Hamilton theorem is verified.

Multiplying the above equation with A^{-1}, we get

$\therefore\ -A^2 + 13A - 54I + 72A^{-1} = 0 \quad \Rightarrow 72.\ A^{-1} = A^2 - 13A + 54I$

$$\Rightarrow\ 72.A^{-1} = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - 13\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow\ 72.A^{-1} = \begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 12 & -3 & -3 \\ 2 & 7 & 1 \\ -2 & 2 & 8 \end{bmatrix}$$

Example 6 : Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}. **[JNTU 2002, (A) May 2011]**

Solution : Characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_2 \to C_2 + C_3$ gives $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$ i.e . $(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(1-\lambda)(2-\lambda-3)+2\right] = 0$ [Expanding by C_1]

$\Rightarrow (1-\lambda)\,[\lambda^2 - 1 + 2] = 0 \Rightarrow (1-\lambda)\,(\lambda^2+1) = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$

By Cayley - Hamilton theorem, we have $A^3 - A^2 + A - I = O$

Now $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ $\quad \therefore\ A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

Substituting the values of A, A^2, A^3, we have

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Multiplying $A^3 - A^2 + A - I = O$ with A^{-1}, we get $A^2 - A + I = A^{-1}$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Example 7 : If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a polynomial in A.

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0$

$\therefore\ A^2 - 4A + 5I = O \Rightarrow A^2 = 4A - 5I$

$\therefore\ A^3 = 4A^2 - 5A,\ A^4 = 4A^3 - 5A^2,\ A^5 = 4A^4 - 5A^3,\ A^6 = 4A^5 - 5A^4$

Now $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = (4A^5 - 5A^4) - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= 3(4A^3 - 5A^2) - 12A^3 + 14A^2$

$= 12A^3 - 15A^2 - 12A^3 + 14A^2 = -A^2$

$= -4A + 5I \quad (\because A^2 - 4A + 5I = O)$

which is a linear polynomial in A.

Alternate Method :

The characteristic equation of A is $\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ By Cayley-Hamilton theorem, $A^2 - 4A + 5I = O$

We can rewrite the given expression as

$A^6 - 4A^5 + 5A^4 + 3A^4 - 12A^3 + 14A^2 = A^4(A^2 - 4A + 5I) + 3A^2[A^2 - 4A + 5I] - A^2$

$= O + O - A^2 = 5I - 4A \quad (\text{using } A^2 - 4A + 5I = O)$

Example 8 : If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ write $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Solution : We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Its characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda + 7 = 0$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation.

Therefore, we must have $A^2 - 5A + 7I = O$ ...(1)

$\Rightarrow A^2 = 5A - 7I$

Multiplying both sides with A, we get

$A^3 = 5A^2 - 7A$ ...(2) $A^4 = 5A^3 - 7A^2$ ...(3) $A^5 = 5A^4 - 7A^3$ ...(4)

Consider $2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$

$= 7A^4 - 14A^3 + A^2 - 4I = 7(5A^3 - 7A^2) - 14A^3 + A^2 - 4I$

$$= 21A^3 - 48A^2 - 4I \qquad = 21(5A^2 - 7A) - 48A^2 - 4I$$
$$= 57A^2 - 147A - 4I \qquad = 57(5A - 7I) - 147A - 4I$$
$$= 138A - 403I \text{ which is a linear polynomial in } A.$$

Aliter: We have $A^2 - 5A + 7I = O$

$$2A^5 - 3A^4 + A^2 - 4I = 2A^5 - 10A^4 + 14A^3 + 7A^4 - 114A^3 + A^2 - 4I$$
$$= 2A^3(A^2 - 5A + 7I) + 7A^4 - 114A^3 + A^2 - 4I$$
$$= 0 + 7A^4 - 35A^3 + 49A^2 + 21A^3 - 48A^2 - 4I \text{, using (1)}$$
$$= 7A^2(A^2 - 5A + 7I) + 21A^3 - 105A^2 + 147A + 57A^2 - 147A - 4I$$
$$= 0 + 21A(A^2 - 5A + 7I) + 57A^2 - 147A - 4I \text{, using (1)}$$
$$= 0 + 57A^2 - 147A - 4I \text{, using (1)}$$
$$= 57(A^2 - 5A + 7I) + 138A - 403I$$
$$= 57(0) + 138A - 403I \text{, using (1)}$$
$$= 138A - 403I.$$

Example 9 : Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. **[JNTU (A) May 2013]**

Solution : We have $|A - \lambda| = \begin{vmatrix} 0-\lambda & c & -b \\ -c & 0-\lambda & a \\ b & -a & 0-\lambda \end{vmatrix}$

$$= -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda)$$
$$= -\lambda^3 - \lambda(a^2 + b^2 + c^2)$$

$\therefore$ The characteristic equation of matrix A is given as

$$\lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$$

To verify Cayley-Hamilton theorem, we have to prove that $A^3 + (a^2 + b^2 + c^2)A = O$

We have $A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$= \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-b^2c-a^2c & bc^2+b^3+a^2b \\ c^3+a^2c+b^2c & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+c^3 & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)\, A$$

$\therefore A^3 + (a^2+b^2+c^2)A = O$

$\therefore$ A satisfies Cayley-Hamilton theorem.

Example 10 : If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find the value of the matrix

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e. \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad i.e.\ \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

$\therefore$ By Cayley–Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = O$ …(1)

We can rewrite the given expression as

$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I$

$= A[(A^3 - 5A^2 + 7A - 3I) + (A + I)] + I$

$= O + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$, using (1)

$= A^2 + A + I$

But $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$\therefore$ Value of the given matrix $= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 11 : Find the eigen values of A and hence find A^n (n is a + ve integer) if

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda = -1, 5$

When λ^n is divided by $(\lambda^2 - 4\lambda - 5)$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + (b)$

Then $\lambda^n = (\lambda^2 - 4\lambda - 5)\, Q(\lambda) + (a\lambda + b)$ ...(1)

Putting $\lambda = -1$ in (1), we get

$-a + b = (-1)^n$ ...(2)

Putting $\lambda = 5$ in (1), we get $5a + b = 5^n$ ...(3)

Solving (2) and (3), we obtain

$$a = \frac{1}{6}\left[5^n - (-1)^n\right] \text{ and } b = \frac{1}{6}\left[5^n + 5(-1)^n\right]$$

Replacing λ by A in (1), we get

$$A^n = (A^2 - 4A - 5I)\, Q(A) + aA + bI$$

$$= O \times Q(A) + aA + bI = aA + bI$$

$$= \frac{1}{6}\left[5^n - (-1)^n\right]\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \frac{1}{6}\left[5^n + 5(-1)^n\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 12 : Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

[JNTU 2005S, 2006S, (H) June 2011 (Set No. 3)]

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)\,[(-3-\lambda)\,(1-\lambda) - 8] - 4\,[-8\,(1-\lambda) + 8] + 3\,[16 - 2\,(-3-\lambda)] = 0$

$\Rightarrow (8-\lambda)\,[\lambda^2 + 2\lambda - 11] - 4\,[8\lambda] + 3\,[2\lambda + 22] = 0 \Rightarrow \lambda^3 - 6\lambda^2 - \lambda + 22 = 0$

Cayley - Hamilton theorem states that every square matrix satisfies its characteristic equation.

To verify Cayley - Hamilton theorem, we have to prove that $A^3 - 6A^2 - A + 22I = O$

Now $A^2 = A \cdot A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$

and $A^3 = A \cdot A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$

Now $A^3 - 6A^2 - A + 22I$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6 \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence Cayley - Hamilton theorem is verified.

Example 13 : Verify Cayley -Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}. **[JNTU 2005S, (A) May 2012 (Set No. 2)]**

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(4-\lambda)(6-\lambda) - 25] - 2[2(6-\lambda) - 15] + 3[10 - 3(4-\lambda)] = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 10\lambda - 1] - 2[-2\lambda - 3] + 3[3\lambda - 2] = 0$

$\Rightarrow -\lambda^3 + 10\lambda^2 + \lambda + \lambda^2 - 10\lambda - 1 + 4\lambda + 6 + 9\lambda - 6 = 0 \Rightarrow \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$

Cayley - Hamilton theorem states that every square matrix satisfies its characteristic equation.

To verify Cayley - Hamilton theorem, we have to prove that $A^3 - 11A^2 - 4A + I = O$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$

$A^3 = A \cdot A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$

$A^3 - 11A^2 - 4A + I$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

To find A^{-1} :

$A^3 - 11A^2 - 4A + I = O \Rightarrow I = -A^3 + 11A^2 + 4A$

Multiplying by A^{-1}, we get

$A^{-1} = -A^2 + 11A + 4I$

$$= \begin{bmatrix} -14 & -25 & -31 \\ -25 & -45 & -56 \\ -31 & -56 & -70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Example 14 : Using Cayley - Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

[JNTU July-2003]

Solution : Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5 = 0$

By Cayley - Hamilton theorem, A satisfies its characteristic equation. So we must have $A^2 = 5I$.

$\therefore \quad A^8 = 5A^6 = 5(A^2)(A^2)(A^2)$

$\quad = 5(5I)(5I)(5I) = 625I.$

Example 15 : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Cayley-Hamilton theorem. Find A^4 and A^{-1} using Cayley-Hamilton theorem. **[JNTU Sep. 2006 (Set No. 4)]**

Solution : Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_3$, we get $\begin{vmatrix} 3-\lambda & 0 & -\lambda \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)\left[(1-\lambda)^2 - 4\right] - \lambda\left[-4 - 2(1-\lambda)\right] = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$ ... (1)

By Cayley-Hamilton theorem, matrix A should satisfy its characteristic equation.

i.e, $A^3 - 3A^2 - 3A + 9I = O$ ... (2)

Now $A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$

$\therefore \quad A^3 - 3A^2 - 3A + 9I$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence Cayley - Hamilton Theorem is verified.

To find A^{-1}.

Multiplying equation. (2) with A^{-1} on both sides, we get

$A^{-1} [A^3 – 3A^2 \ 3A + 9I] = A^{-1} (O)$

$\Rightarrow A^2 - 3A - 3I + 9A^{-1} = 0$

$\Rightarrow 9A^{-1} = 3A + 3I - A^2$

$\therefore A^{-1} = \frac{1}{9}(3A + 3I - A^2)$

$$= \frac{1}{9}\left\{\begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Now $A[A^3 - 3A^2 - 3A + 9I] = 0 \Rightarrow A^4 - 3A^3 - 3A^2 + 9A = 0$

$\Rightarrow A^4 = 3A^3 + 3A^2 - 9A$

$$= \begin{bmatrix} 9 & 72 & -63 \\ 18 & 63 & -72 \\ 18 & -18 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ 0 & 27 & -18 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 18 & -9 \\ 18 & 9 & -18 \\ 18 & -18 & 9 \end{bmatrix}$$

$$\therefore \quad A^4 = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}.$$

Example 16 : Show that the Matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}. **[JNTU May 2007 (Set No. 3)]**

Solution : Given

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & 2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(2-\lambda)^2 + 3] + 2[(2-\lambda) - 0] + 2(-1) = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 13\lambda - 9 = 0.$

By Cayley - Hamilton theorem, we must have

$A^3 - 5A^2 + 13A - 9I = O.$

$$\text{Now } A^2 = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{pmatrix}; \; A^3 = \begin{pmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{pmatrix}$$

$\therefore \quad A^3 - 5A^2 + 13A - 9I$

$$= \begin{pmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{pmatrix} - \begin{pmatrix} -5 & -40 & 0 \\ 15 & -5 & 70 \\ -5 & -20 & 5 \end{pmatrix} + \begin{pmatrix} 13 & -26 & 26 \\ 13 & 26 & 39 \\ -0 & -13 & 26 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

$\therefore \quad A^3 - 5A^2 + 13A - 9I = O. \qquad \ldots (1)$

This verifies Cayley - Hamilton Theorem.

To find A^{-1}:

Multiplying (1) with A^{-1}, we get

$$A^2 - 5A + 13I - 9A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{9}(A^2 - 5A + 13I)$$

$$\therefore \quad A^{-1} = \frac{1}{9}\begin{pmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

Example 17 : Verify the Cayley-Hamilton theorem and find the characteristic roots, where $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **[JNTU (A)June 2009 (Set No.4)]**

Solution : Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$R_1 - R_3$ gives $\begin{vmatrix} -1-\lambda & 0 & 1+\lambda \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$

$C_3 + C_1$ gives $(1+\lambda)\begin{vmatrix} -1 & 0 & 0 \\ 2 & 1-\lambda & 4 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1+\lambda)(-1)[(1-\lambda)(3-\lambda) - 8] = 0$

$\Rightarrow (1+\lambda)(-\lambda^2 + 4\lambda + 5) = 0 \Rightarrow -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = 0$

$\Rightarrow 1 + \lambda = 0; -\lambda^2 + 4\lambda + 5 = 0$

$\Rightarrow \lambda = -1; 5; -1$ are charecteristic roots.

$\therefore$ Characteristic equation of A is $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

Verification : we must have $A^3 - 3A^2 - 9A - 5I = O$

L. H. S. : $A^3 - 3A^2 - 9A - 5I$

$$= \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 24 \\ 24 & 27 & 24 \\ 24 & 24 & 27 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 18 \\ 18 & 9 & 18 \\ 18 & 18 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 41-27-9-5 & 42-24-18+0 & 42-24-18-0 \\ 42-24-18-0 & 41-27-9-5 & 42-24-18-0 \\ 42-24-18-0 & 42-24-18-0 & 41-27-9-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R. H. S.}$$

Hence the Cayley - Hamilton theorem is satisfied.

Example 18 : Verify Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$. Hence find A^{-1}

[JNTU (K) June 2009 (Set No.1)]

Solution : Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 & 1 \\ 2 & 1-\lambda & 6 \\ -1 & 4 & 7-\lambda \end{vmatrix} = 0$$

$R_1 - R_3$ gives

$$\begin{vmatrix} 4-\lambda & 0 & -6+\lambda \\ 2 & 1-\lambda & 6 \\ -1 & 4 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 122 = 0$$

Cayley - Hamilton Theorem says "every square matrix satisfies its characteristic equation".

Now we have to verify that $A^3 - 11A^2 + 122\,I = O$.

$$A^2 = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9+8-1 & 12+4+4 & 3+24+7 \\ 6+2-6 & 8+1+24 & 2+6+42 \\ -3+8-7 & -4+4+28 & -1+24+49 \end{pmatrix} = \begin{pmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 48+40-34 & 64+20+136 & 16+120+238 \\ 6+66-50 & 8+33+200 & 2+198+350 \\ -6+56-72 & -8+28+288 & -2+168+504 \end{pmatrix} = \begin{pmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{pmatrix}$$

Now $A^3 - 11A^2 + 122\,I$

$$= \begin{pmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{pmatrix} - \begin{pmatrix} 176 & 220 & 374 \\ 22 & 363 & 550 \\ -22 & 308 & 792 \end{pmatrix} + \begin{pmatrix} 122 & 0 & 0 \\ 0 & 122 & 0 \\ 0 & 0 & 122 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

$\therefore$ Cayley-Hamilton theorem is verified.

Take $A^3 - 11A^2 + 122\,I = O$

Multiplying with A^{-1}, we get

$$\Rightarrow A^{-1} = \frac{-A^2 + 11A}{122} = \frac{1}{122}\begin{pmatrix} -17 & -24 & 23 \\ -20 & 22 & -16 \\ 9 & -16 & -5 \end{pmatrix}$$

Example 19 : Verify Cayley - Hamilton theorem and find the inverse of $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

[JNTU (H) June 2010 (Set No. 1)]

Solution : Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & -1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 1 - 1) + 3(-2 + 1 + \lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2) + 3(\lambda - 1) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 5) = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

By Cayley-Hamilton theorem, we must have

$$A^3 - A^2 - 5A + 5I = O$$

Verification: $A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix}$$

Now $A^3 - A^2 - 5A + 5I$

$$= \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 15 \\ 10 & -5 & -5 \\ 5 & -5 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ Cayley-Hamilton theorem is verifed.

To find A^{-1}: We have $A^3 - A^2 - 5A + 5I = O$

Multiplying with A^{-1}, we get $A^2 - A - 5 + 5A^{-1} = O$

$\Rightarrow \quad 5A^{-1} = -A^2 + A + 5I$

$\Rightarrow \quad A^{-1} = \frac{1}{5}(-A^2 + A + 5I)$

$$\therefore \quad A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} -4 & 3 & -6 \\ 1 & -2 & -6 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -7 \\ 1 & -1 & 1 \end{bmatrix}$$

Example 20 : Verify Cayley Hamilton theorem and find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

[JNTU (H) Jan. 2012 (Set No. 3)]

Solution : Given matrix is $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(3-\lambda)(-4-\lambda)-12]-1[(-4-\lambda)-6]+3[-4+6-2\lambda]=0$

$\Rightarrow (1-\lambda)(\lambda^2+\lambda-24)-1(-\lambda-10)+3(-2\lambda+2)=0$

$\Rightarrow \lambda^2+\lambda-24-\lambda^3-\lambda^2+24\lambda+\lambda+10-6\lambda+6=0$

$\Rightarrow -\lambda^3+20\lambda-8=0 \cdot$

By Cayley - Hamilton, we must have $-A^3+20A+8I=O$.

Verification :

$$\text{Now } A^2=\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}=\begin{bmatrix}-4&-8&-12\\10&22&6\\2&2&22\end{bmatrix}$$

$$\text{and } A^3=\begin{bmatrix}-4&-8&-12\\10&22&6\\2&2&22\end{bmatrix}\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}=\begin{bmatrix}12&20&60\\20&52&-60\\-40&-80&-88\end{bmatrix}$$

$$\therefore\ -A^3+20A-16I=\begin{bmatrix}-12&-20&-60\\-20&-52&60\\40&80&88\end{bmatrix}+\begin{bmatrix}20&20&60\\20&60&-60\\-40&-80&-80\end{bmatrix}-\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Hence Cayley - Hamilton theorem is verified.

Example 21 : Verify Cayley-Hamilton Theorem and find A^{-1} for $A=\begin{bmatrix}3&0&0\\5&4&0\\3&6&1\end{bmatrix}$

[JNTU (H) Dec. 2012]

Solution : $A=\begin{bmatrix}3&0&0\\5&4&0\\3&6&1\end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}3-\lambda&0&0\\5&4-\lambda&0\\3&6&1-\lambda\end{vmatrix}=0$$

$\Rightarrow (3-\lambda)[(4-\lambda)(1-\lambda)]=0$

$\therefore\ \lambda=3,4,1$ are the roots.

$\Rightarrow (3-\lambda)(\lambda^2-5\lambda+4)=0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda^2 - 15\lambda + 12 = 0$

$\Rightarrow -\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$

By Cayley- Hamilton theorem, $-A^3 + 8A^2 - 19A + 12\,I = O$

Verification :

$$A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 185 & 64 & 0 \\ 279 & 126 & 1 \end{bmatrix}$$

$$-A^3 + 8A^2 - 19A + 12\,I = \begin{bmatrix} -27 & 0 & 0 \\ -185 & -64 & 0 \\ -279 & -116 & -1 \end{bmatrix} + \begin{bmatrix} 72 & 0 & 0 \\ 280 & 128 & 0 \\ 336 & 240 & 8 \end{bmatrix} - \begin{bmatrix} 57 & 0 & 0 \\ 95 & 76 & 0 \\ 57 & -114 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus Cayley - Hamilton theorem is verified.

To find A^{-1}

We have $-A^3 + 8A^2 - 19A + 12\,I = O$

Multiplying with A^{-1}, we have

$$-A^2 + 8A - 19I + 12A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{12}(A^2 - 8A + 19\,I)$$

$$= \frac{1}{12}\left[\begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 40 & 32 & 0 \\ 24 & 48 & 8 \end{bmatrix} + \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}\right]$$

$$= \frac{1}{12}\begin{bmatrix} 4 & 0 & 0 \\ -5 & 3 & 0 \\ 18 & -18 & 12 \end{bmatrix}$$

Example 22 : Verify Cayley - Hamilton theorem and hence find A^{-1} for the matrix

$$A = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

[JNTU (H) May 2012 (Set No. 4)]

Solution : Given matrix is $A=\frac{1}{4}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \left|\frac{1}{4}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \frac{1}{4}\begin{vmatrix}2-4\lambda & -1 & -1\\ -1 & 2-4\lambda & -1\\ 1 & -1 & 2-4\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-4\lambda)\left[(2-4\lambda)^2-1\right]+1\left[-(2-4\lambda)+1\right]-1\left[1-(2-4\lambda)\right]=0$$

$$\Rightarrow (2-4\lambda)(4+16\lambda^2-16\lambda-1)=0$$

$$\Rightarrow 8+32\lambda^2-32\lambda-2-16\lambda-64\lambda^3+64\lambda^2+4\lambda=0$$

$$\Rightarrow -64\lambda^3+96\lambda^2-44\lambda+6=0$$

$$-32\lambda^3+48\lambda^2-22\lambda+3=0$$

By Cayley - Hamilton theorem, we must have

$$-32A^3+48A^2-22A+3I=O$$

Verification :

$$A^2=\frac{1}{16}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\frac{1}{16}\begin{bmatrix}(4+1-1) & (-2-2+1) & (-2+1-2)\\ (-2-2-1) & (1+4+1) & (1-2-2)\\ (2+1+2) & (-1-2-2) & (-1+1+4)\end{bmatrix}$$

$$=\frac{1}{16}\begin{bmatrix}4 & -3 & -3\\ -5 & 6 & -3\\ 5 & -5 & 4\end{bmatrix}$$

$$A^3=\frac{1}{64}\begin{bmatrix}4 & -3 & -3\\ -5 & 6 & -3\\ 5 & -5 & 4\end{bmatrix}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\frac{1}{64}\begin{bmatrix}(8+3-3) & (-4-6+3) & (-4+3-6)\\ (-10-6-3) & (5+12+3) & (5-6-6)\\ (10+5+4) & (-5-10-4) & (-5+5+8)\end{bmatrix}$$

$$=\frac{1}{64}\begin{bmatrix} 8 & -7 & -7 \\ -19 & 20 & -7 \\ 19 & -19 & 8 \end{bmatrix}$$

$-32A^3+48A^2-22A+3I$

$$=\frac{1}{2}\begin{bmatrix} 8 & -7 & -7 \\ -19 & 20 & -7 \\ 19 & -19 & 8 \end{bmatrix}+3\begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix}-\frac{11}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} (-8+24-22+6) & (7-18+11+0) & (7-18+11) \\ (19-30+11+0) & (-20+36-22-6) & (7-18+11+0) \\ (-19+30+1) & (19-30+11) & (-8+24-22+6) \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=O$$

Thus Cayley- Hamilton Theorem is verified.

Consider, $-32A^3+48A^2-22A+3I=O$ multiplying with A^{-1},

$-32A^2+48A-22I+3A^{-1}=O$

$$\Rightarrow A^{-1}=\frac{32A^2-48A+22I}{3}$$

$$=\frac{32}{16}\begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix}-\frac{48}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+\begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}$$

$$=\frac{1}{3}\left\{2\begin{bmatrix} 4 & -3 & -3 \\ -5 & 6 & -3 \\ 5 & -5 & 4 \end{bmatrix}-12\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+\begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}\right\}$$

$$=\frac{1}{3}\begin{bmatrix} (8-24+22) & (-6+12+0) & (-6+12+0) \\ (-10+12+0) & (12-24+22) & (-6+12+0) \\ (10-12+0) & (-10+12+0) & (8-24+22) \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 6 & 6 & 6 \\ 2 & 10 & 6 \\ -2 & 2 & 6 \end{bmatrix}$$

Verification :

$$AA^{-1} = \frac{1}{12}\begin{bmatrix} 6 & 6 & 6 \\ 2 & 10 & 6 \\ -2 & 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} (12-6+6) & (-6+12-6) & (-6-6+12) \\ (4-10+6) & (-2+20-6) & (-2-10+12) \\ (-4-2-16) & (2+4-6) & (2-2+12) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

EXERCISE 2.2

1. Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies Cayley-Hamilton theorem.

2. Show that $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ satisfies the characteristic equation. Hence find A^{-1}.

3. Show that $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}.

[JNTU 2000S, 2002]

4. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ using Cayley-Hamilton theorem.

ANSWERS

2. $\frac{1}{11}\begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$ **3.** $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **4.** $-\frac{1}{6}\begin{bmatrix} 4 & 3 & -2 \\ 2 & -2 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

OBJECTIVE TYPE QUESTIONS

1. The characteristic polynomial of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ is

(a) $\lambda^3 + 7\lambda^2 - 16\lambda + 12$ (b) $\lambda^3 - 7\lambda^2 + 16\lambda - 12$

(c) $\lambda^3 - 6\lambda^2 + 15\lambda - 9 = 0$ (d) $\lambda^3 - 7\lambda^2 - 16\lambda = 12$

QUADRATIC FORMS

3.1 QUADRATIC FORM

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

Note: A homogeneous expression of second degree means each and every term in any expression should have degree two.

e.g. 1. $3x^2 + 5xy - 2y^2$ is a quadratic form in two variables x and y.

e.g. 2. $2x^2 + 3y^2 - 4z^2 + 2xy - 3yz + 5zx$ is a quadratic form in three variables x, y and z.

e.g. 3. $x_1^2 - 2x_2^2 + 4x_3^2 - x_1x_2 + x_2x_3 + 2x_1x_4 - 5x_3x_4$ is a quadratic form in four variables x_1, x_2, x_3, x_4.

The most general form of Quadratic form in n variables is defined as follows:

Def. An expression of the form $Q = X^TAX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$, where a_{ij}'s are constants, is called a quadratic form in n variables $x_1, x_2, \ldots, x_n$. If the constants a_{ij}'s are real numbers it is called a real quadratic form.

$$\begin{aligned} i.e., X^TAX &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n + a_{21}x_2x_1 + a_{22}x_2^2 + \ldots \\ &\quad + a_{2n}x_2x_n + \ldots + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn}x_n^2 \\ &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + \ldots + (a_{1n} + a_{n1})x_1x_n + a_{22}x_2^2 \\ &\quad + (a_{23} + a_{32})x_2x_3 + \ldots + (a_{2n} + a_{n2})x_2x_n + \ldots + a_{nn}x_n^2. \end{aligned}$$

This quadratic form can be written as

$$\begin{aligned} Q &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j = [x_1\ x_2 \ldots x_n]\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \\ &= X^TAX \end{aligned}$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ and A is known as the matrix of the quadratic form.

Ex.: If $A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X^T=[x \;\; y \;\; z]$ then the quadratic form is given by

$X^TAX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$

Theorem : Every real quadratic form in n variables $x_1, x_2, \ldots, x_n$ can be expressed in the form $X^T AX$ where $X=[x_1 \; x_2 \cdots x_n]^T$ is a column matrix and A is a real symmetric matrix of order n.

[The matrix A is called the matrix of the quadratic form $X^T AX$]

3.2 QUADRATIC FORM CORRESPONDING TO A REAL SYMMETRIC MATRIX

Let $A=[a_{ij}]_{n\times n}$ be a real symmetric matrix and let $X = [x_1 \; x_2 \; \ldots \; x_n]^T$ be a column matrix.

Then X^TAX will determine a quadratic form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$. On expanding this we seen to be the quadratic form

$$\sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i}\sum_{j} a_{ij}\, x_ix_j \quad (i \neq j)$$

Note 1 : Consider the quadratic form $a_{11}\, x_1^2 + 2a_{12}\, x_1\, x_2 + a_{22}\, x_2^2$

We write this as $a_{11}\, x_1^2 + (a_{12}+a_{21})\, x_1x_2 + a_{22}\, x_2^2$ where $a_{12}=a_{21}$

This is seen to be $(x_1 \; x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Note 2 : Consider the quadratic form

$$a_{11}\, x_1^2 + a_{22}\, x_2^2 + a_{33}\, x_3^2 + 2a_{12}\, x_1\, x_2 + 2a_{13}\, x_1\, x_3 + 2a_{23}\, x_2\, x_3$$

Write $a_{12}=a_{21}, a_{13}=a_{31}, a_{23}=a_{32}$

With this the above quadratic form $= (x_1, x_2, x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

i.e. $X^T AX$, where A is a symmetric matrix. Quadratic forms in more variables can similarly be written in the form $X^T AX$ by suitably defining A.

Note: 1: To find the quadratic form of a matrix, it must be symmetric otherwise we make it symmetric by defining a new matrix B as follows:

$$b_{ii}=a_{ii} \text{ and } b_{ij}=b_{ji}=\frac{1}{2}(a_{ij}+a_{ji})$$

Note 2 : Rank of the symmetric matrix A is called the rank of the quadratic form.

3.3 MATRIX OF QUADRATIC FORM

We know that any quadratic form Q can be expressed as $Q = X^TAX$

The symmetric matrix A is called the matrix of the quadratic form Q and $|A|$ is called the discriminant of the quadratic form.

If $|A| = 0$, the quadratic form is called singular, otherwise non-singular. In other words, if the rank of A is $r < n$ (number of variables) then the quadratic form is singular otherwise non-singular.

To write the matrix of quadratic form, follow the diagram given below.

For example, consider the quadratic form

$$x^2 + 2y^2 + 7z^2 + 2xy + 6xz + 10yz$$

Write the coefficients of square terms along the diagonal and divide the coefficients of the product terms xy, xz, yz by 2 and write them at the appropriate places.

	x	y	z
x	x^2	$\frac{xy}{2}$	$\frac{xz}{2}$
y	$\frac{yx}{2}$	y^2	$\frac{yz}{2}$
z	$\frac{zx}{2}$	zy	z^2

} write only their coefficients

Thus the matrix of the above quadratic form is

$$\begin{bmatrix} 1 & 2/2 & 6/2 \\ 2/2 & 2 & 10/2 \\ 6/2 & 10/2 & 7 \end{bmatrix} \text{ i.e., } \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

Rule to write the matrix of a Quadratic Form

Step 1: In the first row, write the coefficient of $x_1^2 (= x_1 x_1)$, $\frac{1}{2}$ coefficient of $x_1 x_2$, $\frac{1}{2}$ coefficient of $x_1 x_3$ etc.

Step 2: In the second row, write $\frac{1}{2}$ coefficient of $x_2 x_1 (= x_1 x_2)$, coefficient of $x_2^2 (= x_2 x_2)$, $\frac{1}{2}$ coefficient of $x_2 x_3$, etc.

Step 3: In the third row, write $\frac{1}{2}$ coefficient of $x_3 x_1 (= x_1 x_3)$, $\frac{1}{2}$ coefficient of $x_3 x_2 (= x_2 x_3)$, coefficient of $x_3^2 (= x_3 x_3)$ etc. and so on.

SOLVED EXAMPLES

Example 1 : Find the symmetric matrix corresponding to the quadratic form $x_1^2 + 6x_1x_2 + 5x_2^2$.

Solution : Given quadratic form can be written as

$$x_1 . x_1 + 3x_1 . x_2 + 3x_2 . x_1 + 5x_2 . x_2$$

Let A be the symmetric matrix of this quadratic form. Then $A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $X^T = [x_1 \; x_2]$

We have $X^T AX = [x_1 \; x_2] \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 6x_1 x_2 + 5x_2^2$

Example 2 : Write the matrix relating to the quadratic form $ax^2 + 2hxy + by^2$.

Solution : The given quadratic form can be written as $(x \; y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore$ The corresponding matrix is $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$.

Example 3 : Find the symmetric matrix corresponding to the quadratic form $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$.

Solution : The above quadratic form can be written as

$$x \cdot x + 2xy + 3xz + 2yx + 2y.y + \frac{5}{2} yz + 3zx + \frac{5}{2} yz + 3z.z$$

Let A be the symmetric matrix of this quadratic form. Then

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$. Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $X^T = [x \quad y \quad z]$

and we can verify that $X^T AX = x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$.

Alternate Method: Given quadratic form is in three variables x, y and z.

Now write the coefficients of square terms. i.e., x^2, y^2, z^2 along the diagonal and also write the coefficients of product terms xy, xz and yz divided by 2 at the appropriate places as shown below.

	x	y	z
x	x^2	$xy/2$	$xz/2$
y	$yx/2$	y^2	$yz/2$
z	$zx/2$	$zy/2$	z^2

$\therefore$ The matrix of the given quadratic form is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$

Example 4 : Find the symmetric matrix corresponding to the quadratic form

$$x_1^2 + 2x_2^2 + 4x_2 x_3 + x_3 x_4.$$

Solution : The above quadratic form can be written as

$$x_1.x_1 + 0.x_1x_2 + 0.x_1x_3 + 0.x_1x_4 + 0x_2.x_1 + 2.x_2.x_2 + 2.x_2.x_3 + 0.x_2.x_4$$
$$+ 0.x_3x_1 + 2x_3x_2 + 0.x_3x_3 + \frac{1}{2}x_3x_4 + 0.x_4x_1 + 0.x_4x_2 + \frac{1}{2}x_3x_4 + 0.x_4.x_4.$$

$\therefore$ The matrix relating to the above quadratic form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$

Alternate Method :

Given quadratic form is in four variables x_1, x_2, x_3 and x_4.

Now write the coefficients of square terms along the diagonal and also write the coefficients of product terms at the appropriate places as shown.

	x_1	x_2	x_3	x_4
x_1	x_1^2	$\frac{x_1x_2}{2}$	$\frac{x_1x_3}{2}$	$\frac{x_1x_4}{2}$
x_2	$\frac{x_2x_1}{2}$	x_2^2	$\frac{x_2x_3}{2}$	$\frac{x_2x_4}{2}$
x_3	$\frac{x_3x_1}{2}$	$\frac{x_3x_2}{2}$	x_3^2	$\frac{x_3x_4}{2}$
x_4	$\frac{x_4x_1}{2}$	$\frac{x_4x_2}{2}$	$\frac{x_4x_3}{2}$	x_4^2

Hence the matrix of the given quadratic form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Example 5 : Find the quadratic form corresponding to the matrix $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

Solution : Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$. Then the quadratic form is

$$X^TAX = [x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + by^2 + cz^2 + 2fyz + 2yzx + 2hxy$$

Example 6 : Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$.

Solution : Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

The qudratic form corresponding to the symmetric matrix A is given by

$$X^T AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 \cdot x + 5 \cdot y + (-1)z \\ 5x + 1 \cdot y + 6 \cdot z \\ (-1)x + 6 \cdot y + 2 \cdot z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 5y - z \\ 5x + y + 6z \\ -x + 6y + 2z \end{bmatrix}$$

$$= x(5y - z) + y(5x + y + 6z) + z(-x + 6y + 2z)$$

$$= 5xy - xz + 5xy + y^2 + 6yz - zx + 6zy + 12z^2$$

$$\therefore \quad X^T AX = y^2 + 12z^2 + 10xy + 12yz - 2zx$$

Example 7 : Find the quadratic form relating to the symmetric matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$.

Solution : The Quadratic form related to the given matrix is $X^T AX$ where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \text{ and } X^T = [x \quad y \quad z]$$

$$\therefore \quad \text{Required quadratic form} = X^T AX = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x + 2y + 3z \\ 2x + y + 3z \\ 3x + 3y + z \end{bmatrix}$$

$$= x(x + 2y + 32) + y(2x + y + 3z) + z(3x + 3y + z)$$

$$= x^2 + 2xy + 3zx + 2xy + y^2 + 3yz + 3zx + 3yz + z^2$$

$$= x^2 + y^2 + z^2 + 4xy + 6yz + 6zx$$

Alternate Method:

Here $a_{11} = 1, a_{22} = 1, a_{33} = 1; \; a_{12} = a_{21} = 2; \; a_{13} = a_{31} = 3; a_{23} = a_{32} = 3.$

Then the corresponding Quadratic form is

$a_{11}\, x^2 + a_{22}\, y^2 + a_{33}\, z^2 + (a_{12} + a_{21})\, xy + (a_{13} + a_{31})\, zx + (a_{23} + a_{32})\, yz$

This simplifies to $x^2 + y^2 + z^2 + 4xy + 6xz + 6yz$

Example 8 : Find the quadratic form relating to the matrix diag $[\lambda_1\ \lambda_2\ \ldots\ \lambda_n]$.

Solution : Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ so that $X^T = [x_1\ \ x_2\ \ \ldots\ \ x_n]$

Let $A = \text{diag}\ [\lambda_1\ \lambda_2\ \ldots\ \lambda_n] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$

The required quadratic form is $X^T AX = \lambda_1 x_1{}^2 + \lambda_2 x_2{}^2 + \ldots + \lambda_n x_n{}^2$

Example 9 : Find the Quadratic Form corresponding to the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

[JNTU (H) May 2012 (Set No. 4)]

Solution : Given matrix is $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

From this Q. F. $= (0.x^2 + 1.xy + 2xz + 3xt) + (xy + 2y^2 + 3yz + 4yt)$

$$+ (2zx + 3zy + 4z^2 + 5zt) + (3xt + 4ty + 5zt + 6t^2)$$

$$= 2y^2 + 4z^2 + 6t^2 + 2xy + 6yz + 4zx + 6xt + 8yt + 10zt$$

This is the required Quadratic Form

EXERCISE 3.1

Write down the (symmetric) matrix of the following quadratic forms:

1. (*i*) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1\, x_2 + 8x_1\, x_3$

(*ii*) $x_1{}^2 + 2x_2{}^2 - 7x_3{}^2 - 4x_1x_2 + 8x_1x_3 + 5x_2x_3$ **[JNTU 2003 (Set No. 3)]**

2. $2x_1\, x_2 + 6x_1\, x_3 - 4x_2\, x_3$

3. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

Obtain the quadratic form corresponding to the matrix :

4. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ **[JNTU 2003(Set No. 3)]**

ANSWERS

1. (*i*) $\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & \frac{5}{2} \\ 4 & \frac{5}{2} & -7 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$ **4.** $x_1^2 + x_3^2 + 4x_1x_2 + 6x_1x_3 + 6x_2x_3$

5. $2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 + 10x_1x_2 - 4x_2x_3$ **6.** $x^2 + 4z^2 + 6yz + 10zx + 4xy$

3.4 LINEAR TRANSFORMATION OF A QUADRATIC FORM

Let the point $P(x, y)$ with respect to a set of rectangular axes OX and OY be transformed to the point $P'(x', y')$ with respect to a set of rectangular axes OX' and OY' by the following relations:

$$x' = a_1x + a_2y; \quad y' = b_1x + b_2y$$

or $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

or $X = AY$ where $X = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ and $Y = \begin{bmatrix} x \\ y \end{bmatrix}$

Such a transformation is called **Linear transformation** in two dimensions.

Example : Let (x, y) be the coordinates of a point P referred to the coordinate axes OX and OY. Also let (x', y') be the coordinates of the same point referred to the axes OX' and OY', obtained by rotating the axes OX and OY through an angle θ.

Then the transformation is given by

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \dots (1)$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

such transformation as (1) is called **linear transformation** in two dimensions.

Similarly the linear transformation in n-dimensions can be defined.

In general, the relation Y = AX where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

gives a linear transformation from n variables $x_1, x_2, \dots, x_n$ to the variables $y_1, y_2, \dots, y_n$ i.e., the transformation of the vector X to the vector Y.

The transformation is called linear because it holds the linear relations

$A(X_1+X_2) = AX_1 + AX_2$ and $A(KX) = KAX$.

If $X = AY$ and $Y = BZ$ be the two linear transformations, then $X = CZ$, where $C = AB$ is called **composite linear transformation**.

Consider a quadratic form $X^T AX$ ...(1). Let $X = PY$ ...(2) is a non-singular linear transformation, where P is a non-singular matrix.

Substituting X in (1), we get

$$X^T AX = (PY)^T A(PY) = (Y^T P^T) A(PY) = Y^T(P^T AP)Y = Y^T BY \text{ where } B = P^T AP.$$

Here B is also a symmetric matrix, so that $Y^T BY$ is also a quadratic form.

If $X^T AX$ is a quadratic form in n variables $x_1, x_2, \dots, x_n$ and $Y^T BY$ is a quadratic form in n variables $y_1, y_2, \dots, y_n$, we can see that $X = PY$ transform the original quadratic form into a new quadratic form.

If the matrix P is singular, the transformation is said to be singular otherwise non-singular. A non-singular transformation is also called a **regular transformation**. In this case the inverse transformation exists *i.e.,* $Y = P^{-1}X$.

3.5 ORTHOGONAL TRANSFORMATION

If A is an orthogonal matrix and X, Y are two column vectors, then the transformation

$$Y = AX \qquad \dots (1)$$

is called an **orthogonal transformation**.

From (1) it follows that

$$Y^T = (AX)^T = X^T A^T = X^T A^{-1} \ [\because \text{ A is orthogonal } A^T = A^{-1}] \qquad \dots (2)$$

From (1) and (2), we obtain

$$Y^T Y = X^T A^{-1} AX = X^T X \qquad \dots (3)$$

$\therefore$ If $Y = AX$ is an orthogonal transformation then $X'X = Y'Y = (AX)'(AX) = X'A'AX$ which is possible only if $A'A = I$.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$, then by (3)

$$[y_1 \; y_2 \; .... \; y_n]\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = [x_1 \; x_2 \; .... \; x_n]\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

or $y_1^2 + y_2^2 + ... + y_n^2 = x_1^2 + x_2^2 + ... + x_n^2$

Thus an orthogonal trannsformation $Y = AX$ transforms $\sum x_i^2$ into $\sum y_i^2$.

SOLVED EXAMPLES

Example 1 : Find the inverse transformation of

$y_1 = 2x_1 + x_2 + x_3, \quad y_2 = x_1 + x_2 + 2x_3, \quad y_3 = x_1 - 2x_3$

Solution : The given transformation can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad i.e., \; Y = AX$$

Now $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = 2(-2-0) - 1\,(-2-2) + 1\,(0-1)$$

$$= -4 + 4 - 1 = -1 \neq 0$$

Thus the matrix A is non-singular and hence the transformation is regular.

$\therefore$ The inverse transformation is given by

$$X = A^{-1}Y$$

i.e., $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

or $x_1 = 2y_1 - 2y_2 - y_3, \quad x_2 = -4y_1 + 5y_2 + 3y_3, \quad x_3 = y_1 - y_2 - y_3$

This is the required inverse transformation.

Example 2 : Show that the transformation $y_1 = x_1 \cos\theta + x_2 \sin\theta, \quad y_2 = -x_1 \sin\theta + x_2 \cos\theta$ is orthogonal.

Solution : The given transformation can be written as $Y = AX$

whre $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Here the matrix of transformation is

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Since $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A^T$, the transformation is orthogonal

3.6 RANK OF A QUADRATIC FORM

Let $X^T AX$ be a quadratic form over R. The rank r of A is called the rank of the quadratic form. If $r < n$(order of A) or $|A| = 0$ or A is singular, the quadratic form is called singular otherwise non-singular.

3.7 CANONICAL FORM (OR) NORMAL FORM OF A QUADRATIC FORM

Def. Let $X^T AX$ be a quadratic form in n variables. Then there exists a real non-singular linear transformation $X = PY$ which transforms X^TAX to another quadratic form of type $Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$, then Y^TDY is called the **Canonical form** of $X^T AX$. Here $D = \text{diag}\,[\lambda_1, \lambda_2, ..., \lambda_n]$.

3.8 INDEX OF A REAL QUADRATIC FORM

When the quadratic form X^TAX is reduced to the canonical form, it will contain only r terms, if the rank of A is r. The terms in the canonical form may be positive, zero or negative.

Def. The number of positive terms in canonical form or normal form of a quadratic form is known as the **index** (denoted by s) of the quadratic form.

We now state a theorem (without proof) about invariance of index of a quadratic form.

Theorem : The number of positive terms in any two normal reductions of a real quadratic form is the same. (or) The index of a real quadratic form is invariant from all normal reductions.

Note : The number of negative terms in any two normal reductions of quadratic form is the same. Also the excess of the number of positive terms over the number of negative terms in any two normal reductiions of a quadratic form is the same.

3.9 SIGNATURE OF A QUADRATIC FORM

If r is the rank of a quadratic form and s is the number of positive terms in its normal form, then the excess number of positive terms over the number of negative terms in a normal form of X^TAX *i.e.* $s - (r - s) = 2s - r$ is called the **signature** of the quadratic form.

In other words, signature of the quadratic form is defined as the difference between the number of positive terms and the number of negative terms in its canonical form.

3.10 NATURE OF QUADRATIC FORMS

The quadratic form X^TAX in n variables is said to be (*i*) **positive definite,** if $r = n$ and $s = n$ (or) if all the eigen values of A are positive.

(*ii*) **negative definite,** if $r = n$ and $s = 0$ (or) if all the eigen values of A are negative.

(*iii*) **positive semidefinite,** if $r < n$ and $s = r$ (or) if all the eigen values of $A \geq 0$ and at least one eigen value is zero.

(*iv*) **negative semidefinite,** if $r < n$ and $s = 0$ (or) if all the eigen values of $A \leq 0$ and at least one eigen value is zero.

(*v*) **indefinite,** in all other cases.

Note: If the quadratic form Q is negative definite (semi-definite) then – Q is positive definite (semi-definite).

3.11 THEOREM : SYLVESTER'S LAW OF INERTIA

The signature of quadratic form is invariant for all normal reductions.

We can conclude from the above two theorems that the signature and index of a quadratic form does not depend on the linear transformation $X = PY$ and the resulting normal form.

SOLVED EXAMPLES

Example 1 : Identify the nature of the quadratic form

$$x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$$

Solution : Given quadratic forms is $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$

The matrix of the given quadratic form is $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

The characteristic equation is $\begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$

Applying $R_1 - R_3$, we get

$$\begin{vmatrix} -\lambda & 0 & \lambda \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda \begin{vmatrix} -1 & 0 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$ [Taking λ common from R_1]

$\Rightarrow$ $\lambda[-1\{(4-\lambda)(1-\lambda)-4\}+1(4-4+\lambda)] = 0$ [Expand by R_1]

$\Rightarrow$ $\lambda[-(4-4\lambda-\lambda+\lambda^2-4)+\lambda] = 0$

$\Rightarrow$ $\lambda[5\lambda-\lambda^2+\lambda] = 0$

$\Rightarrow$ $\lambda^2(\lambda-6) = 0 \Rightarrow \lambda^2 = 0$ or $\lambda - 6 = 0$

$\therefore$ Eigen values are $\lambda = 0, 0, 6$, which are positive. and two values are zero .

$\therefore$ The quadratic form is positive semi definite.

Example 2 : Discuss the nature of the quadratic form $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$.

[JNTU 2008 (Set No.1)]

Solution : Given quadratic form is

$$x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$$

This can be written in matrix form as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)[(-1 - \lambda)(4 - \lambda) - 1] - 2(8 - 2\lambda - 3) + 3(2 + 3 + 3\lambda) = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 - 15\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 4\lambda - 15) = 0$

$\Rightarrow \lambda = 0 \text{ (or) } \lambda^2 - 4\lambda - 15 = 0$

$\therefore \lambda = 0,\ 2 + \sqrt{19},\ 2 - \sqrt{19}$

Thus the given quadratic form is indefinite.

Methods of Reduction of Quadratic Form to Canonical Form (or Sum of Squares Form)

Any quadratic form may be reduced to canonical form by means of the following methods:

1. Diagonalisation (Reduction to canonical form using Linear transformation or Linear transformation of Quadratic form).
2. Orthogonalisation (Reduction to canonical form using orthogonal transformation)
3. Lagrange's reduction.

Reduction to canonical form using (Diagonalisation of a symmetric matrix) Linear Transformation:

Let X^TAX be a quadratic form where A is the matrix of the quadratic form.

Let $X = PY$ be the non-singular linear transformation.

Then we have

$$\begin{aligned} X^TAX &= (PY)^TA(PY) \\ &= (Y^TP^T)A(PY) \\ &= Y^T(P^TAP)Y \\ &= Y^TDY \text{ where } D = P^TAP \text{ (Transformed quadratic form)} \end{aligned}$$

Here Y^TDY is also a quadratic form in variables $y_1, y_2, ..., y_n$ and known as canonical form.

Y^TDY is the linear transform of the quadratic form X^TAX under the linear transformation $X = PY$ and $D = P^TAP$ (The matrix D is symmetric).

Congruent Matrices: The matrices D and A are congruent matrices and the transformation X = PY is known as congruent transformation.

In the linear transformation X = PY to reduce the quadratic form to canonical form the matrix P is obtained by congruence transformation as follows:

Working Rule to Reduce Quadratic Form to canonical Form by Diagonalization:

Step 1: Write the symmetric matrix a of the given quadratic form

Step 2: Write the matrix A in the following relation : $A_{n\times n} = I_n A I_n$.

Step 3: Reduce the matrix A on left hand side to a diagonal matrix (*i*) by applying elementary row operations on the left identity matrix and on A on left hand side (*ii*) by applying elementary column operations on the right identity matrix and on A on left hand side.

Step 4: By these operations, A = IAI will be reduced to the form

$D = P^TAP$

where D is the diagonal matrix with elements $d_1, d_2, ..., d_n$ and P is the matrix used in the linear transformation.

The canonical form is given by

$$Y^TDY = [y_1 \quad y_2 \quad \ldots \quad y_n]\begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix}$$

$$= d_1 y_1^2 + d_2 y_2^2 + \ldots + d_n y_n^2$$

Here some of the coefficients $d_1, d_2, ..., d_n$ may be zeros.

SOLVED EXAMPLES

Example 1 : Reduce the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to a diagonal form and interpret the result in terms of quadratic form.

Solution : We write $A = I_3 AI_3 \Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We will now reduce A to diagonal form by applying elementary row operations and elementary column operations. The corresponding row operations will be applied on the prefactor I_3 and corresponding column operations will be applied on postfactor I_3 on R.H.S.

Applying $R_2 \to R_2 + \frac{1}{3}R_1$; $C_2 \to C_2 + \frac{1}{3}C_1$

and $R_3 \to R_3 - \frac{1}{3}R_1$; $C_3 \to C_3 - \frac{1}{3}C_1$, we get

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + \frac{1}{7}R_2$, $C_3 \to C_3 + \frac{1}{7}C_2$, we obtain

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$i.e.\ D = diag\left(6, \frac{7}{3}, \frac{16}{7}\right) = P^T AP$$

where $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$ so that $P^T = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix}$.

We know that Quadratic form corresponding to A is

$$X^T AX = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3 \quad \text{... (1)}$$

Non-singular transformation corresponding to the matrix P is given by $X = PY$.

$$i.e., \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

which can be written as $x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3$; $x_2 = y_2 + \frac{1}{7}y_3$; $x_3 = y_3$ .... (2)

The canonical form of the quadratic form given by (1) is

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2 \text{ (sum of squares)} \quad \text{... (3)}$$

Note : Here the rank of Quadratic form is 3 and it is reduced to sum of three squares (canonical form).

Example 2 : Discuss the nature of the quadratic form $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$.

[JNTU 2008 (Set No.1)]

Solution : Given quadratic form is $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$

This can be written in matrix form as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1 - \lambda) [(- 1 - \lambda) (4 - \lambda) - 1] - 2 (8 - 2\lambda - 3) +3 (2 + 3 + 3\lambda) = 0$

$\Rightarrow$ $\lambda^3 - 4\lambda^2 - 15 \lambda = 0$

$\Rightarrow$ $\lambda (\lambda^2 - 4\lambda - 15) = 0$

$\Rightarrow$ $\lambda = 0$ (or) $\lambda^2 - 4\lambda - 15 = 0$

$\therefore$ $\lambda = 0,\ 2+\sqrt{19},\ \ 2-\sqrt{19}$.

Thus the given quadratic form is indefinite.

Example 3 : Find the nature of the quadratic form $2x^2 + 2y^2 + 2z^2 + 2yz$.

[JNTU (H) Jan. 2012 (Set No. 3)]

Solution : Given Q. F. is $2x^2 + 2y^2 + 2z^2 + 2yz$

Matrix of the Q. F. is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Characteristic equation is $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\,[(2-\lambda)^2 - 1] = 0$

$\Rightarrow (2-\lambda)\,(\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow (2-\lambda)\,(\lambda - 3)(\lambda - 1) = 0$

$\therefore$ The roots of the characteristic equation are 1, 2, 3.

All the roots are positive. The Q. F. is + *ve* definite.

Example 4 : Find the rank, signature and index of the quadratic form $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 4x_1x_3 - 8x_2x_3$ by reducing it to canonical form or normal form. Also write the linear transformation which brings about the normal reduction.

Solution : The given quadratic form is $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 4x_1x_3 - 8x_2x_3$

Its matrix is given by $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$

We write $A = I_3 AI_3$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We apply elementary operations on A in L. H. S. We apply the same row operations on the prefactor and column operations on the postfactor in R. H. S.

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 + R_1$, and $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 + C_1$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again applying $R_3 \to R_3 + \frac{2}{17}R_2, C_3 \to C_3 + \frac{2}{17}C_2$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & \frac{-81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11/17 & 2/17 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 11/17 \\ 0 & 1 & 2/17 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{R_1}{\sqrt{2}}, \frac{C_1}{\sqrt{2}}, \frac{R_2}{\sqrt{17}}, \frac{C_2}{\sqrt{17}}, R_3\sqrt{\frac{17}{81}}, C_3\sqrt{\frac{17}{81}},$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ \frac{11}{17}\sqrt{\frac{17}{81}} & \frac{2}{17}\sqrt{\frac{17}{81}} & \sqrt{\frac{17}{81}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{17}} & \frac{11}{17}\sqrt{\frac{17}{81}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{17}\sqrt{\frac{17}{81}} \\ 0 & 0 & \sqrt{\frac{17}{81}} \end{bmatrix}$$

Thus the given quadratic form is reduced to the normal form ${y_1}^2 - {y_2}^2 - {y_3}^2$ ... (2) by the linear transformation $X = PY$.

