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Table of Contents

Ordinary First Order Differential Equations

10.0 INTRODUCTION

In the physical world nothing is permanent except change. Differential equations are of fundamental importance because they express relationships involving rate of change. These relationships form the basis for studying the phenomena in a variety of fields in Science and engineering.

In fact, many practical laws are expressed mathematically in the form of differential equations. The primary use of differential equation is to serve as a tool for the study of change in the physical world.

Definition 10.1 Differential equation

A differential equation is an equation involving one dependent variable and its derivatives with respect to one or more independent variables.

Differential equations are of two types

(i) Ordinary differential equations (ii) Partial differential equations

EXAMPLES:

1. $\frac{dy}{dx}+4y=\tan x$

2. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{-2x}$

3. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

4. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x+y$

Definition 10.2 Ordinary differential Equation

An ordinary differential equation is an equation involving one dependent variable and its derivatives with respect to only one independent variable.

Equations (1) and (2) are ordinary differential equations.

But (3) and (4) are partial differential equations.

In this chapter we consider only ordinary differential equations.

Definition 10.3 Order and degree

The **order** of a differential equation is the order of the highest ordered derivative involved in the equation.

The **degree** of a differential equation is the degree of the highest ordered derivative involved in the equation, after the equation has been cleared off the radicals and fractions, so far as the derivatives are concerned.

Definition 10.4 Linear and non linear differential equations

A linear differential equation is one in which the dependent variable and its derivatives with respect to the independent variable occur with first degree and no product of the dependent variable and derivative or product of derivatives occur.

A differential equation which is not linear is called a **non-linear differential equation**.

Examples of ordinary differential equations.

(1) $x\frac{dy}{dx}+3y=x^2$

(2) $\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-y=xe^x$

(3) $\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+5y=0$

(4) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y=0$

(5) $y\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\cos x$

(6) $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

Equation (1) is linear, first order and first degree
Equation (2) is linear, second order and first degree.
Equation (3) is linear, second order and first degree.
Equation (4) is linear, second order and first degree.
Equation (5) is non-linear, second order and first degree.
Equation (6) is non-linear, second order and second degree.

Definition 10.5 Solution of a differential equation

The solution of a differential equation is a relation between the dependent and independent variables, not containing derivatives or differentials, which satisfy the equation.

The solution of a differential equation is also known as **integral of the differential equation**.

Definition 10.6 The general solution or complete integral

The solution of an ordinary differential equation which contains as many independent arbitrary constants as the order of the equation is called **the general solution or Complete integral**.

Note

(1) Solve a differential equation means finding the general solution
(2) The general solution does not mean that it includes all possible solutions of the differential equation

There may exist other solutions which cannot be deduced from the general solution (or not included in the general solution). Such solutions not containing arbitrary constants, are called **Singular solutions**.

10.1 FORMATION OF DIFFERENTIAL EQUATIONS

A differential equation is formed by eliminating arbitrary constants from an ordinary relation between the variables.

WORKED EXAMPLES

EXAMPLE 1

Form the differential equation by eliminating the constant from $y = 1 + x^2 + C\sqrt{1 + x^2}$.

Solution.

The given equation is $y=1+x^2+C\sqrt{1+x^2}$ (1)

Since one constant is to be eliminated, we differentiate (1) w. r. to x, once

$$\therefore \quad \frac{dy}{dx} = 2x + \frac{c}{2\sqrt{1+x^2}} \cdot 2x$$

$$= 2x + \frac{cx}{\sqrt{1+x^2}} \Rightarrow \frac{cx}{\sqrt{1+x^2}} = \frac{dy}{dx} - 2x \Rightarrow c = \frac{\sqrt{1+x^2}}{x}\left(\frac{dy}{dx} - 2x\right)$$

Substituting in (1), we get

$$y = 1 + x^2 + \frac{\sqrt{1+x^2}}{x}\left(\frac{dy}{dx} - 2x\right)\sqrt{1+x^2}$$

$$= (1+x^2) + \left(\frac{dy}{dx} - 2x\right)\left(\frac{1+x^2}{x}\right)$$

$$\Rightarrow \quad y = \frac{1+x^2}{x}\left[x + \frac{dy}{dx} - 2x\right]$$

$$\Rightarrow \quad y = \frac{1+x^2}{x}\left[\frac{dy}{dx} - x\right] \Rightarrow xy = (1+x^2)\frac{dy}{dx} - x(1+x^2)$$

$$\therefore \quad (1+x^2)\frac{dy}{dx} - xy = x(1+x^2)$$

which is the required differential equation.

EXAMPLE 2

Form the differential equation by eliminating a and b from $y = a \tan x + b \sec x$.

Solution.

The given equation is $y = a \tan x + b \sec x$ (1)

Since two constants a and b are to be eliminated, we differentiate (1) w. r. to x, twice

$$\therefore \quad \frac{dy}{dx} = a \sec^2 x + b \sec x \tan x \qquad (2)$$

and $$\frac{d^2y}{dx^2} = a \cdot 2 \sec x \sec x \tan x + b[\sec x \cdot \sec^2 x + \tan x \sec x \tan x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2a \sec^2 x \tan x + b \sec x[\sec^2 x + \tan^2 x] \qquad (3)$$

(1) × $\tan x \Rightarrow$ $$y \tan x = a \tan^2 x + b \sec x \tan x \qquad (4)$$

(2) − (4) $\Rightarrow$ $$\frac{dy}{dx} - y \tan x = a \sec^2 x - a \tan^2 x = a[\sec^2 x - \tan^2 x] = a \qquad [\because \sec^2 x - \tan^2 x = 1]$$

$$\therefore \quad a = \frac{dy}{dx} - y \tan x$$

Substituting in (1), we get

$$y = \left(\frac{dy}{dx} - y\tan x\right)\tan x + b\sec x = \frac{dy}{dx}\tan x - y\tan^2 x + b\sec x$$

$$\Rightarrow \quad y(1+\tan^2 x) = \frac{dy}{dx}\tan x + b\sec x$$

$$\Rightarrow \quad y\sec^2 x = \frac{dy}{dx}\tan x + b\sec x \quad \Rightarrow \quad b\sec x = y\sec^2 x - \frac{dy}{dx}\tan x$$

Substituting in (3), we get

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - y\tan x\right)\sec^2 x\tan x + \left(y\sec^2 x - \frac{dy}{dx}\tan x\right)(\sec^2 x + \tan^2 x)$$

$$= 2\sec^2 x\tan x\frac{dy}{dx} - 2y\sec^2 x\tan^2 x + y\sec^2 x(\sec^2 x + \tan^2 x) - \frac{dy}{dx}\tan x(\sec^2 x + \tan^2 x)$$

$$= \frac{dy}{dx}\tan x\left[2\sec^2 x - (\sec^2 x + \tan^2 x)\right] + y\sec^2 x\left[\sec^2 x + \tan^2 x - 2\tan^2 x\right]$$

$$= \frac{dy}{dx}\tan x[\sec^2 x - \tan^2 x] + y\sec^2 x[\sec^2 x - \tan^2 x]$$

$$= \frac{dy}{dx}\tan x + y\sec^2 x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx}\tan x - y\sec^2 x = 0$$

which is the required differential equation.

Aliter:

The eliminant of a and b from (1), (2) and (3) is

$$\begin{vmatrix} y & \tan x & \sec x \\ \frac{dy}{dx} & \sec^2 x & \sec x\tan x \\ \frac{d^2y}{dx^2} & 2\sec^2 x\tan x & \sec x(\sec^2 x + \tan^2 x) \end{vmatrix} = 0$$

Expanding by first column, we get

$$y[\sec^3 x(\sec^2 x + \tan^2 x) - 2\sec^3 x\tan^2 x] - \frac{dy}{dx}[\tan x\sec x(\sec^2 x + \tan^2 x) - 2\sec^3 x\tan x]$$
$$+ \frac{d^2y}{dx^2}[\sec x\tan^2 x - \sec^3 x] = 0$$

$$\Rightarrow \quad y\sec^3 x[\sec^2 x + \tan^2 x - 2\tan^2 x] - \frac{dy}{dx}\tan x\sec x[\tan^2 x - \sec^2 x]$$
$$+ \frac{d^2y}{dx^2}\sec x[\tan^2 x - \sec^2 x] = 0$$

$$\Rightarrow \quad y\sec^3 x[\sec^2 x - \tan^2 x] + \frac{dy}{dx}\sec x\tan x[\sec^2 x - \tan^2 x] - \frac{d^2y}{dx^2}\sec x[\sec^2 x - \tan^2 x] = 0$$

$$\Rightarrow \quad y\sec^3 x + \frac{dy}{dx}\sec x\tan x - \frac{d^2y}{dx^2}\sec x = 0 \qquad [\because \sec^2 x - \tan^2 x = 1]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx}\tan x - y\sec^2 x = 0$$

which is the required differential equation.

EXAMPLE 3

Find the differential equation of all parabolas each of which has latus rectum 4*a* and whose axis is parallel to the *x*-axis.

Solution.

We know that, the equation of the parabola with vertex (h, k), axis parallel to the x-axis and latus rectum $4a$ is $(y-k)^2 = 4a(x-h)$ (1)

where h and k are arbitrary constants and a is a fixed constant.

We differentiate twice to eliminate h and k. Differentiating w.r.to x, we get

$$2(y-k)\frac{dy}{dx} = 4a \quad \Rightarrow \quad (y-k)\frac{dy}{dx} = 2a \qquad (2)$$

Differentiating again w.r.to x

$$(y-k)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx} = 0 \quad \Rightarrow \quad (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad (3)$$

Now (2) $\Rightarrow$

$$y - k = \frac{2a}{\frac{dy}{dx}}$$

Substituting in (3), we get

$$\frac{2a}{\frac{dy}{dx}}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \Rightarrow \quad 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

which is the required differential equation.

EXERCISE 10.1

Form the differential equation by eliminating the arbitrary constants from the following

1. $y = ax^3 + bx^2$.
2. $y = C_1e^x + C_2\cos x$.
3. $y = Cx + C - C^3$.
4. $\sin^{-1}x + \sin^{-1}y = C$.
5. $xy = Ae^x + Be^{-x} + x^2$.
6. $y = C_1e^{2x} + C_2e^{3x}$.
7. $y = a\,e^{2x} + b\,e^{-3x}$.

8. Find the differential equation of the family of parabolas with their focii at the origin and their axes along the x-axis.
9. Find the differential equation of all parabolas having their axes parallel to the y-axis.
10. Form the differential equation with general solution is $ay^2 = (x - c)^2$, where a and c are arbitrary constants.

ANSWERS TO EXERCISE 10.1

1. $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$
2. $(\sin x + \cos x)\frac{d^2y}{dx^2} - 2\cos x\frac{dy}{dx} + (\cos x - \sin x)y = 0$
3. $\left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx}(x+1) + y = 0$
4. $\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$
5. $x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} - xy = 0$
6. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$
7. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$
8. $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$
9. $\frac{d^3y}{dx^3} = 0$
10. $3y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

10.2 FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

The general form of a first order and first degree differential equation is

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Generally, it is written as $F\left(x, y, \frac{dy}{dx}\right) = 0$

10.2.1 Type I Variable Separable Equations

If the differential equation $\frac{dy}{dx} = f(x, y)$ can be rewritten as $F(x)dx + G(y)dy = 0$, then the differential equation is said to be variable separable.

The general solution is got by integration.

$$\therefore \qquad \int F(x)dx + \int G(y)dy = 0$$

$$\Rightarrow \qquad \phi(x) + \psi(y) = C$$

where C is an arbitrary constant.

WORKED EXAMPLE

EXAMPLE 1

Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

Solution.

The given differential equation is

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$$

Dividing by $\tan x \tan y$, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = 0$$

$\Rightarrow$ $$\log_e \tan x + \log_e \tan y = \log_e C \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$\Rightarrow$ $$\log_e(\tan x \cdot \tan y) = \log_e C \qquad \Rightarrow \qquad \tan x \cdot \tan y = C$$

which is the general solution.

EXAMPLE 2

Solve the differential equation $xdy + ydx + 4\sqrt{1 - x^2y^2}\,dx = 0$.

Solution.

The given differential equation is $xdy + ydx + 4\sqrt{1-x^2y^2}\,dx = 0$

$\Rightarrow$ $$d(xy) + 4\sqrt{1-(xy)^2}\,dx = 0$$

Dividing by $\sqrt{1-(xy)^2}$, we get $$\frac{d(xy)}{\sqrt{1-(xy)^2}} + 4dx = 0$$

Integrating, we get $$\int \frac{d(xy)}{\sqrt{1-(xy)^2}} + 4\int dx = 0$$

$\Rightarrow$ $$\sin^{-1}(xy) + 4x = C \qquad \left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\right]$$

which is the general solution.

EXAMPLE 3

Solve $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$.

Solution.

Given $$ydx - xdy + 3x^2y^2e^{x^3}dx = 0$$

Dividing by y^2, we get

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0$$

$\Rightarrow \quad d\left(\frac{x}{y}\right)+d\left(e^{x^3}\right)=0 \qquad \left[\because d(e^{x^3})=3x^2e^{x^3} \quad \text{and} \quad d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}\right]$

Integrating, we get $\int d\left(\frac{x}{y}\right)+\int d\left(e^{x^3}\right)=0 \quad \Rightarrow \quad \frac{x}{y}+e^{x^3}=C$

which is the general solution.

EXAMPLE 4

Solve $(x+y)^2\dfrac{dy}{dx}=a^2$.

Solution.

The given differential equation is $(x+y)^2\dfrac{dy}{dx}=a^2$ (1)

Put $u=x+y$ $\quad \therefore \quad \dfrac{du}{dx}=1+\dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dy}{dx}=\dfrac{du}{dx}-1$

$\therefore$ (1) $\Rightarrow \quad u^2\left(\dfrac{du}{dx}-1\right)=a^2 \quad \Rightarrow \quad \dfrac{du}{dx}-1=\dfrac{a^2}{u^2}$

$$\Rightarrow \quad \frac{du}{dx}=1+\frac{a^2}{u^2}=\frac{u^2+a^2}{u^2}$$

$$\Rightarrow \quad \frac{u^2}{u^2+a^2}du=dx$$

Integrating both sides, we get

$$\int\frac{u^2}{u^2+a^2}du=\int dx$$

$$\Rightarrow \quad \int\frac{(u^2+a^2)-a^2}{u^2+a^2}du=\int dx$$

$$\Rightarrow \quad \int\left(1-\frac{a^2}{u^2+a^2}\right)du=\int dx$$

$$\Rightarrow \quad \int du-a^2\int\frac{du}{u^2+a^2}=\int dx$$

$$\Rightarrow \quad u-a^2\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)=x+C$$

$$\Rightarrow \quad x+y-a\tan^{-1}\left(\frac{x+y}{a}\right)=x+C$$

$$\Rightarrow \quad y-a\tan^{-1}\left(\frac{x+y}{a}\right)=C$$

which is the general solution.

EXAMPLE 5

Solve $y^2 \cos\sqrt{x}dx - 2\sqrt{x}e^{\frac{1}{y}}dy = 0$.

Solution.

The given differential equation is $y^2 \cos\sqrt{x}dx - 2\sqrt{x}e^{\frac{1}{y}}dy = 0$

$$\Rightarrow \quad y^2 \cos\sqrt{x}dx = 2\sqrt{x}e^{\frac{1}{y}}dy \quad \Rightarrow \quad \frac{\cos\sqrt{x}}{2\sqrt{x}}dx = \frac{e^{\frac{1}{y}}}{y^2}dy \qquad (1)$$

Put $u = \sqrt{x} \quad \therefore \quad du = \frac{1}{2\sqrt{x}}dx$

and $v = e^{\frac{1}{y}} \quad \therefore \quad dv = e^{\frac{1}{y}}\left(-\frac{1}{y^2}\right)dy = -\frac{e^{\frac{1}{y}}}{y^2}dy$

Substituting in (1), we get

$$\cos u\, du = -\, dv$$

Integrating both sides, we get

$$\int \cos u\, du = -\int dv$$

$$\Rightarrow \quad \sin u = -v + C \quad \Rightarrow \quad \sin\sqrt{x} = -e^{\frac{1}{y}} + C \quad \Rightarrow \quad \sin\sqrt{x} + e^{\frac{1}{y}} = C.$$

which is the general solution.

EXERCISE 10.2

Solve the following differential equations

1. $x\sqrt{1+y^2} + y\sqrt{1+x^2}\frac{dy}{dx} = 0$.
2. $\frac{dy}{dx} - x\tan(y-x) = 1$.
3. $2x(e^y - 1)dx + e^{x+y}dy = 0$.
4. $(e^x + 1)ydy + (y+1)dx = 0$.
5. $\frac{dy}{dx} = (4x + y + 3)^2$.
6. $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$.
7. $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$.
8. $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$.
9. $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$.
10. $\frac{dy}{dx} = \frac{y^2 - 2y + 5}{x^2 - 2x + 2}$.

ANSWERS TO EXERCISE 10.2

1. $\sqrt{1+x^2} + \sqrt{1+y^2} = C$
2. $\sin(y-x) = \frac{C}{2}e^x$
3. $\log_e(e^y - 1) - (2x+1)e^{-x} = C$
4. $(y+1)(1+e^{-x}) = ce^y$
5. $\tan^{-1}\frac{(4x+y+3)}{2} = 2x + C$
6. $(x+1)(2-e^y) = C$
7. $y\sin y = x^2\log x + C$
8. $x^2 + 2y^2 = 3\log(x^2 + y^2 + 2) + C$
9. $4e^y = 2e^{2x} + x^4 + C$
10. $\tan^{-1}(x-1) - \frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) = C$

10.2.2 Type II Homogeneous Equation

The general differential equation is $\dfrac{dy}{dx} = f(x, y)$ (1)

If this equation can be rewritten as $\dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree, then the differential equation is said to be homogenous.

Definition 10.7

A function $f(x, y)$ is said to be a homogeneous function of degree n if $f(tx, ty) = t^n f(x, y)$ for any $t > 0$.

Solution of Homogeneous differential equation

Let the homogeneous differential equation be

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \qquad (1)$$

To find the solution, put $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{du}{dx}$$

Substituting in (1), it reduces to variable separable type. Hence, we find the solution as in type I.

WORKED EXAMPLES

EXAMPLE 1

Solve $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$.

Solution.

The given differential equation is $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$ (1)

Here $f_1(x, y) = x^2 y$ and $f_2(x, y) = x^3 + y^3$ are homogeneous functions of the same degree 3.

For, $f_1(tx, ty) = (tx)^2 \cdot ty = t^3 x^2 y = t^3 f_1(x, y)$

$f_2(tx, ty) = (tx)^3 + (ty)^3 = t^3 (x^3 + y^3) = t^3 f_2(x, y)$

$\therefore$ the given equation is homogeneous

Put $y = vx$ $\qquad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3 x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx^3}{x^3(1+v^3)} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v(1+v^3)}{1+v^3} = -\frac{v^4}{1+v^3}$$

$$\therefore \qquad \frac{1+v^3}{v^4}dv = -\frac{dx}{x} \quad \Rightarrow \quad \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\int\frac{dx}{x}$$

$$\int v^{-4}dv + \int\frac{1}{v}dv = -\int\frac{dx}{x}$$

$$\Rightarrow \qquad \frac{v^{-3}}{-3} + \log_e v = -\log_e x + C$$

$$\Rightarrow \qquad -\frac{1}{3v^3} + \log_e v + \log_e x = C$$

$$\Rightarrow \qquad -\frac{1}{3\left(\frac{y}{x}\right)^3} + \log_e vx = C \quad \Rightarrow \quad -\frac{x^3}{3y^3} + \log_e y = C \quad \Rightarrow \quad 3y^3\log_e y - x^3 = 3Cy^3$$

which is the general solution of (1).

EXAMPLE 2

Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$.

Solution.

The given differential equation is

$$(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$$

$$\Rightarrow \qquad (x^2 - 4xy - 2y^2)dx = -(y^2 - 4xy - 2x^2)dy$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{(x^2 - 4xy - 2y^2)}{y^2 - 4xy - 2x^2}$$

Here $f_1(x, y) = x^2 - 4xy - 2y^2$ and $f_2(x, y) = y^2 - 4xy - 2x^2$ are homogeneous functions of the same degree 2.

$\therefore$ the given differential equation is homogeneous.

$\therefore$ to solve, put $y = vx$ $\qquad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting in (1), we get

$$v + x\frac{dv}{dx} = -\frac{(x^2 - 4x\cdot vx - 2v^2x^2)}{v^2x^2 - 4x\cdot vx - 2x^2}$$

$$= \frac{x^2(2v^2+4v-1)}{x^2(v^2-4v-2)} = \frac{2v^2+4v-1}{v^2-4v-2}$$

$$\therefore \qquad x\frac{dv}{dx} = \frac{2v^2+4v-1}{v^2-4v-2} - v$$

$$= \frac{2v^2 + 4v - 1 - v(v^2 - 4v - 2)}{v^2 - 4v - 2}$$

$$= \frac{2v^2 + 4v - 1 - v^3 + 4v^2 + 2v}{v^2 - 4v - 2}$$

$$= \frac{-v^3 + 6v^2 + 6v - 1}{v^2 - 4v - 2} = \frac{-(v^3 - 6v^2 - 6v + 1)}{v^2 - 4v - 2}$$

$\therefore$ $$\frac{v^2 - 4v - 2}{v^3 - 6v^2 - 6v + 1} dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{v^2 - 4v - 2}{v^3 - 6v^2 - 6v + 1} dv = -\int \frac{dx}{x}$$

Now $$\frac{d}{dv}(v^3 - 6v^2 - 6v + 1) = 3v^2 - 12v - 6 = 3(v^2 - 4v - 2)$$

$\therefore$ $$\frac{1}{3}\int \frac{3(v^2 - 4v - 2)}{v^3 - 6v^2 - 6v + 1} dv = -\int \frac{dx}{x}$$

$\Rightarrow$ $$\frac{1}{3}\log_e(v^3 - 6v^2 - 6v + 1) = -\log_e x + \log_e C = \log_e \frac{C}{x}$$

$\Rightarrow$ $$\log_e\left(\frac{y^3}{x^3} - 6\frac{y^2}{x^2} - 6\frac{y}{x} + 1\right) = 3\log_e \frac{C}{x}$$

$\Rightarrow$ $$\log_e\left(\frac{y^3 - 6xy^2 - 6x^2y + x^3}{x^3}\right) = \log_e \frac{C^3}{x^3}$$

$\Rightarrow$ $$\frac{y^3 - 6xy^2 - 6x^2y + x^3}{x^3} = \frac{C^3}{x^3} \quad \Rightarrow \quad y^3 - 6xy^2 - 6x^2y + x^3 = c^3$$

which is the general solution of (1).

EXAMPLE 3

Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$

Solution.

The given differential equation is $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$ (1)

$$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy \quad \Rightarrow \quad \frac{dx}{dy} = -\frac{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}}$$

Put $\frac{x}{y} = u \Rightarrow x = uy \quad \therefore \quad \frac{dx}{dy} = u + y\frac{du}{dy}$

$$\therefore \quad u + y\frac{du}{dy} = -\frac{e^u(1-u)}{1+e^u}$$

$$\Rightarrow \quad y\frac{du}{dy} = -\frac{e^u(1-u)}{1+e^u} - u$$

$$= -\frac{\left[e^u(1-u)+u(1+e^u)\right]}{1+e^u}$$

$$= -\frac{\left[e^u - ue^u + u + ue^u\right]}{1+e^u} = -\frac{(u+e^u)}{1+e^u}$$

$$\Rightarrow \quad \frac{1+e^u}{u+e^u}du = -\frac{dy}{y}$$

Integrating both sides

$$\int \frac{1+e^u}{u+e^u}du = -\int \frac{dy}{y}$$

$$\Rightarrow \quad \log_e(u+e^u) = -\log_e y + \log_e C \qquad \left[\because \frac{d}{du}(u+e^u) = 1+e^u\right]$$

$$\Rightarrow \quad \log_e(u+e^u) = \log\frac{C}{y}$$

$$\Rightarrow \quad u+e^u = \frac{C}{y} \quad \Rightarrow \quad \frac{x}{y} + e^{\frac{x}{y}} = \frac{C}{y} \quad \Rightarrow \quad x + ye^{\frac{x}{y}} = C$$

which is the general solution of (1).

EXERCISE 10.3

Solve the following homogeneous equations

1. $\frac{dy}{dx} = \frac{x-y}{x+y}$.
2. $(x^2+y^2)\frac{dy}{dx} = xy$.
3. $x(y-x)dy = y(y+x)dy$.
4. $(xy-2y^2)dx-(x^2-3xy)dy = 0$.
5. $(x^2y-2xy^2)dx-(x^3-3x^2y)dy = 0$.
6. $ydx - xdy = \sqrt{x^2+y^2}\,dx$.
7. $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}dy = 0$.
8. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$.
9. $x^2\frac{dy}{dx} = y^2 + 2xy$, given that y = 1, when x = 1.
10. $(y^2-2xy)dx = (x^2-2xy)dy$.
11. $(x^2+y^2)dx - 2xydy = 0$.

ANSWERS TO EXERCISE 10.3

1. $y^2 - x^2 + 2xy = C$
2. $\log_e Cy = \dfrac{x^2}{2y^2}$
3. $y = x \log_e C\,xy$
4. $y^3 = Cx^2 \cdot e^{-\frac{x}{y}}$
5. $\dfrac{x}{y} + 3\log_e \dfrac{y}{x} + \log x = C$
6. $y + \sqrt{x^2 + y^2} = C$
7. $x \tan\left(\dfrac{y}{x}\right) = C$
8. $x\left(1 + \cos\dfrac{y}{x}\right) = \sin\dfrac{y}{x}$
9. $2y = x(x + y)$
10. $xy(y - x) = C$
11. $x^2 - y^2 = Cx$.

10.2.3 Type III Non-Homogenous Differential Equations of the First Degree

Let us consider the non-homogeneous differential equation $\dfrac{dy}{dx} = \dfrac{ax + by + c}{Ax + By + C}$ (1)

where a, b, c, A, B, C are constants

Case (i) Let $\dfrac{a}{A} \neq \dfrac{b}{B}$

We shall reduce (1) to a homogenous differential equation by the transformation

$$x = X + h \quad \text{and} \quad y = Y + k$$

This transformation represents the shifting of the origin to the point (h, k),

Since $dx = dX$ and $dy = dY$. $\therefore \dfrac{dy}{dx} = \dfrac{dY}{dX}$

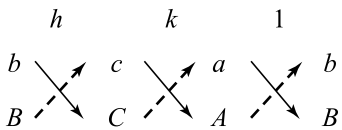
Then equation (1) becomes

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{A(X + h) + B(Y + k) + C}$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + C} \qquad (2)$$

We shall reduce the equation (2) into a homogeneous differential equation.
For this, choose h, k such that $ah + bk + c = 0$ and $Ah + Bk + C = 0$
By the rule of cross multiplication

$$\frac{h}{bC - Bc} = \frac{k}{cA - aC} = \frac{1}{aB - bA}$$



$$\therefore \quad h = \frac{bC - Bc}{aB - bA} \quad \text{and} \quad k = \frac{cA - aC}{aB - bA}$$

$$\therefore \quad \frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

which is a homogeneous differential equation in X and Y

It can be solved by type II by putting $Y = vX$.

Case (ii) Let $\frac{a}{A} = \frac{b}{B} = K(\text{say}) \quad \Rightarrow \quad a = AK,\ b = BK$

$\therefore$ the equation (1) becomes

$$\frac{dy}{dx} = \frac{AKx + BKy + c}{Ax + By + C} = \frac{K(Ax + By) + c}{Ax + By + C}$$

Put $\quad u = Ax + By \quad \therefore \frac{du}{dx} = A + B\frac{dy}{dx}$ and $\frac{dy}{dx} = \frac{Ku + c}{u + C}$

$$\therefore \quad \frac{du}{dx} = A + B\frac{(Ku + c)}{u + C}$$

$$= \frac{A(u + C) + B(Ku + c)}{u + C} = \frac{(A + BK)u + AC + Bc}{u + C}$$

$$\Rightarrow \quad \frac{u + C}{(A + BK)u + AC + Bc} \cdot du = dx$$

Integrating this, we get the solution.

Note Some Equations of type (1) can also be solved by grouping the terms.

Now $$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

$$\Rightarrow \quad (Ax + By + C)dy = (ax + by + c)dx$$

$$\Rightarrow \quad (By + C)dy + Axdy = (ax + c)dx + bydx$$

$$\Rightarrow \quad (By + C)dy - (ax + c)dx + Axdy - bydx = 0$$

Suppose $A = -b$, then we get

$$(By + C)dy - (ax + c)dx + Axdy + Aydx = 0$$

$$\Rightarrow \quad (By + C)dy - (ax + c)dx + A(xdy + ydx) = 0$$

$$\Rightarrow \quad (By + C)dy - (ax + c)dx + Ad(xy) = 0$$

Integrating, we get

$$\int (By + C)dy - \int (ax + c)dx + A\int d(xy) = 0$$

$$\Rightarrow \quad B\frac{y^2}{2} + Cy - \left(a\frac{x^2}{2} + cx\right) + Axy = K$$

where K is an arbitrary constant. This is solution of (1).

Thus, the grouping method is used if $A = -b$.

WORKED EXAMPLES

EXAMPLE 1

Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$.

Solution.

The given differential equation is $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ (1)

Here $\frac{a}{A} = \frac{1}{2}, \quad \frac{b}{B} = 2 \quad \therefore \quad \frac{a}{A} \neq \frac{b}{B}$

$\therefore$ put $x = X + h, y = Y + k \quad \therefore \quad dx = dX, dy = dY$ and $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ equation (1) becomes

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3} \quad \Rightarrow \quad \frac{dY}{dX} = \frac{X+2Y+h+2k-3}{2X+Y+2h+k-3} \tag{2}$$

Choose h and k such that

$h + 2k - 3 = 0$ (3) and $2h + k - 3 = 0$ (4)

(3) × 2 ⇒ $2h + 4k - 6 = 0$ (5)

(5) − (4) ⇒ $3k - 3 = 0 \Rightarrow 3k = 3 \Rightarrow k = 1$

$\therefore \quad h + 2 - 3 = 0 \Rightarrow h = 1$

$\therefore \quad x = X + 1$ and $y = Y + 1 \Rightarrow X = x - 1$ and $Y = y - 1$

$\therefore$(2) becomes $\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$

It is a homogeneous differential equation in X and Y

$\therefore$ Put $Y = vX \quad \therefore \quad \frac{dY}{dX} = v + X\frac{dv}{dX}$

$$\therefore \quad v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{X(1+2v)}{X(2+v)} = \frac{1+2v}{2+v}$$

$$\Rightarrow \quad X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$= \frac{1+2v-v(2+v)}{2+v} = \frac{1+2v-2v-v^2}{2+v} = \frac{1-v^2}{2+v}$$

$$\Rightarrow \quad X\frac{dv}{dX} = -\frac{(v^2-1)}{v+2}$$

Separating the variables, we get

$$\Rightarrow \quad \frac{v+2}{v^2-1}dv = -\frac{dX}{X}$$

Integrating both sides, we get

$$\int \frac{v+2}{v^2-1}dv = -\int \frac{dX}{X}$$

$$\Rightarrow \quad \int \frac{v}{v^2-1}dv + 2\int \frac{dv}{v^2-1} = -\int \frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\log_e(v^2-1) + 2\frac{1}{2}\log\frac{v-1}{v+1} = -\log_e X + \log_e C$$

$$\Rightarrow \quad \log_e(v^2-1) + 2\log\frac{v-1}{v+1} = -2\log_e X + 2\log_e C$$

$$\Rightarrow \quad \log_e(v^2-1) + \log\frac{(v-1)^2}{(v+1)^2} = -\log_e X^2 + \log_e C^2$$

$$\Rightarrow \quad \log_e(v^2-1)\cdot\frac{(v-1)^2}{(v+1)^2} = \log_e\frac{C^2}{X^2}$$

$$\Rightarrow \quad (v^2-1)\frac{(v-1)^2}{(v+1)^2} = \frac{C^2}{X^2}$$

$$\Rightarrow \quad (v+1)(v-1)\frac{(v-1)^2}{(v+1)^2} = \frac{C^2}{X^2}$$

$$\Rightarrow \quad \frac{(v-1)^3}{v+1} = \frac{C^2}{X^2}$$

$$\Rightarrow \quad \frac{\left(\frac{Y}{X}-1\right)^3}{\left(\frac{Y}{X}+1\right)} = \frac{C^2}{X^2}$$

$$\Rightarrow \quad \frac{(Y-X)^3}{X^2(Y+X)} = \frac{C^2}{X^2}$$

$$\Rightarrow \quad (Y-X)^3 = C^2(Y+X)$$

$$\Rightarrow \quad [y-1-(x-1)]^3 = C^2(y-1+x-1) \quad \Rightarrow \quad (y-x)^3 = C^2(y+x-2)$$

which is the solution of the equation (1).

EXAMPLE 2

Solve $\frac{dy}{dx} = \frac{4x-3y-1}{3x+4y-7}$.

Solution.

Given $\frac{dy}{dx} = \frac{4x-3y-1}{3x+4y-7}$ Here $A = 3,\ b = -3$ $\therefore$ $A = -b$

$\therefore$ we can find the solution by grouping method.

$$(3x+4y-7)dy = (4x-3y-1)dx$$

$$\Rightarrow \quad (4y-7)dy + 3xdy = (4x-1)dx - 3ydx$$

$$\Rightarrow \quad (4y-7)dy - (4x-1)dx + 3xdy + 3ydx = 0$$

$$\Rightarrow \quad (4y-7)dy - (4x-1)dx + 3d(xy) = 0$$

Integrating, we get

$$\int (4y-7)dy - \int (4x-1)dx + 3\int d(xy) = 0$$

$$4\frac{y^2}{2} - 7y - \left(4\frac{x^2}{2} - x\right) + 3xy = C$$

$$\Rightarrow \quad 2y^2 - 7y - 2x^2 + x + 3xy = C \quad \Rightarrow \quad 2(x^2 - y^2) - 3xy - x + 7y + C = 0$$

which is the solution of (1).

EXAMPLE 3

Solve $(2x + y + 1)dy = (x + y + 1)dx$.

Solution.

The given differential equation is $(2x + y + 1)dy = (x + y + 1)dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y+1}{2x+2y+1} \tag{1}$$

Here $\frac{a}{A} = \frac{1}{2}, \frac{b}{B} = \frac{1}{2}$ $\therefore$ $\frac{a}{A} = \frac{b}{B}$ [Case (ii)]

$$\therefore \quad \frac{dy}{dx} = \frac{x+y+1}{2(x+y)+1} \tag{2}$$

Put $u = x + y$

$$\therefore \quad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

and (2) $\Rightarrow$ $$\frac{dy}{dx} = \frac{u+1}{2u+1}$$

$$\therefore \quad \frac{du}{dx} = 1 + \frac{u+1}{2u+1} = \frac{2u+1+u+1}{2u+1} = \frac{3u+2}{2u+1}$$

$$\Rightarrow \quad \frac{2u+1}{3u+2}du = dx$$

Integrating both sides, we get

$$\int \frac{2u+1}{3u+2}\,du = \int dx$$

$$\Rightarrow \qquad \int \frac{\frac{2}{3}(3u+2)-\frac{1}{3}}{3u+2}\,du = \int dx$$

$$\Rightarrow \qquad \int \frac{2}{3}\,du - \frac{1}{3}\int \frac{du}{3u+2} = \int dx$$

$$\Rightarrow \qquad \frac{2}{3}u - \frac{1}{9}\log(3u+2) = x + C$$

$$\Rightarrow \qquad \frac{2}{3}(x+y) - \frac{1}{9}\log\left(3(x+y)+2\right) = x + C \quad \Rightarrow \quad \frac{2}{3}y - \frac{1}{9}\log(3x+3y+2) = \frac{1}{3}x + C$$

EXAMPLE 4

Solve $(2x^2 + 3y^2 - 7)x\,dx = (3x^2 + 2y^2 - 8)y\,dy$.

Solution.

The given differential equation is $(2x^2 + 3y^2 - 7)x\,dx = (3x^2 + 2y^2 - 8)y\,dy$

$$\Rightarrow \qquad \frac{y}{x}\frac{dy}{dx} = \frac{(2x^2+3y^2-7)}{(3x^2+2y^2-8)} \qquad (1)$$

Put $x^2 = X, y^2 = Y$ $\quad\therefore\quad$ $2xdx = dX,$ and $2ydy = dY$

$$\therefore \qquad \frac{2y}{2x}\frac{dy}{dx} = \frac{dY}{dX} \quad \Rightarrow \quad \frac{y}{x}\frac{dy}{dx} = \frac{dY}{dX}$$

$\therefore$ equation (1) becomes,

$$\frac{dY}{dX} = \frac{2X+3Y-7}{3X+2Y-8} \qquad (2)$$

This is non homogeneous type.

Here $\frac{a}{A} = \frac{2}{3}$, $\frac{b}{B} = \frac{3}{2}$ $\quad\therefore\quad$ $\frac{a}{A} \neq \frac{b}{B}$

$\therefore$ put $\quad X = X' + h, \quad Y = Y' + k$

$\therefore \quad dX = dX', \quad dY = dY'$ and $\frac{dY}{dX} = \frac{dY'}{dX'}$

$\therefore$ the equation (2) becomes

$$\frac{dY'}{dX'} = \frac{2(X'+h)+3(Y'+k)-7}{3(X'+h)+2(Y'+k)-8} = \frac{2X'+3Y'+2h-3k-7}{3X'+2Y'+3h+2k-8}$$

Choose h, k such that

$$2h + 3k - 7 = 0 \quad \text{and} \quad 3h + 2k - 8 = 0$$

By the rule of cross multiplication

h		k		1	
3	-7	2	3		
2	-8	3	2		

$$\frac{h}{-24+14}=\frac{k}{-21+16}=\frac{1}{4-9}$$

$$\Rightarrow \quad \frac{h}{-10}=\frac{k}{-5}=\frac{1}{-5} \Rightarrow \frac{h}{2}=\frac{k}{1}=1 \Rightarrow h=2, k=1$$

$$\therefore \quad X=X'+2 \Rightarrow X'=X-2 \quad \text{and} \quad Y=Y'+1 \Rightarrow Y'=Y-1$$

$$\therefore \quad \frac{dY'}{dX'}=\frac{2X'+3Y'}{3X'+2Y'}$$

It is homogeneous differential equation in X' and Y'

To solve, put $Y'=vX'$

$$\therefore \quad \frac{dY'}{dX'}=v+X'\frac{dv}{dX'}$$

$$\therefore \quad v+X'\frac{dv}{dX'}=\frac{2X'+3vX'}{3X'+2vX'}=\frac{X'(2+3v)}{X'(3+2v)}=\frac{2+3v}{3+2v}$$

$$\Rightarrow \quad X'\frac{dv}{dX'}=\frac{2+3v}{3+2v}-v=\frac{2+3v-v(3+2v)}{3+2v}=\frac{2+3v-3v-2v^2}{3+2v}$$

$$\Rightarrow \quad X'\frac{dv}{dX'}=\frac{2(1-v^2)}{3+2v} \Rightarrow \left(\frac{3+2v}{1-v^2}\right)dv=2\frac{dX'}{X'}$$

Integrating both sides, we get

$$\int\left(\frac{3+2v}{1-v^2}\right)dv=2\int\frac{dX'}{X'}$$

$$\Rightarrow \quad \int\frac{3}{1-v^2}dv+\int\frac{2v\,dv}{1-v^2}=2\int\frac{dX'}{X'}$$

$$\Rightarrow \quad 3\cdot\frac{1}{2}\log_e\frac{1+v}{1-v}-\log_e(1-v^2)=2\log_e X'+\log_e C$$

$$\Rightarrow \quad \log_e\left(\frac{1+v}{1-v}\right)^{\frac{3}{2}}-\log_e(1-v^2)=\log_e(X')^2+\log_e C$$

$$\Rightarrow \quad \log_e\left[\left(\frac{1+v}{1-v}\right)^{\frac{3}{2}}\cdot\frac{1}{(1-v^2)}\right]=\log_e C(X')^2$$

$$\Rightarrow \quad \left(\frac{1+v}{1-v}\right)^{\frac{3}{2}}\cdot\frac{1}{1-v^2}=C(X')^2$$

$$\Rightarrow \quad \frac{(1+v)^{\frac{3}{2}}}{(1-v)^{\frac{3}{2}}}\cdot\frac{1}{(1+v)(1-v)}=C(X')^2 \Rightarrow \frac{(1+v)^{\frac{1}{2}}}{(1-v)^{\frac{5}{2}}}=C(X')^2$$

Squaring,

$$\frac{(1+v)}{(1-v)^5} = C^2(X')^4 \quad \Rightarrow \quad \frac{1+\frac{Y'}{X'}}{\left(1-\frac{Y'}{X'}\right)^5} = C^2(X')^4$$

$$\Rightarrow \qquad \frac{(X'+Y')(X')^4}{(X'-Y')^5} = C^2(X')^4$$

$$\Rightarrow \qquad \frac{X'+Y'}{(X'-Y')^5} = C^2$$

$$\Rightarrow \qquad X-2+Y-1 = C^2\left[X-2-(Y-1)\right]^5$$

$$\Rightarrow \qquad X+Y-3 = C^2\left[X-Y-1\right]^5 \quad \Rightarrow \quad x^2+y^2-3 = C^2(x^2-y^2-1)^5$$

Which is the general solution of (1)

EXERCISE 10.4

Solve the following non-homogeneous differential equations

1. $\dfrac{dy}{dx} = \dfrac{2x+5y+1}{5x+2y-1}$.
2. $\dfrac{dy}{dx} + \dfrac{10x+8y-12}{7x+5y-9} = 0$.
3. $\dfrac{dy}{dx} = \dfrac{6x+5y-7}{2x+18y-14}$.
4. $\dfrac{dy}{dx} + \dfrac{12x+58y-9}{5x+2y-4} = 0$.
5. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$.
6. $\dfrac{dy}{dx} = \dfrac{x-y-1}{y-x-1}$.
7. $\dfrac{dy}{dx} = \dfrac{2x+3y+4}{4x+6y+5}$.
8. $ydy(x^2+y^2+a^2)+xdx(x^2+y^2-a^2)=0$.

ANSWERS TO EXERCISE 10.4

1. $(x+y)^7 = C\left(x-y-\frac{2}{3}\right)^3$
2. $(x+y-1)^2(y+2x-3)^3 = C$
3. $(2x-3y+1)^2(x+2y-2) = C$
4. $6x^2-9x+5xy+y^2-4y = C$
5. $(x-y)+\log(x+y) = C$
6. $\log(x-y) = x+y+C$
7. $14(x+2y)-9\log(14x+21y+22)-7x = C$
8. $(x^2+y^2)(x^2+y^2+a^4) = 2a^4x^2+a^2C$

10.2.4 Type IV Linear Differential Equation

The general form of the first order linear differential equation in the dependent variable y is

$$a_0(x)\frac{dy}{dx}+a_1(x)y=a_2(x) \qquad (1)$$

where $a_0(x)\neq 0$

Dividing by $a_0(x)$, we get

$$\frac{dy}{dx}+\frac{a_1(x)}{a_0(x)}y=\frac{a_2(x)}{a_0(x)}$$

$$\Rightarrow \qquad \frac{dy}{dx}+P(x)y=Q(x) \qquad (2)$$

where $P(x)=\dfrac{a_1(x)}{a_0(x)}$ and $Q(x)=\dfrac{a_2(x)}{a_0(x)}$

The equation (2) is the **standard form** of the linear differential equation in y or **Leibnitz's linear** equation.

Solution of the linear differential equation (2)

Equation (2) is $$\frac{dy}{dx}+Py=Q$$

Multiply (2) by $e^{\int Pdx}$, we get

$$e^{\int Pdx}\frac{dy}{dx}+e^{\int Pdx}py=Qe^{\int Pdx}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(y\cdot e^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

Integrating w. r. to x, we get

$$y\cdot e^{\int Pdx}=\int Q\cdot e^{\int Pdx}dx+C$$

This is the general solution of (2).

Note (1) When the equation (2) is multiplied by $e^{\int Pdx}$, the L. H. S becomes an exact differential and so $e^{\int Pdx}$ is called an **integrating factor of the equation (2)**.

(2) But $e^{\int Pdx}$ is not an integrating factor of equation (1).

An integrating factor of (1) is $\dfrac{1}{a_0(x)}e^{\int Pdx}$. But solutions of (1) and (2) are same.

(3) Sometimes, it is convenient to rewrite the equation as $\dfrac{dx}{dy}+px=Q$, where P and Q are functions of y, which is linear in x.

Its solution is $$x\, e^{\int Pdy}=\int Q\, e^{\int Pdy}dy+C$$

WORKED EXAMPLES

EXAMPLE 1

Solve the differential equation $(1-x^2)\dfrac{dy}{dx} - xy = 1$.

Solution.

The given differential equation is $(1-x^2)\dfrac{dy}{dx} - xy = 1$

Dividing by $(1-x^2)$, we get

$$\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

This is linear in y. Here $P = -\dfrac{x}{1-x^2}$ and $Q = \dfrac{x}{1-x^2}$.

$\therefore$ the general solution is $$y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

Now $$\int P dx = \int -\frac{x}{1-x^2}\, dx = \frac{1}{2}\log_e(1-x^2) = \log_e \sqrt{1-x^2}$$

$\therefore$ $$e^{\int P dx} = e^{\log_e \sqrt{1-x^2}} = \sqrt{1-x^2} \qquad [\because e^{\log x} = x]$$

and $$\int Q \cdot e^{\int P dx} dx = \int \frac{1}{1-x^2}\sqrt{1-x^2}\, dx = \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$$

$\therefore$ the solution is $$y\sqrt{1-x^2} = \sin^{-1} x + C$$

EXAMPLE 2

The gradient of a curve which passes through the point (4, 0) is defined by the equation $\dfrac{dy}{dx} - \dfrac{y}{x} + \dfrac{5x}{(x+2)(x-3)} = 0$. Find the equation of the curve and find the value of y when $x = 5$.

Solution.

The given differential equation is $\dfrac{dy}{dx} - \dfrac{y}{x} + \dfrac{5x}{(x+2)(x-3)} = 0$

$\Rightarrow$ $$\frac{dy}{dx} - \frac{1}{x}y = -\frac{5x}{(x+2)(x-3)}$$

This is linear in y.

Here $P = -\dfrac{1}{x}$ and $Q = -\dfrac{5x}{(x+2)(x-3)}$.

$\therefore$ the solution is $$y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

Now $$\int P dx = \int -\frac{1}{x}\, dx = -\log_e x = \log_e x^{-1} = \log_e \frac{1}{x}$$

$$\therefore \qquad e^{\int P dx} = e^{\log_e \frac{1}{x}} = \frac{1}{x}$$

and $$\int Q \cdot e^{\int P dx} dx = -\int \frac{5x}{(x+2)(x-3)} \frac{1}{x}\, dx = -5\int \frac{1}{(x+2)(x-3)} dx$$

Let $$\frac{1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$\Rightarrow \qquad 1 = A(x-3) + B(x+2)$$

Put $x = -2$, then $1 = A(-2-3) = -5A \quad \Rightarrow \quad A = -\frac{1}{5}$

Put $x = 3$, then $1 = B(3+2) = 5B \quad \Rightarrow \quad B = \frac{1}{5}$

$$\therefore \qquad \frac{1}{(x+2)(x-3)} = -\frac{1}{5(x+2)} + \frac{1}{5(x-3)}$$

$$\therefore \qquad \int \frac{dx}{(x+2)(x-3)} = -\frac{1}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{dx}{x-3}$$

$$= -\frac{1}{5}\log_e (x+2) + \frac{1}{5}\log_e (x-3)$$

$$= \frac{1}{5}\{\log_e (x-3) - \log_e (x+2)\} = \frac{1}{5}\log_e \left(\frac{x-3}{x+2}\right)$$

$$\therefore \qquad \int Q e^{\int P dx} dx = -5 \cdot \frac{1}{5}\log_e \left(\frac{x-3}{x+2}\right) = -\log_e \left(\frac{x-3}{x+2}\right)$$

$\therefore$ the solution is $$y \cdot \frac{1}{x} = -\log_e \left(\frac{x-3}{x+2}\right) + C$$

The curve passes through (4, 0). So, $x = 4$ and $y = 0$.

$$\therefore \qquad 0 = -\log_e \left(\frac{4-3}{4+2}\right) + C \quad \Rightarrow \quad C = \log_e \frac{1}{6}$$

$\therefore$ the solution is $$\frac{y}{x} = -\log_e \left(\frac{x-3}{x+2}\right) + \log \frac{1}{6}$$

$$\Rightarrow \qquad \frac{y}{x} = \log_e \frac{\frac{1}{6}}{\frac{x-3}{x+2}} = \log_e \left[\frac{x+2}{6(x-3)}\right]$$

$$y = x \log_e \left[\frac{x+2}{6(x-3)}\right]$$

which is the equation of the curve.

When $x = 5$, $$y = 5\log_e \left[\frac{5+2}{6(5-3)}\right] = 5\log_e \frac{7}{12} \simeq -4.377.$$

For practical purposes, the approximate value of y is –4.38.

EXAMPLE 3

Solve $2\cos x\dfrac{dy}{dx} + 4\sin x \cdot y - \sin 2x = 0$ given that $y = 0$ when $x = \dfrac{\pi}{3}$. Further find the maximum value of y.

Solution.
The given differential equation is

$$2\cos x\frac{dy}{dx} + 4\sin x \cdot y = \sin 2x$$

Dividing by $2\cos x$, we get

$$\Rightarrow \quad \frac{dy}{dx} + 2\frac{\sin x}{\cos x}y = \frac{\sin 2x}{2\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} + 2\frac{\sin x}{\cos x}y = \frac{2\sin x\cos x}{2\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} + 2\tan x\ y = \sin x.$$

This is linear in y. Here $P = 2\tan x$ and $Q = \sin x$

$\therefore$ the general solution is $\quad y\, e^{\int Pdx} = \int Q\, e^{\int Pdx}\, dx + C$

Now $\quad \int Pdx = \int 2\tan x\, dx = 2\log\sec x = \log_e \sec^2 x$

$\therefore \quad e^{\int Pdx} = e^{\log_e \sec^2 x} = \sec^2 x$

and $\quad \int Q\, e^{\int Pdx}\, dx = \int \sec^2 x \sin x\, dx$

$$= -\int (\cos x)^{-2}(-\sin x)\, dx = \frac{-(\cos x)^{-1}}{-1} = \frac{1}{\cos x} = \sec x$$

$\therefore$ the general solution is

$$y\sec^2 x = \sec x + C$$

$$\Rightarrow \quad y = \frac{1}{\sec x} + \frac{C}{\sec^2 x} = \cos x + C\cos^2 x$$

When $x = \dfrac{\pi}{3}$, $y = 0$, we get

$$0 = \cos\frac{\pi}{3} + C\cdot\cos^2\frac{\pi}{3} = \frac{1}{2} + C\cdot\frac{1}{4} \quad \Rightarrow \quad \frac{1}{4}C = -\frac{1}{2} \Rightarrow C = -2$$

$\therefore$ the solution is $\quad y = \cos x - 2\cos^2 x \qquad (1)$

To find the maximum value

$$y = \cos x - 2\cos^2 x$$

$$\therefore \quad \frac{dy}{dx} = -\sin x - 2\cdot 2\cos x(-\sin x)$$

$$= -\sin x + 2\cdot 2\sin x\cos x = -\sin x + 2\sin 2x$$

and $$\frac{d^2y}{dx^2} = -\cos x + 2\cdot 2\cos 2x \quad = -\cos x + 4(2\cos^2 x - 1)$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$\Rightarrow \quad -\sin x + 2\sin 2x = 0 \quad \Rightarrow \quad -\sin x + 4\sin x\cos x = 0 \quad \Rightarrow \quad \sin x(-1 + 4\cos x) = 0$

$\Rightarrow \quad \sin x = 0 \quad \Rightarrow x = 0, \pi$ and $4\cos x - 1 = 0 \quad \Rightarrow \quad \cos x = \frac{1}{4}$

$\therefore$ when $\sin x = 0$, $\quad \cos x = \pm 1$

When $\cos x = 1$, $\quad \frac{d^2y}{dx^2} = -1 + 4(2\cdot 1 - 1) = -1 + 4 = 3 > 0$

and $\cos x = -1$, $\quad \frac{d^2y}{dx^2} = -(-1) + 4(2\cdot 1 - 1) = 1 + 4 = 5 > 0$

$\therefore$ when $\cos x = \pm 1$, y is minimum,

When $\cos x = \frac{1}{4}$, $\quad \frac{d^2y}{dx^2} = -\frac{1}{4} + 4\left(2\cdot\frac{1}{16} - 1\right) = -\frac{1}{4} + \frac{1}{2} - 4 = -\frac{15}{4} < 0$

$\therefore$ when $\cos x = \frac{1}{4}$, $\quad y$ is maximum.

Maximum value of $y = \frac{1}{4} - 2\cdot\frac{1}{16} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.

EXAMPLE 4

Solve $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution.

The given differential equation is $(1 + y^2)dx = (\tan^{-1} y - x)dy$

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} + x = \tan^{-1} y$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1 + y^2}x = \frac{\tan^{-1} y}{1 + y^2}$$

This is linear in x. Here $P = \frac{1}{1 + y^2}$ and $Q = \frac{\tan^{-1} y}{1 + y^2}$

$\therefore$ the solution is

$$x\, e^{\int P dy} = \int Q\, e^{\int P dy}\, dy + C$$

Now $\int P dy = \int \frac{1}{1+y^2} dy = \tan^{-1} y$ $\quad \therefore \quad e^{\int P dy} = e^{\tan^{-1} y}$

and $\int Q e^{\int P dy} dy = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} dy$

Put $t = \tan^{-1} y$ $\quad \therefore dt = \frac{1}{1+y^2} dy$

$\therefore \quad \int Q e^{\int P dy} dy = \int t e^t \, dt = t e^t - 1 \cdot e^t = e^t (t-1) = e^{\tan^{-1} y} (\tan^{-1} y - 1)$

$\therefore$ the general solution is

$$x e^{\tan^{-1} y} = (\tan^{-1} y - 1) e^{\tan^{-1} y} + C \quad \Rightarrow \quad x = \tan^{-1} y - 1 + C e^{-\tan^{-1} y}$$

EXERCISE 10.5

Solve the following linear equation

1. $\frac{1}{x}\frac{dy}{dx} + \frac{y}{x}\tan x = \cos x.$
2. $\frac{dy}{dx} - \sin 2x = y \cot x.$
3. $\frac{dy}{dx} + \frac{3x^2}{x^3+5} y = \frac{\cos^2 x}{x^3+5}.$
4. $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}.$
5. $(1+x^2)\tan^{-1} x \frac{dy}{dx} + y = x$, given that $y = -1$ when $x = -1$.
6. $x(x^2-1)\frac{dy}{dx} + (4x^2-2)y + 5x^3 = 0.$
7. $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x.$
8. $\frac{dy}{dx} + \frac{3x^2 y}{1+x^3} = \frac{1+x^2}{1+x^3}.$
9. $(x+2y^3)\frac{dy}{dx} = y.$
10. $\frac{dy}{dx} - 3y\cot x = \sin 2x$ and $y = 2$ when $x = \frac{\pi}{2}$. Show that the minimum value of y is in the range $0 \le x \le \pi$ is $-\frac{2}{27}$.

ANSWERS TO EXERCISE 10.5

1. $y = C\cos x + \frac{x^2}{2}\cos x$
2. $y = C\sin x + 2\sin^2 x$
3. $y(x^3+5) = C + \frac{x}{2} + \frac{\sin 2x}{4}$
4. $y(x^2+1)^2 = x + C$
5. $y\tan^{-1} x = \frac{\pi}{4} - \frac{1}{2}\log 2 + \frac{1}{3}\log(1+x^2)$
6. $y\, x^2(x^2-1) = C - x^3$
7. $y = \sin x - 1 + C e^{-\sin x}$
8. $y(1+x^3) = \frac{x^3}{3} + x + C$
9. $x = y^3 + Cy$

10.2.5 Type V Bernoulli's Equation

The first order first degree differential equation of the form $\dfrac{dy}{dx} + Py = Qy^n$ (1)

where P and Q are function of x is known as **Bernoulli's Equation**.

When $n = 0$, it reduces to a linear differential equation when $n = 1$, it reduces to variable separable type. For any other value of n, the equation is non-linear.

In this case, the equation (1) can be re-written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \quad \Rightarrow \quad y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $z = y^{1-n}$ $\quad\therefore\quad \dfrac{dz}{dx} = (1-n)y^{1-n-1}\dfrac{dy}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

$$\Rightarrow \quad \frac{1}{(1-n)}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

Substituting in (2), we get

$$\frac{1}{(1-n)}\frac{dz}{dx} + Pz = Q \quad \Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is Leibnitz's linear equation in z and hence can be solved.

WORKED EXAMPLES

EXAMPLE 1

Solve $\cos x \dfrac{dy}{dx} - y \sin x = y^3 \cos^2 x$.

Solution.

The given differential equation is $\cos x \dfrac{dy}{dx} - y \sin x = y^3 \cos^2 x$

Dividing by $\cos x$, we get

$$\frac{dy}{dx} - y\frac{\sin x}{\cos x} = y^3 \cos x \quad \Rightarrow \quad \frac{dy}{dx} - \tan x \cdot y = y^3 \cos x \qquad (1)$$

This is Bernoulli's form.

Dividing by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} - \tan x \cdot \frac{1}{y^2} = \cos x \quad \Rightarrow \quad y^{-3}\frac{dy}{dx} - \tan x \cdot y^{-2} = \cos x \qquad (2)$$

Put $y^{-2} = z$ $\quad\therefore\quad -2y^{-3}\dfrac{dy}{dx} = \dfrac{dz}{dx} \quad \Rightarrow \quad y^{-3}\dfrac{dy}{dx} = -\dfrac{1}{2}\dfrac{dz}{dx}$

$\therefore$ the equation (2) becomes

$$-\frac{1}{2}\frac{dz}{dx} - \tan x \cdot z = \cos x \quad \Rightarrow \quad \frac{dz}{dx} + 2\tan x \cdot z = -2\cos x$$

This is Leibnitz's linear equation in z.

Here $P = 2\tan x,\ Q = -2\cos x$

$\therefore$ the general solution is

$$z\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

Now $$\int P dx = \int 2\tan x\, dx = 2\log_e \sec x = \log_e \sec^2 x$$

$\therefore$ $$e^{\int P dx} = e^{\log_e \sec^2 x} = \sec^2 x$$

and $$\int Q\, e^{\int P dx} dx = \int -2\cos x \cdot \sec^2 x\, dx$$

$$= -2\int \cos x \cdot \frac{1}{\cos^2 x}\, dx$$

$$= -\int \frac{dx}{\cos x} = -2\int \sec x\, dx = -2\log_e(\sec x + \tan x)$$

$\therefore$ the general solution is

$$z\sec^2 x = -2\log_e(\sec x + \tan x) + C$$

$\Rightarrow$ $$y^{-2}\cdot \sec^2 x = -2\log_e(\sec x + \tan x) + C$$

$\Rightarrow$ $$\frac{1}{y^2}\cdot \sec^2 x = -2\log_e(\sec x + \tan x) + C$$

EXAMPLE 2

Solve $\mathbf{2y\cos y^2 \dfrac{dy}{dx} - \dfrac{2}{x+1}\sin y^2 = (x+1)^3}$.

Solution.

The given differential equation is $\quad 2y\cos y^2 \dfrac{dy}{dx} - \dfrac{2}{x+1}\sin y^2 = (x+1)^3$ (1)

Put $\quad \sin y^2 = u \quad \therefore \quad \cos y^2 \cdot 2y\dfrac{dy}{dx} = \dfrac{du}{dx} \quad \Rightarrow \quad 2y\cos y^2 \dfrac{dy}{dx} = \dfrac{du}{dx}$

$\therefore$ the equation (1) becomes

$$\frac{du}{dx} - \frac{2}{x+1}u = (x+1)^3$$

This is linear equation in u. Here $\quad P = -\dfrac{2}{x+1} \quad$ and $\quad Q = (x+1)^3$

$\therefore$ the general solution is

$$u\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

Now $$\int P dx = \int -\frac{2}{x+1}\, dx = -2\log_e(x+1) = \log_e(x+1)^{-2} = \log_e \frac{1}{(x+1)^2}$$

$\therefore \qquad e^{\int P dx} = e^{\log \frac{1}{(x+1)^2}} = \frac{1}{(x+1)^2}$

and $\quad \int Q\, e^{\int P dx} dx = \int (x+1)^3 \cdot \frac{1}{(x+1)^2}\, dx = \int (x+1)\, dx = \frac{(x+1)^2}{2}$

$\therefore$ the general solution is

$$u \frac{1}{(x+1)^2} = \frac{(x+1)^2}{2} + C$$

$\Rightarrow \qquad \sin y^2 \cdot \frac{1}{(x+1)^2} = \frac{(x+1)^2}{2} + C \qquad \Rightarrow \qquad 2\sin y^2 = (x+1)^2[(x+1)^2 + 2C]$

EXAMPLE 3

Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution.

The given differential equation is $\qquad \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y \qquad$ (1)

Dividing by $\cos^2 y$, we get

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x \cdot \frac{2\sin y \cos y}{\cos^2 y} = x^3$$

$\Rightarrow \qquad \sec^2 y \frac{dy}{dx} + 2\tan y \cdot x = x^3$

Put $u = \tan y \qquad \therefore \frac{du}{dx} = \sec^2 y \frac{dy}{dx}$

$\therefore$ the equation (1) becomes

$$\frac{du}{dx} + 2x \cdot u = x^3$$

This is Leibnitz's linear equation in u. Here $P = 2x$ and $Q = x^3$

$\therefore$ the general solution is

$$u\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

Now $\qquad \int P dx = \int 2x\, dx = 2\frac{x^2}{2} = x^2 \qquad \therefore \quad e^{\int P dx} = e^{x^2}$

and $\qquad \int Q\, e^{\int P dx} dx = \int x^3 e^{x^2}\, dx$

Put $t = x^2 \qquad \therefore dt = 2x\, dx \quad \Rightarrow \quad \frac{1}{2} dt = x\, dx$

$\therefore \qquad \int x^3 e^{x^2}\, dx = \int t\, e^t \cdot \frac{1}{2} dt = \frac{1}{2}\left[t\, e^t - 1 \cdot e^t\right] = \frac{e^t}{2}(t-1) = \frac{1}{2} e^{x^2}(x^2 - 1)$

$\therefore$ the general solution is

$$u\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1)+C$$

$$\Rightarrow \qquad \tan y \cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1)+C \quad \Rightarrow \quad \tan y = \frac{1}{2}(x^2-1)+C\,e^{-x^2}$$

EXERCISE 10.6

Solve the following equations

1. $\dfrac{dy}{dx}+\dfrac{x}{1-x^2}y = 3\cdot y^{\frac{1}{2}}.$
2. $(x+2y^3)\dfrac{dy}{dx} = y.$
3. $\dfrac{dy}{dx}-2y\tan x = y^2\tan^2 x.$
4. $(1-x^2)\dfrac{dy}{dx}-xy = x^2y^2.$
5. $x\dfrac{dy}{dx}+y = y^2\log_e x.$
6. $3x(1-x^2)y^2\dfrac{dy}{dx}+(2x^2-1)y^3 = ax^3.$
7. $(x+1)\dfrac{dy}{dx}+1 = 2e^{-y}.$
8. $(y\log_e x-2)y\,dx - x\,dy = 0.$
9. $(1-x^2)\dfrac{dy}{dx}+xy = y^2\sin^{-1}x.$
10. $3\dfrac{dy}{dx}+y = e^{3x}y^4.$

ANSWERS TO EXERCISE 10.6

1. $y^{\frac{1}{2}} = C(1-x^2)^{\frac{1}{4}}-(1-x^2)$
2. $x = Cy+y^3$
3. $\dfrac{\sec^2 x}{y}+\dfrac{1}{3}\tan^3 x = C$
4. $\dfrac{1}{y} = \sqrt{1+x^2}(C+\sin^{-1}x)-x$
5. $\dfrac{1}{y} = Cx+\log x+1$
6. $y^3 = ax+cx\sqrt{1-x^2}$
7. $e^y(x+1) = 2x+C$
8. $\dfrac{1}{y} = \dfrac{1}{2}\log_e x+\dfrac{1}{4}+Cx^2$
9. $(1-x^2) = (C+2\cos x)y^2$
10. $y^{-3} = Ce^x-\dfrac{1}{2}e^{3x}$

10.2.6 Type VI Riccati Equation

We shall now consider a special type of first order differential equation known as **Riccati equation**, named after the Italian mathematician J.F Riccati who introduced this equation.

Definition 10.8 A first order differential equation of the form $\dfrac{dy}{dx} = P(x)y^2+Q(x)y+R(x)$ (1)

is called a **Riccati equation**.
If $P(x) = 0$, then the Riccati equation reduces to the first order **linear equation**.

If $R(x) = 0$, then the Riccati equation reduces to the **Bernoulli's equation**.
When $R(x) \neq 0$, the equation (1) cannot be solved by elementary methods.
But if we can find a solution $u(x)$ of (1) by inspection or otherwise, then by change of variables

$$y = u(x) + \frac{1}{z(x)} \tag{2}$$

we can reduce the Riccati equation to a linear equation and hence, it can be solved.

Differentiating (2) w.r.to x, we get $\frac{dy}{dx} = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx}$

Substituting in (1), we get

$$\frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} = P\left(u + \frac{1}{z}\right)^2 + Q\left(u + \frac{1}{z}\right) + R$$

$$= P\left(u^2 + \frac{2u}{z} + \frac{1}{z^2}\right) + Qu + \frac{Q}{z} + R$$

$$= \left(Pu^2 + Qu + R\right) + P\left(\frac{2u}{z} + \frac{1}{z^2}\right) + \frac{Q}{z} \tag{3}$$

But u is a solution of (1).

$\therefore$ $$\frac{du}{dx} = Pu^2 + Qu + R$$

$\therefore$ (3) $\Rightarrow$ $$\frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} = \frac{du}{dx} + P\left(\frac{2u}{z} + \frac{1}{z^2}\right) + \frac{Q}{z}$$

$\Rightarrow$ $$-\frac{1}{z^2}\frac{dz}{dx} = P\left(\frac{2u}{z} + \frac{1}{z^2}\right) + \frac{Q}{z}$$

$\Rightarrow$ $$-\frac{1}{z^2}\frac{dz}{dx} = \frac{1}{z^2}[P(2uz+1) + Qz]$$

$\Rightarrow$ $$-\frac{dz}{dx} = P(2uz+1) + Qz = (2uP+Q)z + P$$

$\Rightarrow$ $$\frac{dz}{dx} + (2uP+Q)z = -P \tag{4}$$

which is first order linear equation in z, from which we can find the solution $z = G(x)$, say

$\therefore$ the solution of the Riccati equation is $y = u(x) + \frac{1}{G(x)}$

Working rule

Step 1: Find a particular solution $u(x)$ by inspection or otherwise.

Step 2: Put $y = u(x) + \frac{1}{z(x)}$.

Step 3: Solve $\frac{dz}{dx} + (2Pu+Q)z = -P$ and get the solution.

Note Suppose the transformation or change of variable is $y = u(x) + z(x)$, then the Reccati equation reduces to the Bernoulli's equation and hence, it can be solved.

For, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dz}{dx}$

(1) becomes

$$\frac{du}{dx} + \frac{dz}{dx} = P(u+z)^2 + Q(u+z) + R$$
$$= P(u^2 + z^2 + 2uz) + Qu + Qz + R$$

But u is a solution of (1)

$$\therefore \quad \frac{du}{dx} = Pu^2 + Qu + R$$

$$\therefore \quad Pu^2 + Qu + R + \frac{dz}{dx} = Pu^2 + Pz^2 + 2Puz + Qu + Qz + R$$

$$\Rightarrow \quad \frac{dz}{dx} = (2Pu + Q)z + Pz^2$$

This is Bernoulli's equation in z.
Since linear equation is easier to solve than Bernoulli's equation, we shall use the transformation (2) and the linear equation is (4).

WORKED EXAMPLES

EXAMPLE 1

Solve the Riccati equation $x\frac{dy}{dx} = y^2 + 2y - 3$.

Solution.
The given equation is

$$x\frac{dy}{dx} = y^2 + 2y - 3 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{x} + \frac{2y}{x} - \frac{3}{x} \qquad (1)$$

which is Riccati equation. Here $P = \frac{1}{x}$, $Q = \frac{2}{x}$, $R = -\frac{3}{x}$

Step 1: By inspection, we find $y = 1$ is a solution $\quad \therefore u(x) = 1$ (2)

Step 2: Put $y = u(x) + \frac{1}{z(x)} = 1 + \frac{1}{z}$

Step 3: Then we get, $\frac{dz}{dx} + (2Pu + Q)z = -P$

$$\Rightarrow \quad \frac{dz}{dx} + \left(2\frac{1}{x} + \frac{2}{x}\right)z = -\frac{1}{x} \quad \Rightarrow \quad \frac{dz}{dx} + \frac{4}{x}z = -\frac{1}{x}$$

This is linear in z and can be written as $\frac{dz}{dx} + Pz = Q$, where $P = \frac{4}{x}$ and $Q = -\frac{1}{x}$

$\therefore$ the general solution is

$$z\,e^{\int P dx} = \int Q\,e^{\int P dx}\,dx + C$$

Now $$\int P\,dx = \int \frac{4}{x}\,dx = 4\log_e x$$

$\therefore$ $$e^{\int P dx} = e^{4\log_e x} = e^{\log_e x^4} = x^4$$

$\therefore$ $$\int Q\,e^{\int P dx}\,dx = \int -\frac{1}{x}x^4 dx = -\int x^3 dx = -\frac{x^4}{4}$$

$\therefore$ the general solution is

$$zx^4 = -\frac{x^4}{4} + C \quad \Rightarrow \quad z = -\frac{1}{4} + \frac{C}{x^4}$$

$\therefore$ the solution of the given equation is

$$y = u(x) + \frac{1}{z} = 1 + \frac{1}{-\frac{1}{4} + \frac{C}{x^4}}$$

$$= 1 + \frac{4x^4}{4C - x^4} = \frac{4C - x^4 + 4x^4}{4C - x^4} = \frac{4C + 3x^4}{4C - x^4}$$

where C is an arbitrary constant.

EXAMPLE 2

Solve $y' = -e^{-x}y^2 + y + e^x$.

Solution.

The given equation is

$$y' = -e^{-x}y^2 + y + e^x \tag{1}$$

which is Riccati's equation. Here $P = -e^{-x}$, $Q = 1$, $R = e^x$

Step 1: By inspection, we find that $y = e^x$ is a solution of (1)

[$\because$ $e^x = -e^{-x}\cdot e^{2x} + e^x + e^x = -e^x + e^x + e^x = e^x$ which is true]

$\therefore$ $$u(x) = e^x$$

Step 2: Put $y = u(x) + \frac{1}{z} = e^x + \frac{1}{z}$

Step 3: The equation reduces to $\frac{dz}{dx} + (2Pu + Q)z = -P$

$\Rightarrow$ $$\frac{dz}{dx} + [2e^x(-e^{-x}) + 1]z = e^{-x} \quad \Rightarrow \quad \frac{dz}{dx} - z = e^{-x}$$

This is linear in z and can be written as

$$\frac{dz}{dx} + Pz = Q \quad \text{where } P = -1 \text{ and } Q = e^{-x}$$

$\therefore$ the general solution is

$$z\,e^{\int P dx} = \int Q\,e^{\int P\,dx}\,dx + C$$

Now $$\int P\,dx = \int -1\,dx = -x$$

$\therefore$ $$e^{\int P\,dx} = e^{-x}$$

and $$\int Q\,e^{\int P\,dx} dx = \int e^{-x}\cdot e^{-x}\,dx = \int e^{-2x}\,dx = \frac{e^{-2x}}{-2}$$

$\therefore$ the general solution is

$$ze^{-x} = \frac{e^{-2x}}{2} + C \quad \Rightarrow \quad z = -\frac{e^{-x}}{2} + Ce^{x}$$

$\therefore$ the solution of the given equation is

$$y = u(x) + \frac{1}{z} = e^{x} + \frac{1}{-\frac{1}{2}e^{-x} + C\,e^{x}}$$

$$= e^{x} + \frac{1}{-\frac{1}{2\,e^{x}} + C\,e^{x}}$$

$$= e^{x} + \frac{2\,e^{x}}{2C\,e^{2x} - 1}$$

$$= \frac{e^{x}[-1 + 2C\,e^{2x}] + 2\,e^{x}}{2C\,e^{2x} - 1} = \frac{-e^{x} + 2C\,e^{3x} + 2\,e^{x}}{2C\,e^{2x} - 1} = \frac{2C\,e^{3x} + e^{x}}{2C\,e^{2x} - 1}$$

where C is an arbitrary constant.