Where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{17}} & \frac{11}{17}\sqrt{\frac{17}{81}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{17}\sqrt{\frac{17}{81}} \\ 0 & 0 & \sqrt{\frac{17}{81}} \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

We have the rank of quadratic form, r = No of non-zero terms in the normal form $\left(y_1^2 - y_2^2 - y_3^2 \right) = 3$

Index of the quadratic form, s = 1 and signature = 2s – r = 2(1) – 3 = – 1

Example 5 : Reduce the following quadratic form to canonical form and find its rank and signature.

$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt.$

Solution : Given quadratic form is

$$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt \quad \text{....(1)}$$

The matrix of the quadratic form is given by

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ -2 & 4 & -6 & 0 \\ 3 & -6 & 9 & -3 \\ -1 & 0 & -3 & 1 \end{bmatrix}$$

We write $A = I_4 A I_4$

We will perform elementary operations on A in LHS. The corresponding row operations will be performed on prefactor and corresponding column operations will be performed on postfactor in RHS.

Applying $R_1 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 + R_1$,
and $C_2 \to C_2 + 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 + C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing $R_2 \to R_2 + R_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing corresponding column operations $C_2 \to C_2 + C_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Performing $R_4 \to R_4 - \frac{1}{2}R_2, C_4 \to C_4 - \frac{1}{2}C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 3 & -3 & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{2}, C_2 \to \frac{1}{2}C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ -3 & 0 & 1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 3/2 & -3 & \frac{-1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

This is same as $D = P^T AP$.

The canonical form is

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = y_1^2 - y_2^2 + y_4^2 \text{ (sum of squares)}$$

The linear transformation is $X = PY$.

Rank of the given quadratic form r = number of non-zero terms in its normal form = 3.

The index of the quadratic form is the number of positive terms in its normal form s = 2.

The signature of the given quadratic form is the excess of the number of positive terms and the number of negative terms in the normal form (*i.e.* difference between the no. of +*ve* terms and no. of –*ve* terms) = 2 – 1 = 1 (or) $2s - r = 4 - 3 = 1$.

Example 6 : Reduce the quadratic form $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz$ to the canonical form. **[JNTU May 2007 (Set No. 1)]**

Solution : Given quadratic form is $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz$

Its matrix is given by

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}.$$

We write $A = I_3 AI_3$

$$\Rightarrow \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We apply elementary operations on A in LHS to reduce it to diagonal form. We apply same row operations on pre factor and same column operations on post-factor of A in RHS.

Applying $R_2 \to 7R_2 + 2R_1, C_2 \to 7C_2 + 2C_1$, we get

$$\begin{bmatrix} 7 & -2 & 0 \\ 0 & 38 & -14 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to 7R_2 + 2R_1, C_2 \to 7C_2 + 2C_1$, we get

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 266 & -14 \\ 0 & -14 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{R_1}{\sqrt{7}}, \frac{C_1}{\sqrt{7}}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 & 0 \\ \frac{2}{\sqrt{14}} & \frac{7}{\sqrt{14}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{14}} & 0 \\ 0 & \frac{7}{\sqrt{14}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2, C_3 \to C_3 + C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 & 0 \\ \frac{2}{\sqrt{14}} & \frac{7}{\sqrt{14}} & 0 \\ \frac{2}{\sqrt{14}} & \frac{7}{\sqrt{14}} & 1 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ 0 & \frac{7}{\sqrt{14}} & \frac{7}{\sqrt{14}} \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{R_3}{2}, \frac{C_3}{2}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 & 0 \\ \frac{2}{\sqrt{14}} & \frac{7}{\sqrt{14}} & 0 \\ \frac{2}{2\sqrt{14}} & \frac{7}{2\sqrt{14}} & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{14}} & \frac{2}{2\sqrt{14}} \\ 0 & \frac{7}{\sqrt{14}} & \frac{7}{2\sqrt{14}} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

i.e. The canonical form of the given quadratic form is

$$\mathrm{Y^TDY} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + y_2^2 + y_3^2$$

$\therefore$ The normal form of the Quadratic form is $y_1^2 + y_2^2 + y_3^2$.

Example 7 : Find nature of the quadratic form, index and signature of $10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$. **[JNTU APril 2007 (Set No. 4), Sep. 2008 (Set No.1)]**

Solution : The given quadratic form is $10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$

Its matrix is given by $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

We write $A = I_3 A I_3$ and follow the procedure described earlier.

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \left[\text{Applying } \begin{matrix} R_2 \to 5R_2 + R_1, C_2 \to 5C_2 + C_1, \\ R_3 \to 2R_3 + R_1, C_3 \to 2C_3 + C_1 \end{matrix}\right]$$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ -1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} (\text{Applying } R_3 \to 2R_3 - R_2, C_3 \to 2C_3 - C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & 0 \\ \frac{1}{\sqrt{40}} & \frac{5}{\sqrt{40}} & 0 \\ -1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{40}} & -1 \\ 0 & \frac{5}{\sqrt{40}} & -5 \\ 0 & 0 & 4 \end{bmatrix} \left[\text{Applying } \begin{matrix} \frac{R_1}{\sqrt{10}}, \frac{C_1}{\sqrt{10}}, \\ \frac{R_2}{\sqrt{40}}, \frac{C_2}{\sqrt{40}} \end{matrix}\right]$$

Thus the normal form of the quadratic form is $y_1^2 + y_2^2$.
This is by the linear transformation X = PY;

Here, $P = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{40}} & -1 \\ 0 & \frac{5}{\sqrt{10}} & -5 \\ 0 & 0 & 4 \end{bmatrix}$

and new quadratic form is $Y = P^T AP$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Here rank , $r = 2$, $n = 3$

$\therefore$ Index, s = no.of positive terms in normal form = 2

Signature = $2s - r = 2(2) - 2 = 2$.

Since $r = s < n$, the quadratic form is positive simidefinite

Example 8 : Find the transformation which will transform $4x^2 + 3y^2 + z^2 - 8xy - 6yz + 4zx$ into a sum of squares and find the reduced form.

[JNTU 2008 (Set No.4)]

Solution : Given quadratic form can be written as

$$4x^2 - 4xy + 2xz - 4xy + 3y^2 - 3yz + 2zx - 3yz + z^2$$

The corresponding matrix is

$$A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

We write $A = I_3 A I_3$.

We apply elementary operations on A of L.H.S and we apply the same row operations on the prefactor and column operations on the post factor.

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 + R_1$, $2R_3 - R_1$, $C_2 + C_1$ $2C_3 - C_1$, we get

$$\begin{bmatrix} 4 & -4 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Applying $2R_2 - R_3$, $R_1 - 2R_3$, $2C_2 - C_3$, $C_1 - 2C_3$, we get

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 3 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 3 & 3 & -1 \\ 0 & 2 & 0 \\ -4 & -2 & 2 \end{bmatrix}$$

Applying $R_1 + R_2$, $C_1 + C_2$, we get

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -6 \\ 3 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 6 & 3 & -1 \\ 2 & 2 & 0 \\ -6 & -2 & 2 \end{bmatrix}$$

Applying $\frac{R_1}{2}, \frac{R_2}{\sqrt{2}}, \frac{R_3}{\sqrt{2}}. \frac{C_1}{2}, \frac{C_2}{\sqrt{2}}, \frac{C_3}{\sqrt{2}}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{2} & \frac{2}{2} & \frac{-6}{2} \\ 3/\sqrt{2} & 2/\sqrt{2} & -2/\sqrt{2} \\ -1/\sqrt{2} & 0 & 2/\sqrt{2} \end{bmatrix} A \begin{bmatrix} 6/2 & 3/\sqrt{2} & -1/\sqrt{2} \\ 2/2 & 2/\sqrt{2} & 0 \\ -6/2 & -2/\sqrt{2} & 2/\sqrt{2} \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{6}{2} & \frac{2}{2} & \frac{-6}{2} \\ -1/\sqrt{2} & 0 & 2/\sqrt{2} \\ 3/\sqrt{2} & 2/\sqrt{2} & -2/\sqrt{2} \end{bmatrix} A \begin{bmatrix} 6/2 & -1/\sqrt{2} & 3/\sqrt{2} \\ 2/2 & 0 & 2/\sqrt{2} \\ -6/2 & 2/\sqrt{2} & -2/\sqrt{2} \end{bmatrix}$$

i.e., $D = P^T AP$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 6/2 & -1/\sqrt{2} & 3/\sqrt{2} \\ 2/2 & 0 & 2/\sqrt{2} \\ -6/2 & 2/\sqrt{2} & -2/\sqrt{2} \end{bmatrix}$

The linear transformation is $X = PY$ *i.e.* $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6/2 & -1/\sqrt{2} & 3/\sqrt{2} \\ 2/2 & 0 & 2/\sqrt{2} \\ -6/2 & 2/\sqrt{2} & -2/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

i.e., $x = 3y_1 - \frac{1}{\sqrt{2}} y_2 + \frac{3}{\sqrt{2}} y_3$

$y = y_1 + \sqrt{2} y_3$

$z = -3y_1 + \sqrt{2} y_2 - \sqrt{2} y_3$

The canonical form is reduced to $X^T AX = (PY)^T A(PY) = Y^T (P^T AP)Y = Y^T DY$

$$\therefore \quad Y^T DY = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 - y_2^2 - y_3^2$$

Thus the given quadratic form is reduced to normal form ${y_1}^2 - {y^2}_2 - {y_3}^2$ by the linear transformation $X = PY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $P = \begin{bmatrix} 3 & -1/\sqrt{2} & 3/\sqrt{2} \\ 1 & 0 & 2/\sqrt{2} \\ -3 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Example 9 : Find nature of the quadratic form, index and signature of $10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$. **[JNTU APril 2007 (Set No. 4), Sep. 2008 (Set No.1)]**

(or) Find the transformation that will transform $10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xy$ into a sum of squares and find its reduced form. **[JNTU (K) 2010]**

Solution : The given quadratic form is $10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$.

Its matrix is given by $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

We write $A = I_3 A I_3$

$$i.e. \quad \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Applying $R_2 \to 5R_2 + R_1$; $R_3 \to 2R_3 + R_1$)

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(Applying $C_2 \to 5C_2 + C_1$; $C_3 \to 2C_3 + C_1$)

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{(Applying } R_3 \to 2R_3 - R_2\text{)}$$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \text{(Applying } C_3 \to 2C_3 - C_2\text{)}$$

Thus the given quadratic form is reduced into normal form.

i.e., $D = P^T AP$ where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

The linear transformation is $X = PY$ *i.e.*, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

i.e. $x = y_1 + y_2 + y_3$; $y = 5y_2 - 5y_3$; $z = 4y_3$

The given quadratic form is reduced to

$X^T AX = (PY)^T A(PY) = Y^T (P^T AP)Y = Y^T DY$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 10y_1^2 + 40y_2^2$$

Nature of the quadratic form is positive semi - definite (Here $r = 2$, $n = 3$, $s = 2$)

Index, s = no.of positive terms in normal form = 2

Signature = $2s - r = 2(2) - 2 = 4$.

Example 10 : Reduce the quadratic form to the canonical form $x^2 + y^2 + 2z^2 - 2xy + 4zx + 4yz$ **[JNTU (H) June 2010 (Set No. 3)]**

Solution : Given quadratic form is $x^2 + y^2 + 2z^2 - 2xy + 4zx + 4yz$

The matrix of the quadratic form is $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

We wirte $A = I_3AI_3$. We will apply row and column operations and reduce A on the L.H.S. to the diagonal form. Same row operations will be applied on prefactor, and same column operations will be applied on post factor is R.H.S. and reduce R.H.S. to the form P^TAP.

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + R_1 \\ C_2 \to C_2 + C_1 \end{matrix} \; gives \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_3 - 2R_1 \\ C_3 - 2C_1 \end{matrix} \; gives \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 + 2R_3 \\ C_2 + 2C_3 \end{matrix} \; gives \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix}\frac{R_2}{\sqrt{8}}\\ \frac{C_2}{\sqrt{8}}\\ \frac{R_3}{\sqrt{2}}\\ \frac{C_3}{\sqrt{2}}\end{matrix}\; gives \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -3/\sqrt{8} & 1/\sqrt{8} & 2/\sqrt{8}\\ \frac{-2}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix} A \begin{bmatrix}1 & -\sqrt{3}/8 & \frac{-2}{\sqrt{2}}\\ 0 & 1/\sqrt{8} & 0\\ 0 & 2/\sqrt{8} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

The normal form is $y_1^2 + y_2^2 - y_3^2$.

This is done by the linear transformation X = PY where $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$P = \begin{bmatrix}1 & -3/\sqrt{8} & \frac{-2}{\sqrt{2}}\\ 0 & 1/\sqrt{8} & 0\\ 0 & 2/\sqrt{8} & \frac{1}{\sqrt{2}}\end{bmatrix} \text{ and } Y = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

Example 11 : Reduce the quadratic form to the canonical form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$. **[JNTU (H) June 2010 (Set No. 4)]**

Solution : We write $A = I_3 A I_3$. We will perform elementary operations on A in L.H.S. The corresponding row operations will be performed on prefactor and the corresponding column operations will be performed on the prefactor in R.H.S.

$$\begin{bmatrix}3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Applying $R_2 \to 3R_2 + R_1;\; C_2 \to 3C_2 + C_1$, we get

$$\begin{bmatrix}3 & 0 & 0\\ 0 & 5 & -3\\ 0 & -3 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Applying $5R_3 + 3R_2, 5C_3 + 3C_2$, we get

$$\begin{bmatrix}3 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 3 & 9 & 5\end{bmatrix} A \begin{bmatrix}1 & 1 & 3\\ 0 & 3 & 9\\ 0 & 0 & 5\end{bmatrix}$$

Applying $\frac{R_1}{\sqrt{3}}, \frac{C_1}{\sqrt{3}}, \frac{R_2}{\sqrt{5}}, \frac{C_2}{\sqrt{5}}, \frac{R_3}{\sqrt{6}}, \frac{C_3}{\sqrt{6}}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{5}} & 0 \\ \frac{3}{\sqrt{6}} & \frac{9}{\sqrt{6}} & \frac{5}{\sqrt{6}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{5}} & \frac{9}{\sqrt{6}} \\ 0 & 0 & \frac{5}{\sqrt{6}} \end{bmatrix}$$

This is of the form P^TAP. The normal form is $y_1^2 + y_2^2 + y_3^2$, where this is done using the linear transformation X = PY,

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{5}} & \frac{9}{\sqrt{6}} \\ 0 & 0 & \frac{5}{\sqrt{6}} \end{bmatrix}$

Example 12 : Reduce the quadratic form to the canonical form $2x^2 + 5y^2 + 3z^2 + 4xy$.

[JNTU (H) June 2011 (Set No. 1)]

Solution : Given quadratic form is $2x^2 + 5y^2 + 3z^2 + 4xy$

The matrix of the quadratic form is given by $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

We write $A = I_3 A I_3$

We will perform elementary operations on A in L. H. S. The corresponding row operations will be performed on pre factor of A and coresponding column operatiions wll be performed on post factor of A in R. H. S.

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 - R_1; C_2 - C_1$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{R_1}{\sqrt{2}}, \frac{C_1}{\sqrt{2}}; \frac{R_2}{\sqrt{3}}, \frac{C_2}{\sqrt{3}}; \frac{R_3}{\sqrt{3}}, \frac{C_3}{\sqrt{3}}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

This is of the form $D = P^T AP$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix and

$$P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

The canonical form is $y_1^2 + y_2^2 + y_3^2$ which is given $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Example 13 : Reduce the quadratic form to the canonical form

$3x^2 - 3y^2 - 5z^2 - 2xy - 6yz - 6xz$ **[JNTU (H) June 2011 (Set No. 4)]**

Solution : Given quadratic form is $3x^2 - 3y^2 - 5z^2 - 2xy - 6yz - 6zx$

Matrix of the quadratic form is $A = \begin{bmatrix} 3 & -1 & -3 \\ -1 & -3 & -3 \\ -3 & -3 & -5 \end{bmatrix}$

We write $A = I_3 A I_3$

We apply elementary row and column opeation on A in LHS. We apply the same row operations as prefactor of A and column operations on postfactor of A in RHS.

$$\begin{bmatrix} 3 & -1 & -3 \\ -1 & -3 & -3 \\ -3 & -3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -3 \\ 0 & -10 & -12 \\ -3 & -3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Applying } 3R_2 + R_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & -3 \\ 0 & -30 & -12 \\ -3 & -12 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Applying } 3C_2 + C_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & -3 \\ 0 & -30 & -12 \\ 0 & -12 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Applying } R_3 + R_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -30 & -12 \\ 0 & -12 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Applying } C_3 + C_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -30 & -12 \\ 0 & 0 & -16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & -6 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\text{Applying } 5R_3 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -30 & 0 \\ 0 & 0 & -80 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & -6 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 5 \end{bmatrix} (\text{Applying } 5C_3 - 2C_2)$$

Applying $\frac{R_1}{\sqrt{3}}, \frac{C}{\sqrt{3}}, \frac{R_2}{\sqrt{30}}, \frac{C_2}{\sqrt{30}}, \frac{R_3}{\sqrt{80}}, \frac{C_3}{\sqrt{80}}$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{30}} & \frac{3}{\sqrt{30}} & 0 \\ \frac{3}{\sqrt{80}} & \frac{-6}{\sqrt{80}} & \frac{5}{\sqrt{80}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{30}} & \frac{3}{\sqrt{80}} \\ 0 & \frac{3}{\sqrt{30}} & -\frac{6}{\sqrt{80}} \\ 0 & 0 & \frac{5}{\sqrt{80}} \end{bmatrix}$$

The canonical form of the quadratic form is $y_1^2 - y_2^2 + y_3^2$.

Example 14 : Reduce the quadratic form to the canonical form $3x^2 - 3y^2 - 5z^2 - 2xy - 6zx - 6yz$ **[JNTU (H) June 2012 (Set No. 2)]**

Solution : Given Q. F. is $3x^2 - 3y^2 - 5z^2 - 2xy - 6zx - 6yz$

The matrix of Q. F. is $A=\begin{bmatrix}3 & -1 & -3\\ -1 & -3 & -3\\ -3 & -3 & -5\end{bmatrix}$

We write $A = I_3 A I_3$.

We will perform elementary operations on A in L. H. S. and the corresponding row operations will be performed on pre factor and corresponding column operations will be performed on post facter of A in R. H. S.

We will reduce A in L. H. S. to diagonal form

$$\begin{bmatrix}3 & -1 & -3\\ -1 & -3 & -3\\ -3 & -3 & -5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{matrix}3R_2+R_1\\ R_3+R_1\end{matrix}\text{ gives }\begin{bmatrix}3 & -1 & -3\\ 0 & -10 & -12\\ 0 & -4 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 1 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{matrix}3C_2+C_1\\ C_3+C_1\end{matrix}\text{ gives }\begin{bmatrix}3 & 0 & 0\\ 0 & -30 & -12\\ 0 & -12 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 1 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$5R_3-2R_2\text{ gives }\begin{bmatrix}3 & 0 & 0\\ 0 & -30 & -12\\ 0 & 0 & 16\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 3 & -6 & 5\end{bmatrix}A\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$5C_3-2C_2\text{ gives }\begin{bmatrix}3 & 0 & 0\\ 0 & -30 & 0\\ 0 & 0 & 80\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 3 & -6 & 5\end{bmatrix}A\begin{bmatrix}1 & 1 & 3\\ 0 & 3 & -6\\ 0 & 0 & 5\end{bmatrix}$$

$\frac{R_1}{\sqrt{3}},\frac{C_1}{\sqrt{3}},\frac{R_2}{\sqrt{30}},\frac{C_2}{\sqrt{30}},\frac{R_3}{\sqrt{80}},\frac{C_3}{\sqrt{80}}$ gives

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{3}} & 0 & 0\\ \frac{1}{\sqrt{30}} & \frac{3}{\sqrt{30}} & 0\\ \frac{3}{\sqrt{80}} & \frac{-6}{\sqrt{80}} & \frac{5}{\sqrt{80}}\end{bmatrix}A\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{30}} & \frac{3}{\sqrt{80}}\\ 0 & \frac{3}{\sqrt{30}} & -\frac{6}{\sqrt{80}}\\ 0 & 0 & \frac{5}{\sqrt{80}}\end{bmatrix}$$

This is in the form $D = P^T AP$ where D is a diagonal matrix.

The canonical form of the Q. F. is $y_1^2 - y_2^2 + y_3^2$.

Example 15 : Find the rank and signature of the quadratic form $x_1x_2 - 4x_1x_4 - 2x_2x_3 + 12x_3x_4$ **[JNTU (H) Dec. 2012, (A) Nov. 2012 (Set No. 3)]**

Solution : Given quadratic form is $x_1x_2 - 4x_1x_4 - 2x_2x_3 + 12x_3x_4$

The matrix of Q. F. is, $A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -2 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & -1 & 0 & 6 \\ -2 & 0 & 6 & 0 \end{bmatrix}$

We write $A = I_4AI_4$

We will perform elementary operations on A in L. H. S. The corresponding row operation will be performed on the pre-factor and corresponding column operations will be performed on the post-factor in R. H.S. We will reduce L. H. S to diagonal form.

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & -2 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & -1 & 0 & 6 \\ -2 & 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2; C_1 \leftrightarrow C_2$ gives

$$\begin{bmatrix} 0 & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 0 & -2 \\ -1 & 0 & 0 & 6 \\ 0 & -2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 + 2R_2; C_3 + 2C_2$ gives

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 + R_3 ; C_2 + C_3$ gives

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sqrt{2}R_1, \sqrt{2}R_2, \frac{R_3}{\sqrt{2}}, \frac{R_4}{\sqrt{2}}, \sqrt{2}C_1, \sqrt{2}C_2, \frac{C_3}{\sqrt{2}}, \frac{C_4}{\sqrt{2}}$ gives

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 \\ 3\sqrt{2} & 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 0 & 3\sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$R_1 \leftrightarrow R_2 ; R_3 \leftrightarrow R_4$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} A \begin{bmatrix} 3\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

This is of the form $I_4 = P^T AP$ where $P = \begin{bmatrix} 3\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

The cononical form is $y_1^2 + y_2^2 + y_3^2 + y_4^2$ with the linear transform $X = PY$ when

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$r =$ Rank of the Q. F. $=$ number of non-zero terms in the normal form $= 4$

$s =$ index of the Q. F. $=$ number of positive terms in the normal form $= 4$

Signature $= 2s - r = 8 - 4 = 4$

Example 16 : Determine the matrix, index and signature of the quadratic form $x_1^2 + 2x_2^2 + 3x_0^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$ **[JNTU (H) May 2012 (Set No. 1)]**

Solution : The matrix of the given Q. F. is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

We will reduce the matrix to a diagonal form using linear transformations.

We write $A = I_3 A I_3$. We will reduce A in L. H. S. to diagonal form by using elementary matrix operations. The corresponding row operations will be performed on pre-factor and corresponding column operations will be performed on post-factor in R. H. S.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_2 - R_1; C_2 - C_1; R_3 + R_1; C_3 + C_1$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3, C_2 - C_3$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 + 2R_2; C_3 + 2C_2$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ -3 & 2 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

This is of the form $D = P^T AP$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a diagonal matrix and

$$P = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$$

The non-singular on transform $X = PY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

gives the canonical form of the Q. F. as $y_1^2 - y_2^2 + 2y_3^2$.

Here rank of the Quadratic form $= r = 3$, Index $= s =$ no. of the terms $= 2$,

Signature $= 2s - r = 4 - 3 = 1$.

Here $r = n$ and $s < n$.

$\therefore$ The Quadratic form is indifinite.

3.12 REDUCTION TO NORMAL FORM BY ORTHOGONAL TRANSFORMATION

If, in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduced to the cononical form by an **orthogonal transformation.**

Consider the quadratic form

$$a_{11}\, x_1^2 + a_{22}\, x_2^2 + a_{33}\, x_3^2 + (a_{12} + a_{21})\, x_1\, x_2 + (a_{13} + a_{31})\, x_1\, x_3 + (a_{23} + a_{32})\, x_2\, x_3$$

where $a_{ij} = a_{ji}$ for all i, j and a_{ij}'s are all real.

This is same as $X^T AX$, where $X^T = (x_1 \quad x_2 \quad x_3)$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with $a_{ij} = a_{ji}$.

The following points may be carefully noted when we propose to reduce $X^T AX$ to its canonical form (or normal form).

(1) If A has eigen values $\lambda_1, \lambda_2, \lambda_3$ (not-necessarily distinct) and X_1, X_2, X_3 are three eigen vectors which are linearly independent, we can construct normalized eigen vectors e_1, e_2, e_3 corresponding to $\lambda_1, \lambda_2, \lambda_3$ which are pairwise orthogonal.

Then we define

$P = (e_1\ e_2\ e_3)$ where $e_1 = X_1/\|X_1\|,\ e_2 = X_2/\|X_2\|,\ e_3 = X_3/\|X_3\|$.

(**Note:** If $X = (a, b, c)$ be a vector then $\| X \| = \sqrt{a^2 + b^2 + c^2}$)

This P will be an orthogonal matrix.

i.e., $P^T P = PP^T = I$ and $P^T AP = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

Then defining $X = PY$, we get

$$X^T AX = Y^T P^T APY = Y^T DY, \text{ where } D = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$$
$$= \lambda_1 {y_1}^2 + \lambda_2 {y_2}^2 + \lambda_3 {y_3}^2.$$

This is the required normal form.

(2) If A is an nth order square real symmetric matrix, the above results can be generalized.

(3) If A is of order n and it is not possible to have n linearly independent pairwise orthogonal eigen vectors, the above procedure does not work.

Procedure to Reduce Quadratic form to canonical form By Orthogonal Transformation :

1. Write the coefficient matrix A associated with the given quadratic form.
2. Find the eigen values of A.
3. Write the canonical form using $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$.
4. Form the matrix P containing the normalized eigen vectors of A as column vectors. Then $X = PY$ gives the required orthogonal transformation which reduces Quadratic form to Canonical form.

SOLVED EXAMPLES

Example 1 : Reduce the quadratic form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$ to the normal form by orthogonal transformation. **[JNTU (A) Nov. 2010 (Set No. 4), (H) Dec. 2011, (Set No. 1)]**

Solution : The matrix A of the quadratic form is $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

i.e.,
$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

which gives $(\lambda - 3)(\lambda - 1)(\lambda - 4) = 0$ so that the eigen values are $\lambda = 3,\ \lambda = 1,\ \lambda = 4$ which are all different.

When $\lambda = 3$, we have
$$-y = 0$$
$$-x - y - z = 0$$
$$-y = 0$$

$\therefore\ y = 0, x = -z.$ The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

When $\lambda = 1$, the vector is given by
$$2x - y = 0$$
$$-x + y - z = 0$$
$$-y + 2z = 0$$

Let $z = k$ so that $y = 2k$ and $2x = 2k \Rightarrow x = k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus the corresponding eigen vector is $X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Similarly for $\lambda = 4$, we have eigen vector is $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

We observe that these 3 vectors are mutually orthogonal. We normalize these vectors and obtain

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, e_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, e_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Let P = The modal matrix in normalised form $= [e_1 \ \ e_2 \ \ e_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Diagonalization: Since P is orthogonal matrix $P^{-1} = P^T$ so $P^TAP = D$ where D is the diagonal matrix.

$$\therefore \ D = P^TAP = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$\therefore \ D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ = diag [3, 1, 4] and the quadratic form will be reduced to the normal form

$Y^TDY = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 + y_2^2 + 4y_3^2$ by the orthogonal transformation $X = PY$.

i.e.,
$$x = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{3}}$$
$$y = \frac{2}{\sqrt{6}} y_2 - \frac{1}{\sqrt{3}} y_3$$
$$z = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{3}} y_3$$

Example 2 : Find the eigen vectors of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ and hence reduce $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ to a sum of squares.

[JNTU 2000, (A) June 2009, Nov. 2010 (Set No. 2), (H) June 2011 (Set No. 2)]

Solution : Let A be the given matrix. Its characteristic equation is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

On simplification, this becomes

$$(\lambda - 2)^2 (\lambda - 8) = 0 \quad \therefore \lambda = 2, 2, 8$$

Hence 2, 2, 8 are the eigen values of the matrix A.

Eigen vector corresponding to $\lambda = 2$

Consider $\begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(*i.e.*) $\left.\begin{aligned} 4x - 2y + 2z &= 0 \\ -2x + y - z &= 0 \\ 2x - y + z &= 0 \end{aligned}\right\}$

Solving these, we get $y = 2x + z$

Take $x = \alpha$, $z = \delta$. Then we have

$$x = \alpha$$
$$y = 2\alpha + \delta$$
$$z = \delta$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are two eigen vectors corresponding to $\lambda = 2$.

Eigen vector corresponding to $\lambda = 8$

Consider the equations

$$\begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(*i.e.*) $-2x - 2y + 2z = 0$

$-2x - 5y - 2 = 0$

$2x - y - 5z = 0.$

The augmented matrix is

$$\left(\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ -2 & -5 & -1 & 0 \\ 2 & -1 & -5 & 0 \end{array}\right)$$

$$\simeq \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -5 & -1 & 0 \\ 2 & -1 & -5 & 0 \end{array}\right) \text{ by } R_1 \to \frac{R_1}{-2}$$

$$\simeq \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \end{pmatrix} \text{ by } R_2 \to R_2 + 2R_1 \text{ and } R_3 \to R_3 - 2R_1$$

$$\simeq \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \text{ by } R_2 \to \frac{R_2}{-3} \text{ and } R_3 \to \frac{R_3}{-3}$$

$$\simeq \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ by } R_1 \to R_1 - R_2 \text{ and } R_3 \to R_3 - R_2$$

$\Rightarrow x - 2z = 0$ and $y + z = 0$

$\Rightarrow x = 2z$ and $y = -z$

Take $z = \beta \cdot$ Then $x = 2\beta,\ y = -\beta$ and $z = \beta$

Hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \beta \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 8$.

Taking $\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 8$, corresponding eigen vectors are

$X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ linearly independent

(these are not pairwise orthogonal)

Define $P = (X_1\ X_2\ X_3) = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Calculating P^{-1}, we get $P^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -2 \\ -2 & 1 & 5 \\ 2 & -1 & 1 \end{pmatrix}$ (work out ?)

Then $P^{-1}AP = \text{diag}\,(2 \quad 2 \quad 8)$

Hence the above matrix P is a modal matrix matrix of the matrix A.

Since P is not an orthogonal matrix this is not useful to transform $X^T AX$ to normal form.

Hence consider the eigen vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Corresponding to $\lambda = 2$, $\alpha \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

is also an eigen vector corresponding to $\lambda = 2$.

Let us choose α, δ such that this is orthogonal to $\begin{pmatrix}0\\1\\1\end{pmatrix}$

(*i.e.*) $\begin{pmatrix}\alpha\\2\alpha+\delta\\\delta\end{pmatrix}$ is orthogonal to $\begin{pmatrix}0\\1\\1\end{pmatrix}$

Hence $0+2\alpha+\delta+\delta=0 \quad \Rightarrow \alpha=-\delta$

(*i.e.*) $\begin{pmatrix}-\delta\\-\delta\\\delta\end{pmatrix} \simeq \begin{pmatrix}1\\1\\-1\end{pmatrix}$ is also an eigen vector corresponding to $\lambda = 2$.

Thus $X_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}, X_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, X_3 = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

are pairwise orthogonal eigen vectors of A corresponding to $\lambda = 2, 2, 8$.

The normalized eigen vectors are

$$e_1 = \frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}, e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

Hence the normalized modal matrix is

$$P = \begin{pmatrix}\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

We can verify that $P^TP = I = PP^T$ and $P^TAP = \text{diag}\,(2, 2, 8)$

Consider the orthogonal transformation $X = PY$. Sustituting it, we get

$$X^TAX = Y^TP^TAPY = Y^T(P^TAP) = Y^TDY$$

$$= Y^T\begin{pmatrix}2&0&0\\0&2&0\\0&0&8\end{pmatrix}Y = \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&8\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = 2y_1^2+2y_2^2+8y_3^2$$

This is the required normal form.

Note : Sum of squares $= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 2y_1^2 + 2y_2^2 + 8y_3^2$

r = rank = no. of non - zero terms = 3

s = index = no. of + ve terms = 3

Signature = $2s - r = 6 - 3 = 3$

Example 3 : Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal reduction.

[JNTU 2005S, 2006S, 2008 (Set No. 3), (K) June 2009, May 2010 (Set No.2), (A) May 2011]

Solution : Comparing the given quadratic form with

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

we have $a_{11} = 3;\ a_{22} = 5;\ a_{33} = 3$

$2a_{12} = -2 \Rightarrow a_{12} = -1 = a_{21}$

$2a_{13} = 2 \Rightarrow a_{13} = 1 = a_{31}$

$2a_{23} = -2 \Rightarrow a_{23} = -1 = a_{32}$

The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[-1(3-\lambda)+1]+1[1-(5-\lambda)]=0$

$\Rightarrow 3-\lambda[\lambda^2 - 8\lambda + 14]+[\lambda-2]+[\lambda-4]=0$

$\Rightarrow 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$

$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0 \Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\Rightarrow (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow (\lambda-2)(\lambda-3)(\lambda-6) = 0$

$\therefore\ \lambda = 2, 3, 6$

The eigen values of A are 2, 3, 6

The corresponding eigen vectors are given by (A – λI) X = O

$$i.e., \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Case (1) : Let λ = 2. Then

$$(1) \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 + x_3 = 0 \quad \ldots (2)$

$-x_1 + 3x_2 - x_3 = 0 \quad \ldots (3)$

$x_1 - x_2 + x_3 = 0 \quad \ldots (4)$

Solve (2) & (3)

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{matrix}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} = k_1.$$

or $\dfrac{x_1}{-1} = \dfrac{x_2}{0} = \dfrac{x_3}{1} = k_1.$

$$\therefore \quad X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \hat{X}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Case (2) : Let $\lambda = 3$. Then

$$(1) \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -x_2 + x_3 = 0 \quad \text{... (5)}$$

$$-x_1 + 2x_2 - x_3 = 0 \quad \text{... (6)}$$

$$x_1 - x_2 = 0 \quad \text{... (7)}$$

Solve (5) & (6)

$$\begin{matrix} -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{matrix}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} = k_2.$$

$$\therefore X_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \ \hat{X}_2 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

Case (3) : Let $\lambda = 6$. Then

$$(1) \Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -3x_1 - x_2 + x_3 = 0 \quad \text{... (8)}$$

$$-x_1 - x_2 - x_3 = 0 \quad \text{... (9)}$$

$$x_1 - x_2 - 3x_3 = 0 \quad \text{... (10)}$$

Solve (8) & (9)

$$\begin{matrix} -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} = k_3. \quad \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3.$$

$$\therefore \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ \hat{X}_3 \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Here $\hat{X}_1, \hat{X}_2, \hat{X}_3$ are pairwise orthogonal.

The modal matrix is $[X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

The normalised modal matrix is $P = [\hat{X}_1 \ \hat{X}_2 \ \hat{X}_3] \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

This is an orthogonal matrix.

Diagonalised matrix $D = P^{-1}AP = P^T AP$

$$\therefore \quad D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\text{or} \quad D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & \sqrt{3} & \sqrt{6} \\ 0 & \sqrt{3} & -2\sqrt{6} \\ \sqrt{2} & \sqrt{3} & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Canonical form = $Y^T DY$

$$= [y_1 \ \ y_2 \ \ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [y_1 \ \ y_2 \ \ y_3] \begin{bmatrix} 2y_1 \\ 3y_2 \\ 6y_3 \end{bmatrix} = 2y_1^2 + 3y_2^2 + 6y_3^2$$

which is the required canonical form.

The orthogonal transformation which reduces the quadratic form $X^T AX$ to canonical form is given by $X = PY \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Example 4 : Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into sum of squares form by an orthogonal transformation and give the matrix of transformaton.
[JNTU 2003 (Set No.3), 2005 (Set No.1), Sep. 2008 (Set No.2)]

Solution : The given quadratic form is $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

The matrix of the given transformation is $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)[(3-\lambda)(3-\lambda)-1]-1[(3-\lambda)+1]+1[-1-(3-\lambda)]=0$

$\Rightarrow (\lambda-1)(\lambda-4)^2 = 0$

$\therefore \lambda = 1, 4, 4$

Let $\lambda = 1$. Then the eigen vector is given by $(A-\lambda)X = O \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to 2R_2 - R_1, R_3 \to 2R_3 - R_1$ gives $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$ gives $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad 2x_1 + x_2 + x_3 = 0$

and $3x_2 - 3x_3 = 0 \Rightarrow x_2 = x_3$

Let $x_3 = k \Rightarrow x_2 = k$ and $x_1 = -k$ .Then

$X_1 = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

Similarly put $\lambda = 4$, we get $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$ gives $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1 + x_2 + x_3 = 0$ or $x_1 - x_2 - x_3 = 0$

Let $x_3 = k_1$ and $x_2 = k_2$. Then $x_1 = k_1 + k_2$

$\therefore \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Hence $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = 4$.

Consider $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

i.e., $\begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow a + b + a = 0 \Rightarrow 2a + b = 0 \Rightarrow b = -2a$

Required vector is

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a - 2a \\ -2a \\ a \end{bmatrix} = \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ The vector orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Hence $\hat{X}_1 = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \hat{X}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \hat{X}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$ are the three normalised vectors.

Take, $P = [\hat{X}_1 \;\; \hat{X}_2 \;\; \hat{X}_3] = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 0 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$

Since P is orthogonal, we have $P^T = P^{-1}$

Thus $D = P^{-1}AP = P^T AP$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 0 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{diag}\,(1,4,4)$$

$$Q = Y^T DY = [y_1 \;\; y_2 \;\; y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [y_1 \;\; 4y_2 \;\; 4y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 4y_2^2 + 4y_3^2$$

This is the required canonical form.

The Orthogonal transformation which reduces the quadratic form to canonical form is given by X = PY

i.e., $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\Rightarrow \; x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3 \; ; \quad x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3 ; \quad x_3 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

Here P is the matrix of transformation.

Example 5 : Find the orthogonal transformation which transforms the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to the canonical form.

Solution : Comparing the given quadratic form with

$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$, we have

$a_{11} = 1;\ a_{22} = 3;\ a_{33} = 3$

$2a_{12} = 0 \Rightarrow a_{12} = 0 = a_{21}$

$2a_{13} = 0 \Rightarrow a_{13} = 0 = a_{31}$

$2a_{23} = -2 \Rightarrow a_{23} = -1 = a_{32}$

The matrix of the given quadratic form is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\,[(3-\lambda)^2 - 1] = 0 \Rightarrow (1-\lambda)\,(\lambda-4)\,(\lambda-2) = 0$

$\therefore\ \lambda = 1, 2, 4$

$\therefore$ The eigen values of A are 1, 2, 4.

The eigen vector of A corresponding to $\lambda = 1$ is given by

$(A - I)X = O$

$i.e.\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$i.e.\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (Applying $R_3 \to 2R_3 + R_2$)

$\Rightarrow 2x_2 - x_3 = 0, 3x_3 = 0 \Rightarrow x_3 = 0, x_2 = 0$

Let $x_1 = k_1$. Then

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 1$,

where k_1 is an arbitrary constant.

The eigen vector of A corresponding to $\lambda = 2$ is given by

$(A - 2I)X = O \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$i.e.\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (Applying $R_3 \to R_3 + R_2$)

$-x_1 = 0,\ x_2 - x_3 = 0 \Rightarrow x_1 = 0$ and $x_2 = x_3$

Let $x_3 = k_2 \Rightarrow x_2 = k_2$. Then

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ is the eigen vector.}$$

The eigen vector of A corresponding to $\lambda = 4$ is given by

$$(A-4I)X = O \Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{Applying } R_3 \to R_3 - R_2)$$

$\Rightarrow -3x_1 = 0 \Rightarrow x_1 = 0$; and $-x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$

Let $x_3 = k_3 \Rightarrow x_2 = -k_3$. Then

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ is the eigen vector}$$

$$\text{Modal matrix } = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{Normalised modal matrix } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Diagonalised matrix, $D = P^{-1}AP = P^T AP$

$$\therefore \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & \sqrt{2} & 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The canonical form of the given quadratic form is

$$Y^T DY = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 \quad 2y_2 \quad 4_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2$$

The othogonal transformation which reduces the quadratic form to canonical form is given by X = PY

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow x_1 = y_1, \; x_2 = \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{2}} y_3 = \frac{1}{\sqrt{2}} [y_2 - y_3] \text{ and } x_3 = \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{2}} y_3 = \frac{1}{\sqrt{2}} [y_2 + y_3]$$

This is the orthogonal transformation which reduces the given quadratic form to canonical form.

Example 6 : Reduce the quadratic form to canonical form by an orthogonal reduction and state the nature of the quadratic form $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$.

[JNTU 2008 (Set No. 2), Nov. 2010 (Set No. 4)]

Solution : The quadratic form can be written in matrix form as

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2 - \lambda) [(2 - \lambda)^2 - 1] + 1 [\lambda - 2 - 1] - 1 [1 - \lambda + 2] = 0$

$\Rightarrow \quad (2 - \lambda) [4 + \lambda^2 - 4\lambda - 1] + \lambda - 3 + \lambda - 3 = 0$

$\Rightarrow \quad 6 + 2\lambda^2 - 8\lambda - 3\lambda - \lambda^2 + 4\lambda^2 + 2\lambda - 6 = 0$

$\Rightarrow\ -\lambda^3 + 6\lambda^2 - 11\lambda + 2\lambda = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda = 0$

$\Rightarrow\ \lambda(\lambda^2 - 6\lambda + 9) = 0 \Rightarrow \lambda = 0$ or $\lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow\ \lambda = 0$ or $(\lambda - 3)(\lambda - 3) = 0.$

$\therefore\ \lambda = 0, 3, 3,$ are the eigen values of A.

Hence the given quadratic form is positive semi-definite.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to λ. Then

$(A - \lambda I)X = O$

$$i.e., \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Case (1) : Let $\lambda = 0$. Then

$$(1) \Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow\ 2x_1 - x_2 - x_3 = 0 \quad \ldots (2)$$

$$-x_1 + 2x_2 - x_3 = 0 \quad \ldots (3)$$

$$-x_1 - x_2 + 2x_3 = 0 \quad \ldots (4)$$

Solve (2) & (3),
$$\begin{matrix} -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 \end{matrix}$$

$$\therefore\ \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} = k_1 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k_1.$$

$$\therefore\ X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{or}\quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad [\because \text{Put } k_1 = 1]$$

Normalised eigen vector is, $\hat{X}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Case (2) : $\lambda = 3$

$$(1) \Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -x_1 - x_2 - x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \end{aligned}\right\} \qquad \text{.... (5)}$$

These three are identical and give only one independent equation $x_1 + x_2 + x_3 = 0$.

Let $x_2 = k_2$ and $x_3 = k_3$. Then

$x_1 = -k_3 - k_2$

$$\text{Now} \quad X_2 = \begin{bmatrix} -k_2 - k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Consider $a\,(X_2) + b\,(X_3)$ orthogonal to X_2

i.e., $a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ orthogonal to $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

i.e., $\begin{bmatrix} -a-b \\ a \\ b \end{bmatrix}$ orthogonal to $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

$\Rightarrow a + b + a = 0 \Rightarrow b = -2a$

$\therefore$ The required vector is $a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 2a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a \\ -2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Thus $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ are the eigen vectors corresponding to $\lambda = 3$.

Normalised eigen vectors are $\hat{X}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\hat{X}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}$

The normalised modal matrix is $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$.

Since P is orthogonal, $P^T = P^{-1}$ and

$D = P^{-1}AP = P^TAP$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -3/\sqrt{2} & 3/\sqrt{2} & 0 \\ 3/\sqrt{6} & 3/\sqrt{6} & -6/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The canonical form is $Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_2^2 + 3y_3^2$

This is possible through the transformation $X = PY$

i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Example 7 : Reduce the quadratic form to canonical form by an orthogonal reduction and state the nature of the quadratic form $5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy$

[JNTU (H) June 2009 (Set No.3)]

Solution : The matrix of the given quadratic form is

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

We will find characteristic roots and vectors.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 3 & 7 \\ 3 & 26-\lambda & 2 \\ 7 & 2 & 10-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)[(26-\lambda)(10-\lambda)-4] - 3[30-3\lambda-14] + 7[6-182+7\lambda] = 0$$

$$\Rightarrow -\lambda^3 + 41\lambda^2 - 378\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 41\lambda + 378) = 0$$

$$\therefore \lambda = 0, \lambda = 14, \lambda = 27.$$

Eigen values are all different.

Since one eigen value is zero and other eigen values are positive, the given quadratic form is positive semi-definite.

Case (*i*) : Let $\lambda = 0$. Then

$$(A - \lambda I)X = O$$

$$\Rightarrow AX = O$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5R_2 - 3R_1; 5R_3 - 7R_1$ gives

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dfrac{R_2}{-11}$ gives

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & -11 & 1 \\ 0 & -11 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$ gives

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & -11 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $5x + 3y + 7z = 0$

$-11y + z = 0$

Let $z = k$. Then $y = \frac{z}{11} = \frac{k}{11}$

$$\Rightarrow \quad x = \frac{-16}{11}k$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{16}{11}k \\ \frac{k}{11} \\ k \end{bmatrix} = \frac{k}{11}\begin{bmatrix} -16 \\ 1 \\ 11 \end{bmatrix}$$

$\therefore$ $X_1 = \begin{bmatrix} -16 \\ 1 \\ 11 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 0$.

Case (*ii*) : $\lambda = 14$.

The corresponding eigen vector is given by $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 5-14 & 3 & 7 \\ 3 & 26-14 & 2 \\ 7 & 2 & 10-14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -9 & 3 & 7 \\ 3 & 12 & 2 \\ 7 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3R_2 + R_1 ; 9R_3 + 7R_1$ gives

$$\begin{bmatrix} -9 & 3 & 7 \\ 0 & 39 & 13 \\ 0 & 39 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$ gives

$$\begin{bmatrix} -9 & 3 & 7 \\ 0 & 39 & 13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{13}$ gives

$$\begin{bmatrix} -9 & 3 & 7 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -9x + 3y + 7z = 0$$
$$3y + z = 0$$

Let $z = k \Rightarrow y = -\frac{k}{3}$

$-9x = -3y - 7z = k - 7k = -6k \Rightarrow x = \frac{2}{3}k$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}k \\ -\frac{k}{3} \\ k \end{bmatrix} = \frac{k}{3}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\therefore \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 14$.

Case (*iii*): $\lambda = 27$. The corresponding eigen vector is given by $(A - 27I)X = O$

i.e., $\begin{bmatrix} 5-27 & 3 & 7 \\ 3 & 26-27 & 2 \\ 7 & 2 & 10-27 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -22 & 3 & 7 \\ 3 & -1 & 2 \\ 7 & 2 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$22R_2 + 3R_1; 22R_3 + 7R_1$ gives

$$\begin{bmatrix} -22 & 3 & 7 \\ 0 & -13 & 65 \\ 0 & 65 & -325 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{13}; \frac{R_3}{-65}$ gives

$$\begin{bmatrix} -22 & 3 & 7 \\ 0 & -1 & 5 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$ gives

$$\begin{bmatrix} -22 & 3 & 7 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k$

$-y + 5z = 0 \;\Rightarrow\; y = 5z = 5k$

Now $22x = 3y + 7z = 15k + 7k = 22k \;\Rightarrow\; x = k$

$$\therefore \quad X = \begin{bmatrix} k \\ 5k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

$\therefore \quad X_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 27$

Thus $X_1 = \begin{bmatrix} -16 \\ 1 \\ 11 \end{bmatrix}, \; X_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \; X_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

We observe that the three vectors are mutually orthogonal.
We normalize these eigen vectors.

$$e_1 = \begin{bmatrix} \frac{-16}{\sqrt{378}} \\ \frac{1}{\sqrt{378}} \\ \frac{11}{\sqrt{378}} \end{bmatrix}, \; e_2 = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}, \; e_3 = \begin{bmatrix} \frac{1}{\sqrt{27}} \\ \frac{5}{\sqrt{27}} \\ \frac{1}{\sqrt{27}} \end{bmatrix}$$

The normalised Modal matrix is $P = [e_1 \;\; e_2 \;\; e_3] = \begin{bmatrix} \frac{-16}{\sqrt{378}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{27}} \\ \frac{1}{\sqrt{378}} & -\frac{1}{\sqrt{14}} & \frac{5}{\sqrt{27}} \\ \frac{11}{\sqrt{378}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{27}} \end{bmatrix}$

Diagonalization: Since P is orthogonal matrix, $P^{-1} = P^T$ so $P^T AP = D$ where D is the diagonal matrix.

$$\Rightarrow \quad D = P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 27 \end{bmatrix} \quad \textit{i.e.}, D = \text{diag}(0, 14, 27)$$

$\therefore$ The Quadratic form reduces to canonical form = $Y^TDY = 14y_2^2 + 27y_3^2$

Index of Quadratic Form = no. of + ve terms in normal form = 2.

Example 8 : Reduce the quadratic form to the canonical form $2x^2 + 2y^2 + 2z^2 - 2xy + 2zx - 2yz$ **[JNTU (H) Jan 2012 (Set No. 4)]**

Solution : Given Q. F. is $2x^2 + 2y^2 + 2z^2 - 2xy + 2zx - 2yz$

The matrix of the Q. F. is $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1] + 1[\lambda - 2 + 1] + 1(1 - 2 + \lambda)$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$ or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\lambda = 1, 1, 4$ are the roots.

Eigen vector corresponding to the value $\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0$. Take $x_3 = k_1$ and $x_2 = k_2$

Then $x_1 = x_2 - x_3 = k_2 - k_1$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 - k_1 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. But these are not orthogonal.

Consider $a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -a+b \\ b \\ a \end{bmatrix}$ orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$a-b+a=0$ or $2a=b$

$$a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}+2a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Take normalized vectors $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ and $\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ for the value $\lambda=1$

For $\lambda=4$, the corresponding eigen vector is given by

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_3+R_2 gives $\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -3x_2-3x_3=0 \Rightarrow x_2=-x_3$

Let $x_2=k \Rightarrow x_3=-k$

$-2x_1-x_2+x_3=0 \Rightarrow 2x_1=-x_2+x_3=-k-k=-2k \Rightarrow x_1=-k$

$$\therefore X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -k \\ k \\ -k \end{bmatrix}=-k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Normalized vector is $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Consider $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Since P is orthogonal matrix, $P^T = P^{-1}$, we have $P^T AP = D$

i.e., $\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \text{diag}\,(1,1,4)$

The required canonical form is $(y_1^2 + y_2^2 + 4y_3^2)$

Example 9 : Reduce the quadratic form $x^2 + 4xy + y^2$ to the canonical form by orthogonal reduction. Find the index, signature and nature of the quadratic form.

[JNTU (A) May 2011 (Set No. 3)]

Solution : Given Q. F. is $x^2 + 4xy + y^2$

Its matrix is $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Charecteristic polynomial is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)^2 - 4$$

$$= 1 + \lambda^2 - 2\lambda - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

$\therefore \quad \lambda = 3, -1$ are the eigen values,

Case I : Let $\lambda = 3$. Then

$(A - 3I)X = O$

$\Rightarrow \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x+2y=0$

$2x-2y=0$

Let $x=y=k$

$$X=\begin{bmatrix}-x\\y\end{bmatrix} = k\begin{bmatrix}1\\1\end{bmatrix}$$

$X_1=\begin{bmatrix}1\\1\end{bmatrix}$ is the corresponding eigen vector.

Case II : Let $\lambda=-1$. Then

$(A-I)(X)=O$

$$\Rightarrow \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow 2x+2y=0 \Rightarrow x+y=0$

Let $x=k \Rightarrow y=-k$

$$X=\begin{bmatrix}x\\y\end{bmatrix}=k\begin{bmatrix}1\\-1\end{bmatrix}$$

$X_2=\begin{bmatrix}1\\-1\end{bmatrix}$ is the corresponding eigen vector.

X_1 and X_2 are orthogonal.

Normalizing the eigen vectors,

$$e_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}, e_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$$

Let $P=[e_1\ e_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$. Then

$$P^{-1}=P^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Diagonal Matrix $D=P^TAP$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \text{diag}(3, -1)$$

The normal form is $3y_1^2 - y_2^2$

$$Y^T DY = [y_1 \; y_2] \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

index = number of positive terms in its normal form

$\therefore$ Index $s = 1$

$r =$ rank $=$ no. of non-zero terms in the normal form $= 2$

Signature $= 2s - r = 2 - 2 = 0$ and $n = 2$

Here $r = n$ and $s = 0$.

$\therefore$ The quadritic form is negative definite.

Example 10 : Reduce the quadratic form $q = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_2x_3$ into a canonical form by Orthogonal reduction. Find the index, signature and nature of the quadratic form.

[JNTU (A) May 2012 (Set No. 1)]

Solution : Given quadratic form is $q = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_2x_3$

The matrix of the Q. F is $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)^2 - 1] = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$$\Rightarrow (2-\lambda)(\lambda - 3)(\lambda - 1) = 0$$

$\therefore$ $\lambda = 1, \lambda = 2, \lambda = 3$ are the eigen values

Case I : $\lambda = 1$. The eigen vector is given by $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the matrix, $x_1 = 0; x_2 + x_3 = 0$

Let $x_3 = k, x_2 = -k$. Then

$X = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

Normalized Eigen vector is $\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Case II : $\lambda = 2$. The eigen vector is given by $(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the matrix, $x_2 = 0, x_3 = 0$, Let $x_1 = k$

$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the normalised vector

Case III : $\lambda = 3$. The eigen vector is given by $(A - 3I)X = O$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0$ and $-x_2 + x_3 = 0 \Rightarrow x_2 = x_3$.

$\therefore \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding $\lambda = 3$. $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ is the normalised vector

Take, $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ as the modal matrix

This is an orthogonal matrix . $\therefore\ P^{-1} = P^T$

Then $P^{-1} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Then $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The canonical form is $y_1^2 + 2y_2^2 + 3y_3^2$

For the quadratic form ,

$s =$ Index $=$ no. of positive terms $= 3$

$r =$ rank $=$ number of non-zero terms $= 3$

Signature $= 2s - r = 3$

Example 11 : Reduce the following quadratic form by orthogonal reduction and obtain the corresponding transformation. Find the index, signature and nature of the quadratic form $q = 2xy + 2yz + 2zx$. **[JNTU (A) May 2012 (Set No. 2)]**

Solution : Given quadratic form is $q = 2xy + 2yz + 2zx$

Matrix of the Q. F. is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characterstic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda^3 + 3\lambda + 2 = 0$

$\therefore\ \lambda = -1, -1, 2$ are the eigen values.

Case I : For $\lambda = -1$ the characterstic vector is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 0$

Let $x_3 = k_1$ and $x_2 = k_2$. Then $x_1 = -k_1 - k_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are the eigen vectors, corresponding to $\lambda = -1$.

But these two vectors are not orthogonal .

Let , $a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -a-b \\ b \\ a \end{bmatrix}$ is orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$a + b + a = 0 \Rightarrow b = -2a$

$\therefore a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 2a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+2a \\ -2a \\ a \end{bmatrix} = \begin{bmatrix} a \\ -2a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is the required vector.

$\therefore$ Normalized eigen vector $= \begin{bmatrix} \frac{1}{\sqrt{4}} \\ -\frac{2}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-2}{2} \\ \frac{1}{2} \end{bmatrix}$

Case II : $(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + R_2 \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_2 + 3x_3 = 0 \Rightarrow x_2 = x_3$

Let $x_2 = x_3 = k$

$-2x_1 + x_2 + x_3 = 0 \Rightarrow 2x_1 = x_2 + x_3 = 2k$

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized eigen vector is $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Normalised modal matrix, $P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Since P is orthogonal $P^{-1} = P^T$

Thus we have $P^T AP = D$ where D is the diagonal matrix.

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{-2}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} A \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The canonical form of Q. F. is $-x_1^2 - x_2^2 + 2x_3^2$

$r =$ Rank $=$ no. of non-zero terms $= 3$

s $=$ index $=$ no. of positive terms $= 1$

Signature $=$ $2s - r = 2 - 3 = -1$

$X = PY$ is the linear transformation

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Thus taking,

$$x = \frac{-1}{\sqrt{2}} x_1 + \frac{1}{2} x_2 + \frac{1}{\sqrt{3}} x_3$$

$$y = \frac{-1}{2} x_2 + \frac{1}{\sqrt{3}} x_3$$

$z = \frac{1}{\sqrt{2}} x_1 + \frac{1}{2} x_2 + \frac{1}{\sqrt{3}} x_3$, we reduced the given Q. F. into canonical form.

Example 12 : Reduce the quadratic form, $q = 3x^2 - 2y^2 - z^2 - 4xy + 12yz + 8xz$ to the canonical form by orthogonal reduction. Find its rank, index and signature. Find also the corresponding transformation. **[JNTU (A) May 2012 (Set No. 3)]**

Solution : Given quadratic form is $q = 3x^2 - 2y^2 - z^2 - 4xy + 12yz + 8xz$

The matrix of the Q. F. is $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & -2-\lambda & 6 \\ 4 & 6 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)[(2+\lambda)(1+\lambda) - 36] + 2[(2+2\lambda) - 24] + 4[-12 + 8 + 4\lambda] = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 + 3\lambda + 2 - 36) + 2[2\lambda - 22] + 4(4\lambda - 4) = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 + 3\lambda - 34) + 4\lambda - 44 + 16\lambda - 16 = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 + 3\lambda - 34) - 20(3-\lambda)$$

$$(3-\lambda)(\lambda^2 + 3\lambda - 54) = 0$$

$$(3-\lambda)(\lambda + 9)(\lambda - 6) = 1$$

$\lambda = 3, \lambda = 6, \lambda = -9$ are the roots.

Eigen vector corresponding to $\lambda = 3$ is given by

$(A - 3I)X = O$

$$\Rightarrow \begin{bmatrix} 0 & -2 & 4 \\ -2 & -5 & 6 \\ 4 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 2R_2$ gives, $\begin{bmatrix} 0 & -2 & 4 \\ -2 & -5 & 6 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 - 2R_1$ gives, $\begin{bmatrix} 0 & -2 & 4 \\ -2 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_2 + 4x_3 = 0 \Rightarrow x_2 = 2x_3$

Let $x_3 = k$. Then $x_2 = 2k$

and $-2x_1 - 5x_2 + 6x_3 = 0$

$$\Rightarrow \quad 2x_1 = -5x_2 + 6x_3$$
$$= -10k + 6k$$
$$= -4k$$

$\Rightarrow \quad x_1 = -2k$

$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Thus

$\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

The normalized eigen vector is $\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$.

Eigen vector corresponding to $\lambda = 6$

$$\begin{bmatrix} -3 & -2 & 4 \\ -2 & -8 & 6 \\ 4 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3R_2 - 2R_1$; $3R_3 + 4R_1$ gives $\begin{bmatrix} -3 & -2 & 4 \\ 0 & -20 & 10 \\ 0 & 10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$10x_2 - 5x_3 = 0 \Rightarrow x_3 = 2x_2$

Let $x_2 = k \Rightarrow x_3 = 2k$

$-3x_1 - 2x_2 + 4x_3 = 0$

$-3x_1 - 2k + 8k = 0 \Rightarrow x_1 = 2k$

$-3x_1 + 6k = 0$

$$\begin{bmatrix} 2k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

$\therefore \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is the corresponding vector and the normalized eigen vector is $\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

Eigen vector corresponding to $\lambda = -9$

$$\begin{bmatrix} 12 & -2 & 4 \\ -2 & 7 & 6 \\ 4 & 6 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{matrix} 6R_2 + R_1; \\ 3R_3 - R_1 \end{matrix}$ gives $\begin{bmatrix} 12 & -2 & 4 \\ 0 & 40 & 40 \\ 0 & 20 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

Let $x_3 = k$. Then $x_2 = -k$.

$12x_1 - 2x_2 + 4x_3 = 0 \Rightarrow 12x_1 + 2x + 4k = 0 \Rightarrow 12x_1 = -6k \Rightarrow x_1 = -\frac{1}{2}k$.

$\therefore$ The eigen vector is $\begin{bmatrix} -\frac{1}{2}k \\ -k \\ k \end{bmatrix} = \frac{-k}{2}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\therefore \begin{bmatrix} 1/3 \\ 2/3 \\ -(2/3) \end{bmatrix}$ is the normalized eigen vector.

Consider $P = \begin{bmatrix} \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$ as modal matrix.

Since P is orthogonal, we have $P^{-1} = P^T$

Then we have $P^T AP = D$

$$\Rightarrow \begin{bmatrix} \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} A \begin{bmatrix} \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} = \text{diag}(3, 6, -9)$$

The row form of the Q. F. is $3y_1^2 + 6y_2^2 - 9y_3^2$.

This is from done by $X = PY$

$$i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Rank of the Q. F. $= r =$ no. of non-zero terms $= 3$

Index $= s =$ no. of $+ve$ terms $= 2$

Signature $= 2s - r = 1$

Example 13 : Reduce the quadratic form $q = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 - 4x_2x_3 + 6x_3x_1$ into a canonical form by diagonalising the matrix of the quadratic form.

[JNTU (A) May 2012 (Set No. 4)]

Solution : Given Q. F. is $q = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 - 4x_2x_3 + 6x_3x_1$

The matrix of the Q. F. is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & -2 & 1 \end{bmatrix}$

We write $A = I_3 A I_3$.

We will use elementary matrix operators to reduce the matrix in LHS to diagonal form.

We will apply same row operations on the prefactor of A and same column operation on post factor of A in RHS.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 - 2R_1; \\ R_3 - 3R_1 \end{matrix} \text{ gives } \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -8 \\ 0 & -8 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1;$
$C_3 - 3C_1$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -8 \\ 0 & -8 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3R_3 - 8R_2$ gives, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -8 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -8 & 3 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3C_3 - 8C_2$ gives, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -8 & 3 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}$.

This is of the form $D = PAP^T$.
The matrix on LHS is in diagonal form. Using this diagonal matrix,
the new form of the Q. F. is $y_1^2 - 3y_2^2 + 40y_3^2$

The modal matrix is, $P = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}$

$X = PY$ is the transform which changes given Q. F. to canonical form

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow x = y_1 - 2y_2 + 7y_3; y = y_2 - 8y_3, z = 3y_3$
Rank of the Q. F. $= r =$ no. of non-zero terms $= 3$
Index $=$ no. of $+ve$ terms $= s = 2$
Signature $= 2s - r = 4 - 3 = 1$

3.13 REDUCTION TO CANONICAL FORM USING LAGRANGE'S METHOD

Procedure to Reduce Quadratic Form to canonical form:

1. Take the common terms from product terms of given quadratic form.
2. Make perfect squares suitably by regrouping the terms
3. The resulting relation gives the required canonical form.

SOLVED EXAMPLES

Example 1 : Reduce the quadratic form $x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz$ to canonical form by Lagrange's reduction. **[JNTU 2006 (Set No.2)]**

Solution :

Given Quadratic Form $= x^2 - 2xy + 4xz + y^2 + 2z^2 + 4yz$

$$= (x-y+2z)^2 - 2z^2 + 8yz$$

$$= (x-y+2z)^2 - 2(z^2 - 4yz)$$

$$= (x-y+2z)^2 - 2(z-2y)^2 + 8y^2$$

$$= y_1^2 - 2y_2^2 + 8y_3^2$$

where $y_1 = x - y + 2z$, $y_2 = z - 2y$ and $y_3 = y$
which is the required canonical form.

Example 2 : Using Lagrange's reduction, transform $x_1^2 - 4x_2^2 + 5x_3^2 + 2x_1x_2 - 4x_1x_3 + 2x_4^2 - 6x_3x_4$ to canonical form.

Solution :

Given Quadratic form $= (x_1^2 + 2x_1x_2 - 4x_1x_3) - 4x_2^2 + 6x_3^2 + 2x_4^2 - 6x_3x_4$

$$= (x_1 + x_2 - 2x_3)^2 + x_2^2(-4-1) + x_3^2(6-4) + 2x_4^2 - 6x_3x_4 + 4x_2x_3$$

$$= (x_1 + x_2 - 2x_3)^2 - 5x_2^2 + 4x_2x_3 + 2x_3^2 + 2x_4^2 - 6x_3x_4$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(25x_2^2 - 20x_2x_3) + 2x_3^2 + 2x_4^2 - 6x_3x_4$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(5x_2 - 2x_3)^2 + x_3^2(2 + \frac{4}{5}) + 2x_4^2 - 6x_3x_4$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(5x_2 - 2x_3)^2 + \frac{14}{5}x_3^2 + 2x_4^2 - 6x_3x_4$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(5x_2 - 2x_3)^2 + \frac{5}{14}\left[\left(\frac{14}{5}\right)^2 x_3^2 - 6\cdot\frac{14}{5}x_3x_4\right] + 2x_4^2$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(5x_2 - 2x_3)^2 + \frac{5}{14}\left(\frac{14}{5}x_3 - 3x_4\right)^2 - 9\cdot\frac{5}{14}x_4^2 + 2x_4^2$$

$$= (x_1 + x_2 - 2x_3)^2 - \frac{1}{5}(5x_2 - 2x_3)^2 + \frac{5}{14}\left(\frac{14}{5}x_3 - 3x_4\right)^2 - \frac{17}{4}x_4^2$$

$$= y_1^2 - \frac{1}{5}y_2^2 + \frac{5}{14}y_3^2 - \frac{17}{4}y_4^2$$

where $y_1 = x_1 + x_2 - 2x_3$, $y_2 = 5x_2 - 2x_3$, $y_3 = \frac{14}{5}x_3 - 3x_4$ and $y_4 = x_4$ which is the required canonical form.

Example 3 : Reduce the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ to canonical form by Lagrange's reduction.

Solution : Given Quadratic form

$$= 6\left[x_1^2 - \frac{2}{3}x_1(x_2 - x_3)\right] + 3x_2^2 + 3x_3^2 - 2x_2x_3$$

$$= 6\left[x_1 - \frac{1}{3}(x_2 - x_3)\right]^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 - \frac{2}{3}(x_2 - x_3)^2$$

$$= 6(x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3)^2 + \frac{7}{3}x_2^2 - \frac{2}{3}x_2x_3 + \frac{7}{3}x_3^2$$

$$= 6(x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3)^2 + \frac{7}{3}(x_2^2 - \frac{2}{7}x_2x_3) + \frac{7}{3}x_3^2$$

$$= 6(x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3)^2 + \frac{7}{3}\left(x_2 - \frac{1}{7}x_3\right)^2 + \frac{7}{3}x_3^2 - \frac{7}{3} \times \frac{1}{49}x_3^2$$

$$= 6(x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3)^2 + \frac{7}{3}(x_2 - \frac{1}{7}x_3)^2 + \frac{16}{7}x_3^2$$

$$= 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2 \text{ which is in the canonical form}$$

where $y_1 = x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3$, $y_2 = x_2 - \frac{1}{7}x_3$ and $y_3 = x_3$

Example 4 : Reduce the following quadratic form to canonical form by Lagrange's reduction. $x^2 - 14y^2 + 2z^2 + 4xy + 16yz + 2zx$ and hence find the index, signature and nature of the quadratic form. **[JNTU (A) Nov. 2011]**

Solution : Giving Q. F. is $x^2 - 14y^2 + 2z^2 + 4xy + 16yz + 2zx$

$$= x^2 + 4xy + 2zx + 2z^2 - 14y^2 + 16yz$$

$$= (x + 2y + z)^2 + z^2 - 18y^2 + 12yz$$

$$= (x + 2y + z)^2 + (z + 6y)^2 - 54y^2$$

Taking $x + 2y + z = x_1$, $z + 6y = x_2$, $y = x_3$

We get $x_1^2 + x_2^2 - 54x_3^2$ as the new form of the Q. F.

$s =$ index $=$ no. of $+ve$ terms $= 2$

$r =$ rank $=$ no. of non-zero terms $= 3$

Signature $= 2s - r = 4 - 3 = 1$

The Q. F. is indefinite.

EXERCISE 3.2

1. Reduce the following quadratic forms to canonical form by linear transformation.

(i) $x^2+4y^2+z^2+4xy+6yz+2zx$ (ii) $2x^2+9y^2+6z^2+8xy+8yz+6zx$

(iii) $x^2+y^2+2z^2-2xy+zx$

2. Reduce the following quadratic forms to canonical form by orthogonal transformation.

(i) $2x^2+2y^2+2z^2-2xy+2zx-2yz$ **[JNTU 2006S, 2008 (Set No.2)]**

(ii) $4x^2+3y^2+z^2-8xy-6yz+4zx$

(iii) $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$

(iv) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$ **[JNTU 2004S(Set No. 3), 2005S (Set No. 2,4)]**

(v) $3x_1^2+5x_2^2+3x_3^2-2x_2x_3+2x_1x_3-2x_1x_2$ **[JNTU 2005 (Set No. 3)]**

(vi) $6x^2+3y^2+3z^2-4yz+4xz-2xy$

3. Identify the nature of the quadratic form

(i) $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ (ii) $-3x_1^2-3x_2^2-3x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$

4. Reduce the following quadratic form to sum of squares by linear transformation $x^2+4y^2+z^2+4xy+6yz+2zx$.

5. Reduce the following quadratic form to canonical form and find its rank and signature $6x_1^2+3x_2^2+14x_3^2+4x_2x_3+18x_3x_1+4x_1x_2$. **[JNTU 2001]**

6. Reduce the quadratic form : $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form. Also specify the matrix of transformation. **[JNTU 2003 (Set No. 2),2005S (Set No. 3)]**

7. Using Lagrange's reduction, transform

(i) $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3$ to canonical form **[JNTU 2003S (Set No. 3)]**

(ii) $2x_1^2+7x_2^2+5x_3^2-8x_1x_2-10x_2x_3+4x_1x_3$ to canonical form.

8. By Lagrange's reduction transform the quadratic form X^TAX to sum of squares form for $A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & -2\\ 4 & -2 & 18\end{bmatrix}$. **[JNTU 2003S, 2006 (Set No. 1)]**

9. Reduce the quadratic form $3x^2-2y^2-z^2+12yz+8zx-4xy$ to canonical form by an orthogonal reduction and state the nature of the quadratic form.

[JNTU 2008S (Set No.2)]

10. Reduce the quadratic form $8x^2+7y^2+3z^2-12xy-8yz+4zx$ into a 'sum of squares' by an orthogonal transformation and give the matrix of transformation. Also state the nature. **[JNTU 2008S(Set No.3)]**

11. Reduce the quadratic form to canonical form by an orthogonal reduction and state the nature of the quadratic form.

$x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$. **[JNTU (H) June 2009 (Set No.2)]**

12. Discuss the nature of the quadratic form and reduces it to canonical form

$x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ **[JNTU (A) June 2009 (Set No.2)]**

ANSWERS

1. (i) $y_1^2 + y_2^2 + y_3^2$ (ii) $2y_1^2 - 7y_2^2 - \frac{-13}{1k}y_3^2$ (iii) $y_1^2 + y_2^2 + y_3^2$

2. (i) $4y_1^2 + y_2^2 + y_3^2$ (ii) $4y_1^2 - y_2^2 + y_3^2$ (iii) $y_1^2 + 4y_2^2 + 4y_3^2$

(iv) $y_1^2 + 2y_2^2 + 4y_3^2$ (v) $2y_1^2 + 3y_2^2 + 6y_3^2$ (vi) $4x^2 + y^2 + z^2$

3. (i) Positive definite (ii) Negative definite

5. $r = 3,\ s = -1$

7. (i) $(x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$ (ii) $2(x_1 - 2x_2 - x_3)^2 - (x_2 + x_3)^2 + 4x_3^2]$

8. $[x_1 + 2(x_2 + 2x_3)]^2 + 2(x_2 - 5x_3)^2 - 48x_3^2$

3.14 SYLVESTER'S THEOREM

This theorem is useful in finding the approximate value of a matrix to a higher power and functions of matrices.

If the square matrix A has n distinct eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ and P(A) is a polynomial of the form $P(A) = C_0A^n + C_1A^{n-1} + C_2A^{n-2} + ... + C_{n-1}A + C_n.I_n$ where $C_0, C_1, C_2, ..., C_n$ are constants then the polynomial P(A) can be expressed in the following form:

$$P(A) = \sum_{r=1}^{n} P(\lambda_r) \cdot Z(\lambda_r) = P(\lambda_1) \cdot Z(\lambda_1) + P(\lambda_2) \cdot Z(\lambda_2) + P(\lambda_3) \cdot Z(\lambda_3) + ...$$

where $Z(\lambda_r) = \dfrac{[f(\lambda_r)]}{f'(\lambda_r)}$

Here $f(\lambda) = |\lambda I - A|$

$[f(\lambda)] =$ Adjoint of the matrix $[\lambda I - A]$

and $f'(\lambda_r) = \left(\dfrac{df}{d\lambda}\right)_{\lambda = \lambda_r}$

SOLVED EXAMPLES

Example 1 : If $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find A^{50}

[JNTU 2008, (H)2009,(K)May2010(Set No.4)]

Solution : Consider the polynomial $P(A) = A^{50}$

Now $[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}$

$\therefore \quad f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix}$

$= (\lambda-1)(\lambda-3) = \lambda^2 - 4\lambda + 3$ ... (1)

$\therefore$ Eigen values of $f(\lambda)$ are $\lambda_1 = 1$ and $\lambda_2 = 3$

From (1), $f'(\lambda) = 2\lambda - 4$ ... (2)

$f'(1) = 2 - 4 = -2, \quad f'(3) = 6 - 4 = 2$

$[f(\lambda)]$ = Adjoint matrix of the matrix $[\lambda I - A]$

$= \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix}$ ... (3)

Now $\quad Z(\lambda_r) = \dfrac{[f(\lambda_r)]}{f'(\lambda_r)}$, r = 1, 2 we get

$Z(\lambda_1) = \dfrac{[f(\lambda_1)]}{f'(\lambda_1)}$ and $Z(\lambda_2) = \dfrac{[f(\lambda_2)]}{f'(\lambda_2)}$

$\therefore \quad Z(\lambda_1) = Z(1) = \dfrac{[f(1)]}{f'(1)} = -\dfrac{1}{2}\begin{bmatrix} 1-3 & 0 \\ 0 & 1-1 \end{bmatrix}$, using (2) and (3)

$= -\dfrac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

and $\quad Z(\lambda_2) = Z(3) = \dfrac{[f(3)]}{f'(3)} = \dfrac{1}{2}\begin{bmatrix} 3-3 & 0 \\ 0 & 3-1 \end{bmatrix}$, using (2) and (3)

$= \dfrac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Thus by sylvester's theroem, we get

$$P(A) = P(\lambda_1)\cdot Z(\lambda_1) + P(\lambda_2)\cdot Z(\lambda_2)$$

i.e., $A^{50} = \lambda_1^{50}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \lambda_2^{50}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ $[\because\ P(A) = A^{50}]$

$= 1^{50}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 3^{50}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\0 & 3^{50}\end{bmatrix}$

or $\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}^{50} = \begin{bmatrix}1 & 0\\0 & 3^{50}\end{bmatrix}$

Example 2 : Using sylvester's theorem, find A^{200}, if $A = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$

Solution : Consider a special polynomial $P(A) = A^{200}$

$$[\lambda I - A] = \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}\lambda-2 & 0\\0 & \lambda-1\end{bmatrix}$$

$$\therefore \quad f(\lambda) = |\lambda I - A| = \begin{vmatrix}\lambda-2 & 0\\0 & \lambda-1\end{vmatrix} = (\lambda-1)(\lambda-2)$$

$$= \lambda^2 - 3\lambda + 2 \qquad \text{... (1)}$$

$\therefore$ Eigen values of $f(\lambda)$ are $\lambda_1 = 1$ and $\lambda_2 = 2$

$\therefore$ $[f(\lambda)]$ = Adjoint of the matrix $[\lambda I - A]$

$$= \begin{bmatrix}\lambda-1 & 0\\0 & \lambda-2\end{bmatrix} \qquad \text{... (2)}$$

$\therefore$ From (1), $f'(\lambda) = 2\lambda - 3$ ... (3)

Now, $Z(\lambda_r) = \dfrac{[f(\lambda_r)]}{f'(\lambda_r)}$, $r = 1, 2$. We get

$$Z(\lambda_1) = Z(1) = \frac{1}{f'(\lambda_1)}[f(\lambda_1)] = \frac{1}{f'(1)}[f(1)]$$

$$= \frac{1}{-1}\begin{bmatrix}1-1 & 0\\0 & 1-2\end{bmatrix}, \text{ using (2) and (3)}$$

$$= \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$Z(\lambda_2) = Z(2) = \frac{1}{f'(2)}[f(2)] = \frac{1}{1}\begin{bmatrix}2-1 & 0\\0 & 2-2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

Thus by sylvester's theorem, we get

$$A^{200} = P(\lambda_1)\cdot Z(\lambda_1) + P(\lambda_2)\cdot Z(\lambda_2) = \lambda_1^{200}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \lambda_2^{200}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$= 1^{200}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + 2^{200}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}2^{200} & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}2^{200} & 0\\0 & 1\end{bmatrix}$$

EXERCISE 3.3

Using sylvester's theorem, find

1. A^{256}, if $A = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$ **[JNTU(K) Nov.2009S(Set No.4)]**

2. A^{100}, if $A = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ **[JNTU(K) Nov.2009S(Set No.3)]**

3. A^{150}, if $A = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$

4. A^{200}, if $A = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$

ANSWERS

1. $\begin{bmatrix}1 & 0\\0 & 3^{256}\end{bmatrix}$ **2.** $\begin{bmatrix}2^{100} & 0\\0 & 1\end{bmatrix}$ **3.** $\begin{bmatrix}2^{150} & 0\\0 & 1\end{bmatrix}$ **4.** $\begin{bmatrix}2^{200} & 0\\0 & 2^{200}\end{bmatrix}$

OBJECTIVE TYPE QUESTIONS

1. The symmetric matrix associated with the quadratic form $x^2 + 3y^2 - 8xy$ is

(a) $\begin{bmatrix}1 & 4\\4 & -3\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}$ (c) $\begin{bmatrix}1 & -4\\-4 & 3\end{bmatrix}$ (d) $\begin{bmatrix}1 & -8\\-8 & 3\end{bmatrix}$

2. The symmetric matrix associated with the quadratic form $x_1^2 - 2x_1x_2 + 2x_2^2$ is

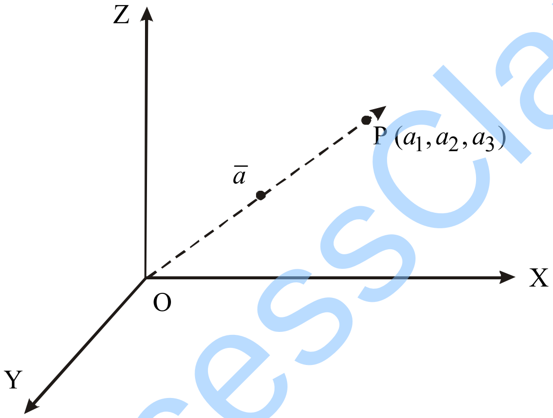
(a) $\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 1\\1 & 2\end{bmatrix}$ (c) $\begin{bmatrix}1 & -1\\-1 & 1/2\end{bmatrix}$ (d) $\begin{bmatrix}-1 & -1\\2 & 2\end{bmatrix}$

3. The symmetric matrix associated with the quadriatic form $ax^2 + 2hxy + by^2$ is

(a) $\begin{bmatrix}a & 2h\\2h & b\end{bmatrix}$ (b) $\begin{bmatrix}a & -h\\-h & b\end{bmatrix}$ (c) $\begin{bmatrix}a & 0\\0 & b\end{bmatrix}$ (d) $\begin{bmatrix}a & h\\h & b\end{bmatrix}$

Vector Spaces

1.1. Let P be a point (a_1, a_2, a_3) with respect to the frame of reference OXYZ. Let $\overline{OP}$ be denoted by the vector $\bar{a}$. Now the vector $\bar{a}$ is associated with the point P given by the ordered trial (a_1, a_2, a_3). Conversely the ordered triad (a_1, a_2, a_3) defines a point P associated with the vector $\bar{a}$.



This shows that the set of all points in 3-D space has a one to one correspondence with the set of all vectors starting from the origin. Thus each vector is representable as an ordered triad of three real numbers. This enables us to write $\bar{a} = (a_1, a_2, a_3)$.

From this we can visualise the 3.dimensional space as an ordered set of triads (a_1, a_2, a_3) where a_1, a_2, a_3 are real numbers.

This space is denoted by R^3.

1.2. ALGEBRAIC STRUCTURE OF 3 - D VECTORS

Let V be the set of 3.D vectors and F be the field of scalars.

Now it is easy to verify the following algebraic structure with the vector addition (+) and scalar multiplication $(\bullet)$ of a vector.

G :(1) Closure : $\alpha + \beta \in V \ \forall \ \alpha, \beta \in V$

(2) Associativity : $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \ \forall \ \alpha, \beta, \gamma \in V$

(3) Identity : $\alpha + \bar{O} = \bar{O} + \alpha = \alpha \ \forall \ \alpha \in V$ null vector $\bar{O}$ is the additive in V.

(4) Inverse : $\alpha + (-\alpha) = \bar{O} = (-\alpha) + \alpha$. Every vector in V has the additive inverse.

(5) Commutativity : $\alpha + \beta = \beta + \alpha \ \forall \ \alpha, \beta \in V$

$\therefore$ (V, +) is an abelian group.

A : *Admission of scalar multiplication in V*

(1) $a\alpha \in V \ \forall \ a \in F$ and $\alpha \in V$

(2) $1.\alpha = \alpha \ \ \forall \ \alpha \in V$, where 1 is the unity of F.

S : *Scalar multiplication*

(1) $a(b\alpha) = (ab)\alpha \qquad \forall \ a, b \in F$ and $\alpha \in V$

Scalar multiplication of vectors is associative

(2) $a(\alpha + \beta) = a\alpha + a\beta \quad \forall \ a \in F$ and $\alpha, \beta \in V$

Multiplication by scalars is distributive over vector addition.

(3) $(a+b)\alpha = a\alpha + b\alpha \quad \forall \ a, b \in F$ and $\alpha \in V$

Multiplication by vectors is distributive over scalar addition.

The above verified **GAS** structure leads us to define abstract vector spaces or simply vector spaces or linear spaces.

1.3. THE *n* - DIMENSIONAL VECTORS

Now we can generalise the 3-D vector concepts to n-dimensional vectors. Thereby, a point in the n-dimensional vector space has n coordinates of the form $(a_1, a_2, ..., a_n)$. Hence we define an n-dimensional vector $\bar{a}$ as an ordered n-tuple, such as $\bar{a} = (a_1, a_2, a_3, ..., a_n)$

where each a_i is a number, real or complex. This space is denoted by $\mathbf{R}^n$ or $\mathbf{C}^n$.

1.4. SCALAR MULTIPLICATION OF A VECTOR

Let $\alpha = (a_1, a_2, ..., a_n)$ be a vector in R^n and k be a scalar.

Then we define $k\alpha = (ka_1, ka_2, ..., ka_n)$.

1.5. ADDITION OF VECTORS

Let $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$ be two vectors of R^n.

Then we define $\alpha + \beta = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$

1.6. INTERNAL COMPOSITION

Let F be a set. If $aob \in F$ for all $a, b \in F$. Then 'o' is said to be an internal composition in the set F.

1.7. EXTERNAL COMPOSITION

Let V and F be two sets. If $a\,o\,\alpha \in V$ for all $a \in F$ and for all $\alpha \in V$, then 'o' is said the be an external composition in V over F. Here the resulting element $a\,o\,\alpha$ is an element of the set V.

1.8. VECTOR SPACES

Definition. *Let V be a non-empty set whose elements are called vectors. Let F be any set whose elements are called scalars where* $(F, +, \bullet)$ *is a field.*

The set V is said to be a vector space if

(1) *there is defined an internal composition in V called addition of vectors denoted by +, for which (V, +) is an abelian group.*

(2) *there is defined an external composition in V over F, called the scalar multiplication in which* $a\alpha \in V$ *for all* $a \in F$ *and* $\alpha \in V$.

(3) *the above two compositions satisfy the following postulates*

(*i*) $a(\alpha+\beta) = a\alpha + a\beta$ (*ii*) $(a+b)\alpha = a\alpha + b\alpha$ (*iii*) $(ab)\alpha = a(b\alpha)$ (*iv*) $1\alpha = \alpha$

$\forall\ a, b \in F$ and $\alpha, \beta \in V$ and 1 is the unity element of F. Instead of saying that V is a vector space over the field F, we simply say V(F) is a vector space. Sometimes if the field F is understood, then we simply say that V is a *'vector space'*.

1.9. VECTOR SPACE

If R is a field of real numbers, then V(R) is called the real vector space.

If C is the field of complex numbers, then V(C) is called the complex vector space.

Note.1. In the above definition (V, +) is an abelian group implies that for all $\alpha, \beta, \gamma \in V$

(*a*) $\alpha+\beta \in V$ (*b*) $\alpha + (\beta+\gamma) = (\alpha+\beta)+\gamma$

(*c*) there exists an element $\bar{O} \in V$ such that $\alpha + \bar{O} = \alpha$

(*d*) To every vector $\alpha \in V$ there exists $-\alpha \in V$ such that $\alpha + (-\alpha) = 0$

(*e*) $\alpha+\beta = \beta+\alpha$

2. The use of symbol + for two different compositions (for the addition in the field F and for the internal composition in V) should not cause any confusion. It should be understood according to the context in which it is used.

3. In vector space, two types of zero elements come into operation. One is the zero vector $\bar{O}$ of V and the other is the zero scalar 0 of the field F.

1.10. NULL SPACE OR ZERO VECTOR SPACE

The vector space having only one zero vector $\bar{O}$ is called the zero vector space or null space. It clearly satisfies all the postulates on any field of scalars.

Theorem : ***Let V(F) be a vector space and*** $O, \bar{O}$ ***be the zero scalar and zero vector respectively. Then***

(1) $a\bar{O} = \bar{O}\ \forall\ a \in F$

(2) $0\alpha = \bar{O}\ \forall\ \alpha \in V$

(3) $a(-\alpha) = -(a\alpha)\ \forall\ a \in F,\ \forall\ \alpha \in V$

(4) $(-a)\alpha = (-a\alpha)\ \forall\ a \in F,\ \forall\ \alpha \in V$

(5) $a\alpha = \bar{O} \Rightarrow a = 0$ or $\alpha = \bar{O}$

(6) $a(\alpha-\beta) = a\alpha - a\beta\ \forall\ a \in F,\ \forall\ \alpha, \beta \in V$

(7) $(-a)(-\alpha) = a\alpha\ \forall\ a \in F,\ \forall\ \alpha \in V$

Proof. (1) $a\bar{O} = a(\bar{O}+\bar{O}) = a\bar{O} + a\bar{O}$

$\therefore\ a\bar{O} + \bar{O} = a\bar{O} + a\bar{O}$ (Distributive Law)

$\bar{O} = a\bar{O}$ (Left Cancellation Law)

(2) $0\,\alpha = (0+0)\,\alpha = 0\,\alpha + 0\,\alpha$

$\therefore\ \bar{O} + 0\,\alpha = 0\,\alpha + 0\,\alpha \Rightarrow \bar{O} = 0\,\alpha$ (Cancellation Law)

(3) $a\,[\alpha + (-\alpha)] = a\bar{O} \Rightarrow a\,\alpha + a(-\alpha) = \bar{O}$

$\Rightarrow a(-\alpha)$ is the additive inverse of $a\,\alpha \Rightarrow a\,(-\alpha) = -(a\,\alpha)$

(4) $[a + (-a)]\,\alpha = 0\,\alpha \Rightarrow a\,\alpha + (-a)\,\alpha = \bar{O}$

$\Rightarrow (-a)\,\alpha$ is the additive inverse of $a\,\alpha$

$\Rightarrow (-a)\,\alpha = -(a\,\alpha)$

(5) $a\,\alpha = \bar{O}$ and $a = 0$. Then there is nothing to prove. $a\,\alpha = \bar{O}$ and $a \neq 0$

Now $a \neq 0, a \in F \Rightarrow$ there exists. $a^{-1} \in F$ such that $a\,a^{-1} = a^{-1}a = 1$

Now $a\,\alpha = \bar{O} \Rightarrow a^{-1}(a\,\alpha) = a^{-1}\,\bar{O} \Rightarrow 1\,\alpha = \bar{O} \Rightarrow \alpha = \bar{O}$

$\therefore\ a\alpha = \bar{O} \Rightarrow a = 0$ or $\alpha = \bar{O}$

(6) $a\,(\alpha - \beta) = a\,[\alpha + (-\beta)] = a\,\alpha + a\,(-\beta) = a\,\alpha - a\,\beta$

(7) $(-a)\,(-\alpha) = -[a\,(-\alpha)] = -[-(a\,\alpha)] = a\,\alpha$

1.11. Theorem. ***Let V(F) be a vector space***

(1) If $a, b \in F$ ***and*** $\alpha \in V$ ***where*** $\alpha \neq \bar{O}$ ***then*** $a\,\alpha = b\,\alpha \Rightarrow a = b$

(2) If $a \in F$ ***where*** $a \neq 0$ ***and*** $\alpha, \beta \in V$ ***then*** $a\,\alpha = a\,\beta \Rightarrow \alpha = \beta$.

Proof : (1) $a\,\alpha = b\,\alpha \Rightarrow a\,\alpha + (-b\,\alpha) = b\,\alpha + (-b\,\alpha) \Rightarrow (a-b)\,\alpha = \bar{O}$

$\Rightarrow a - b = 0$ as $\alpha \neq \bar{O} \Rightarrow a = b$

(2) $a\,\alpha = a\,\beta \Rightarrow a\,\alpha - a\,\beta = \bar{O} \Rightarrow a\,(\alpha - \beta) = \bar{O} \Rightarrow \alpha - \beta = \bar{O}$ as $a \neq 0 \Rightarrow \alpha = \beta$.

SOLVED PROBLEMS

Ex. 1. *The set* C_n *of all n-tuples of complex numbers with addition as the external composition and scalar multiplication of complex numbers by complex numbers is a vector space over the field of complex numbers with the following definitions.*

(*i*) If $\alpha, \beta \in C_n$ and $\alpha = (a_1, a_2, ..., a_n)$ and $\beta = (b_1, b_2, ..., b_n)$ for all $a_k, b_k \in C$

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \in C_n$

(*ii*) $x\,\alpha = (x\,a_1, x\,a_2, ..., x\,a_n)\ \forall\ x \in C$

Sol. (*i*) By definition $x + y$ is an *n*-tuple of complex numbers. Hence C_n is closed.

(*ii*) $(\alpha + \beta) + \gamma = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) + (c_1, \ldots, c_n)$

where $\gamma = (c_1, c_2, ..., c_n)$ with c_i's $\in C$

$= \{(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, ..., (a_n + b_n) + c_n\}$

$= \{a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots, a_n + (b_n + c_n)\}$ since + is associative in C

$= (a_1, a_2, ..., a_n) + (b_1 + c_1, b_2 + c_2, ..., b_n + c_n) = \alpha + (\beta + \gamma)$

$\therefore$ Vector addition is associative in C_n.

(*iii*) $\alpha+\bar{O}=(a_1, a_2, \ldots, a_n)+(0, 0, \ldots, 0) = (a_1+0, a_2+0, \ldots, a_n+0) = (a_1, a_2, \ldots, a_n) = \alpha$

Similarly $\bar{O}+\alpha=\alpha \Rightarrow \alpha+\bar{O}=\bar{O}+\alpha=\alpha$

$\therefore$ The n-tuple $\bar{O}=(0, 0, \ldots, 0)$ is the identity in C_n.

(*iv*) $\alpha+(-\alpha)=(a_1, a_2, \ldots, a_n)+(-a_1, -a_2, \ldots, -a_n)$

$=(a_1-a_1, a_2-a_2, \ldots, a_n-a_n) = (0, 0, 0, \ldots, 0)=\bar{O}$. $\quad\therefore \alpha+(-\alpha)=\bar{O}$

$\Rightarrow (-\alpha) \in C$, is the additive inverse of α in C

(*v*) $\alpha+\beta=(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$

$=(b_1+a_1, a_2+b_2, \ldots, b_n+a_n)$ ($\because$ + is commutative in C)

$=\beta+\alpha$.

(*vi*) By definition $x \in C, \alpha \in C_n \Rightarrow x\alpha \in C_n$

(*vii*) Let $x, y \in C$. $\quad x(y\alpha)=x(ya_1, ya_2, \ldots, ya_n) = \{x(ya_1), x(ya_2), \ldots, x(ya_n)\}$

$=\{(xy)a_1, (xy)a_2, \ldots, (xy)a_n\} = xy(a_1, a_2, \ldots, a_n) = (xy)\alpha$

(*viii*) $x \in C$ and $\alpha, \beta \in C_n$

$\Rightarrow x(\alpha+\beta)=x(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$

$=\{x(a_1+b_1), x(a_2+b_2), \ldots, x(a_n+b_n)\} = (xa_1+xb_1, xa_2+xb_2, \ldots, xa_n+xb_n)$

$=(xa_1, xa_2, \ldots, xa_n)+(xb_1, xb_2, \ldots, xb_n) = x\alpha+x\beta$

(*ix*) $x, y \in C$ and $\alpha \in C_n$

$\Rightarrow (x+y)\alpha=(x+y)(a_1, a_2, \ldots, a_n) = \{(x+y)a_1, (x+y)a_2, \ldots, (x+y)a_n\}$

$=\{(xa_1+ya_1), (xa_2+ya_2), \ldots, (xa_n+ya_n)\}$ (Since distributive law is true in C)

$=(xa_1, xa_2, xa_n)+(ya_1, ya_2, \ldots, ya_n) = x\alpha+y\alpha$

(*x*) $1.\alpha=1(a_1, a_2, \ldots, a_n)=(a_1, a_2, \ldots, a_n)=\alpha$

Since all the postulates are verified C_n (C) is a vector space.

Ex.2. *Prove that the set of all real valued continuous functions defined in the open interval (0,1) is a vector space over the field of real numbers, with respect to the operations of addition and scalar multiplication defined as*

$(f+g)(x)=f(x)+g(x)$ ... (1)

$(af)(x)=af(x)$. a is real ... (2) where $0<x<1$

Sol: Let V be the set of all such real values continuous functions and R be the field of real numbers.

(*i*) The sum of two continuous functions is again a continuous function

$(f+g) \in V \;\forall\; f, g \in V$.

(*ii*) $\{(f+g)+h\}(x)=(f+g)x+h(x)$ (by def. (1))

$= f(x)+g(x)+h(x) = f(x)+(g+h)(x) = \{f+(g+h)\}(x)$

$\Rightarrow (f+g)+h = f+(g+h)$. $\quad\therefore$ V is associative

(*iii*) Let the function O be defined as $\bar{O}(x)=0$

$\therefore (0+b)(x)=\bar{O}(x)=f(x)$ (by def (1))

$=0+f(x)=f(x)$ as the real number 0 is the additive identity in R.

$\therefore \bar{O}+f=f, \ \forall f\in V$

(*iv*) $\{f+(-f)\}(x)=f(x)+(-f(x)\} = f(x)-f(x)=0=\bar{O}(x)$

The function $(-f)$ is the additive inverse of f

$\therefore f+(-f)=\bar{O}$ (the identity function)

(*v*) $(f+g)(x)=f(x)+g(x)$

$=g(x)+f(x)$ (real numbers are commutative under addition)

$=(g+f)(x)$ by definition (1)

Thus $f+g=g+f, \ \forall \ g, f\in V$. $\quad\therefore$ (V, +) is an abelian group

(*vi*) For all $a\in R$ and $f, g\in V$, $\ a(f+g)=af+ag$

Now $[a(f+g)](x)=a[(f+g)(x)]=a[f(x)+g(x)] = af(x)+ag(x)$

$=(af)(x)+(ag)(x) \quad =(af+ag)(x) \quad \therefore a(f+g)=af+ag$

(*vii*) If $a, b\in R$ and $f\in V$ then, $\{(a+b)f\}(x)=(a+b)f(x)$ (by (2))

$=a\,f(x)+b\,f(x)$ (as $f(x)$ is real)

$=(a\,f)(x)+(b\,f)(x)=(af+bf)(x)$ (by (1))

$\Rightarrow (a+b)f=af+bf$

(*viii*) If $a, b\in R$ and $f\in V$ then, $\{a(bf)\}(x)$ (by (2))

$=a\{(bf)(x)\}=a\{bf(x)\}$

$=(a\,b)f(x)$ as $f(x)$ is real $=\{(a\,b)f\}(x) \Rightarrow a(bf)=(ab)f$

(*ix*) Since 1 is the identity of the field R and $f\in V$,

we have $(1\,f)(x)=1\,f(x)=f(x)$ $\quad\therefore 1\,f=f$

All the postulates of vector space are verified. Hence V (R) is a vector space.

Ex. 3. *V is the set of all $m\times n$ matrices with real entries and R is the field of real numbers. "Addition of matrices' is the internal composition and 'multiplication of a matrix by a real number' an external composition in V. Show that V (R) is a vector space.*

Sol: Let $\alpha, \beta, \gamma\in V$ and $x, y\in R$ where $\alpha=[a_{ij}], \ \beta=[b_{ij}], \gamma=[c_{ij}]$ for $a's, b's, c's \in R$

(*i*) Addition of two matrices is a matrix. $\quad\therefore \alpha+\beta\in V \ \forall \ \alpha, \beta\in V$

(*ii*) $\alpha+(\beta+\gamma)=[a_{ij}]+[b_{ij}+c_{ij}]=[a_{ij}+(b_{ij}+c_{ij})] \quad =[(a_{ij}+b_{ij})+c_{ij}]$

(Since real numbers are associative in R)

$=[a_{ij}+b_{ij}]+[c_{ij}] \quad =(\alpha+\beta)+\gamma$ $\quad\therefore$ V is associative.

(*iii*) If $O=[o_{ij}]$ is the null matrix then $\alpha+O=[a_{ij}]+[o_{ij}]=[a_{ij}+o_{ij}] \quad =[a_{ij}]=\alpha$

Thus $\alpha+O=O+\alpha=\alpha$. $\Rightarrow$ The null matrix O is the additive identity in V.

(*iv*) $\alpha+(-\alpha)=[a_{ij}]+[-a_{ij}] \quad =[a_{ij}-a_{ij}]=[-o_{ij}]=0$

$\therefore$ $(-\alpha)$ is the additive inverse of every α in V

(*v*) Clearly $\alpha+\beta=\beta+\alpha$ $\quad\therefore$ (V, +) is an abelian group.

(*vi*) For $x\in R$ and $\alpha\in V$. $\quad x\alpha=x[a_{ij}]=[xa_{ij}]\Rightarrow x\alpha\in V$

(*vii*) For $x, y\in R$ and $\alpha\in V$

$x(y\alpha)=x(y[a_{ij}])=x[ya_{ij}]=[xya_{ij}]=(xy)[a_{ij}]=(xy)\alpha$

(*viii*) $x(\alpha+\beta)=[a_{ij}+b_{ij}]=[x(a_{ij}+b_{ij})]=[xa_{ij}+xb_{ij}] \quad =[xa_{ij}]+[xb_{ij}]$

$=x[a_{ij}]+x[b_{ij}]=x\alpha+x\beta$

(*ix*) $(x+y)\,\alpha=(x+y)\,[a_{ij}] \quad =[(x+y)\,a_{ij}]=[x\,a_{ij}+y\,a_{ij}]$

$=[x\,a_{ij}]+[y\,a_{ij}] \quad =x[a_{ij}]+y[a_{ij}]=x\,\alpha+y\,\alpha$

(*x*) $1.\alpha=1[a_{ij}]=[a_{ij}]=\alpha$ $\quad$ Hence V (R) is a vector space.

Ex.4. *Let V be the set of all pairs (a, b) of real numbers and R be the field of real numbers. Show that with the operation* $(a_1,b_1)+(a_2,b_2)=(a_1+a_2,o)$, $c(a_1,b_1)=(ca_1,b_1)$ *V (R) is not a vector space.*

Sol: If any one of the postulates of a vector space is not satisfied, then V (R) cannot be a vector space. Let (x, y) be the identity in V.

Now $(a,b)+(x,y)=(a,b)\ \forall\ a,b\in V \Rightarrow (a+x,0)=(a,b)$ $\quad$ ... (1)

If $b\neq 0$, then we cannot have the equality (1).

Thus there exists no element $(x,y)\in V$ such that $(a,b)+(x,y)=(a,b)\ \forall\ a,b\in V$

$\therefore$ Additive identity does not exist in V and hence V (R) is not a vector space.

EXERCISE 1 (*a*)

1. Show that the set of all triads (x_1, x_2, x_3) where x_1, x_2, x_3 are real numbers forms a vector space over the field of real numbers with respect to the operations of addition and scalar multiplication defined as

(*i*) $(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$ and

(*ii*) $c\,(x_1,x_2,x_3)=(c\,x_1,c\,x_2,c\,x_3)$ c is a real number.

2. Let P be the set of all polynomials in one indeterminate with real coefficients. Show that P (R) is vector space, if in P the addition of polynomials is taken as the internal composition and the multiplication of polynomial by a constant polynomial (*i.e.* by an element *k*) as scalar multiplication.

3. If F is a field, the F (F) is a vector space, if in F the addition of field F is taken as the internal composition and the multiplication of the field F is taken as the external composition in F over F.

4. Let R^n be the set of all ordered *n*-tuples of real numbers given by

$R^n = [(x_1, x_2, \dots, x_n) \mid x_i \in R]$

The internal and external composition in R^n are defined by

(*i*) $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

(*ii*) $a\,(x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n),\ a \in R$. Show that R^n (R) is a vector space.

5. Let F be any field and K any subfield of F. Then F (K) is a vector space, if the addition of the field F is taken as the internal composition in F, and the field multiplication is taken as the external composition in F over K.

6. Let V (R) be a vector space and $W = \{(x, y) \mid x, y \in V\}$. For (x_1, y_1) and (x_2, y_2) in W we have $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and if $\alpha = a_1 + ia_2 \in C$, we define

$a\,(x_1, y_1) = (a_1x_1 - a_2x_2, a_2x_1 + a_1x_2)$. Prove that W (C) is a vector space.

7. The set of all real valued differentiable or integrable functions defined in some interval [0,1] is a vector space.

8. Let V be the set of all pairs of real numbers and let F be the field of real numbers with the definition $(x_1, y_1) + (x_2, y_2) = (3y_1 + 3y_2, -x_1 - x_2)$, $c\,(x_1, y_1) = (3\,c\,y_1, -c\,x_1)$

Show that V (F) is not a vector space.

9. V is the set of all polynomials over real numbers of degree atmost one and F = R.

If $f(t) = a_0 + a_1 t$ and $g(t) = b_0 + b_1(t)$ in V ; define

$f(t) + g(t) = (a_0 + b_0) + (a_0b_1 + a_1b_0)\,t$ and $kf(t) = (ka_0) + (ka_1)\,t,\ k \in F$

Show that V (F) is not a vector space. (**Hint:** Additive identity does not exist)

10. C is the field of complex numbers and R is the field of real numbers. Show that C (R) is a vector space and R (C) is not a vector space.

11. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$, $c\,(x, y) = (cx, y)$

Show that V (F) is not a vector space.

1.12. VECTOR SUBSPACES

Definition. *Let V (F) be a vector space and* $W \subseteq V$. *Then W is said to be a subspace of V if W itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.*

Note.1. If W (F) is a subspace of V (F) then W is a sub-group of V.

Note.2. Let V (F) be a vector space. The zero vector space $\{\bar{O}\} \subseteq V$ and $V \subseteq V$.

$\therefore$ $\{\bar{O}\}$ and V are the trivial subspaces of V.

1.13. Theorem. ***Let V (F) be a vector space and let*** $W \subseteq V$. ***The necessary and sufficient conditions for W to be a subspace of V are***

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$ ***(ii)*** $a \in F, \alpha \in W \Rightarrow a\alpha \in W$. ***(S.V. U 2011)***

Proof. *Conditions are necessary*

(*i*) W is a vector subspace of V

$\Rightarrow$ W is a subgroup of (V, +) $\Rightarrow$ (W, +) is a group

$\Rightarrow$ if $a, \beta \in W$ then $a - \beta \in W$.

(*ii*) W is a subspace of V

$\Rightarrow$ W is closed under scalar multiplication $\Rightarrow$ for $a \in F$, $\alpha \in W$; $a\alpha \in W$

Conditions are sufficient.

Let W be a nonempty subset of V satisfying the two given conditions

$\alpha \in W$, $\alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \bar{O} \in W$ (by (*i*))

$\therefore$ The zero vector of V is also the zero vector of W

$\bar{O} \in W$, $\alpha \in W \Rightarrow \bar{O} - \alpha \in W \Rightarrow (-\alpha) \in W$ (by (*i*))

$\Rightarrow$ additive inverse of each element of W is also in W

Again $\alpha \in W$, $\beta \in W \Rightarrow \alpha \in W$, $(-\beta) \in W \Rightarrow \alpha - (-\beta) \in W$ (by (*i*))

$\Rightarrow \alpha + \beta \in W$

i.e., W is closed under vector addition

As $W \subseteq V$, all the elements of W are also the elements of V. Thereby vector addition in W will be associative and commutative. This implies that (W, +) is an abelian group.

Further by (*ii*), W is closed under scalar multiplication and the other postulates of vector space hold in w as $W \subseteq V$.

$\therefore$ W itself is a vector space under the operations of V.

Hence W (F) is a vector subspace of V (F)

A more useful and abridged form of the above theorem is the following.