EXAMPLE 3

Solve $y' = x^3y^2 + x^{-1}y - x^5$.

Solution.
The given equation is

$$y' = x^3y^2 + x^{-1}y - x^5 \tag{1}$$

which is Riccati equation. Here $P = x^3,\ Q = x^{-1},\ R = -x^5$

Step 1: By inspection, we find $y = x$ is a solution of (1) $\quad \therefore u(x) = x$

Step 2: Put $y = u(x) + \dfrac{1}{z(x)} = x + \dfrac{1}{z}$

Step 3: The given equation reduces to

$$\frac{dz}{dx} + (2Pu + Q)z = -P$$

$$\Rightarrow \quad \frac{dz}{dx} + (2x^3x + x^{-1})z = -x^3 \quad \Rightarrow \quad \frac{dz}{dx} + (2x^4 + x^{-1})z = -x^3$$

This is linear in z and can be written as

$$\frac{dz}{dx} + Pz = Q$$

where $\quad P = 2x^4 + x^{-1}$ and $Q = -x^3$

$\therefore$ solution is

$$z\,e^{\int P dx} = \int Q\,e^{\int P\,dx}\,dx + C$$

Now $\quad \int P\,dx = \int (2x^4 + x^{-1})\,dx = \int\left(2x^4 + \frac{1}{x}\right)dx = 2\frac{x^5}{5} + \log_e x$

$$\therefore \quad e^{\int P dx} = e^{2\frac{x^5}{5} + \log_e x} = e^{2\frac{x^5}{5}} \cdot e^{\log_e x} = x e^{2\frac{x^5}{5}}$$

and $\quad \int Q\,e^{\int P dx}\,dx = \int -x^3 \cdot x e^{2\frac{x^5}{5}}\,dx = -\int x^4 \cdot e^{\frac{2}{5}x^5}\,dx$

Put $t = \frac{2}{5}x^5 \qquad \therefore \quad dt = \frac{2}{5}\cdot 5x^4\,dx \quad \Rightarrow \quad x^4 dx = \frac{1}{2}dt$

$$\therefore \quad \int Q\,e^{\int P dx}\,dx = -\int \frac{1}{2}e^t\,dt = -\frac{1}{2}e^t = -\frac{1}{2}e^{\frac{2}{5}x^5}$$

$\therefore$ the general solution is

$$z\,x\,e^{\frac{2}{5}x^5} = -\frac{1}{2}e^{\frac{2}{5}x^5} + C \quad \Rightarrow \quad z = -\frac{1}{2x} + \frac{C}{x}e^{-\frac{2}{5}x^5}$$

$\therefore$ the solution of the given equation is

$$y = u(x) + \frac{1}{z} = x + \frac{1}{-\frac{1}{2x} + \frac{C}{x}e^{-\frac{2}{5}x^5}}$$

$$= x + \frac{2x}{-1 + 2Ce^{-\frac{2}{5}x^5}} = \frac{-x + 2Cxe^{-\frac{2}{5}x^5} + 2x}{2Ce^{-\frac{2}{5}x^5} - 1} = \frac{x + 2Cxe^{-\frac{2}{5}x^5}}{2Ce^{-\frac{2}{5}x^5} - 1}$$

EXERCISE 10.7

Solve the following Riccati equations

1. $x\frac{dy}{dx} = y^2 + y - 2.$
2. $xy' + y^2 = 2y.$
3. $y' = 2y^2 - 3y + 1.$
4. $\frac{dy}{dx} = 2e^{-x}y^2 + 3y - 4e^x.$
5. $y' + xy = xy^2$
6. $y' = xy^2 - (2x - 1)y + x - 1$

ANSWERS TO EXERCISE 10.7

1. $y = 1 + \frac{3x^3}{3C - x^3}$
2. $y = 2 + \frac{2}{Cx^2 - 1}$
3. $y = 1 + \frac{e^x}{C - 2e^x}$
4. $y = e^x + \frac{3}{Ce^{-7x} - e^{-x}}$
5. $y = \frac{C}{C - e^{\frac{x^2}{2}}}$
6. $y = 1 + \frac{1}{Ce^{-x} - [x - 1]}$

10.2.7 Type VII First Order Exact Differential Equations

A first order ordinary differential equation is of the form $\frac{dy}{dx} = f(x, y)$

In a more symmetric form it can be written as $Mdx + Ndy = 0$,
where M and N are functions of x and y

Definition 10.9

The first order differential equation $Mdx + Ndy = 0$ is said to be an **exact differential equation** if there exists a function $u(x, y)$ such that $du = Mdx + Ndy$.

Then the equation is $du = 0$

Integrating, $u(x, y) = c$ is the general solution.

For example: $xdy + ydx = 0$ is an exact differential equation, since $xdy + ydx = 0 \Rightarrow d(xy) = 0$

Integrating, $xy = c$ is the general solution.

Theorem 10.1 A necessary and sufficient condition that the differential equation $Mdx + Ndy = 0$ be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof **Condition is necessary**

If the equation $Mdx + Ndy = 0$ is exact, then there exists a function $u(x, y)$ such that

$$du = Mdx + Ndy \tag{1}$$

But
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2}$$

$\therefore$
$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Since dx and dy are independent increments, we have

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

$\therefore$
$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

If M and N have continuous partial derivatives, then

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Condition is sufficient

Assume
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{3}$$

Then we prove that the equation is exact. That is $Mdx + Ndy$ is exact.

Let $F = \int_{y \text{ constant}} M dx$ be the partial integral of $M\, dx$.

That is the integral is obtained keeping y as constant.

Then $$\frac{\partial F}{\partial x} = M$$

$$\therefore \quad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{(by (3))}$$

But $$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}(M) = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial^2 F}{\partial x \partial y} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(N - \frac{\partial F}{\partial y}\right) = 0$$

Integrating partially w.r.to x,

$$N - \frac{\partial F}{\partial y} = \phi(y) \quad \Rightarrow \quad N = \phi(y) + \frac{\partial F}{\partial y}$$

Let $$u(x,y) = \int\left[\phi(y) + \frac{\partial F}{\partial y}\right]dy = \int \frac{\partial F}{\partial y} dy + \int \phi(y) dy = F + \int \phi(y) dy$$

Then $$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial}{\partial y}\int \phi(y) dy = \frac{\partial F}{\partial y} + \phi(y) = N$$

Since $$u = F + \int \phi(y) dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} \quad \therefore \quad M = \frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}.$$

Then $$Mdx + Ndy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow \quad Mdx + Ndy = du$$

So, the equation is exact.

Procedure to find the solution of $Mdx + Ndy = 0$

1. First, check the condition for exactness. i.e., to check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Treating y as a constant, integrate M w.r.to x. Let $F = \int_{y \text{ constant}} Mdx$.
3. Find $N - \frac{\partial F}{\partial y}$ and compute $G = \int\left(N - \frac{\partial F}{\partial y}\right)dy$.
4. The general solution is

$$\int_{y \text{ constant}} Mdx + \int\left(N - \frac{\partial F}{\partial y}\right)dy = C$$

$$\Rightarrow \quad F + G = C \qquad \text{(I)}$$

Note Sometimes the general solution is stated as below

$$\int_{y\,\text{constant}} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = C \text{ (II) i.e., } G = \int (\text{terms of } N \text{ not containing } x)dy$$

In some cases this formula fails to give the correct solution. So, it is not advisable to use formula II.
Refer worked example 1.

However in the exact equation $Mdx + Ndy = 0$, if N does not contain constant term, it is found that (II) also gives the correct answer.

WORKED EXAMPLES

EXAMPLE 1

Solve $\dfrac{dy}{dx} = \dfrac{y \sin 2x}{y^2 + \cos^2 x}$.

Solution.

The given equation is $\dfrac{dy}{dx} = \dfrac{y\sin 2x}{y^2 + \cos^2 x}$

$\Rightarrow \quad (y^2 + \cos^2 x)dy = y\sin 2x\, dx \quad \Rightarrow \quad y\sin 2x\, dx - (y^2 + \cos^2 x)dy = 0$

This is of the form $Mdx + Ndy = 0$. Here $M = y\sin 2x$ and $N = -(y^2 + \cos^2 x)$

$\therefore \quad \dfrac{\partial M}{\partial y} = \sin 2x$ and $\dfrac{\partial N}{\partial x} = -2\cos x(-\sin x) = \sin 2x$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, the equation is exact.

To find the solution

Treating y as constant, integrate M w.r.to x

$$\therefore \quad F = \int Mdx = \int y\sin 2x dx = y\left(-\frac{\cos 2x}{2}\right) = -\frac{y}{2}\cos 2x$$

$$\therefore \quad \frac{\partial F}{\partial y} = -\frac{1}{2}\cos 2x$$

and
$$N - \frac{\partial F}{\partial y} = -y^2 - \cos^2 x + \frac{1}{2}\cos 2x = -y^2 - \cos^2 x + \frac{1}{2}(2\cos^2 x - 1) = -y^2 - \frac{1}{2}$$

$$\therefore \quad G = \int\left(N - \frac{\partial F}{\partial y}\right)dy = \int\left(-y^2 - \frac{1}{2}\right)dy = -\left[\frac{y^3}{3} + \frac{y}{2}\right]$$

$\therefore$ the general solution is

$$F + G = C$$

$$\Rightarrow \quad -\frac{y}{2}\cos 2x - \left[\frac{y^3}{3} + \frac{y}{2}\right] = C \quad \Rightarrow \quad \frac{y}{2}(1 + \cos 2x) + \frac{y^3}{3} + C = 0.$$

Note Suppose we find the general solution by the formula (II)

$F+\int(\text{terms of } N \text{ not containing } x)dy = C$

$$\Rightarrow \quad -\frac{y}{2}\cos 2x + \int -y^2 dy = C \quad \Rightarrow \quad -\frac{y}{2}\cos 2x - \frac{y^3}{3} = C \quad \Rightarrow \quad \frac{y}{2}\cos 2x + \frac{y^3}{3} + C = 0$$

We notice that the term $\frac{y}{2}$ is missing in this solution.

In the given problem $N = -(y^2+\cos^2 x) = -\left(y^2 + \frac{1+\cos 2x}{2}\right) = -y^2 - \frac{1}{2} - \frac{1}{2}\cos 2x,$

which contains a constant term. Because of this, the formula (II) fails to give the correct answer.

EXAMPLE 2

Solve the differential equation $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$.

Solution.

The given equation is

$$(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0 \qquad (1)$$

It is of the form $Mdx + Ndy = 0$

Here $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$\therefore \quad \frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ and $\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the equation (1) is exact.

To find the solution

Treating y as constant, integrate M w.r.t. to x

$$\therefore \quad F = \int Mdx = \int (5x^4 + 3x^2y^2 - 2xy^3)dx$$

$$= 5\frac{x^5}{5} + 3y^2\frac{x^3}{3} - 2y^3\frac{x^2}{2} = x^5 + x^3y^2 - x^2y^3$$

$$\therefore \quad \frac{\partial F}{\partial y} = 2x^3y - 3x^2y^2$$

and $$N - \frac{\partial F}{\partial y} = 2x^3y - 3x^2y^2 - 5y^4 - 2x^3y + 3x^2y^2 = -5y^4$$

$$\therefore \quad G = \int\left(N - \frac{\partial F}{\partial y}\right)dy = \int -5y^4 dy = -\frac{5y^5}{5} = -y^5$$

$\therefore$ the general solution is $F + G = C \quad \Rightarrow \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C$

$$\Rightarrow \quad x^3(x^2+y^2) - y^3(x^2+y^2) = C \quad \Rightarrow \quad (x^2+y^2)(x^3-y^3) = C$$

Note In this problem, N does not contain constant term and so we can find G by (II) is $G = \int -5y^4 \, dy = -y^5$.

EXAMPLE 3

Solve the differential equation $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0.$

Solution.
The given equation is

$$(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$$

It is of the form $Mdx + Ndy = 0$

Here $M = y^2e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = y^2 \cdot e^{xy^2} 2xy + e^{xy^2} \cdot 2y = (2xy^3 + 2y)e^{xy^2}$$

and $$\frac{\partial N}{\partial x} = 2y[xe^{xy^2} \cdot y^2 + e^{xy^2} \cdot 1] = [2xy^3 + 2y]e^{xy^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, the equation is exact.
To find the solution, integrate M w. r. to x, treating y as constant.

$$\therefore \quad F = \int Mdx = \int (y^2e^{xy^2} + 4x^3)dx = y^2\frac{e^{xy^2}}{y^2} + 4\frac{x^4}{4} = e^{xy^2} + x^4$$

$$\therefore \quad \frac{\partial F}{\partial y} = e^{xy^2} \cdot 2xy = 2xye^{xy^2} \text{ and } N - \frac{\partial F}{\partial y} = 2xye^{xy^2} - 3y^2 - 2xye^{xy^2} = -3y^2$$

$$\therefore \quad G = \int \left(N - \frac{\partial F}{\partial y} \right) dy = \int -3y^2 dy = -3\frac{y^3}{3} = -y^3$$

$\therefore$ the general solution is

$$F + G = C \quad \Rightarrow \quad e^{xy^2} + x^4 - y^3 = C$$

EXERCISE 10.8

Solve the following differential equations.

1. $(x^2 - 2xy + 3y^2)dx + (y^2 + 6xy - x^2)dy = 0.$
2. $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0.$
3. $(2y\sin x + \cos x)dx = (x\sin y + 2\cos x + \tan y)dy.$
4. $x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x.$
5. $(e^y + y\cos y)dx + (xe^y + x\cos xy)dy = 0.$
6. $(xe^{xy} + 2y)\frac{dy}{dx} + ye^{xy} = 0.$

7. $(2xy^3 + y\cos x)dx + (3x^2y^2 + \sin x)dy = 0.$

8. $\sin x \sin^2 y dx - (\cos x \cos y \tan y + \cos x \tan y)dy = 0.$

9. $\left(1 + e^{\frac{x}{y}}\right)dx + \left(1 - \frac{x}{y}\right)e^{\frac{x}{y}} dy = 0.$

10. $x^2 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x.$

11. $y(x^2 + y^2 + a^2)\frac{dy}{dx} + x(x^2 + y^2 - a^2) = 0.$

12. $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0.$

ANSWERS TO EXERCISE 10.8

1. $x^3 - 3x^2y + 9xy^2 + y^3 = C$
2. $x^3 - 6x^2y - 6xy^2 + y^3 = C$
3. $x\cos y - 2y\cos x - \log_e \sec y = C$
4. $x^3 \tan y - \sin x = C$
5. $xe^y + \sin xy = C$
6. $e^{xy} + y^2 = C$
7. $x^2y^3 + y\sin x = C$
8. $\sin^2 y \cos x = C$
9. $x + ye^{\frac{x}{y}} = C$
10. $x^3 \tan y - \sin x = C$
11. $x^4 + y^4 - 2a^2x^2 + 2y^2(a^2 + x^2) = C$
12. $x^2y + (y - \tan y)x + \tan y = C$

10.3 INTEGRATING FACTORS

Sometimes the equation $Mdx + Ndy = 0$ may not be exact, but it can be made exact by multiplying it by a suitable function $\boldsymbol{\mu}(x, y)$. Such a function is called an **integrating factor** (I.F).

So, $\boldsymbol{\mu}(Mdx + Ndy) = 0$ is an exact differential equation.

Though there are standard techniques of finding the integrating factors of $Mdx + Ndy = 0$, in some cases it is possible to find an I.F by inspection, after grouping the terms suitably.

The following list of exact differentials will be useful to recognize an **integrating factor.**

(1) $xdy + ydx = d(xy)$

(2) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

This means that $ydx - xdy = 0$ becomes exact by multiplying by $\frac{1}{y^2}$.

So, $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

Similarly,

(3) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(4) $\frac{xdy + ydx}{xy} = d(\log_e xy)$

[Integrating factor of $xdy + ydx = 0$ is $\frac{1}{xy}$]

INTEGRATING FACTORS

Sometimes the equation $Mdx + Ndy = 0$ may not be exact, but it can be made exact by multiplying it by a suitable function $\mu(x, y)$. Such a function is called an **integrating factor** (I.F).

So, $\mu(Mdx + Ndy) = 0$ is an exact differential equation.

Though there are standard techniques of finding the integrating factors of $Mdx + Ndy = 0$, in some cases it is possible to find an I.F by inspection, after grouping the terms suitably.

The following list of exact differentials will be useful to recognize an **integrating factor.**

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This means that $ydx - xdy = 0$ becomes exact by multiplying by $\frac{1}{y^2}$.

So, $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

Similarly,

(3) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(4) $\frac{xdy + ydx}{xy} = d(\log_e xy)$

[Integrating factor of $xdy + ydx = 0$ is $\frac{1}{xy}$]

(5) $xdx + ydy = \frac{1}{2}d(x^2 + y^2)$

(6) $\frac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

(7) $\frac{ydx - xdy}{xy} = d\left(\log_e \frac{x}{y}\right)$

(8) $\frac{xdy - ydx}{x^2 - y^2} = \frac{1}{2}d\left[\log_e \frac{x+y}{x-y}\right]$

(9) $\frac{xdy + ydx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$

Note From this list, it is seen that the simple differential equation $ydx - xdy = 0$ has $\frac{1}{x^2}$, $\frac{1}{y^2}$, $\frac{1}{x^2+y^2}$, $\frac{1}{xy}$ as integrating factors and so the equation can be solved in different ways.

Hence, I.F is not unique. If $\mu\,(x, y)$ is an I.F, then k $\mu\,(x, y)$ is also an I.F for any non-zero constant k.

WORKED EXAMPLES

EXAMPLE 1

Solve $ydx - xdy + 3x^2y^2e^{x^3} = 0$.

Solution.

The given equation is $ydx - xdy + 3x^2y^2e^{x^3} = 0$

Dividing by y^2,

$$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3} = 0 \quad \Rightarrow \quad d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$

Integrating, we get

$$\int d\left(\frac{x}{y}\right) + \int d(e^{x^3}) = 0 \quad \Rightarrow \quad \frac{x}{y} + e^{x^3} = C \quad \Rightarrow \quad x + ye^{x^3} = Cy$$

which is the general solution

EXAMPLE 2

Solve $xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$.

Solution.

The given equation is

$$xdx + ydy = \frac{a^2(xdx - ydx)}{x^2 + y^2} \quad \Rightarrow \quad xdx + ydy = a^2 d\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$$

Integrating, we get

$$\int xdx + \int ydy = a^2 \int d\left(\tan^{-1}\frac{x}{y}\right)$$

$$\Rightarrow \qquad \frac{x^2}{2}+\frac{y^2}{2}=a^2\tan^{-1}\left(\frac{x}{y}\right)+C \quad \Rightarrow \quad x^2+y^2=2a^2\tan^{-1}\left(\frac{x}{y}\right)+2C$$

$\therefore$ the general solution is

$$x^2+y^2=2a^2\tan^{-1}\left(\frac{x}{y}\right)+C', \quad \text{where } C'=2C$$

EXAMPLE 3

Solve $x\cos\left(\frac{y}{x}\right)[ydx+xdy]=y\sin\left(\frac{y}{x}\right)[xdy-ydx]$.

Solution.

The given equation is

$$x\cos\left(\frac{y}{x}\right)[ydx+xdy]=y\sin\left(\frac{y}{x}\right)[xdy-ydx]$$

Dividing by x^2y, we get

$$\frac{x\cos\left(\frac{y}{x}\right)}{x^2y}d(xy)=\sin\left(\frac{y}{x}\right)\left[\frac{xdy-ydx}{x^2}\right]$$

$$\Rightarrow \qquad \cos\left(\frac{y}{x}\right)\frac{d(xy)}{xy}=\sin\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad \frac{d(xy)}{xy}=\frac{\sin\frac{y}{x}}{\cos\frac{y}{x}}d\left(\frac{y}{x}\right)$$

Integrating, we get

$$\int\frac{d(xy)}{xy}=\int\frac{\sin\frac{y}{x}}{\cos\frac{y}{x}}d\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad \log_e xy=-\log_e\cos\left(\frac{y}{x}\right)+\log_e C$$

$$\Rightarrow \qquad \log_e xy+\log_e\cos\frac{y}{x}=\log_e C \quad \Rightarrow \quad \log_e xy\cos\left(\frac{y}{x}\right)=\log_e C \quad \Rightarrow \quad xy\cos\left(\frac{y}{x}\right)=C$$

which is the general solution of the given equation.

EXAMPLE 4

Solve $(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0$.

Solution.
The given equation is

$$(xy^2 - e^{\frac{1}{x^3}})dx - x^2 ydy = 0$$

$$\Rightarrow \quad xy^2 dx - x^2 ydy - e^{\frac{1}{x^3}} dx = 0$$

$$\Rightarrow \quad xy[ydx - xdy] - e^{\frac{1}{x^3}} dx = 0$$

$$\Rightarrow \quad x^3 y \frac{ydx - xdy}{x^2} - e^{\frac{1}{x^3}} dx = 0$$

$$\Rightarrow \quad -x^3 y \left[\frac{xdy - ydx}{x^2}\right] - e^{\frac{1}{x^3}} dx = 0$$

$$\Rightarrow \quad x^3 y \, d\left(\frac{y}{x}\right) + e^{\frac{1}{x^3}} dx = 0$$

$$\Rightarrow \quad x^4 \left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) + e^{\frac{1}{x^3}} dx = 0 \quad \Rightarrow \quad \left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) + \frac{1}{x^4} e^{\frac{1}{x^3}} dx = 0 \qquad \text{[dividing by } x^4\text{]}$$

Integrating, we get

$$\int \frac{y}{x} d\left(\frac{y}{x}\right) + \int e^{\frac{1}{x^3}} \cdot \frac{1}{x^4} dx = 0 \quad \Rightarrow \quad \frac{\left(\frac{y}{x}\right)^2}{2} + \int e^{\frac{1}{x^3}} \frac{1}{x^4} dx = 0$$

Let
$$I = \int e^{\frac{1}{x^3}} \frac{1}{x^4} dx$$

Put $t = \dfrac{1}{x^3}$ $\quad \therefore \quad dt = -\dfrac{3}{x^4} dx \quad \Rightarrow \quad -\dfrac{1}{3} dt = \dfrac{1}{x^4} dx$

$$\therefore \quad I = -\frac{1}{3} \int e^t dt = -\frac{1}{3} e^t = -\frac{1}{3} e^{\frac{1}{x^3}}$$

$$\therefore \quad \frac{1}{2}\left(\frac{y}{x}\right)^2 - \frac{1}{3} e^{\frac{1}{x^3}} = c \quad \Rightarrow \quad 3y^2 - 2x^2 \, e^{\frac{1}{x^3}} = 6cx^2.$$

which is the general solution of the given equation.

10.3.1 Rules for Finding the Integrating Factor for Non-Exact Differential Equation *Mdx* + *Ndy* = 0

Rule 1. If the equation $Mdx + Ndy = 0$ is homogeneous, that is M and N are homogeneous functions in x and y of the same degree, then $\dfrac{1}{Mx + Ny}$ is an integrating factor if $Mx + Ny \neq 0$

Rule 2. If $Mdx + Ndy = 0$ is of the form $f_1(x, y) y \, dx + f_2(x, y) x \, dy = 0$,

that is $M = f_1(x, y) y$, $N = f_2(x, y) x$, then $\dfrac{1}{Mx - Ny}$ is an I.F if $Mx - Ny \neq 0$

Rule 3. If $Mdx + Ndy = 0$ is such that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x, say $f(x)$, then $e^{\int f(x)dx}$ is an I.F

Rule 4. If $Mdx + Ndy = 0$ is such that $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y, say $g(y)$, then $e^{\int g(y)dy}$ is an I.F.

Rule 5. If the differential equation is of the type

$$x^a y^b (mydx + nxdy) + x^{a_1} y^{b_1} (m_1 ydx + n_1 xdy) = 0,$$

where a, b, a_1, b_1, m, n, m_1, n_1 are constants and $mn_1 - m_1 n \neq 0$, has an I.F of the form $x^h y^k$ where the constants h and k are given by

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a_1+h+1}{m_1} = \frac{b_1+k+1}{n_1}$$

Equivalently, we have

$$nh - mk = (m-n) + (mb - na) \text{ and } n_1 h - m_1 k = (m_1 - n_1) + (m_1 b_1 - n_1 a_1)$$

WORKED EXAMPLES

Problems using Rule 1

EXAMPLE 1

Show that the equation $(2x - y)dy + (2y + x)dx = 0$ can be made exact by the integrating factor $\frac{1}{x^2 + y^2}$ and hence, solve the equation.

Solution.

Given $$(2x - y)dx + (2y + x)dy = 0 \quad (1)$$

It is of the form $$Mdx + Ndy = 0.$$

Here $M = 2x - y$, $N = 2y + x$, which are homogeneous functions of the same degree 1.

Now $Mx + Ny = (2x - y)x + (2y + x)y = 2x^2 + 2y^2 - xy + xy = 2(x^2 + y^2) \neq 0$

$\therefore \quad \frac{1}{Mx + Ny} = \frac{1}{2(x^2 + y^2)}$ is an integrating factor.

Hence, $\frac{1}{x^2 + y^2}$ is an integrating factor omitting the constant factor 2, by Rule 1.

Multiplying (1) by $\frac{1}{x^2 + y^2}$, it will become exact.

$$\therefore \quad \frac{2x - y}{x^2 + y^2}dx + \frac{2y + x}{x^2 + y^2}dy = 0 \text{ is exact.}$$

Here $M = \frac{2x - y}{x^2 + y^2}$ and $N = \frac{2y + x}{x^2 + y^2}$

To find the solution, integrate M w.r.t. to x, treating y as constant.

$$\therefore \quad F = \int Mdx = \int \frac{2x - y}{x^2 + y^2}dx = \int \frac{2x}{x^2 + y^2}dx - y\int \frac{dx}{x^2 + y^2}$$

$$= \log_e (x^2 + y^2) - y \cdot \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) = \log_e (x^2 + y^2) - \tan^{-1}\left(\frac{x}{y}\right)$$

In $N = \dfrac{2y+x}{x^2+y^2}$, there is no term without x and there is no constant term. $\therefore\ G = 0$

$\therefore$ the general solution is $F+G=C \quad \Rightarrow \quad \log_e(x^2+y^2) - \tan^{-1}\dfrac{x}{y} = C.$

EXAMPLE 2

Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0.$

Solution.

The given equation is

$$(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0 \qquad (1)$$

It is of the form $Mdx + Ndy = 0$, where $M = y^3 - 2yx^2$ and $N = 2xy^2 - x^3$ are homogenous functions of degree 3.

Now $\qquad Mx + Ny = (y^3 - 2yx^2)x + (2xy^2 - x^3)y$

$$= xy^3 - 2x^3y + 2xy^3 - x^3y = 3xy^3 - 3x^3y = 3xy(y^2 - x^2) \neq 0 \qquad (\text{if } y \cdot x \neq 0)$$

$\therefore \qquad \dfrac{1}{Mx+Ny} = \dfrac{1}{3xy(y^2-x^2)}$ is an integrating factor by Rule 1.

Hence, $\dfrac{1}{xy(y^2-x^2)}$ is an integrating factor.

Multiplying (1) by $\dfrac{1}{xy(y^2-x^2)}$, it will be exact.

$\therefore \qquad \dfrac{(y^3-2yx^2)}{xy(y^2-x^2)}dx + \dfrac{(2xy^2-x^3)}{xy(y^2-x^2)}dy = 0$ is exact.

$$\Rightarrow \qquad \frac{y(y^2-2x^2)}{xy(y^2-x^2)}dx + \frac{x(2y^2-x^2)}{xy(y^2-x^2)}dy = 0 \quad \Rightarrow \quad \frac{y^2-2x^2}{x(y^2-x^2)}dx + \frac{2y^2-x^2}{y(y^2-x^2)}dy = 0$$

For this exact equation $\qquad M = \dfrac{y^2-2x^2}{x(y^2-x^2)} \qquad$ and $\qquad N = \dfrac{2y^2-x^2}{y(y^2-x^2)}$

To find the solution, integrate M w.r.to x, treating y as constant.

$$\therefore \qquad F = \int M dx = \int \frac{(y^2-2x^2)}{x(y^2-x^2)}dx = \int \frac{(y^2-x^2)-x^2}{x(y^2-x^2)}dx$$

$$= \int\left(\frac{1}{x} - \frac{x}{y^2-x^2}\right)dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{y^2-x^2}dx$$

$$= \int \frac{1}{x}dx + \frac{1}{2}\int \frac{-2x}{y^2-x^2}dx$$

$$= \log_e x + \frac{1}{2}\log_e(y^2-x^2) = \log_e x + \log_e\sqrt{y^2-x^2}$$

But $N = \dfrac{y^2 + (y^2 - x^2)}{y(y^2 - x^2)} = \dfrac{y}{y^2 - x^2} + \dfrac{1}{y}$,

which does not contain a constant term.
So, integrating the terms of N not containing x w.r.to y, we get

$$G = \int \frac{1}{y} dy = \log_e y$$

$\therefore$ the general solution is $\quad F + G = C$

$$\Rightarrow \quad \log_e x + \log_e \sqrt{y^2 - x^2} + \log_e y = \log_e C'$$

$$\Rightarrow \quad \log xy\sqrt{y^2 - x^2} = \log_e C' \quad \Rightarrow \quad xy\sqrt{y^2 - x^2} = C'$$

Problems using Rule 2

EXAMPLE 3

Solve $(xy^2 + y)dx + (x - x^2y)dy = 0.$

Solution.

The given equation is $(xy^2 + y)dx + (x - x^2y)dy = 0$ (1)

It is of the form $\quad Mdx + Ndy = 0$

where $\quad M = xy^2 + y = (xy + 1)y = f_1(x, y)y$

and $\quad N = x - x^2y = (1 - xy)x = f_2(x, y)x$

Now $\quad Mx - Ny = (xy + 1)y \cdot x - (1 - xy)x \cdot y = x^2y^2 + xy - xy + x^2y^2 = 2x^2y^2 \neq 0$

$\therefore \quad \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2y^2}$ is an integrating factor by Rule 2.

Hence, $\dfrac{1}{x^2y^2}$ is an I.F

Multiplying (1) by $\dfrac{1}{x^2y^2}$, we get $\dfrac{xy^2 + y}{x^2y^2}dx + \dfrac{x - x^2y}{x^2y^2}dy = 0$ is exact.

$$\Rightarrow \quad \left(\frac{1}{x} + \frac{1}{yx^2}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

For this exact equation, $M = \dfrac{1}{x} + \dfrac{1}{yx^2}$ and $N = \dfrac{1}{xy^2} - \dfrac{1}{y}$.

Treating y as constant, integrate M w.r. to x.

$$\therefore \quad F = \int Mdx = \int \left(\frac{1}{x} + \frac{1}{yx^2}\right)dx = \int \frac{1}{x}dx + \frac{1}{y}\int x^{-2}dx$$

$$= \log_e x + \frac{1}{y}\left[\frac{x^{-1}}{-1}\right] = \log_e x - \frac{1}{xy}$$

But N does not contain a constant term.

Now integrate w.r.to y the terms in N not containing x.

In N, the term not containing x is $-\frac{1}{y}$ $\quad \therefore \quad G = \int -\frac{1}{y}\,dy = -\log_e y$

$\therefore$ the general solution is $\quad F + G = C$

$$\Rightarrow \qquad \log_e x - \frac{1}{xy} - \log_e y = C \quad \Rightarrow \quad \log_e \frac{x}{y} - \frac{1}{xy} = C$$

EXAMPLE 4

Solve $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$.

Solution.

The given equation is

$(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

It is of the form $\quad Mdx + Ndy = 0$

where $\quad M = (x^2y^2 + xy + 1)y = f_1(x, y)y$

and $\quad N = (x^2y^2 - xy + 1)x = f_2(x, y)x$

Now $\quad Mx - Ny = (x^2y^2 + xy + 1)yx - (x^2y^2 - xy + 1)xy$

$$= xy[x^2y^2 + xy + 1 - x^2y^2 + xy - 1] = xy \cdot 2xy = 2x^2y^2 \neq 0$$

$\therefore \quad \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$ is an I.F and hence, $\frac{1}{x^2y^2}$ is an I.F, by Rule 2.

Multiplying (1) by $\frac{1}{x^2y^2}$, it becomes exact.

$\therefore \quad \frac{(x^2y^2 + xy + 1)y}{x^2y^2}dx + \frac{(x^2y^2 - xy + 1)x}{x^2y^2}dy = 0$ is an exact equation

$$\Rightarrow \left(1 + \frac{1}{xy} + \frac{1}{x^2y^2}\right)y\,dx + \left(1 - \frac{1}{xy} + \frac{1}{x^2y^2}\right)x\,dy = 0$$

For this exact equation, $\quad M = \left(1 + \frac{1}{xy} + \frac{1}{x^2y^2}\right)y$

and $\quad N = \left(1 - \frac{1}{xy} + \frac{1}{x^2y^2}\right)x = x - \frac{1}{y} + \frac{1}{xy^2}$

Treating y as constant, integrate M w.r.to x

$$\therefore F = \int Mdx = \int\left(1 + \frac{1}{xy} + \frac{1}{x^2y^2}\right)y\,dx = y\int dx + \int \frac{1}{x}dx + \frac{1}{y}\int x^{-2}dx$$

$$= y \cdot x + \log_e x + \frac{1}{y}\frac{x^{-1}}{(-1)} = xy + \log_e x - \frac{1}{xy}$$

N does not contain constant term.

Now integrate w.r. to y the terms of N not containing x

In N, the terms not containing x is $-\frac{1}{y}$ $\quad\therefore\quad G=\int -\frac{1}{y}dy=-\log_e y$

$\therefore$ the general solution is $F+G=C$

$$\Rightarrow \quad xy+\log_e x-\frac{1}{xy}-\log_e y=C \quad \Rightarrow \quad xy+\log_e\left(\frac{x}{y}\right)-\frac{1}{xy}=C$$

Problems using Rule 3

EXAMPLE 5

Solve $(x^3+xy^4)dx+2y^3dy=0$.

Solution.

The given equation is $(x^3+xy^4)dx+2y^3dy=0$ (1)

It is of the form $Mdx+Ndy=0$, where $M=x^3+xy^4$ and $N=2y^3$

$$\therefore \quad \frac{\partial M}{\partial y}=4xy^3 \text{ and } \frac{\partial N}{\partial x}=0$$

$$\therefore \quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \text{ and } \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=4xy^3$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2y^3}4xy^3=2x=f(x)$$

$$\therefore \quad \text{I.F}=e^{\int f(x)dx}=e^{\int 2x\,dx}=e^{x^2} \qquad \text{[by Rule 3]}$$

Multiplying (1) by e^{x^2}, it becomes exact.

$\therefore\ e^{x^2}(x^3+xy^4)dx+e^{x^2}2y^3dy=0$ is exact differential equation.