1.14. Theorem. ***Let V (F) be a vector space. A non-empty set*** $W \subseteq V$.
The necessary and sufficient condition for W to be a subsapce of V is
$a, b \in F$ ***and*** $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in W$ ***... (I)***

Proof. *Condition is necessary.*

W (F) is a subspace of V (F) $\Rightarrow$ W (F) is a vector space

$\therefore$ $a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$ and $b \in F,\ \beta \in W \Rightarrow b\beta \in W$

Now $a\alpha \in W,\ b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Condition is sufficient.

Let W be the non-empty subset of V satisfying the given condition

i.e., $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ ... (I)

Taking $a = 1, b = -1$ and $\alpha, \beta \in W \Rightarrow 1\alpha + (-1)\beta \in W$

$\Rightarrow \alpha - \beta \in W$ $[\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

($H \subseteq G$ and $a, b \in H \Rightarrow a\,o\,b^{-1} \in H$ then (H, o) is subgroup of (G, 0)).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +).

$\Rightarrow$ (W, +) is an abelian group. Again taking $b = 0$

$a, 0 \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + 0\beta \in W \Rightarrow a\alpha \in W \Rightarrow a \in F$ and $\alpha \in W \Rightarrow a\alpha \in W$

$\therefore$ W is closed under scalar multiplication.

The remaining postulates of vector space hold in W as $W \subseteq V$.

$\therefore$ W (F) is a vector subspace of V (F).

1.15. Theorem. ***A non-empty set W is a subset of vector space V (F). W is a subspace of W if and only if $a \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + \beta \in W$.***)

Proof. *Condition is necessary.* W is a subspace of V (F)

$\Rightarrow$ W (F) is a vector space. $\therefore a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Further $a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$

Condition is sufficient. W is a non-empty subset of V satisfying the condition.

$a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

(*i*) Now taking $a = -1$, for $\alpha \in W$ we have $(-1)\alpha + \alpha \in W \Rightarrow \bar{O} \in W$

(*ii*) Again $a \in F, \alpha, \bar{O} \in W \Rightarrow a\alpha + \bar{O} \in W \Rightarrow a\alpha \in W$

$\therefore$ W is closed with respect to scalar multiplication.

(*iii*) $-1 \in F$ and $\alpha, \bar{O} \in W \Rightarrow (-1)\alpha + \bar{O} \in W \Rightarrow -\alpha \in W$. $\therefore$ Inverse exists in W.

The remaining postulates of vector space hold good in W since they hold in V of which W is a subset.

Hence W is a subspace of V (F).

SOLVED PROBLEMS

Ex.1. *The set W of ordered triads (x, y, 0) where $x, y \in F$ is a subspace of V_3 (F).*

Sol: Let $\alpha, \beta \in W$ where $\alpha = (x_1, y_1, 0)$ and $\beta = (x_2, y_2, 0)$ for some $x_1, y_1, x_2, y_2 \in F$.

Let $a, b \in F$. $\therefore a\alpha + b\beta = a(x_1, y_1, 0) + b(x_2, y_2, 0)$

$$= (ax_1, ay_1, 0) + (bx_2, by_2, 0) = (ax_1 + bx_2, ay_1 + by_2, 0)$$

Clearly $ax_1 + bx_2, ay_1 + by_2 \in F \Rightarrow a\alpha + b\beta \in W$ for all $a, b \in F$ and $\alpha, \beta \in W$

Hence W is subspace of V_3 (F).

Ex.2. *Let p, q, r be the fixed elements of a field F. Show that the set W of all triads* (x, y, z) *of elements of F, such that* $px+qy+rz=0$ *is a vector subspace of* V_3 *(F).*

Sol: By definition $W \neq \phi$. Let $\alpha, \beta \in W$ where $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in F$.

Then by definition $px_1 + qy_1 + rz_1 = 0$ ... (1) $px_2 + qy_2 + rz_2 = 0$ ... (2)

If $a, b \in F$, then we have $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $p(ax_1 + bx_2) + q(ay_1 + by_2) + r(az_1 + bz_2)$

$= a(px_1 + qy_1 + rz_1) + b(px_2 + qy_2 + rz_2) = a.0 + b.0 = 0$ (by (1) and (2))

$\therefore a\alpha + b\beta = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \in W$. Hence W is a subspace of V_3 (F).

Ex. 3. *Let R be the field of real numbers and W = { (x, y, z) / x, y, z are rational numbers }. Is W a subspace of* V_3 (R).

Sol: Let $\alpha = (2,3,4)$ be an element of W ; $a = \sqrt{7}$ is an element of R.

Now $a\alpha = \sqrt{7}(2,3,4) = (2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}) \notin W$.

($\therefore 2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}$ not rational numbers).

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of V_3 (R).

Ex. 4. *Let V be the vector space of all polynomials in an indeterminate x over the field F. Let W be a subset of V consisting of all polynomials of degree* $\leq n$. *Then W is a subspace of V.*

Sol: Let $\alpha, \beta \in W$. Then α, β are polynomials over F of degree $\leq n$.

If $a, b \in F$ then $a\alpha + b\beta$ will also be a polynomial of degree $\leq n$.

$\therefore a\alpha + b\beta \in W \Rightarrow$ W is a subspace of V.

Ex. 5. *Let V be the set of all* $n \times n$ *matrices and F be the field. If W is the subset of* $n \times n$ *symmetric matrices in V, show that W is a subspace of V (F).*

Sol: $O_{n\times n} = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \end{vmatrix}$ is a symmetric matrix

since $i-j$ th element of $O = 0 = j-i$ th element of 0

$\therefore W \neq \phi$. Let $a, b \in F$ and $P, Q \in W$ where $P = [p_{ij}], Q = [q_{ij}]$

$\because$ P, Q are symmetric, $p_{ij} = p_{ji}$ and $q_{ij} = q_{ji}$ ... (1)

$\therefore aP + bQ = a[p_{ij}] + b[q_{ij}] = [ap_{ij}] + [bq_{ij}] = [ap_{ij} + bq_{ij}]_{n\times n}$

The $i-j$ th element in $a\text{P}+b\text{Q} = ap_{ij}+bq_{ij} \quad = ap_{ji}+bq_{ji}$ (by(1))

$= j-i$ th element of $a\text{P}+b\text{Q}$

$\Rightarrow$ $a\text{P}+b\text{Q}$ is a symmetric $n\times n$ matrix $\Rightarrow$ $a\text{P}+b\text{Q}\in\text{W}$

$\therefore$ W is a vector subspace of V (F).

Ex. 6. *Let F be a field and A be a* $m\times n$ *matrix over F.*

$\text{F}_{1\times m}$ *is the set of all* $1\times m$ *matrices defined over f forming the vector space* $\text{F}_{1\times m}(\text{F})$.

Define $\text{W} = \{\text{X}=[x_1, x_2, \ldots x_n]\in\text{F}_{1\times m}\ 1\times\text{A}=\text{O}_{1\times n}\}$.

Prove that W is a subspace of $\text{F}_{1\times m}(\text{F})$.

Sol: Let $\text{X},\text{Y}\in\text{W}$ and $a,b\in\text{F}$. By def. of W, $\text{XA}=\text{O}_{1\times n}$ and $\text{YA}=\text{O}_{1\times n}$

Now $(a\text{X}+b\text{Y})\,\text{A} = (a\text{X})\,\text{A}+(b\text{Y})\,\text{A} = a\,(\text{XA})+b\,(\text{YA}) = a\,\text{O}_{1\times n}+b\,\text{O}_{1\times n} = \text{O}_{1\times n}$

$\therefore\ a\text{X}+b\text{Y}\in\text{W}$ $\qquad\therefore$ W is a subspace of $\text{F}_{1\times m}$ (F).

Note. W is called a solution space of XA = O.

EXERCISE 1 (*b*)

1. Let R be the field of real numbers. Show the set of triads that
(1) $\{(x,2y,3z)\,/\,x,y,z\in\text{R}\}$ ***(S. K. U. 2013)***
(2) $\{(x,x,x)\,/\,x\in\text{R}\}$ form the subspaces of $\text{R}^3(\text{R})$.

2. Let $\text{V}=\text{R}^3=\{(x,y,z)\,/\,x,y,z\in\text{R}\}$ and W be the set of triads (x,y,z) such that $x-3y+4z=0$. Show that W is a subspace of V (R).

3. Show that the set W of the elements of the vector space $\text{V}_3(\text{R})$ of the form $(x+2y,\,y,\,-x+3y)$ where $x,y\in\text{R}$ is a subspace of V_3 (R).

4. Prove that the set of solutions (x,y,z) of the equation $x+y+2z=0$ is a subspace of the space R^3 (R).

5. Show that the solutions of the differential equation $(\text{D}^2-5\text{D}+6)\,y=0$ is a subspace of the vector space of all real-valued continuous functions over R.
[**Hint.** Let $y=f(x)$ and $y=g(x)$ be two solutions.
$\therefore\ \text{D}^2f(x)-5\text{D}f(x)+6f(x)=0\ \ldots(1),\quad \text{D}^2g(x)-5\text{D}g(x)+6g(x)=0\ \ldots(2)$
(1) a + (2) b gives : $\text{D}^2[af(x)+bg(x)]-5\text{D}\,[af(x)+bg(x)]+6\,[af(x)+bg(x)]=0$
$\therefore\ af(x)+bg(x)$ is also a solution of D.E. etc.]

6. Show that the subset W defined below is not a subspace of R^3 (R).
(*i*) $\text{W}=\{(a,b,c)\,|\,a,b,c$ are rationals $\}$ (*ii*) $\text{W}=\{(a,b,c)\,|\,a^2+b^2+c^2\le 1\}$

7. Let V be the vector of $n\times n$ matrices over a field F. Show that W the set of matrices which commute with a given matrix T is a subspace of V.
$\text{W}=\{\text{A}=[a_{ij}]\in\text{V}:\text{AT}=\text{TA}\}$

8. V is the vector space of 2×2 matrices over a field F. Show that W is not a subspace of V where (*i*) $W = \{A \in V \mid \det. A = 0\}$ (*ii*) $W = \{A \in V \mid A^2 = A\}$

[**Hint.** (*i*) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in W.

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with det. $(A+B) \neq 0$ etc.

[**Hint.** (*ii*) Clearly $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$. As $(2I)^2 \neq 2I$ scalar multiplication in W fails].

9. Let V be the set of all real valued functions defined on $[-1,1]$. Which of the following are subspaces of V (R).

(*i*) $W_1 = \{f \in V \mid f(0) = 0\}$ (*ii*) $W_2 = \{f \in V \mid f(1) = f(-1)\}$

(*iii*) $W_3 = \{f \in V \mid f(x) = 0 \text{ if } x < 0\}$ (*iv*) $W_4 = \{f \in V \mid f(x) = f(-x)\}$

10. W is the subset of C^3. Which of the following are the subsapces of C^3 (C).

(*i*) $W = \{(\alpha_1, \alpha_2, \alpha_3) \in C^3 \mid \alpha_1 \text{ is real}\}$

(*ii*) $W = \{(\alpha_1, \alpha_2, \alpha_3) \in C^3 \mid \text{either } \alpha_1 = 0 \text{ or } \alpha_2 = 0\}$

11. Let x be an indeterminate over the field F. Which of the following are subspaces of F [x] over F ?

(*i*) All monic polynomials of degree atmost 10.

(*ii*) All polynomials having α and β in F as roots. (*iii*) All polynomials divisible by x.

(*iv*) All polynomials $f(x)$ such that $2f(0) = f(1)$.

12. Prove that $W = \{\lambda(1,1,1) \mid \lambda \in R\}$ is a subspace of R^3.

ALGEBRA OF SUBSPACES

1.16. Theorem. ***The intersection of any two subspaces W_1 and W_2 of vector space V (F) is also a subspace.***

Proof. W_1 and W_2 are subspaces V (F)

$\Rightarrow \bar{O} \in W_1$ and $\bar{O} \in W_2 \Rightarrow \bar{O} \in W_1 \cap W_2$ $\quad\therefore W_1 \cap W_2 \neq \phi$

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$ $\quad\therefore \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Now $a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

$\therefore a\alpha + b\beta \in W_1 \cap W_2$ $\quad\therefore W_1 \cap W_2$ is a subspace of V (F)

1.17. Theorem. ***The intersection of any family of subspaces of a vector space is also a subspace.***

Proof. Let V (F) be a vector space.

Let $W_1, W_2, W_3 \dots W_n$ be the n subspaces of V (F).

Let $W = \bigcap_{i=1}^{n} W_i = \{\alpha \in V : \alpha \in W_n \ \forall\, n\}$

$\bar{O} \in W_i$ for $i = 1, 2, \dots, n \Rightarrow \bar{O} \bigcap_{1}^{n} W_i$ and $\bigcap_{i=1}^{n} W_i \neq \phi$

Let $\alpha, \beta \in \bigcap_{i=1}^{n} W_i \Rightarrow \alpha, \beta \in W_n \ \forall\, n$

Since each W_n is a subspace we have $a, b \in F$ and $\alpha, \beta \in W_n \Rightarrow a\alpha + b\beta \in W_n \ \forall\, n$

$\Rightarrow a\alpha + b\beta \in \bigcap_{1}^{n} W_i$. Hence $\bigcap_{i=1}^{n} W_i$ is a subspace of V (F).

Note : The union of two subspaces of V (F) may not be a subspace of V (F).

e.g. Let W_1 and W_2 be two subspaces of $V_3(R)$ given by

$$W_1 = \{(0, y, 0) \mid y \in R\}, \quad W_2 = \{(0, 0, z) \mid z \in R\}$$

$\therefore\ W_1 \cup W_2 \{(0, y, 0) \cup (0, 0, z) \mid y, z \in R\}$

Now $(0, y, 0) + (0, 0, z) = (0, y, z) \notin W_1 \cup W_2$

Since neither $(0, y, z) \in W_1$ nor $(0, y, z) \in W_2$

Thus $W_1 \cup W_2$ is not closed under vector addition.

$\therefore\ W_1 \cup W_2$ is not a subspace of V (F).

1.18. Theorem. ***The union of two subspaces is a subspace if and only if one is contained in the other.***

Proof. Let W_1 and W_2 be two subspaces of V (F).

Condition is necessary. Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\therefore\ W_1 \cup W_2 = W_2$ or $W_1 \Rightarrow W_1 \cup W_2$ is a subspace of V (F)

Condition is sufficient. Let $W_1 \cup W_2$ be a subspace.

Let us suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

Now $W_1 \not\subset W_2 \Rightarrow$ there exists $x \in W_1$ and $x \notin W_2$ ... (1)

$W_2 \not\subset W_1 \Rightarrow$ there exists $y \in W_2$ and $y \notin W_1$ ... (2)

$\therefore\ x \in W_1 \cup W_2$ and $y \in W_1 \cup W_2$

$\Rightarrow x + y \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow x + y \in W_1$ or $x + y \in W_2$.

Now $x + y, x \in W_1$ (subspace) $\Rightarrow 1(x + y) + (-1)\, x \in W_1 \Rightarrow y \in W_1$ ... (3)

Similarly $x + y, y \in W_2$ (subspace) $\Rightarrow 1(x + y) + (-1)\, y \in W_2 \Rightarrow x \in W_2$ ... (4)

Thus (3) and (4) contradict (2) and (1). $\therefore$ Either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

1.19. LINEAR SUM OF TWO SUBSPACES

Definition. *Let* W_1 *and* W_2 *be two subspaces of the vector space V (F). Then the linear sum of the subspaces* W_1 *and* W_2*, denoted by* $W_1 + W_2$*, is the set of all sums* $\alpha_1 + \alpha_2$ *such that* $\alpha_1 \in W_1$, $\alpha_2 \in W_2$ *i.e.,* $W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$

1.20. Theorem. ***If*** W_1 ***and*** W_2 ***are any two subspaces of a vector space (F) then***

(i) $W_1 + W_2$ ***is a subspace of V (F).***

(ii) $W_1 \subseteq W_1 + W_2$ ***and*** $W_2 \subseteq W_1 + W_2$.

Proof. (*i*) Let $\alpha, \beta \in W_1 + W_2$. Then

$\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ where $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$.

If $a, b \in F$ then $a\alpha_1, b\beta_1 \in W_1$ ($\because$ W_1 subspace)

and $a\alpha_2, b\beta_2 \in W_2$ ($\because$ W_2 subspace)

Now $a\alpha + b\beta = a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) = (a\alpha_1, b\beta_1) + (a\alpha_2, b\beta_2) \in W_1 | W_2$

$\therefore$ $a, b \in F$ and $\alpha, \beta \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V (F).

(*ii*) $\alpha_1 \in W_1$ and $\bar{O} \in W_2 \Rightarrow \alpha_1 + \bar{O} \in W_1 + W_2$. $\therefore$ $\alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$

$\Rightarrow W_1 \subseteq W_1 + W_2$. Similarly, $W_2 \subseteq W_1 + W_2$. Hence $W_1 \cup W_2 \subseteq W_1 + W_2$.

SOLVED PROBLEMS

Ex.1. *Let V be the vector space of all functions from R into R. Let* S_e *be the subset of even functions,* $f(-x) = f(x)$ S_o *be the subset of odd functions,* $f(-x) = -f(x)$.

Prove that (1) S_e *and* S_o *are subspaces of V* (2) $S_e + S_o = V$ (3) $S_e \cap S_o = \{\bar{o}\}$

Sol: (1) Let $f_e, g_e \in S_e$ and $a, b \in R$

$\therefore$ $(af_e + bg_e)(-x) = af_e(-x) + bg_e(-x) = af_e(x) + bg_e(x) = (af_e + bg_e)(x)$

$\Rightarrow$ $af_e + bg_e$ is an even function.

$\therefore$ $f_e, g_e \in S_e, a$ and $b \in R \Rightarrow af_e + bg_e \in S_e$

Hence S_e is a subspace of V. Similarly we can prove that S_o is a subspace of V.

(2) Since S_e and S_o are subspaces of V. $S_e + S_o$ is also a subspace of V

$\therefore$ $S_e + S_o \subseteq V$. Let $g_e(x) = \frac{1}{2}[f(x) + f(-x)]$, $h_o(x) = \frac{1}{2}[f(x) - f(-x)]$

Clearly g_e is an even function and h_o is an odd function

Now $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = g_e(x) + h_o(x) = (g_e + h_o)(x)$

$\Rightarrow$ $f = g_e + h_o$ where $g_e \in S_e, h_o \in S_o$

Thus $f \in V \Rightarrow f \in S_e + S_o$ $\therefore$ $V \subseteq S_e + S_o$. Hence $S_e + S_o = V$.

(3) Let $\bar{O}$ denote the zero function *i.e.*, $\bar{O}(x) = 0 \; \forall \, x \in R$

Let $f \in S_e \cap S_o$ then $f(-x) = f(x)$ and $f(-x) = -f(x)$

$\therefore \; f(x) = -f(x) \Rightarrow \; 2\,f(x) = 0$

$\Rightarrow f(x) = 0 = \bar{O}(x) \Rightarrow f = \bar{O}$. Hence $S_e \cap S_o = \{\bar{O}\}$.

Ex.2. *Let* W_1 *and* W_2 *be subspaces of a vector space V such that* $W_1 + W_2 = V$ *and* $W_1 \cap W_2 = \{\bar{O}\}$. *Prove that for each vector* α *in V there are unique vectors* $\alpha_1 \in W_1, \alpha_2 \in W_2$ *such that* $\alpha = \alpha_1 + \alpha_2$.

Sol: $\alpha \in V \Rightarrow \alpha \in W_1 + W_2$ $(\because \; V = W_1 + W_2)$

$\Rightarrow \; \alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1, \alpha_2 \in W_2$

If possible let $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W_1, \beta_2 \in W_2$

$\therefore \; \alpha_1 + \alpha_2 = \beta_1 + \beta_2$. $\alpha_1 - \beta_1 = \beta_2 - \alpha_2$

Now $\alpha_1 - \beta_1 \in W_1$ and $\beta_2 - \alpha_2 \in W_2$

$\Rightarrow \; \alpha_1 - \beta_1 \in W_1$ and $\alpha_1 - \beta_1 \in W_2$

$\Rightarrow \; \alpha_1 - \beta_1 \in W_1 \cap W_2 \; \Rightarrow \; \alpha_1 - \beta_1 = \bar{O}$ $(\because \; W_1 \cap W_2 = \{\bar{O}\})$

$\Rightarrow \; \beta_2 - \alpha_2 = \bar{O}$

$\Rightarrow \; \alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Hence $\alpha = \alpha_1 + \alpha_2$ is unique.

1.21. LINEAR COMBINATION OF VECTORS

Definition. *Let V (F) be a vector space. If* $\alpha_1, \alpha_2, ..., \alpha_n \in V$ *then any vector* $\gamma = a_1 \, \alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ *where* $a_1, a_2, ..., a_n \in F$ *is called a linear combination of the vectors* $\alpha_1, \alpha_2, ..., \alpha_n$.

Note.1. γ is a vector belonging to V (F).

1.22. LINEAR SPAN OF A SET

Definition. *Let S be a non-empty subset of a vector space V (F). The linear span of S is the set of all possible linear combinations of all possible finite subsets of S.*

The linear span of S is denoted by L (S)

$\therefore \; L(S) = \{ \gamma : \gamma = \sum a_i \alpha_i, \, a_i \in F, \alpha_i \in S \}$

Note.1. S may be a finite set but L (S) is infinite set.

L (S) is said to be generated or spanned by S

2. If S is an empty subset of V then we define L (S) = {0}

3. $S \subseteq L\{S\}$

1.23. Theorem. ***The linear span L(S) of any subset S of a vector space V (F) is a subspace of v (F).***

Proof. Let $\alpha, \beta \in L(S)$ and $a, b \in F$

$\therefore\ \alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m, \qquad \beta = b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n$

where $a'_i s, b'_i s \in$ F and $\alpha'_i s, \beta'_i s \in$ S

$\therefore\ a\,\alpha + b\,\beta = a\,(a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m) + b\,(b_1\beta_1 + b_2\beta_2 + ... + b_n\beta_n)$

$= (a\,a_1)\,\alpha_1 + (a\,a_2)\,\alpha_2\ ... + (aa_m)\alpha_m + (b\,b_1)\beta_1 + (b\,b_2)\beta_2 + ... + (bb_n)\beta_n$

$\Rightarrow a\,\alpha + b\,\beta$ is a linear combination of finite set

$\alpha_1, \alpha_2 ... \alpha_m, \beta_1, \beta_2 ... \beta_n$ of the elements of S. $\Rightarrow\ (a\,\alpha + b\,\beta) \in$ L (S)

Thus $a, b \in$ F and $\alpha, \beta \in$ L (S) $\Rightarrow\ (a\,\alpha + b\,\beta) \in$ L (S) $\quad\therefore$ L (S) is a subspace of V (F).

1.24. Theorem. ***Let S be a non-empty subset of the vector space V (F). The linear span L (S) is the intersection of all subspaces of V which contain S.***

Proof. Let W be the subspace of V (F) containing S. Every linear combination of finite set of elements of S is an element of W as W is closed.

But the set of every linear combination of finite set of elements of S is L (S).

$\therefore$ L (S) $\subseteq$ W

$\therefore$ L (S) $\subseteq$ Intersection of all subspaces of V containing S $\quad(\because$ S $\subseteq$ L (S))

The intersection of all subspaces of V containing S $\subseteq$ L (S)

Hence L (S) is the intersection of all subspaces of V containing S.

SOLVED PROBLEMS

Ex.1. *Express the vector* $\alpha = (1, -2, 5)$ *as a linear combination of the vectors* $e_1 = (1,1,1), e_2 = (1,2,3)$ *and* $e_3 = (2, -1, 1)$

Sol: Let $\alpha = (1, -2, 5) = x\,(1,1,1) + y\,(1,2,3) + z\,(2, -1, 1)$

$= (x + y + 2z,\ x + 2y - z,\ x + 3y + z)$

Hence $x + y + 2z = 1,\ x + 2y - z = -2,\ x + 3y + z = 5$

Reducing to echelon form we get $\alpha + \beta + 2\gamma = 1,\ \beta - 3\gamma = -3,\ 5\gamma = 10$

These equations are consistant and have a solution given by $\alpha = -6, \beta = 3, \gamma = 2$

Hence $\alpha = -6\,e_1 + 3e_2 + 2e_3$

Ex.2. *Show that the vector* $\alpha = (2, -5, 3)$ *in* R^3 *cannot be expressed as a linear combination of the vectors* $e_1 = (1, -3, 2), e_2 = (2, -4, -1)$ *and* $e_3 = (1, -5, 7)$

Sol: Let $\alpha = ae_1 + be_2 + ce_3$

i.e., $(2, -5, 3) = a\,(1, -3, 2) + b\,(2, -4, -1) + c\,(1, -5, 7)$

$= (a + 2b + c,\ -3a - 4b - 5c,\ 2a - b + 7c)$

$\Rightarrow \quad a + 2b + c = 2, \quad -3a - 4b - 5c = -5, \quad 2a - b + 7c = 3$

Now reducing to echelan form we get $a + 2b + c = 2,\ 2b - 2c = 1,\ 0 = 3$

As the system is inconsistant, the equations have no solution. Hence α cannot be expressed as a $l.c$ of e_1, e_2, e_3.

Ex.3. *The subset* $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ *of* $V_3(R)$ *generates the entire vector space* $V_3(R)$.

Sol: Let $(a,b,c) \in V$ then we can write $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\Rightarrow (a,b,c) \in L(S) \;\Rightarrow\; V \subseteq L(S)$. But $L(S) \subseteq V$ Hence $L(S) = V$.

Ex.4. *In the vector space* $R^3(R)$. *Let* $\alpha = (1,2,1), \beta = (3,1,5), \gamma = (3,-4,5)$. *Show that subspace spanned by* $S = \{\alpha, \beta\}$ *and* $T = \{\alpha, \beta, \gamma\}$ *are the same.*

Sol: $L(T) = l.c.$ of the vectors of $T = \{a\alpha + b\beta + c\gamma \,/\, a, b, c \in R\}$

Let $\gamma = x\alpha + y\beta \Rightarrow (3,-4,5) = x(1,2,1) + y(3,1,5)$

$\therefore\; x + 3y = 3, 2x + y = -4, x + 5y = 5$

$x = -3, y = 2$ satisfy all the three equations. $\therefore\; (3,-4,5) = -3(1,2,1) + 2(3,1,5)$

$\therefore\; c(3,-4,5) = -3c(1,2,1) + 2c(3,1,5) \Rightarrow c\gamma = -3c\alpha + 2c\beta$

$\therefore\; a\alpha + b\beta + c\gamma = a\alpha + b\beta - 3c\alpha + 2c\beta$

$= (a-3c)\alpha + (b+2c)\beta = p\alpha + q\beta$ when $p, q \in R$

$\therefore$ But $p\alpha + q\beta \in L(S)$. $\therefore\; L(T) = L(S)$.

Ex.5. *In the vector space* $R^3(R)$ *let S = {91, 0, 0), (0, 1, 0)}. Find L (S).*

Sol: $L(S) = \{\alpha / \alpha = a(1,0,0) + b(0,1,0), a, b \in R\} = \{\alpha / \alpha = (a,b,0); a, b \in R\}$

Geometrically the linear span is the plane $Z = 0$.

1.25. Theorem. ***If*** W_1 ***and*** W_2 ***are two subspaces of a vector space V (F) then*** $L(W_1 \cup W_2) = W_1 + W_2$.

Proof. Let $\alpha_1 \in W_1$ and $\bar{O} \in W_2$ $\therefore\; \alpha_1 \in W_1$ and $\bar{O} \in W_2 \Rightarrow \alpha_1 + \bar{O} \in W_1 + W_2$

$\therefore$ Every $\alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore\; W_1 \subseteq W_1 + W_2$. Similarly $W_2 \subseteq W_1 + W_2$. Hence $W_1 \cup W_2 \subseteq W_1 + W_2$

Also W_1 and W_2 subspaces of V $\Rightarrow W_1 \cup W_2$ subspace of $W_1 + W_2$.

$\therefore\; L(W_1 \cup W_2)$ is a subspace of $W_1 + W_2$. Again let $\alpha \in W_1 + W_2$

$\therefore$ By def. $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ $\therefore\; \alpha_1, \alpha_2 \in W_1 \cup W_2$

Now, $\alpha = \alpha_1 + \alpha_2 = 1.\alpha_1 + 1.\alpha_2 = l.c.$ of elements of $W_1 \cup W_2$

$\therefore\; \alpha \in L(W_1 \cup W_2)$ $\therefore\; W_1 + W_2 \subseteq L(W_1 \cup W_2)$ ... (*i*)

Also we know that $L(W_1 \cup W_2)$ is the smallest subsapce containing $W_1 \cup W_2$ and $W_1 \cup W_2 \subseteq W_1 + W_2$. $\therefore\; L(W_1 \cup W_2) \subseteq W_1 + W_2$ ... (*ii*)

From (*i*) and (*ii*) $L(W_1 \cup W_2) = W_1 + W_2$

1.26. Theorem. ***If S is a subset of a vector space V (F) then prove that***
(I) ***S is a subspace of V*** $\Leftrightarrow$ ***L (S) = S*** ***(S. V. U. 2001/O)*** **(II)** ***L (L (S)) = L (S).***

Proof. I. (*i*) Let S be a subspace of V,

Let $\alpha \in L(S)$. Then $\alpha = a_1 \alpha_1 + a_2 \alpha_2 + ... + a_m \alpha_m$ where $a_1, a_2, ... a_m \in F$

and $\alpha_1, \alpha_2 ... \alpha_m \in S$

$\because$ S is a subspace of V, it is closed w.r. to scalar multiplication and vector addition.

$\therefore a_1\alpha_1 + a_2\alpha_2 ... + a_m\alpha_m = \alpha \in S$

Thus $\alpha \in L(S) \Rightarrow \alpha \in S$ $\therefore L(S) \subseteq S$

Let $\beta \in S$. Now $\beta = 1 . \beta = l.c.$ of infinite elements of S

$\Rightarrow \beta \in L(S)$ $\therefore S \subseteq L(S)$. Hence L (S) = S.

(*ii*) Suppose L (S) = S.

We know that L (S) is a subspace of *v*. $\Rightarrow$ S is a subspace of V.

II. We know that L (S) is a subspace of V. By **I**, L (L (S)) = L (S).

1.27. Theorem. ***If S, T are the subsets of a vector space V (F), then***
(I) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ **(II)** $L(S \cup T) = L(S) + L(T)$

Proof. (I) Let $\alpha \in L(S)$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ where $a_i's \in F$ and $\alpha_i's \in S$

$\because S \subseteq T$, then $\{\alpha_1, \alpha_2 ... \alpha_n\} \subseteq T$

$\therefore \alpha = l.c.$ of finite subset of T $\Rightarrow \alpha \in L(T)$

Hence $L(S) \subseteq L(T)$.

(II) Let $\alpha \in L(S \cup T)$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 ... + b_n\beta_n$ where $a_1, a_2 ... a_m, b_1, b_2 ... b_n \in F$

$\alpha_1, \alpha_2 ... \alpha_m \in S$ and $\beta_1, \beta_2 ... \beta_n \in T$.

(*i*) But $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m \in L(S)$, $b_1\beta_1, b_2\beta_2 + ... + \beta_n \in L(T)$

$\therefore \alpha =$ an element of L (S) + an element of L (T)

$\therefore \alpha \in L(S) + L(T) \Rightarrow L(S \cup T) \subseteq L(S) + L(T)$ ... (1)

(*ii*) Let $\alpha \in L(S) + L(T)$

$\therefore \alpha = \gamma + \delta$ where $\gamma \in L(S)$, $\delta \in L(T)$ $\therefore \gamma =$ *l.c.* of a finite no. of elements of S

$\delta =$ *l.c.* of a finite no. of elements of T

$\therefore \alpha = \gamma + \delta =$ *l.c.* of a finite no. of elements of $S \cup T \Rightarrow \alpha = \gamma + \delta \in L(S \cup T)$

$\therefore L(S) + L(T) \subseteq L(S \cup T)$ ... (2)

Hence from (1) and (2) : $L(S \cup T) = L(S) + L(T)$.

EXERCISE 1 (*c*)

1. Show that each of the following set of vectors generates $R^3(R)$.

(*i*) $\{(1,0,0), (1,1,0), (1,1,1)\}$ (*ii*) $\{(1,2,3), (0,1,2), (0,0,1)\}$ (*iii*) $\{(1,2,1), (2,1,0), (1,-1,2)\}$

2. If $\alpha = (1,2,-1), \beta = (2,-3,2), \gamma = (4,1,3)$ and $\delta = (-3,1,2)$ are the vectors of $V_3(R)$ show that $L(\{\alpha, \beta\}) \neq L(\{\gamma, \delta\})$.

3. Prove that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S. **(*N. U.*)**

1.28. LINEAR DEPENDENCE OF VECTORS

Definition. *Let V (F) be a vector space. A finite subset* $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *of vectors of V is said to be a linearly dependent (L.D.) set if there exist scalars* $\alpha_1, \alpha_2, ..., \alpha_n \in F$, *not all zero, such that* $a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O}$.

1.29. LINEAR INDEPENDENCE OF VECTORS

Definition. Let V (F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be linearly independent (L.I,) if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O}, a_i's \in F$$

$$\Rightarrow a_1 = 0, a_2 = 0, ..., a_n = 0.$$

1.30. Theorem. ***Every superset of a linearly dependent (L.D) set of vectors is linearly dependent (L.D.).***

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a linearly dependent set of vectors.

$\therefore$ There exist scalars $a_1, a_2, ..., a_n \in F$, not all zero such that

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \bar{O} \quad ... (1)$$

Let $S' = \{\alpha_1, \alpha_2 ... \alpha_n, \beta_1, \beta_2 ... \beta_n\}$ be a super set of S. then (1) can be re-written as

$$a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n + 0\beta_1, 0\beta_2 + ... + 0\beta_n = \bar{O} \quad ... (2)$$

In (2) all the scalars are not zero $\Rightarrow$ S' is linearly dependent (L.D.)

Hence any super set of an L.D. set is L.D.

1.31. Theorem. ***Every non-empty subset of a linearly independent (L.I.) set of vectors is linearly independent (L.I.)***

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a L.I. set of vectors

Let us consider the subset $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ where $1 \leq k \leq m$.

Now $a_1\alpha_1 + a_2\alpha_2 + ... + a_k\alpha_k = \bar{O}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_m = \bar{O}$

$\Rightarrow a_1 = 0, a_2 = 0, a_k = 0$ ($\because$ S is L.I. set.)

Hence the subset $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is L.I.

1.32. Theorem. ***A set of vectors which contains atleast one zero vector is linearly dependent.***

Proof. Let $\alpha_1 = \bar{O}$ and $\alpha_2, \alpha_3 \ldots \alpha_m \neq \bar{O}$

Then $1\,\alpha_1 + 0\alpha_2 + 0\alpha_3 \ldots + \bar{O}\alpha_m = \bar{O},\ a_1 = 1 \in$ F with atleast one scalar $a_1 \neq 0$

$\Rightarrow \{\bar{O}, \alpha_2, \ldots, \alpha_m\}$ L.D. set.

Hence the set of vectors containing zero vector is linearly dependent (L.D.).

Note. A L.I. subset of a vector space V (F) does not contain zero vector.

1.33. Theorem. ***A single non - zero vector forms a linearly independent (L.I.) set.***

Proof. Let L = $\{\alpha\}$ be a subset of V (F) where $a \neq \bar{O}$

If $a \in$ F then $a\alpha = \bar{O} \Rightarrow a = 0$. $\therefore$ The set S is linearly independent.

Note. $\{\alpha\}$ is a L.D. set.

SOLVED PROBLEMS

Ex.1. *Show that the system of vectors (1,3,2),* $(1,-7,-8), (2,1,-1)$ *of* $V_3(R)$ *is linearly dependent.*

Sol. Let $a, b, c \in R$, then $a\,(1,3,2) + b\,(1,-7,-8) + c\,(2,1,-1) = \bar{O}$

$\Rightarrow (a+b+2c, 3a-7b+c, 2a-8b-c) = (0,0,0)$

$\Rightarrow a+b+2c = 0, 3a-7b+c = 0, 2a-8b-c = 0$

$\Rightarrow a = 3, b = 1, c = -2$ $\therefore$ The given vectors are linearly dependent.

Ex.2. *Show that the system of vectors* $(1,2,0), (0,3,1), (-1,0,1)$ *of* $V_3(Q)$ *is L.I. where Q is the field of rational numbers.*

Sol. Let $x, y, z, \in Q$ then $x\,(1,2,0) + y\,(0,3,1) + z\,(-1,0,1) = \bar{O}$

$\Rightarrow (x-z, 2x+3y, y+z) = (0,0,0)$

$\Rightarrow x-z = 0, 2x+3y = 0, y+z = 0 \Rightarrow x = 0, y = 0, z = 0$

Hence the system is L.I.

Ex.3. *If two vectors are linearly dependent, prove that one of them is a scalar multiple of the other.*

Sol: Let α, β be two L.D. vectors of V (F)

Then there exist $a, b \in F$, not both zero such that $a\alpha + b\beta = \bar{O}$

Let $a \neq 0$, then $\alpha = \left(-\frac{b}{a}\right)\beta$ $\Rightarrow$ α is a scallar multiple of β

If $b \neq 0$, similarly we get $\beta = a$ scalar multiple of α.

Ex.4. *Prove that the vectors* (x_1, y_1) *and* (x_2, y_2) *of* $V_2(F)$ *are L.D. if* $x_1 y_2 - x_2 y_1 = 0$.

Sol. Let $a, b \in F$ then $a(x_1 y_1) + b(x_2 y_2) = (0,0)$

$\Rightarrow \ (a x_1 + b x_2, a y_1 + b y_2) = (0,0) \ \Rightarrow \ a x_1 + b x_2 = 0, a y_1 + b y_2 = 0$

These equations are consistant.

If det. $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = 0 \ \Rightarrow \ x_1 y_2 - x_2 y_1 = 0$

Ex.5. *In the vector space* $V_n(F)$, *the system of n vectors* $e_1 = (1,0,0\dots 0)$, $e_2 = (0,1,0\dots 0)$, ..., $e_n = (0,0,\dots 1)$ *is linearly independent where 1 is unity of f.*

Sol. Let $a_1, a_2, \dots, a_n \in F$, then $a_1 e_1 + a_2 e_2 + \dots + a_n e_n = \bar{O}$

$\Rightarrow \ a_1(1,0,\dots,0) + a_2(0,1,0\dots)\dots + a_n(0,0,\dots,0,1) = \bar{O}$

$\Rightarrow \ (a_1, a_2, \dots, a_n) = (0,0,\dots,0) \Rightarrow \ a_1 = 0, a_2 = 0, \ \dots a_n = 0$

$\Rightarrow$ The given set of vectors is linearly independent.

Ex.6. *Prove that the four vector* $\alpha = (1,0,0), \beta = (0,1,0)$, $\gamma = (0,0,1)$, $\delta = (1,1,1)$ *in* $V_3(C)$ *form L.D. set, but any three of them are L.I.*

Sol. Let $a, b, c, d \in C$ $\quad \therefore \ a\alpha + b\beta + c\gamma + d\delta = \bar{O}$

$\Rightarrow \ a(1,0,0) + b(0,1,0) + c(0,0,1) + d(1,1,1) = \bar{O}$

$\Rightarrow \ (a+d, b+d, c+d) = (0,0,0)$

$\Rightarrow \ a+d = 0, \ b+d = 0, c+d = 0 \qquad \therefore \ a = -d, \ b = -d, \ c = -d$

Thus if $d = -k$, then $a = k, b = k, c = k$, showing that $\alpha + \beta + \gamma - \delta = \bar{O}$

$\therefore$ The four vectors $\alpha, \beta, \gamma, \delta$ are L.D.

(*ii*) But $a\alpha + b\beta + c\delta = \bar{O} \ \Rightarrow \ a(1,0,0) + b(0,1,0) + c(1,1,1) = \bar{O}$

$\Rightarrow \ (a+c, b+c, c) = (0,0,0) \Rightarrow \ a+c = 0, b+c = 0, c = 0$,

$\Rightarrow \ a = 0, b = 0, c = 0 \ \Rightarrow$ the vectors α, β, δ are L.I.

Similarly we can show that any other three vectors are L.I.

Ex.7. *If* α, β, γ *are linearly independent vectors of V (R) show that* $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ *are also L.I.*

Sol. Let $a, b, c \in R$.

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = \bar{O} \ \Rightarrow \ (a+c)\alpha + (a+b)\beta + (b+c)\gamma = \bar{O}$

Given α, β, γ are linearly independent.

$\Rightarrow \ a + 0.b + c = 0, \quad a + b + 0.c = 0, \quad 0.a + b + c = 0$.

Now the coefficient matrix A of these equations is $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$,

where rank of A = 3 *i.e.,* equal to the no. of unknowns.

$\Rightarrow$ $a = 0, b = 0, c = 0$ is the only solution of the given equations.

$\therefore$ $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also L.I.

Ex.8. *Let* $F(x)$ *be the vector space of all polynomials over the field F. Show that the infinite set* $S = \{1, x, x^2, ...\}$ *is linearly independent.*

Sol. Let $S' = \{x^{m_1}, x^{m_2}, ..., x^{m_n}\}$ be any finite subset of having a vectors, where $m_1, m_2, ... m_n$ are non-negative integers.

Let $a_1, a_2 ... a_n \in F$

$\therefore$ $a_1 x^{m^1} + a_2 x^{m^2} + ... + a_n x^{m^n} = \bar{O}$ (*i.e.* zero polynomial)

Then by definition of equality of two vectors we have $a_1 = 0, a_2 = 0, ..., a_n = 0$

$\Rightarrow$ Every finite subset of S is L.I. and hence s is linearly independent.

Ex.9. *Let V be the vector space of* 2×3 *matrices over R. Show that the vectors A, B, C form L.I set where*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

Sol. Let $a, b, c \in R$ then for L.I.

$a A + b B = c C = 0 \quad \Rightarrow \quad a = 0, b = 0, c = 0$

Now $a A + b B = c C = 0$

$$\Rightarrow a \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a-2c & 4a+5b-3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $2a+b+4c = 0$, $a+b-c = 0$, $-a-3b+2c = 0$

$3a-2b+c = 0$, $-2a-2c = 0$, $4a+5b-3c = 0$

These equations have only one solution $a = 0, b = 0, c = 0$.

Ex.10. *Prove that the set* $\{1, i\}$ *is L.D. in the vector space C (C) but is L.I. in the vector space C (R).*

Sol. Since $(-i)\, i = 1$ with $(-i) \in C$, (one vector is a multiple of other)

$\Rightarrow$ the set $\{1, i\}$ in C (C) is L.D.

There exists no real number a such that $a\,i = 1$.

Hence $\{1, i\}$ is not L.D and so L.I in C (R).

Ex. 11. *Prove that the set* $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ *is linearl*

Sol. Let $\alpha_1 = (1,0,0,-1), \alpha_2 = (0,1,0,-1), \alpha_3 = (0,0,1,-1), \alpha_4 = (0,0,0,1)$

Let $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = \bar{O}, \quad \bar{O} \in V_4(R)$

$\Rightarrow a_1(1,0,0,-1) + a_2(0,1,0,-1) + a_3(0,0,1,-1) + a_4(0,0,0,1) = (0,0,0,0)$

$\Rightarrow (a_1, a_2 + a_2, a_3, -a_1 - a_2 - a_3 + a_4) = (0,0,0,0)$

$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

Since all the co-ordinates are equal to zer0, the set is linearly independent.

1.34. Theorem. ***Let V (F) be a vector space and S = $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of non-zero vectors of V (F). Then S is linearly dependent if and only if some vector $\alpha_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceding vectors.***

Proof. Given S = $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n\}$ is L.D. Then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + .. \, a_n\alpha_n = \bar{O}$

Let k be the greatest suffix of a for which $\alpha_k \ne 0$.

Then $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_n = \bar{O} \Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k = \bar{O}$

Now suppose $k = 1$, (1) $\Rightarrow a_1\alpha_1 = \bar{O}$

But $a_1 \ne 0$, so $\alpha_1 = 0$, which contradicts that each element of S is a non-zero vector.

Hence $k > 1$, *i.e.*, $2 \le k \le n$.

Again from (1) we have $a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 \ldots . - a_{k-1}\alpha_{k-1}$

$\Rightarrow \quad a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 \ldots . - a_{k-1}\alpha_{k-1})$

$\Rightarrow \quad \alpha_k = (-a_k^{-1}a_1)\,\alpha_1 + (-a_k^{-1}a_2)\,\alpha_2 + \ldots . + (-a_k^{-1}\alpha_{k-1})\,\alpha_{k-1}$

= (L.C. of its preceding vectors).

Conversely. Let some $\alpha_p \in S$ be expressible as a linear combination of its preceding vectors *i.e.*, for $b_1, b_2, \ldots, b_{p-1} \in F$.

$\alpha_p = b_1\,\alpha_1 + b_2\,\alpha_2 + \ldots + b_{p-1}\alpha_{p-1} \Rightarrow b_1\,\alpha_1 + b_2\,\alpha_2 + \ldots + b_{p-1}\alpha_{p-1} + (-1)\,\alpha_p = \bar{O}$

$\Rightarrow$ the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is L.D. ... [(– 1) is a non-zero coefft.]

Hence the superset S = $\{\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots, \alpha_n\}$ is L.D.

Note. If β is a linear combination of the set of vectors $\alpha_1, \alpha_2, ..., \alpha_n$, then the set of vectors $\{\beta, \alpha_1, \alpha_2, ..., \alpha_n\}$ is L.D.

1.35. Theorem. ***Let*** $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ ***be a subset of the vector space V (F). If*** $\alpha_i \in S$ ***is a linear combination of its preceding vectors then L (S) = (S') where*** $S' = \{\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n\}$

Proof. Clearly $S' \subset S \Rightarrow L(S') \subset L(S)$. Let $\beta \in L(S)$ then for $a_1, a_2 ... a_n \in F$.

$\therefore \ \beta = a_1 \alpha_1 + a_2 \alpha_2 + ... + a_i \alpha_i + ... + a_n \alpha_n$

But given that $\alpha_i = b_1 \alpha_1 + b_2 \alpha_2 ... + b_{i-1} \alpha_{i-1} \ \forall \ b's \in F$

$\therefore \ \beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_i (b_1\alpha_1 + b_2\alpha_2 ... + b_{i-1} \alpha_{i-1}) + a_{i+1} \alpha_{i+1} + ... + a_n \alpha_n$

$= (a_1 + a_i b_1) \alpha_1 + (a_2 + a_i b_2) \alpha_2 + ... + (a_{i-1} + a_i b_{i-1}) \alpha_{i-1} + a_{i+1}\alpha_{i+1} ... + a_n\alpha_n$

$=$ L.C. of the elements of S'

$\therefore \ \beta \in L(S') \Rightarrow L(S) \subset L(S')$. Hence $L(S) = L(S')$.

EXERCISE 1 (*d*)

1. Determine whether the following sets of vectors are L.D. or L.I.

(*a*) $\{(2,-3), (6,-9)\}$ in $R^2(R)$ (*b*) $\{(4,3,-2), (2,-6,7)\}$ in $V_3(R)$.

(*c*) $\{A, B\}$ where $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 & 12 \\ 9 & 3 & -3 \end{bmatrix}$ in the vector space of 2×3 matrices over R.

(*d*) $\{p_1(x), p_2(x)\}$ where $p_1(x) = 1 - 2x + 3x^2$ and $p_2(x) = 2 - 3x$ in the vector space of all polynomials over R.

2. Determine whether the following sets of vectors in $R^3(R)$ are L.D or L.I.

(*a*) $\{(1,-2,1), (2,1,-1), (7,-4,1)\}$ (*b*) $\{(2,0,5), (3,-5,8), (4,-2,1), (0,0,1)\}$

(*c*) $\{(-1,2,1), (3,0,-1), (-5,4,3)\}$

3. Determine whether each of the following sets of vectors $V_4(Q)$ is or L.D. or L.I., Q is the field of rational numbers.

(*a*) $\{(2,1,1,1), (1,3,1,-2), (1,2,-1,3)\}$ (*b*) $\{(0,1,0,1), (1,2,3,-1), (1,0,1,0), (0,3,2,0)\}$

(*c*) $\{(1,2,-1,1), (0,1,-1,2), (2,1,0,3), (1,1,0,0)\}$

(*d*) $\{(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)\}$

4. Show that the set $\{1, x, x - x^2\}$ is L.I., set of vectors in F (x) over the field of real numbers.

5. If α, β, γ are the vectors of V (F), and $a, b \in F$ show that the set $\{\alpha, \beta, \gamma\}$ is L.D, if the set $\{\alpha + a\beta + b\gamma, \beta, \gamma\}$ is linearly dependent.

6. If $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is any linearly independent subset of a vector space V and $\beta \notin L(S)$, then prove that $\alpha_1, \alpha_2 \ldots \alpha_n, \beta\}$ is linearly independent.

7. Under what conditions on the scalar β is the subset $\{(\beta, 1, 0), (1, \beta, 1), (0, 1, \beta)\}$ of R^3, linearly dependent ?

8. If $f(x)$ is a polynomial of degree n with real coefficients prove that

$\left\{ f(x), \frac{d}{dx} f(x), \frac{d^2}{dx^2} f(x), \ldots \frac{d^n}{dx^n} f(x) \right\}$ is a linearly independent set.

Basis and Dimension

2.1. BASIS OF VECTOR SPACE

Definition. *A subset S of a vector space V (F) is said to be the basis of V, if*

(i) S is *linearly independent.* (ii) *the linear span of* S is V *i.e.,* $L(S) = V$

Note: A vector space may have more than one basis.

2.2. FINITE DIMENSIONAL SPACE.

Definition. *A vector space V (F) is said to be finite dimensional if it has a finite basis.* (OR)

A vector space V (F) is said to be finite dimensional if there is a finite subset S in V such that L (S) = V.

Ex. 1. *Show that the set of n vectors* $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), e_n = (0, 0, \ldots, 1)$ *is a basis of* $V_n(F)$.

Sol. Let S = $\{e_1, e_2, \ldots, e_n\}$. ($i$) It can be easily verified that the set S is L.I,

(ii) Now any vector $\alpha = (a_1, a_2 \ldots a_n) \in V_n(F)$ can be put in the form

$(a_1, a_2, \ldots, a_n) = a_1(1, 0, \ldots, 0) + a_2(0, 1, \ldots, 0) \ldots + a_n(0, \ldots, 1) = a_1e_1 + a_2e_2 + \ldots + a_ne_n$

$=$ *l. c.* of elements of the set S $\Rightarrow \alpha \in L(S)$.

$\therefore \alpha \in V \Rightarrow \alpha \in L(S)$ $\therefore$ V = L (S). Hence S is a basis of $V_n(F)$

Note 1. The set $S = \{e_1, e_2, \ldots, e_n\}$ is called the standard basis of $V_n(F)$ or F^n

Note 2. The set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the standard basis of $V_3(R)$ or R^3.

Ex. 2. *Show that the infinite set* $S = \{1 . x, x^2, \ldots, x^n, \ldots\}$ *is a basis of the vector space* $F[x]$ *over the field F.*

Sol. (i) The set S is clearly L. I. (ii) Let $f(x) = a_0 + a_1x + \ldots + a_nx^n \in F[x]$

$\therefore$ where $a_0, a_1, \ldots, a_n \in F$

$\therefore f(x) \in L(S) \Rightarrow F[x] = L(S)$ $\therefore$ S is a basis of $F[x]$

Let $S' = \{1, x, x^2, \ldots, x^m\}$ be a finite subset of S. If $g(x) = c_0 + c_1 x + c_2 x^2 \ldots + c_nx^n$ and $n > m$ it is not possible to write $g(x)$ as a *l.c.* of the element of S'.

Hence the vector space $F[x]$ is not finite dimensional.

2.3 Theorem. ***IF V(F) is a finite dimensional vector space, then there exists a basis set of V.***

Proof. Since V (F) is finite dimensional, there exists a finite set S such that, L (S) = V

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$

We may assume that S does not contain the $\bar{O}$ vector.

If S is L. I., then S is a basis set of V. If S is L. D. set, then there exists a vector $\alpha_i \in S$ which can be expressed as a linear combination of the preceding vectors,.

Omitting α_i from S, let $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n\} \Rightarrow S_1 \subset S$

By previous theorem $L(S_1) = L(S)$. Now $L(S) = V \Rightarrow L(S_1) = V$

If S_1 is L.I set then S_1 will be a basis of V.

If S_1 is linearly dependent, then proceeding as above for a finite number of steps, we will be left with a L. I. set S_k and $L(S_k) = V$. Hence S_k will be the basis of V (F)

Thus there exists a basis set for V (F)

2.4. BASIS EXTENSION

Theorem. ***Let V (F) be a finite dimensional vector space and*** $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ ***a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.***)

Proof. Given $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ is L. I. subset of V.

Since V (F) is finite dimensional it has a finite basis B. Let $B = \{\beta_1, \beta_2 ... \beta_n\}$

Now consider the set $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ written in this order. Clearly L (S) = V.

Each α can be expressed as a l.c. of β's as B is the basis of V $\Rightarrow S_1$ is L. D.