For this exact equation $M=e^{x^2}(x^3+xy^4)$ and $N=e^{x^2}\cdot 2y^3$

Treating y as constant, integrate M w.r.to x

$$\therefore \quad F=\int M\,dx=\int e^{x^2}(x^3+xy^4)dx=\int e^{x^2}(x^2+y^4)xdx$$

Put $t=x^2$ $\therefore dt=2x\,dx \Rightarrow x\,dx=\frac{dt}{2}$

$$\therefore \quad F=\int e^{x^2}(x^2+y^4)x\,dx=\int e^t(t+y^4)\frac{dt}{2}=\frac{1}{2}\int te^t dt+\frac{y^4}{2}\int e^t dt=\frac{1}{2}[te^t-1\cdot e^t]+\frac{y^4}{2}\cdot e^t$$

$$=\frac{e^t}{2}[(t-1)+y^4]=\frac{e^{x^2}}{2}[x^2-1+y^4]$$

In N, there is no term without x and it does not contain constant term. $\therefore G=0$

$\therefore$ the general solution is $F+G=C \Rightarrow \frac{e^{x^2}}{2}\left(x^2-1+y^4\right)=C \Rightarrow e^{x^2}(x^2-1+y^4)=2C=C'$

EXAMPLE 6

Solve $(x^2 + y^2 + x)dx + xy\,dy = 0$.

Solution.

The given equation is $(x^2 + y^2 + x)dx + xydy = 0$ (1)

It is of the form $Mdx + Ndy = 0$,

where $M = x^2 + y^2 + x$ and $N = xy$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{and} \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}y = \frac{1}{x} = f(x)$$

$\therefore$ $\text{I.F} = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log_e x} = x$, by Rule 3.

Multiplying (1) by x, it becomes exact differential equation

$$\therefore \quad x(x^2 + y^2 + x)dx + x^2 y\,dy = 0 \text{ is exact}$$

For this exact equation $M = x(x^2 + y^2 + x)$ and $N = x^2 y$

Treating y as constant, integrate M w.r.to x.

$$\therefore \quad F = \int M\,dx = \int x(x^2 + y^2 + x)\,dx = \int x^3 dx + y^2 \int x\,dx + \int x^2 dx$$

$$= \frac{x^4}{4} + y^2 \cdot \frac{x^2}{2} + \frac{x^3}{3} = \frac{1}{12}[3x^4 + 6x^2y^2 + 4x^3]$$

In N, there is no term without x and there is no constant term. $\therefore\ G = 0$

$\therefore$ the general solution is $F + G = C$

$$\Rightarrow \quad \frac{1}{12}[3x^4 + 6x^2y^2 + 4x^3] = C \quad \Rightarrow \quad 3x^4 + 6x^2y^2 + 4x^3 = 12C = C'$$

Problems using Rule 4

EXAMPLE 7

Solve $(y\log_e y)\,dx + (x - \log_e y)dy = 0$.

Solution.

The given equation is

$$(y\log_e y)\,dx + (x - \log_e y)dy = 0 \quad (1)$$

This is of the form $Mdx + Ndy = 0$

where $M = y\log_e y$ and $N = x - \log_e y$

$\therefore \quad \dfrac{\partial M}{\partial y} = y \cdot \dfrac{1}{y} + \log_e y \cdot 1 = 1 + \log_e y$ and $\dfrac{\partial N}{\partial x} = 1$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

and $$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - 1 - \log_e y = -\log_e y$$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{\log_e y}{y\log_e y} = -\frac{1}{y} = g(y)$$

I.F is $e^{\int g(y)dy}$, by rule 4.

But $\int g(y)\,dy = \int -\dfrac{1}{y}dy = -\log_e y = \log_e\left(\dfrac{1}{y}\right) \quad \therefore \quad e^{\int g(y)dy} = e^{\log_e \frac{1}{y}} = \dfrac{1}{y}$

Multiplying (1) by $\dfrac{1}{y}$, it becomes exact differential equation.

$$\therefore \quad \frac{1}{y}(y\log_e y)dx + \frac{1}{y}(x - \log_e y)dy = 0 \text{ is exact.}$$

$$\Rightarrow \quad \log_e y\,dx + \left(\frac{x}{y} - \frac{1}{y}\log_e y\right)dy = 0.$$

For this exact equation,

$$M = \log_e y \quad \text{and} \quad N = \frac{x}{y} - \frac{1}{y}\log_e y$$

To find the solution, integrate M w.r.to x, treating y as a constant.

$$\therefore \quad F = \int M\,dx = \int \log_e y\,dx = \log_e y \cdot x = x\log_e y$$

In N, there is no constant term and integrate w.r.to y the terms not containing x in N.

In N, the term not containing x is $-\dfrac{1}{y}\log_e y$

$$\therefore \quad G = \int -\frac{1}{y}\log_e y\,dy = -\int \log_e y\,\frac{1}{y}dy = -\frac{(\log_e y)^2}{2}$$

$\therefore$ the general solution is $F + G = C \quad \Rightarrow \quad x\log_e y - \dfrac{(\log_e y)^2}{2} = C$

EXAMPLE 8

Solve $(xy^3 + y)dx + (2x^2y^2 + 2x)dy = 0$.

Solution.

The given equation is

$$(xy^3 + y)dx + (2x^2y^2 + 2x)dy = 0 \qquad (1)$$

This is of the form $Mdx + Ndy = 0$

Here $M = xy^3 + y$ and $N = 2x^2y^2 + 2x$

$\therefore \quad \frac{\partial M}{\partial y} = 3xy^2 + 1$ and $\frac{\partial N}{\partial x} = 4xy^2 + 2$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy^2 + 2 - 3xy^2 - 1 = xy^2 + 1$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{xy^2 + 1}{y(xy^2+1)} = \frac{1}{y} = g(y)$$

$\therefore$ I.F is $e^{\int g(y)dy}$, by Rule 4

Now $\int g(y)\,dy = \int \frac{1}{y}dy = \log_e y \qquad \therefore \quad e^{\int g(y)dy} = e^{\log_e y} = y$

Multiplying (1) by y, it becomes an exact differential equation.

$$\therefore \quad y(xy^3 + y)dx + y(2x^2y^2 + 2x)dy = 0 \text{ is exact.}$$

$$\Rightarrow \quad y^2(xy^2 + 1)dx + y(2x^2y^2 + 2x)dy = 0$$

For this exact equation $M = y^2(xy^2 + 1)$ and $N = y(2x^2y^2 + 2x)$

Treating y as constant, integrate M w.r.to x.

$$\therefore \quad F = \int M\,dx = \int y^2(xy^2 + 1)\,dx = y^2\int xy^2\,dx + y^2\int dx$$

$$= y^4\int x\,dx + y^2x = y^4 \cdot \frac{x^2}{2} + y^2x = \frac{x^2y^4}{2} + xy^2$$

In $N = y(2x^2y^2 + 2x)$, there is no term without x and no constant term. $\therefore \quad G = 0$

$\therefore$ the general solution is $F + G = C$

$$\Rightarrow \quad \frac{x^2y^4}{2} + xy^2 = C \quad \Rightarrow \quad x^2y^4 + 2xy^2 = 2C = C'$$

Problems using Rule 5

EXAMPLE 9

Solve $(2x^2y - y^3)dx + (x^3 - 2xy^2)dy = 0$.

Solution.

The given equation is $(2x^2y - y^3)dx + (x^3 - 2xy^2)dy = 0$ (1)

Method1: It is homogeneous equation, use Rule 1 and find the solution.
[Same as example 2 page 10.47]
Method 2: Regrouping the terms, we get

$$2x^2y\,dx - y^3dx + x^3dy - 2xy^2dy = 0$$

$\Rightarrow \quad x^2(2y\,dx + x\,dy) - y^2(ydx + 2xdy) = 0$

$\Rightarrow \quad x^2(2ydx + xdy) + y^2(-ydx - 2xdy) = 0$

This is of the type

$$x^a y^b (mydx + nxdy) + x^{a_1} y^{b_1}(m_1 ydx + n_1 xdy) = 0$$

So, $x^h y^k$ is an I.F, by rule 5

Here $a = 2,\ b = 0,\ m = 2,\ n = 1$ and $a_1 = 0,\ b_1 = 2,\ m_1 = -1,\ n_1 = -2$

$\therefore \quad \frac{2+h+1}{2} = \frac{0+k+1}{1} \Rightarrow h+3 = 2k+2 \Rightarrow h-2k = -1 \Rightarrow h = 2k-1$ (2) [by formula]

and $\frac{0+h+1}{-1} = \frac{2+k+1}{-2} \Rightarrow \frac{h+1}{1} = \frac{k+3}{2} \Rightarrow 2h+2 = k+3 \Rightarrow 2h-k = 1$ (3)

$(2)\times 2 \quad 2h - 4k = -2$ (4)

Subtracting (4) from (3) $\quad -3k = -3 \Rightarrow k = 1$ and $h = 2k-1 = 2-1 = 1$

$\therefore$ integrating factor is $\boldsymbol{xy}$

Multiplying (1) by xy, it becomes exact differential equation.

$\therefore \quad xy(2x^2y - y^3)dx + xy(x^3 - 2xy^3)dy = 0$ is exact.

$\Rightarrow \quad (2x^3y^2 - xy^4)dx + (x^4y - 2x^2y^3)dy = 0$

For this exact equation $M = 2x^3y^2 - xy^4$, $N = x^4y - 2x^2y^3$

Treating y as constant, integrate M w.r.to x.

$$\therefore \quad F = \int Mdx = \int (2x^3y^2 - xy^4)dx = 2y^2\int x^3dx - y^4\int x\,dx$$

$$= 2y^2\cdot\frac{x^4}{4} - y^4\cdot\frac{x^2}{2} = \frac{1}{2}[x^4y^2 - x^2y^4]$$

In $N = x^4y - 2x^2y^3$, there is no term without x and there is no constant term. $\therefore \quad G = 0$

$\therefore$ the general solution is $F + G = C$

$$\Rightarrow \quad \frac{1}{2}(x^4y^2 - x^2y^4) = C \Rightarrow x^2y^2(x^2 - y^2) = 2C = C_1$$

EXAMPLE 10

Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0.$

Solution.

The given equation is $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ (1)

$\Rightarrow \quad y^2dx - xydy + 2x^2ydx + 2x^3dy = 0 \Rightarrow y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$

This is of the form $x^a y^b(mydx+nxdy)+x^{a_1}y^{b_1}(m_1 y\,dx+n_1 xdy)=0$

So, $x^h y^k$ is an I.F, by rule 5.

Here $a=0,\quad b=1,\quad m=1,\quad n=-1$ and $a_1=2,\quad b_1=0,\quad m_1=2,\quad n_1=2$

$$\therefore\quad \frac{0+h+1}{1}=\frac{1+k+1}{-1} \Rightarrow h+1=-k-2 \Rightarrow h+k=-3 \qquad \text{[By formula]}$$

$$\text{and}\quad \frac{2+h+1}{2}=\frac{0+k+1}{2} \Rightarrow h+3=k+1 \Rightarrow h-k=-2$$

Adding, $2h=-5 \Rightarrow h=-\frac{5}{2}$ and $k=-3-h=-3+\frac{5}{2}=-\frac{1}{2}$

$\therefore$ integrating factor is $\boldsymbol{x^{-\frac{5}{2}}y^{-\frac{1}{2}}}$

Multiplying (1) by $x^{-\frac{5}{2}}y^{-\frac{1}{2}}$, it becomes an exact differential equation.

$$\therefore\quad x^{-\frac{5}{2}}y^{-\frac{1}{2}}(y^2+2x^2y)dx+x^{-\frac{5}{2}}y^{-\frac{1}{2}}(2x^3-xy)dy=0 \text{ is exact}$$

$$\Rightarrow\quad \left(x^{-\frac{5}{2}}y^{\frac{3}{2}}+2\cdot x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dx+\left(2x^{\frac{1}{2}}y^{-\frac{1}{2}}-x^{-\frac{3}{2}}y^{\frac{1}{2}}\right)dy=0$$

For this exact equation, $M=x^{-\frac{5}{2}}\cdot y^{\frac{3}{2}}+2x^{-\frac{1}{2}}\cdot y^{\frac{1}{2}}$ and $N=2x^{\frac{1}{2}}y^{-\frac{1}{2}}-x^{-\frac{3}{2}}y^{\frac{1}{2}}$

Treating y as constant, integrate M w.r.t. to x

$$\therefore\quad F=\int M\,dx=\int\left(x^{-\frac{5}{2}}y^{\frac{3}{2}}+2x^{-\frac{1}{2}}\cdot y^{\frac{1}{2}}\right)dx$$

$$=y^{\frac{3}{2}}\int x^{-\frac{5}{2}}dx+2y^{\frac{1}{2}}\int x^{-\frac{1}{2}}dx$$

$$=y^{\frac{3}{2}}\cdot\frac{x^{-\frac{5}{2}+1}}{-\frac{5}{2}+1}+2y^{\frac{1}{2}}\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}=-\frac{2}{3}y^{\frac{3}{2}}x^{-\frac{3}{2}}+4y^{\frac{1}{2}}x^{\frac{1}{2}}$$

In $N=\left(2x^{\frac{1}{2}}\cdot y^{-\frac{1}{2}}-x^{-\frac{3}{2}}\cdot y^{\frac{1}{2}}\right)$, there is no term without x and there is no constant term. $\therefore\quad G=0$

$\therefore$ the general solution is $F+G=C$

$$\Rightarrow\quad -\frac{2}{3}y^{\frac{3}{2}}x^{-\frac{3}{2}}+4y^{\frac{1}{2}}x^{\frac{1}{2}}=C \Rightarrow 12x^{\frac{1}{2}}y^{\frac{1}{2}}-2x^{-\frac{3}{2}}y^{\frac{3}{2}}=3C=C'$$

EXERCISE 10.9

Solve the following equations

1. $(x+2y)dx+(2x+y)dy=0.$
2. $y(y^3-x)dx+x(y^3+x)dy=0.$
3. $(y^2+2x^2y)dx+(2x^3-xy)dy=0.$
4. $(x^2+y^2)dx-2xydy=0.$
5. $(xy^2-e^{\frac{1}{x^3}})dx-x^2y\,dy=0.$
6. $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy=0.$
7. $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0.$
8. $(3xy-2y^2)dx+(x^2-2xy)dy=0.$
9. $y^2dx+(x^2-xy-y^2)dy=0.$
10. $(x^2-yx^2)dy+(y^2+x^2y^2)dx=0.$
11. $(2xy+e^x)dx-e^xdy=0.$
12. $xdx+ydy+4y^3(x^2+y^2)dy=0.$
13. $(2y-3xy^2)dx-xdy=0$ given that is has an integrating factor of the form x^hy^k.
14. Solve $(5x^2+12xy-3y^2)dx+(3x^2-2xy)dy=0$ given that x^h is an integrating factor.
15. Solve $(3xy-2ay^2)dx+(x^2-2ay)dy=0$ given that is has an integrating factor of the form x^h.

ANSWERS TO EXERCISE 10.9

1. $x^2+y^2+4xy=c$
2. $2xy^3-x^2=cy^2$
3. $6(xy)^{\frac{1}{2}}-x^{-\frac{3}{2}}y^{\frac{3}{2}}=c$
4. $x^2-y^2=cx$
5. $\frac{1}{3}e^{x^{-3}}-\frac{1}{2}\frac{y^2}{x^2}=c$
6. $xy-\frac{1}{xy}-2\log y=c$
7. $x^3y^3+x^2=cy$
8. $x^3y-x^2y^2=c$
9. $(x-y)y^2=c(x+y)$
10. $\frac{1}{x}+\frac{1}{y}+\log y-x=c$
11. $x^2y+e^x=cy$
12. $\log(x^2+y^2)+2y^4=c$
13. $\frac{x^2}{y}-x^3=c$
14. $x^5-x^3y^2+3x^4y=c$
15. $x^3y-ax^2y^2=c$

10.4 ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BUT OF DEGREE HIGHER THAN ONE

We shall now consider differential equations of the first order, but of degree higher than one.

For convenience $\frac{dy}{dx}$ is denoted by p.

The general form of the differential equation of the first order and n^{th} degree is

$$p^n+P_1p^{n-1}+P_2p^{n-2}+\ldots+P_{n-1}p+P_n=0 \tag{1}$$

where $P_1, P_2, \ldots, P_n$ are given functions of x and y.

In the general form, the equation (1) is not solvable. But there are some special cases which can be solved.

The following are the special types of equations:

1. **Equations solvable for p**
2. **Equations solvable for y**
3. **Equations solvable for x**
4. **Clairaut's equation**

10.4.1 Type 1 Equations Solvable for p

Suppose the equation

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0 \quad (1)$$

of n^{th} degree in p can be resolved into n linear factors, then it can be put into the form

$$(p - a_1)(p - a_2)\cdots(p - a_n) = 0 \quad (2)$$

where $a_1, a_2, \ldots, a_n$ are functions of x and y.

Equating the factors of (2) to zero, we get n differential equations each of the first order and first degree.

The component equation are

$$p - a_1 = 0, \quad p - a_2 = 0, \ \ldots, \quad p - a_n = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = a_1, \quad \frac{dy}{dx} = a_2, \ldots, \quad \frac{dy}{dx} = a_n$$

These equations can be solved by methods discussed earlier.

$\therefore$ the solutions are $\phi_1(x, y, c_1) = 0$, $\phi_2(x, y, c_2), \ldots, \phi_n(x, y, c_n) = 0$,

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

All possible solutions will be included in the solution

$$\phi_1(x, y, c_1) \cdot \phi_2(x, y, c_2) \ldots \phi_n(x, y, c_n) = 0$$

Since the equation is of first order, it can have only one constant.

So, we take $c_1 = c_2 = c_3 \cdots = c_n = c$

$\therefore$ the general solution is $\phi_1(x, y, c) \cdot \phi_2(x, y, c) \ldots \phi_n(x, y, c) = 0$

where c is arbitrary

WORKED EXAMPLES

EXAMPLE 1

Solve $p^2 + 2xp - 3x^2 = 0$.

Solution.

The given equation is $p^2 + 2xp - 3x^2 = 0 \quad \Rightarrow \quad (p - x)(p + 3x) = 0$

The component equations are $p = x$ and $p = -3x$

Now $\quad p = x \quad \Rightarrow \quad \dfrac{dy}{dx} = x \quad \Rightarrow \quad dy = x\,dx$

$$\Rightarrow \quad \int dy = \int x\,dx \quad \Rightarrow \quad y = \frac{x^2}{2} + c \quad \Rightarrow \quad y - \frac{x^2}{2} - c = 0$$

and $\quad p=-3x \quad \Rightarrow \quad \frac{dy}{dx}=-3x \quad \Rightarrow \quad dy=-3x\,dx$

$\Rightarrow \quad \int dy=-3\int x\,dx \quad \Rightarrow \quad y=-3\frac{x^2}{2}+c \quad \Rightarrow \quad y+3\frac{x^2}{2}-c=0$

$\therefore$ the general solution is

$$\left(y-\frac{x^2}{2}-c\right)\left(y+3\frac{x^2}{2}-c\right)=0$$

EXAMPLE 2

Solve $p^3+2xp^2-y^2p^2-2xy^2p=0$.

Solution.

The given equation is

$$p^3+2xp^2-y^2p^2-2xy^2p=0$$

$$\Rightarrow \quad p\left[p^2+2xp-y^2p-2xy^2\right]=0$$

$$\Rightarrow \quad p\left[p(p+2x)-y^2(p+2x)\right]=0 \quad \Rightarrow \quad p(p+2x)(p-y^2)=0$$

$\therefore$ the component equations are

$$p=0, \quad p+2x=0 \quad \text{and} \quad p-y^2=0$$

Now $\quad p=0 \quad \Rightarrow \quad \frac{dy}{dx}=0 \quad \Rightarrow \quad y=c \quad \Rightarrow \quad y-c=0$

$p+2x=0 \quad \Rightarrow \quad p=-2x \quad \Rightarrow \quad \frac{dy}{dx}=-2x \quad \Rightarrow \quad dy=-2x\,dx$

$\therefore \quad \int dy=-2\int x\,dx \quad \Rightarrow \quad y=-2\cdot\frac{x^2}{2}+c \quad \Rightarrow \quad y=-x^2+c \quad \Rightarrow \quad y+x^2-c=0$

and $p-y^2=0 \quad \Rightarrow \quad \frac{dy}{dx}=y^2 \quad \Rightarrow \quad \frac{dy}{y^2}=dx$

$\therefore \quad \int y^{-2}dy=\int dx \quad \Rightarrow \quad \frac{y^{-1}}{-1}=x+c \quad \Rightarrow \quad -\frac{1}{y}=x+c \quad \Rightarrow \quad y(x+c)+1=0$

$\therefore$ the general solution is

$$(y-c)(y+x^2-c)\{y(x+c)+1\}=0$$

EXAMPLE 3

Solve $x^2p^2-2xyp+2y^2-x^2=0$.

Solution.

The given equation is

$$x^2p^2-2xyp+(2y^2-x^2)=0$$

Treating as a quadratic in p, we get

$$p=\frac{2xy\pm\sqrt{4x^2y^2-4x^2(2y^2-x^2)}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^4}}{2x^2}$$

$$= \frac{2xy \pm \sqrt{4x^4 - 4x^2y^2}}{2x^2} = \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{2x^2} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$

This is homogeneous equation, since $f_1(x, y) = y \pm \sqrt{x^2 - y^2}$ and $f_2(x, y) = x$ are homogeneous of the same degree 1

To solve, put $y = vx$ $\therefore$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ $$v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x} = \frac{vx \pm x\sqrt{1 - v^2}}{x} = v \pm \sqrt{1 - v^2}$$

$\therefore$ $$x\frac{dv}{dx} = \pm\sqrt{1 - v^2} \Rightarrow \frac{dv}{\sqrt{1 - v^2}} = \pm\frac{dx}{x}$$ [Separating the variables]

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 - v^2}} = \pm\int \frac{dx}{x} \Rightarrow \sin^{-1} v = \pm\log_e x + c \Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \pm\log_e x + c$$

$\therefore$ $\sin^{-1}\left(\frac{y}{x}\right) - \log_e x - c = 0$ and $\sin^{-1}\left(\frac{y}{x}\right) + \log_e x - c = 0$

$\therefore$ the general solution is

$$\left[\sin^{-1}\left(\frac{y}{x}\right) - \log_e x - c\right]\left[\sin^{-1}\left(\frac{y}{x}\right) + \log_e x - c\right] = 0$$

EXERCISE 10.10

Solve the following differential equations

1. $p^2 - 5p + 6 = 0.$
2. $xp^2 - 2yp + x = 0.$
3. $yp^2 + (x - y)p - x = 0.$
4. $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}.$
5. $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0.$
6. $xyp^2 - (x^2 + y^2)p + xy = 0.$
7. $x^2p^2 + 3xyp + 2y^2 = 0.$
8. $p^2 + (x + y - 2\frac{y}{x})p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0.$
9. $p^2 - p(e^x + e^{-x}) + 1 = 0.$
10. $2p^2 - (x + 2y^2)p + xy^2 = 0.$

ANSWERS TO EXERCISE 10.10

1. $(y-3x-c)(y-2x-c)=0$
2. $(y+\sqrt{y^2-x^2}-c)(y+\sqrt{y^2-x^2}-cx^2)=0$
3. $(y-x-c)(y^2+x^2-c)=0$
4. $(xy-c)(x^2-y^2-c)=0$
5. $(y-cx^2)(y^2+3x^2-c)=0$
6. $(y-cx)(y^2-x^2-c)=0$
7. $(xy-c)(yx^2-c)=0$
8. $(ye^x-cx)(y+x^2-cx)=0$
9. $(y-e^x-c)(y+e^{-x}-c)=0$
10. $(4y-x^2-c)(xy+cy+1)=0$

10.4.2 Type 2 Equations Solvable for *y*

Let the first order differential equation $f(x, y, p)=0$

where $p=\dfrac{dy}{dx}$, be solvable for y and let $\quad y=F(x,p) \quad (1)$

Differentiating w.r.t. to x, we get

$$\frac{dy}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial p}\cdot\frac{dp}{dx} \qquad \left[\because dy=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial p}dp\right]$$

$$\Rightarrow \quad p=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial p}\cdot\frac{dp}{dx}$$

$$\Rightarrow \quad p=\phi\left(x, p, \frac{dp}{dx}\right) \qquad (2)$$

This is a first order equation in p and solving it, we obtain the solution

$$\psi(x, p, c)=0 \qquad (3)$$

Eliminating p using (1) and (3), we get the relation between x, y and c, which is the general solution of (1).

Note

(1) Sometimes the elimination of p using (1) and (3) may be tedious or impossible.

(2) In such cases we may rewrite the solution as parametric equations $x = x(p, c)$ and $y = y(p, c)$, treating p as parameter.

(3) When p is eliminated, the solution may result in a relation between the variables without any arbitrary constant.

This solution is called the *singular solution*.

Remark: If the given equation is a quadratic in p, the discriminant = 0 is called the p-discriminant relation.

If the general solution $\phi(x, y, c) = 0$ is a quadratic in c, its discriminant = 0 is called the c-discriminant relation and it gives the singular solution. When p-discriminant and c-discriminant are given, their common factor = 0 is the singular solution.

The discriminant is taken as $B^2 - 4AC = 0$.

Type 2 Equations Solvable for *y*

Let the first order differential equation $f(x, y, p) = 0$

where $p = \dfrac{dy}{dx}$, be solvable for y and let $\quad y = F(x, p) \quad (1)$

Differentiating w.r.t. to x, we get

$$\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \cdot \frac{dp}{dx} \qquad \left[\because dy = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial p} dp\right]$$

$$\Rightarrow \quad p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} \cdot \frac{dp}{dx}$$

$$\Rightarrow \quad p = \phi\left(x, p, \frac{dp}{dx}\right) \quad (2)$$

This is a first order equation in p and solving it, we obtain the solution

$$\psi(x, p, c) = 0 \quad (3)$$

Eliminating p using (1) and (3), we get the relation between x, y and c, which is the general solution of (1).

Note

(1) Sometimes the elimination of p using (1) and (3) may be tedious or impossible.

(2) In such cases we may rewrite the solution as parametric equations $x = x(p, c)$ and $y = y(p, c)$, treating p as parameter.

(3) When p is eliminated, the solution may result in a relation between the variables without any arbitrary constant.

This solution is called the *singular solution*.

Remark: If the given equation is a quadratic in p, the discriminant = 0 is called the p-discriminant relation.

If the general solution $\phi(x, y, c) = 0$ is a quadratic in c, its discriminant = 0 is called the c-discriminant relation and it gives the singular solution. When p-discriminant and c-discriminant are given, their common factor = 0 is the singular solution.

The discriminant is taken as $B^2 - 4AC = 0$.

WORKED EXAMPLES

EXAMPLE 1

Solve $xp^2 - 2yp + x = 0$.

Solution.

The given equation is

$$xp^2 - 2yp + x = 0 \tag{1}$$

Solving for y, we get

$$2yp = x + xp^2 \quad \Rightarrow \quad y = \frac{x(1+p^2)}{2p}$$

Differentiate w.r. to x, we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{p\left[x \cdot 2p\frac{dp}{dx} + (1+p^2)\cdot 1\right] - x(1+p^2)\cdot\frac{dp}{dx}}{p^2}$$

$$\Rightarrow \qquad p = \frac{1}{2}\frac{\left\{2p^2x\frac{dp}{dx} + p(1+p^2) - x(1+p^2)\frac{dp}{dx}\right\}}{p^2}$$

$$\Rightarrow \qquad 2p^3 = 2p^2x\frac{dp}{dx} + p(1+p^2) - x\frac{dp}{dx} - xp^2\frac{dp}{dx}$$

$$\Rightarrow \qquad 2p^3 = p^2x\frac{dp}{dx} + p(1+p^2) - x\frac{dp}{dx}$$

$$\Rightarrow \qquad 2p^3 - p(1+p^2) = x\frac{dp}{dx}(p^2-1)$$

$$\Rightarrow \qquad p^3 - p = x\frac{dp}{dx}(p^2-1)$$

$$\Rightarrow \qquad p(p^2-1) = x\frac{dp}{dx}\cdot(p^2-1)$$

$$\Rightarrow \qquad p(p^2-1) - (p^2-1)x\frac{dp}{dx} = 0$$

$$\Rightarrow \qquad \left(p - x\frac{dp}{dx}\right)(p^2-1) = 0 \quad \Rightarrow \quad p - x\frac{dp}{dx} = 0 \quad \text{or} \quad p^2 - 1 = 0$$

Case 1

Take $p - x\dfrac{dp}{dx} = 0.$

$$\Rightarrow \quad x\frac{dp}{dx} = p \quad \Rightarrow \quad \frac{dp}{p} = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{dp}{p} = \int \frac{dx}{x} \quad \Rightarrow \quad \log_e p = \log_e x + \log_e c$$

$$\Rightarrow \quad \log_e p = \log_e cx \quad \Rightarrow \quad p = cx \qquad (2)$$

Substituting for p in (1), we get

$$x \cdot c^2x^2 - 2ycx + x = 0 \quad \Rightarrow \quad c^2x^2 - 2cy + 1 = 0 \quad \Rightarrow \quad 2cy = c^2x^2 + 1$$

which is the general solution.

Case 2

Let $p^2 - 1 = 0 \quad \Rightarrow \quad p = \pm 1$

Substituting in (1), we get

$$x(\pm 1)^2 - 2y(\pm 1) + x = 0 \quad \Rightarrow \quad x \pm 2y + x = 0 \quad \Rightarrow \quad 2x \pm 2y = 0$$

$$\therefore \quad x \pm y = 0 \quad \Rightarrow \quad x + y = 0 \text{ or } x - y = 0$$

which are the singular solutions, since they do not contain any arbitrary constants.

Aliter: The general solution is $x^2c^2 - 2yc + 1 = 0$.

Here $A = x^2$, $B = -2y$, $C = 1$

c-discriminant is $B^2 - 4AC = 0 \Rightarrow 4y^2 - 4x^2 \cdot 1 = 0$

$\Rightarrow \quad y^2 - x^2 = 0 \quad \Rightarrow \quad (y - x)(y + x) = 0$

$\Rightarrow \quad x - y = 0$ or $x + y = 0$

are the singular solutions.

EXAMPLE 2

Solve $y + px = x^4p^2$.

Solution.

The given equation is

$$y + px = x^4p^2 \quad \Rightarrow \quad y = -px + x^4p^2$$

$\therefore$ differentiating w.r. to x, we get

$$\frac{dy}{dx} = -\left(p \cdot 1 + x\frac{dp}{dx}\right) + x^4 2p\frac{dp}{dx} + p^2 \cdot 4x^3$$

$$\Rightarrow \quad p = -p + (2x^4p - x)\frac{dp}{dx} + 4p^2x^3$$

$$\Rightarrow \quad 2p - 4p^2x^3 = x(2px^3 - 1)\frac{dp}{dx}$$

$\Rightarrow \qquad 2p(1-2px^3) = x(2px^3-1)\dfrac{dp}{dx}$

$\Rightarrow \qquad 2p(1-2px^3) - x(2px^3-1)\dfrac{dp}{dx} = 0$

$\Rightarrow \qquad (2px^3-1)\left[2p + x\dfrac{dp}{dx}\right] = 0 \quad \Rightarrow \quad 2px^3 - 1 = 0 \text{ or } 2p + \dfrac{xdp}{dx} = 0$

Case (i)

Let $\quad 2p + x\dfrac{dp}{dx} = 0 \quad \Rightarrow \quad x\dfrac{dp}{dx} = -2p \quad \Rightarrow \quad \dfrac{dp}{p} = -2\dfrac{dx}{x}$

$\therefore \quad \int \dfrac{dp}{p} = -2\int \dfrac{dx}{x} \quad \Rightarrow \quad \log_e p = -2\log_e x + \log c \quad \Rightarrow \quad \log_e p = \log c\, x^{-2} \quad \Rightarrow \quad p = c\, x^{-2}$

Substituting for p in (1), we get

$$y + cx^{-2} \cdot x = x^4 c^2 x^{-4} \quad \Rightarrow \quad y + cx^{-1} = c^2 \quad \Rightarrow \quad y + \frac{c}{x} = c^2$$

which is the general solution

Case (ii)

Let $\quad 2px^3 - 1 = 0 \quad \Rightarrow \quad 2px^3 = 1 \quad \Rightarrow \quad p = \dfrac{1}{2x^3}$

Substituting for p in (1), we get

$$y + \frac{1}{2x^3}x = x^4 \cdot \frac{1}{4x^6} \quad \Rightarrow \quad y + \frac{1}{2x^2} = \frac{1}{4x^2}$$

Multiplying by $4x^2$, we get

$$4x^2 y + 2 = 1 \quad \Rightarrow \quad 4x^2 y + 1 = 0,$$

which is the singular solution, since it does not contain any arbitrary constant.

Aliter: General solution is $\quad y + \dfrac{c}{x} = c^2 \quad \Rightarrow \quad xc^2 - c - xy = 0$

Here $A = x, \quad B = -1, \quad C = -xy$

$\therefore B^2 - 4AC = 0 \quad \Rightarrow \quad 1 - 4x(-xy) = 0 \quad \Rightarrow \quad 1 + 4x^2y = 0$ is the singular solution.

EXAMPLE 3

Solve $y^2 = 1 + p^2$.

Solution.

The given equation is $y^2 = 1 + p^2 \quad \Rightarrow \quad y = \pm\sqrt{1+p^2}$ (1)

Take $\quad y = \sqrt{1+p^2}$ (2)

This is solvable for y.

Differentiating w.r.to x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+p^2}} \cdot 2p\frac{dp}{dx} \quad \Rightarrow \quad p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx} \quad \Rightarrow \quad p\left(\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} - 1\right) = 0$$

$$\Rightarrow \qquad p = 0 \quad \text{or} \quad \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} - 1 = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx} = 1$$

Case (i) Take $\dfrac{dp}{\sqrt{1+p^2}} = dx$

$$\Rightarrow \qquad \int \frac{dp}{\sqrt{1+p^2}} = \int dx \quad \Rightarrow \quad \log_e\left(p + \sqrt{1+p^2}\right) = x + c$$

$$\therefore \qquad x = \log_e\left(p + \sqrt{1+p^2}\right) - c \quad \text{and} \quad y = \sqrt{1+p^2}$$

which give the solutions.

Case (ii) If $p = 0$, then $y^2 = 1$ is the singular solution.

Similarly, taking $y = -\sqrt{1+p^2}$, we can find the solution, $x = -\log\left(P + \sqrt{1+p^2}\right) + c'$, $y = -\sqrt{1+p^2}$

10.4.3 Type 3 Equations Solvable for x

Let the first order differential equation $f(x, y, p) = 0$ be solvable for x and let $x = F(y, p)$ (1)
Differentiating w.r.to y, we get

$$\frac{dx}{dy} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} \quad \Rightarrow \quad \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y}$$

This is of the form $\phi\left(y, p, \dfrac{\partial p}{\partial y}\right) = 0$ (2)

It is a first order differential equation in p.

Let the solution be $\psi(y, p, c) = 0$ (3)

Eliminating p using (1) and (3), we get the general solution.

Note

(1) Sometimes the elimination of p using (1) and (3) may be tedious or impossible.
In such cases we write the solution as parametric equation $x = x(p, c)$, $y = y(p, c)$, treating p as the parameter.

(2) When p is eliminated the solution may result in a relation between the variables x and y without any arbitrary constant.
This solution is called singular solution.

WORKED EXAMPLES

EXAMPLE 1

Solve $p^2y + 2px = y$.

Solution

The given equation is $p^2y + 2px = y$ (1)

Solving for x, we get $2px = y - p^2y \Rightarrow x = \dfrac{y(1-p^2)}{2p}, \quad p \neq 0$

Differentiating w.r.to y, we get

$$\frac{dx}{dy} = \frac{1}{2}\left[\frac{1-p^2}{p}\cdot 1 + y\left[\frac{p\left(-2p\cdot\frac{dp}{dy}\right)-(1-p^2)\frac{dp}{dy}}{p^2}\right]\right]$$

$$\Rightarrow \quad \frac{1}{p} = \frac{1}{2p} - \frac{p}{2} - \frac{y}{2p^2}\left[(2p^2+1-p^2)\frac{dp}{dy}\right]$$

$$\Rightarrow \frac{1}{p} - \frac{1}{2p} + \frac{p}{2} = -\frac{y}{2p^2}(1+p^2)\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1+p^2}{2p} = -\frac{y}{2p^2}(1+p^2)\frac{dp}{dy} \Rightarrow 1 = -\frac{y}{p}\frac{dp}{dy} \Rightarrow \frac{dy}{y} = -\frac{dp}{p}$$

$$\therefore \quad \int\frac{dy}{y} = -\int\frac{dp}{p} \Rightarrow \log_e y = -\log_e p + \log_e c \Rightarrow \log_e y = \log_e\frac{c}{p} \Rightarrow y = \frac{c}{p} \quad (2)$$

Substituting for p in (1) we get

$$\frac{c^2}{y^2}\cdot y + 2\cdot\frac{c}{y}x = y \Rightarrow \frac{c^2}{y} + \frac{2cx}{y} = y \Rightarrow y^2 = c^2 + 2cx$$

which is the general solution of (1)

EXAMPLE 2

Solve $y^2 \log_e y = xyp + p^2$.

Solution.

The given equation is $y^2\log_e y = xyp + p^2$ (1)

Solving for x, we get

$$xyp = y^2\log_e y - p^2$$

$$\Rightarrow \quad x = \frac{y^2\log_e y}{yp} - \frac{p^2}{yp} = \frac{y\log_e y}{p} - \frac{p}{y}$$

Differentiating w.r.to y, we get

$$\frac{dx}{dy} = \frac{p\left[y\frac{1}{y} + \log_e y\right] - y\log_e y\frac{dp}{dy}}{p^2} - \frac{\left(y\frac{dp}{dy} - p\cdot 1\right)}{y^2}$$

$$\Rightarrow \quad \frac{1}{p} = \frac{p(1+\log_e y)}{p^2} - \frac{y\log_e y \frac{dp}{dy}}{p^2} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \quad \frac{1}{p} = \frac{1}{p} + \frac{1}{p}\log_e y - \left(\frac{y\log_e y}{p^2} + \frac{1}{y}\right)\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \quad \left(\frac{y^2\log_e y + p^2}{p^2 y}\right)\frac{dp}{dy} = \frac{1}{p}\log_e y + \frac{p}{y^2}$$

$$\Rightarrow \quad \frac{(y^2\log_e y + p^2)}{p^2 y}\frac{dp}{dy} = \frac{(y^2\log_e y + p^2)}{py^2}$$

$$\Rightarrow \quad (y^2\log_e y + p^2)\frac{dp}{dy} = \frac{p}{y}(y^2\log_e y + p^2)$$

$$\Rightarrow \frac{dp}{dy}(y^2\log_e y + p^2) - \frac{p}{y}(y^2\log_e y + p^2) = 0$$

$$\Rightarrow \quad (y^2\log_e y + p^2)\left(\frac{dp}{dy} - \frac{p}{y}\right) = 0 \quad \Rightarrow \quad y^2\log_e y + p^2 = 0 \quad \text{or} \quad \frac{dp}{dy} - \frac{p}{y} = 0$$

Case (i) Let $\frac{dp}{dy} = \frac{p}{y} \quad \Rightarrow \quad \frac{dp}{p} = \frac{dy}{y}$

$$\therefore \quad \int \frac{dp}{p} = \int \frac{dy}{y} \quad \Rightarrow \quad \log_e p = \log_e y + \log_e c = \log_e cy$$

$$\Rightarrow \quad p = cy \qquad (2)$$

Substituting for p in (1), we get

$$y^2\log_e y = xy\cdot cy + c^2y^2 \quad \Rightarrow \quad \log_e y = cx + c^2$$

which is the general solution of (1)

Case (ii) Let $y^2\log_e y + p^2 = 0 \quad \Rightarrow \quad p^2 = -y^2\log_e y \qquad (3)$

Substituting in (1), we get

$$y^2\log_e y = xyp - y^2\log_e y \quad \Rightarrow \quad xyp = 2y^2\log_e y \quad \Rightarrow \quad p = \frac{2y\log_e y}{x}$$

Squaring, $p^2 = \frac{4y^2}{x^2}(\log y)^2 \quad \Rightarrow \quad -y^2\log y = \frac{4y^2}{x^2}(\log y)^2$ [using (3)]

$$\Rightarrow \quad -1 = \frac{4}{x^2}\log_e y \quad \Rightarrow \quad x^2 + 4\log_e y = 0 \qquad [\because y^2\log_e y \neq 0]$$

which is the singular solution, since it does not contain the arbitrary constant.

Aliter: To find the singular solution

$$c^2 + cx - \log_e y = 0$$

Here A = 1, B = x, $C = \log_e y$

$\therefore$ $$B^2 - 4AC = 0 \Rightarrow x^2 - 4(-\log_e y) = 0 \Rightarrow x^2 + 4\log_e y = 0$$

which is the singular solution.

EXERCISE 10.11

Solve the following equations

1. $y = 3x + \log_e p$.
2. $4y = x^2 + p^2$.
3. $xp^2 + 2px - y = 0$.
4. $y = (1+p)x + p^2$.
5. $x = p + p^4$.
6. $y = 2px + y^2 p^3$.
7. $x = \tan^{-1} p + \dfrac{p}{1+p^2}$.
8. $3px - y + 6p^2 y^2 = 0$.

ANSWERS TO EXERCISE 10.11

1. $y = 3x + \log_e \dfrac{3}{1 - ce^{3x}}$
2. $y = \dfrac{x^2 + p^2}{4}$ and $\log_e \dfrac{c(p-x)}{x^2} = \dfrac{x}{p-x}$
3. $(y-c)^2 = 4cx$ is the general solution and $x + y = 0$ is the singular solution.
4. $x = 2(1-p) + ce^{-p}$, $y = 2 - p^2 + (1+p)ce^{-p}$
5. $y = \dfrac{p^2}{2} + 4\dfrac{p^5}{5} + c$, $x = p + p^4$
6. $y^2 = 2cx + c^3$ is the general solution and $27y^4 + 32x^3 = 0$ is the singular solution.
7. $x - \sqrt{\dfrac{y+1-c}{c-y}}x(c-y) = \tan^{-1}\sqrt{\dfrac{y+1-c}{c-y}}$
8. $y^3 = 3c(x+2c)$

10.4.4 Type 4 Clairaut's Equation

The first order differential equation $y = px + f(p)$ is known as Clairaut's equation, where f is known function.

The General and the singular solutions of Clairaut's equation

The clairaut's equation is $$y = px + f(p) \tag{1}$$

Differentiating w.r.to x,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$\Rightarrow$ $$p = p + \{x + f'(p)\}\frac{dp}{dx}$$

$$\Rightarrow \qquad \{x+f'(p)\}\frac{dp}{dx}=0 \quad \Rightarrow \quad \frac{dp}{dx}=0 \quad \text{or} \quad x+f'(p)=0$$

Case (i) Let $\quad \frac{dp}{dx}=0 \quad \Rightarrow \quad p=c$

$\therefore$ substituting in (1), we get $\quad y=cx+f(c)$
which is the general solution of (1).

Case (ii) Let $\quad x+f'(p)=0 \qquad (2)$

Eliminating p using (1) and (2), we get a solution without any arbitrary constant and which cannot be obtained from the general solution. Hence, it is the singular solution of the differential equation.

Note (1) The singular solution is the envelope of the family of straight lines $y=cx+f(c)$ given by the general solution.

(2) **To find the general solution replace p by c in the given clairaut's equation.**

Extended form of Clairaut's Equation

The extended form of Clairaut's equation is

$$y=xg(p)+f(p) \qquad (3)$$

It is also known as Lagrange's equation of first order.

Differentiating w.r.to x, it will reduce to a linear equation in p and its solution is

$$F(x, p, c)=0 \qquad (4)$$

Eliminating p using (3) and (4), we get the general solution.

WORKED EXAMPLES

EXAMPLE 1

Find the general and singular solutions of $y=(x-a)p-p^2$.

Solution.

The given equation is

$$y=(x-a)p-p^2. \qquad (1)$$

It is Clairaut's form. Replacing p by c, the general solution is

$$y=(x-a)c-c^2 \quad \Rightarrow \quad c^2-(x-a)\,c+y=0 \qquad (2)$$

To find the singular solution

The singular solution is the envelope of (2)

Since (2) is a quadratic in c the envelope is $B^2-4AC=0$

Here $A=1$, $B=-(x-a)$, $C=y$

$$\therefore \qquad (x-a)^2-4y=0 \quad \Rightarrow \quad 4y=(x-a)^2,$$

which is the singular solution.

EXAMPLE 2

Solve $y = 2px + y^2p^3$.

Solution.

The given equation is

$$y = 2px + y^2p^3 \tag{1}$$

It is not Clairaut's form, but it can be reduced to Clairaut's form.

Multiplying (1) by y, we get

$$y^2 = 2\,pxy + y^3p^3 \tag{2}$$

Put $y^2 = Y$, then $2y\dfrac{dy}{dx} = \dfrac{dY}{dx} \quad \Rightarrow \quad 2yp = \dfrac{dY}{dx} \quad \Rightarrow \quad yp = \dfrac{1}{2}\dfrac{dY}{dx} = \dfrac{1}{2}P$, where $P = \dfrac{dY}{dx}$

$\therefore$ (2) becomes

$$Y = xP + \left(\frac{1}{2}P\right)^3 \quad \Rightarrow \quad Y = xP + \frac{P^3}{8}$$

This is in Clairaut's form

Replacing P by c, the general solution is $Y = cx + \dfrac{c^3}{8} \quad \Rightarrow \quad y^2 = cx + \dfrac{c^3}{8}$ (3)

To find the singular solution

The singular solution is the envelope of (3)

Differentiating (3) partially w.r.to c,

$$0 = x + \frac{3}{8}c^2 \quad \Rightarrow \quad c^2 = -\frac{8x}{3} \tag{4}$$

Substituting in (3)

$$y^2 = cx + \frac{1}{8}c\left(-\frac{8x}{3}\right) \quad \Rightarrow \quad cx - \frac{cx}{3} = \frac{2}{3}cx \quad \Rightarrow \quad c = \frac{3y^2}{2x}$$

$\therefore$ (4) becomes $\quad \left(\dfrac{3y^2}{2x}\right)^2 = -\dfrac{8x}{3} \quad \Rightarrow \quad 9\dfrac{y^4}{4x^2} = -\dfrac{8x}{3} \quad \Rightarrow \quad 27y^4 = -32x^3$

$\Rightarrow \quad 27y^4 + 32x^3 = 0$ which is the singular solution.

EXAMPLE 3

Solve $y = 2px + yp^2$.

Solution.

The given system is $y = 2px + yp^2$ (1)

It is not Clairaut's form. It can be reduced to clairaut's form.

Put $\quad X = 2x \quad$ and $\quad Y = y^2 \quad \therefore \quad dX = 2dx \quad$ and $\quad dY = 2ydy$

$$\therefore \quad \frac{dY}{dX} = \frac{2y}{2}\cdot\frac{dy}{dx} = y\frac{dy}{dx} \quad \Rightarrow \quad P = yp, \quad \text{where } P = \frac{dY}{dX}, \quad p = \frac{dy}{dx}$$

Multiplying (1) by y, we get

$$y^2 = 2pxy + y^2p^2 \quad \Rightarrow \quad y^2 = 2x(yp) + (yp)^2 \quad \Rightarrow \quad Y = XP + P^2$$

which is Clairaut's form in P.

$\therefore$ the general solution is $Y = Xc + c^2 \quad \Rightarrow \quad y^2 = 2cx + c^2$ (2)

To find the singular solution

The singular solution is the envelope (2)

Differentiating (2) partially w.r.to c, we get $0 = 2x + 2c \quad \Rightarrow \quad c = -x$

Substituting in (2) we get $y^2 = -2x^2 + x^2 = -x^2 \quad \Rightarrow \quad x^2 + y^2 = 0$

which is the singular solution.

EXAMPLE 4

Find the general and singular solution of $x^2(y - px) = yp^2$.

Solution.

The given equation is $x^2(y - px) = yp^2$ (1)

It is not Clairaut's form

But it can be reduced to Clairaut's form

Put $x^2 = X$ and $y^2 = Y$ $\quad \therefore \quad 2xdx = dX$ and $2ydy = dY$

$$\therefore \quad \frac{dY}{dX} = \frac{2y}{2x} \cdot \frac{dy}{dx} = \frac{y}{x} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad P = \frac{y}{x}p \quad \Rightarrow \quad p = \frac{x}{y}P, \text{ where } P = \frac{dY}{dX} \text{ and } p = \frac{dy}{dx}$$

$\therefore$ equation (1) becomes $X\left(y - x \cdot \frac{x}{y}P\right) = y \cdot \frac{x^2}{y^2}P^2 \quad \Rightarrow \quad X\frac{(y^2 - x^2P)}{y} = \frac{x^2}{y}P^2$

$$\Rightarrow \quad X(Y - XP) = XP^2 \quad \Rightarrow \quad Y - XP = P^2 \quad \Rightarrow \quad Y = XP + P^2$$

This is Clairaut's form in P

$\therefore$ the general solution is $Y = cX + c^2 \quad \Rightarrow \quad y^2 = cx^2 + c^2$ (2)

To find the singular solution

The singular solution is the envelope of (2)

Differentiating (2) partially w.r.to c, $\quad 0 = x^2 + 2c \quad \Rightarrow \quad c = -\frac{x^2}{2}$

Substituting in (2), we get $y^2 = -\frac{x^2}{2} \cdot x^2 + \left(-\frac{x^2}{2}\right)^2 = -\frac{x^4}{2} + \frac{x^4}{4} = -\frac{x^4}{4} \quad \Rightarrow \quad 4y^2 + x^4 = 0$

This is the singular solution.

Aliter:

(2) is $c^2 + cx^2 - y^2 = 0$.

It is quadratic in c. Here A = 1, B = x^2, C = $-y^2$

$\therefore$ the envelope of (2) is $B^2 - 4AC = 0 \quad \Rightarrow \quad x^4 - 4 \cdot 1(-y^2) = 0 \quad \Rightarrow \quad x^4 + 4y^2 = 0$

This is the singular solution.

Ordinary Second and Higher Order Differential Equations

11.0 INTRODUCTION

An important class of differential equations is the class of linear differential equation of the second and higher order with constant coefficients. Such differential equations arise in modelling physical and engineering problems such as the theory of electric circuits, mechanical vibrations, biological problems etc. The techniques used for second order linear differential equations can also be extended to higher order linear differential equations with constant coefficients.

Some standard form of linear differential equations with variable coefficients can be reduced to linear differential equations with constant coefficients and hence solved by using the methods of second and higher order linear differential equations with constant coefficients.

11.1 LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

The general form of the n^{th} order linear ordinary differential equation with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = Q(x) \tag{1}$$

where $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ are constants with $a_0 \neq 0$

If $Q(x) = 0$, then the equation (1) becomes

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{2}$$

which is called the **homogeneous equation** corresponding to (1).

The general solution of (2) is called the complementary function of (1) and is denoted by y_c. The general solution of (2) contains n arbitrary constants. Particular solution is a solution which does not contain any arbitrary constants.

If y_p is a particular solution of (1), then the general solution of (1) is

$$y = y_c + y_p$$

Note The general solution of an ordinary linear differential equation is also known as complete solution.

11.1.1 Complementary Function

We denote $D = \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2},\ D^3 = \frac{d^3}{dx^3},\ \cdots,\ D^n = \frac{d^n}{dx^n}$

Then equation (1) can be written as

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = Q(x) \tag{3}$$

To find the complementary function, we solve

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = 0 \tag{4}$$

Replacing D by m in (4), we get the equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \tag{5}$$

This equation (5) is called the **auxiliary equation** of (4).

Let $m_1, m_2, \ldots, m_n$ be the roots of (5).

Case (i): If the roots $m_1, m_2, \ldots, m_n$ are real and different, then the

$$\text{C.F is } y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

Case (ii): If some of the roots are real and equal, say

$$m_1 = m_2 = m_3 = \cdots = m_r = m, \quad r < n$$

and others are different, then the

$$\text{C.F is } y_c = (C_1 + C_2 x + C_3 x^2 + \cdots + C_r x^{r-1}) e^{mx} + C_{r+1} e^{m_{r+1} x} + \cdots + C_n e^{m_n x}$$

In particular, if 2 roots are equal, i.e., if $r = 2$, then the

C.F is $y_c = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

Case (iii): If two roots are complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ and the others are real and different, then the

C.F is $y_c = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x] + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

Case (iv): If $m_1 = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$ and the other roots are real and different, then the

C.F is $y_c = e^{\alpha x}[(C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x] + C_5 e^{m_5 x} + \cdots + C_n e^{m_n x}$

11.1.2 Particular Integral

Equation (3) can be written as

$$f(D)y = Q(x), \text{ where } f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

$$\therefore \qquad \text{P.I} = \frac{1}{f(D)} Q(x)$$

Depending upon the nature of the function $Q(x)$, we have different methods of finding the particular integral.

We shall consider here rules for finding the particular integral, when $Q(x)$ is of the form e^{ax}, $\sin ax$, $\cos ax$, x^n and $e^{ax} g(x)$, where $g(x)$ is x^n, $\sin ax$, $\cos ax$, $x^m \sin ax$ and $x^m \cos ax$.

TYPE 1: $Q(x) = e^{ax}$

$$\text{P.I} = \frac{1}{f(D)} e^{ax}$$

(1) If $f(a) \neq 0$, then P.I $= \dfrac{e^{ax}}{f(a)}$ [replace D by a]

(2) If $f(a) = 0$, then $f(D) = (D - a)^r g(D)$, where $g(a) \neq 0$

$$\therefore \text{ P.I} = \frac{1}{(D-a)^r g(D)} e^{ax} = \frac{1}{g(a)} \cdot \frac{1}{(D-a)^r} e^{ax} = \frac{1}{g(a)} \cdot \frac{x^r e^{ax}}{r!}$$

In particular, if $r = 1, 2, 3, \ldots$

$$\frac{1}{D-a} e^{ax} = \frac{x}{1!} e^{ax} = x e^{ax}, \quad \frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax} = \frac{x^2 e^{ax}}{2}$$

$$\frac{1}{(D-a)^3} e^{ax} = \frac{x^3}{3!} e^{ax} = \frac{x^3 e^{ax}}{6} \text{ etc.}$$

WORKED EXAMPLES

EXAMPLE 1

Solve the differential equation $(D^2 - 4D + 3)\,y = 0$.

Solution.

The given equation is $(D^2 - 4D + 3)y = 0$

Auxiliary equation is $m^2 - 4m + 3 = 0 \Rightarrow (m - 1)(m - 3) = 0 \Rightarrow m = 1, 3$

The roots are real and different.

$\therefore$ the solution is $y = C_1 e^x + C_2 e^{3x}$.

EXAMPLE 2

Solve the differential equation $(4D^2 - 4D + 1)\,y = 0$.

Solution.

The given equation is $(4D^2 - 4D + 1)\,y = 0$

Auxiliary equation is $4m^2 - 4m + 1 = 0 \Rightarrow (2m - 1)^2 = 0 \Rightarrow m = \dfrac{1}{2}, \dfrac{1}{2}$

The roots are real and equal,

$\therefore$ the solution is $y = (C_1 + C_2 x)e^{\frac{x}{2}}$

EXAMPLE 3

Solve the differential equation $(D^2 - 2D + 2)\,y = 0$.

Solution.

The given equation is $(D^2 - 2D + 2)\,y = 0$

Auxiliary equation is $m^2 - 2m + 2 = 0$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

The roots are complex numbers with $\alpha = 1$, $\beta = 1$

$\therefore$ the solution is $y = e^x\,[C_1 \cos x + C_2 \sin x]$

EXAMPLE 4

Solve $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = e^{2x}$.

Solution.
The given equation is

$$(D^2 + 6D + 5)y = e^{2x}$$

To find the complementary function, solve $(D^2 + 6D + 5)y = 0$
Auxiliary equation is $m^2 + 6m + 5 = 0$

$\Rightarrow$ $(m + 5)(m + 1) = 0 \Rightarrow m = -5$ or -1.

The roots are real and different.

$\therefore$ $$\text{C.F} = C_1 e^{-5x} + C_2 e^{-x}$$

and $$\text{P.I} = \frac{1}{D^2 + 6D + 5} e^{2x} = \frac{e^{2x}}{2^2 + 6\cdot 2 + 5} = \frac{e^{2x}}{4 + 12 + 5} = \frac{e^{2x}}{21} \qquad [\because f(a) \neq 0, \text{ here } a = 2]$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = C_1 e^{-5x} + C_2 e^{-x} + \frac{e^{2x}}{21}$$

EXAMPLE 5

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$.

Solution.
The given equation is

$$(D^2 + 4D + 5)y = -2\cosh x = -2\left[\frac{e^x + e^{-x}}{2}\right] = -[e^x + e^{-x}]$$

To find the complementary function, solve $(D^2 + 4D + 5)y = 0$
Auxiliary equation is $m^2 + 4m + 5 = 0$

$\Rightarrow$ $$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

The roots are complex numbers with $\alpha = -2$ and $\beta = 1$.

$\therefore$ $$\text{C.F} = e^{-2x}[C_1 \cos x + C_2 \sin x]$$

$$\text{P.I} = \frac{1}{D^2 + 4D + 5}\left[-(e^x + e^{-x})\right]$$

$$= \frac{1}{D^2 + 4D + 5}(-e^x) + \frac{1}{(D^2 + 4D + 5)}(-e^{-x}) = \text{P.I}_1 + \text{P.I}_2$$

$$\text{P.I}_1 = \frac{1}{D^2+4D+5}(-e^x) = -\frac{e^x}{1+4+5} = -\frac{e^x}{10}$$

$$\text{P.I}_2 = \frac{1}{D^2+4D+5}(-e^{-x}) = -\frac{e^{-x}}{1-4+5} = -\frac{e^{-x}}{2}$$

$\therefore$ $$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 = -\frac{e^x}{10} - \frac{e^{-x}}{2}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = e^{-2x}[C_1\cos x + C_2\sin x] - \frac{e^x}{10} - \frac{e^{-x}}{2}$$

EXAMPLE 6

Solve $(D^3 - 12D + 16)y = (e^x + e^{-2x})^2$.

Solution.
The given equation is

$$(D^3 - 12D + 16)y = (e^x + e^{-2x})^2$$

$\Rightarrow$ $$(D^3 - 12D + 16)y = e^{2x} + 2e^{-x} + e^{-4x}$$

To find the complementary function, solve $(D^3 - 12D + 16)y = 0$
Auxiliary equation is $m^3 - 12m + 16 = 0$
By trial $m = 2$ is a root.

2	1	0	−12	16
	0	2	4	−16
	1	2	−8	0

$\therefore$ the other roots are given by $m^2 + 2m - 8 = 0$

$\Rightarrow$ $(m+4)(m-2) = 0 \quad \Rightarrow \quad m = -4,\ m = 2$

$\therefore$ the roots are $m = 2, 2, -4$ and two roots are equal.

$\therefore$ $\text{C.F} = C_1 e^{-4x} + (C_2 + C_3 x)e^{2x}$

$$\text{P.I}_1 = \frac{1}{D^3 - 12D + 16}e^{2x}$$

$$= \frac{1}{(D+4)(D-2)^2}e^{2x}$$

$$= \frac{1}{6}\cdot\frac{1}{(D-2)^2}e^{2x} \quad = \frac{1}{6}\cdot\frac{x^2}{2!}e^{2x} \quad = \frac{x^2e^{2x}}{12}$$

$$\left[\because f(a) = f(+2) = 0, \text{ we use } \frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2!}e^{ax}\right]$$

$$\text{P.I}_2 = \frac{1}{D^3 - 12D + 16}2\cdot e^{-x} = 2\frac{e^{-x}}{(-1)^3 - 12(-1) + 16} = \frac{2e^{-x}}{-1+12+16} = \frac{2e^{-x}}{27}$$

$$\text{P.I}_3 = \frac{1}{D^3 - 12D + 16}e^{-4x}$$

$$= \frac{1}{(D-2)^2(D+4)}e^{-4x} \quad = \frac{1}{(-4-2)^2}xe^{-4x} = \frac{x}{36}e^{-4x}$$

$\left[\text{Here } f(a) = f(-4) = 0, \text{ we use } \frac{1}{D-a}e^{ax} = xe^{ax}\right]$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 + \text{P.I}_3$

$$\Rightarrow \quad y = C_1 e^{-4x} + (C_2 + C_3 x)e^{2x} + \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$$

TYPE 2: $Q(x)$ = sin ax or cos ax, where a is a constant

$$\text{P.I} = \frac{1}{f(D)}\sin ax = \frac{1}{f(D^2)}\sin ax \qquad [\text{rewriting in terms of } D^2]$$

$$= \frac{\sin ax}{f(-a^2)} \qquad \text{if } f(-a^2) \neq 0 \text{ i.e., replacing } D^2 \text{ by } -a^2$$

If $f(-a^2) = 0$, then $\text{P.I} = \frac{1}{f(D^2)}\sin ax = x\frac{1}{f'(D^2)}\sin ax$

where $f'(D^2)$ is the derivative of $f(D^2)$ w. r. to D

and $$\text{P.I} = \frac{x \sin ax}{f'(-a^2)} \quad \text{if } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then $$\text{P.I} = x^2\frac{1}{f''(D^2)}\sin ax = \frac{x^2 \sin ax}{f''(-a^2)} \text{ and so on.}$$

Similarly, we get for cos ax replacing sin ax

In particular, $\frac{1}{D^2+a^2}\sin ax = \frac{x}{2}\int \sin ax\, dx = -\frac{x\cos ax}{2a}$

and $$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\int \cos ax\, dx = \frac{x \sin ax}{2a}, \text{ if } f(-a^2) = 0$$

Aliter If $Q(x) = \sin ax$ or $\cos ax$

$$\text{P.I} = \frac{1}{f(D)}\sin ax = \text{I.P of } \frac{1}{f(D)}e^{iax} \qquad [\text{I.P} = \text{Imaginary Part}]$$

and $$\text{P.I} = \frac{1}{f(D)}\cos ax = \text{R.P of } \frac{1}{f(D)}e^{iax} \qquad [\text{R.P} = \text{Real Part}]$$

Now apply Type 1 procedure.

WORKED EXAMPLES

EXAMPLE 7

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution.

The given equation is

$$(D^2 - 4D + 3)y = \sin 3x \cos 2x$$

$$\Rightarrow \quad (D^2 - 4D + 3)y = \frac{1}{2}[\sin 5x + \sin x] = \frac{1}{2}\sin 5x + \frac{1}{2}\sin x$$

To find the complementary function, solve $(D^2 - 4D + 3)y = 0$

Auxiliary equation is $m^2 - 4m + 3 = 0$

$$\Rightarrow \quad (m-1)(m-3) = 0 \quad \Rightarrow m = 1 \text{ or } 3$$

The roots are real and different.

$$\therefore \quad \text{C.F} = C_1 e^x + C_2 e^{3x}$$

$$\text{P.I}_1 = \frac{1}{2} \cdot \frac{1}{(D^2 - 4D + 3)} \sin 5x$$

$$= \frac{1}{2} \frac{1}{(-5^2 - 4D + 3)} \sin 5x \qquad \text{[replacing } D^2 \text{ by } -5^2\text{]}$$

$$= -\frac{1}{4(2D + 11)} \sin 5x$$

$$= -\frac{2D - 11}{4(4D^2 - 121)} \sin 5x \qquad \text{[Multiplying Nr and Dr by } 2D - 11\text{]}$$

$$= -\frac{(2D - 11)\sin 5x}{4[4(-5^2) - 121]} = -\frac{(2\cos 5x \cdot 5 - 11 \sin 5x)}{4[-100 - 121]} = \frac{10\cos 5x - 11\sin 5x}{884}$$

$$\text{P.I}_2 = \frac{1}{2} \cdot \frac{1}{(D^2 - 4D + 3)} \sin x$$

$$= \frac{1}{2} \cdot \frac{1}{(-1^2 - 4D + 3)} \sin x$$

$$= \frac{1}{2} \cdot \frac{1}{(2 - 4D)} \sin x$$

$$= \frac{1}{4} \cdot \frac{1}{(1 - 2D)} \sin x$$

$$= \frac{1}{4} \cdot \frac{1 + 2D}{(1 - 4D^2)} \sin x \qquad \text{[Multiplying Dr and Nr by } 1 + 2D\text{]}$$

$$= \frac{(1+2D)\sin x}{4[1-4(-1^2)]} = \frac{(\sin x + 2\cos x)}{4(1+4)} = \frac{\sin x + 2\cos x}{20}$$

$\therefore$ $$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 = \frac{(10\cos 5x - 11\sin 5x)}{884} + \frac{(\sin x + 2\cos x)}{20}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = C_1 e^{x} + C_2 e^{3x} + \frac{(10\cos 5x - 11\sin 5x)}{884} + \frac{(\sin x + 2\cos x)}{20}$$

EXAMPLE 8

Solve $(D^2 + 1)y = \sin^2 x$.

Solution.

The given equation is

$$(D^2 + 1)y = \sin^2 x$$

$\Rightarrow$ $$(D^2+1)y = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$$

To find the complementary function, solve $(D^2 + 1)y = 0$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

The roots are complex numbers with $\alpha = 0$ and $\beta = 1$

$\therefore$ $$\text{C.F} = e^{0x}(C_1\cos x + C_2\sin x) = C_1\cos x + C_2\sin x$$

$$\text{P.I}_1 = \frac{1}{D^2+1}\cdot\frac{1}{2} = \frac{1}{2(D^2+1)}e^{0x} = \frac{1}{2(0+1)} = \frac{1}{2}$$

$$\text{P.I}_2 = \frac{1}{2}\cdot\frac{1}{D^2+1}\cos 2x = \frac{1}{2}\cdot\frac{\cos 2x}{(-2^2+1)} = -\frac{\cos 2x}{6}$$

$\therefore$ $$\text{P.I} = \frac{1}{2} - \left(-\frac{\cos 2x}{6}\right) = \frac{1}{2} + \frac{\cos 2x}{6}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = C_1\cos x + C_2\sin x + \frac{1}{2} + \frac{\cos 2x}{6}$$

EXAMPLE 9

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$.

Solution.

The given equation is

$$(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$$

To find the complementary function, solve $(D^2 - 3D + 2)y = 0$

Auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow \quad (m-2)(m-1)=0 \quad \Rightarrow \quad m=2, 1$

The roots are real and different.

$\therefore$ C.F $= C_1e^x + C_2e^{2x}$

$$\text{P.I} = \frac{1}{D^2-3D+2}(2\cos(2x+3)+2e^x)$$

$$= \frac{1}{D^2-3D+2}2\cos(2x+3) + \frac{1}{D^2-3D+2}2e^x = \text{P.I}_1 + \text{P.I}_2$$

$$\text{P.I}_1 = \frac{1}{D^2-3D+2}2\cos(2x+3) = 2\cdot\frac{1}{-2^2-3D+2}\cos(2x+3) = 2\frac{1}{(-2-3D)}\cos(2x+3)$$

$$= -2\frac{1}{3D+2}\cos(2x+3) = -2\frac{(3D-2)}{9D^2-4}\cos(2x+3) \qquad \text{[Replace } D^2 \text{ by } -2^2]$$

$$= -2\frac{\left(3D\cos(2x+3)-2\cos(2x+3)\right)}{9(-2^2)-4}$$

$$= -2\frac{[-3\sin(2x+3)\cdot 2-2\cos(2x+3)]}{-36-4}$$

$$= \frac{1}{20}[-6\sin(2x+3)-2\cos(2x+3)]$$

$$= -\frac{1}{10}[3\sin(2x+3)+\cos(2x+3)]$$

$$\text{P.I}_2 = \frac{1}{D^2-3D+2}\cdot 2e^x = \frac{1}{(D-1)(D-2)}2e^x \qquad [\because f(a)=0]$$

$$= 2\cdot\frac{1}{(1-2)}\frac{1}{(D-1)}e^x = -2\cdot xe^x \qquad \left[\because \frac{1}{D-a}e^{ax} = xe^{ax}\right]$$

$$\therefore \text{P.I} = \text{P.I}_1 + \text{P.I}_2 = -\frac{1}{10}[3\sin(2x+3)+\cos(2x+3)] - 2xe^x$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = C_1e^x + C_2e^{2x} - \frac{1}{10}[3\sin(2x+3)+\cos(2x+3)] - 2xe^x$$

TYPE 3: $Q(x) = x^m$, where m is a positive integer

$$\text{P.I} = \frac{1}{f(D)}x^m$$

Take out the lowest degree term of D in $f(D)$ and write the other terms as $1+g(D)$ or $1-g(D)$

$$\therefore \quad \text{P.I} = \frac{1}{D^k[1\pm g(D)]}x^m = \frac{1}{D^k}(1\pm g(D))^{-1}x^m$$

Since $D^m x^m$ = constant and $D^{m+1}x^m = 0$, expand $[1\pm g(D)]^{-1}$ upto D^m, using binomial series expansion.
To find the particular integral, we use the binominal series expansions of

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
and $$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

WORKED EXAMPLES

EXAMPLE 10

Solve $(D^3 - D^2 - D + 1)y = 1 + x^2$.

Solution.
The given equation is

$$(D^3 - D^2 - D + 1)y = 1 + x^2$$

To find the complementary function, solve $(D^3 - D^2 - D + 1)y = 0$
Auxiliary equation is $m^3 - m^2 - m + 1 = 0$

$\Rightarrow$ $m^2(m-1) - (m-1) = 0$

$\Rightarrow$ $(m-1)(m^2-1) = 0$

$\Rightarrow$ $(m-1)(m-1)(m+1) = 0 \quad \Rightarrow \quad m = +1, +1, -1$

The roots are real with two roots equal and the third different.

$\therefore$ $$\text{C.F} = C_1 e^{-x} + (C_2 + C_3 x)e^{x}$$

$$\text{P.I} = \frac{1}{D^3 - D^2 - D + 1}(1+x^2) = \frac{1}{1-(D+D^2-D^3)}(1+x^2)$$
$$= [1-(D+D^2-D^3)]^{-1}(1+x^2)$$
$$= [1+(D+D^2-D^3)+(D+D^2-D^3)^2+\cdots](1+x^2)$$
$$= (1+D+2D^2)(1+x^2) \qquad [\text{Taking terms upto } D^2]$$
$$= 1+x^2+D(1+x^2)+2D^2(1+x^2)$$
$$= 1+x^2+2x+2\cdot 2 = x^2+2x+5$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = C_1 e^{-x} + (C_2 + C_3 x)e^{x} + x^2 + 2x + 5$$

EXAMPLE 11

Solve $(D^2 + 4)y = x^4 + \cos^2 x$.

Solution.
The given equation is

$$(D^2+4)y = x^4 + \cos^2 x = x^4 + \frac{1+\cos 2x}{2} = \left(x^4 + \frac{1}{2}\right) + \frac{\cos 2x}{2}$$

To find the complementary function, solve $(D^2+4)y = 0$
Auxiliary equation is $m^2 + 4 = 0 \Rightarrow m^2 = -4 \quad \Rightarrow \quad m = \pm 2i$

The roots are complex with $\alpha = 0$ and $\beta = 2$

$\therefore$ $$\text{C.F} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{P.I}_1 = \frac{1}{D^2+4}\left(x^4+\frac{1}{2}\right) = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}\left(x^4+\frac{1}{2}\right)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}\right]\left[x^4+\frac{1}{2}\right]$$

$$= \frac{1}{4}\left[x^4+\frac{1}{2}-\frac{1}{4}D^2\left(x^4+\frac{1}{2}\right)+\frac{1}{16}D^4\left(x^4+\frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left[x^4+\frac{1}{2}-\frac{4\cdot 3}{4}x^2+\frac{1}{16}4\cdot 3\cdot 2\cdot 1\right]$$

$$= \frac{1}{4}\left[x^4-3x^2+\frac{1}{2}+\frac{3}{2}\right] = \frac{1}{4}[x^4-3x^2+2]$$

$$\text{P.I}_2 = \frac{1}{2}\cdot\frac{1}{D^2+4}\cos 2x = \frac{1}{2}\cdot\frac{x}{2}\int \cos 2x\, dx = \frac{x}{4}\left[\frac{\sin 2x}{2}\right] = \frac{x\sin 2x}{8} \qquad [\because\ f(-2)^2 = 0]$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$\Rightarrow$ $$= C_1\cos 2x + C_2 \sin 2x + \frac{1}{4}(x^4-3x^2+2)+\frac{x\sin 2x}{8}$$

EXAMPLE 12

Solve $(D^3 + 3D^2 + 2D)y = x^2 + 1$.