Hence some vector in S_1 can be expressed as a *l.c.* of its proceeding vectors. This vector cannot be any of α's, since S is L. I. So this vector must be some β_i.

Consider now the set.

$S_2 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_{i-1}, \beta_{i+1}, ..., \beta_n\} = S_1 - \{\beta_i\}$ obviously $L(S_n) = L(S_1) = V$.

If S_2 is L. I., then S_2 forms a basis of V and it is the extended set.

If S_2 is L. D. then continue this procedure till we get a set $S_k \subset S$ such that S_k is L. I.

$\therefore L(S_k) = L(S) = V$

S_k will be extended set of S forming a basis of V.

Note. 1. Every basis is a spanning set but every spanning set is not a basis.

2. If a basis of V(F) contains n elements, then $m \le n$.

2.5. Theorem. *Let* $S=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ *be a basis set of a finite dimensional vector space V (F). Then for every* $\alpha \in V$ *there exists a unique set of scalars* $a_1, a_2, ..., a_n \in F$ *such that* $\alpha = a_1\,\alpha_1 + a_2\,\alpha_2 + \ldots + a_n\,\alpha_n$

Proof. If possible let there exist another set of scalars $b_1, b_2 \ldots b_n \in F$ such that

$\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

$\therefore\ a_1\,\alpha_1 + a_2\,\alpha_2 + \ldots + a_n\,\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

$\Rightarrow\ (a_1 - b_1)\,\alpha_1 + (a_2 - b_2)\,\alpha_2 \ldots + (a_n - b_n)\,\alpha_n = \bar{O}$

Since the set $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is L. I.

we have $a_1 - b_1 = 0, a_2 - b_2 = 0 \ldots a_n - b_n = 0 \Rightarrow a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$

2.6. COORDINATES

Definition. *Let* $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ *be the basis set of a finite dimensional vector space* $V(F)$. *Let* $\beta \in V$ *be given by* $\beta = a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n$ *for* $a_1, a_2, \ldots, a_n \in F$ *then the scalars* $(a_1, a_2, \ldots, a_n)$ *are called the coordinates.*

Note. Coordinates change with the change of basis.

SOLVED PROBLEMS

Ex. 1. *Show that the vector (1,1,2), (1,2,5), (5,3,4) of* $R^3(R)$ *do not form a basis set of* $R^3(R)$.

Sol. Writing these vectors as rows of a matrix we have $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$

Reducing to echelon form, we get $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

But in the last matrix there are only two non-zero rows, hence the given vectors are L.D. Therefore they ean not form a basis of $R^3(R)$.

Ex. 2. *Show that the set* $\{(1,0,0),(1,1,0),(1,1,1)\}$ *is a basis of* $C^3(C)$. *Hence find the coordinates of the vector* $(3+4i, 6i, 3+7i)$ *in* $C^3(C)$.

Sol. Let S = {(1,0,0), (1,1,0), (1,1,1)}

Keeping these vectors as rows of a matrix we get $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Reducing to echelen form, $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.,* $R_2 - R_1$ and $R_3 - R_2$

$\therefore$ rank A = no. of unknowns. $\Rightarrow$ The given set is L. I.

Let $z \in C^3$ be $z = (a, b, c)$ where $a, b, c \in C$

Now $(a, b, c) = p\,(1,0,0) + q\,(1,1,0) + r\,(1,1,1)$ for $p, q, r \in C$ $\quad = (p+q+r, q+r, r)$

$\Rightarrow \quad a = p+q+r, \quad b = q+r, c = r. \qquad \Rightarrow \quad r = c, q = b-c, p = a-b$

$\therefore \quad z = (a-b)\,(1,0,0) + (b-c)\,(1,1,0) + c\,(1,1,0)$

$= l.c.$ of the given vectors of S $\Rightarrow z \in L\,(S)$. $\qquad \therefore$ S is a basis of $C^3(C)$

Now if $(a, b, c) = (3+4i, 6i, 3+7i)$ then $p = 3-2i, q = -3-i, r = 3+7i$ which are the coordinates of the given vector.

Ex. 3. *The set* $S_4 = \{\alpha, \beta, \gamma, \delta\}$ *where* $\alpha = (1,0,0), \beta = (1,1,0), \gamma = (1,1,1)$ *and* $\delta = (0,1,0)$ *is a spaning set of* $R^3(R)$ *but not a basis of set.*

Sol. Keeping the vectors in matrix form $\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

δ - row reduced to zero row.

$\Rightarrow$ δ can be expressed as $l.c$ of α, β, γ $\Rightarrow$ S is L. D.

Consider $S_3 = \{\alpha, \beta, \gamma\}$ $\quad \therefore \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Whenever $[\alpha\,\beta\,\gamma]$ can be reduced to the standard basis set, S_3 will be the basis of $V_3(F)$ *i.e.,* $L\,(S_3) = V$. As $S_3 \subset S_4, L\,(S_3) = L\,(S_4) = V$

Thus S_4 is a spaning set of $V_3(F)$ but not a basis set.

Ex. 4. *If* $\alpha = (1,-1,0), \beta = (2,1,3)$ *find a basis for* R^3 *containing* α *and* β.

Sol. Let $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ and $S = \{e_1, e_2, e_3\}$ we know that S is a standard basis for R^3

We have $\beta \in R^3$. Let $S_1 = \{\beta, e_1, e_2, e_3\}$ then $L\,(S_1) = L\,(S) = R^3$

Now $\beta = 2e_1 + e_2 + 3e_3$

$\Rightarrow e_3 = \frac{1}{3}\beta - \frac{2}{3}e_1 - \frac{1}{3}e_2$ which shows that e_3 is a L. C. of β, e_1, e_2

Let $S_1 = \{\beta, e_1, e_2\}$ then we have $L(S_1) = L(S) = R^3$. Again $\alpha \in R^3$

Let $S_2 = \{\alpha, \beta, e_1, e_2\}$ then $L(S_2) = L(S_1) = R^3$

We have $\alpha = e_1 - e_2 \Rightarrow e_2 = -\alpha + 0\beta + e_1$ which show that e_2 is a L. C.of α, β, e_1

Remove e_2 from S_2 and let $S_3 = \{\alpha, \beta, e_1\}$ then $L(S_3) = L(S) = R^3$

Since the dimension of R^3 is 3. We have S_3 is a basis of R^3.

$\therefore \{(1,-1,0)\ (2,1,3)\ (1,0,0)\}$ is a basis of R^3.

Note: We can prove that $\{\alpha, \beta, e_2\}$ and $\{\alpha, \beta, e_3\}$ are also basis of R^3.

Ex. 5. *Show that* $(2,2,1)(2,1,1)(2,1,0)$ *from a basis of* $(1,2,1)$ *in* $V_3(F)$

Sol. Let $(1,2,1) = \alpha(2,2,1) + \beta(2,1,1) + \gamma(2,1,0)$

$= (2\alpha + 2\beta + 2\gamma, 2\alpha + \beta + \gamma, \alpha + \beta)$

Comparing, $2\alpha + 2\beta + 2\gamma = 1$ ... (1)

$2\alpha + 4\gamma = 2$ ... (2)

$\alpha + \beta = 1$ ... (3)

Solving we get $\alpha = 3/2, \beta = -1/2, \gamma = -1/2$

$\therefore (1,2,1) = \frac{3}{2}(2,2,1) - \frac{1}{2}(2,1,1) - \frac{1}{2}(2,1,0)$

Ex. 6. *Find a basis for* R^4 *write* $(3,2,5,4), (6,3,5,2)$ *as numbers.*

Sol. Let $\alpha_1 = (3,2,5,4), \alpha_2 = (6,3,5,2)$

$e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)$

We know that $\{e_1, e_2, e_3, e_4\}$ is a basis of R^4

we have $\alpha_2 = 6e_1 + 3e_2 + 5e_3 + 2e_4$

$\therefore \alpha_2, e_1, e_2, e_3, e_4$ are L.D. *i.e.* e_4 is a L.C. of α_2, e_1, e_2, e_3

Suppose we remove e_4 from the set $\{\alpha_2, e_1, e_2, e_3, e_4\}$ then $\{\alpha_2, e_1, e_2, e_3\}$ is L.I. set

Consider $\{\alpha_1, \alpha_2, e_1, e_2, e_3\}$. Since the dimension of R^4 is 4 this must be a L.D. set.

Consider $x_1\alpha_1 + x_2\alpha_2 + x_3e_1 + x_4e_2 + x_5e_3 = 0$

then $3x_1 + 6x_2 + x_3 = 0$ ... (1) $2x_1 + 3x_2 + x_4 = 0$ ... (2)

$5x_1 - 5x_2 + x_5 = 0$ ... (3) $4x_1 + 2x_2 = 0$ ... (4)

(4) $\Rightarrow$ $x_2 = -2x_1$ substituting (I), (2), (3) we get

$$-9x_1 + x_3 = 0 \Rightarrow x_1 = \frac{1}{9}x_3;\quad -4x_1 + x_4 = 0 \Rightarrow x_1 = \frac{1}{4}x_4;\quad -5x_1 + x_5 = 0 \Rightarrow x_1 = \frac{1}{5}x_5$$

$\therefore$ $x_1 = 1, x_2 = -2, x_3 = 9, x_4 = 4, x_5 = 5$

$\therefore$ $\alpha_1 - 2\alpha_2 + 9e_1 + 4e_2 + 5e_3 = 0$ *i.e.,* $e_3 = \frac{-1}{5}\alpha_1 + \frac{2}{5}\alpha_2 - \frac{9}{5}e_1 - \frac{4}{5}e_2$

which proves that e_3 is a L. C. of $\alpha_1, \alpha_2, e_1, e_2$

We can prove that $\{\alpha_1, \alpha_2, e_1, e_2\}$ is L.I. set, which gives a basis for R^4.

Ex. 7. *If* $\alpha_1 = (1,2,-1), \alpha_2 = (-3,-6,3), \alpha_3 = (2,1,3)$ *and* $\alpha_4 = (8,7,7)$ *and if* $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *is such that L (S) = W, find a basis by reducing S.*

Sol. We observe $\alpha_2 = -3\alpha_1 = -3\alpha_1 + 0\alpha_3 + 0\alpha_4$

$\therefore$ α_2 is a L. C. of $\alpha_1, \alpha_3, \alpha_4$

Let $S_1 = \{\alpha_1, \alpha_3, \alpha_4\}$. We have $L(S_1) = L(S) = W$

Again $\alpha_4 = 2\alpha_1 + 3\alpha_3$. Thus α_4 is L. C. of α_1 and α_3

Let $S_2 = \{\alpha_1, \alpha_3\}$ then $L(S_2) = L(S_1) = W$

Now $a\alpha_1 + b\alpha_2 = 0 \;\Rightarrow\; a\,(1,2,-1) + b\,(2,1,3) = 0$

$\Rightarrow$ $(a+2b,\, 2a+b,\, -a+3b) = (0,0,0) \Rightarrow$ $a + 2b = 0$

$2a + b = 0$, $-a + 3b = 0$

on solving : $a = 0 = b$ is the only solution which proves that S_2 is L.I. set

Thus S_2 is a basis for W

Note : In every step we can remove one vector which is a L. C. of other vectors.
In the above problem any subset of S containing 2 vectors will become a basis of W.

Ex. 8. *Find the co-ordinates of* $(2,3,4,-1)$ *with respect to the basis of* $V_4(R)$, $B = \{(1,1,1,2), (1,-1,0,0), (0,0,1,1), (0,1,0,0)\}$

Sol. Let $(2,3,4,-1) = a\,(1,1,1,2) + b\,(1,-1,0,0) + c\,(0,0,1,1) + d\,(0,1,0,0)$

$= (a+b, a-b+d, a+c, 2a+c)$

Equating the components, we get $a + b = 2, a - b + d = 3, a + c = 4, 2a + c = -1$

Solving we get $a = -5, b = 7, c = 9, d = 15$

$\therefore$ The co-ordinates are $(-5,7,9,15)$

EXERCISE 2 (*a*)

1. Show that the set $\{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $C^3(C)$ but not a basis of $C^3(R)$.

2. Find the coordinates of α with respect to the basis set $\{x, y, z\}$ where

(*i*) $\alpha=(4,5,6), x=(1,1,1), y=(-1,1,1), z=(1,0,-1)$

(*ii*) $\alpha=(1,0,-1), x=(0,1,-1), y=(1,1,0), z=(1,0,2)$

(*iii*) $\alpha=(2,1,3), x=(1,1,1), y=(-1,1,0), z=(1,0,-1)$

3. Show that the four vectors $\{(2,1,0),(0,1,2),(-7,2,5),(8,0,0)\}$ is not a basis of $R^3(R)$

4. Show that if $\{\alpha_1,\alpha_2,\alpha_3\}$ is a basis of $R^3(R)$ then the set $\{\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1\}$ is also a basis of $R^3(R)$

5. (*i*) Show that the set $\{(1,2,1),(2,1,0),(1,-1,2)\}$ forms a basis of $V_3(F)$

(*ii*) Show that the set of vectors $\{(2,1,4),(1,-1,2),(3,1,-2)\}$ form a basis for R^3.

6. Show that the set $\{(1,i,0),(2i,1,1),(0,1+i,1-i)\}$ is a basis of $V_3(C)$ and find the coordinates of the vectors $(1,0,0),(0,1,0),(0,0,1)$ with respect to this basis.

7. Show that the set $\{(3-i,2+2i,4),(2,2+4i,3),(1-i,-2i,1)\}$ a basis of $V_3(C)$ and find the coordinates of the vectors (I, 0, 0) and (0, 0, I) with respect to this basis.

8. Show that the vectors (1,1,2), (1,2,5), (5,3,4) in $R^3(R)$ do not fonn a basis set of R^3.

9. Find the coordinates of (*i*) $(2i,3+4i,5)$ (*ii*) $(6i,7,8i)$ (*iii*) $(3+4i,6i,3+7i)$ with respect to the basis set $\{(1,0,0),(1,1,0),(1,1,1)\}$ of $C^3(C)$.

10. Under what conditions on the scalar $a\in R$ is the set $\{(0,1,a),(a,0,1),(a,1,1+a)\}$ a basis of $R^3(R)$.

2.7 Invariance Theorem . ***Let V(F) be a finite dimensional vector space. Then any two bases of V have the same number of elements.***

Proof. Let S_m and S_n be the two bases of V(F) where $S_m=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$, $S_n=\{\beta_1,\beta_2,\ldots,\beta_n\}$

Obviously both S_m and S_n are L.I. subsets of V.

(*i*) Consider S_m as the basis of V and S_n and L. I. set.

$\Rightarrow \quad L(S_m)=V$ and $n(S_m)=m$

$\therefore$ S_n can be extended to be a basis of V $\Rightarrow$ $n \le m$

(*ii*) Consider S_n as the basis of V and S_m as L.I. set $\Rightarrow$ $L(S_n) = V$ and $n(S_n) = n$

$\therefore$ S_m can be extended to form a basis of V $\Rightarrow$ $m \le n$

But both S_m and S_n are bases of V. $\therefore$ $n \le m$ and $m \le n$ $\Rightarrow$ $m = n$

Thus any two bases of V have the same number of elements.

2.8. DIMENSION OF A VECTOR SPACE

Definition. *Let V (F) be a finite dimensional vector space. The number of elements in any basis of V is called the dimension of V and is denoted by dim V.*

Note: The dimension of zero vector space $\{\bar{O}\}$ is said be zero.

e.g. The set S = {(1,0,0), (0,1,0), (0,0,1)} is a basis of $V_3(R)$

$\therefore$ dim V = no. of elements of S = 3.

2.9. Theorem. ***Every set of (n+1) or more vectors in an n dimensional vector space is linearly dependent.***

Proof. Let V (F) be the vector space with V = *n*

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_{n+1}\}$ be any subset consisting of $(n+1)$ vectors of V (F)

If S is L I. then S itself is the basis or can be extended to be a basis of V (F).

In any of these two cases S will have more than or equal to $(n+1)$ vectors. But every basis of V must contain exactly *n* vectors. Therefore S can not be L.I.

Hence S (*i.e.* every $n+1$ or more vectors in V) is L.D.

Note. The largest L.I. subset of a finite dimensional vector space of dimension *n* is a basis.

2. 10. Theorem. ***Let V (F) be a finite dimensional vector space of dimension n. Then any set of n linearly independent vectors in V forms a basis of V.***

Proof. Given dim V = *n*.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be L. I. set of *n* vectors in V.

If S is not a basis of V, then it can be extended to form a basis of V. In such a case the basis will contain more than *n* vectors.

But every basis of V must contain exactly *n* vectors. Therefore our presumption is wrong and S must be a basis of V.

2. 11. Theorem. ***Let V (S) be a finite dimensional vector space of dimension n. Let S be a set of n vectors of V such that L (S) = V. Then S is a basis of V (F).***

Proof. Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the subset of vector space V (F).

If S is L.I. and because L (S) = V, the set S becomes the basis of V.

If S is L.D. then there exists a proper subset of S forming a basis of V. In such case we get a basis of V consisting less than a vectors.

As every basis of V must contain exactly n vectors, S cannot be L.D. Hence S is a basis of V.

Note. If V is a finite dimensional vector space of dimension n, then V cannot be generated by a set of vectors whose number of elements is less than n.

2.12.Theorem. ***V is a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ..., \beta_n$ then any independent set of vectors in V is finite and contains no more than n elements.***

Proof. Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a linearly independent subset of vector space V (F)

Let $S = \{\beta_1, \beta_2, ..., \beta_n\}$ and L (S) = V.

Then any vector of V is *l.c.* of the elements of S.

Let $\alpha_m \in V$ be a linear combination of the elements of S.

$\Rightarrow S' = \{\alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ is L. D.

$\therefore$ There exists a vector in S' which is a linear combination of its preceeding vectors. This vector must be one among $\beta' s$.

Let it be β_i. $\therefore$ $\beta_i = l.c.$ of $\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}$

If $S_1 = \{\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$. Then $L(S_1) = L(S) = V$

Again the vector $\alpha_{m-1} \in V$ is a *l.c.* of the elements of S_1

$\therefore \alpha_{m-1}, \alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n$ are L.D.

$\therefore$ There exists a vector in this set which is a *l.c.* of its predecessors.

Such vector is one of $\beta' s$ as α_{m-1}, α_m are L. I. Let it be β_k.

If $S_2 = \{\alpha_{m-1}, \alpha_m, \beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$ then

$\beta_k = l.c$ of the elements of S_3. $\therefore$ $L(S) = L(S_1) = L(S) = V$

Continuing this process further for $m-3$ times

We get $S_{m-1} = \{\alpha_2, \alpha_3, ..., \alpha_m, \beta_{i1}, \beta_{i2}, \beta_{in-m+1}\}$. So that $L(S_{m-1}) = V$.

This set consists of atleast one β_i. Otherwise $S_{m-1} = \{\alpha_2, \alpha_3 ... \alpha_m\}$

So that α_1 is a *l.c.* of $\alpha_2, \alpha_3 ... \alpha_m$

This can not happen as the set $\alpha_2, \alpha_3 ..., \alpha_m$ is L. I.

Hence S_{m-1} consists of atleast one $\beta_i \Rightarrow n - m + 1 \geq 1 \Rightarrow n \geq m$

$\therefore$ The no. of elements of the L. I. set in $V \leq n$.

DIMENSION OF A SUBSPACE

2.13. Theorem. ***Let V(F) be a finite dimensional vector space of dimension n and W be the subspace of V. Then W is a finite dimensional vector space with dim $W \leq n$.***

Proof. Dim V = $n \Rightarrow$ each $(n+1)$ or more vectors of V form an L.D. set.

Given W is a subspace of V(F)

$\Rightarrow$ each set of $(n+1)$ vectors in W is a subset of V and hence L.D.

Thus any LI. set of vectors in W can contain at the most n vectors .

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the largest L.I. subset of W, where $m \le n$.

Now we shall prove that S is the basis of W.

For any $\beta \in W$ consider, $S_1 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta\}$

Since S is the largest set of L.I. vectors, S_1 is L.D.

$\therefore$ Therefore there exists $a_1, a_2, ..., a_m,\ b \in F$ not all zero such that

$a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b\beta = 0$

Let $b = 0$, then we have $a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = 0$

$\Rightarrow a_1 = 0, a_2 = 0, ..., a_m = 0$ as S is L. I.

This proves that S_1 is L.I which is a contradiction.

$\therefore\ b \ne 0$. Therefore there exists $b^{-1} \in F$ such that $bb^{-1} = 1$

$\therefore\ a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b\beta = 0 \Rightarrow b\beta = -a_1\alpha_1 - a_2\alpha_2 ... a_m\alpha_m$

$\Rightarrow \beta = (-b^{-1}a_1) + (-b^{-1}a_2)\,\alpha_2 + ... + (-b^{-1}a_m)\,\alpha_m$

$\Rightarrow \beta = a$ linear combination of elements of S. $\Rightarrow \beta \in L(S)$

Also S is L.I. Hence S is the basis of W.

$\therefore$ W is a finite dimensional vector space with dim $W \le n$.

2.14. Theorem. ***Let*** W_1 ***and*** W_2 ***be two subspaces of a finite dimensional vector space V (F). Then*** $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

Proof. Since W_1 and W_2 are subspace of V, $W_1 + W_2$ and $W_1 \cap W_2$ are also subspace of V.

Let $\dim(W_1 \cap W_2) = k$ and $S = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ be a basis of $W_1 \cap W_2$.

Clearly $S \subseteq W_1$ and $S \subseteq W_2$ and S is L.I.

Since S is L.I. and $S \subseteq W_1$, the set S can be extended to form a basis of W_1.

Let $B_1 = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1. $\therefore$ $\dim W_1 = k + m$.

Again since S is L.I. and $S \subseteq W_2$, the set S can be extended to form a basis of W_2.

Let $B_2 = \{\gamma_1, \gamma_2, ..., \gamma_k, \beta_1, \beta_2, ..., \beta_t\}$ be a basis of W_2. $\quad \therefore$ $\dim W_2 = k + t$

$\therefore$ $\dim W_1 + \dim W_2 - \dim (W_1 \cap W_2) = (k+m) + (k+t) - k = k + m + t$

Now we shall prove that the set

$S' = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_t\}$ is a basis of $W_1 + W_2$ and hence $\dim (W_1 + W_2) = k + m + t$.

(*i*) *To prove that S' is L.I.*

Now $c_1\gamma_1 + c_2\gamma_2 + ... + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = 0$ ... (1)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = -c_1\gamma_1 - ... - c_k\gamma_k - a_1\alpha_1 - ... - a_m\alpha_m$

$=$ *l.c.* of elements of B_1 and $\quad \therefore \in W_1$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1$

Again $0\gamma_1 + 0\gamma_2 + ... 0\gamma_k + b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t$

$=$ *l.c.* of elements of B_2 and $\therefore \in W_2$ $\quad \Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_2$

Therefore by (1) and (2) $\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1 \cap W_2$

Hence it can be expressed as a *l.c.* of the elements of the basis S of $W_1 \cap W_2$

Let $b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + ... + d_k\gamma_k$

$\therefore b_1\beta_1 + b_2\beta_2 ... + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 ... - d_k\gamma_k = \bar{O} \Rightarrow$ (*l.c.* of elements of basis B_2) $= \bar{O}$

$b_1 = 0, b_2 = 0 ... b_t = 0, d_1 = 0, ... k_k = 0$

Substituting in I we have $c_1\gamma_1 + c_2\gamma_2 + ... + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 ... + a_m\alpha_m = \bar{O}$

$\Rightarrow$ (*l.c.* of elements of basis B_1) $= \bar{O}$ $\Rightarrow c_1 = 0, c_2 = 0 ... c_k = 0, a_1 = 0, a_2 = 0 ... a_m = 0$

Thus the relation I implies that $c_1 = c_2 = ... = c_k = a_1 = a_2 = ... \alpha_m = o, b_1 = b_2 = ... = b_t = o$

$\therefore$ S' is L.I. set.

(*ii*) *To prove* $L(S') = W_1 + W_2$

Every vector of S' is a vector of $W_1 + W_2$. $\quad \therefore L(S') \subseteq W_2 + W_2$

Let $\delta \in W_1 + W_2$. $\quad \therefore \delta = \alpha + \beta$ where $\alpha \in W_1, \beta \in W_2$

$\delta =$ (*l.c.* of elements of B_1) + (*l.c.* of elements of B_2)

$=$ (*l.c.* of γ's and α's) + (*l.c.* of γ's and β's) $=$ *l.c.* of γ's, α's and β's.

$=$ *l.c.* of elements of S'.

$\therefore \delta \in L(S')$ $\quad \therefore W_1 + W_2 \subseteq L(S')$

$\therefore L(S') = W_1 + W_2$. $\quad$ Hence S' is the basis of $W_1 + W_2$.

$\therefore \dim (W_1 + W_2) = k + m + t$. $\quad$ Hence the theorem.

SOLVED PROBLEMS

Ex. 1. *Let* W_1 *and* W_2 *be two subspaces of* R^4 *given by*

$W_1 = \{(a, b, c, d) : b - 2c + d = 0\}$, $W_2 = \{(a, b, c, d) : a = d, b = 2c\}$. *Find the basis and dimension of* (*i*) W_1 (*ii*) W_2 (*iii*) $W_1 \cap W_2$ *and hence find* $\dim(W_1 + W_2)$.

Sol. (i) Given $V = \{(a, b, c, d) : b - 2c - d\}$

Let $(a, b, c, d) \in V$ then $(a, b, c, d) = (a, 2c - d, c, d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$, $\quad \therefore (a, b, c, d) = l.c.$ of L.I. set

$\{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$ which forms the basis of W_1 $\quad \therefore \dim W_1 = 3$.

(*ii*) Given $W_2 = \{(a, b, c, d) : a = d, b = 2c\}$.

Let $\alpha \in W_2 \Rightarrow \alpha = (a, b, c, d)$ where $a = d, b = 2c$

$\therefore \alpha = (d, 2c, c, d) = d(1,0,0,1), + c(0,2,1,0) \Rightarrow \alpha = l.c$ of L.I. set $\{(1,0,0,1), (0,2,1,0)\}$

$\therefore$ It forms a basis $\qquad \therefore \dim W_2 = 2$.

(*iii*) $W_1 \cap W_2 = \{(a,b,c,d) / b - 2c + d = 0, a = d, b = 2c\}$

Now $b - 2c + d = 0, a = d, b = 2c$ gives $b = 2c, a = 0, d = 0$

$\therefore (a,b,c,d) = (0,2c,c,0) = c(0,2,1,0) \quad \Rightarrow (a,b,c,d) =$ multiple of the vector (0,2,1,0)

$\therefore$ Basis of $W_1 \cap W_2 = (0,2,1,0) \quad \Rightarrow \dim W_1 \cap W_2 = 1$

(*iii*) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 = 3 + 2 - 1 = 4$

Ex. 2. *If W is the subspace of* $V_4(R)$ *generated by the vectors* $(1,-2,5,-3), (2,3,1,-4)$ *and* $(3,8,-3,-5)$ *find a basis of W and its dimension.*

Sol. Arranging the given vectors as rows of a matrix

Let $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$. Reducing to echelon form,

$R_2 - 2R_1, R_2 - 3R_1; A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$. Again by $R_3 - 2R_2$, $A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence the two non-zero rows *viz*; $(1,-2,5,-3), (0,7,-9,2)$ form the least L. I. set and hence a basis of W. $\quad \therefore \dim W = 2$

Ex. 3. *V is the space generated by the polynomials* $\alpha = x^3 + 2x^2 - 2x + 1$, $\beta = x^3 + 3x^2 - x + 4$, $\gamma = 2x^3 + x^2 - 7x - 7$. *Find a basis of V and its dimension.*

Sol. Now V is the polynomial space generated by $\{\alpha, \beta, \gamma\}$ given above

$\alpha = x^3 + 2x^2 - 2x + 1$

$\Rightarrow$ the coordinates of are $(1, 2, -2, 1)$ w. r. to the base $\{x^3, x^2, x, 1\}$

Similarly the coordinates of β and γ w. r. to the same base are $(1, 3, -1, 4)$ and $(2, 1, -7, -7)$.

Forming the matrix A with these co-ordinate as rows

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$ Reducing to echilon form,

by $R_2 - R_1, R_3 - 2R_1, \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$

again by $R_3 + 3R_1$, $A \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence the basis of V is formed by the coordinate vectors $(1, 2, -2, 1)$ and $(0, 1, 1, 3)$

Now the polynomials are $\alpha = x^3 - 2x^2 - 2x + 1$ and $\delta = x^2 + x + 3$

Thus the base of V is $\{\alpha, \delta\}$ $\therefore$ dim W = 2

Ex. 4. *V is the vector space of polynomials over R.* W_1 *and* W_2 *are the subspaces generated by* $\{x^3 + x^2 - 1, x^3 + 2x^2 + 3x, 2x^3 + 3x^2 + 3x - 1\}$ *and* $\{x^3 + 2x^2 + 2x - 2, 2x^3 + 3x^2 + 2x - 3, x^3 + 3x^2 + 4x - 3\}$ *respectively.*

Find (i) $\dim(W_1 + W_2)$ *(ii)* $\dim(W_1 \cap W_2)$

Sol. The coordinates of the polynomials w.r.to the basis $\{x^3, x^2, x, 1\}$ are respectively $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$.

(*i*) Arranging the vectors generating W_1 as rows of a matrix and reducing to echelon form $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ the basis of W_1 contains 2 vectors $\Rightarrow$ $\dim W_1 = 2$ similarly we get $\dim W_2 = 2$.

(*ii*) Now the subspace $W_1 + W_2$ is by all the six vectors. Hence arranging them in rows of a matrix and reducing to echelon form.

$$P = \begin{Bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{Bmatrix} \sim \begin{Bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix} = Q$$

$\Rightarrow$ The basis of $W_1 + W_2$ consists of the three non zero row vectors in Q.

$\Rightarrow \dim(W_1 + W_2) = 3$.

Now we have the formula $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = 2 + 2 - 3 = 1$

Ex. 5. *If* $U = \{(1,2,1),(0,1,2)\}, W = \{(1,0,0),(0,1,0)\}$ *determine the dimension of* $U + W$

Sol. Given $U = \{(1,2,1),(0,1,2)\}$, $W = \{(1,0,0),(0,1,0)\}$ are vector spaces

We can observe that, $\dim U = 2, \dim W = 2$

To find a basis for $U \cap W$

Let $x \in U \cap W \quad \Rightarrow x \in U$ and $x \in W$

$\Rightarrow x = a(1,2,1) + b(0,1,2) = (a, 2a+b, a+2b)$ and $x = c(1,0,0) + d(0,1,0) = (c,d,0)$

Equating the values $a = c, 2a + b = d$ and $a + 2b = 0$

Solving we get, $a = 2/3d, b = -1/3d, c = 2/3d$

$\therefore\ x = (c,d,0) = \left(\frac{2}{3}d, d, 0\right) = \frac{1}{3}d\,(2,3,0) \qquad \therefore\ U \cap W$ is generated by $(2,3,0)$

$\dim U \cap W = 1$. $\quad \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 2 + 2 - 1 = 3$

EXERCISE 2 (*b*)

1. P and Q are the two subspaccs of R^4 defined by $P = \{(a,b,c,d) : b + c + d = 0\}$; $Q = \{(a,b,c,d) : a + b = 0, c = 2d\}$. Find the dimension and basis of P, Q and $P \cap Q$.

2. Let W_1 and W_2 be the subspaces of R^4 generated by $\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ respectively. Find the dimension of (*i*) W_1 (*ii*) W_2 (*iii*) $W_1 + W_2$ (*iv*) $W_1 \cap W_2$.

QUOTIENT SPACE

2.15. Coset : *Let W be a subspace of a vector space V (F) then for any element* $\alpha \in V$, *the set* $W+\alpha = \{x+\alpha / x \in W\}$ *is called the right coset of W in V, generated by* α.

Similarly the set $\alpha + W = \{\alpha + x / x \in W\}$ *is called the left coset of W in V, generated by* α.

Since (W, +) is a subgroup of the abelian group (V. +), by commutative property

$x+\alpha = \alpha + x \quad \forall \alpha \in V,\ x \in W$

Hence $W+\alpha$ is called simply the coset of W in V, generated by α.

Note 1. For $\bar{O} \in V$, $\bar{O} + W = W$. $\therefore$ W is itself a coset in V, generated by $\bar{O}$.

Note. 2. For $x \in W$, $x + W = W$. $\therefore$ Coset $x + W =$ coset W.

Note. 3. Any two cosets of W in V are either identical or disjoint.

i.e. Either $\alpha + W = \beta + W$ or $(\alpha + W) \cap (\beta + W) = \phi$

Note 4. If $\alpha + W$ and $\beta + W$ are two cosets of W in V then

$\alpha + W = \beta + W \Leftrightarrow \alpha - \beta \in W$

2.16. QUOTIENT SET

Let W be a subspace of V (F). Then the set of all cosets of W in V denoted by

V/W = $\{W+\alpha\ \forall\ \alpha \in V\}$ is called Quotient set.

2.17. Theorem. ***Let W be a subspace of vector V (F). Then the set V /W is a vector space over F for the vector addition and scalar multiplication defined by***

$(W+\alpha)+(W+\beta) = W+(\alpha+\beta)\ \forall\ \alpha, \beta \in V$ ... (1)

$a\,(W+\alpha) = W + a\alpha\ \forall\ a \in F,\ \alpha \in V$ ... (2)

Proof. First of all let us prove that the above two compositions are well defined.

(*i*) Let $W+\alpha = W+\alpha'$ then it $\Rightarrow \alpha - \alpha' \in W$

$W+\beta = W+\beta'$ then it $\Rightarrow \beta - \beta' \in W$. Hence $(\alpha - \alpha') + (\beta - \beta') \in W$

$\Rightarrow (\alpha+\beta) - (\alpha' + \beta') \in W \Rightarrow W + (\alpha+\beta) = W + (\alpha' + \beta')$

$\therefore$ Addition of cosets is well defined

(*ii*) $W+\alpha = W+\alpha' \Rightarrow \alpha - \alpha' \in W$

$\alpha - \alpha' \in W, a \in F \Rightarrow a(\alpha - \alpha') \in W \Rightarrow a\alpha - a\alpha' \in W \Rightarrow W + a\alpha = W + a\alpha'$

$\therefore$ Scalar multiplication is well defined.

Now we shall show that V/W is a vector space.

1. Closure axiom. $\alpha, \beta \in V \Rightarrow \alpha + \beta \in V$ and $W+\alpha, W+\beta \in V/W$

$\therefore (W+\alpha)+(W+\beta) = W+(\alpha+\beta)$ by (1)

As $W+(\alpha+\beta) \in V/W$, it is closed under addition.

2. Associativity. Let $W+\alpha, W+\beta, W+\gamma \in V/W$ for $\alpha, \beta, \gamma \in V$.

$\therefore\ [(W+\alpha)+(W+\beta)]+(W+\gamma) = [W+(\alpha+\beta)]+(W+\gamma)$

$W+(\alpha+\beta)+\gamma = W+\alpha+(\beta+\gamma)$

$[W+\alpha]+[W+(\beta+\gamma)] = (W+\alpha)+[(W+\beta)+(W+\gamma)]$

$\therefore$ V/W is associative.

3. Identity. $W+\bar{O}, W+\alpha \in V/W$ for $\alpha \in V$

$\therefore\ (W+\bar{O})+(W+\alpha) = W+(\bar{O}+\alpha) = W+\alpha$ ($\bar{O}$ being the identity of V)

and $(W+\alpha)+(W+\bar{O}) = W+\alpha \Rightarrow W+\bar{O}$ is additive identity in V /W

4. Inverse. $\alpha \in V \Rightarrow -\alpha \in V$

$\therefore\ (W+\alpha)+[W+(-\alpha)] = W+(\alpha-\alpha) = W+\bar{O}$ = additive identity

$\Rightarrow$ additive inverse exists in V /W.

5. Commutative. $(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$ for $\alpha, \beta \in V$

$= W+(\beta+\alpha) = (W+\beta)+(W+\alpha) \Rightarrow$ addition is commutative

$\therefore$ V/W is an abelian group for addition defined in (1).

6. $a \in F$ and $W+\alpha, W+\beta \in V/W \Rightarrow a[(W+\alpha)+(W+\beta)] = a[W+\alpha+\beta)]$

$= W+a(\alpha+\beta) = W+(a\alpha+a\beta) = (W+a\alpha)+W+a\beta = a(W+\alpha)+a(W+\beta)$

7. $a, b \in F$ and $W+\alpha \in V/W \Rightarrow (a+b)(W+\alpha) = W+(a+b)\alpha$

$= W+(a\alpha+b\alpha) = (W+a\alpha)+(W+b\alpha) = a(W+\alpha)+b(W+\alpha)$

8. $a, b \in F$ and $W+\alpha \in V/W$

$(a\,b)(W+\alpha) = W+(a\,b)\alpha = W+a(b\alpha) = a[W+b\alpha] = a[b(W+\alpha)]$

9. $W+\alpha \in V/W$ and the unity $1 \in F$, $\Rightarrow 1(W+\alpha) = W+1.\alpha = W+\alpha$

This shows that all the postulates of vector space are satisfied. Hence V/W is a vector space.

The vector space V /W is called the Quotient space of V relative to W.

DIMENSION OF QUOTIENT SPACE

2.18. Theorem. ***Let W be a sub space of a finite dimensional vector space V(F), then* dim V/W = dim V – dim W.**

Proof. Since V is finite dimensional, W is also finite dimensional.

Let the set $B = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be the basis of W. $\therefore$ dim $W = m$.

Since the set B is L.I. it can be extended to form a basis of V.

Let the set $S = \{\alpha_1, \alpha_2, \ldots \alpha_m, \beta_1, \beta_2 \ldots \beta_l\}$ be the basis of V. $\therefore$ $\dim V = m + l$

$\therefore$ $\dim V - \dim W = (m + l) - m = l$

Now we shall prove that the set $S' = \{W + \beta_1, W + \beta_2 \ldots W + \beta_t\}$ is a basis of V/W and hence $\dim V/W = l$.

(*i*) *To prove S′ is L.I.*

The zero vector of V/W is W

Now $b_1 (W + \beta_1) + b_2 (W + \beta_2) + \ldots + b_t (W + \beta_t) = W$ .... (1)

$\Rightarrow (W + b_1\beta_1) + (W + b_2\beta_2) + \ldots + (W + b_t\beta_t) = W$

$\Rightarrow W + (b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t) = W \Rightarrow (b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t) \in W$

But any vector of W is *l.c.* of elements of B.

Let $b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t = a_1\alpha_1 + a_2\alpha_2 \ldots + a_m\alpha_m$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t - a_1\alpha_1 - a_2\alpha_2 \ldots - a_m\alpha_m = \bar{O}$

$\Rightarrow$ (l.c. of elements of L.I set) = $\bar{O}$

$\Rightarrow b_1 = b_2 = \ldots = b_l = 0, a_1 = a_2 = \ldots = a_m = 0 \Rightarrow$ The set S' is L.I.

(*ii*) *To Prove L (S′) = V/W.*

Since S is the basis of V, $\alpha \in V$ can be expressed as

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_m\alpha_m + d_1\beta_1 + d_2\beta_2 \ldots + d_l\beta_l$

$= \gamma + d_1\beta_1 + d_2\beta_2 \ldots + d_l\beta_l$ where $c's, d's \in F$ where $\gamma = c_1\alpha_1 + c_2\alpha_2 \ldots c_m\alpha_m$

= (*l.c.* of the elements of B) $\Rightarrow \gamma \in W$

For $\alpha \in V$, $W + \alpha \in V/W$

Now $W + \alpha = W + \gamma + d_1\beta_1 + d_2\beta_2 \ldots + d_l\beta_l$

$= W + d_1\beta_1 + d_2\beta_2 \ldots + d_l\beta_l \quad [\gamma \in W \Rightarrow W + \gamma = W]$

$= (W + d_1\beta_1) + (W + d_2\beta_2) \ldots + (W + d_l\beta_l) = d_1(W + \beta_1) + d_2(W + \beta_2) \ldots + d_l(W + \beta_l)$

(*l.c.* of elements of S')

$\therefore W + \alpha \in L(S')$ $\therefore L(S') = V/W$

Therefore S' is the basis of V/W and hence $\dim V/W = l = \dim V - \dim W$.

Ex. *If W is a subspace of V (F) then for* $a, b \in V$ *show that*

(i) $\alpha \in (\beta + W)$ *(ii)* $\beta \in (\alpha + W)$ *are equivalent.*

Sol. $\alpha \in \beta + W \Rightarrow \alpha = \beta + \gamma$ for some $\gamma \in W$ $\Rightarrow \alpha - \beta \in W$

For $(-1) \in F, (-1)(\alpha - \beta) \in W \Rightarrow \beta - \alpha \in W \Rightarrow (\beta - \alpha + \alpha) \in W + \alpha \Rightarrow \beta \in W + \alpha$.

Linear Transformations

3.1. VECTOR SPACE HOMOMORPHISM

Definition. *Let U and V be two vector spaces over the same field F. Thus the mapping* $f : U \to V$ *is called a homomorphism from U into V if*

(*i*) $f(\alpha + \beta) = f(\alpha) + f(\beta)\ \forall\ \alpha, \beta \in U$ (*ii*) $f(a\alpha) = af(\alpha)\ \forall a \in F; \forall\ \alpha \in U$

Note. 1. If *f* is onto function then V is called the homomorphic image of *f*.

2. If f is one - one onto function then f is called an isomorphism. Thus it is said that U is isomorphic to V denoted by $U \cong V$.

3. The two conditions of homomorphism are combined into a single condition, called the linear property to define the linear transformation as below.

3.2. LINEAR TRANSFORMATION

Definition. *Let U(F) and V(F) be two vector spaces. Then the function.* $T : U \to V$ *is called a linear transformation of U into V if* $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)\ \forall a, b \in F; \alpha, \beta \in U$.

Clearly the vector space homomorphism is equivalent to linear transformation.

Linear Operator : Definition. *If* $T : U \to U$ *(i.e T transforms U into itself) then T is called a linear operator on U.*

Linear Functional : Definition. *If* $T : U \to F$ *(i.e. T transforms U into the field F) then T is called a linear functional on U.*

3.3. ZERO TRANSFORMATION

Theorem. ***Let U(F) and V(F) be two vector spaces. Let the mapping*** $\mathbf{T : U \to V}$ ***be defined by*** $T(\alpha) = \hat{O}\ \forall\ \alpha \in U$ ***where*** $\hat{O}$ ***(zero crown) is the zero vector of V. Then T is a linear transformation.***

Proof. For $a, b \in F$ and $\alpha, \beta \in U \Rightarrow a\alpha + b\beta \in U$ ($\because$ U is V.S.)

By definition we have $T(a\alpha + b\beta) = \hat{O} = a\hat{O} + b\hat{O} = aT(\alpha) + bT(\beta)$

$\therefore$ By the definition of linearity T is a linear transformation.

Such a L.T. is called the zero transformation and is denoted by O.

3.4. IDENTITY OPERATOR

Theorem. ***Let V(F) be a vector spaee and the mapping*** $\mathbf{I : V \to V}$ ***be defined by*** $\mathbf{I(\alpha) = \alpha\ \ \forall\ \alpha \in V}$. ***Then, I is a linear operator from V into itself.***

Proof. $a, b \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in V$ ($\because$ V is L.S.)

By definition we have $I(a\alpha + b\beta) = a\alpha + b\beta = a\,I(\alpha) + b\,I(\beta)$ (by def.)

$\therefore$ I is a L. T from V into itself and I is called the **Identity Operator.**

3.5. NEGATIVE OF TRANSFORMATION

Theorem. *Let U (F) and V (F) be two vector spaces and* $T : U \to V$ *be a linear transformation. Then the mapping* $(-T)$ *defined by* $(-T)(\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ *is a linear transformation.*

Proof. $a, b \in F$ and $\alpha, \beta \in U \Rightarrow a\alpha + b\beta \in U$ ($\because$ U is V.S.)

Now by definition $(-T)(a\alpha + b\beta) = -[T(a\alpha + b\beta)]$

$= -[a\,T(\alpha) + b\,T(\beta)] = -a\,T(\alpha) - b\,T(\beta)$

$= a[-T(\alpha)] + b[-T(\beta)] = a[(-T)(\alpha)] + b[(-T)(\beta)]$

$\Rightarrow -T$ is a linear transformation.

PROPERTIES OF LINEAR TRANSFORMATIONS

3.6. Theorem. *Let* $T : U \to V$ *is a linear transformation from the vector space U(F) to the vector space V(F). Then* (*i*) $T(\bar{O}) = \hat{O}$, *where* $\bar{O} \in U$ *and* $\hat{O} \in V$

(*ii*) $T(-\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ (*iii*) $T(\alpha - \beta) = T(\alpha) - T(\beta)\ \forall\ \alpha, \beta \in U$

(*iv*) $T(a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) \ldots + a_nT(\alpha_n)\ \forall\ a_i \in F$ *and* $\alpha's \in U$.

Proof. (*i*) $\alpha, \bar{O} \in U \Rightarrow T(\alpha), T(\bar{O}) \in V$

Now $T(\alpha) + T(\bar{O}) = T(\alpha + \bar{O})$ (T is L.T.) $= T(\alpha) = T(\alpha) + \hat{O}$ ($\bar{O} \in V$)

By cancellation law $T(\bar{O}) = \hat{O}$

(*ii*) $T(-\alpha) = T(-1 . \alpha) = (-1)\,T(\alpha) = -T(\alpha)$

(*iii*) $T(\alpha - \beta) = T[\alpha + (-1)\beta] = T(\alpha) + (-1)T(\beta) = T(\alpha) - T(\beta)$ ($\because$ T is L.T.)

(*iv*) For $n = 1$, $T(a_1\alpha_1) = a_1\,T(\alpha_1)$ ($\because$ T is L.T.)

$n = 2$, $T(a_1\alpha_1 + a_2\alpha_2) = a_1T(\alpha_1) + a_2T(\alpha_2)$

Let this be true for $n = m$

$\therefore T(a_1\alpha_1 + a_2\alpha_2 + a_m\alpha_m) = a_1T(\alpha_1) + a_2T(\alpha_2) \ldots + a_mT(\alpha_m)$ ... (1)

Now $T[a_1\alpha_1 + \ldots + a_m\alpha_m) + a_{m+1}\alpha_{m+1}] = T(a_1\alpha_1 + \ldots + a_m\alpha_m) + T(a_{m+1}\alpha_{m+1})$

$= a\,T(\alpha_1) + \ldots + a_mT(\alpha_m) + a_{m+1}T(\alpha_{m+1})$ $\therefore$ The relation is true for $n = m + 1$

Hence it is true for all integral values of *n*.

DETERMINATION OF LINEAR TRANSFORMATION

3.7. Theorem. *Let U(F) and V(F) be two vector spaces and S =* $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *be a basis of U. Let* $\{\delta_1, \delta_2, ..., \delta_n\}$ *be a set of n vectors in V. Then there exists a unique linear transformation* $T : U \to V$ *such that* $T(\alpha_i) = \delta_i$ *for i = 1, 2, ... n.*

Proof. Let $\alpha \in U$. Since S = $\{\alpha_1, \alpha_2 .. \alpha_n\}$ is a basis of U, there exist unique scalars $a_1, a_2 ... a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$

(*i*) Existence of T, $\delta_1, \delta_2, ..., \delta_n \in V \Rightarrow (a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n) \in V$

We define $T : U \to V$, such that $T(\alpha) = a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n$

$\therefore$ T is a mapping from U into V.

Now $\alpha_i = 0 . \alpha_1 + 0 . \alpha_1 + 0 . \alpha_2 .. + 1 . \alpha_i + 0 . \alpha_{i+1} + ... + 0 . \alpha_n$

$\therefore$ by the definition of T mapping

$T(\alpha_i) = 0 . \delta_1 + 0 . \delta_1 + 0 . \delta_2 .. + \delta_i + 0 . \delta_{i+1} ... + 0 . \delta_n \Rightarrow T(\alpha_i) = \delta_i, \ \forall i = 1, 2, ... n$

(*ii*) To show that T is L.T:

Let $a, b \in F$ and $\alpha, \beta \in U$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$; $\quad \beta = b_1\alpha_1 + b_2\alpha_2 + ... + b_n\alpha_n$

$\therefore T(\alpha) = a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n$; $\quad T(\beta) = b_1\delta_1 + b_2\delta_2 + ... + b_n\delta_n$

$\therefore a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + ... + b_n\alpha_n)$ (by def.)

$= (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + ... + (aa_n + bb_n)\alpha_n$

$\therefore T(a\alpha + b\beta) = T[(aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + ... + (aa_n + bb_n)\alpha_n]$

$= (aa_1 + bb_1)\delta_1 + (aa_2 + bb_2)\delta_2 + ... + (aa_n + bb_n)\delta_n$

$= a(a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n) + b(b_1\delta_1 + ... + b_n\delta_n) = a\,T(\alpha) + b\,T(\beta)$

$\therefore$ T is a L.T.

(*iii*) To show that T is unique.

Let $T' : U \to V$ be another L.T. so that $T(\alpha_i) = \delta_i$ for $i = 1, 2, ... n$

If $\alpha = a_1\alpha_1 + a_2\alpha_2 ... + a_n\alpha_n$

then $T'(\alpha) = T'(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n) = a_1T'(\alpha_1) + a_2T'(\alpha_2) + ... + a_nT'(\alpha_n)$

$= a_1\delta_1 + a_2\delta_2 + ... + a_n\delta_n = T(\alpha)$

$\therefore$ T' = T and hence T is unique.

Note. In determining the L.T. the assumption that $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a basis of U is essential.

3.8. Theorem. *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $S' = \{\beta_1, \beta_2, ..., \beta_n\}$ be two ordered bases of n, dimensional vector space V(F). Let $\{a_1, a_2, ..., a_n\}$ be an ordered set of n scalars such that $\alpha = a_1\alpha_1 + a_2\alpha_2, ..., a_n\alpha_n$ and $\beta = a_1\beta_1 + a_2\beta_2, ..., a_n\beta_n$. Show that $T(\alpha) = \beta$ where T is the linear operator on V defined by $T(\alpha_i) = \beta_i$, $i = 1, 2, ..., n$.*

Proof. Now $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$

$$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + ... + a_n T(\alpha_n) \qquad \text{(T is L.T.)}$$

$$= a_1\beta_1 + a_2\beta_2 + ... + a_n\beta_n = \beta$$

SOLVED PROBLEMS

Ex. 1. *The mapping $T : V_3(R) \to V_2(R)$ is defined by $T(x, y, z) = (x - y, x - z)$. Show that T is a linear transformation.*

Sol. Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be two vectors of $V_3(R)$

For $a, b \in R$

$T[a\alpha + b\beta] = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)] = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$= ((ax_1 + bx_2) - (ay_1 + by_2), ax_1 + bx_2 - (az_1 + bz_2))$

$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$

$= (a(x_1 - y_1), a(x_1 - z_1) + (b(x_2 - y_2), b(x_2 - z_2))$

$= a(x_1 - y_1, x_1 - z_1) + b(x_2 - y_2, x_2 - z_2)$

$= a\,T(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2) = a\,T(\alpha) + b\,T(\beta)$

$\Rightarrow$ T is a linear transformation from $V_3(R)$ to $V_2(R)$.

Ex. 2. *If $T : V_3(R) \to V_2(R)$ is defined as $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$ Prove that T is a linear transformation.*

Sol. Let $\alpha = (p_1, q_1, r_1), \beta = (p_2, q_2, r_2)$ be two vectors of $V_3(R)$. Let $a, b \in R$.

Let $T(a\alpha + b\beta) = T[a(p_1, q_1, r_1) + b(p_2, q_2, r_2)]$

$= T[ap_1 + bp_2 . \overline{aq_1 + bq_2}, ar_1 + br_2]$

$= (ap_1 + bp_2 - \overline{aq_1 + bq_2}, ap_1 + bp_2 + ar_1 + br_2)$

$= (a(p_1 - q_1) + b(p_2 - q_2), a(p_1 + r_1) + b(p_2 + r_2))$

$= a(p_1 - q_1, p_1 + r_1) + b(p_2 - q_2, p_2 + r_2)$

$= aT(p_1, q_1, r_1) + bT(p_2, q_2, r_2) = aT(\alpha) + bT(\beta)$

Similarly we can prove $T(c\alpha) = cT(\alpha)$ for $c \in R$

Thus, $T : V_3(R) \to V_2(R)$ is a linear transformation.

Ex. 3. *The mapping* $T : V_3(R) \to V_1(R)$ *is defined by* $T(a,b,c) = a^2 + b^2 + c^2$; *Can T be a linear transformation ?*

Sol. Let $\alpha = (a,b,c)$ and $\beta = (x,y,z)$ be two vectors of $V_3(R)$.

For $p, q \in R$, $T(pa+qx,\ pb+qy,\ pc+qz) = (pa+qx)^2 + (pb+qy)^2 + (pc+qz)^2$

Now $= p\,T(\alpha) + q\,T(\beta) = p\,T(a,b,c) + q\,T(x,y,z)$

$= p(a^2+b^2+c^2) + q(x^2+y^2+z^2) = T(p\alpha + q\beta) \neq p\,T(\alpha) + q\,T(\beta)$

$\therefore$ T is not a L.T from $V_3(R)$ to $V_1(R)$.