Solution.
The given equation is

$$(D^3+3D^2+2D)y = x^2+1$$

To find the complementary function, solve $(D^3+3D^2+2D)y = 0$

Auxiliary equation is $m^3 + 3m^2 + 2m = 0$

$\Rightarrow$ $m(m^2+3m+2) = 0$

$\Rightarrow$ $m(m+1)(m+2) = 0 \quad \Rightarrow \quad m = 0, -1, -2$

The roots are real and different.

$\therefore$ $$\text{C.F} = C_1e^{0x} + C_2e^{-x} + C_3e^{-2x} = C_1 + C_2e^{-x} + C_3e^{-2x}$$

$$\text{P.I} = \frac{1}{D^3+3D^2+2D}(x^2+1) = \frac{1}{2D\left[1+\frac{1}{2}(3D+D^2)\right]}(x^2+1)$$

$$= \frac{1}{2D}\left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1}(x^2+1)$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{3D + D^2}{2}\right)^2 + \cdots\right](x^2 + 1)$$

$$= \frac{1}{2D}\left[1 - \frac{3D}{2} - \frac{D^2}{2} + \frac{9D^2}{4}\right](x^2 + 1)$$

$$= \frac{1}{2D}\left[1 - \frac{3D}{2} + \frac{7D^2}{4}\right](x^2 + 1)$$

$$= \frac{1}{2D}\left[x^2 + 1 - \frac{3}{2} \cdot D(x^2 + 1) + \frac{7}{4} D^2 (x^2 + 1)\right]$$

$$= \frac{1}{2D}\left[x^2 + 1 - \frac{3}{2} \cdot 2x + \frac{7}{4} \cdot 2\right]$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{9}{2}\right]$$

$$= \frac{1}{2}\int\left(x^2 - 3x + \frac{9}{2}\right)dx \quad = \frac{1}{2}\left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{9x}{2}\right) \quad \left[\because \frac{1}{D} f(x) = \int f(x)\, dx\right]$$

∴ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = C_1 + C_2 e^{-x} + C_3 e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{9x}{2}\right]$$

Note It may be taken as a general rule to perform the operation $\frac{1}{D}$ last, because it is simpler.

TYPE 4: If $Q(x) = e^{ax} g(x)$, where $g(x)$ may be x^m or sin bx or cos bx

then $$\text{P.I} = \frac{1}{f(D)} e^{ax} g(x) = e^{ax} \frac{1}{f(D + a)} g(x)$$

The effect of taking the exponential function e^{ax} outside of the operator shifts the operator D to $D + a$. This process is called the **exponential shift**.

Now $\frac{1}{f(D + a)} g(x)$ can be evaluated by using the methods in the types 1, 2, 3, depending upon the type of the function.

TYPE 5: If $Q(x) = x^m \cos ax$ or $x^m \sin ax$

then $$\text{P.I} = \frac{1}{f(D)} x^m \cos ax \text{ or } x^m \sin ax$$

Since $$e^{iax} = \cos ax + i \sin ax$$

$$\cos ax = \text{R.P of } e^{iax} \quad \text{and} \quad \sin ax = \text{I.P of } e^{iax}$$

$$\therefore \quad \frac{1}{f(D)} x^m \cos ax = \text{R.P. of } \frac{1}{f(D)} e^{iax} \cdot x^m = \text{R.P. of } e^{iax} \frac{1}{f(D + ia)} x^m$$

$$\frac{1}{f(D)}x^m \sin ax = \text{I.P. of } \frac{1}{f(D)}e^{iax}\cdot x^m = \text{I.P. of } e^{iax}\frac{1}{f(D+ia)}x^m$$

These can be evaluated using Type 3.

WORKED EXAMPLES

EXAMPLE 13

Solve $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$.

Solution.

The given equation is

$$(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$$

To find the complementary function, solve $(D^2 - 2D + 2)y = 0$

Auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

The roots are complex numbers with $\alpha = 1$ and $\beta = 1$

$$\therefore \quad \text{C.F} = e^x[C_1 \cos x + C_2 \sin x]$$

$$\text{P.I}_1 = \frac{1}{D^2 - 2D + 2}e^x x^2$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}x^2 \qquad \text{[exponential shifting]}$$

$$= e^x \cdot \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2}x^2$$

$$= e^x \cdot \frac{1}{D^2 + 1}x^2$$

$$= e^x(1 + D^2)^{-1}x^2 = e^x(1 - D^2)x^2 = e^x[x^2 - D^2(x^2)] \qquad [\because \; D(x^2) = 2x;\; D^2(x^2) = 2]$$

$$\Rightarrow \quad \text{P.I}_1 = e^x[x^2 - 2]$$

$$\text{P.I}_2 = \frac{1}{D^2 - 2D + 2}5 = \frac{1}{D^2 - 2D + 2}5\cdot e^{0x} = \frac{5}{2}$$

$$\text{P.I}_3 = \frac{1}{D^2 - 2D + 2}e^{-2x} = \frac{e^{-2x}}{4 - 2(-2) + 2} = \frac{e^{-2x}}{10}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 + \text{P.I}_3$

$$\Rightarrow \quad y = e^x[C_1 \cos x + C_2 \sin x] + e^x[x^2 - 2] + \frac{5}{2} + \frac{e^{-2x}}{10}$$

EXAMPLE 14

Solve $(D^2 - 2D)y = x^2e^x \cos x$.

Solution.

The given equation is

$$(D^2 - 2D)y = x^2e^x \cos x$$

To find the complementary function, solve $(D^2 - 2D)y = 0$

Auxiliary equation is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

The roots are real and different.

$\therefore$ $\quad \text{C.F} = C_1 + C_2 e^{2x}$

$$\text{P.I} = \frac{1}{D^2 - 2D} x^2 e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1)} x^2 \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2} x^2 \cos x$$

$$= e^x \frac{1}{D^2 - 1} x^2 \cos x = e^x \text{ R.P. of } \frac{1}{D^2 - 1} x^2 e^{ix}$$

$$= e^x \text{ R.P. of } e^{ix} \frac{1}{(D+i)^2 - 1} x^2$$

$$= e^x \text{ R.P. of } e^{ix} \frac{1}{D^2 + 2iD - 1 - 1} x^2$$

$$= e^x \text{ R.P. of } e^{ix} \frac{1}{D^2 + 2iD - 2} x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } e^{ix} \frac{1}{\left[1 - \left(iD + \frac{D^2}{2}\right)\right]} x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } e^{ix} \left[1 - \left(iD + \frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } e^{ix} \left[1 + iD + \frac{D^2}{2} + \left(iD + \frac{D^2}{2}\right)^2\right] x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } e^{ix} \left[1 + iD + \frac{D^2}{2} - D^2\right] x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } e^{ix} \left[1 + iD - \frac{D^2}{2}\right] x^2$$

$$= -\frac{e^x}{2} \text{ R.P. of } (\cos x + i \sin x) \left[x^2 + iD(x^2) - \frac{1}{2} D^2(x^2)\right]$$

$$= -\frac{e^x}{2} \text{ R.P. of } [\cos x + i \sin x][x^2 + 2ix - 1]$$

$$= -\frac{e^x}{2}[\cos x \cdot (x^2 - 1) - 2x \sin x] = \frac{e^x}{2}[(1 - x^2)\cos x + 2x \sin x]$$

∴ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = C_1 + C_2 e^{2x} + \frac{e^x}{2}[(1 - x^2)\cos x + 2x \sin x]$$

EXAMPLE 15

Solve $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x - 1) + 3^x$.

Solution.
The given equation is

$$(D^2 - 4)y = \cosh(2x - 1) + 3^x$$

$$\Rightarrow \quad (D^2 - 4)y = \frac{e^{2x-1} + e^{-(2x-1)}}{2} + 3^x$$

$$\Rightarrow \quad (D^2 - 4)y = \frac{1}{2}e^{2x-1} + \frac{e^{-(2x-1)}}{2} + 3^x$$

To find complementary function, solve $(D^2 - 4)y = 0$
Auxiliary equation is $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$
The roots are real and different.

$$\therefore \quad \text{C.F.} = C_1 e^{-2x} + C_2 e^{2x}$$

$$\text{P.I}_1 = \frac{1}{2}\frac{1}{(D^2 - 4)}e^{2x-1} = \frac{1}{2}\frac{1}{(D-2)(D+2)}e^{-1} \cdot e^{2x}$$

$$= \frac{e^{-1}}{2} \cdot \frac{1}{(2+2)}\frac{1}{(D-2)}e^{2x}$$

$$= \frac{e^{-1}}{8}\frac{1}{D-2}e^{2x} = \frac{e^{-1}}{8} \cdot xe^{2x} = \frac{x}{8}e^{2x-1} \quad \left[\because \frac{1}{D-a}e^{ax} = xe^{ax}\right]$$

$$\text{P.I}_2 = \frac{1}{2}\frac{1}{(D^2 - 4)}e^{-(2x-1)} = \frac{e}{2}\frac{1}{(D-2)(D+2)}e^{-2x}$$

$$= \frac{e}{2}\frac{1}{(-2-2)(D+2)}e^{-2x}$$

$$= -\frac{e}{8}\frac{1}{(D+2)}e^{-2x} = -\frac{e}{8}xe^{-2x} = -\frac{x}{8}e^{-(2x-1)}$$

$$\therefore \quad \text{P.I}_1 + \text{P.I}_2 = \frac{x}{8}[e^{2x-1} - e^{-(2x-1)}] = \frac{x}{4}\sinh(2x - 1)$$

$$\text{P.I}_3 = \frac{1}{D^2-4}3^x = \frac{1}{D^2-4}e^{x\log_e 3} \qquad [\text{we know } 3^x = e^{\log_e 3^x} = e^{x\log_e 3}]$$

$$= \frac{e^{x\log_e 3}}{(\log_e 3)^2-4} = \frac{3^x}{(\log_e 3)^2-4}$$

∴ the general solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 + \text{P.I}_3$$

⇒ $$y = C_1e^{-2x} + C_2e^{2x} + \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log_e 3)^2-4}$$

EXAMPLE 16

Solve $(D^4 - 1)y = e^x \cos x$.

Solution.
The given equation is

$$(D^4-1)y = e^x\cos x$$

To find the complementary function, solve $(D^4-1)y = 0$

Auxiliary equation is $(m^4-1) = 0$

⇒ $(m^2-1)(m^2+1) = 0 \quad \Rightarrow m^2-1=0, \quad m^2+1=0$

⇒ $m = \pm 1, \qquad m = \pm i$

Two roots are real and different and the other two roots are complex numbers with $\alpha = 0$ and $\beta = 1$

∴ $$\text{C.F} = C_1e^{-x} + C_2e^x + C_3\cos x + C_4\sin x$$

$$\text{P.I} = \frac{1}{D^4-1}e^x\cos x = e^x\frac{1}{(D+1)^4-1}\cos x$$

$$= e^x\frac{1}{D^4+4D^3+6D^2+4D+1-1}\cos x$$

$$= e^x\frac{1}{D^4+4D^3+6D^2+4D}\cos x$$

$$= e^x\frac{1}{(-1^2)^2+4D(-1^2)+6(-1^2)+4D}\cos x$$

$$= e^x\frac{1}{1-4D-6+4D}\cos x = \frac{e^x\cos x}{-5} = -\frac{e^x\cos x}{5}$$

∴ the general solution is

$$y = \text{C.F} + \text{P.I.}$$

⇒ $$y = C_1e^{-x} + C_2e^x + C_3\cos x + C_4\sin x - \frac{e^x\cos x}{5}$$

EXAMPLE 17

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$.

Solution.
The given equation is

$$(D^2 + 2D + 1)y = \frac{e^{-x}}{x^2}$$

To find the complementary function, solve $(D^2 + 2D + 1)y = 0$
Auxiliary equation is $m^2 + 2m + 1 = 0 = (m + 1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$
The roots are real and equal.

$\therefore \quad \text{C.F} = (C_1 + C_2x)e^{-x}$

$$\text{P.I} = \frac{1}{D^2 + 2D + 1}\left(\frac{e^{-x}}{x^2}\right) = \frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x^2}\right)$$

$$= e^{-x} \cdot \frac{1}{(D-1+1)^2}\left(\frac{1}{x^2}\right)$$

$$= e^{-x} \cdot \frac{1}{(D-1+1)^2} x^{-2}$$

$$= e^{-x} \cdot \frac{1}{D^2} x^{-2}$$

$$= e^{-x} \frac{1}{D}\int x^{-2}\,dx$$

$$= e^{-x} \frac{1}{D}\left[\frac{x^{-2+1}}{-2+1}\right] = -e^{-x}\frac{1}{D}[x^{-1}] = -e^{-x}\int\frac{dx}{x} = -e^{-x}\log_e x$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = (C_1 + C_2x)e^{-x} - e^{-x}\log_e x = e^{-x}[C_1 + C_2x - \log_e x]$$

EXAMPLE 18

Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax$.

Solution.
The given equation is

$$(D^2 + a^2)y = \tan ax$$

To find the complementary function, solve $(D^2 + a^2)y = 0$
Auxiliary equation is $m^2 + a^2 = 0 \quad \Rightarrow m = \pm ia$
The roots are complex numbers with $\alpha = 0$ and $\beta = a$

$\therefore \quad \text{C.F} = C_1\cos ax + C_2\sin ax$

$$\text{P.I} = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ia)(D-ia)}\tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ia}\tan ax - \frac{1}{D+ia}\tan ax\right]$$

Now $$\frac{1}{D-ai}\tan ax = e^{iax}\int \tan ax\, e^{-iax}dx \qquad \left[\because \frac{1}{D-a}f(x) = e^{ax}\int f(x)\,e^{-ax}dx\right]$$

$$= e^{iax}\int \tan ax(\cos ax - i\sin ax)dx$$

$$= e^{iax}\int\left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\int\left\{\sin ax - i\left(\frac{1-\cos^2 ax}{\cos ax}\right)\right\}dx$$

$$= e^{iax}\int[\sin ax - i(\sec ax - \cos ax)]\,dx$$

$$= e^{iax}\left[\left(\frac{-\cos ax}{a}\right) - i\left\{\frac{1}{a}\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right\}\right]$$

$$= -\frac{e^{iax}}{a}[(\cos ax - i\sin ax) + i\log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a}e^{iax}\left[e^{-iax} + i\log(\sec ax + \tan ax)\right]$$

$$= -\frac{1}{a}\left[1 + ie^{iax}\log(\sec ax + \tan ax)\right]$$

Changing i to $-i$, we have

$$\frac{1}{D+ia}\tan ax = -\frac{1}{a}\left[1 - ie^{-iax}\log(\sec ax + \tan ax)\right]$$

$$\therefore\ \text{P.I} = \frac{1}{2ia}\left[-\frac{1}{a}\left\{1 + ie^{iax}\log(\sec ax + \tan ax)\right\} + \frac{1}{a}\left\{1 - ie^{-iax}\log(\sec ax + \tan ax)\right\}\right]$$

$$= \frac{1}{2ia}\left[-\frac{i}{a}e^{iax}\log(\sec ax + \tan ax) - \frac{i}{a}e^{-iax}\log(\sec ax + \tan ax)\right]$$

$$= -\frac{1}{2a^2}\log(\sec ax + \tan ax)\left[e^{iax} + e^{-iax}\right]$$

$$= -\frac{1}{2a^2}\log(\sec ax + \tan ax)\cdot 2\cos ax$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax)\cdot\cos ax$$

∴ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \cos ax \cdot \log_e(\sec ax + \tan ax)$$

EXERCISE 11.1

Solve the following differential equations.

1. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 13\frac{dy}{dx} = 0$
2. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0$
3. $(D^2 - 5D + 6)y = e^{4x}$
4. $(D^2 + 4D + 8)y = (1 + e^x)^2$
5. $(3D^2 + D - 14)y = 13e^{2x}$
6. $(D^3 - 12D + 16)y = (e^x + e^{-2x})^2$
7. $(D^3 + 3D + 2)y = e^{-x} + e^{-2x}$
8. $(4D^2 - 4D + 1)y = 4$
9. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x}$
10. $(D^2 - 2D + 1)y = x + 1$
11. $(D^3 - D^2 - D + 1)y = 1 + x^2$
12. $(D^2 - 3D + 2)y = \sin 5x$
13. $(D^2 + 4)y = 3\cos^2 x$
14. $(D^2 + 16)y = e^{-3x} + \cos 4x$
15. $(D^2 + 4)y = \cos^2 x$
16. $(D^2 + 16)y = \cos^3 x$
17. $(D^2 + 2D - 3)y = e^{2x}(1 + x^2)$
18. $(D^3 - 3D^2 + 3D - 1)y = x^2e^x$
19. $(D^3 - 2D + 4)y = e^x \cos x$
20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$
21. $(D^2 + D)y = x^2 + 2x + 4$
22. $(D^2 + 2D - 3)y = e^x \cos x + e^{-x} \cdot x^2$
23. $4y'' - 4y' + y = x - \frac{\sin 2x}{e^x}$
24. $(D^2 - 4D + 13)y = e^{2x}\cos 3x$
25. $(D^3 - 7D - 6)y = (1 + x)\, e^{2x}$
26. $(D^2 + 4D + 13)y = e^{-2x} \cos 3x$
27. $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x$
28. $(D^2 - 4D + 4)y = (1 + x)^2\, e^{2x}$
29. $(D^2 + 4)y = \tan 2x$
30. $(D^2 + 1)y = \sec x$
31. $\frac{d^2y}{dx^2} - 4y = x \sinh x$
32. $(D + 2)^2 y = e^{-2x} \sin x$
33. $(D^2 - 3D + 2)y = x \cos x$
34. $(D^2 - 4D + 4)y = x^2e^{2x} \cos 2x$
35. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$
36. $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$
37. $(D^2 + 1)^2 y = x^2 \cos x$
38. $(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$
39. $(D^2 + 4)^2 y = \cos 2x$
40. $(D^2 + 4D + 3)y = e^{-x} \sin x + xe^{3x}$
41. $(D^2 + 4D + 3)y = e^{-x} \sin x$
42. $(D^4 + D^3 + D^2)y = 5x^2 + \cos x$
43. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$
44. $(D^2 + a^2)y = \sec ax$

ANSWERS TO EXERCISE 11.1

1. $y = C_1 + e^{-2x}(C_2 \cos 3x + C_3 \sin 3x)$
2. $y = C_1e^{-2x} + C_2 \cos 2x + C_3 \sin 2x$
3. $y = C_1e^{2x} + C_2e^{3x} + \frac{1}{2}e^{4x}$
4. $y = e^{-2x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8} + \frac{2}{13}e^x + \frac{e^{2x}}{20}$

5. $y = C_1 e^{2x} + C_2 e^{-7x/3} + xe^{2x}$

6. $y = (C_1 + C_2 x)e^{2x} + C_3 e^{-4x} + \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$

7. $y = C_1 e^{-x} + C_2 e^{-2x} + x(e^{-x} - e^{-2x})$

8. $y = (C_1 + C_2 x)e^{x/2} + 4$

9. $y = C_1 + (C_2 + C_3 x)e^{-x} + \frac{e^{2x}}{18}$

10. $y = (C_1 + C_2 x)e^{x} + x + 3$

11. $y = C_1 e^{-x} + (C_2 + C_3 x)e^{x} + x^2 + 2x + 5$

12. $y = C_1 e^{x} + C_2 e^{2x} + \frac{1}{130}(9\cos 3x - 7\sin 3x)$

13. $y = C_1 \cos 2x + C_2 \sin 2x + \frac{3}{8}(1 + x\sin 2x)$

14. $y = C_1 \cos 4x + C_2 \sin 4x + \frac{e^{-3x}}{25} + \frac{x\sin 4x}{8}$

15. $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}(1 + x\sin 2x)$

16. $y = C_1 \cos 4x + C_2 \sin 4x + \frac{1}{28}\cos 3x + \frac{1}{20}\cos x$

17. $y = C_1 e^{-3x} + C_2 e^{x} + \frac{e^{2x}}{5}\left(x^2 - \frac{12x}{5} + \frac{87}{25}\right)$

18. $y = (C_1 + C_2 x + C_3 x^2)e^{x} + \frac{x^5}{60}e^{x}$

19. $y = C_1 e^{-2x} + e^{x}(C_2 \cos x + C_3 \sin x) + \frac{xe^{x}}{20}(3\sin x - \cos x)$

20. $y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{8}\left(x^2 - \frac{7x}{8} + \frac{11}{6}\right) + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

21. $y = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x$

22. $y = C_1 e^{x} + C_2 e^{-3x} + \frac{e^{x}}{17}(4\sin x - \cos x) - \frac{e^{-x}}{4}\left(x^2 + \frac{1}{2}\right)$

23. $y = (C_1 + C_2 x)e^{x/2} + x + 4 + \frac{e^{-x}}{625}(7\sin 2x - 24\cos 2x)$

24. $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + \frac{x}{6}e^{2x}\sin 3x$

25. $y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} - \frac{e^{2x}}{12}\left(x + \frac{17}{5}\right)$

26. $y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x) + \frac{x}{4}e^{-2x}\sin 3x$

27. $y = C_1 e^{-x} + C_2 e^{x} - \frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^{x}}{2}\left(\frac{3x}{2} - \frac{x^2}{2} + \frac{x^3}{3}\right)$

28. $y = (C_1 + C_2 x)e^{2x} + \frac{(x+1)^4}{12}e^{2x}$

29. $y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log_e[\sec 2x + \tan 2x]$

30. $y = C_1 \cos x + C_2 \sin x + x \sin x + \cos x \log_e \cos x$

31. $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{e^x}{6}\left(x + \frac{2}{3}\right) + \frac{e^{-x}}{6}\left[x - \frac{2}{3}\right]$

32. $y = [C_1 + C_2 x - \sin x]e^{-2x}$

33. $y = C_1 e^x + C_2 e^{2x} + \frac{1}{50}[(5x-6)\cos x - (15x+17)\sin x]$

34. $y = (C_1 + C_2 x)e^{2x} + \frac{e^{2x}}{8}\left[4x \sin 2x + (3 - 2x^2)\cos 2x\right]$

35. $y = (C_1 + C_2 x)e^x - e^x[x \sin x + 2\cos x]$

36. $y = C_1 e^{-x} + C_2 e^x - \frac{1}{2}[x \sin x + \cos x] + \frac{e^x}{12}[2x^3 - 3x^2 + 9x]$

37. $y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x + \frac{1}{48}\left[4x^3 \sin x + (9x^2 - x^4)\cos x\right]$

38. $y = (C_1 + C_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$

39. $y = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x - \frac{x^2}{32}\cos 2x$

40. $y = C_1 e^{-3x} + C_2 e^{-x} - \frac{e^{-x}}{5}[2\cos x + \sin x] + \frac{e^{-3x}}{24}\left[x - \frac{5}{12}\right]$

41. $y = C_1 e^{-3x} + C_2 e^{-x} - \frac{e^{-x}}{5}[2\cos x + \sin x]$

42. $y = C_1 + C_2 x + e^{-\frac{1}{2}x}\left[C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x\right] + 5\left[\frac{x^4}{12} - \frac{x^3}{3} - x^2\right] - \sin x$

43. $y = C_1 \cos x + C_2 \sin x + \sin x \log_e \sin x - x \cos x$

44. $y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log_e \cos ax$

11.2 LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

11.2.1 Cauchy's Homogeneous Linear Differential Equations

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \tag{1}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants with $a_0 \neq 0$, is called **Cauchy's homogeneous linear differential equation**. It is also known as **Euler–Cauchy linear equation** or **Euler's linear equation**.
This can be reduced to a linear differential equation with constant coefficients if we put

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Cauchy's Homogeneous Linear Differential Equations

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \qquad (1)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants with $a_0 \neq 0$, is called **Cauchy's homogeneous linear differential equation**. It is also known as **Euler–Cauchy linear equation** or **Euler's linear equation**.
This can be reduced to a linear differential equation with constant coefficients if we put

$$x = e^z \quad \Rightarrow \quad z = \log_e x \qquad \therefore \frac{dz}{dx} = \frac{1}{x}$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{dy}{dz} \cdot \dfrac{1}{x} \quad \Rightarrow \quad x\dfrac{dy}{dx} = \dfrac{dy}{dz}$

If $\quad \boldsymbol{\theta} = \dfrac{d}{dz}$, then $\quad x\dfrac{dy}{dx} = \boldsymbol{\theta} y \quad \Rightarrow \quad xDy = \boldsymbol{\theta} y \quad \Rightarrow \quad xD = \boldsymbol{\theta}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\boldsymbol{\theta} y + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{1}{x} = -\frac{1}{x^2}\boldsymbol{\theta} y + \frac{1}{x^2}\boldsymbol{\theta}^2 y \qquad \left[\because \frac{dz}{dx} = \frac{1}{x}\right]$$

$$\Rightarrow \qquad x^2\frac{d^2y}{dx^2} = \boldsymbol{\theta}^2 y - \boldsymbol{\theta} y \quad \Rightarrow \quad x^2\frac{d^2y}{dx^2} = \boldsymbol{\theta}(\boldsymbol{\theta}-1)y \quad \Rightarrow \quad x^2D^2 = \boldsymbol{\theta}(\boldsymbol{\theta}-1)$$

Similarly, $x^3\dfrac{d^3y}{dx^3} = \boldsymbol{\theta}(\boldsymbol{\theta}-1)(\boldsymbol{\theta}-2)y$ and so on.

Substituting in (1), the equation reduces to linear differential equation with constant coefficients in y and z, which can be solved by using the methods discussed earlier.

Note Remember $x = e^z$ (or $z = \log_e x$)

and $\qquad \boldsymbol{\theta} = \dfrac{d}{dz}$, then $xD = \boldsymbol{\theta}$, $\;x^2D^2 = \boldsymbol{\theta}(\boldsymbol{\theta} - 1)$, $\;x^3D^3 = \boldsymbol{\theta}(\boldsymbol{\theta} - 1)(\boldsymbol{\theta} - 2)$ etc.

WORKED EXAMPLES

EXAMPLE 1

Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

Solution.
The given equation is

$$(x^2D^2 - xD + 1)y = 0 \qquad (1)$$

which is Cauchy's equation.

Put $x = e^z$ and $\boldsymbol{\theta} = \dfrac{d}{dz}$, then $xD = \boldsymbol{\theta}$, $\;x^2D^2 = \boldsymbol{\theta}(\boldsymbol{\theta} - 1)$

∴ the equation (1) is $[\theta(\theta-1)-\theta+1]y=0$

$\Rightarrow \quad (\theta^2-2\theta+1)y=0$

Auxiliary equation is $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

The roots are real and equal.

∴ the general solution is $y=(C_1+C_2z)e^z=(C_1+C_2\log x)x$

EXAMPLE 2

Solve $(x^2D^2-3xD+4)y=x^2$ given that $y(1)=1$ and $y'(1)=0$.

Solution.

The given equation is

$$(x^2D^2-3xD+4)y=x^2 \qquad (1)$$

which is Cauchy's equation.

Put $x=e^z$ and $\theta=\frac{d}{dz}$, then $xD=\theta$, $x^2D^2=\theta(\theta-1)$

∴ the equation (1) is

$$(\theta(\theta-1)-3\theta+4)y=e^{2z}$$

$$\Rightarrow \quad (\theta^2-4\theta+4)y=e^{2z}$$

To find the complementary function, solve $(\theta^2-4\theta+4)y=0$

Auxiliary equation is $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

The roots are real and equal.

∴ $$\text{C.F}=(C_1+C_2z)e^{2z}=(C_1+C_2\log_e x)x^2$$

$$\text{P.I}=\frac{1}{\theta^2-4\theta+4}e^{2z}=\frac{1}{(\theta-2)^2}e^{2z}=\frac{z^2e^{2z}}{2}=\frac{1}{2}(\log_e x)^2x^2$$

∴ the general solution is $y=\text{C.F}+\text{P.I}$

$$\Rightarrow \quad y=(C_1+C_2\log_e x)x^2+\frac{1}{2}x^2(\log_e x)^2$$

When $x=1, y=1$, so we get $C_1=1$

Now $$\frac{dy}{dx}=(C_1+C_2\log_e x)2x+x^2\cdot C_2\frac{1}{x}+\frac{1}{2}\left(x^2 2\log_e x\cdot\frac{1}{x}+(\log_e x)^2\cdot 2x\right)$$

When $x=1$, $\frac{dy}{dx}=0$. So, we get $0=C_1\cdot 2+C_2 \Rightarrow C_2=-2C_1=-2$

∴ the solution is $$y=(1-2\log_e x)x^2+\frac{1}{2}x^2(\log_e x)^2$$

$$\Rightarrow \quad y=x^2\left[1-2\log_e x+\frac{1}{2}(\log_e x)^2\right]$$

EXAMPLE 3

Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x\log_e x.$

Solution.

The given equation is

$$(x^2D^2 + 4xD + 2)y = x\log_e x \tag{1}$$

which is Cauchy's equation.

Put $x = e^z$ and $\theta = \dfrac{d}{dz}$, then $xD = \theta$, $\quad x^2D^2 = \theta(\theta - 1)$

$\therefore$ the equation (1) is $\quad (\theta(\theta-1) + 4\theta + 2)y = e^z \cdot z$

$\Rightarrow \quad (\theta^2 + 3\theta + 2)y = ze^z$

To find the complementary function, solve $(\theta^2 + 3\theta + 2)y = 0$

Auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1$

The roots are real and different.

$$\therefore \quad \text{C.F} = C_1e^{-z} + C_2e^{-2z} = C_1x^{-1} + C_2x^{-2} = \frac{C_1}{x} + \frac{C_2}{x^2}$$

$$\text{P.I} = \frac{1}{\theta^2 + 3\theta + 2}ze^z = e^z\frac{1}{(\theta+1)^2 + 3(\theta+1) + 2}z$$

$$= e^z\frac{1}{\theta^2 + 2\theta + 1 + 3\theta + 3 + 2}z$$

$$= e^z\frac{1}{\theta^2 + 5\theta + 6}z$$

$$= \frac{e^z}{6}\frac{1}{\left(1 + \dfrac{5\theta + \theta^2}{6}\right)}z$$

$$= \frac{e^z}{6}\left[1 + \left(\frac{5\theta + \theta^2}{6}\right)\right]^{-1}z$$

$$= \frac{e^z}{6}\left[1 - \frac{(5\theta + \theta^2)}{6} + \cdots\right]z$$

$$= \frac{e^z}{6}\left[1 - \frac{5}{6}\theta\right]z = \frac{e^z}{6}\left[z - \frac{5}{6}\theta(z)\right] = \frac{x}{6}\left[\log_e x - \frac{5\cdot 1}{6}\right] = \frac{x}{6}\left[\log_e x - \frac{5}{6}\right]$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{x}{6}\left[\log_e x - \frac{5}{6}\right]$$

EXAMPLE 4

Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}.$

Solution.
The given equation is

$$(x^2D^2 + 4xD + 2)y = x^2 + \frac{1}{x^2} \qquad (1)$$

which is Cauchy's equation.

Put $x = e^z$ and $\boldsymbol{\theta} = \frac{d}{dz}$, then $xD = \boldsymbol{\theta}$, $x^2D^2 = \boldsymbol{\theta}(\boldsymbol{\theta} - 1)$

$\therefore$ the equation (1) is $(\boldsymbol{\theta}(\boldsymbol{\theta}-1)+4\boldsymbol{\theta}+2)y = e^{2z} + \frac{1}{e^{2z}}$

$\Rightarrow$ $$(\boldsymbol{\theta}^2 + 3\boldsymbol{\theta} + 2)y = e^{2z} + e^{-2z}$$

To find the complementary function, solve $(\boldsymbol{\theta}^2 + 3\boldsymbol{\theta} + 2)y = 0$

Auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1$

The roots are real and different.

$\therefore$ $$\text{C.F} = C_1e^{-2z} + C_2e^{-z}$$

$$\text{P.I}_1 = \frac{1}{\boldsymbol{\theta}^2 + 3\boldsymbol{\theta} + 2}e^{2z} = \frac{e^{2z}}{4+3\cdot 2+2} = \frac{e^{2z}}{12}$$

$$\text{P.I}_2 = \frac{1}{\boldsymbol{\theta}^2 + 3\boldsymbol{\theta} + 2}e^{-2z} = \frac{1}{(\boldsymbol{\theta}+1)(\boldsymbol{\theta}+2)}e^{-2z} = \frac{1}{(-2+1)(\boldsymbol{\theta}+2)}e^{-2z} = -ze^{-2z}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$

$\Rightarrow$ $$y = C_1e^{-2z} + C_2e^{-z} + \frac{e^{2z}}{12} - ze^{-2z} = \frac{C_1}{x^2} + \frac{C_2}{x} + \frac{x^2}{12} - \frac{1}{x^2}\log_e x$$

EXAMPLE 5

Solve $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$.

Solution.
The given equation is

$$(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2 \qquad (1)$$

which is Cauchy's equation.

Put $x = e^z$ and $\boldsymbol{\theta} = \frac{d}{dz}$, then $xD = \boldsymbol{\theta}$, $x^2D^2 = \boldsymbol{\theta}(\boldsymbol{\theta} - 1)$

$\therefore$ the equation (1) is $(\boldsymbol{\theta}(\boldsymbol{\theta}-1) - \boldsymbol{\theta} + 1)y = \left(\frac{z}{e^z}\right)^2$

$\Rightarrow$ $$(\boldsymbol{\theta}^2 - 2\boldsymbol{\theta} + 1)y = z^2e^{-2z}$$

$\Rightarrow$ $$(\boldsymbol{\theta} - 1)^2y = z^2e^{-2z}$$

To find the complementary function, solve $(\boldsymbol{\theta} - 1)^2y = 0$

Auxiliary equation is $(m-1)^2 = 0 \quad \Rightarrow \quad m = 1, 1$

The roots are real and equal.

$\therefore \quad \text{C.F} = (C_1 + C_2 z)e^z$

$$\text{P.I} = \frac{1}{(\theta-1)^2} z^2 e^{-2z} = e^{-2z} \frac{1}{(\theta-2-1)^2} z^2$$

$$= e^{-2z} \frac{1}{(\theta-3)^2} z^2$$

$$= \frac{e^{-2z}}{9} \frac{1}{\left(1-\frac{\theta}{3}\right)^2} z^2$$

$$= \frac{e^{-2z}}{9}\left(1-\frac{\theta}{3}\right)^{-2} z^2$$

$$= \frac{e^{-2z}}{9}\left[1 + 2\cdot\frac{\theta}{3} + 3\cdot\frac{\theta^2}{9} + \cdots\right] z^2$$

$$= \frac{e^{-2z}}{9}\left[1 + \frac{2\theta}{3} + \frac{1}{3}\theta^2\right] z^2 = \frac{e^{-2z}}{9}\left[z^2 + \frac{2}{3}\cdot 2z + \frac{1}{3}\cdot 2\right] = \frac{e^{-2z}}{9}\left[z^2 + \frac{4}{3}z + \frac{2}{3}\right]$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = (C_1 + C_2 z)e^z + \frac{e^{-2z}}{9}\left[z^2 + \frac{4}{3}z + \frac{2}{3}\right]$$

$$= (C_1 + C_2 \log_e x)x + \frac{1}{9x^2}\left[(\log_e x)^2 + \frac{4}{3}\log_e x + \frac{2}{3}\right]$$

EXAMPLE 6

Solve $x^2y'' + 3xy' + 5y = x\cos(\log x) + 3$.

Solution.

The given equation is

$$(x^2D^2 + 3xD + 5)y = x\cos(\log x) + 3 \qquad (1)$$

which is Cauchy's equation.

Put $x = e^z$ and $\theta = \frac{d}{dz}$, then $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

$\therefore$ the equation (1) is $(\theta(\theta-1) + 3\theta + 5)y = e^z\cos z + 3$

$$(\theta^2 + 2\theta + 5)y = e^z\cos z + 3$$

To find the complementary function, solve $(\theta^2 + 2\theta + 5)y = 0$

Auxiliary equation is $m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

The roots are complex numbers with $\alpha = -1$ and $\beta = 2$

$$\therefore \quad \text{C.F} = e^{-z}[C_1 \cos 2z + C_2 \sin 2z]$$

$$\text{P.I}_1 = \frac{1}{\theta^2 + 2\theta + 5} e^z \cos z$$

$$= e^z \frac{1}{(\theta+1)^2 + 2(\theta+1) + 5} \cos z$$

$$= e^z \frac{1}{\theta^2 + 2\theta + 1 + 2\theta + 2 + 5} \cos z$$

$$= e^z \frac{1}{\theta^2 + 4\theta + 8} \cos z$$

$$= e^z \frac{1}{-1 + 4\theta + 8} \cos z$$

$$= e^z \frac{1}{4\theta + 7} \cos z$$

$$= e^z \frac{4\theta - 7}{16\theta^2 - 49} \cos z$$

$$= e^z \frac{(4\theta - 7)\cos z}{16(-1^2) - 49} = -\frac{e^z}{65}[4(-\sin z) - 7\cos z] = \frac{e^z}{65}[4\sin z + 7\cos z]$$

$$\text{P.I}_2 = \frac{1}{\theta^2 + 2\theta + 5} \cdot 3 = \frac{1}{\theta^2 + 2\theta + 5} 3 \cdot e^{0z} = \frac{3}{0 + 0 + 5} = \frac{3}{5}$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$\Rightarrow \quad y = e^{-z}[C_1 \cos 2z + C_2 \sin 2z] + \frac{e^z}{65}[4\sin z + 7\cos z] + \frac{3}{5}$$

$$= \frac{1}{x}\left[C_1 \cos(2\log_e x) + C_2 \sin(2\log_e x)\right] + \frac{x}{65}[4\sin(\log_e x) + 7\cos(\log_e x)] + \frac{3}{5}$$

EXAMPLE 7

Solve $(x^2D^2 - xD + 4)y = x^2 \sin(\log x)$.

Solution.

The given equation is

$$(x^2D^2 - xD + 4)y = x^2 \sin(\log x) \qquad (1)$$

which is Cauchy's equation.

Put $x = e^z$ and $\theta = \frac{d}{dz}$, then $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

$\therefore$ the equation (1) becomes $(\theta(\theta - 1) - \theta + 4)y = e^{2z}\sin z$

$$\Rightarrow \quad (\theta^2 - 2\theta + 4)y = e^{2z}\sin z$$

To find the complementary function, solve $(\theta^2 - 2\theta + 4)y = 0$

Auxiliary equation is $\quad m^2 - 2m + 4 = 0$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4-16}}{2} \quad = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

The roots are complex numbers with $\alpha = 1$ and $\beta = \sqrt{3}$

$$\text{C.F} = e^z\left[C_1 \cos\sqrt{3}z + \sin\sqrt{3}z\right]$$

$$\text{P.I} = \frac{1}{\theta^2 - 2\theta + 4} e^{2z} \sin z$$

$$= e^{2z} \frac{1}{(\theta+2)^2 - 2(\theta+2) + 4} \sin z$$

$$= e^{2z} \frac{1}{\theta^2 + 4\theta + 4 - 2\theta - 4 + 4} \sin z$$

$$= e^{2z} \frac{1}{\theta^2 + 2\theta + 4} \sin z$$

$$= e^{2z} \frac{1}{-1^2 + 2\theta + 4} \sin z$$

$$= e^{2z} \frac{1}{2\theta + 3} \sin z$$

$$= e^{2z} \frac{2\theta - 3}{4\theta^2 - 9} \sin z = e^{2z} \frac{(2\theta - 3)\sin z}{4(-1^2) - 9} = -\frac{e^{2z}}{13}(2\cos z - 3\sin z)$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = e^z\left[C_1 \cos\sqrt{3}z + \sin\sqrt{3}z\right] - \frac{e^{2z}}{13}[2\cos z - 3\sin z]$$

$$= x[C_1 \cos\left(\sqrt{3}\log_e x\right) + C_2 \sin\left(\sqrt{3}\log_e x\right) - \frac{x^2}{13}\left[2\cos(\log_e x) - 3\sin(\log_e x)\right]$$

EXAMPLE 8

Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$.

Solution.

The given equation is $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x \qquad (1)$

$$(x^2D^2 + 4xD + 2)y = e^x$$

which is Cauchy's equation.

Put $x = e^z$ and $\theta = \dfrac{d}{dz}$, then $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

$\therefore$ the equation (1) becomes $(\theta(\theta - 1) + 4\theta + 2)y = e^{e^z}$

$\Rightarrow$ $(\theta^2 + 3\theta + 2)y = e^{e^z}$

To find the complementary function, solve $(\theta^2 + 3\theta + 2)y = 0$

Auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

The roots are real and different.

$\therefore$ $\text{C.F} = C_1 e^{-z} + C_2 e^{-2z}$

$$\text{P.I} = \frac{1}{\theta^2 + 3\theta + 2} e^{e^z} = \frac{1}{(\theta+1)(\theta+2)} e^{e^z}$$

$$= \left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right] e^{e^z} \quad \text{[Splitting into partial fractions]}$$

$$= \frac{1}{\theta+1} e^{e^z} - \frac{1}{\theta+2} e^{e^z} = \text{P.I}_1 - \text{P.I}_2$$

$$\text{P.I}_1 = \frac{1}{\theta+1} e^{e^z} = \frac{1}{\theta+1} e^{-z} \cdot e^z \cdot e^{e^z} = e^{-z} \frac{1}{\theta - 1 + 1} e^z e^{e^z} = e^{-z} \frac{1}{\theta} e^z e^{e^z} = e^{-z} \int e^z e^{e^z}\, dz$$

But $x = e^z \quad \therefore \quad dx = e^z dz$ (2)

$\therefore$ $$\text{P.I}_1 = e^{-z} \int e^x\, dx = e^{-z} e^x = \frac{1}{x} e^x$$

$\therefore$ $$\text{P.I}_2 = \frac{1}{\theta+2} e^{e^z} = \frac{1}{\theta+2} e^{-2z} \cdot e^{2z} \cdot e^{e^z} = e^{-2z} \frac{1}{\theta - 2 + 2} e^{2z} e^{e^z}$$

$$= e^{-2z} \frac{1}{\theta} e^{2z} e^{e^z}$$

$$= \frac{1}{x^2} \int x e^x\, dx \quad \text{[Using (2)]}$$

$$= \frac{1}{x^2}[x e^x - 1 \cdot e^x] = \frac{(x-1)}{x^2} e^x = \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x$$

$\therefore$ $$\text{P.I} = \text{PI}_1 - \text{P.I}_2 = \frac{1}{x} e^x - \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x = \frac{e^x}{x^2}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = C_1 e^{-z} + C_2 e^{-2z} + \frac{e^x}{x^2} = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{e^x}{x^2}$$

11.2.2 Legendre's Linear Differential Equation

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (ax+b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = Q(x) \quad (1)$$

where a, b and a_i's are constants, is called **Legendre's linear differential equation**.

It can be reduced to a linear differential equation with constant coefficients by the substitution.

$$ax + b = e^z \quad \text{and} \quad \theta = \frac{d}{dz} \quad \Rightarrow \quad z = \log(ax+b)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{ax+b}\cdot a \quad \Rightarrow \quad (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = a\theta y \quad \Rightarrow \quad (ax+b)D = a\theta$$

Similarly, $\quad (ax+b)^2 D^2 = a^2\theta(\theta-1)$ and so on.

Substituting in (1) we get an equation with constant coefficient, which can be solved by using the methods discussed to solve the equations of the type 2.1.

WORKED EXAMPLES

EXAMPLE 9

Solve $(x+2)^2\dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4.$

Solution.

The given equation is

$$[(x+2)^2D^2 - (x+2)D + 1]y = 3x + 4 \qquad (1)$$

which is Legendre's equation.

Put $\quad x + 2 = e^z \quad \Rightarrow \quad x = e^z - 2 \quad$ and $\quad \theta = \dfrac{d}{dz}$

then $\quad (x+2)D = \theta \quad$ and $\quad (x+2)^2D^2 = \theta(\theta - 1) \qquad [\because a = 1]$

$\therefore$ the equation (1) is $\quad [\theta(\theta-1) - \theta + 1)y = 3(e^z - 2) + 4$

$\Rightarrow \quad (\theta^2 - 2\theta + 1)y = 3e^z - 2$

To find the complementary function, solve $(\theta^2 - 2\theta + 1)y = 0$

Auxiliary equation is $\quad m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m-1)^2 = 0 \quad \Rightarrow \quad m = 1, 1$

The roots are real and equal.

$$\text{C.F} = (C_1 + C_2 z)e^z$$

$$\text{P.I} = \frac{1}{(\theta-1)^2}(3e^z - 2) = \frac{1}{(\theta-1)^2}3e^z - 2\frac{1}{(\theta-1)^2}\cdot e^{0z} = 3\frac{z^2}{2}e^z - 2 \quad \left[\because \frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2}e^{ax}\right]$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = (C_1 + C_2 z)e^z + \frac{3}{2}z^2e^z - 2$$

$$= \left\{(C_1 + C_2 z) + \frac{3}{2}z^2\right\}e^z - 2$$

$$= \left\{C_1 + C_2\log_e(x+2) + \frac{3}{2}\left(\log_e(x+2)\right)^2\right\}(x+2) - 2$$

EXAMPLE 10

Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log_e(1+x)]$.

Solution.
The given equation is

$$[(1+x)^2D^2 + (1+x)D + 1]y = 2\sin(\log_e(1+x)) \quad (1)$$

which is Legendre's equation.

Put $1+x = e^z \Rightarrow x = e^z - 1$ and $\theta = \frac{d}{dz}$

then $(1+x)D = \theta$ and $(1+x)^2D^2 = \theta(\theta-1)$ $[\because a = 1]$

$\therefore$ the equation (1) is $(\theta(\theta-1)+\theta+1)y = 2\sin z \Rightarrow (\theta^2+1)y = 2\sin z$

To find the complementary function, solve $(\theta^2+1)y = 0$

Auxiliary equation is $m^2+1=0 \Rightarrow m = \pm i$

The roots are complex numbers with $\alpha = 0$ and $\beta = 1$

$\therefore$ C.F $= C_1\cos z + C_2\sin z$

$$\text{P.I} = \frac{1}{\theta^2+1}2\sin z = 2\cdot\frac{z}{2}\int \sin z\,dz = z(-\cos z) = -z\cos z \qquad [f(-a^2) = f(-1^2) = 0]$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow y = C_1\cos z + C_2\sin z - z\cos z$$

$$= C_1\cos[\log_e(1+x)] + C_2\sin[\log_e(1+x)] - \log_e(1+x)\cdot\cos(\log_e(1+x))$$

EXAMPLE 11

Solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

Solution.
The given equation is

$$[(3x+2)^2D^2 + (3x+2)D - 36]y = 3x^2+4x+1 \quad (1)$$

which is Legendre's equation.

Put $3x+2 = e^z \Rightarrow x = \frac{e^z-2}{3}$ and $\theta = \frac{d}{dz}$

Then $(3x+2)D = 3\cdot\theta$ and $(3x+2)^2D^2 = 3^2\cdot\theta(\theta-1) = 9\theta(\theta-1)$ $[\because a = 3]$

$\therefore$ the equation (1) is

$$(9\theta(\theta-1)+3\cdot 3\theta - 36)y = 3\cdot\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right)+1$$

$$\Rightarrow (9\theta^2-36)y = \frac{1}{3}(e^{2z}-4e^z+4)+\frac{4}{3}e^z - \frac{8}{3}+1$$

$$\Rightarrow 9(\theta^2-4)y = \frac{1}{3}[e^{2z}-1] \Rightarrow (\theta^2-4)y = \frac{1}{27}[e^{2z}-1]$$

To find the complementary function, solve $(\theta^2 - 4)y = 0$

Auxiliary equation is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

The roots are real and different.

$$\therefore \qquad \text{C.F} = C_1 e^{2z} + C_2 e^{-2z}$$

$$\text{P.I} = \frac{1}{\theta^2 - 4}\left[\frac{1}{27}(e^{2z} - 1)\right] = \frac{1}{27}\left[\frac{1}{\theta^2 - 4}e^{2z} - \frac{1}{\theta^2 - 4}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(\theta+2)(\theta-2)}e^{2z} - \left(\frac{-1}{4}\right)\right]$$

$$= \frac{1}{27}\left[\frac{1}{2+2}\cdot z e^{2z} + \frac{1}{4}\right] = \frac{1}{108}[ze^{2z} + 1]$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \qquad y = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108}[ze^{2z} + 1]$$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108}[(3x+2)^2 \log_e (3x+2) + 1]$$

EXERCISE 11.2

Solve the following Cauchy's linear differential equations.

1. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x$
2. $(x^2D^2 - 2xD - 4)y = 32 (\log x)^2$
3. $(x^2D^2 - 4xD + 6)y = x^2 + 2 \log x$
4. $x^2 y'' - 2xy' - 4y = x^4$
5. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2 + \cos(\log x)$
6. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\sin(\log x)}{x}$
7. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$
8. $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \frac{1}{x}$
9. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \cdot \sin(\log x)$
10. $(x^2D^2 + 2xD - 20)y = (1 + x)^2$
11. $x^2 y'' - xy' + y = x$
12. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log_e x$
13. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4\sin(\log x)$
14. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 12\frac{\log x}{x^2}$

Solve the following Legendre equations.

15. $[(x + 1)^2D^2 + (x + 1)D + 1]y = 4 \cos[\log(1 + x)]$

16. $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=x+2$

17. $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$

18. $(x+a)^2\frac{d^2y}{dx^2}-4(x+a)\frac{dy}{dx}+6y=x$

19. $(2x+3)^2y''-(2x+3)y'-12y=6x$

ANSWERS TO EXERCISE 11.2

1. $y=C_1\cos(\log x)+C_2\sin(\log x)+\log x$

2. $y=C_1x^4+C_2x^{-1}-8\left[(\log x)^2-\frac{3}{2}\log x+\frac{13}{8}\right]$

3. $y=C_1x^4+C_2x^{-1}-\frac{x^2}{6}-\frac{1}{2}\log x+\frac{3}{8}$

4. $y=C_1x^4+C_2x^{-1}+\frac{x^4}{5}\log_e x$

5. $y=(C_1+C_2\log x)x^2+\frac{(x\log x)^2}{2}+\frac{1}{25}[3\cos(\log x)-4\sin(\log x)]$

6. $y=x^2(C_1x^{\sqrt{3}}+C_2x^{-\sqrt{3}})+\frac{1}{61x}[6\cos(\log x)+5\sin(\log x)]$

7. $y=C_1x^{-1}+C_2x^3-\frac{x^3}{3}\left(\log x+\frac{2}{3}\right)$

8. $y=C_1x^{-1}+C_2x^{-2}+\frac{x}{6}+\frac{1}{x}\log x$

9. $y=C_1\cos(\log x)+C_2\sin(\log x)+\frac{1}{4}\log x\sin(\log x)-\frac{1}{4}(\log x)^2\cos(\log x)$

10. $y=C_1x^4+C_2x^{-5}-\frac{x^2}{14}-\frac{x}{9}-\frac{1}{20}$

11. $y=(C_1+C_2\log_e x)x+\frac{x}{2}(\log_e x)^2$

12. $y=C_1x^3+\frac{C_2}{x}-\frac{x^2}{3}\left[\log_e x+\frac{2}{3}\right]$

13. $y=C_1\cos(\log_e x)+C_2\sin(\log_e x)-2\log_e\cdot\cos(\log_e x)$

14. $y=2(\log x)^3+C_1(\log x)+C_2$

15. $y=C_1\cos[\log(x+1)]+C_1\sin[\log(x+1)]+2\log(x+1)\sin[\log(x+1)]$

16. $y=(x+2)[C_1+C_2\log(x+2)]+\frac{1}{2}(x+2)[\log(x+2)]^2$

17. $y=(1+2x)^2\{[\log(1+2x)]^2+C_1\log(1+2x)+C_2\}$

18. $y=C_1(x+a)^2+C_2(x+a)^3+\frac{1}{2}(x+a)-\frac{a}{6}$

19. $y=C_1(2x+3)^{\frac{(3+\sqrt{57})}{4}}+C_2(2x+3)^{\frac{(3+\sqrt{57})}{4}}-\frac{3}{14}(2x+3)+\frac{3}{4}$

11.3 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

So far, we have considered single differential equation with one independent variable x and one dependent variable y. We shall now consider linear differential equations with one independent variable t and two dependent variables x and y. To solve for two variables x and y, we need two differential equations. That is, we have a system of two linear differential equations, known as simultaneous linear differential equations, for which the solution is to be determined. The equations need not be of the same order. In this section, we shall consider only first order linear differential equations with constant coefficients and we shall consider the following three types of equations.

Type I: $a_1 \frac{dx}{dt} + b_1 y = f(t)$ and $a_2 \frac{dy}{dt} + b_2 x = g(t)$

Type II: $a_1 \frac{dx}{dt} + b_1 x + c_1 y = f(t)$ and $a_2 \frac{dy}{dt} + b_2 x + c_2 y = g(t)$

Type III: $a_1 \frac{dx}{dt} + b_1 \frac{dy}{dt} + c_1 x = f(t)$ and $a_2 \frac{dx}{dt} + b_2 \frac{dy}{dt} + c_2 y = g(t)$

To solve these three types of equations, first eliminate one of the variables, say y, from the two equations and obtain a second order linear differential equation with constant coefficients in x and t, from which x can be determined by using the methods discussed earlier. Then y can be determined by using the given equations. The solution will be $x = F(t)$; $y = G(t)$.

WORKED EXAMPLES

TYPE I

EXAMPLE 1

Solve $\frac{dx}{dt} - y = t; \frac{dy}{dt} + x = t^2.$

Solution.
The given equations are

$$\frac{dx}{dt} - y = t \quad (1) \qquad \text{and} \qquad \frac{dy}{dt} + x = t^2 \quad (2)$$

First we eliminate y.

Differentiate (1) w.r.to t. $\therefore \frac{d^2x}{dt^2} - \frac{dy}{dt} = 1$

From (2), $\frac{dy}{dt} = t^2 - x$

$$\therefore \quad \frac{d^2x}{dt^2} - (t^2 - x) = 1 \Rightarrow \frac{d^2x}{dt^2} + x = 1 + t^2 \Rightarrow D^2x + x = 1 + t^2, \quad \text{where } D = \frac{d}{dt}$$

$$\Rightarrow \quad (D^2 + 1)x = 1 + t^2$$

This is a second order linear differential equation with constant coefficients in x.
To find the complementary function, solve $(D^2 + 1)x = 0$

Auxiliary equation is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$\therefore \quad \text{C.F} = C_1 \cos t + C_2 \sin t \qquad [\alpha = 0; \beta = 1]$

$$\text{P.I} = \frac{1}{D^2+1}(1+t^2) = (1+D^2)^{-1}(1+t^2)$$
$$= (1 - D^2 + D^4 - \cdots)(1+t^2)$$
$$= (1-D^2)(1+t^2) = 1+t^2 - D^2(1+t^2) = 1+t^2-2 = t^2-1$$

$\therefore \quad x = \text{C.F} + \text{P.I}$

$\Rightarrow \quad x = C_1 \cos t + C_2 \sin t + t^2 - 1 \qquad (3)$

From (1), $\quad y = \frac{dx}{dt} - t = \frac{d}{dt}[C_1 \cos t + C_2 \sin t + t^2 - 1] - t$

$$= -C_1 \sin t + C_2 \cos t + 2t - t = C_2 \cos t - C_1 \sin t + t \qquad (4)$$

$\therefore$ the solution is $\quad x = C_1 \cos t + C_2 \sin t + t^2 - 1$ and $y = C_2 \cos t - C_1 \sin t + t$

EXAMPLE 2

Solve $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given $x = 2, y = 0$ at $t = 0$.

Solution.
The given equations are

$$\frac{dx}{dt} + y = \sin t \quad (1) \qquad \text{and} \qquad \frac{dy}{dt} + x = \cos t \quad (2)$$

First we eliminate y.

Differentiate (1) w.r.to t. $\quad \therefore \quad \frac{d^2x}{dt^2} + \frac{dy}{dt} = \cos t$

From (2), $\quad \frac{dy}{dt} = \cos t - x$

$$\therefore \quad \frac{d^2x}{dt^2} + \cos t - x = \cos t \quad \Rightarrow \quad \frac{d^2x}{dt^2} - x = 0 \quad \Rightarrow \quad (D^2-1)x = 0, \quad \text{where } D = \frac{d}{dt} \qquad (3)$$

This is a second order linear homogeneous differential equation with constant coefficients.

Auxiliary equation is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$\therefore \quad x = C_1 e^t + C_2 e^{-t}$

From (1), $y = -\frac{dx}{dt} + \sin t = -\frac{d}{dt}[C_1 e^t + C_2 e^{-t}] + \sin t$

$$= -C_1 e^t - C_2 e^{-t}(-1) + \sin t = C_2 e^{-t} - C_1 e^t + \sin t$$

Given when $t = 0, x = 2, y = 0$

$\therefore \quad C_1 + C_2 = 2$ and $C_2 - C_1 = 0 \quad \Rightarrow \quad C_1 = C_2$

$\therefore \quad 2C_2 = 2 \quad \Rightarrow \quad C_2 = 1$ and $C_1 = 1$

$\therefore \quad$ the solution is $x = e^t + e^{-t}$ and $y = e^{-t} - e^t + \sin t$

EXAMPLE 3

Solve $\frac{dx}{dt} + 2y = 5e^t$; $\frac{dy}{dt} - 2x = 5e^t$, given $x = -1$ and $y = 3$, when $t = 0$.

Solution.

The given equations are

$$\frac{dx}{dt} + 2y = 5e^t \quad (1) \quad \text{and} \quad \frac{dy}{dt} - 2x = 5e^t \quad (2)$$

Differentiating (1) w.r.to t, we get

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} = 5e^t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + 2[2x + 5e^t] = 5e^t \Rightarrow D^2x + 4x + 10e^t = 5e^t \Rightarrow (D^2 + 4)x = -5e^t \quad \text{[Using (2)]}$$

This is a second order linear differential equation with constant coefficients in x

To find the complementary function, solve $(D^2 + 4)x = 0$

Auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$\therefore \quad \text{C.F} = C_1 \cos 2t + C_2 \sin 2t \quad [\because \alpha = 0, \beta = 0]$$

$$\text{P.I} = \frac{1}{D^2 + 4}(-5e^t) = -\frac{5e^t}{1+4} = -e^t$$

$\therefore$ solution is $x = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad x = C_1 \cos 2t + C_2 \sin 2t - e^t$$

(1) is $\frac{dx}{dt} + 2y = 5e^t$

$$\Rightarrow \quad 2y = -\frac{d}{dt}[C_1 \cos 2t + C_2 \sin 2t - e^t] + 5e^t$$

$$\Rightarrow \quad 2y = -(-2C_1 \sin 2t + 2C_2 \cos 2t - e^t) + 5e^t$$

$$\Rightarrow \quad 2y = 2C_1 \sin 2t - 2C_2 \cos 2t + 6e^t$$

$$\Rightarrow \quad y = C_1 \sin 2t - C_2 \cos 2t + 3e^t$$

Given when $t = 0$, $x = -1$ and $y = 3$

$$\therefore \quad -1 = C_1 \cos 0 + C_2 \sin 0 - e^0 \Rightarrow -1 = C_1 - 1 \Rightarrow C_1 = 0$$

and $3 = C_1 \sin 0 - C_2 \cos 0 + 3e^0 \Rightarrow 3 = -C_2 + 3 \Rightarrow C_2 = 0$

$\therefore$ the solution is $x = -e^t$ and $y = 3e^t$

TYPE II

EXAMPLE 4

Solve $\frac{dx}{dt} + 5x - 2y = t$; $\frac{dy}{dt} + 2x + y = 0$ given $x = y = 0$ when $t = 0$.

Solution.

The given equations are

$$\frac{dx}{dt} + 5x - 2y = t \quad \text{and} \quad \frac{dy}{dt} + 2x + y = 0$$

Let $D = \frac{d}{dt}$, then $(D+5)x - 2y = t$ (1)

$2x + (D+1)y = 0$ (2)

First we eliminate y.
We solve (1) and (2) as simultaneous algebraic equations, making y terms equal.
Operate $(D+1)$ on (1) and multiply (2) by 2.

$\therefore$ $(D+1)(D+5)x - 2(D+1)y = (D+1)t$

$\Rightarrow$ $(D^2+6D+5)x - 2(D+1)y = 1+t$ (3)

and $4x + 2(D+1)y = 0$ (4)

(3) + (4) $\Rightarrow$ $(D^2+6D+5)x + 4x = 1+t$

$\Rightarrow$ $(D^2+6D+9)x = 1+t$

This is a second order linear differential equation with constant coefficients in x.
To find the complementary function, solve $(D^2+6D+9)x = 0$
Auxiliary equation is $m^2+6m+9=0 \Rightarrow (m+3)^2 = 0 \Rightarrow m = -3, -3$
The roots are real and equal.

$\therefore$ $\text{C.F} = (C_1 + C_2 t)e^{-3t}$

$$\begin{aligned}\text{P.I} &= \frac{1}{D^2+6D+9}(1+t)\\ &= \frac{1}{9}\cdot\frac{1}{\left[1+\frac{(6D+D^2)}{9}\right]}(1+t)\\ &= \frac{1}{9}\left[1+\frac{(6D+D^2)}{9}\right]^{-1}(1+t)\\ &= \frac{1}{9}\left[1-\frac{(6D+D^2)}{9}\right](1+t)\\ &= \frac{1}{9}\left[1-\frac{6D}{9}\right](1+t) = \frac{1}{9}\left(1+t-\frac{6}{9}\right) = \frac{1}{9}\left[t+\frac{1}{3}\right] = \frac{1}{27}(3t+1)\end{aligned}$$

$\therefore$ $x = \text{C.F} + \text{P.I} = (C_1 + C_2 t)e^{-3t} + \frac{1}{27}(3t+1)$ (5)

From (1), $2y = (D+5)x - t = Dx + 5x - t$

$$\begin{aligned}&= \frac{d}{dt}\left[(C_1+C_2t)e^{-3t} + \frac{1}{27}(3t+1)\right] + 5\left[(C_1+C_2t)e^{-3t} + \frac{1}{27}(3t+1)\right] - t\\ &= [C_1 + C_2 t](-3e^{-3t}) + e^{-3t}(C_2) + \frac{3}{27} + 5(C_1+C_2t)e^{-3t} + \frac{5}{9}t + \frac{5}{27} - t\\ &= e^{-3t}[-3C_1 - 3C_2t + C_2 + 5C_1 + 5C_2t] - \frac{4}{9}t + \frac{5}{27} + \frac{1}{9}\\ &= e^{-3t}[2C_1 + C_2 + 2C_2t] - \frac{4}{9}t + \frac{8}{27}\end{aligned}$$

$\Rightarrow$ $y = \frac{1}{2}\left\{e^{-3t}[2C_1 + C_2 + 2C_2t] - \frac{4}{9}t + \frac{8}{27}\right\}$ (6)

When $t = 0, x = 0, y = 0.$

(5) $\Rightarrow$ $C_1 + \frac{1}{27} = 0 \Rightarrow C_1 = -\frac{1}{27}$

(6) $\Rightarrow$ $\frac{1}{2}\left\{[2C_1 + C_2] + \frac{8}{27}\right\} = 0 \Rightarrow 2\cdot\left(-\frac{1}{27}\right) + C_2 + \frac{8}{27} = 0 \Rightarrow C_2 = -\frac{6}{27} = -\frac{2}{9}$

$\therefore$
$$x = \left(-\frac{1}{27} - \frac{2}{9}t\right)e^{-3t} + \frac{1}{27}(3t+1)$$
$$= -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(3t+1) = \frac{1}{27}[1+3t-(1+6t)e^{-3t}]$$

and
$$y = \frac{1}{2}\left[e^{-3t}\left(\frac{-2}{27} - \frac{4}{9}t - \frac{2}{9}\right) - \frac{4t}{9} + \frac{8}{27}\right]$$
$$= \frac{1}{2}\left[-e^{-3t}\left(\frac{8}{27} + \frac{4}{9}t\right) - \frac{4}{27}(3t-2)\right]$$
$$= \frac{4}{2\times 27}\left[-e^{-3t}(3t+2) + (2-3t)\right] = \frac{2}{27}\left[2-3t-e^{-3t}(3t+2)\right]$$

$\therefore$ the solution is $x = \frac{1}{27}[1+3t-(1+6t)e^{-3t}]$ and $y = \frac{2}{27}\left[2-3t-e^{-3t}(3t+2)\right]$

EXAMPLE 5

Solve $\frac{dx}{dt} + 2x - 3y = t; \frac{dy}{dt} - 3x + 2y = e^{2t}$.

Solution.
The given equations are

$$\frac{dx}{dt} + 2x - 3y = t \quad \text{and} \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$$

$\Rightarrow$ $(D+2)x - 3y = t,$ where $D = \frac{d}{dt}$ (1)

and $-3x + (D+2)y = e^{2t}$ (2)

First we eliminate y.

Operate $(D+2)$ on (1) and multiply (2) by 3.

$\therefore$ $(D+2)^2x - 3(D+2)y = (D+2)t$

$\Rightarrow$ $(D+2)^2x - 3(D+2)y = 1 + 2t$ (3)

and $-9x + 3(D+2)y = 3e^{2t}$ (4)

(3) + (4) $\Rightarrow$ $(D+2)^2x - 9x = 1 + 2t + 3e^{2t}$

$\Rightarrow$ $(D^2 + 4D + 4 - 9)x = 1 + 2t + 3e^{2t}$

$\Rightarrow$ $(D^2 + 4D - 5)x = 1 + 2t + 3e^{2t}$

This is a second order linear differential equation with constant coefficients in x.

To find the complementary function, solve $(D^2 + 4D - 5)x = 0$

Auxiliary equation is $m^2 + 4m - 5 = 0 \quad \Rightarrow \quad (m+5)(m-1) = 0 \quad \Rightarrow \quad m = -5, 1$

The roots are real and different.

$$\therefore \qquad \text{C.F} = C_1 e^{-5t} + C_2 e^t$$

$$\text{P.I}_1 = \frac{1}{D^2 + 4D - 5}(1+2t)$$

$$= -\frac{1}{5}\frac{1}{\left[1 - \frac{(4D+D^2)}{5}\right]}(1+2t)$$

$$= -\frac{1}{5}\left[1 - \frac{(4D+D^2)}{5}\right]^{-1}(1+2t)$$

$$= -\frac{1}{5}\left[1 + \frac{4D+D^2}{5}\right][1+2t]$$

$$= -\frac{1}{5}\left[1 + \frac{4D}{5}\right](1+2t)$$

$$= -\frac{1}{5}\left[1 + 2t + \frac{4}{5}D(1+2t)\right] = -\frac{1}{5}\left[1 + 2t + \frac{4}{5}\cdot 2\right] = -\frac{1}{5}\left[2t + \frac{13}{5}\right]$$

$$\text{P.I}_2 = \frac{1}{D^2 + 4D - 5}3e^{2t} = 3\frac{e^{2t}}{4 + 4\cdot 2 - 5} = \frac{3e^{2t}}{7}$$

$\therefore$ solution $\quad x = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 = C_1 e^{-5t} + C_2 e^t - \frac{1}{5}\left(2t + \frac{13}{5}\right) + \frac{3e^{2t}}{7}$

From (1), $\quad 3y = (D+2)x - t$

$$\Rightarrow \qquad 3y = \frac{dx}{dt} + 2x - t$$

$$= \frac{d}{dt}\left[C_1 e^{-5t} + C_2 e^t - \frac{1}{5}\left(2t + \frac{13}{5}\right) + \frac{3e^{2t}}{7}\right] + 2\left[C_1 e^{-5t} + C_2 e^t - \frac{1}{5}\left(2t + \frac{13}{5}\right) + \frac{3e^{2t}}{7}\right] - t$$

$$= C_1 e^{-5t}(-5) + C_2 e^t - \frac{2}{5} + \frac{6e^{2t}}{7} + 2C_1 e^{-5t} + 2C_2 e^t - \frac{4t}{5} - \frac{26}{25} + \frac{6e^{2t}}{7} - t$$

$$\Rightarrow \qquad 3y = -3C_1 e^{-5t} + 3C_2 e^t - \frac{9}{5}t + \frac{12}{7}e^{2t} - \frac{36}{25}$$

$$\Rightarrow \qquad y = -C_1 e^{-5t} + C_2 e^t - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$

$\therefore$ the solution is $x = C_1 e^{-5t} + C_2 e^t - \frac{1}{5}\left(2t + \frac{13}{5}\right) + \frac{3e^{2t}}{7}$ and $y = -C_1 e^{-5t} + C_2 e^t - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$

TYPE III

EXAMPLE 6

Solve $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$.

Solution.
The given equations are

$$Dx - Dy + 2y = \cos 2t, \quad \text{where } D = \frac{d}{dt}$$

$$\Rightarrow \quad Dx - (D-2)y = \cos 2t \quad (1)$$

and $$Dx + Dy - 2x = \sin 2t \quad \Rightarrow \quad (D-2)x + Dy = \sin 2t \quad (2)$$

First we eliminate y.

Operate D on (1) and $(D-2)$ on (2)

$$\therefore \quad D^2x - D(D-2)y = -2\sin 2t \quad (3)$$

and $$(D-2)^2x + D(D-2)y = (D-2)\sin 2t$$

$$\Rightarrow \quad (D-2)^2x + D(D-2)y = 2\cos 2t - 2\sin 2t \quad (4)$$

$$(3)+(4) \quad \Rightarrow \quad D^2x + (D-2)^2x = 2\cos 2t - 4\sin 2t$$

$$\Rightarrow \quad (2D^2 - 4D + 4)x = 2[\cos 2t - 2\sin 2t]$$

$$\Rightarrow \quad (D^2 - 2D + 2)x = \cos 2t - 2\sin 2t$$

This is a second order linear differential equation with constant coefficients in x.
To find the complementary function, solve $(D^2 - 2D + 2)x = 0$

Auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$

The roots are complex numbers with $\boldsymbol{\alpha} = 1$ and $\boldsymbol{\beta} = 1$

$$\text{C.F} = e^t[C_1 \cos t + C_2 \sin t]$$

$$\therefore \text{P.I}_1 = \frac{1}{D^2 - 2D + 2}\cos 2t = \frac{1}{-4 - 2D + 2}\cos 2t$$

$$= -\frac{1}{2(D+1)}\cos 2t$$

$$= -\frac{(D-1)}{2(D^2-1)}\cos 2t$$

$$= \frac{-(D-1)\cos 2t}{2(-4-1)}$$

$$= \frac{1}{10}[D(\cos 2t) - \cos 2t] = \frac{1}{10}[-2\sin 2t - \cos 2t] = -\frac{1}{10}[2\sin 2t + \cos 2t]$$

and $$\text{P.I}_2 = \frac{1}{D^2 - 2D + 2}(-2\sin 2t)$$

$$= (-2)\frac{1}{-4 - 2D + 2}\sin 2t$$

$$= -2 \cdot \frac{1}{-2(D+1)}\sin 2t$$

$$= \frac{D-1}{D^2-1}\sin 2t$$

$$= \frac{(D-1)\sin 2t}{-4-1} = -\frac{1}{5}[2\cos 2t - \sin 2t] = -\frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t$$

$\therefore \qquad x = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$

$$= e^t[C_1\cos t + C_2\sin t] - \frac{1}{10}[2\sin 2t + \cos 2t] - \frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t$$

$$= e^t[C_1\cos t + C_2\sin t] - \frac{1}{2}\cos 2t$$

Now adding the given equations, we get

$$2\frac{dx}{dt} + 2y - 2x = \cos 2t + \sin 2t$$

$\Rightarrow \qquad 2y = \cos 2t + \sin 2t - 2\frac{dx}{dt} + 2x$

$$= \cos 2t + \sin 2t - 2\frac{d}{dt}\left[e^t(C_1\cos t + C_2\sin t) - \frac{1}{2}\cos 2t\right] + 2\left[e^t(C_1\cos t + C_2\sin t) - \frac{1}{2}\cos 2t\right]$$

$$= \cos 2t + \sin 2t - 2\left[e^t(-C_1\sin t + C_2\cos t) + (C_1\cos t + C_2\sin t)e^t - \frac{1}{2}(-2\sin 2t)\right] + 2\left[e^t(C_1\cos t + C_2\sin t) - \frac{1}{2}\cos 2t\right]$$

$$= 2e^t[C_1\sin t - C_2\cos t - C_1\cos t - C_2\sin t + C_1\cos t + C_2\sin t] + \cos 2t - \cos 2t + \sin 2t - 2\sin 2t$$

$\Rightarrow \qquad 2y = 2e^t[C_1\sin t - C_2\cos t] - \sin 2t \qquad \Rightarrow \qquad y = e^t[C_1\sin t - C_2\cos t] - \frac{1}{2}\sin 2t$

$\therefore$ the solution is $x = e^t[C_1\cos t + C_2\sin t] - \frac{1}{2}\cos 2t$ and $y = e^t[C_1\sin t - C_2\cos t] - \frac{1}{2}\sin 2t$

EXAMPLE 7

Solve $2\frac{dx}{dt} + \frac{dy}{dt} - 3x = e^t$; $\frac{dx}{dt} + \frac{dy}{dt} + 2y = \cos 2t$.