Ex. 4. *Let V be the vector space of polynomials in the variable x over R. Let* $f(x) \in V(R)$; *show that*

(*i*) $D : V \to V$ *defined by* $Df(x) = \dfrac{df(x)}{dx}$

(*ii*) $I : V \to V$ *defined by* $If(x) = \displaystyle\int_0^x f(x)\,dx$ *are linear transformations.*

Sol. Let $f(x), g(x) \in V(R)$ and $a, b \in R$

(*i*) $D[af(x)+bg(x)] = \dfrac{d}{dx}[af(x)+bg(x)] = \dfrac{d}{dx}[af(x)] + \dfrac{d}{dx}[bg(x)]$

$$= a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)] = a\,D\,f(x) + b\,D\,g(x)$$

$\therefore$ D is a linear transformation and D is called a differential operator.

(ii) $I[af(x)+bg(x)] = \displaystyle\int_0^x [af(x)+bg(x)]\,dx = a\int_0^x f(x)\,dx + b\int_0^x g(x)\,dx = a\,I\,f(x) + b\,I\,g(x)$

$\therefore$ I is a linear transformation and is called Integral transformation.

Ex. 5. *Let* $P_n(R)$ *be the vector. space of all polynomials of degree n over a field R. If a linear operator T on* $P_n(R)$ *is such that* $Tf(x) = f(x+1),\ f(x) \in P_n(R)$.

Show that $T = 1 + \dfrac{D}{1!} + \dfrac{D^2}{2!} + \dfrac{D^3}{3!} + \ldots + \dfrac{D^n}{n!}$

Sol. Let $f(x) = a_0 + a_1x + a_2x^2 \ldots + a_nx^n \quad \forall\ a_i \in R$

$$\left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \ldots + \frac{D^n}{n!}\right] f(x) = a_0 + a_1x + a_2x^2 \ldots + a_nx^n$$

$$+\frac{1}{1!}(0 + a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}) + \frac{1}{2!}(0 + 0 + 2a_2 + 6a_3 + \ldots + n(n-1)a_nx^{n-2})$$

$+ \; ...... \; ...... \; ...... \; ..... + \frac{1}{n!}(0+0+0 \ldots + a_n \, n!)$

$= a_0 + a_1 (x+1) + a_2 (x+1)^2 + \ldots + a_n (x+1)^n$

$= f(x+1) = \mathrm{T} \, f(x)$ (by def.)

$\therefore \; \mathrm{T} = \left(1 + \frac{D}{1!} + \frac{D^2}{2!} + \ldots + \frac{D^n}{n!}\right).$

Thus T is a linear operator from $\mathrm{P}_n(\mathrm{R})$ into $\mathrm{P}_n(\mathrm{R})$.

Ex. 6. *Is the mapping* $\mathrm{T} : \mathrm{R}^3 \to \mathrm{R}^2$ *defined by* $T(x, y, z) = (|x|, 0)$ *a linear transformation.*

Sol. We have $\mathrm{T} : \mathrm{R}^3 \to \mathrm{R}^2$ defined by $\mathrm{T}(x, y, z) = (|x|, 0)$

Let $\alpha, \beta \in \mathrm{R}^3$ where $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

For $a, b \in \mathrm{R}$, $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$\therefore \; \mathrm{T}(a\alpha + b\beta) = \mathrm{T}(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = (|ax_1 + bx_2|, 0)$

And $a\,\mathrm{T}(\alpha) + b\,\mathrm{T}(\beta) = a\,\mathrm{T}(x_1, y_1, z_1) + b\,\mathrm{T}(x_2, y_2, z_2)$

$= a(|x_1|, 0) + b(|x_2|, 0) = (a|x_1| + b|x_2|, 0)$

Clearly $\mathrm{T}(a\alpha + b\beta) \neq a\mathrm{T}(\alpha) + b\,\mathrm{T}(\beta)$. Hence T is not a linear transformation.

Ex. 7. *Let T be a linear transformation on a vector space U into V. Prove that the vectors* $x_1, x_2, \ldots x_n \in \mathrm{U}$ *are linearly independent if* $\mathrm{T}(x_1), \mathrm{T}(x_2) \ldots \mathrm{T}(x_n)$ *are L.I.*

Sol. Given $\mathrm{T} : \mathrm{U(F)} \to \mathrm{V(F)}$ is a L.T. and $x_1, x_2, \ldots x_n \in \mathrm{U}$.

Let there exist $a_1, a_2 \ldots a_n \in \mathrm{F}$ such that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = \bar{\mathrm{O}}$ ... (1) $(\bar{\mathrm{O}} \in \mathrm{U})$

$\therefore \; \mathrm{T}(a_1x_1 + a_2x_2 + \ldots + a_nx_n = \mathrm{T}(\bar{\mathrm{O}}) \Rightarrow a_1 \, \mathrm{T}(x_1) + a_2 \, \mathrm{T}(x_2) \ldots + a_n \, \mathrm{T}(x_n) = \hat{\bar{\mathrm{O}}}$ $(\bar{\mathrm{O}} \in \mathrm{V})$

But $\mathrm{T}(x_1), \mathrm{T}(x_2), \ldots, \mathrm{T}(x_n)$ are L. I. $\therefore \; a_1 = a_2 = \ldots = a_n = 0$.

$\therefore$ From (1) $x_1, x_2 \ldots x_n$ are L. I.

Ex. 8. *Let V be a vector space of* $n \times n$ *matrices over the field F. M is a fixed matrix in V. The mapping* $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ *is defined by T (A) = AM + MA where* $\mathrm{A} \in \mathrm{V}$. *Show that T is linear.*

Sol. Let $a, b \in \mathrm{F}$ and $\mathrm{A}, \mathrm{B} \in \mathrm{V}$. Then T (A) = AM + MA and T (B) = BM + MB

$\therefore \; \mathrm{T}(a\mathrm{A} + b\mathrm{B}) = (a\mathrm{A} + b\mathrm{B})\,\mathrm{M} + \mathrm{M}\,(a\mathrm{A} + b\mathrm{B}) = a\,(\mathrm{AM} + \mathrm{MA}) + b\,(\mathrm{BM} + \mathrm{MB})$

$= a\,\mathrm{T}(\mathrm{A}) + b\,\mathrm{T}(\mathrm{B})$. T is a linear transformation.

Ex. 9. *Describe explicitly the linear transformation* $T : R^2 \to R^2$ *such that* *T (2,3) = (4,5) and T (1,0) = (0, 0).*

Sol. First of all we have to show that the vectors (2,3) and (1, 0) are L.I.

Let $= a\,(2,3) + b\,(1,0) = \bar{O}$

$\Rightarrow\ (2a+b, 3a+0) = (0,0)\ \Rightarrow\ 2a+b=0, 3a=0 \quad \Rightarrow\ 2a=0, b=0$

$\therefore\ S = \{(2,3), (1,0)\}$ is L.I. Let us prove that L (S) = R^2

Let $(x, y) \in R^2$ and $(x, y) = a\,(2,3) + b\,(1,0) = (2a+b, 3a)$

$\Rightarrow\ 2a+b = x\,,\ 3a = y \quad \Rightarrow\ a = \frac{y}{3};\ b = \frac{3x-2y}{3}$. Hence S spans R^2

Now $T\,(x, y) = T\left[\frac{y}{3}(2,3) + \frac{3x-2y}{3}(1,0)\right]$

$$= \frac{y}{3}T\,(2,3) + \frac{3x-2y}{3}T\,(1,0) = \frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$$

$\therefore$ This is the required transformation.

Ex. 10. *Find* $T\,(x, y, z)$ *where* $T : R^3 \to R$ *is defined by T (1,1,1) = 3,* $T\,(0,1,-2) = 1$, $T\,(0,0,1) = -2$.

Sol. Let $S = \{(1,1,1), (0,1,-2), (0,0,1)\}$

(*i*) Let $a\,(1,1,1) + b\,(0,1,-2) + c\,(0,0,1) = \bar{O}$

$\Rightarrow\ (a, a+b, a-2b+c) = (0,0,0) \quad (\because\ \bar{O} \in R^3)$

$\Rightarrow\ a = 0, a+b = 0, a-2b+c = 0 \Rightarrow\ a = 0, b = 0, c = 0 \quad (\therefore\ $ S is L.I. set)

(*ii*) Let $(x, y, z) \in R^3$

$(x, y, z) = a\,(1,1,1) + b\,(0,1,-2) + c\,(0,0,1) = (a, a+b, a-2b+c)$

$\Rightarrow\ a = x, a+b = y, a-2b+c = z \ \Rightarrow\ a = x, b = y-x, c = z+2y-3x$

$\therefore$ S spans R^3

Hence $T\,(x, y, z) = T\,[x\,(1,1,1) + (y-x)\,(0,1,-2) + (z+2y-3x)\,(0,0,1)]$

$= x\,T\,(1,1,1) + (y-x)\,T\,(0,1,-2) + (z+2y-3x)\,T\,(0,0,1)$

$= x\,(3) + (y-x)\,(1) + (z+2y-3x)\,(-2)$

$= 8x - 3y - 2z$ which is the required linear functional.

Ex. 11. *Let* $T:U\to V$ *be a linear transformation* $\{(1,2,1),(2,1,0),(1,-1,-2)\}$ *and* $\{(1,0,0),(0,1,0),(1,1,1)\}$ *are the basis of U and V. Find T for the transformation of the basis of U to the basis of V.*)

Sol. Let $\alpha_1=(1,2,1),\alpha_2=(2,1,0),\alpha_3=(1,-1,-2)$ and let $\{\alpha_1,\alpha_2,\alpha_3\}$ form a basis for U.

Let $\beta_1=(1,0,0),\beta_2=(0,1,0),\beta_3=(1,1,1)$ and $\{\beta_1,\beta_2,\beta_3\}$ form a basis for V.

We define $T:U\to V$ as follows : $T(\alpha_i)=\beta_i, i=1,2,3$

Let $\alpha\in U$. Let $\alpha=a_1\alpha_1+a_2\alpha_2+a_3\alpha_3$

$=a_1(1,2,1)+a_2(2,1,0)+a_3(1,-1,-2)$

$=(a_1+2a_2+a_3, 2a_1+a_2-a_3, a_1-2a_3)=(x,y,z)$ (say)

Then $x=a_1+2a_2+a_3; y=2a_1+a_2-a_3; z=a_1-2a_3$

Equating the components, $x=a_1+2a_2+a_3, y=2a_1+a_2-a_3, z=a_1-2a_3$

Solving we get, $a_1=\frac{1}{3}(-2x+4y-3z); a_2=x-y+z; a_3=\frac{1}{3}(-x+2y-3z)$

$T(\alpha)=T(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3)$

$=a_1T(\alpha_1)+a_2T(\alpha_2)+a_3T(\alpha_3)$ ($\because$ T is a L. T.)

$=a_1\beta_1+a_2\beta_2+a_3\beta_3$ (by the dfinition of T)

$=a_1(1,0,0)+a_2(0,1,0)+a_3(1,1,1)$

$=(a_1+a_3, a_2+a_3, a_3)$

$\therefore$ $T(\alpha)=T(x,y,z)=(a_1+a_3,a_2+a_3,a_3)$

$$=\left(-x+2y-2z,\frac{2x-y}{3},\frac{-x-2y+3z}{3}\right)$$

This gives the required linear transformation.

Ex. 12. *Show that the transformation* $T:R^3\to R^3$ *defined by* $T(x,y,z)=(x-y,0,y+z)$ *is a linear transformation*

Sol. $T:R^3\to R^3$ is defined as $T(x,y,z)=(x-y,0,y+z)$

$\alpha=(x_1,y_1,z_1)$ and $\beta=(x_2,y_2,z_2)\in R^3$ (domain)

then $\alpha+\beta=(x_1+x_2,y_1+y_2,z_1+z_2)$ and $c\alpha=(cx_1,cy_1,cz_1)$

$T(\alpha+\beta)=T(x_1+x_2,y_1+y_2,z_1+z_2)$

$$= (x_1 + x_2 - \overline{y_1 + y_2}, 0, y_1 + y_2 + \overline{z_1 + z_2})$$

$$= (x_1 - y_1 + x_2 - y_2, 0, y_1 + z_1 + y_2 + z_2)$$

$$= (x_1 - y_1, 0, y_1 + z_1) + (x_2 - y_2, 0, y_2 + z_2)$$

$$= T(\alpha) + T(\beta)$$

$\therefore$ T $(\alpha + \beta) =$ T $(\alpha) +$ T (β) ... (1)

$T(c\alpha) = T(cx_1, cy_1, cz_1)$

$= (cx_1 - cy_1, 0, cy_1 + cz_1) = c\,(x_1 - y_1, 0, y_1 + z_1) = cT(\alpha)$

$T(c\alpha) = cT(\alpha)$ ... (2)

by (1) & (2) T is a linear transformation.

EXERCISE 3 (*a*)

1. Which of the following maps are linear transformations ,

(*a*) $T : V_1 \to V_3$ defined by $T(x) = (x, 2x, 3x)$.

(*b*) $T : R^2 \to R^2$ defined by $T(a, b) = (2a + 3b, 3a - 4b)$.

(*c*) $T : R^3 \to R^2$ defined by $T(x, y, z) = (x - y, x - z)$.

(*d*) $T : P \to P$ defined by $T(x) = x^2 + x$.

(*e*) $T : P \to P$ defined by $T\,p(x) = p(0) + xp'(0) + \dfrac{x^2}{2!} p''(0)$.

(*f*) $T : R^3 \to R^3$ defined by $T(x, y, z) = (x + 1, y, z)$.

(*g*) $T : R^2 \to R^2$ defined by $T(x, y) = (x^3, y^3)$.

2. Let V be the space of $m \times n$ matrices over the field F. Let P be a fixed $m \times m$ matrix and Q is a fixed $n \times n$ matrix over F.

$T : V \to V$ is defined by T (A) = PAQ. Then show that T is a linear transformation.

{ **Hint :** $T(aA + bB) = P(aA + bB)Q = (a\,PA + b\,PB)\,Q$

$= a\,PAQ + b\,PBQ = a\,T(A) + b\,T(B)$).

Find a linear transformation.

3. $T : R^2 \to R^2$ such that T (1,0) = (1,1) and T (0,1) = (−1,2)

4. $T : V_2 \to V_2$ such that T (1,2) = (3,0) and T (2,1) = (1,2)

5. $T : V_3 \to V_3$ such that T (0,1,2) = (3,1,2) and T (1,1,1) = (2,2,2).

6. $T : V_2(R) \to V_3(R)$ such that T (1,2) = (3,−1,5) and T (0,1) = (2,1,−1)

7. $T : R^2 \to R^3$ such that $T(2,-5) = (-1,2,3)$ and $T(3,4) = (0,1,5)$

8. Find a linear transformation. $T : R^2 \to R^2$, such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$

Prove that T maps the square with vertices (0,0), (1,0), (1,1), (0,1) into a parallelogram.

SUM OF LINEAR TRANSFORMATONS

3.9. Definition. *Let T_1 and T_2 be two linear transformations from U (F) into V (F). Then their sum $T_1 + T_2$ is defined by* $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\ \alpha \in U$

3.10. Theorem. ***Let U(F) and V(F) be two linear transformations. Let T_1 and T_2 be two linear transformations from U into V. Then the mapping $T_1 + T_2$ defined by $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\ \alpha \in U$ is a linear tranformation.***

Proof. Given $T_1 : U \to V$ and $T_2 : U \to V$, $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\ \alpha \in U$

$T_1(\alpha) \in V$ and $T_2(\alpha) \in V \Rightarrow T_1(\alpha) + T_2(\alpha) \in V$. Hence $(T_1+T_2) : U \to V$

Let $a, b \in F$ and $\alpha, \beta \in U$. Then $(T_1+T_2)(a\alpha + b\beta) = T_1(a\alpha + b\beta) + T_2(a\alpha + b\beta)$ (by def.)

$= a\,T_1(\alpha) + b\,T_1(\beta) + a\,T_2(\alpha) + b\,T_2(\beta) = a[T_1(\alpha) + T_2(\alpha)] + b[T_1(\beta) + T_2(\beta)]$

$= a(T_1+T_2)(\alpha) + b(T_1+T_2)(\beta)$ $\therefore$ $T_1 + T_2$ is a L. T. from U into V.

3.11. SCALAR MULTIPLICATION OF A L. T.

Theorem. ***Let $T : U(F) \to V(F)$ be a linear transformation and $a \in F$. Then the function aT defined by $(aT)(\alpha) = a\,T(\alpha)\ \forall\ \alpha \in U$ is a linear transformation.***

Proof. Given $T : U(F) \to V(F)$ and $(aT)(\alpha) = a\,T(\alpha)$, $a \in F$, $\alpha \in U$

Now $T(\alpha) \in V \Rightarrow a\,T(\alpha) \in V$. $\therefore$ (aT) is a mapping from U into V

For $c, d \in F$ and $\alpha, \beta \in U$

$(aT)[c\alpha + d\beta] = a\,T(c\alpha + d\beta)$ (by def)

$= a[cT(\alpha) + d\,T(\beta)] = acT(\alpha) + ad\,T(\beta) = c(aT)(\alpha) + d(aT)(\beta)$

Hence aT is a L.T. from U into V.

SOLVED PROBLEMS

Ex. 1. *Let* $T : V_3(R) \to V_2(R)$ *and* $H : V_3(R) \to V_2(R)$ *be the two linear transformations defined by* $T(x,y,z) = (x-y, y+z)$ *and* $H(x,y,z) = (2x, y-z)$

Find (i) $H + T$ *(ii)* aH

Sol. (*i*) $(H+T)(x,y,z) = H(x,y,z) + T(x,y,z) = (2x, y-z) + (x-y, y+z) = (3x-y, 2y)$

(*ii*) $(aH)(x,y,z) = aH(x,y,z) = a(2x, y-z) = (2ax, ay-az)$

Ex. 2. *Let* $G : V_3 \to V_3$ *and* $H : V_3 \to V_3$ *be two linear operators defined by* $G(e_1) = e_1 + e_2, G(e_2) = e_3, G(e_3) = e_2 - e_3$ *and* $H(e_1) = e_3, H(e_2) = 2e_2 - e_3, H(e_3) = 0$ *where* $\{e_1, e_2, e_3\}$ *is the standard basis of* $V_3(R)$.

Find (*i*) G + H (*ii*) 2 G

Sol. Let S = {(1,0,0), (0,1,0), (0,0,1)} be the standard basis of $V_3(R)$ so that

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$G(e_1) = e_1 + e_2 \Rightarrow G(1,0,0) = (1,1,0)$, $G(e_1) = e_3 \Rightarrow G(0,1,0) = (0,0,1)$

$G(e_3) = e_2 - e_3 \Rightarrow G(0,0,0) = (0,1,-1)$

Again $H(e_1) = e_3 \Rightarrow H(1,0,0) = (0,0,1)$, $H(e_2) = 2e_2 - e_3 \Rightarrow H(0,1,0) = (0,2,-1)$

$H(e_3) = \bar{o} \Rightarrow H(0,0,1) = (0,0,0)$

(*i*) $(G + H)(e_1) = G(e_1) + H(e_1) = e_1 + e_2 + e_3 \Rightarrow (G + H)(1,0,0)=(1,1,1)$

$(G + H)(e_2) = G(e_2) + H(e_2) = 2e_2 \Rightarrow (G + H)(0,1,0) = (0,2,0)$

$(G + H)(e_3) = G(e_3) + H(e_3) = e_2 - e_3 \Rightarrow (G + H)(0,0,1) = (0,1,-1)$

(*ii*) $2G(e_1) = 2G(e_1) = 2e_1 + e_2$, $2G(e_2) = 2G(e_2) = 2e_3$, $2G(e_3) = 2G(e_3) = 2e_2 - 2e_3$ etc.

PRODUCT OF LINEAR TRANSFORMATIONS

3.12. Theorem. ***Let U(F), V (F) and W(F) are three vector spaces and*** $T : U \to W$ ***and*** $H : U \to V$ ***are two linear transformations. Then the composite function TH (called the product of linear transformations) defined by*** $(TH)(\alpha) = T[H(\alpha)]\ \forall\ \alpha \in U$ ***is a linear transformation from U into W.***

Proof. Given $H : U(F) \to V(F)$ and $T : V(F) \to W(F)$

For $a \in U \Rightarrow H(\alpha) \in V$

Again $H(\alpha) \in V \Rightarrow T[H(\alpha)] \in W \Rightarrow (TH)(\alpha) \in W$

$\therefore$ TH is a mapping from U into W

Now Let $a, b \in F$, $\alpha, \beta \in U$. Then $(TH)[a\alpha + b\beta] = T[H(a\alpha + b\beta)]$ (by def.)

$= T[aH(\alpha) + bH(\beta)]$ (H is L.T.)

$= a(TH)(\alpha) + b(TH)(\beta)$

$\therefore$ TH is a LT. from U to W.

Note. The range of H is the domain of T.

3.13. Theorem. ***Let H, H' be two linear transformations from U(F) to V(F). Let T, T' be the linear transformations from V(F) to W(F) and*** $a \in F$.

Then (i) T(H+H') = TH + TH' (ii) (T + T')H = TH + T'H

(iii) $a(TH) = (aT)H = T(aH)$

Proof. (*i*) Let $\alpha \in U$. Then $T(H+H')(\alpha) = T[H(\alpha)+H'(\alpha)]$

$$= TH(\alpha) + TH'(\alpha) = (TH+TH')(\alpha)$$

(*ii*) Similar to (*i*)

(*iii*) Let at $\alpha \in U$. $\quad a(TH)(\alpha) = aT[H(\alpha)] = [(aT)H](\alpha)$

Again $[T(aH)](\alpha) = T[aH(\alpha)] = [a(TH)](\alpha)$. $\quad \therefore \; a(TH)] = (aT)H = T(aH)$

3.14. ALGEBRA OF LINEAR OPERATORS

Theorem. ***Let A, B, C be linear operators on a vector space V(F). Also let O be the zero operator and I the identity operator on V. Then***

(*i*) AO = OA = O **(*ii*) AI = IA = A**

(*iii*) A (B+C) = AB + AC **(*iv*) (A + B)C = AC + BC**

(*v*) A(BC) = (AB)C

Proof. Let $\alpha \in V$

(*i*) $AO(\alpha) = A[O(\alpha)] = A(\bar{O})$ (by def. of 0)

$= \bar{O}$ (A is L.T.) $= O(\alpha)$ $(\forall \; \alpha \in U)$

Similarly $OA(\alpha) = O[A(\alpha) = \bar{O} = O(\alpha) \Rightarrow OA = O$. Thus AO = OA = O.

(*ii*) Similar to (*i*)

(*iii*) $[A(B+C)](\alpha) = A[(B+C)(\alpha)] = A[B(\alpha)+C(\alpha)] = AB(\alpha) + AC(\alpha) = (AB+AC)(\alpha)$

$$\therefore \; A(B+C) = AB + AC$$

(*iv*) Similar to (*iii*)

(*v*) $[A(BC)](\alpha) = A[(BC)(\alpha)] = A[B\{C(\alpha)\}] = (AB)[C(\alpha)] = [(AB)C](\alpha)$

$\therefore \quad A(BC) = (AB)C$

SOLVED PROBLEMS

Ex. 1. *Let* $T: R^3 \to R^2$ *and* $H: R^3 \to R^2$ *be defined by* $T(x, y, z) = (3x, y+z)$ *and* $H(x, y, z) = (2x-x, y)$. *Compute* (*i*) T + H (*ii*) $4T-5H$ (*iii*) T H (*iv*) H T

Sol. Since T and H map V, the linear transformations T + H and $4T-5H$ are defined.

(*i*) $(T+H)(x, y, z) = T(x,y,z) + H(x,y,z) = (3x, y+z) + (2x-z, y) = (5x-z, 2y+z)$

(*ii*) $(4T-5H)(x, y, z) = 4T(x,y,z) - 5H(x,y,z)$

$$= 4(3x, y+z) - 5(2x-z, y) = (2x+5z, -y+4z)$$

(*iii*) and (iv) both TH and HT are not defined because the range of T is not equal to the domain of H and vice versa.

Ex. 2. *Let* $T_1: R^3 \to R^2$ *and* $T_2: R^2 \to R^2$ *are two linear transformations defined by* $T_1(x,y,z) = (3x, 4y-z)$, $T_2(x, y) = (-x, y)$. *Compute* T_1T_2 *and* T_2T_1.

Sol. (*i*) Since the range of T_2 *i.e.* R^2 is not equal to the domain of T_1 *i.e.,* R^3, T_1T_2 is not defined.

(*ii*) But the range of T_1 *i.e.* R^3 is equal to the domain of T_2, T_2T_1 is defined.

$\therefore \quad (T_2T_1)(x, y, z) = T_2[T_1(x, y, z)] = T_2(3x, 4y - z) = (-3x, 4y - z)$

Ex. 3. *Let P(R) be the vector space of all polynomials in x and D, T be two linear operators on P defined by* $D[f(x)] = \dfrac{df}{dx}$ *and* $T[f(x)] = x\,f(x)\ \forall\ f(x) \in V$

Show (*i*) $TD \neq DT$ (*ii*) $(TD)^2 = T^2D^2 + TD$

Sol. (*i*) $(DT)\,f(x) = D[T\,f(x)] = D[x\,f(x)] = f(x) + x\,f'(x)$

$$(TD)\,f(x) = T[D\,f(x)] = T\left[\frac{df}{dx}\right] = x\,f'(x)$$

Clearly $DT \neq TD$. Also $(DT)\,f(x) - (TD)\,f(x) = f(x)$

$\Rightarrow (DT - TD)\,f(x) = I\,f(x)$ (I is identity) $\Rightarrow DT - TD = I$

(*ii*) $$(TD)^2\ f(x) = (TD)\ [(TD)\,f(x)] = (TD)\left[x\frac{df}{dx}\right] = T\left[D\left(x\frac{df}{dx}\right)\right]$$
$$= T\left[\frac{df}{dx} + x\frac{d^2f}{dx^2}\right] = x\frac{df}{dx} + x^2\frac{d^2f}{dx^2}$$

Now $$(T^2D^2)\,f(x) = T^2D\,\{Df(x)\} = T^2\left[D\left(\frac{df}{dx}\right)\right] = T^2\left(\frac{d^2f}{dx^2}\right)$$
$$= T\left[T\left(\frac{d^2f}{dx^2}\right)\right] = T\left[x\frac{d^2f}{dx^2}\right] = x^2\frac{d^2f}{dx^2}$$

$$\therefore\ (T^2D^2 + TD)\,f(x) = (T^2D^2 + TD)\,f(x) + (TD)\,f(x) = x^2\frac{d^2f}{dx^2} + x\frac{df}{dx}$$

$\therefore\ (TD)^2\ f(x) = (T^2D^2 + TD)\ f(x) \quad \forall\ f(x) \in P$. Hence $(TD)^2 = T^2D^2 + TD$.

TRANSFORMATIONS AS VECTORS

3.15. Theorem. ***Let L (U, V) be the set of all linear transformations from a vector space U (F) into a vector space V (F). Then L (U, V) be a vector space relative to the operations of vector addition and scalar multiplication defined as***

(i) $(T + H)(\alpha) = T(\alpha) + H(\alpha)$ ***(ii)*** $(aT)(\alpha) = aT(\alpha)$ ***for all*** $\alpha \in U . a \in F$ ***and*** $T, H \in L(U, V)$. ***The set L (U, V) is also denoted by Hom (U, V)***

Proof. We have already proved that for all $T, H \in L(U, V)$ and $a \in F$,

$(T + H)$ and (aT) or (aH) are linear transformations and hence $\in L(U, V)$.

Now we verify the remaining properties of vector space.

(*i*) Associativity.

$[T+(H+G)](\alpha) = T(\alpha)+(H+G)(\alpha) = T(\alpha)+[H(\alpha)+G(\alpha)]$

$= [T(\alpha)+H(\alpha)]+G(\alpha)$ (addition in V associative)

$= (T+H)(\alpha)+G(\alpha) = [(T+H)+G](\alpha)$ $\therefore$ $T+(H+G) = (T+H)+G$

(*ii*) Additive identity in L (U, V)

Let O be the zero transformation from U into V *i.e.,* $O(\alpha) = \hat{\hat{O}}$ $\forall$ $\alpha \in U, \hat{\hat{O}} \in V$

$\therefore$ $O \in L(U, V)$

Now $(O+T)(\alpha) = O(\alpha)+T(\alpha) = \hat{\hat{O}}+T(\alpha) = T(\alpha)$ ($\hat{\hat{O}}$ is additive indentity in V)

$\therefore$ $O+T = T$ $\forall$ $T \in L(U, V)$

$\Rightarrow$ O is the additive identity in L (U, V)

(*iii*) Additive inverse.

For $T \in L(U, V)$ let us define $(-T)$ as $(-T)(\alpha) = -T(\alpha)$ $\forall$ $\alpha \in U$

Then $(-T) \in L(U, V)$.

Now $(-T+T)(\alpha) = (-T)(\alpha)+T(\alpha) = -T(\alpha)+T(\alpha) = \hat{\hat{O}}$ $(\hat{\hat{O}} \in V)$

$\Rightarrow$ $-T+T = O$ for all $T \in L(U, V)$.

(*iv*) Commutativity

$\Rightarrow$ $(T+H)(\alpha) = T(\alpha)+H(\alpha) = H(\alpha)+T(\alpha)$ (addition in V is commutative)

$= (H+T)(\alpha) \Rightarrow (T+H) = (H+T)$

$\therefore$ L (U, V) is an abelian group w.r. to addition

(*v*) Let $a \in F$ and $T, H \in L(U, V), \alpha \in U$

Then $[a(T+H)](\alpha) = a[(T+H)(\alpha)$ (by def.)

$= a[T(\alpha)+H(\alpha)]$ (by def.)

$= a\,T(\alpha)+a\,H(\alpha) = (a\,T)(\alpha)+(a\,H)(\alpha)$ (by def.)

$= (a\,T+a\,H)(\alpha) \Rightarrow a(T+H) = a\,T+a\,H$

(*vi*) Let $a, b \in F$ and $T \in L(U, V)$. Then $[(a+b)\,T](\alpha) = (a+b)\,T(\alpha)$ (by def.)

$= a\,T(\alpha)+b\,T(\alpha)$ ($\because$ V is a vector space)

$= (a\,T)(\alpha)+(b\,T)(\alpha)$ (by def.) $= (a\,T+b\,T)(\alpha) \Rightarrow (a+b)\,T = a\,T+b\,T$

(*vii*) $[(a\,b)\,T]\,\alpha = (a\,b)\,T(\alpha) = a[b\,T(\alpha)]$ ($\because$ V is vector space)

$= a[(b\,T)(\alpha)]$ (by def.) $= [a(b\,T)(\alpha)]$ $\Rightarrow$ $(a\,b)\,T = a(b\,T)$

(viii) Let $1 \in F$ and $T \in L(U, V)$.

Then $(1 . T)(\alpha) = 1 . T(\alpha) = T(\alpha)$ (by def.) $\Rightarrow 1 . T = T$.

Hence L (U, V) is a vector space over the field F.

3.16. Theorem. ***Let L (U, V) be the vector space of all linear transformations from U(F) to V (F) such that dim U = n and dim V = m. Then dim L (U, V) = mn.***

Proof. Let $B_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, ..., \beta_n\}$ be the ordered basis of U and V respectively. By the previous theorems there exists uniquely a linear transformation T_{ij} from U to V such that

$$T_{ij}(\alpha_1) = \beta_1, T_{ij}(\alpha_2) = \hat{\hat{O}}, ..., T_{ij}(\alpha_1) = \hat{\hat{O}} \text{ where } \beta_1, \hat{\hat{O}} \in V.$$

i.e. $T_{ij}(\alpha_i) = \beta_j$, $i = 1, 2 ... n$ $j = 1, 2 ... m$ and $T_{ij}(\alpha_k) = \hat{\hat{O}}, k \neq i$.

Thus there are mn such T_{ij}'$s \in L(U, V)$. We shall show that $S = \{T_{ij}\}$ of mn elements is a basis for L (U, V)

(*i*) *To prove S is L.I.*

For a_{ij}'$s \in F$ let us suppose that $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij} = 0$ $(O \in L(U, V))$

For $\alpha_k \in U$, $k = 1, 2 ... n$ we get $\left[\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}\right](\alpha_k) = O(\alpha_k)$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k) = \hat{\hat{O}} \Rightarrow \sum_{j=1}^{m} a_{kj} T_{kj}(\alpha_k) = \hat{\hat{O}} \qquad (\hat{\hat{O}} \in V)$$

$$\Rightarrow a_{k1}\alpha_1 + a_{k2}\alpha_2 + ... + a_{km}\alpha_m = \hat{\hat{O}} \Rightarrow a_{k1} = 0, a_{k2} = 0 ... a_{km} = 0 \qquad (\because B_1 \text{ is L.I.})$$

Hence $S = \{T_{ij}\}$ is an L.I. set.

(*ii*) *To show that* L (S) = L (U,V).

Let $T \in L(U, V)$. The vector $T(\alpha_1) \in V$ can be expressed as

$T(\alpha_1) = b_{11}\beta_1 + b_{21}\beta_2 + ... + b_{m1}\beta_m$

In general for $i = 1, 2 ... m$

$$T(\alpha_i) = b_{i1}\beta_1 + b_{2i}\beta_2 + ... + b_{mi}\beta_m \qquad ... (1)$$

Consider the linear transformation $H = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} T_{ij}$

Clearly H is a linear combination of $S = \{T_{ij}\}$; therefore $H \in L(U, V)$.

Let $\alpha_k \in U$ for $k = 1, 2 \ldots n$. Since $T_{ij}(\alpha_k) = \hat{O}$ for $k \neq i$, We have $T_{kj}(\alpha_k) = \beta_j$

Consider $H(\alpha_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} T_{ij}(\alpha_k) = \sum_{j=1}^{m} b_{kj} T_{kj}(\alpha_k) = \sum_{j=1}^{m} b_{kj} \beta_j$

i.e. $H(\alpha_k) = b_{k1}\beta_1 + b_{k2}\beta_2 + \ldots + b_{km}\beta_m = T(\alpha_k)$ [by (1)]

Hence $H(\alpha_k) = T(\alpha_k)$ for each k. $\therefore$ H = T

Thus T is a linear combination of elements of S. *i.e.*, L (S) = L (U, V).

$\therefore$ S is a basis set of L (U, V) $\therefore$ dim L(U, V) = mn.

EXERCISE 3 (*b*)

1. Let $T : R^3 \to R^2$ and $H : R^2 \to R^3$ be two linear transformation defined by

   $T(x, y, z) = (x - 3y - 2z, y - 4z)$ and $H(x, y) = (2x, 4x - y, 2x + 3y)$

   Find HT and TH. Is product commutative ?

2. Define on R^2 linear operators H and T as follows $H(x, y) = (0, x)$ and $T(x, y) = (x, 0)$

   and show that $TH = 0, HT \neq TH$ and $T^2 = T$.

3. Give an example of a linear operator T on R^3 such that $T \neq O, T^2 \neq O$ but $T^2 = O$.

   [**Hint:** $T : R^3 \to R^3$ where $T(x, y, z) = (0, x, y)$]

4. Let P be the polynomial space in one indeterminate x with real coefficients. Let $D : P \to P$ and $S : P \to P$ be two linear operators defined by .

   $$D f(x) = \frac{df}{dx} \text{ and } S f(x) = \int_0^x f(x)\, dx \quad \forall f(x) \in P$$

   Show that DS = I and $SD \neq I$ where I is the identity transformation.

5. Let $T : V_3(R) \to V_3(R)$ be defined by $T(a, b, c) = (3a, a - b, 2a + b + c)$

   Prove that $(T^2 - I)(T - 3I) = O$.

3.17. RANGE AND NULL SPACE OF A LINEAR TRANSFORMATION

RANGE. Definition. *Let U (F) and V (F) be two vector spaces and let* $T : U \to V$ *be a linear transformation. The range of T is defined to be the set Range* $(T) = R(T) = \{T(\alpha) : \alpha \in U\}$.

Obviously the range of T is a subset of V. *i.e.* $R(T) \subseteq V$.

3.18. Theorem. ***Let U(F) and V(F) be two vector spaces. Let $T: U(F) \to V(F)$ be a linear transformation. Then the range set R (T) is a subspace of V (F).***

Proof. For $\bar{O} \in U \Rightarrow T(\bar{O}) = O \in R(T)$ $\therefore$ R (T) is non-empty set and $R(T) \subseteq V$

Let $\alpha_1, \alpha_2 \in U$ and $\beta_1, \beta_2 \in R(T)$ be such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$

For $a, b \in F$, $a\alpha_1 + b\alpha_2 \in U$ $(\because U$ is V.S.) $\Rightarrow T(a\alpha_1 + b\alpha_2) \in R(T)$

But $T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2) = a\beta_1 + b\beta_2 \in R(T)$ $(\because$ T is L.T.)

Thus $a, b \in F$ and $\beta_1, \beta_2 \in R(T) \Rightarrow a\beta_1 + b\beta_2 \in R(T)$

$\therefore$ R (T) is a subspace of V(F). R (T) is called the range space.

3.19. NULL SPACE OR KERNEL

Definition. *Let U (F) and V (F) be two vector spaces and $T: U \to V$ be a linear transformation. The null space denoted by N (T) is the set of all vectors $\alpha \in U$ such that $T(\alpha) = \bar{O}$ (zero vector of V).*

The null space of N (T) is also called the kernel of T *i.e.*, $N(T) = \{\alpha \in U : T(\alpha) = \bar{O} \in V\}$.

Obviously the null space $N(T) \subseteq U$. ***(S.V. U. 2011)***

3.20. Theorem. ***Let U (F) and V (F) be two vector spaces and $T: U \to V$ is a linear transformation. Then null space N (T) is a subspace of U (F).***

)

Proof. Let $N(T) = \{\alpha \in U : T(\alpha) = \bar{O} \in V\}$

$\because T(o) = \hat{O} \Rightarrow \hat{O} \in N(T)$ $(\bar{O} \in U, \hat{O} \in V)$

$\therefore$ N(T) is a non-empty subset of U.

Now $\alpha, \beta \in N(T) \Rightarrow T(\alpha) = \hat{O}, T(\beta) = \hat{O}$

For $a, b \in F$. $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) = a.\hat{O} + b.\hat{O} = \hat{O}$ (T is L.T.)

$\therefore T(a\alpha + b\beta) = \hat{O}$

By definition $a\alpha + b\beta \in N(T)$

Thus $a, b \in F$ and $\alpha, \beta \in N(T) \Rightarrow a\alpha + b\beta \in N(T)$

$\therefore$ Null space N (T) is a subspace of U (F).

3.21. Theorem. ***Let $T: U(F) \to V(F)$ be a linear transformation. If U is finite dimensional then the range space (T) is a finite dimensional subspace of V (F).***

Proof. Given U is finite dimensional

$\therefore$ Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be the basis set of U (F).

Let $\beta \in R(T)$, the range space of T.

Then there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

$\therefore$ $\alpha = a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n$ for $a's \in F$

$\Rightarrow$ $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 \dots a_n\alpha_n) \Rightarrow \beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) \dots + a_n T(\alpha_n)$

But $S' = \{T(\alpha_1), T(\alpha_2) \dots T(\alpha_n)\} \in R(T)$

Now $\beta \in R(T)$ and $\beta = l.c$ of elements of $S' \Rightarrow \beta \in L(S')$.

Thus R(T) is spanned by a finite set S'.

$\therefore$ R(T) is finite dimensionalsubspace of V(F).

DIMENSION OF RANGE AND KERNEL

3.22. Definition. Let $T : U(F) \to V(F)$ be *a linear transformation where U is finite dimensional vector space.*

Rank : *Then the rank of T denoted by* $\rho(T)$ *is the dimension of range space R(T).*

$$\rho(T) = \dim R(T)$$

Nullity : *The nullity of T denoted by* $\nu(T)$ *is the dimension of null space N (T).*

$$\nu(T) = \dim N(T)$$

RANK - NULLITY THEOREM :

3.23. Theorem. ***Let U(F) and V(F) be two vector spaces and*** $T : U \to V$ ***be a linear transformation. Let U be finite dimensional then*** $\rho(T) + \nu(T) = \dim U$

i.e. rank (T) + nullity (T) = dim U.

Proof. The null space N (T) is a subspace of finite dimensional space U (F).

$\Rightarrow$ N (T) is finite dimensional

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_k$ be a basis set of N (T). $\therefore$ dim N(T) = ν (T) = k.

$\therefore$ $T(\alpha_1) = \hat{\hat{O}}, T(\alpha_2) = \hat{\hat{O}} \dots T(\alpha_k) = \hat{\hat{O}}$ ... (1) $(\hat{\hat{O}} \in V)$

As S is L.I. it can be extended to form a basis of U (F)

Let $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_k, \theta_1, \theta_2 \dots \theta_m\}$ be the extended basis of U (F). $\therefore$ $\dim U = k + m$

Now we show that the set of images of additional vectors

$S_2 = \{T(\theta_1), T(\theta_2) \dots T(\theta_m)\}$ is a basis of R (T). Clearly $S_2 \subseteq R(T)$

(*i*) *To prove* S_2 *is L. I.*

Let $a_1, a_2 \dots a_m \in F$ be such that, $a_1 T(\theta_1) + a_2 T(\theta_2) + \dots + a_m T(\theta_m) = \hat{\hat{O}}$

$\Rightarrow$ $T[a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m] = \hat{\hat{O}}$

$\Rightarrow$ $a_1\theta_1 + a_2\theta_2 \dots + a_m\theta_m \in N(T)$, null space of T.

But each vector in N(T) is a *l.c.* of elements of basis S

$\therefore$ For some $b_1, b_2 \dots b_m \in F$, let $a_1\theta_1 + a_2\theta_2 \dots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \dots b_k\alpha_k$

$\Rightarrow$ $a_1\theta_1 + \dots + a_m\theta_m - b_1\alpha_1 - \dots - b_k\alpha_k = \hat{\hat{O}}$

$\Rightarrow$ $a_1 = 0, a_2 = 0 \dots a_m = 0, b_1 = 0 \dots b_k = 0$ $\quad$ ($\because$ S_1 is L.I.) $\quad$ $\Rightarrow$ S_2 is L.I. set

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space R (T), then there exists $\alpha \in U$ such that $T(\alpha) = \beta$

Now $\alpha \in U$ $\Rightarrow$ there exist $c's, d's \in F$ such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 \dots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 \dots + d_m\theta_m$$

$\Rightarrow$ $T(\alpha) = T[c_1\alpha_1 + \dots + c_k\alpha_k + d_1\theta_1 + \dots + d_m\theta_m]$

$$= c_1 T(\alpha_1) + \dots + c_k T(\alpha_k) + d_1 T(\theta_1) + \dots + d_m T(\theta_m)$$

$\Rightarrow$ $\beta = d_1 T(\theta_1) + d_2 T(\theta_2) \dots + d_m T(\theta_m)$ $\quad$ (by (1)) $\quad$ $\Rightarrow$ $\beta \in L(S_2)$

$\therefore$ S_2 is a basis of R(T) and dim R(T) = m.

Thus dim R(T) + dim N(T) = $m + k$ = dim U $\quad$ *i.e.*, $\rho(T) + \nu(T) = \dim U$.

SOLVED PROBLEMS

Ex. 1. *If* $T : V_4(R) \to V_3(R)$ *is a linear transformation defined by*

$T(a,b,c,d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d)$ *for* $a,b,c,d \in R$,

then verify $\rho(T) + \nu(T) = \dim V_4(R)$

Sol. Let $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ be the basis set of $V_4(R)$.

$\therefore$ The transformation T on B will be $T(1,0,0,0) = (1,1,1)$, $T(0,1,0,0) = (-1,0,1)$

$T(0,0,1,0) = (1,2,3)$, $\quad T(0,0,0,1) = (1,-1,-3)$

Let $S_1 = \{(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)\}$ $\quad$ $\therefore$ $S_1 \subseteq R(T)$

Now we verify whether S_1 is L.I. or not,. If not, we find the least L.I. set by forming

the matrix, $S_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$ applying $R_2 + R_1, R_3 - R_1, R_4 - R_1$

$S_1 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$ Again apply $R_4 + 2R_3, R_3 - R_2$, $S_1 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The non - zero rows of vectors $\{(1,1,1), (0,1,2)\}$

Constitute the L.I. set forming the basis of R (T) $\Rightarrow$ dim R (T) = 2

Basis for null space of T. $\alpha \in N(T) \Rightarrow T(\alpha) = \hat{\hat{O}}$

$\therefore T(a,b,c,d) = \hat{\hat{O}}$ where $\hat{\hat{O}} = (0,0,0) \in V_3(R)$

$\Rightarrow (a-b+c+d, a+2c-d, a+b+3c-3d) = (0,0,0)$

$\Rightarrow a-b+c+d = 0$, $a+2c-d = 0$

$a+b+3c-3d = 0$, we have to solve these for *a, b, c, d.*

Coefficient matrix $= \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$, by $R_2 - R_1, R_3 - R_1 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$

by $R_3 - 2R_2$, the echelon form is $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The equivalent system of equations are

$a-b+c+d = 0$, $b+c-2d = 0$ $\Rightarrow b = 2d - c$, $a = d - 2c$

The number of free variables is 2 namely *c, d* and the values of *a*, *b* depend on these and hence nullity (T) = dim N(T) = 2.

Choosing $c = 1, d = 0$, we get $a = -2, b = -1$, $\therefore (a,b,c,d) = (-2,-1,1,0)$

Choosing $c = 0, d = 1$, we get $a = 1, b = 2$. $\therefore (a,b,c,d) = (1,2,0,1)$

$\therefore \{(-2,-1,1,0), (1,2,0,1)\}$ constitute a basis of N(T)

$\therefore$ dim R(T) + dim N(T) = 2 + 2 = 4 = dim $V_4(R)$.

Ex. 2. *Find a linear transformation* $T: R^3 \to R^3$ *whose range is spanned by* $(1,2,0,-4)$, $(2,0,-1,-3)$.

SoL Given R (T) is spanned by $\{(1,2,0,-4), (2,0,-1,-3)\}$

Let us include a vector (0, 0, 0, 0) in this set which will not effect the spanning property so that

$S = \{(1,2,0,-4), (2,0,-1,-3), (0,0,0,0)\}$

Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of R^3. We know there exists a transformation T such that

$T(\alpha_1) = (1,2,0,-4)$; $T(\alpha_2) = (2,0,-1,-3)$ and $T(\alpha_3) = (0,0,0,0)$

Now if $\alpha \in R^3 \Rightarrow \alpha = (a,b,c) = a\,\alpha_1 + b\,\alpha_2 + c\,\alpha_3$

$$\therefore\ T(a,b,c) = T(a\,\alpha_1 + b\,\alpha_2 + c\,\alpha_3) = a\,T(\alpha_1) + b\,T(\alpha_2) + c\,T(\alpha_3)$$

$$= a(1,2,0,-4) + b(2,0,-1,-3) + c(0,0,0,0)$$

$\therefore\ T(a,b,c) = (a+2b,\ 2a,\ -b,\ -4a-3b)$ is the required transformation.

Ex. 3: *Find* $F : R^3 \to R^4$ *is a linear transformation whose range is spanned by* $(1,-1,2,3)$ *and* $(2,3,-1,0)$.

Sol. Consider the standard basis for R^3 e_1, e_2, e_3 where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Then $F(e_1) = (1,-1,2,3)$, $F(e_2) = (2,3,-1,0)$ and $F(e_3) = (0,0,0,0)$.

We know that, $(x,y,z) = xe_1 + ye_2 + ze_3$

$\therefore\ F(x,y,z) = F(xe_1 + ye_2 + ze_3) = xF(e_1) + yF(e_2) + zF(e_3)$ ($\because$ F is a linear transformation)

$$= (x, -x, 2x, 3x) + (2y, 3y, -y, 0) + (0,0,0,0) \quad = (x+2y,\ -x+3y,\ 2x-y,\ 3x)$$

Ex. 4. *Let V be a vector space of* 2×2 *matrices over reals. Let P be a fixed matrix of V ;* $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ *and* $T : V \to V$ *be a linear operator defined by T(A) = PA,* $A \in V$. *Find the nullity T.*

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

The null space N (T) is the set of all 2×2 matrices whose T - image is $\hat{O}$

$$\Rightarrow\ T(A) = PA = \hat{O}\ \Rightarrow\ T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow a-c=0,\ b-d=0 \Rightarrow a=c,\ b=d,$

The free variables are *c* and *d*. Hence dim N(T) = 2.

Ex. 5. *Find the null space, range, rank and nullity of the transformation* $T : R^2 \to R^3$ *defined by* $T(x,y) = (x+y,\ x-y,\ y)$. ***(S. V. U. M12, O2000, K. U. M 2013)***

Sol. (*i*) Let $\alpha = (x,y) \in R^2$. Then $\alpha \in N(T) \Rightarrow T(a) = \hat{O}$ $\qquad (\hat{O} \in R^3)$

i.e., $T(x,y) = (0,0,0) \Rightarrow (x+y,\ x-y,\ y) = (0,0,0)$

$\Rightarrow \quad x+y=0, x-y=0, y=0 \Rightarrow \quad x=0, y=0 \qquad \therefore \quad \alpha=(0,0)=(\hat{\bar{O}} \in R^2)$

Thus the null space of T consists of only zero vector of R^2

$\therefore$ nullity T = dim N(T) = 0

(*ii*) Range space of $T=\{\beta \in R^2 : T(\alpha)=\beta$ for $\alpha \in R^2\}$

$\therefore$ The range space consists of all vectors of the type $(x+y, x-y, y)$ for all $(x, y) \in R^2$

(*iii*) dim R(T) + dim N(T) = dim $R^2 \Rightarrow$ dimR(T) + 0 = 2 $\Rightarrow$ rank of T = 2.

Ex. 6. *Let V (F) be a vector space and T be a linear operator on V. Prove that the following statements are true*

(*i*)*The intersection of the range of T and null space of T is the zero subspace of T.*

i.e., $R(T) \cap N(T)=\{\bar{O}\}$ (*ii*) *If* $T[T(\alpha)]=\bar{O}$, *then* $T(\alpha)=\bar{O}$

Sol. (*i*) $\Rightarrow$ (*ii*)

Let $R(T) \cap N(T)=\{\bar{O}\}$. Let $T(\alpha)=\beta \quad \therefore \beta \in R(T)$ ... (1)

Now $T[T(\alpha)]=\bar{O} \Rightarrow T(\beta)=\bar{O} \Rightarrow \beta \in N(T)$ ... (2)

From(1) and (2) $\beta \in R(T) \cap N(T)$. But $R(T) \cap N(T)=\{\bar{O}\} \Rightarrow \beta=\bar{O} \quad \Rightarrow T(\alpha)=\bar{O}$

Thus $T[T(\alpha)]=\bar{O} \Rightarrow T(\alpha)=\bar{O}$

(*ii*) $\Rightarrow$ (*i*) Given $T[T(\alpha)]=\bar{O} \Rightarrow T(\alpha)=\bar{O}$

Let $\beta \in R(T) \cap N(T)$. $\quad \therefore \beta \in R(T)$ and $\beta \in N(T)$

Now $\beta \in R(T) \Rightarrow T(\alpha)=\beta$ for some $\alpha \in V$ and $\beta \in N(T) \Rightarrow T(\beta)=\bar{O} \Rightarrow T[T(\alpha)]=\bar{O}$

$\Rightarrow T(\alpha)=\bar{O} \Rightarrow \beta=\bar{O}$. Thus $R(T) \cap N(T)=\bar{O}$

Ex. 7. *Verify the Rank-Nullity theorem for the linear map* $T: V_4 \rightarrow V_3$ *defined by* $T(e_1)=f_1+f_2+f_3$, $T(e_2)=f_1-f_2+f_3$, $T(e_3)=f_1$, $T(e_4)=f_1+f_3$ *when* $\{e_1, e_2, e_3\}$ *and* $\{f_1, f_2, f_3\}$ *are standard basis* V_4 *and* V_3 *respectively*

Sol.:Let $e_1=(1,0,0,0), e_2=(0,1,0,0) \quad e_3=(0,0,1,0), e_4=(0,0,0,1)$ and

$f_1=(1,0,0), f_2=(0,1,0), f_3=(0,0,1)$

$\{e_1, e_2, e_3, e_4\}, \{f_1, f_2, f_3\}$ are the standard basis of V_4 and V_3 respectively.

we have $T(e_1)=f_1+f_2+f_3 \Rightarrow T(1,0,0,0)=(1,0,0)+(0,1,0)+(0,0,1)=(1,1,1)$

$T(e_2)=f_1-f_2+f_3 \Rightarrow T(0,1,0,0)=(1,0,0)-(0,1,0)+(0,0,1)=(1,-1,1)$

$T(e_3)=f_1 \Rightarrow T(0,0,1,0)=(1,0,0)$; $\quad T(e_4)=f_1+f_3 \Rightarrow T(0,0,0,1)=(1,0,0)+(0,0,1)$

Let $\alpha \in V_4$. The α can be written as $\alpha = ae_1 + be_2 + ce_3 + de_4$

Then $T(\alpha) = a\,T(e_1) + b\,T(e_2) + c\,T(e_3) + d\,T(e_4)$

$= a(1,1,1) + b(1,-1,1) + c(1,0,0) + d(1,0,1) = (a+b+c+d,\ a-b,\ a+b+d)$

Consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_4 = R_4 - R_1 \\ R_3 = R_3 - R_1 \\ R_2 = R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$R_4 = R_4 + R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_1 + R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 - R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then $\{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ is a basis set for R(T).

Thus dim R(T) = 3. Suppose $T(\alpha) = \bar{0} \Rightarrow (a+b+c+d,\ a-b,\ a+b+d) = (0,0,0)$

Which gives $a+b+c+d = 0,\ a-b = 0,\ a+b+d = 0$.

From this we have $c = 0, b = a$ and $d = -2a$. Thus $[a,b,c,d] = [1,1,0,-2]$

$\therefore$ Rank of the null space N(T) = 1. $\therefore$ Rank of T = dim R(T) = 3

Nullity = dim N(T) = 1. Dim $V_4 = 4$

Thus Rank + Nullity = Dimension, is verified

Ex. 8. *Verify Rank - nullity theorem for the linear transformation* $T: R^3 \to R^3$ *defined by* $T(x,y,z) = (x-y, 2y+z, x+y+z)$

Sol. Given $T: R^3 \to R^3$ defined by $T(x,y,z) = (x-y, 2y+z, x+y+z)$ is a linear transformation.

We know that dimension of $R^3 = 3$ ... (1)

Let $\alpha = (x,y,z) \in R^3$. If $\alpha \in N(T)$ then $T(\alpha) = \bar{0}$

$\Rightarrow T(x,y,z) = \bar{0} \quad \Rightarrow (x-y, 2y+x, x+y+z) = (0,0,0)$

Comparing the components, $x-y=0; 2y+z=0; x+y+z=0$

taking $y = k$ we get $x = k$ and $z = -2k$

$\therefore (x,y,z) = (k,k,-2k) = k(1,1,-2)$

Thus every element in N(T) is generated by the vector $(1,1,-2)$.

Thus dim (N(T)) = 1 ... (2)

Again $T(x,y,z) = (x-y, 2y+z, x+y+z)$

from this $T(1,0,0)=(1,0,1); T(0,1,0)=(-1,2,1)$ and $T(0,0,1)=(0,1,1)$

Let $S=\{(1,0,0),(-1,2,1),(0,1,1)\}$ and let $A=\begin{bmatrix}1 & 0 & 1\\ -1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}$

$$R_2+R_1 \text{ gives } \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 2\\ 0 & 1 & 1\end{bmatrix}$$

$$\frac{R_2}{2} \text{ gives } \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

$$R_3-R_2 \text{ gives } \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Thus the set $\{(1,0,0),(0,1,1)\}$ constitute the basis of $R(T)$ *i.e.* range of T.

Thus, $\dim(R(T))=2$ .... (3)

Substituting in rank - nullity theorem, rank + nullity = dimension

$\Rightarrow 1+2=3$. This verifies the theorem.

Ex. 9. *$T:R^3\to R^2$ be the linear transformation defined as $T(x_1,x_2,x_3)$ $=(x_1-x_2,x_1+x_3)$. Find the Rank (T), and Nullity (T).*

Sol. Let $T(x_1,x_2,x_3)=(0,0)=(x_1-x_2,x_1+x_3)$

$x_1-x_2=0$ and $x_1+x_3=0$; $x_1=x_2$ and $x_1=-x_3$. $\therefore x_1=-x_3$

Let $x_1=k, x_2=k$ and $x_3=-k$

$\therefore (x_1,x_2,x_3)=(k,k,-k)=k(1,1,-1)$ $\therefore (1,1,-1)$ constitutes a basis for $N(T)$

$\dim\{N(T)\}=1=$ Nullity. We know that, $\dim(R^3)=3$

from rank - nullity theorem, we have rank + nullity = dimension $\Rightarrow$ rank (T) $=3-1=2$

Ex. 10. *Find the Kernal of the linear transformation $T:R^2\to R^2$ defined as $T(1,0)=(1,1)$ are $T(0,1)=(-1,2)$*

Sol. We know that $\{(1,0),(0,1)\}$ is a standard basis for R^2

Let $(x,y)\in R^2$. Let $\alpha=(x,y)=x(1,0)+y(0,1)$

Now $\mathrm{T}(x, y) = x\mathrm{T}(1,0) + y\mathrm{T}(0,1) = x\,(1,1) + y\,(-1,2)$

Thus, $\mathrm{T}\,(x, y) = (x - y, x + 2y)$

To find the null space

If $(x, y) \in \mathrm{N(T)}, \mathrm{T}(x, y) = \overline{0} \;\Rightarrow (x - y, x + 2y) = (0,0)$

Comparing components, $x + y = 0, x + 2y = 0$

Solving $x = 0, y = 0$. $\quad\therefore\;$ $(0,0)$ is the only element in null space.

Thus $\mathrm{N(T)} = \{(0,0)\}$

Ex. 11. *If $T : R^3 \to R^3$ be a linear transformation defined by $\mathrm{T}(x, y, z) = (x - y, y - z, z - x)$ then show that T is a linear transformation and find its rank.*

Sol. Given $\mathrm{T} : \mathrm{R}^3 \to \mathrm{R}^3$ is defined as $\mathrm{T}(x, y, z) = (x - y, y - z, z - x)$

Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

$$\mathrm{T}(\alpha + \beta) = \mathrm{T}(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (x_1 + x_2 - \overline{y_1 + y_2}, y_1 + y_2 - \overline{z_1 + z_2}, z_1 + z_2 - \overline{x_1 + x_2})$$

$$= \left[(x_1 - y_1) + (x_2 - y_2), (y_1 - z_1) + (y_2 - z_2), (z_1 - x_1, z_2 - x_2)\right]$$

$$= \left[(x_1 - y_1, y_1 - z_1, x_1 - x_1)\right] + \left[(x_2 - y_2, y_2 - z_2, z_2 - x_2)\right]$$

$$= \mathrm{T}(\alpha) + \mathrm{T}(\beta)$$

$$\mathrm{T}(c\alpha) = \mathrm{T}(cx_1, cy_1, cz_1) = (cx_1 - cy_1, cy_1 - cz_1, cz_1 - cx_1)$$

$$= c\left[(x_1 - y_1), (y_1 - z_1), (z_1 - x_1)\right]$$

$$= c\mathrm{T}(\alpha)$$

$\therefore\;\mathrm{T} : \mathrm{R}^3 \to \mathrm{R}^3$ is a linear transformation.

We have $\mathrm{T}(\alpha) = (x - y, y - z, z - x)$

$$= x(1, 0, -1) + y(-1, 1, 0) + z(0, -1, 1)$$

$\therefore\;\mathrm{R(T)} = \mathrm{L}\{(1, 0, -1), (-1, 1, 0), (0, -1, 1)\}$

Let $(1, 0, -1) = a\,(-1, 1, 0) + b\,(0, -1, 1) = (-a, a - b, b)$

Comparing $a = -1, b = -1$

$\therefore\;(1, 0, -1) = 1\,(-1, 1, 0) - 1\,(-1, 1, 0)$

$\therefore\;$ R(T) is generated by $\{(-1, 1, 0), (0, -1, 1)\}$

Rank of $\mathrm{T} = 2$.

EXERCISE 3 (*c*)

1. Let $T : V_4 \to V_3$ be a linear transformation defined by $T(\alpha_1) = (1,1,1)$; $T(\alpha_2) = (1,-1,1)$; $T(\alpha_3) = (1,0,0)$, $T(\alpha_4) = (1,0,1)$. Then verify that $\rho(T) + \nu(T) = \dim V_4$

2. Describe explicitly the linear transformation $T : R^3 \to R^3$ whose range space is spanned by $\{(1,0,-1), (1,2,2)\}$.

3. Let $T : R^3 \to R^2$ be a linear transformation defined by $T(x,y,z) = (x+y, y+z)$

Find a basis, dimension of each of the range and null space of T.

4. Let $T : R^3 \to R^3$ be the linear mapping defined by

$T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

Find the rank, nullity and find a basis for each of the range and null space of T.

5. Let $T : R^3 \to R^3$ be defined by $T(x,y,z) = (x-y+2z, 2x+y-z, -x-2y)$

Find the null space of T.

6. $T : V_3(R) \to V_2(R)$ is defined by $T(a,b,c) = (a,b)\ \forall\ (a,b,c) \in R^3$

Prove that T is a linear transformation. Find the kernel of T.

7. Let V (F) be an *n* dimensional vector space and let T be a L.T. from V into V such that range and null space of T are identical. Prove that *n* is even.

Vector Space Isomorphism

4.1. Definition. *Let U(F) and V(F) be two vector spaces. The one-one onto transformation* $T : U \to V$ *is called the isomorphism and is denoted by* $U(F) \cong V(F)$.

Now we prove some more properties of vector space isomorphism.

4.2. Theorem. ***Two finite dimensional vector spaces U and V over the same field F are isomorphic if and only if they have the same dimension.***

i.e. $\mathbf{U(F) \cong V(F) \Leftrightarrow \dim U = \dim V}$

Proof. Let U(F) and V(F) be finite dimensional and $U(F) \cong V(F)$. Then there exists an one-one onto transformation $T : U \to V$

To prove that $\dim U = \dim V$.

Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of U. $\therefore \dim U = n$

Let $S' = \{T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)\}$ be the set of T-images of S $\Rightarrow S' \subseteq V$

(*i*) To show that S' is L.I.

Consider the equation $a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_n T(\alpha_n) = \hat{O}$ $\quad a's \in F \quad \hat{O} \in V$

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n] = T(\bar{O})$ (T is L.T. ; $\bar{O} \in U$)

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n = \bar{O}$ ($\because$ T is one-one)

$\Rightarrow a_1 = 0, a_2 = 0, \ldots a_n = 0$ ($\because$ S is L.I.)

$\Rightarrow$ S' is L.I. set.

(*ii*) To show that L (S') = V

Let $\beta \in V$. Since T is onto there exists; $\alpha \in U$ such that $T(\alpha) = \beta$

But $\alpha \in U \Rightarrow \alpha = l.c$ of elements of basis

$$= b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n \qquad (b's \in F)$$

$\Rightarrow T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n)$

$\Rightarrow \beta = b_1 T(\alpha_1) + b_2 T(\alpha_2) \ldots + b_n T(\alpha_n)$ (T is L.T.)

$\Rightarrow$ = *l.c.* of elements of S' $\Rightarrow \beta \in L(S')$

$\therefore$ L(S') = V. As S' is also L..I. ; S' forms a basis of V. $\therefore \dim V = n = \dim U$

Converse. Let dim U = dim V. To prove $U \cong V$.

Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ and $S' = \{\beta_1, \beta_2 \ldots \beta_n\}$ be the bei of U and V respectively so that dim U = n = dim V.

$\therefore\ \alpha \in U \ \Rightarrow\ \alpha = c_1\alpha_1 + c_2\alpha_2 \ldots + c_n\alpha_n$ for some $c's \in F$

Now define $T' : U \to V$ such that $T'(\alpha) = c_1\beta_1 + c_2\beta_2 \ldots + c_n\beta_n = \sum c_i\beta_i$

(*a*) *To show T′ is one - one*

Let $\theta \in U$ such that, $\theta = d_1\alpha_1 + \ldots + d_n\alpha_n \quad d's \in F$

$\Rightarrow\ T'(\theta) = d_1\beta_1 + \ldots + d_n\beta_n$ (by def.)

Now $T'(\alpha) = T'(\theta) \ \Rightarrow\ c_1\beta_1 + c_2\beta_2 \ldots + c_n\beta_n = d_1\beta_1 + d_2\beta_2 \ldots + d_n\beta_n$

$\Rightarrow\ (c_1 - d_1)\beta_1 + \ldots + (c_n - d_n)\beta_n = \hat{O}$ $(\hat{O} \in V)$

$\Rightarrow\ c_1 - d_1 = 0, \ldots c_n - d_n = 0$ (S' is L.I.)

$\Rightarrow\ c_1 = d_1, \ldots c_n = d_n$

$\therefore\ c_1\alpha_1 + c_2\alpha_2 \ldots + c_n\alpha_n = d_1\alpha_1 + d_2\alpha_2 \ldots + d_n\alpha_n$

$\Rightarrow\ \alpha = \theta$ $\quad\therefore$ T' is one-one.

(*b*) *To show T′ is onto*

For $\delta \in V$, we can express $\delta = c_1\beta_1 + c_2\beta_2 \ldots + c_n\beta_n$ for $c's \in F$

If $\gamma = e_1\alpha_1 + e_2\alpha_2 \ldots + e_n\alpha_n$ by definition $T'(\gamma) = \delta$

$\therefore$ For $\delta \in V$ there exists $\gamma \in U$ such that $T'(\gamma) = \delta$ $\quad\therefore$ T' is onto.

(*c*) *To show that T′ is L. T.*

Let $a, b \in F$ and $\alpha, \theta \in U$, then $T'(a\alpha + b\theta) = T'[a\,\Sigma\, c_i\alpha_i + b\,\Sigma\, d_i\alpha_i]$

$= T'[\Sigma\,(a\,c_i + b\,\Sigma\, d_i)\,\alpha_i] = \Sigma\,(a\,c_i + b\,d_i)\,\beta_i$ (by def.)

$= a\,\Sigma\, c_i\beta_i + b\,\Sigma\, d_i\beta_i = a\,T'(\alpha) + b\,T'(\theta)$

$\therefore$ T' is a linear transformation.

Thus $T' : U \to V$ is one-one onto linear transformation.

Hence T' is an isomorphism $\quad\therefore\ U \cong V$

Corollary. ***The image set of a basis set under an isomorphism is a basis set.***

The proof of this statement is the first part of the above proof.

4.3. FUNDAMENTAL THEOREM OF HOMOMORPHISM

Theorem. ***Let U(F) and V(F) be two vector spaces and $T : U \to V$ is a onto linear transformation. N is the null space of T. Then $U/N \cong V$***

Proof. N is the null space of T $\Rightarrow$ $N \subseteq U$

$\therefore$ The quotient space, $U/N = \{N+\alpha : \alpha \in U\}$ is the set of all cosets of N in U.

We know that $(N+\alpha_1)+(N+\alpha_2) = N+(\alpha_1+\alpha_2)\ \forall\ \alpha_i \in U$, $a(N+\alpha) = N+a\alpha\ \forall\ a \in F$

Let a function $f : U/N \to V$ be defined such that $f(N+\alpha) = T(\alpha)\ \forall\ \alpha \in U$

Now to prove f is an homomorphism.

Let $\alpha_1, \alpha_2 \in U$ and $a, b \in F$, then $f[a(N+\alpha_1)+b(N+\alpha_2)] = f[(N+a\alpha_1)+(N+b\alpha_2)]$

$= f[(N+(a\alpha_1+b\alpha_2)] = T(a\alpha_1+b\alpha_2)$

$= a\,T(\alpha_1)+b\,T(\alpha_2) = af(N+\alpha_1)+b(N+\alpha_2)$

$\therefore$ f is a homomorphism.

(*i*) *To prove f is one-one*

$f(N+\alpha_1) = f(N+\alpha_2) \Rightarrow T(\alpha_1) = T(\alpha_2) \Rightarrow T(\alpha_1)-T(\alpha_2) = \hat{0}$

$\Rightarrow T(\alpha_1-\alpha_2) = \hat{0} \Rightarrow \alpha_1-\alpha_2 \in N \Rightarrow N+\alpha_1 = N+\alpha_2$

(*ii*) *To prove f is onto*

Since $T : U \to V$ is onto for every $\beta \in V$, there exists some $\alpha \in U$ such that $T(\alpha) = \beta$

$\therefore$ For this $\alpha \in U$, $(N+\alpha) \in U/N$. Hence $f(N+\alpha) = T(\alpha) = \beta$.

$\therefore$ For all $\beta \in V$, there exists $N+\alpha \in U/N$, so that $f(N+\alpha) = \beta$

$\Rightarrow$ f is onto, thus f is one-one homomorphism

$\therefore$ f is an isomorphism from U/N to V *i.e.* $U/N \cong V$.

Note. In the above theorem, if T is not given as onto then the statement will be as :

Let U(F) and V(F) be two vector spaces and $T : U \to V$ be a linear transformation.

Then $\frac{U}{N} \cong T(U)$ where N is the null space of T.

4.4. Theorem. ***Every n-dimensional vector space V(F) is isomorphic to*** F^n.

Proof. Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be the basis of the n-dimensional vector space V(F)

For $\alpha \in V$ there exist $a_1, a_2 \ldots a_n \in F$ such that $\alpha = a_1\alpha_1+a_2\alpha_2 \ldots + a_n\alpha_n$

Let a mapping $T : U \to F^n$ be defined by $T(\alpha) = (\alpha_1, \alpha_2 \ldots \alpha_n)$

i.e., then the T-image of α is the n-tuple of the coordinates of α

(*i*) *To show that T is one-one*

For $\alpha, \beta \in V$, $a's, b's \in F$

Let $\alpha = a_1\alpha_1+a_2\alpha_2 \ldots + a_n\alpha_n = \Sigma\, a_i\alpha_i$; $\beta = b_1\alpha_1+b_2\alpha_2 \ldots + b_n\alpha_n = \Sigma\, b_i\alpha_i$

Now $T(\alpha) = T(\beta) \Rightarrow (a_1, a_2 \dots a_n) = (b_1, b_2 \dots b_n)$

$\Rightarrow a_1 = b_1, a_2 = b_2, \dots a_n = b_n \Rightarrow \Sigma a_i\alpha_i = \Sigma b_i\alpha_i \Rightarrow \alpha = \beta$. $\therefore$ T is one-one

(*ii*) To show that T is onto

For given any $(c_1, c_2 \dots c_n) \in F^n$ there exists $\gamma \in V$ such that $\gamma = (c_1\alpha_1 + c_2\alpha_2 \dots + c_n\alpha_n)$

$\therefore T(\gamma) = (c_1, c_2 \dots c_n)$ (by def.) $\therefore$ T is onto

(*iii*) *To show that T is linear*

Let $a, b \in F$ and $\alpha, \beta \in V$

$$T(a\alpha + b\beta) = [a\,\Sigma\, a_i\alpha_i + b\,\Sigma\, b_i\alpha_i] = T[\Sigma\,(aa_i + bb_i)\alpha_i]$$

$$= (aa_1 + bb_1, aa_2 + bb_2, \dots aa_n + bb_n)$$

$$= a\,(a_1, a_2 \dots a_n) + b\,(b_1, b_2 \dots b_n) = aT(\alpha) + bT(\beta)$$

$\therefore$ T is a linear transformation.

Thus T is an isomorphism from U to F^n *i.e.,* $U \cong F^n$

SOLVED PROBLEMS

Ex. 1. *Let* $T_k : V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x, y, kz), k \neq 0, k \in R$ *show that* T_k *is an isomorphism. What about* T_o *if* $T_o : V_3(R) \to V_3(R)$ *defined by* $T(x, y, z) = (x, y, 0)$.

Sol. Let $\alpha, \beta \in V_3(R)$ where $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$\therefore T(\alpha) = (x_1, y_1, z_1), T(\beta) = (x_2, y_2, z_2)$

(*i*) For $a, b \in R$, $T_k(a\alpha + b\beta) = T_k[a(x_1, y_1, z_1) + b\,(x_2, y_2, z_2)]$

$= T_k(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = (ax_1 + bx_2, ay_1 + by_2, k(az_1 + bz_2))$

$= (ax_1, ay_1, akz_1) + (bx_2, by_2, bkz_2) = a(x_1, y_1, kz_1) + b(x_2, y_2, kz_2)$

$= aT_k(\alpha) + bT_k(\beta)$. $\therefore T_k$ is a linear transformation

(*ii*) *To prove* T_k *is one one*

Now $T_k(x_1, y_1, z_1) = T_k(x_2, y_2, z_2) \Rightarrow (x_1, y_1, kz_1)(x_2, y_2, kz_2)$

$\Rightarrow kz_1 = kz_2 \Rightarrow z_1 = z_2$ $\because k \neq 0$

$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$ $\therefore$ T is one one.

(*iii*) *To prove T is onto*

Since $k \neq 0$, for every vector $(x_1, y_1, z_1) \in R^3$ there exists a vector $\left(x_1, y_1, \frac{z_1}{k}\right)$ in R^3 such that $T_k\left(x_1, y_1, \frac{z_1}{k}\right) = (x_1, y_1, z_1)$ $\therefore$ T is onto.

Thus T is one-one onto linear transformation from R^3 onto itself and hence t is an isomorphism.

Clearly T is L.T. $T_o(x_1, y_1, z_1) = T_o(x_2, y_2, z_2) \Rightarrow (x_1, y_1, 0) = (x_2, y_2, 0)$

does not $\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$ $\therefore$ T_o is not one one

Also T_o is not onto and hence T_o is not an isomorphism

Ex. 2. *If A and B are subspaces of a vector space V over a field F, then prove that* $\frac{A+B}{B} \cong \frac{A}{A \cap B}$.

Sol. We know that $A+B$ is a subspace of V containing B

$\therefore$ $\frac{A+B}{B}$ is a vector space over F.

Also $A \cap B$ is a subspace of A. $\therefore$ $\frac{A}{A \cap B}$ is a vector space over F.

An element of $\frac{A+B}{B}$ is of the form $\alpha + \beta + B$ where $\alpha \in A, \beta \in B$.

But $\beta + B = B$. $\therefore$ An element of $\frac{A+B}{B}$ is of the form $\alpha + \beta$

Define a map $T : A \to \frac{A+B}{B}$ by $T(\alpha) = \alpha + B \ \forall \alpha \in A$

Clearly T is well defined and onto.

Let $a, b \in F$ and $\alpha_1, \alpha_2 \in A$. $\therefore$ $a\alpha_1 + b\alpha_2 \in A$

Now $T(a\alpha_1 + b\alpha_2) = (a\alpha_1 + b\alpha_2) + B = (a\alpha_1 + B) + (b\alpha_2 + B)$

$= a(\alpha_1 + B) + b(\alpha_2 + B) = aT(\alpha_1) + bT(\alpha_2)$

$\therefore$ T is a linear transformation.

Hence (By Theorem 4.3) $\frac{A}{\text{Ker T}} \cong \frac{A+B}{B}$

Now $\text{Ker T} = \left\{\alpha \in A/T(\alpha) = 0 \text{ of } \frac{A+B}{B}\right\} = \{\alpha \in A/\alpha + B = B\} = A \cap B$.

Thus $\frac{A}{A \cap B} \cong \frac{A+B}{B}$ *i.e.,* $\frac{A+B}{B} \cong \frac{A}{A \cap B}$

4.5. DIRECT SUMS

Definition. *Let $U_1, ..., U_n$ be subspaces of a vector space V (F). Then V is said to be the internal direct sum of $U_1, ..., U_n$ if every $v \in V$ can be written in one and only one way as $v = u_1 + u_2 + ... + u_n$ where $u_i \in U_i \ \forall \ i$.*

Now we introduce another concept, known as the external direct sum as follows:

Let $V_1, ..., V_n$ *be any finite number of vector spaces over a field F. Let* $V = \{(v_1, ..., v_n) / v_i \in V_i\}$. *We take two elements* $(v_1, ..., v_n)$ *and* $(v'_1, ..., v'_n)$ *of V to be equal if and only if for each* $i, v_i = v'_i$. *We define addition on V as* $(v_1, ..., v_n) + (v'_1, ..., v'_n) = (v_1 + v'_1, ..., v_n + v'_n)$. *Finally we define scalar multiplication on V as* $a(v_1, ..., v_n) = (a\, v_1, ..., a\, v_n)$ *where* $a \in F$.

We can easily see that V is a vector space over F for the operations defined above (by checking the vector space axioms). V is called the external direct sum of $V_1, ..., V_n$ *is denoted by writing* $V = V \oplus ..... \oplus V_n$

Theorem. ***If V is the internal direct sum of*** $\mathbf{U_1, ..., U_n}$***, then*** $\mathbf{V \cong U_1 \oplus ..... \oplus U_n}$.

Proof. Let $v \in V$. Since V is the internal direct sum of $U_1, ..., U_n$, v can be written in one and only one way as $v = u_1 + ... + u_n$ ... (1) where $u_i \in U_i \ \forall i$

Define $T : V \to U_1 \oplus ..... \oplus U_n$ by $T(v) = (u_1, ..., u_n)$

T is well defined since $v \in V$ has a unique representation of the form (1).

(*i*) If $(u'_1 + ... + u'_n)$ is any element of $U_1 + ... + U_n$ then

$v' = u'_1 + ... + u'_n \in V$ and is such that $T(v'_i) = (u'_1 ...... u'_n)$ $\therefore$ T is onto.

(*ii*) Let $a, b \in F$ and $v, v' \in V$

Let $v = u'_1 + ... + u'_n$ and $v' = u'_1 + ... u'_n$ where $v'_i, u'_i \in U_i$.

Then $T(v) = T(v')$

$\Rightarrow (u_1, ..., u_n) = (u'_1, ..., u'_n) \Rightarrow u_1 = u'_1, ..., u_n = u'_n$

$\Rightarrow u_1 + ... + u_n = u'_1 + ... + u'_n \Rightarrow v = v'$ $\therefore$ T is 1–1.

(*iii*) Let $a, b \in F$ and $v, v' \in V$ $\therefore\ av + bv' \in V$

$\therefore\ T(av + bv') = T[a(u_1 + ... + u_n) + b(u'_1 + ... + u'_n)] \in V$

$= T[(au_1 + bu'_1) + ... + (au_n + bu'_n)] = (au_1 + bu'_1, ..., au_n + bu'_n)$

$= (au_1, ..., au_n) + (bu'_1, ..., bu'_n) = a(u_1, ..., u_n) + b(u'_1, ..., u'_n)$

$= aT(v) + bT(v')$. $\therefore$ T is a homomorphism.

Thus T is an isomorphism and hence $V \cong U_1 \oplus ..... \oplus U_n$.

Note. *Because of the isomorphism proved above, we shall henceforth merely refer to a direct sum, not qualifying that it be internal or external.*

4. 6. DIRECT SUM OF TWO SUBSPACES.

Definition. *Let W_1 and W_2 be two subspaces of the vector space* $V(F)$. *Then V is said to be the direct sum of the subspaces W_1 and W_2 if every element* $\alpha \in V$ *can be uniquely written as* $\alpha = \alpha_1 + \alpha_2$ *where* $\alpha_1 \in W_1$ *and* $\alpha_2 \in W_2$.

Thus $V = W_1 \oplus W_2$ and every element of V can be uniquely written as sum of an element of W_1 and an element of W_2.

We denote "V is the direct sum of subspaces W_1, W_2" as $V = W_1 \oplus W_2$.

DISJOINT SUBSPACES :

Definition. *Two subspaces* W_1 *and* W_2 *of the vector space* $V(F)$ *are said to be disjoint if their intersection is the zero subspace i.e. if* $W_1 \cap W_2 = \{\bar{O}\}$.

Theorem. ***The necessary and sufficient conditions for a vector space V(F) to be a direct sum of its subspaces W_1 and W_2 are that (i) $V = W_1 + W_2$ and (ii) $W_1 \cap W_2 = \{\bar{O}\}$ i.e. W_1 and W_2 are disjoint.***

Proof. The conditions are necessary.

V is the direct sum of its subspaces $\Rightarrow$ Each element of V can be uniquely written as sum of an element of W_1 and an element of $W_2 \Rightarrow V = W_1 + W_2$.

If possible, let $\bar{O} \neq \alpha \in W_1 \cap W_2$.

$\therefore\ \alpha \in W_1, \alpha \in W_2 \Rightarrow \alpha \in V$ and $\alpha = \bar{O} + \alpha$ where $\bar{O} \in W_1, \alpha \in W_2$

and $\alpha = \alpha + \bar{O}$ where $\alpha \in W_1, \bar{O} \in W_2$

Thus an element in V can be written in at least two different ways as sum of an element of W_1 and an element of W_2. This is a contradiction of the hypothesis. Hence $\bar{O}$ is the only element of V common to both W_1 and W_2 *i.e.* $W_1 \cap W_2 = \{\bar{O}\}$.

Thus the conditions (*i*) and (*ii*) are necessary.

The conditions are sufficient.

Let $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\bar{O}\}$

$V = W_1 + W_2 \Rightarrow$ Each element of *V* can be written as sum of an element of W_1 and an element of $W_2 \Rightarrow \alpha = \alpha_1 + \alpha_2$ where $\alpha \in V$ and $\alpha_1 \in W_1, \alpha_2 \in W_2$.

If posaible, let $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W_1, \beta_2 \in W_2$.

$\therefore\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \Rightarrow \alpha_1 - \beta_1 = \beta_2 - \alpha_2 \in W_1 \cap W_2$

Since $\alpha_1 \in W_1, \beta_1 \in W_1 \Rightarrow \alpha_1 - \beta_1 \in W_1$

$\alpha_2 \in W_2, \beta_2 \in W_2 \Rightarrow \alpha_2 - \beta_2 \in W_2$

Since $W_1 \cap W_2 = \{\bar{o}\}, \alpha_1 - \beta_1 = \bar{o} = \alpha_2 - \beta_2 \Rightarrow \alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$

$\Rightarrow \alpha \in V$ is uniquely written as an element of W_1 and an element of W_2

$\Rightarrow V = W_1 \oplus W_2$

Thus the conditions (*i*) and (*ii*) are sufficient.

SINGULAR AND NON-SINGULAR TRANSFORMATIONS

4. 7. SINGULAR TRANSFORMATION

Definition. *A linear transformation $T: U(F) \to V(F)$ is said to be singular if the null space of T consists of atleast one non-zero vector.*

i.e. If there exists a vector $\alpha \in U$ such that $T(\alpha) = \bar{O}$ for $\alpha \neq \bar{O}$ then T is singular.

4. 8. NON - SINGULAR TRANSFORMATION

Definition. *A linear transformation $T: U(F) \to V(F)$ is said to be non-singular if the null space consists of one zero vector alone.*

i.e., $\alpha \in U$ and $T(\alpha) = \hat{O} \Rightarrow \alpha = \hat{O} \Rightarrow N(T) = \{\bar{O}\}$

4. 9. Theorem. ***Let U(F) and V(F) be two vector spaces and $T: U \to V$ be a linear transformation. Then T is non-singular if f, the set of images of a linearly independent set, is linearly independent.***

Proof. (*i*) Let T be non-singular and let $S = \{\alpha_1, \alpha_2, ... \alpha_n\}$

be a L.I. subset of U. Then its T - images set be $S' = \{T(\alpha_1), T(\alpha_2), ... T(\alpha_n)\}$

Now to prove S' is L.I. For some $a_1, a_2 ... a_n \in F$

Let $a_1 T(\alpha_1) + a_2 T(\alpha_2) + ... + a_n T(\alpha_n) = \hat{O}$ $\quad (\hat{O} \in V)$

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n] = \hat{O}$ $\quad (\because$ T is L.T.)

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n = \hat{O}$ $\quad (\because$ T is non- singular)

$\Rightarrow a_1 = 0, a_2 = 0, ..., a_n = 0$ $\quad (\because$ S is L.I.)

Thus S' is linearly independent.

(*ii*) Let the T - image of any L.I. set be L.I., then to prove T is non-singular.

Let $\alpha \in U$ and $\alpha \neq \bar{O}$. Then the set $B = \{\alpha\}$ is L.I. set and its image set $B' = \{T(\alpha)\}$ given to be L.I.

$\Rightarrow T(\alpha) \neq \hat{O}$ $\quad (\because \{\hat{O}\}$ vectors is L.D.)

Thus $\alpha \neq \bar{O} \Rightarrow T(\alpha) \neq \hat{O}$. $\quad \therefore$ T is non-singular.

4.10. Theorem. *Let U(F) and V(F) be two finite dimensional vector spaces. The linear transformation* $\mathbf{T:U\to V}$ *is an isomorphism iff T is non singular.*

Proof. (*i*) Let $T:U(F)\to V(F)$ be an isomorphism so that T is one- one onto.

To prove T is non-singular

Let $\alpha\in U$, then $T(\alpha)=\hat{O}\Rightarrow T(\alpha)=T(\bar{O})$ $\quad(\because T(\bar{O})=\hat{O}$ in any L.T.)

$\Rightarrow \alpha=\bar{O}$ $\quad$ (T is one-one)

$\therefore$ T is non-singular

(*ii*) Let T be non-singular .

i.e. $\alpha\in U, T(\alpha)=\hat{O}\ \Rightarrow\ \alpha=\hat{O}$; $N(T)=\{\bar{O}\}$, $\therefore$ dim N(T) = 0

For $\alpha_1,\alpha_2\in U, T(\alpha_1)=T(\alpha_2)\Rightarrow T(\alpha_1)-T(\alpha_2)=\hat{O}$ $\quad[\hat{O}\in V]$

$\Rightarrow T(\alpha_1-\alpha_2)=\hat{O}$ $\quad(\because$ T is L.T.)

$\Rightarrow \alpha_1-\alpha_2=\bar{O}\ \Rightarrow\ \alpha_1=\alpha_2$ $\quad(\because$ T is non-singular)

$\therefore$ T is one-one

(*iii*) dim U = dim R(T) + dim N(T) = dim R(T) $\quad(\because$ dim N(T) = 0)

Also $T:U\to V$ is one-one by (*ii*) $\Rightarrow$ V = R(T). $\Rightarrow$ T is an onto mapping

Again dim U = dim V. Hence T is an isomorphism.

SOLVED PROBLEMS

Ex. 1. *A linear mapping* $T:R^3\to R^3$ *is defined by*

$T(x,y,z)=(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta, z)$. *Show that T is non-singular.*

Sol. Let $T(x,y,z)=\hat{O}$

$\Rightarrow (x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta, z)=(0,0,0)\ \Rightarrow\ x\cos\theta-y\sin\theta=0$

$x\sin\theta+y\cos\theta=0$, $\quad z=0$

Squaring and adding $x^2+y^2=0\ \Rightarrow\ x=0, y=0$

Thus $T(x,y,z)=\hat{O}\ \Rightarrow\ (x,y,z)=(0,0,0)$. $\quad\therefore$ T is non singular.

Ex. 2. *Show that a linear transformation* $T:U\to V$ *over the field F is non-singular if and only if T is one-one.*

Sol. (*i*) Let T be non-singular *i.e.*, $\alpha\in U, T(\alpha)=\bar{O}\ \Rightarrow \alpha=\bar{O}$

Now for $\alpha_1,\alpha_2\in U$, $T(\alpha_1)=T(\alpha_2)$

$\Rightarrow T(\alpha_1)-T(\alpha_2)=\hat{O}$ $\quad(\hat{O}\in V)$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{O}$ ($\because$ T is L.T.)

$\Rightarrow \alpha_1 - \alpha_2 = 0$ ($\because$ T is non-singular)

$\Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one.

(*ii*) Let T be one-one

$\therefore$ zero element $\hat{O}$ of V is the T - image of only one element $\in U$.

$\Rightarrow$ null space of U consists of only one element.

$\because$ null space $N(T) \subseteq U$, it must consist of $\bar{O}$.

$\Rightarrow$ null space N(T) consists of only one $\bar{O}$ element.

$\Rightarrow N(T) = \{\bar{O}\}$ $\Rightarrow$ T is non-singular.

Ex. 3. *Let* $T : U \to V$ *be a linear transformation of U (F) into V (F) where U (F) is finite dimensional. Prove that U and the range space of T have the same dimnesion iff T is non-singular.*

Sol. (*i*) Let dim U = dim (Range T) = dim R(T)

$\because$ dim U = dim R(T) + dim N(T) $\Rightarrow$ dim N(T) = 0

$\Rightarrow$ The null space of T is the zero space $\{\bar{O}\}$

$\Rightarrow$ Hence T is non-singular

(*ii*) Let T be non-singular. Then $N(T) = \{\bar{O}$ and nullity T = 0.

As dim U = dim R(T) + dim N(T) = dim R(T) + 0

$\Rightarrow$ dim U = dim R(T).

Ex. 4. *If U and V are finite dimensional vector spaces of the same dimension, then a linear mapping* $T : U \to V$ *is one-one iff it is onto.*

Sol. T is one-one $\Leftrightarrow N(T) = \{\bar{O}\}$ $\Leftrightarrow$ dim N(T) = 0

$\Leftrightarrow$ dim R(T) + dim N(T) = dim U = dim V

$\Leftrightarrow$ R (T) = V $\Leftrightarrow$ T is onto.

Note. In view of the above examples, we can state the following theorem.

Theorem. Let U and V are vector spaces of equal (finite) dimension, and let $T : U \to V$ is a linear transformation. Then the following are equivalent.

(*a*) T is one - to one (*b*) T is onto (*c*) rank (T) = dim (U)

We note that linearity of T is essential in the above theorems.

Ex. 5. Let $T : P_2(R) \to P_3(R)$ be the linear transformation

defined by $T\{f(x)\} = 2f'(x) + \int_0^x 3f(t)\,dt$

Now $T(1) = 0 + \int_0^x 3\,dt = 3x$, $T(x) = 2 + \int_0^x 3(t)\,dt = 2 + \frac{3}{2}x^2$, $T(x^2) = 4x + \int_0^x 3t^2\,dt = 4x + x^3$

$\therefore R(T) = \text{span}(\{T(1), T(x), T(x^2)\}) = \text{span}\left(3x, 2 + \frac{3}{2}x^2, 4x + x^3\right)$

Since $\left\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\right\}$ is linearly independent.

We have rank (T) = 3. Also $\dim P_3(R) = 4$, T is not onto.

From Rank Nullity theorem, nullity (T) + 3 = 3. Thus nullity (T) = 0.

So N (T) = $\{0\}$. We have T is one to one.

Ex. 6. Let $T : F^2 \to F^2$ be the linear transformation defined by $T(a_1, a_2) = (a_1 + a_2, a_1)$.

We can easily see that $N(T) = \{0\}$. Then T is one - to - one. T must be onto.

4. 11. INVERSE FUNCTION

Definition. *Let* $T : U \to V$ *be a one-one onto mapping. Then the mapping* $T^{-1} : U \to V$ *defined by* $T^{-1}(\beta) = \alpha \Leftrightarrow T(\alpha) = \beta$, $\alpha \in U, \beta \in V$ *is called the inverse mapping of T.*

Note: If $T : U \to V$ is one-one onto mapping then the mapping $T^{-1} : V \to U$ is also one-one onto.

4. 12. Theorem. ***Let U (F) and V (F) be two vector spaces and*** $T : U \to V$ ***be a one-one onto linear transformation. Then*** T^{-1} ***is a linear transformation and that T is said to be invertihle.***

Proof. Let $\beta_1, \beta_2 \in V$ and $a, b, \in F$

Since T is one-one onto function there exist unique vectors $\alpha_1, \alpha_2 \in U$ such that

$T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$. Hence by the definition T^{-1},

we have $\alpha_1 = T^{-1}(\beta_1)$ and $\alpha_2 = T^{-1}(\beta_2)$. Also $\alpha_1, \alpha_2 \in U$ and $a, b, \in F \Rightarrow a\,\alpha_1 + b\,\alpha_2 \in U$

$\therefore T(a\,\alpha_1 + b\,\alpha_2) = a\,T(\alpha_1) + b\,T(\alpha_2) = a\,\beta_1 + b\,\beta_2$ ($\because$ T is L.T.)

$\therefore$ By the definition of inverse $T^{-1}(a\,\beta_1 + b\,\beta_2) = a\,\alpha_1 + b\,\alpha_2 = a\,T^{-1}(\beta_1) + b\,T^{-1}(\beta_2)$

$\therefore T^{-1}$ is a linear transformation from V into U.

4. 13. Theorem. ***A linear transformation T on a finite dimensional vector space is invertible if and only if T is non-singular.***

Proof. Let U (F) and V (F) be two vector spaces.

Let $T : U \to V$ be a linear transformation.

(*i*) Let T be non-singular *i.e.* For $\alpha \in U, T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now to prove T is invertible, it is enough to show that T is one - one onto. For this refer proof of theorem 4.8.

(*ii*) Let T be invertible so that T is one - one onto. Now to prove T is non-singular.

For $\alpha \in U, T(\alpha) = \hat{0} = T(\bar{0})$ $(\because$ T is L. T.)

$\Rightarrow \alpha = \bar{0}$ T is one-one. $\therefore$ T is non-singular.

4.14. Theorem. ***Let U(F) and V(F) be two finite dimensional vector spaces such that dim U = dim V. If $T: U \to V$ is a linear transformation then the following are equivalent : -***

(1) *T is invertible* **(2) *T is non-singular*** **(3) *The range of T is V***

(4) *If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is any basis of U, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V.*

(5) *There is some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of U such that $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ basis of V.*

Proof. Here we shall prove a series of implications

viz., (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1)

Now (1) $\Rightarrow$ (2)

If T is invertible then T is one-one and therefore $T(\alpha) = \hat{0} = T(\bar{O}) \Rightarrow \alpha = 0$.

Hence T is nonsingular.

(*ii*) $\Rightarrow$ (*iii*)

Let T be non -singular. Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of U.

Then S is L.I. set. Since T is non-singular the set $S' = \{T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)\}$ is a linearly independent set in V. But dim V = *n*, hence the set S' is a basis of V. Then a vector $\beta \in V$ can be expressed as $\beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_n T(\alpha_n)$ for some a_i's $\in F$

$= T[a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n] = T(\alpha), \ \alpha \in V$ $(\because$ T is L.T.)

$\Rightarrow \beta \in$ range of T.

Thus every vector in V is in the range of T. $\therefore$ R (T) = V

(3) $\Rightarrow$ (4)

Let the range of T be V *i.e.* T is onto. If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of U, the T-images of these vectors $S' = \{T(\alpha_1), T(\alpha_2) \ldots T(\alpha_n)\}$ span the range of T *i.e.* V.

$\therefore$ L(S') = V.

Since S' is a L.I. set of *n* vectors and dim V = *n,* the set S' is a basis of V.

(4) $\Rightarrow$ (5). This is obvious in the above proof. (5) $\Rightarrow$ (1)

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a basis of U such that $S' = \{T(\alpha_1), T(\alpha_2) \dots T(\alpha_n)\}$ is a basis of V.

Since L (S') = R (T) it is clear that R (T) = V. *i.e.* T is onto.

Let $\alpha \in$ null space of T *i.e.* N(T) then $\alpha \in U$

$\therefore\ \alpha = b_1\alpha_1 + b_2\alpha_2 \dots + b_n\alpha_n$ for some $b_i's \in F$.   Hence $T(\alpha) = \hat{\bar{O}}$.

$\Rightarrow\ T[b_1\alpha_1 + b_2\alpha_2 \dots + b_n\alpha_n] = \hat{\bar{O}}$

$\Rightarrow b_1 T(\alpha_1) + b_2 T(\alpha_2) \dots + b_n T(\alpha_n) = \hat{\bar{O}}$ ($\because$ T is L.T.)

$\Rightarrow b_1 = 0, b_2 = 0 \dots b_n = 0 \Rightarrow \alpha = \bar{O}$ ($\because$ S' is L.I.)

$\therefore$ This shows that T is non-singular and one-one. Hence T is invertible.

SOLVED PROBLEMS

Ex. 1. *If* $T : R^3 \to R^3$ *is invertible operator defined by*

$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. *Find* T^{-1}.

Sol. Since T is invertible. $T(x, y, z) = (a, b, c) \Rightarrow T^{-1}(a, b, c) = (x, y, z)$

$\therefore\ (2x, 4x - y, 2x + 3y - z) = (a, b, c) \Rightarrow 2x = a, 4x - y = b, 2x + 3y - z = c$

Solving $x = \frac{a}{2}, y = 2a - b, z = 7a - 3b - c$.   Hence $T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a - b, 7a - 3b - c\right)$

Ex. 2. *The set* $\{e_1, e_2, e_3\}$ *is the standard basis of* $V_3(R)$. $T : V_3(R) \to V_3(R)$ *is a linear operator defined by* $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$. *Show that T is non-singular and find its inverse.*

Sol. Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$

Now $T : R^3 \to R^3$ is defined by $T(e_1) = e_1 + e_2 \Rightarrow T(1, 0, 0) = (1, 1, 0)$;

$T(e_2) = e_2 + e_3 \Rightarrow T(0, 1, 0) = (0, 1, 1)$; $T(e_3) = e_1 + e_2 + e_3 \Rightarrow T(0, 0, 1) = (1, 1, 1)$

Let $\alpha = (x, y, z) \in V_3(R)$

$\therefore\ \alpha = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$\therefore\ T(\alpha) = x\,T(1, 0, 0) + y\,T(0, 1, 0) + z\,T(0, 0, 1) = x(1, 1, 0) + y(0, 1, 1) + z(1, 1, 1)$

$\therefore$ The transformation is given by $T(x, y, z) = (x + z, x + y + z, y + z)$

Now If $T(x, y, z) = \hat{\bar{O}}$, then $(x + z, x + y + z, y + z) = (0, 0, 0)$

$\Rightarrow x + z = 0, x + y + z = 0, y + z = 0 \Rightarrow x = y = z = 0$.

$\therefore\ T(\alpha) = \hat{\bar{O}} \Rightarrow \alpha = \bar{O}$.   Hence T is non-singular and therfore T^{-1} exists.

Let $T(x, y, z) = (a, b, c)$ $\Rightarrow (x + z, x + y + z, y + z) = (a, b, c)$

$\Rightarrow\ x+z=a$ Solving $x=b-c$

$x+y+z=b$, $y+z=c$, $y=b-a$, $z=a-b+c$

$\therefore\ T^{-1}(a,b,c)=(x,y,z)=(b-c,b-a,a-b+c)$

Ex. 3. *A linear transformation T is defined on* $V_3(C)$ *by* $T(a,b)=(a\alpha+b\beta, a\gamma+b\delta)$ *where* $\alpha,\beta,\gamma,\delta$ *are fixed elements of C. Prove that T is invertible if and only if* $\alpha\delta-\beta\gamma\neq 0$.

Sol. $T:V_2(C)\to V_2(C)$ is a L.T. and $\dim V_2(C)=2$

T is invertible if and only if T is one-one onto.

T is onto iff the range of T is the whole set V_2 *i.e.* $R(T)=V_2(C)$

Now $S=\{(1,0),(0,1)\}$ is a basis of $V_2 \Rightarrow L(S)=V_2$.

$T(1,0)=(1.\alpha+0.\beta, 1.\gamma+0.\delta)=(\alpha,\gamma)$; $T(0,1)=(0.\alpha+1.\beta, 0.\gamma+1.\delta)=(\beta,\delta)$

$\therefore$ T is invertible iff $S'=\{(\alpha,\gamma),(\beta,\delta)\}$ span $V_2(C)$.

As $\dim V_2(C)=2$, the set S' containing two vectors will span V_2 if S' is L.I.

For $x,y\in C$, $x(\alpha,\gamma)+y(\beta,\delta)=(0,0)$

$$\Rightarrow (x\alpha+y\beta, x\gamma+y\delta)=(0,0) \Rightarrow x\alpha+y\beta=0,\ x\gamma+y\delta=0$$

These equations will have the only solution $x=0, y=0$ iff

$$\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}\neq 0 \ \text{i.e.,}\ \ \alpha\delta-\beta\gamma\neq 0$$

$\therefore$ T is invertible $\Leftrightarrow\ \alpha\delta-\beta\gamma\neq 0$

Ex. 4. *Show that the linear operator T defined on* R^3 *by* $T(x,y,z)=(x+z,x-z,y)$ *is invertible and hence find* T^{-1} ***(O. U. 2011)***

Sol. Let $(x,y,z)\in N(T)\Rightarrow T(x,y,z)=\bar{0}\ \Rightarrow (x+z,x-z,y)=(0,0,0)$

$\Rightarrow x+z=0, x-z=0, y=0$. Solving $x=0,y=0,z=0\Rightarrow N(T)=\{\bar{0}\}$

Hence T is non-singular and so it is invertible.

(2) Let $T(x,y,z)=(a,b,c)\Rightarrow T^{-1}(a,b,c)=(x,y,z)$

$\therefore\ (x+z,x-z,y)=(a,b,c)\ \Rightarrow x+z=a; x-z=b; y=c$

from this we get, $2x=a+b\Rightarrow x=\dfrac{a+b}{2}$. $z=a-\left(\dfrac{a+b}{2}\right)=\dfrac{a-b}{2}$

$T^{-1}(a,b,c)=\left(\dfrac{a+b}{2},c,\dfrac{a-b}{2}\right)$

EXERCISE 4

1. Show that each of the following linear operators T on R^3 is invertible and find T^{-1}.

(*a*) $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$ (*b*) $T(a, b, c) = (a - 3b - 2c, b - 4c, c)$

(*c*) $T(a, b, c) = (3a, a - b, 2a + b + c$

(*d*) $T(x, y, z) = (x + y + z, y + z, z)$ (*e*) $T(a, b, c) = (a - 2b - c, b - c, a)$

2. The set $\{e_1, e_2, e_3\}$ is the standard basis set of $V_3(R)$. The linear operator $T : R^3 \to R^3$ is defined below. Show that T is invertible and find T^{-1}

(*i*) $T(e_1) = e_1 + e_2$, $T(e_2) = e_1 - e_2 + e_3$, $T(e_3) = 3e_1 + 4e_3$

(*ii*) $T(e_1) = e_1 - e_2$, $T(e_2) = e_2$, $T(e_3) = e_1 + e_2 - 7e_3$

(*iii*) $T(e_1) = e_1 - e_2 + e_3$, $T(e_2) = 3e_1 - 5e_3$, $T(e_3) = 3e_1 - 2e_3$

3. Let $T : U \to V$ be a non. singular linear transformation then prove that $(T^{-1})^{-1} = T$.

4. 15. Theorem. *The necessary and sufficient condition for a linear operator T on a vector space V (F) to be invertible is that there exists a linear transformation H on V such that TH = HT = I.*

Proof. Given $T : V(F) \to V(F)$ is a L.T.

(*i*) Let T be invertible. Hence T^{-1} exists and is one-one onto.

Let $\alpha \in V$ and $T(\alpha) = \beta$ so that $\alpha = T^{-1}(\beta)$

$$(TT^{-1})(\beta) = T[T^{-1}(\beta)] = T(\alpha) = \beta = I(\beta) \Rightarrow TT^{-1} = I$$

Again $(T^{-1}T)(\alpha) = T^{-1}[T(\alpha)] = T^{-1}(\beta) = \alpha = I(\alpha) \Rightarrow T^{-1}T = I$

$\therefore T^{-1}T = T^{-1}T = I$

Taking $H = T^{-1}$ we get TH = HT = I such that $H : V \to V$ is one - one onto L.T.

(*ii*) Let there exist a linear operator $H : V \to V$ such that TH = HT = I.

To prove T is invertible.

For $\alpha_1, \alpha_2 \in V$ $T(\alpha_1) = T(\alpha_2)$

$\Rightarrow H[T(\alpha_1)] = H[T(\alpha_2)] \Rightarrow (HT)(\alpha_1) = (HT)(\alpha_2)$

$\Rightarrow I(\alpha_1) = I(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$ $(\because HT = I = TH)$

$\therefore$ T is one-one

For $\beta \in V$ there exists $\alpha \in V$ such that $H(\beta) = \alpha$ $(\because H : V \to V)$

$\Rightarrow TH(\beta) = T(\alpha) \Rightarrow I(\beta) = T(\alpha) \Rightarrow \beta = T(\alpha)$

$\therefore$ For any $\alpha \in V$ there exists $\beta \in V$ such that $T(\alpha) = \beta$, therefore T is onto.

Thus T is one-one onto and hence T is invertible.

4.16. UNIQUENESS OF INVERSE

Theorem. ***Let T be an invertible linear operator on a vector space V(F). Then T possesses unique inverse.***

Proof. Let Hand G be two inverses of T. Then

HT = TH = I and GT = TG = I

Now G (TH) = GI = G. $\therefore$ again G (TH) = (GT) H = IH = H

Since product of linear transformation,s is associative

$\therefore$ G = G (TH) = (GT) H = H $\Rightarrow$ G = H. Hence the inverse of T is unique.

4.17. Theorem. ***Let T be an invertible operator oa a vector space V (F). Then show that (i) a T is invertible linear operator. where $a \neq 0$ and $a \in F$.***

(ii) $(a\,T)^{-1} = \frac{1}{a} T^{-1}$ ***(iii)*** T^{-1} ***is invertible and*** $(T^{-1})^{-1} = T$

Proof. T is an invertible operator T is one-one onto and $TT^{-1} = T^{-1}T = I$

For $a \neq 0$ and $a \in F \Rightarrow a^{-1} = \frac{1}{a} \in F$

T is a linear operator $\Rightarrow$ (a T) is a linear operator.

Also $(aT)\,(a^{-1}T^{-1}) = a\,[T(a^{-1}T^{-1})] = a\,[a^{-1}(TT^{-1})] = (aa^{-1})\,(TT^{-1}) = 1\,.\,I = I$

For $\alpha_1, \alpha_2 \in F$ $(a\,T)\,(\alpha_1) = (a\,T)\,(\alpha_2)$

$\Rightarrow a\,T(\alpha_1) = a\,T(\alpha_2) \Rightarrow T(a\alpha_1) = T(a\alpha_2)$

$\Rightarrow a\,\alpha_1 = a\,\alpha_2 \Rightarrow \alpha_1 = \alpha_2$ ($\because$ T is is one-one)

$\therefore$ a T is one-one. Also T is onto $\Rightarrow$ a T is onto.

Hence (a T) is invertible operator. For $\alpha, \beta \in V$ and $c, d \in F$

$(aT)\,(c\,\alpha + d\,\beta) = aT\,(c\,\alpha + d\,\beta) = T\,(ac\,\alpha + ad\,\beta)$

$= acT\,(\alpha) + adT\,(\beta) = c\,(aT)\,(\alpha) + d\,(aT)\,(\beta)$

Hence (a T) is a linear transformation.

Thus (a T) is a one-one onto linear operator on the vector space V (F).

Hence (a T) is invertible.

(ii) Since $TT^{-1} = T^{-1}T = I$

we have $(aa^{-1})\,(TT^{-1}) = (a^{-1}a)\,(T^{-1}T) = I$ ($\because aa^{-1} = a^{-1}a = 1$)

$\Rightarrow (aT)\,(a^{-1}T^{-1}) = (a^{-1}T^{-1})\,(aT) = I$ $\Rightarrow (aT)^{-1} = a^{-1}T^{-1} = \frac{1}{a} T^{-1}$

(*iii*) Again $TT^{-1} = T^{-1}T = I$

$\Rightarrow$ that the inverse of T^{-1} is T. *i.e.* $(T^{-1})^{-1} = T$

4.18. Theorem. ***If T and U are invertible linear operators on a vector space V(F) then show that TH is invertible and $(TH)^{-1} = H^{-1}T^{-1}$***

Proof. Given T and H are invertible $\Rightarrow$ T^{-1}, H^{-1} exist and $TT^{-1} = T^{-1}T = I$

$$HH^{-1} = H^{-1}H = I$$

Now $(H^{-1}T^{-1})(TH) = H^{-1}(T^{-1}T)H = H^{-1}IH = H^{-1}H = I$

Again $(TH)(H^{-1}T^{-1}) = T(HH^{-1})T^{-1} = TIT^{-1} = TT^{-1} = I$

Thus $(TH)(H^{-1}T^{-1}) = (H^{-1}T^{-1})(TH) = I \Rightarrow (TH)^{-1} = H^{-1}T^{-1}$

$\therefore$ T H is invertible and $(TH)^{-1} = H^{-1}T^{-1}$

SOLVED PROBLEMS

Ex. 1. *If A. B, C are linear transformations on a vector space V (F) such that* $AB = CA = I$ *then show that A is invertible and* $A^{-1} = B = C$

Sol. (*i*) *To prove* A^{-1} *exists*

For $\alpha_1, \alpha_2 \in V$ $A(\alpha_1) = A(\alpha_2) \Rightarrow CA(\alpha_1) = CA(\alpha_2)$

$\Rightarrow I(\alpha_1) = I(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$ $\therefore$ A is one one.

Let $\beta \in V$. Since $B: V \to V$, then $B(\beta) \in V$

For some $\alpha \in V$ let $B(\beta) = \alpha$ $\Rightarrow A[B(\beta)] = A(\alpha) \Rightarrow (AB)(\beta) = A(\alpha)$

$\Rightarrow I(\beta) = A(\alpha) \Rightarrow \beta = A(\alpha)$

Thus for some $\beta \in V$ there exists $\alpha \in V$ such that $A(\alpha) = \beta$. $\therefore$ A is onto

Thus A is one one onto $\Rightarrow A^{-1}$ exists.

(*ii*) *To prove* $A^{-1} = B = C$

Now $AB = I$. $\Rightarrow A^{-1}(AB) = A^{-1}I \Rightarrow (A^{-1}A)B = A^{-1} \Rightarrow IB = A^{-1} \Rightarrow B = A^{-1}$

Again $CA = I \Rightarrow (CA)A^{-1} = IA^{-1} \Rightarrow C(AA^{-1}) = A^{-1} \Rightarrow CI = A^{-1} \Rightarrow C = A^{-1}$

Hence $A^{-1} = B = C$.

Ex. 2. *If T is a linear operator on a vector space V (F) such that* $T^2 - T + I = 0$, *then show that T is invertible.*

Sol. If $T^2 - T + I = 0$, then $T - T^2 = I$

(*i*) *To prove T is one one*

Let $\alpha_1, \alpha_2 \in V$, then $T(\alpha_1) = T(\alpha_2)$ .... (1)

$\Rightarrow$ $T[T(\alpha_1)] = T[T(\alpha_2)] \Rightarrow T^2(\alpha_1) = T^2(\alpha_2)$ .... (2)

Subtracting (2) from (1) $\Rightarrow$ $T(\alpha_1) - T^2(\alpha_1) = T(\alpha_2) - T^2(\alpha_2)$

$\Rightarrow (T - T^2)(\alpha_1) = (T - T^2)(\alpha_2) \quad \Rightarrow I(\alpha_1) = I(\alpha_2) \quad \Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one one

(*ii*) *To prove T is onto*

For $\alpha \in V$, $T(\alpha) \in V$ and $\alpha - T(\alpha) \in V$

Now $T[\alpha - T(\alpha)] = T(\alpha) - T^2(\alpha) = (T - T^2)(\alpha) = I(\alpha) = \alpha$

Thus for $\alpha \in V$ there $\alpha - T(\alpha) \in V$ such that $T[\alpha - T(\alpha)] = \alpha$

$\therefore$ T is onto. Thus T is one one onto and hence invertible.

Ex. 3. *If A and B are linear transformations on a finite dimensional vector space V (F) and if A B = I, then show that A and B are invertible.*

Sol. (*i*) *To prove B is invertible*

Let $\alpha_1, \alpha_2 \in V(F)$, then $B(\alpha_1) = B(\alpha_2)$

$\Rightarrow A[B(\alpha_1)] = A[B(\alpha_2)] \Rightarrow (AB)(\alpha_1) = (AB)(\alpha_2)$

$\Rightarrow I(\alpha_1) = I(\alpha_2) \quad \Rightarrow \alpha_1 = \alpha_2$ $\therefore$ B is one one.

Since B is a. L.T on a finite dimensional veotor space such that B is one-one, B will be onto.

Now B is one-one onto $\Rightarrow$ B is invertible $\Rightarrow B^{-1}$ exists

(*ii*) *To prove A is invertible*

Now AB = I $\Rightarrow (AB)B^{-1} = IB^{-1} \Rightarrow A(BB^{-1}) = B^{-1}$

$\Rightarrow AI = B^{-1} \Rightarrow A = B^{-1}$

Hence $B^{-1}B = BB^{-1} = I$ $\Rightarrow$ AB = BA = I

$\Rightarrow$ A is invertible and $A^{-1} = B$.

Ex. 4. *Let* $A = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ *and* $B = \{\beta_1, \beta_2 \dots \beta_n\}$ *be two ordered bases of a finite dimensional vector space V (F). Prove that there exists an invertible linear operator T on V such that* $T(\alpha_i) = \beta_i$.

Sol. Already we have proved in a previous theorem that there exists a linear transformation T on V such that $T(\alpha_i) = \beta_i$ for *i* = 1, 2, ...*n*.

Now we prove that T is invertible. This is equivalent to proving T is non-singular.

For $\alpha \in V \Rightarrow \alpha = l.c$ of elements of A.

$= a_1\alpha_1 + a_2\alpha_2 \dots + a_n\alpha_n$ for $a's \in F$ $\therefore T(\alpha) = \bar{0}$

$\Rightarrow T(a_1\alpha_1 + a_2\alpha_2 \dots + a_n\alpha_n) = \bar{0} \quad \Rightarrow a_1T(\alpha_1) + a_2T(\alpha_2) \dots + a_nT(\alpha_n) = \bar{0}$

$\Rightarrow a_1\beta_1 + a_2\beta_2 \dots + a_n\beta_n = \bar{0}$ $(\because T(\alpha_i) = \beta_i)$

$\Rightarrow a_1 = 0, a_2 = 0 \dots a_n = \bar{0} \quad \Rightarrow \alpha = \bar{0}$ ($\because$ B is L.I.)

Thus $T(\alpha) = \bar{0} \quad \Rightarrow \alpha = \bar{0}$. $\therefore$ T is non-singular and hence invertible.

Matrix of Linear Transformation

5.1. Let U(F) and V(F) be two vector spaces so that dim U = n and dim V = m.Let $T: U \to V$ be a linear transformation.

Let $B_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the ordered basis of U. and $B_2 = \{\beta_1, \beta_2, ..., \beta_m\}$ be the ordered basis of V.

For every $\alpha \in U \Rightarrow T(\alpha) \in V$ and $T(\alpha)$ can be expressed as a $lc.$ of elements of the basis B_2. Let there exist $a's \in F$ such that

$$\left.\begin{array}{ll} T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + ... + a_{m1}\beta_m, & T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + ... + a_{m2}\beta_m \\ ... \quad ... \quad .... \quad ..., & T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + ... + a_{mj}\beta_m \\ ... \quad ... \quad ... \quad ..., & T(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + ... + a_{mn}\beta_m \end{array}\right\} \quad .. (A)$$

Writing the coordinates of $T(\alpha_1), T(\alpha_2) ... T(\alpha_n)$ successively as columns of a matrix

we get $$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ ... & ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mj} & ... & a_{mn} \end{bmatrix}_{m \times n}$$

This matrix represented as $[a_{ij}]_{m \times n}$ is called the matrix of the linear transformation T $w.r.$ to the bases B_1 and B_2. Symbolically $[T : B_1, B_2]$ or $[T] = [a_{ij}]_{m \times n}$

Thus the matrix $[a_{ij}]_{m \times n}$ completely determines the linear transformation through the relations given in (A). Hence the matrix $[a_{ij}]_{m \times n}$ represents the transformation T.

Note. Let $T: V \to V$ be the linear operator so that dim V = n.

If $B_1 = B_2 = B$ (say) then the above said matrix is called the matrix of T relative to the ordered basis B. It is denoted as [T ; B] = $[T]_B = [a_{ij}]_{n \times n}$.

5.2. Corollary. ***Let V (F) be an n dimensional vector space for which B is a basis. Show that***

(i) $[\mathrm{I}:\mathrm{B}] = \mathrm{I}_{n\times n}$ ***(ii)*** $[\mathrm{O}:\mathrm{B}] = \mathrm{O}_{n\times n}$ ***where I and 0 are the identity and zero transformations on V, respectively.***

Proof. Let $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$

(*i*) $I(\alpha_1) = 1.\alpha_1 + 0.\alpha_2 \ldots + 0.\alpha_n$, $I(\alpha_2) = 0.\alpha_1 + 1.\alpha_2 \ldots + 0.\alpha_n$

... ... , $I(\alpha_n) = 0.\alpha_1 + 0.\alpha_2 \ldots + 1.\alpha_n$

$\therefore$ The matrix is the unit matrix

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{n\times n} = [\delta_{ij}]_{n\times n}; \text{ Kronecker delta } \begin{bmatrix} \delta_{ij} = 1, i = j \\ \quad = 0, i \neq j \end{bmatrix}$$

(*ii*) Now $O(\alpha_1) = \bar{O} = 0.\alpha_1 + 0.\alpha_2 \ldots + 0.\alpha_n$, $O(\alpha_2) = \bar{O} = 0.\alpha_1 + 0.\alpha_2 \ldots + 0.\alpha_n$

... ... , $O(\alpha_n) = \bar{O} = 0.\alpha_1 + 0.\alpha_2 \ldots + 0.\alpha_n$

$$\therefore \text{ The matrix of zero transformation } [O:B] = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & & \ldots & \\ 0 & 0 & .. & 0 \end{bmatrix} = O_{n\times n}$$

SOLVED PROBLEMS

Ex. 1. *Let* $T: V_2 \to V_3$ *be defined by* $T(x, y) = (x+y, 2x-y, 7y)$ *Find* $[T: B_1, B_2]$ *where* B_1 *and* B_2 *are the standard bases of* V_2 *and* V_3.

Sol. B_1 is the standard basis of V_2 and B_2 that of V_3

$\therefore B_1 = \{(1,0), (0,1)\}$ $\quad B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\therefore T(1,0) = (1,2,0) = 1(1,0,0) + 2(0,1,0) + 0(0,0,1)$

$T(0,1) = (1,-1,7) = 1(1,0,0) - 1(0,1,0) + 7(0,0,1)$

$$\therefore \text{ The matrix of T relative to } B_1 \text{ and } B_2 \text{ is } [T: B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

Ex. 2. *Let* $T: R^3 \to R^2$ *be the linear transformations defined by* $T(x, y, z) = (3x+2y-4x, x-5y+3x)$. *Find the matrix of T relative to the bases* $B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$, $B_2 = \{(1,3), (2,5)\}$

Sol. Let $(a,b) \in R^2$ and let $(a,b) = p\,(1,3) + q\,(2,5) = (p+2q, 3p+5q)$

$\Rightarrow\ p+2q = a\,,\ 3p+5q = b\,.$ Solving $p = -5a+2b\,.\ q = 3a-b$

$\therefore\ (a,b) = (-5a+2b)\,(1,3) + (3a-b)\,(2,5)$

Now T $(1,1,1) = (1,-1) = -7\,(1,3) + 4\,(2,5)$, T $(1,1,0) = (5,-4) = -33\,(1,3) + 19\,(2,5)$,

T $(1,0,0) = (3,1) = -13\,(1,3) + 8\,(2,5)$

$\therefore$ The matrix of L. T. relative to B_1 and B_2. $[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

Ex. 3. *If the matrix of a linear transformation T on* $V_3(R)$ *w.r. to the standard basis is* $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ *what is the matrix of T w.r.to the basis.* $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$

Sol. (*i*) Let the standard basis of $V_3(R)$ be $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$. $\therefore$ Given $[T]_B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

$\therefore\ T(\alpha_1) = 0\,\alpha_1 + 1, \alpha_2 + (-1)\,\alpha_3 = (0,1,-1)$

$T(\alpha_2) = 1.\,\alpha_1 + 0\,.\,\alpha_2 - 1\,.\,\alpha_3 = (1,0,-1)$, $T(\alpha_3) = 1.\,\alpha_1 - 1\,.\,\alpha_2 + 0\,.\,\alpha_3 = (1,-1,0)$

Let $(a,b,c) \in V_3(R)$ then $(a,b,c) = a\,(1,0,0) + b\,(0,1,0) + c\,(0,0,1) = a\,\alpha_1 + b\,\alpha_2 + c\,\alpha_3$

$\therefore\ T(a,b,c) = a\,T(\alpha_1) + b\,T(\alpha_2) + c\,T(\alpha_3) = a\,(0,1,-1) + b\,(1,0,-1) + c\,(1,-1,0)$

$= (b+c), a-c, -a-b)$ which is the required transformation.

(*ii*) Let $B_2 = \{\beta_1, \beta_2, \beta_3\}$ where $\beta_1 = (0,1,-1), \beta_2 = (1,-1,1), \beta_3 = (-1,1,0)$

Using the transformation $T(a,b,c) = (b+c, a-c, -a-b)$ we have

$T(\beta_1) = T(0,1,-1) = (0,1,-1)\,;\ T(\beta_2) = T(1,-1,1) = (0,0,0),\ T(\beta_3) = T(-1,1,0) = (1,-1,0)$

Now Let $(a,b,c) = x\beta_1 + y\beta_2 + z\beta_3 = x\,(0,1,-1) + y\,(1,-1,1) + z\,(-1,1,0)$

$= (y-z, x-y+z, -x+y)$

$y - z = a, x-y+z = b, -x+y = c \quad \Rightarrow x = a+b, y = a+b+c, z = b+c$

$\therefore\ (a,b,c) = (a+b)\beta_1 + (a+b+c)\beta_2 + (b+c)\beta_3$

$\therefore\ T(\beta_1) = (0,1,-1) = 1\,.\,\beta_1 + 0\,.\,\beta_2 + 0\,.\,\beta_3$

$\therefore\ T(\beta_2) = (0,0,0) = 0\,.\,\beta_1 + 0\,.\,\beta_2 + 0\,.\,\beta_3,\qquad \therefore\ T(\beta_3) = (1,-1,0) = 0\,.\,\beta_1 + 0\,.\,\beta_2 - 1\,.\,\beta_3$

$$\therefore\ [T\,;B_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Ex. 4. *Let* $D : P_3 \to P_2$ *be the polynomial differential transformation* $D(p) = \dfrac{dp}{dx}$.
Find the matrix of D relative to the standard bases. $B_1 = \{1, x, x^2, x^3\}$ *and* $B_2 = \{1, x, x^2\}$

Sol. $D(1) = 0 = 0\,.\,1 + 0\,.\,x + 0\,.\,x^2,\qquad D(x) = 1 = 1\,.\,1 + 0\,.\,x + 0\,.\,x^2$

$D(x^2) = 2x = 0\,.\,1 + 2\,.\,x + 0\,.\,x^2,\quad D(x^3) = 3x^2 = 0\,.\,1 + 0\,.\,x + 3\,.\,x^2$

$\therefore$ The matrix of D relative to B_1 and B_2 is $[T : B_1\,; B_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Ex. 5. $T : R^2 \to R^2$ *such that* $T(1,1) = (2,-3), T(1,-1) = (4,7)$. *Find the matrix of T relative to the basis* $S = \{(1,0),(0,1)\}$

Sol. $T : R^2 \to R^2$ is defined as $T(1,1) = (2,-3)\,;\ T(1,-1) = (4,7)$

Let, $S = \{(1,0),(0,1)\}$ is the ordered basis for R^2. Let, $S' = \{(1,1),(1,-1)\}$

We know that $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)\,;\ (0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

given, T is a linear transformation,

$$T(1,0) = T\left[\frac{1}{2}(1,1) + \frac{1}{2}(1,-1)\right]$$

$$= \frac{1}{2}T(1,1) - \frac{1}{2}T(1,-1) = \frac{1}{2}(2,-3) - \frac{1}{2}(4,7) = (3,2)$$

$$= 3\,(1,0) + 2\,(0,1)$$

$$T(0,1) = T\left[\frac{1}{2}(1,1) - \frac{1}{2}(1,-1)\right]$$

$$= \frac{1}{2}T(1,1) - \frac{1}{2}T(1,-1) = \frac{1}{2}(2,-3) - \frac{1}{2}(4,7) = (-1,-5) = -1\,(1,0) - 5\,(0,1)$$

Matrix of T is $\begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}$

5.3. Theorem. *Let U(F) and V(F) be two vector spaces such that dim U = n and dim V = m. Then corresponding to every matrix* $[a_{ij}]_{m\times n}$ *of mn scalars belonging to F there corresponds a unique linear transformation* $T : U \to V$ *such that* $[T : B ; B'] = [a_{ij}]_{m\times n}$ *where B and B′ are the ordered bases of U and V respectively.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

$$\text{Given } [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots & & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$$

Now $\alpha_j \in B \Rightarrow T(\alpha_j) \in V$. Let $T : U \to V$ be defined by

$$T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \ldots + a_{ij}\beta_i + \ldots + a_{mj}\beta_m = \sum_{i=1}^{m} a_{ij}\beta_i$$

Let $\alpha \in U$ be any vector then $\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

$\therefore\ T(\alpha) = b_1T(\alpha_1) + b_2T(\alpha_2) + \ldots + b_nT(\alpha_n)$

$$= b_1\sum_{i=1}^{m} a_{i1}\beta_i + b_2\sum_{i=1}^{m} a_{i2}\beta_i \ldots + b_n\sum_{i=1}^{m} a_{in}\beta_i = \sum_{j=1}^{n} b_j\left(\sum_{i=1}^{m} a_{ij}\beta_i\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}b_j\right)\beta_i$$

$$= \sum_{i=1}^{m} c_i\beta_i \quad \text{where } c_i = \sum_{i=1}^{n} a_{ij}b_j = c_1\beta_1 + c_2\beta_2 + \ldots + c_m\beta_m$$

$\Rightarrow$ $T(\alpha)$ is expressible uniquely as a *l.c.* of the elements of B′ ($\because$ β'*s* are unique)

$\Rightarrow$ T is unique.

Thus for all $\sum a_{ij}\beta_i \in V$ there is a L.T. from U to V such that

$$T(\alpha_j) = \sum_{i=1}^{m} a_{ij}\beta_i \qquad (j = 1, 2 \ldots n)$$

Thus corresponding to the matrix $[a_{ij}]_{m\times n}$ there corresponds a transformation $T : U \to V$.

5.4. Theorem. 1. *Let U(F) and V(F) be two linear transformations so that* $\dim U = n$ *and* $\dim V = m$. *Let B and B′ be the ordered bases for U and V respectively. Let* $T : U \to V$ *then for all* $\alpha \in U$, $[T : B\,;B'][\alpha]_B = [T(\alpha)]_{B'}$

where $[\alpha]_B$ *is the coordinate matrix of* α *with respect to the basis B and* $[T(\alpha)]_{B'}$ *is the coordinate matrix of* $T(\alpha) \in V$ *with respect to B′.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

Let $A = [a_{ij}]$ be the matrix of T relative to B and B′. Then $[T : B, B'], A = [a_{ij}]_{m \times n}$

$$\Rightarrow \text{For } \alpha_j \in B,\ T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 \ldots + a_{ij}\beta_i + \ldots + a_{mj}\beta_m = \sum_{i=1}^{m} a_{1j}\beta_i$$

For any $\alpha \in U$, $\alpha = b_1\alpha_1 + b_2\alpha_2 \ldots + b_n\alpha_n \quad \forall\ b's \in F$

$$\therefore \text{ The coordinate matrix of } \alpha = [\alpha]_B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\therefore\ T(\alpha) = b_1T(\alpha_1) + b_2T(\alpha_2) \ldots + b_nT(\alpha_n) = \sum_{j=1}^{m} b_j T(\alpha_j)$$

$$= \sum_{j=1}^{n} b_j \left(\sum_{j=1}^{m} a_{ij}\beta_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_j a_{ij} \right) \beta_i$$

$$= \sum_{i=1}^{m} c_i\beta_i \quad \text{where } c_i = \sum_{j=1}^{n} b_j a_{ij} \quad \Rightarrow \quad T(\alpha) = c_1\beta_1 + c_2\beta_2 + \ldots + c_m\beta_m$$

$\therefore$ The coordinate matrix of $T(\alpha)$ *w.r.* to the basis of B′ is

$$[T(\alpha)]B' = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \ldots \\ c_m \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \ldots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \ldots + a_{2n}b_n \\ \ldots \quad \ldots \quad \ldots \\ a_{m1}b_1 + a_{m2}b_2 + \ldots + a_{mn}b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{12} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix} = A\,[\alpha]_B$$

Thus $[T(\alpha)]_{B'} = [T : B, B'][T(\alpha)]_B$

Theorem. 2. *If V is an n dimensional vector space over F and B is an ordered basis of V, then prove that for any linear operator T on V and* $\alpha \in V$, $[T(\alpha)]_B = [T]_B [\alpha]_{B'}$.

Proof. Let $T : V \to V$ is a linear transformation and B is an ordered basis of V.

Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$.

Let $A = [a_{ij}]_{n \times n}$ be the matrix of T relative to B.

Then $[T]_B = A = [a_{ij}]_{n \times n}$

$\therefore \quad \forall \ \alpha_j \in B, \quad T(\alpha_j) = a_{1j}\alpha_1 + a_{2j}\alpha_2 + \ldots + a_{ij}\alpha_i + \ldots + a_{nj}\alpha_n \ = \sum_{i=1}^{n} a_{ij}\ \alpha_i$

For any $\alpha \in V$, $\alpha = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$ for every $b \in F$.

$$\therefore \quad \text{The coordinate matrix of } \alpha = [\alpha]_B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$\therefore \quad T(\alpha) = T(b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n) = b_1 T(\alpha_1) + b_2 T(\alpha_2) + \ldots + b_n T(\alpha_n)$$

$$= \sum_{j=1}^{n} b_j T(\alpha_j) = \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} a_{ij}\ \alpha_i \right) \ = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} b_j a_{ij} \right) \alpha_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}\ b_j \right) \alpha_1$$

$$= \sum_{i=1}^{n} c_i \alpha_i \quad \text{where} \quad c_i = \sum_{j=1}^{n} a_{ij}\ b_j$$

$\therefore$ The coordinate matrix of $T(\alpha)$ *w.r.t.* basis B is

$$[T(\alpha)]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\ b_j \\ \sum_{j=1}^{n} a_{2j}\ b_j \\ \ldots\ldots \\ \ldots\ldots \\ \sum_{j=1}^{n} a_{nj}\ b_j \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \ldots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \ldots + a_{2n}b_n \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{n1}b_1 + a_{n2}b_2 + \ldots + a_{nn}b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ : \\ : \\ b_n \end{bmatrix} = A[(\alpha)]_B = [T]_B [\alpha]_B$$

5.5. Theorem. *Let U(F) and V(F) be two vector spaces so that dim U = n and dim V = m. Let T and H be the linear transformations from U to V. If B and B′ are the ordered bases of U and V respectively then*

(i) $[T + H : B : B'] = [T : B ; B'] + [H : B ; B';]$

(ii) $[c\,T : B ; B'] = c\,[T : B ; B']$ *where* $c \in F$.

Proof. Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and $B' = \{\beta_1, \beta_2 \dots \beta_m\}$

Let $[a_{ij}]_{m \times n}$ be the matrix $[T : B ; B']$. $\quad \therefore \; T(\alpha_j) = \sum_{i=1}^{m} \alpha_{ij}\beta_i$, $\; j = 1, 2 \dots n$

Let $[b_{ij}]_{m \times n}$ be the matrix [H : B ; B']. $\quad \therefore \; T(\alpha_j) = \sum_{i=1}^{m} b_{ij}\beta_i$, $\; j = 1, 2 \dots n$.

(*i*) Now $(T + H)(\alpha_j) = T(\alpha_j) + H(\alpha_j) \quad j = 1, 2 \dots n$

$$= \sum_{i=1}^{n} a_{ij}\beta_i + \sum_{i=1}^{m} b_{ij}\beta_i = \sum_{i=1}^{n} (a_{ij} + b_{ij})\,\beta_i$$

$\therefore$ Matrix of $[T + H : B ; B'] = [T : B, B'] + [H : B ; B']$

(*ii*) $(cT)(\alpha_j) = cT(\alpha_j) \quad (j = 1, 2 \dots n) \qquad = c \sum_{i=1}^{m} a_{ij}\beta_i = \sum_{i=1}^{m} (c\, a_{ij})\,\beta_i$

$\therefore$ The matrix of cT relative to B and B' is

$[cT : B ; B'] = [c\, a_{ij}]_{m \times n} = c\,[a_{ij}]_{m \times n} = c\,[T : B ; B']$

Note. If U = V and the bases are such that B = B'

Then (*i*) $[T + H] = [T] + [H]$ $\quad$ (*ii*) $[c\,T] = c\,[T]$

5.6. Theorem. *Let T and H be linear operators on an n-dimensional vector space V(F). If B is the basis of V then [TH] = [T] [H]*

Proof. Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$. $\quad$ Let $[a_{ik}]_{n \times n}$ be the matrix [T]

$$T(a_k) = \sum_{i=1}^{n} a_{ik}\, \alpha_i \quad k = 1, 2 \dots n.$$

Let $[b_{kj}]_{n\times n}$ be the matrix [H]. Then $H(\alpha_j) = \sum_{k=1}^{n} b_{kj}\,\alpha_k \quad j = 1, 2 \ldots n.$

$\therefore$ Now $(T\,H)(\alpha_j) = T\,[H(\alpha_j)]$

$$= T\left(\sum_{k=1}^{n} b_{kj}\,\alpha_k\right) \qquad = \sum_{k=1}^{n} b_{kj}\,T(\alpha_k) \qquad [\text{T is L.T.}$$

$$= \sum_{k=1}^{n} b_{kj}\left(\sum_{i=1}^{n} a_{ik}\alpha_i\right) \qquad = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)\alpha_i \qquad = \sum_{i=1}^{n} c_{ij}\,\alpha_i \quad \text{where } c_i = \sum_{k=1}^{n} a_{ik}\,b_{kj}$$

$\therefore$ The matrix $[TH] = [c_{ij}]_{n\times n} = [\Sigma\, a_{ik}\, b_{kj}]_{n\times n} = [a_{ik}]\,[b_{kj}] = [T]\,[H].$

5.7. Theorem. ***Let T be a linear operator on an n-dimensional vector space V and let B be an ordered basis for V. Then T is invertible iff*** $[T]_B$ ***is an invertible matrix and*** $[T^{-1}]_B = [[T]_B]^{-1}$

Proof. (*i*) Let T be invertible. $\therefore$ T^{-1} exists and $T^{-1}T = I = TT^{-1}$

$\Rightarrow [T^{-1}T]_B = [I]_B = [TT^{-1}]_B \qquad \Rightarrow [T^{-1}]_B\,[T]_B = I = [T]_B\,[T^{-1}]_B$

$\Rightarrow [T]_B$ is invertible and $[[T]_B]^{-1} = [T^{-1}]_B$.

(*ii*) Let $[T]_B$ be an invertible matrix. $[T]_{B^{-1}}$ exists.

$\therefore$ There exists a linear operator H on V such that $[T]^{-1} = [H]$

$\Rightarrow [H][T] = [I] = [T][H] \qquad \Rightarrow [HT] = [I] = [TH]$

$\Rightarrow$ HT = I = TH $\qquad \Rightarrow$ T is invertible.

CHANGE OF BASIS

5.8. Transition Matrix. Let V(F) be an n-dimensional vector space and $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \ldots \beta_n\}$ be two arbitrary bases of V.

Let us suppose

$\beta_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots a_{n1}\alpha_n$; $\qquad \beta_2 = a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots a_{n2}\alpha_n$

....; $\qquad \beta_n = a_{1n}\alpha_1 + a_{2n}\alpha_2 + \ldots a_{nn}\alpha_n$

The matrix of transformation from B to B' is P = $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n}$

$\therefore$ The matrix P is called the transition matrix from B to B'.

Note. Since $B' = \{\beta_1, \beta_2, \dots \beta_n\}$ is an L.I. set, the matrix P is invertible and P^{-1} is a transition matrix from B' to B.

5.9. Theorem. *Let P be the transition matrix from a basis B to a basis B' in an n-dimensional vector space V(F). Then for* $\alpha \in V$ *(i)* $P[\alpha]_{B'} = [\alpha]_B$ *(ii)* $[\alpha]_{B'} = P^{-1}[\alpha]_B$

Proof. Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ and $B' = \{\beta_1, \beta_2, \dots \beta_n\}$ be two ordered bases of $V_n(F)$

Let the transition matrix $P = [a_{ij}]_{n \times n}$. Then there exists a unique linear transformation T in V such that $T(\alpha_j) = \beta_j \quad j = 1, 2 \dots n.$

Since T is an onto function, then T is invertible.

$\Rightarrow$ $[T]_B = P$ is a unque matrix and hence invertible.

(*i*) Now $T(\alpha_j) = \beta_j = a_{1j}\alpha_1 + a_{2j}\alpha_2 + \dots a_{nj}\alpha_n = \sum_{i=1}^{n} a_{ij}\alpha_i \qquad (j = 1, 2 \dots n)$

For $\alpha \in V$, $\alpha = l.c.$ of elements $B' = b_1\beta_1 + b_2\beta_2 \dots + b_n\beta_n = \sum_{j=1}^{n} b_j \beta_j \qquad (b's \in F)$

$\Rightarrow$ The coordinate matrix of $[\alpha]_{B'} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Also $\alpha = \sum_{j=1}^{n} b_j \beta_j = \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} a_{ij}\alpha_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_j \right) \alpha_i = \sum_{i=1}^{n} (a_{i1}b_1 + a_{i2}b_2 \dots + a_{in}b_n)\,\alpha_i$

$\therefore$ The coordinate matrix of α relative to B

$$[\alpha]_B = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \dots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \dots + a_{2n}b_n \\ \dots \quad \dots \quad \dots \\ a_{n1}b_1 + a_{n2}b_2 + \dots + a_{nn}b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \qquad \therefore\ [\alpha]_B = P[\alpha]_{B'}$$

(*ii*) Pre multiplying by P^{-1}. $\qquad P^{-1}[\alpha]_B = P^{-1}P[\alpha]_{B'}$

$\Rightarrow$ $P^{-1}[\alpha]_B = I[\alpha]_{B'}$ $\qquad \Rightarrow$ $P^{-1}[\alpha]_B = [\alpha]_{B'}$.

5.10. Theorem. *Let P be the transition matrix from a basis B to a basis B' in a vector space* $V_n(F)$. *Then for any linear operator T on V, [T ; B] P = P [T ; B]*

Proof. Let $\alpha \in V$ be arbitary vector. Then by the previous theorems we have

$P[\alpha]_{B'} = [\alpha]_B \quad \dots (1) \quad$ and $\quad [T]_B\,[\alpha]_B = [T(\alpha)]_B \qquad \dots (2)$

Now pre-multiplying (1) by $P^{-1}[T]_B$ we get $P^{-1}[T]_B\, P\, [\alpha]_{B'} = P^{-1}[T]_B\, [\alpha]_B$

$= P^{-1}[T(\alpha)]_B$ (by (2)) $= [T(\alpha)]_{B'}$ (by (1))

$\therefore\ P^{-1}[T]_B\, P\, [\alpha]_{B'} = [T]_{B'}\, [\alpha]_{B'} \Rightarrow P^{-1}[T]_B\, P = [T]_{B'}$

Premultiplying with P $[T]_B\, P = P\, [T]_{B'}$ or $[T:B]\, P = P\, [T:B']$

SOLVED PROBLEMS

Ex. 1. *Let T be the linear operator on* R^2 *defined by* $T(x,y) = (4x-2y, 2x+y)$

Find the matrix of T w.r. to the basis $T\{(1,1), (-1,0)\}$.

Also verify $[T]_B\, [\alpha]_B = [T](\alpha)_B \quad \forall\ \alpha \in R^2$

Sol. Let $(a,b) \in R^2$. Then $(a,b) = p\,(1,1) + q\,(-1,0) = (p-q, p)$

$\Rightarrow a = p - q$ and $b = p \quad \Rightarrow p = b$ and $q = b - a$

$\therefore\ (a,b) = b\,(1,1) + (b-a)\,(-1,0)$ ... (1)

(*i*) Given transformation is $T(x,y) = (4x-2y, 2x+y)$

$\therefore\ T(1,1) = (2,3) = 3\,(1,1) + 1\,(-1,0)$ .... (by (1))

$T(-1,0) = (-4,-2) = -2\,(1,1) + 2\,(-1,0) \qquad \therefore\ [T:B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

(*ii*) Let $\alpha \in R^2$ where $\alpha = (a,b)$

$\therefore\ \alpha = (a,b) = b\,(1,1) + (b-a)\,(-1,0) \qquad \therefore\ [\alpha]_B = \begin{bmatrix} b \\ b-a \end{bmatrix}$

$\therefore\ [T]_B\, [\alpha]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} b \\ b-a \end{bmatrix} = \begin{bmatrix} 2a+b \\ -2a+3b \end{bmatrix}$ ... (1)

Again $T(\alpha) = T(a,b) = (4a-2b, 2a+b)$

Let $(4a-2b, 2a+b) = x\,(1,1) + y\,(-1,0) = (x-y, x)$

$\Rightarrow 4a - 2b = x - y$ and $2a + b = x \quad \Rightarrow x = 2a + b$ and $y = -2a + 3b$

Hence $T(\alpha) = T(a,b) = (2a+b)\,(1,1) + (-2a+3b)\,(-1,0)$

$\therefore$ The matrix of $T(\alpha)$ *w.r.* to the base B is $[T(\alpha)]_B = \begin{bmatrix} 2a+b \\ -2a+3b \end{bmatrix}$ ... (2)

$\therefore$ From (1) and (2) $[T]_B\, [\alpha]_B = [T(\alpha)]_B$.

Ex. 2. *Let T be a linear operator on* $V_3(R)$ *defined by*

$T(x,y,z)=(3x+z,-2x+y,-x+2y+z)$. *Prove that T is invertible and find* T^{-1}.

Sol. Consider the standard basis $B=\{(1,0,0),(0,1,0),(0,0,1)\}$

Given transformation is $T(x,y,z)=(3x+z,-2x+y,-x+2y+z)$

$\therefore\ T(1,0,0)=(3,-2,-1)=3(1,0,0)-2(0,1,0)-1(0,0,1)$

$\therefore\ T(0,1,0)=(0,1,2)=0(1,0,0)+1(0,1,0)+2(0,0,1)$

$\therefore\ T(0,0,1)=(1,0,4)=1(1,0,0)+0(0,1,0)+4(0,0,1)$

$\therefore\ [T]_B=\begin{bmatrix}3&0&1\\-2&1&0\\-1&2&4\end{bmatrix}=P$ (say)   T is invertible if $[T]_B$ is invertible .

$\det P=\begin{vmatrix}3&0&1\\-2&1&0\\-1&2&4\end{vmatrix}=9$ $\because \det P\neq 0$, the matrix P is inversible

Calculating $P^{-1}=\dfrac{adj\ P}{\det P}$ we get $P^{-1}=\dfrac{1}{9}\begin{bmatrix}4&2&-1\\8&13&-2\\-3&-6&3\end{bmatrix}$. $\therefore P^{-1}=[T]_B^{-1}=[T^{-1}]_B$

To find the transformation T^{-1} take $\alpha\in V$ where $\alpha=(a,b,c)$

$\Rightarrow \alpha=a(1,0,0)+b(0,1,0)+c(0,0,1)$ $\Rightarrow (\alpha)_B=\begin{bmatrix}a\\b\\c\end{bmatrix}$

We know that $[T^{-1}(\alpha)]_B=[T^{-1}]_B[\alpha]_B=\dfrac{1}{9}\begin{bmatrix}4&2&-1\\8&13&-2\\-3&-6&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$

$\therefore\ [T^{-1}(\alpha)]_B=\begin{bmatrix}4a+2b-c\\8a+13b-2c\\-3a-6b+3c\end{bmatrix}$

$\Rightarrow T^{-1}(\alpha)=T^{-1}(a,b,c)=(4a+2b-c,\ 8a+13b-2c,\ -3a-6b+3c)$

Ex. 3. *Let* $B=\{(1,0),(0,1)\}$ *and* $B'=\{(1,3),(2,5)\}$ *be the bases of* R^2.
Find the transition matrices from B to B' *and* B' *to B.*

Sol. (*i*) Given $B=\{(1,0),(0,1)\}$ and $B'=\{(1,3),(2,5)\}$

Now $(1,3)=1\,(1,0)+3\,(0,1);(2,5)=2\,(1,0)+5\,(0,1)$

$\therefore$ The transition matrix from B to $B' = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

(*ii*) Again $(1,0)=a\,(1,3)+b\,(2,5)=(a+2b,3a+5b)$

$\therefore\ 1=a+2b$ and $0=3a+5b$ $\Rightarrow a=-5$ and $b=3$

Similarly $(0,1)=x\,(1,3)+y\,(2,5)=(x+2y,3x+5y) \Rightarrow x+2y=0$ and $3x+5y=1$

$\Rightarrow x=2$ and $y=-1$. Hence $(1,0)=-5\,(1,3)+3\,(2,5);(0,1)=2\,(1,3)-1\,(2,5)$.

Ex. 4. *Let T be the linear operator on* $V_3(R)$ *defined by* $T\,(x,y,z)=(2y+z,x-4y,3x)$

(*i*) *Find the matrix of T relative to the basis* $B=\{(1,1,1),(1,1,0),(1,0,0)\}$

(*ii*) *Verify* $[T(\alpha)]_B=[T]_B\,[\alpha]_B$

Sol. Let $\alpha=(a,b,c)\in V_3(R)$

$\therefore\ \alpha=(a,b,c)=p\,(1,1,1)+q\,(1,1,0)+r\,(1,0,0)=(p+q+r,p+q,p)$

$\therefore\ a=p+q+r, b=p+q, c=p \qquad \Rightarrow p=c, q=b-c, r=a-b$

$\therefore \alpha\,(a,b,c)=c\,(1,1,1)+(b-c)\,(1,1,0)+(a-b)\,(1,0,0)$

$\Rightarrow [\alpha]_B=\begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$. Given $T\,(x,y,z)=(2y+z,x-4y,3x)$

$\therefore\ T\,(1,1,1)=(3,-3,3)=l\,(1,1,1)+m\,(1,1,0)+n\,(1,0,0)=(l+m+n,l+m,l)$

$\Rightarrow l+m+n=3, l+m=-3$ and $l=3 \qquad \Rightarrow l=3, m=-6, n=6$

$\therefore\ T\,(1,1,1)=3\,(1,1,1)-6\,(1,1,0)+6\,(1,0,0)$

Similarly $T\,(1,1,0)=(2,-3,3)=3\,(1,1,1)-6\,(1,1,0)+5\,(1,0,0)$

$T\,(1,0,0)=(0,1,3)=3\,(1,1,1)-2\,(1,1,0)-1\,(1,0,0) \qquad \therefore\ [T]_B=\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & 1 \end{bmatrix}$

Now $(T)_B(\alpha)_B=\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & 1 \end{bmatrix}\begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}=\begin{bmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{bmatrix}$ .... (1)

Again $T\,(\alpha)=T\,(a,b,c)=(2b+c,a-4b,3a)$

$=3a\,(1,1,1)+(a-4b-3a)\,(1,1,0)+(2b+c-a+4b)\,(1,0,0)$

$=3a\,(1,1,1)-(2a+4b)\,(1,1,0)+(-a+6b+c)\,(1,0,0)$

$\therefore\ [T(\alpha)]_B = \begin{bmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{bmatrix}$. Hence $[T(\alpha)]_B = [T]_B\,[\alpha]_B$.

Ex. 5. *Let* $T: R^3 \to R^2$ *be a linear transformation defined by* $T(x,y,z) = (2x+y-z, 3x-2y+4z)$.

(*i*) *Obtain the matrix of T relative to the bases*

$B_1 = \{(1,1,1),(1,1,0),(1,0,0)\}$; $B_2 = \{(1,3),(1,4)\}$

(*ii*) *Verify for any vector* $\alpha \in R^3$ $[T:B_1,B_2]\ [\alpha:B_1] = [T(\alpha):B_2]$

Sol. Let $(a,b) \in R^2$ and let $(a,b) = p\,(1,3) + q\,(1,4) \Rightarrow (p+q, 3p+4q)$

$\Rightarrow a = p+q$ and $b = 3p+4q \quad \Rightarrow p = 4a-b$ and $q = b-3a$

$\therefore\ (a,b) = (4a-b)\,(1,3) + (b-3a)\,(1,4)$ .... (1)

Given $T\,(x,y,z) = (2x+y-z,\ 3x-2y+4z)$

$\therefore\ T\,(1,1,1) = (2,5) = (4\,.\,2-5)\,(1,3) + (5-3\,.\,2)\,(1,4) = 3\,(1,3) + (-1)\,(1,4)$

$T\,(1,1,0) = (3,1) = 11\,(1,3) - 8\,(1,4)$

(*ii*) Let $\alpha = (x,y,z) \in R^3$

$\therefore\ (x,y,z) = l\,(1,1,1) + m\,(1,1,0) + n\,(1,0,0) = (l+m+n, l+m, l)$

$\Rightarrow x = l+m+n, y = l+m, z = l \qquad \Rightarrow l = z, m = y-z, n = x-y$

$\therefore\ \alpha = (x,y,z) = z\,(1,1,1) + (y-z)\,(1,1,0) + (x-y)\,(1,0,0)$

$\therefore\ [\alpha]_{B_1} = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} \quad \therefore\ [T:B_1;B_2]\,(\alpha_{B_1}) = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} = \begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$

Also $T(\alpha) = T(x,y,z) = (2x+y-z, 3x-2y+4z)$

$= \{4\,(2x+y-z) - (3x-2y+4z)\,(1,3)\} + \{(3x-2y+4z) - 3\,(2x+y-z)\}\,(1,4)$

$= (5x+6y-8z)\,(1,3) + (-3x-5y+7z)\,(1,4)$

$\therefore\ [T(\alpha)]_{B_2} = \begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$. From I and II : $[T:B_1,B_2]\ [\alpha:B_1] = [T(\alpha):B_2]$

Ex. 6. *Let V(F) be a vector space of polynomials in x of degree atmost 3 and D be the differential operator on V.*

If the basis for V(F) is $B = \{1, x, x^2, x^3\}$ verify $[D:B]\ [\alpha:B] = [D\,(\alpha):B]$.

Sol. The basis $B = \{1, x, x^2, x^3\}$. $\therefore$ $D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$

$D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$; $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$

$D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$

$$\therefore [D : B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\alpha = f(x) = a + bx + cx^2 + dx^3 \in V$

$$\therefore [\alpha : B] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \therefore [D : B]\, [\alpha : B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}$$

Again $D(\alpha) = D\{f(x)\} = b + 2cx + 3dx^2$

$$[D(\alpha) : B] = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}.$$

Hence $[D : B]\, [\alpha : \beta] = [D(\alpha) : B]$

Ex. 7. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Let T be a linear operator on* R^2 *defined by* $T(\alpha) = A\alpha$, *where* α *is written as a column vector. Find the matrix of T relative to the basis* $\{(1,0), (0,1)\}$

Sol. Let $T(1,0) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1(1,0) + 3(0,1)$

$T(0,1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2(1,0) + 4(0,1)$ $\therefore$ The matrix of $T = [T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Ex. 8. *If C(R) is a vector space having the bases* $B_1 = \{1, i\}$ *and* $B_2 = \{1+i, 1+2i\}$, *find the transition matrix of T from* B_1 *to* B_2

Sol. Now $(1+i) = 1(1) + 1(i)$; $(1+2i) = 1(1) + 2(i)$

$\therefore$ The transition matrix from B_1 to B_2 is $[T : B_1; B_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

5.11. We complete this section with the introduction of the "Left-Multiplication Transformation " L_A, where A is $m \times n$ matrix. This transformation is most useful in transfering properties about transformations to analogous properties about matrices and vice versa.

5.12. Def : *Let A be an $m \times n$ matrix with elements from a field F. We denote by L_A, the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = A\,x$ (the matrix product of A and x) for each column vector $x \in F^n$. We call L_A as "Left - Multiplication transformation".*

Ex. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Then $A \in M_{2\times 3}(R)$ and $L_A : R^3 \to R^2$

Take $x = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ then $L_A(x) = Ax = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

5.13. Theorem. ***Let A be an $m \times n$ matrix with elements from F. Then the left multiplication transformation $L_A : F^n \to F^m$ is linear. Further, if B is any other $m \times n$ matrix (with elements from F) and β, γ are the standard ordered bases for F^n and F^m, respectively then we have***

(*i*) $[L_A]_\beta^\gamma = A$ (*ii*) $L_A = L_B$ iff $A = B$ (*iii*) $L_{A+B} = L_A + L_B$, $L_{aA} = aL_A \forall a \in F$

(*iv*) If $T : F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix c such that $T = L_C$ in fact $C = [T]_\beta^\gamma$

(*v*) If E is an $n \times p$ matrix then $L_{AE} = L_A L_E$ (*vi*) If $m = n$, then $LI_n = I_{F^n}$

Proof. Let $a, b \in F$ and x, y are two vectors then we have $L_A(ax + by) = A\,(ax + by)$

$= A\,(ax) + A\,(by) = a\,(Ax) + b\,(Ay)$

$= a\,L_A(x) + b\,L_A(y)$

Thus L_A is a linear transformation.

(*i*) The j^{th} column of $[L_A]_\beta^\gamma = L_A(e_j)$.

However, $L_A(e_j) = A\,(e_j)$. This is the j^{th} column of A.

Thus $[L_A]_\beta^\gamma = A$

(*ii*) If $L_A = L_B$, we have by (*i*) $[L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$ converse is trivial.

(*iii*) $L_{A+B}(x) = (A+B)(x) = Ax + Bx = L_A(x) + L_B(x)$

$\therefore L_{A+B} = L_A + L_B$

$L_{aA}(x) = (aA)x = a(Ax) = a L_A(x)$

$\therefore L_{aA} = a L_A$

(*iv*) Let $C = [T]_\beta^\gamma$. We know that $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$ or $T(x) = C_x = L_C(x) \;\forall\; x \in F^n$

Thus $T = L_C$.

Uniqueness of *c* follows from (*iii*), (*v*), (*vi*) are simple to follow.

As remarked earlier we use the definition of L_A, to prove the properties of matrices. Here we prove the associativity of matrix multiplication.

Theorem. Let A, B, C be the matrices such that A (BC) is defined. Thus (AB) C is also defined and A(BC) = (AB) C. *i.e.* matrix multiplication is associative.

Proof. We can use properties of matrices to show that (AB) C is defined.

Now $L_{A(BC)} = L_A(L_{BC}) = L_A(L_B L_C)$

$= (L_A L_B) L_C = (L_{AB}) L_C = L_{(AB)C}$

EXERSICE 5

1. Find the matrix of linear transformation T on R^3 defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$ with respect to the ordered basis $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

2. If the matrix of T on R^2 relative to the standard basis is $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$ find the matrix of T relative to the basis $\{(1,1), (1,-1)\}$.

3. If the matrix of T on R^3 relative to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ find the matrix relative to the basis $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$

4. If the matrix of transformation T on $V_3(R)$ relative to the standard basis is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ find the matrix of T relative to the basis $S = \{(1,2,2), (1,1,2), (1,2,1)\}$

5. Let T be a linear operator on R^3 defined by $T(x, y) = (2y, 3x - y)$

Find the matrix of T relative to the basis $\{(1,3),(2,5)\}$

6. Let T be a linear transformation from R^3 to R^2 defined by $T(x, y, z) = (x + y, 2z - x)$

Find the matrix of T from the basis $\{(1,0,-1),(1,1,1),(1,0,0)\}$ to the base $\{(0,1),(1,0)\}$

7. If T be linear operation on R^3 defined by

$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 + x_2, -x_1, 2x_2 + 4x_3)$ determine the matrix of T relative to the standard basis of R^3

8. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and T be the linear operator on R^2 defined by $T(\alpha) = A\alpha$, where α is written as a column vector. Find the matrix of T relative to the basis $\{(1,3),(2,5)\}$

9. If C(R) is a vector space having the basis as $B_1 = \{1, i\}$ and $B_2 = \{1+i, 1+2i\}$

Find the transition matrix from B_2 to B_1

10. Let the matrix of T relative to the bases B_1 and B_2 be

Find the linear transformation relative to the bases B_1 and B_2 where

(*i*) B_1 and B_2 are standard bases of V_2 and V_3 respectively.

(*ii*) $B_1 = \{(1,1),(-1,1)\}; B_2 = \{(1,1,1),(1,-1,1),(0,0,1)\}$

(*iii*) $B_1 = \{(1,2),(-2,1)\}; B_2 = \{(1,-1,-1),(1,2,3),(-1,0,2)\}$

11. Given $[T : B_1; B_2] = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ find the linear transformation when

(*i*) B_1 and B_2 are standard bases of V_3 and V_2 respectively.

(*ii*) $B_1 = \{(1,1,1),(1,2,3),(1,0,0)\}$; $B_2 = \{(1,1),(1,-1)\}$

(*iii*) $B_1 = \{(1,-1,1),(1,2,0),(0,-1,0)\}$; $B_2 = \{(1,0),(2,-1)\}$

12. The matrix of linear transformation $T : V_4 \to V_3$

$[T : B_1; B_2] = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 0 \end{bmatrix}$. Find the transformation when

(*i*) B_1 and B_2 are standard bases of V_4 and V_3 respectively.

(*ii*) $B_1 = \{(1,1,1,2),(1,-1,0,0),(0,0,1,1),(0,1,0,0)\}$; $B_2 = \{(1,2,3),(1,-1,1),(2,1,1)\}$

13. Let A be an $m \times n$ matrix. Then A is invertible if and only if L_A is invertible.

Further $(L_A)^{-1} = L_A^{-1}$.

14. Let $T : U \to V$ be a linear transformation from an n - dimensional vector space V to an m dimensional vector space V. Let β and γ are the ordered bases for U and V respectively. Prove that

(*i*) rank (T) = rank (L_A)

(*ii*) Nullity (T) = nullity (L_A) where $A = [T]_\beta$

ANSWERS

1. (*i*) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & 2 \\ 6 & 5 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1/2 & 5/2 \\ -3/2 & 5/2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

5. $\begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$ **6.** $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ **7.** $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$ **8.** $\begin{bmatrix} -5 & -8 \\ 6 & 10 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ **10.** (*i*) $T(x, y) = (x + 2y, y, -x + 3y)$

(*ii*) $T(x, y) = (2y - x, y, 3y - 3x)$ (*iii*) $T(x, y) = \frac{1}{5}(2x + 4y, -x - 2y, -17x + y)$

11. (*i*) $T(a, b, c) = (a - b + 2c, 3a + b)$ (*ii*) $T(a, b, c) = (2a + 8b - 6c, 2a - 8b + 4c)$

(*iii*) $T(a, b, c) = (5a - 2b, -a - 2c)$

12. (*i*) $T(a, b, c, d) = (a + b + 2c + 3d, a + c - d, a + 2b)$

(*ii*) $T(a, b, c, d) = (7a + 2b + 11c - 8d, 11a + 7b + 22c - 19d, 13a + 8b + 30c - 23d)$