Solution.

The given equations are

$$(2D-3)x + Dy = e^t \qquad \left[D = \frac{d}{dt}\right] \qquad (1)$$

and

$$Dx + (D+2)y = \cos 2t \qquad (2)$$

First we eliminate y,

Operate $(D+2)$ on (1) and D on (2)

$\therefore \qquad (D+2)(2D-3)x + (D+2)Dy = (D+2)e^t$

$\Rightarrow$ $$(2D^2 + D - 6)x + (D + 2)Dy = e^t + 2e^t = 3e^t \quad (3)$$

and $$D \cdot Dx + D(D + 2)y = D\,(\cos 2t)$$

$\Rightarrow$ $$D^2x + D(D + 2)y = -2 \sin 2t \quad (4)$$

$(3) - (4) \Rightarrow$ $$(D^2 + D - 6)x = 3e^t + 2 \sin 2t$$

This is a second order linear differential equation with constant coefficients in x.
To find the complementary function, solve $(D^2 + D - 6)x = 0$
Auxiliary equation is $m^2 + m - 6 = 0 \Rightarrow (m + 3)(m - 2) = 0 \Rightarrow m = -3, 2$
The roots are real and different.

$\therefore$ $$\text{C.F} = C_1e^{-3t} + C_2e^{2t}$$

$$\text{P.I}_1 = \frac{1}{D^2 + D - 6}3e^t = \frac{3e^t}{1+1-6} = -\frac{3e^t}{4}$$

and $$\text{P.I}_2 = \frac{1}{D^2 + D - 6}2\sin 2t = 2 \cdot \frac{1}{-4 + D - 6}\sin 2t$$

$$= 2 \cdot \frac{1}{D-10}\sin 2t$$

$$= 2\frac{D+10}{D^2-100}\sin 2t$$

$$= \frac{2(D+10)\sin 2t}{-4-100}$$

$$= -\frac{1}{52}[D(\sin 2t) + 10\sin 2t]$$

$$= -\frac{1}{52}[2\cos 2t + 10\sin 2t] = -\frac{1}{26}[\cos 2t + 5\sin 2t]$$

$\therefore$ $$x = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$= C_1e^{-3t} + C_2e^{2t} - \frac{3e^t}{4} - \frac{1}{26}[\cos 2t + 5\sin 2t]$$

Subtracting the given equations, we get

$$\frac{dx}{dt} - 3x - 2y = e^t - \cos 2t$$

$\Rightarrow$ $$2y = \frac{dx}{dt} - 3x - e^t + \cos 2t$$

$$= \frac{d}{dt}\left[C_1e^{-3t} + C_2e^{2t} - \frac{3e^t}{4} - \frac{1}{26}[\cos 2t + 5\sin 2t]\right]$$

$$-3\left[C_1e^{-3t} + C_2e^{2t} - \frac{3e^t}{4} - \frac{1}{26}\{\cos 2t + 5\sin 2t\}\right] - e^t + \cos 2t$$

$$= -3C_1e^{-3t} + 2C_2e^{2t} - \frac{3}{4}e^t - \frac{1}{26}(-2\sin 2t + 10\cos 2t) - 3C_1e^{-3t}$$
$$-3C_2e^{2t} + \frac{9}{4}e^t + \frac{3}{26}\cos 2t + \frac{15}{26}\cdot\sin 2t - e^t + \cos 2t$$
$$= -6C_1e^{-3t} - C_2e^{2t} + \frac{e^t}{2} + \frac{19}{26}\cos 2t + \frac{17}{26}\sin 2t$$

$\therefore$
$$y = -3C_1e^{-3t} - \frac{C_2}{2}e^{2t} + \frac{e^t}{4} + \frac{19}{52}\cos 2t + \frac{17}{52}\sin 2t$$

$\therefore$ the solution is
$$x = C_1e^{-3t} + C_2e^{2t} - \frac{3e^t}{4} - \frac{1}{26}[\cos 2t + 5\sin 2t]$$

and
$$y = -3C_1e^{-3t} - \frac{C_2}{2}e^{2t} + \frac{e^t}{4} + \frac{19}{52}\cos 2t + \frac{17}{52}\sin 2t$$

EXERCISE 11.3

Solve the following simultaneous linear differential equations.

1. $\frac{dx}{dt} + y = e^t$; $\quad \frac{dy}{dt} - x = e^{-t}$.
2. $Dx + y = \sin 2t$; $\quad -x + Dy = \cos 2t$.
3. $\frac{dx}{dt} + 2y = 5e^t$; $\quad \frac{dy}{dt} - 2x = 5e^t$ given that $x = -1$ and $y = 3$ when $t = 0$.
4. $\frac{dx}{dt} + y = \sin t$; $\quad x + \frac{dy}{dt} = \cos t$ given that $x = 2$ and $y = 0$ at $t = 0$.
5. $\frac{dx}{dt} + y = \sin t + 1$; $\quad \frac{dy}{dt} + x = \cos t$ given that $x = 1$ and $y = 2$ at $t = 0$.
6. $\frac{dx}{dt} + 2y = -\sin t$; $\quad \frac{dy}{dt} - 2x = \cos t$.
7. $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$; $\quad \frac{dy}{dt} + 3x + 2y = 0$.
8. $\frac{dx}{dt} + 2x - 3y = 5t$; $\quad \frac{dy}{dt} - 3x + 2y = 0$ given that $x(0) = 0, y(0) = -1$.
9. $\frac{dx}{dt} + 2x - 3y = t$; $\quad \frac{dx}{dt} - 3x + 2y = e^{2t}$.
10. $\frac{dx}{dt} = 3x + 8y$, $\quad \frac{dy}{dt} = -x - 3y$, $x(0) = 6, y(0) = -2$.
11. $(D + 2)x + 3y = 0$; $\quad 3x + (D + 2)y = 2e^{2t}$.
12. $\frac{dx}{dt} + 2x + 3y = 2e^t$; $\quad \frac{dy}{dt} + 3x + 2y = 0$.

13. $(D-3)x + 2(D+2)y = e^{2t}$; $2(D+1)x + (D-1)y = 0$,   where $D = \dfrac{d}{dt}$.

14. $4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 2x + 31y = e^t$; $3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + x + 24y = 3$.

ANSWERS TO EXERCISE 11.3

1. $x = C_1 \cos t + C_2 \sin t + \sinh t$, $y = C_1 \sin t - C_2 \cos t - \cosh t + e^t$.
2. $x = C_1 \cos t + C_2 \sin t - \frac{1}{3}\cos 2t$, $y = C_1 \sin t - C_2 \cos t + \frac{1}{3}\sin 2t$.
3. $x = -e^t$, $y = 3e^t$.
4. $x = 2(1+t)e^t$, $y = -2te^t + \sin t$.
5. $x = e^{-t}$, $y = 1 + \sin t + e^{-t}$.
6. $x = C_1 \cos 2t + C_2 \sin 2t - \cos t$, $y = C_1 \sin 2t - C_2 \cos 2t - \sin t$.
7. $x = C_1 e^{-5t} + C_2 e^t + \frac{8}{7}e^{2t}$, $y = C_1 e^{-5t} - C_2 e^t - \frac{6}{7}e^{2t}$.
8. $x = \frac{3}{5}e^{5t} + 2e^t - 2t - \frac{13}{5}$, $y = -\frac{3}{5}e^{5t} + 2e^t - 3t - \frac{12}{5}$.
9. $x = C_1 e^t + C_2 e^{-5t} + \frac{3}{7}e^{2t} - \frac{2t}{5} - \frac{13}{25}$, $y = C_1 e^t - C_2 e^{-5t} - \frac{3t}{5} - \frac{12}{25}$.
10. $x = 4e^t + 2e^{-t}$, $y = -[e^t + e^{-t}]$.
11. $x = C_1 e^{-5t} + C_2 e^t - \frac{6e^{2t}}{7}$, $y = C_1 e^{-5t} - C_2 e^t + \frac{8}{7}e^{2t}$.
12. $x = C_1 e^{-5t} + C_2 e^t + \frac{8}{7}e^{2t}$, $y = C_1 e^{-5t} - C_2 e^t - \frac{6}{7}e^{2t}$.
13. $x = C_1 e^{-\frac{t}{3}} + C_2 e^{-5t} - \frac{e^{2t}}{49}$, $y = C_1 e^{-\frac{t}{3}} - \frac{4}{3}C_2 e^{-5t} + \frac{6}{49}e^{2t}$.
14. $x = \dfrac{e^{-4t}}{2}(C_1 \cos t + C_2 \sin t) - \dfrac{e^{-4t}}{2}(C_1 \sin t - C_2 \cos t) + \dfrac{31}{26}e^t - \dfrac{93}{17}$,

    $y = e^{-4t}(C_1 \cos t - C_2 \sin t) + \dfrac{6}{17} - \dfrac{2e^t}{13}$.

11.4 METHOD OF VARIATION OF PARAMETERS

The method of variation of parameters, due to Lagrange, is a powerful method of finding a particular integral to a second order equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \qquad (1)$$

11.4 METHOD OF VARIATION OF PARAMETERS

The method of variation of parameters, due to Lagrange, is a powerful method of finding a particular integral to a second order equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \tag{1}$$

when its complementary function $C_1 y_1(x) + C_2 y_2(x)$ is known where C_1, C_2 are arbitrary constants and $y_1(x)$, $y_2(x)$ are two independent solutions of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{2}$$

The method of variation of parameters replaces C_1 and C_2 in the C.F by functions of x, $u(x)$ and $v(x)$ which are to be determined so that

$$y_p = u(x)y_1 + v(x)y_2 \tag{3}$$

is a particular solution of (1). It can be seen that

$u(x) = -\int \frac{y_2 R(x)}{W} dx, v(x) = \int \frac{y_1 R(x)}{W} dx$ and W is the Wronskian of y_1 and y_2 and $W \neq 0$

i.e.,
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$\therefore$ the general solution of (1) is

$$y = \text{C.F} + y_p$$

$\Rightarrow$
$$y = C_1 y_1 + C_2 y_2 + u(x)y_1 + v(x)y_2$$

11.4.1 Working Rule

Step 1. Rewrite the given equation in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Step 2. Find the C.F $= C_1 y_1 + C_2 y_2$

Step 3. Find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $u(x) = -\int \frac{y_2 R(x)}{W} dx$

and
$$v(x) = \int \frac{y_1 R(x)}{W} dx$$

Step 4. P.I $= u(x)y_1 + v(x)y_2$

Step 5. General solution is

$$y = \text{C.F} + \text{P.I}$$
$$= C_1 y_1 + C_2 y_2 + u(x)y_1 + v(x)y_2$$

WORKED EXAMPLES

EXAMPLE 1

Solve $\frac{d^2y}{dx^2} + a^2 y = \sec ax$.

Solution.
The given equation is

$$(D^2 + a^2)y = \sec ax$$

To find the complementary function, solve $(D^2 + a^2)y = 0$

Auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

The roots are complex numbers with $\alpha = 0$ and $\beta = a$

$\therefore$ $\quad$ C.F $= C_1 \cos ax + C_2 \sin ax$

To find $\quad$ P.I $= u(x)y_1 + v(x)y_2$

Here $\quad y_1 = \cos ax, \quad y_2 = \sin ax, \quad R(x) = \sec ax$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a[\cos^2 ax + \sin^2 ax] = a \neq 0$$

Now $\quad u = u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \frac{\sin ax \cdot \sec ax}{a} dx$

$$= -\frac{1}{a}\int \frac{\sin ax}{\cos ax} dx$$

$$= \frac{1}{a}\int -\frac{\sin ax}{\cos ax} dx = \frac{1}{a}\cdot\frac{1}{a}\log\cos ax = \frac{1}{a^2}\cdot\log\cos ax$$

and $\quad v = v(x) = \int \frac{y_1 R(x)}{W} dx = \int \frac{\cos ax \sec ax}{a} dx = \frac{1}{a}\int dx = \frac{x}{a}$

$\therefore$ $\quad$ P.I $= u(x)y_1 + v(x)y_2$

$$= \frac{1}{a^2}\log\cos ax \cdot \cos ax + \frac{x}{a}\sin ax = \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$\Rightarrow$ $$y = C_1\cos ax + C_2\sin ax + \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax$$

EXAMPLE 2

Solve by method of variation of parameters $\frac{d^2y}{dx^2} + y = x\sin x$.

Solution.

The given equation is

$$(D^2 + 1)y = x \sin x$$

To find the complementary function solve $(D^2 + 1)y = 0$

Auxiliary equation is $m^2 + 1 = 0 \quad m = \pm i \quad [\alpha = 0, \beta = 1]$

$$\text{C.F} = C_1\cos x + C_2\sin x$$

To find $\quad$ P.I $= uy_1 + vy_2$

Here $\quad y_1 = \cos x, \quad y_2 = \sin x, \quad R(x) = x\sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Now $$u = u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \frac{\sin x \cdot x \sin x}{1} dx$$
$$= -\int x \sin^2 x \, dx$$
$$= -\int x \frac{(1-\cos 2x)}{2} dx$$
$$= -\frac{1}{2}\int x \, dx + \frac{1}{2}\int x \cos 2x \, dx$$
$$= -\frac{x^2}{4} + \frac{1}{2}\left[x \frac{\sin 2x}{2} - 1 \cdot \frac{(-\cos 2x)}{2^2}\right]$$
$$= -\frac{x^2}{4} + \frac{1}{8}[2x \sin 2x + \cos 2x]$$

and $$v = v(x) = \int \frac{y_1 R(x)}{W} dx = \int \frac{\cos x \cdot x \sin x}{1} dx$$
$$= \frac{1}{2}\int x \sin 2x \, dx$$
$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1 \cdot \left(-\frac{\sin 2x}{2^2}\right)\right] = \frac{1}{8}[-2x \cos 2x + \sin 2x]$$

$\therefore$ $$\text{P.I} = \left\{-\frac{x^2}{4} + \frac{1}{8}[2x \sin 2x + \cos 2x]\right\}\cos x + \frac{1}{8}[-2x \cos 2x + \sin 2x]\sin x$$
$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x \cos x \sin 2x + \cos x \cos 2x - 2x \sin x \cos 2x + \sin x \sin 2x]$$
$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x\{\sin 2x \cos x - \cos 2x \sin x\} + \cos 2x \cos x + \sin 2x \sin x]$$
$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x \sin(2x-x) + \cos(2x-x)] = -\frac{x^2}{4}\cos x + \frac{1}{8}[2x \sin x + \cos x]$$

$\therefore$ the general solution is
$$y = \text{C.F} + \text{P.I}$$
$$= C_1 \cos x + C_2 \sin x - \frac{x^2}{4}\cos x + \frac{1}{8}[2x \sin x + \cos x]$$
$$= C_3 \cos x + C_2 \sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x, \text{ where } C_3 = C_1 + \frac{1}{8}$$

EXAMPLE 3

Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \cot x$ using method of variation of parameters.

Solution.
The given equation is
$$(D^2+1)y = \operatorname{cosec} x \cot x$$

To find the complementary function solve $(D^2+1)y=0$

Auxiliary equation is $m^2+1=0 \Rightarrow m=\pm i$

The roots are complex numbers with $\alpha=0$ and $\beta=1$.

$\therefore$ $\quad \text{C.F} = C_1\cos x + C_2\sin x$

To find $\quad \text{P.I} = uy_1 + vy_2$

Here $\quad y_1=\cos x, \quad y_2=\sin x, \quad R(x)=\operatorname{cosec} x\cot x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

Now, $u=u(x)=-\int \frac{y_2R(x)}{W}dx = -\int \sin x \operatorname{cosec} x \cot x\, dx = -\int \frac{\cos x}{\sin x}dx = -\log\sin x$

and $\quad v=v(x)=\int \frac{y_1R(x)}{W}dx = \int \cos x \operatorname{cosec} x\cot x\, dx$

$$= \int \frac{\cos x}{\sin x}\cot x\, dx$$

$$= \int \cot^2 x\, dx$$

$$= \int (\operatorname{cosec}^2 x - 1)\, dx = -\cot x - x = -(x+\cot x)$$

$\therefore$ $\quad \text{P.I} = uy_1 + vy_2 = -\log\sin x\cdot\cos x - (x+\cot x)\sin x$

$$= -\cos x\log\sin x - x\sin x - \cos x = -\cos x[1+\log\sin x] - x\sin x$$

$\therefore$ the general solution is

$$y = \text{C.F} + \text{P.I}$$

$\Rightarrow$ $\quad y = C_1\cos x + C_2\sin x - \cos x[1+\log\sin x] - x\sin x$

$$= (C_1-1)\cos x + C_2\sin x - \cos x\log\sin x - x\sin x$$

$$= C_3\cos x + C_2\sin x - \cos x\log\sin x - x\sin x, \qquad \text{where } C_3 = C_1 - 1$$

EXAMPLE 4

Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\log x$ by the method of variation of parameters.

Solution.

The given equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\log x \quad \Rightarrow \quad x^2D^2y + xDy - y = x^2\log x \qquad (1)$$

This is Cauchy's equation.

Put $x=e^z$ and $\theta = \frac{d}{dz}$, then $xD=\theta$, $x^2D^2 = \theta(\theta-1)$

$\therefore$ the equation (1) is $[\theta(\theta-1)+\theta-1]y = ze^{2z}$

$\Rightarrow \qquad (\theta^2 - \theta + \theta - 1)y = ze^{2z} \quad \Rightarrow \quad (\theta^2 - 1)y = ze^{2z}$

which is a second order linear differential equation with constant coefficients in y.

To find the complementary function solve, $(\theta^2 - 1)y = 0$

Auxiliary equation is $\qquad m^2 - 1 = 0 \Rightarrow m = \pm 1$

The roots are real and different.

$$\therefore \qquad \text{C.F} = C_1 e^z + C_2 e^{-z} = C_1 x + C_2 \frac{1}{x}$$

To find the P.I $= uy_1 + vy_2$, we write the equation in the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log_e x \qquad \text{[dividing (1) by } x^2]$$

Here $\qquad y_1 = x, \quad y_2 = \frac{1}{x}, \quad R(x) = \log_e x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0$$

Now
$$u = u(x) = -\int \frac{y_2 R(x)}{W} dx = -\int \frac{\frac{1}{x}\log_e x}{-\frac{2}{x}} dx = \frac{1}{2}\int \log_e x\, dx$$

$$= \frac{1}{2}\left[\log_e x \cdot x - \int \frac{1}{x}\cdot x dx\right] = \frac{1}{2}[x\log_e x - x] = \frac{x}{2}[\log_e x - 1]$$

and
$$v = v(x) = \int \frac{y_1 R(x)}{W} dx = \int \frac{x\log_e x}{-\frac{2}{x}} dx = -\frac{1}{2}\int x^2 \log_e x\, dx$$

$$= -\frac{1}{2}\left[\log_e x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3} dx\right] \qquad [\text{Taking } u = \log x, \; dv = x^2 dx]$$

$$= -\frac{1}{6}\left[x^3 \log_e x - \int x^2 dx\right]$$

$$= -\frac{1}{6}\left[x^3 \log_e x - \frac{x^3}{3}\right] = \frac{x^3}{18}[1 - 3\log_e x]$$

$$\therefore \qquad \text{P.I} = uy_1 + vy_2 = \frac{x}{2}(\log_e x - 1)x + \frac{x^3}{18}(1 - 3\log_e x)\frac{1}{x}$$

$$= \frac{x^2}{18}[9(\log_e x - 1) + 1 - 3\log_e x] = \frac{x^2}{18}(6\log_e x - 8) = \frac{x^2}{9}(3\log_e x - 4)$$

$\therefore \qquad$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad = C_1 x + \frac{C_2}{x} + \frac{x^2}{9}(3\log_e x - 4)$$

EXAMPLE 5

Solve by the method of variations of parameters $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=\dfrac{e^{-x}}{x^2}$.

Solution.

The given equation is

$$(D^2+2D+1)y=\frac{e^{-x}}{x^2}$$

To find the complementary function, solve $(D^2+2D+1)y=0$

Auxiliary equation is $m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1$

The roots are real and equal.

$\therefore$ $\quad$ $\text{C.F}=e^{-x}(C_1+C_2x)=C_1e^{-x}+C_2xe^{-x}$

To find $\quad$ $\text{P.I}=uy_1+vy_2$

$$y_1=e^{-x},\quad y_2=xe^{-x},\quad R(x)=\frac{e^{-x}}{x^2}$$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} \end{vmatrix}=e^{-x}\cdot e^{-x}\begin{vmatrix} 1 & x \\ -1 & 1-x \end{vmatrix}=e^{-2x}(1-x+x)=e^{-2x}\neq 0$$

Now $\quad$ $$u=u(x)=-\int\frac{y_2R(x)}{W}dx=-\int\frac{xe^{-x}}{e^{-2x}}\cdot\frac{e^{-x}}{x^2}dx=-\int\frac{1}{x}dx=-\log_e x$$

and $\quad$ $$v=v(x)=\int\frac{y_1R(x)}{W}dx=\int\frac{e^{-x}}{e^{-2x}}\frac{e^{-x}}{x^2}dx=\int\frac{1}{x^2}dx=\int x^{-2}dx=\frac{x^{-1}}{-1}=-\frac{1}{x}$$

$\therefore$ $\quad$ $$\text{P.I}=(-\log x)e^{-x}+\left(-\frac{1}{x}\right)\cdot xe^{-x}=-e^{-x}(\log x+1)$$

$\therefore$ the general solution is

$$\begin{aligned} y=\text{C.F}+\text{P.I}&=C_1e^{-x}+C_2xe^{-x}-e^{-x}(\log x+1)\\ &=(C_1-1)e^{-x}+C_2xe^{-x}-e^{-x}\log x\\ &=C_3e^{-x}+C_2xe^{-x}-e^{-x}\log x;\qquad C_3=C_1-1\end{aligned}$$

EXAMPLE 6

Find the general solution of $x^2\dfrac{d^2y}{dx^2}-x(x+2)\dfrac{dy}{dx}+(x+2)y=x^3$ given that $y=x, y=xe^x$ are two linearly independent solutions of the corresponding homogeneous equation.

Solution.

The given equation is

$$x^2\frac{d^2y}{dx^2}-x(x+2)\frac{dy}{dx}+(x+2)y=x^3$$

$\Rightarrow$ $$\frac{d^2y}{dx^2}-\frac{x+2}{x}\frac{dy}{dx}+\frac{(x+2)}{x^2}y=x$$

Given $y=x,\quad y=xe^x$ are two independent solutions of

$$\frac{d^2y}{dx^2}-\frac{x+2}{x}\frac{dy}{dx}+\frac{x+2}{x^2}y=0$$

$\therefore$ $\quad$ $\text{C.F} = C_1x + C_2xe^x$

To find $\quad$ $\text{P.I} = uy_1 + vy_2$

Here $\quad$ $y_1 = x, \quad y_2 = xe^x, \quad R(x) = x$

Now

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x(xe^x + e^x) - xe^x = x^2e^x$$

$$u = u(x) = -\int \frac{y_2R(x)}{W}dx = -\int \frac{xe^x \cdot x}{x^2 \cdot e^x}dx = -\int dx = -x$$

and

$$v = v(x) = \int \frac{y_1R(x)}{W}dx = \int \frac{x \cdot x}{x^2e^x}dx = \int e^{-x}\,dx = -e^{-x}$$

$\therefore$ $\quad$ $\text{P.I} = (-x)y_1 + (-e^{-x})y_2 = (-x)x + (-e^{-x})xe^x = -x^2 - xe^0 = -(x^2+x)$

$\therefore$ the general solution is $\quad y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $\quad$ $y = C_1x + C_2xe^x - (x^2+x)$

EXERCISE 11.4

Solve the following equations by the method of variation of parameters.

1. $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9y=\dfrac{e^{3x}}{x^2}$

2. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+2y=e^x\tan x$

3. $\dfrac{d^2y}{dx^2}+y=x\cos x$

4. $2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-y=25e^{-x}$

5. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=x\log x$

6. $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=x^2+\dfrac{1}{x^2}$

7. $(x^2+x)\dfrac{d^2y}{dx^2}+(2-x^2)\dfrac{dy}{dx}-(2+x)y=x(x+1)^2$ if the complementary function is known to be $C_1e^x + C_2x^{-1}$.

8. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=\dfrac{e^{-x}}{x^2}$

9. $\dfrac{d^2y}{dx^2}+y=\sec x$

10. $(D^2-2D)y=e^x\cos x$

11. $\dfrac{d^2y}{dx^2}+4y=\tan 2x$

12. $\dfrac{d^2y}{dx^2}+y=\operatorname{cosec} x$

13. $(D^2+a^2)y=\tan ax$

14. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=\sin(\log x)$

15. Find the general solution of $(2x+1)(x+1)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=(2x+1)^2$ by the method of variation of parameters given that $y=x$ and $y=\frac{1}{x+1}$ are two linearly independent solutions of the corresponding homogeneous equation.

ANSWERS TO EXERCISE 11.4

1. $y=e^{3x}[C_3+C_2x-\log x]$ where $C_3=C_1-1$
2. $y=e^x[C_1\cos x+C_2\sin x]-e^x\cos x\log[\sec x+\tan x]$
3. $y=C_1\cos x+C_3\sin x+\frac{x^2}{4}\sin x+\frac{x}{4}\cos x$ where $C_3=C_2-\frac{1}{8}$
4. $y=C_1e^{3/2x}+C_2e^{-x}-2e^{-x}-5xe^{-x}$
5. $y=C_1x\log x+C_2x+\frac{1}{6}x(\log x)^2$
6. $y=\frac{C_1}{x}+\frac{C_2}{x^2}+\frac{x^2}{12}-\frac{1}{x^2}\log x$
7. $y=C_1e^x+C_2x^{-1}-x-1-\frac{x^3}{3}$
8. $y=(C_1+C_2x)e^{-x}-e^{-x}\log x$
9. $y=C_1\cos x+C_2\sin x-\cos x\log\sec x+x\sin x$
10. $y=C_1+C_2e^{2x}-\frac{1}{2}e^x\cos x$
11. $y=C_1\cos 2x+C_2\sin 2x-\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$
12. $y=C_1\cos x+C_2\sin x-x\cos x+\sin x\log\sin x$
13. $y=C_1\cos ax+C_2\sin ax-\frac{1}{a^2}\cos ax\log(\sec ax+\tan ax)$
14. $y=C_1x^2+C_2x^3+\frac{1}{10}[\sin(\log x)+\cos(\log x)]$
15. $y=C_1x+\frac{C_2}{x+1}+\frac{4x^3+3x^2}{6(x+1)}$

11.5 METHOD OF UNDETERMINED COEFFICIENTS

We have seen solution of non–homogeneous differential equations by finding complementary function and particular integral.

All these methods use operator method for finding the particular integral, except the variation of parameters method.

We shall now discuss another method, the method of underdetermined coefficients to find the particular integral when complementary function is known. This method is applicable only for linear differential equations with constant coefficients.

Consider the linear differential equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny_n=Q(x)$$

where $a_0, a_1, \ldots, a_n$, are constants and $Q(x)$ is of special form having a finite **family of derivatives** consisting of independent functions.

(i) If $Q(x) = x^m$, then

$Q'(x) = mx^{m-1}, \; Q''(x) = m(m-1)x^{m-2}, \ldots, \; Q^m(x) = m!$

Thus, all the derivatives of x^m can be written interms of the finite family of independent functions $\{x^m, x^{m-1}, x^{m-2}, \ldots, x, 1\}$.

So, the particular integral $y_p(x)$ can be written as

$y_p(x) = C_0 x^m + C_1 x^{m-1} + C_2 x^{m-2} + \cdots + C_{m-1}{}^x + C_m$

(ii) If $Q(x) = e^{ax}$, where a is a constant, then

$Q'(x) = ae^{ax}, Q''(x) = a^2 e^{ax}, \cdots$

Thus, the derivatives can be expressed in terms of the finite family $\{e^{ax}\}$.

So, $y_p(x) = Ce^{ax}$

(iii) If $Q(x) = \sin ax$ (or $\cos ax$),

then $Q'(x) = a\cos ax, \; Q''(x) = -a^2 \sin ax, \; Q'''(x) = -a^2 \cos ax$

Thus, derivatives can be expressed interms of the finite family of independent functions $\{\sin ax, \cos ax\}$

So, $y_p(x) = C_1 \sin ax + C_2 \cos ax$

(iv) If $Q(x) = x^m \cdot e^{ax}$, then

$$Q'(x) = x^m ae^{ax} + e^{ax} m x^{m-1} = a x^m e^{ax} + m e^{ax} x^{m-1}$$

$$Q''(x) = a\left\{x^m ae^{ax} + e^{ax} \cdot m x^{m-1}\right\} + m\left\{e^{ax}(m-1)x^{m-2} + x^{m-1} ae^{ax}\right\}$$

$$= e^{ax}\left\{a^2 x^m + 2am\, x^{m-1} + (m)(m-1)x^{m-2}\right\}$$

and so on.

Thus, derivatives can be expressed interms of the finite family $\left\{x^m e^{ax}, x^{m-1} e^{ax}, \cdots, x\, e^{ax}, e^{ax}\right\}$

So, $y_p(x) = e^{ax}\left\{C_0 x^m + C_1 x^{m-1} + \cdots + C_{m-1} x + C_m\right\}$

(v) If $Q(x) = e^{ax} \sin bx$ (or $e^{ax} \cos bx$)

then $Q'(x) = e^{ax}\, b\cos bx + \sin bx \cdot a\, e^{ax} = e^{ax}[b\cos bx + a\sin bx]$

$Q''(x) = e^{ax}\left[-b^2 \sin bx + ab\cos bx\right] + \left[b\cos bx + a\sin bx\right] ae^{ax}$

and so on.

Thus, the derivatives can be expressed interms of the finite family $\left\{e^{ax} \sin bx, e^{ax} \cos bx\right\}$

That is the product of the family $\{e^{ax}\}$ and the family $\{\sin bx, \cos bx\}$

So, $y_p(x) = e^{ax}\left\{C_1 \cos bx + C_2 \sin bx\right\}$

Note However the above method fails if the derivatives involve an infinite family of functions. For example if $Q(x) = \tan x$ or $\sec x$ the method fails.

If $\quad Q(x) = \tan x$, then $Q'(x) = \sec^2 x$

$$Q''(x) = 2\sec^2 x \tan x, \quad Q'''(x) = 2\sec^4 x + \sec^2 x \tan^2 x$$

$$Q^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x + 4\sec^4 x\, 2\tan x \quad \text{and so on.}$$

Thus, $Q^n(x)$ involves $\tan^{(n-1)} x$ and $\sec^{(n)} x$ and so as n increase the terms will increase.
So, it is not possible to write all the derivatives interms of a finite set of independent functions.
We will now list some of the special type of functions with finite family of derivatives.

Table 1

	$Q(x)$	$\text{P.I} = y_p$
1.	K (constant)	C
2.	$K \cdot e^{ax}$	$C e^{ax}$
3.	$K \sin ax$ or $K \cos ax$	$C_1 \cos ax + C_2 \sin ax$
4.	$K x^m$	$C_0 x^m + C_1 x^{m-1} + \cdots + C_{m-1} x + C_m$
5.	$e^{ax} \sin bx$ or $e^{ax} \cos bx$	$e^{ax}(C_1 \cos bx + C_2 \sin bx)$
6.	$K x^m e^{ax}$	$e^{ax}(C_0 x^m + C_1 x^{m-1} + \cdots + C_{m-1} x + C_m)$
7.	$K x^m \sin ax$ (or) $K x^m \cos ax$	$x^m(a_0 \cos ax + b_0 \sin ax) + x^{m-1}(a_1 \cos ax + b_1 \sin ax)$ $+ \cdots + x(a_{m-1} \cos ax + b_{m-1} \sin ax) + (a_m \cos ax + b_m \sin ax)$

Remark

Modification Rule

If any term in the choice of the particular integral is also a term of the complementary function, then multiply this term by x (or x^m if the root of the auxiliary equation is of multiplicity m).

Working Rule

Given $f(D)y = Q(x)$ is a linear differential equation with constant coefficients.

1. Find the complementary function y_c by solving $f(D)y = 0$.
2. Depending upon the nature of $Q(x)$, the particular integral y_p is written as per the above Table 1.
3. The constants occurring in y_p are determined such that y_p satisfies the given equation.
4. The general solution is $y = \text{C.F} + \text{P.I}$

WORKED EXAMPLES

EXAMPLE 1

Solve $(D^2 - 9)y = 9x^2 - 2x$.

Solution.

The given equation is

$$(D^2 - 9)y = 9x^2 - 2x \tag{1}$$

To find the complementary function, solve $(D^2 - 9)y = 0$

Auxiliary equation is $m^2 - 9 = 0 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$

$\therefore$ $\text{C.F} = Ae^{-3x} + Be^{3x}$

To find the P.I

Since $Q(x) = 9x^2 - 2x$, which is a quadratic, assume the particular integral as

$$y = C_0 x^2 + C_1 x + C_2 \qquad (2)$$

Choose C_0, C_1, C_2 such that it satisfies the given equation (1)

Differentiating (2) w.r.to x, we get

$$\frac{dy}{dx} = 2C_0 x + C_1 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2C_0$$

Substituting in the equation (1), we get

$$2C_0 - 9(C_0 x^2 + C_1 x + C_2) = 9x^2 - 2x$$

$$\Rightarrow \quad (2C_0 - 9C_2) - 9C_1 x - 9C_0 x^2 = 9x^2 - 2x$$

Equating like coefficients on both sides, we get

$$2C_0 - 9C_2 = 0, \quad -9C_1 = -2 \quad \Rightarrow \quad C_1 = \frac{2}{9} \quad \text{and} \quad -9C_0 = 9 \quad \Rightarrow \quad C_0 = -1$$

$$\therefore \quad 2(-1) - 9C_2 = 0 \quad \Rightarrow \quad 9C_2 = -2 \quad \Rightarrow \quad C_2 = -\frac{2}{9}$$

$$\therefore \quad \text{P.I} = -x^2 + \frac{2}{9}x - \frac{2}{9}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = Ae^{-3x} + Be^{3x} - x^2 + \frac{2}{9}x - \frac{2}{9}$$

EXAMPLE 2

Solve $\dfrac{d^2y}{dx^2} + y = 2\cos x.$

Solution.

The given equation is $(D^2 + 1)y = 2\cos x \qquad (1)$

To find the complementary function, solve $(D^2 + 1)y = 0$

Auxiliary equation is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$$\therefore \quad \text{C.F} = A\cos x + B\sin x$$

To find the P.I

Since $Q(x) = 2\cos x$, we have to assume the P.I as

$$y = C_1 \cos x + C_2 \sin x$$

But these appear as terms in the C.F.

So, we have to modify the P.I as $y = x(C_1 \cos x + C_2 \sin x)$ (2) **[Refer remark]**

Now choose C_1 and C_2 such that it satisfies the given equation (1).

Differentiating (2) w.r.to x, we get

$$\frac{dy}{dx} = x[-C_1 \sin x + C_2 \cos x] + (C_1 \cos x + C_2 \sin x)\cdot 1$$

and

$$\frac{d^2y}{dx^2} = x[-C_1 \cos x - C_2 \sin x] + [-C_1 \sin x + C_2 \cos x] - C_1 \sin x + C_2 \cos x$$

$$= x[-C_1 \cos x - C_2 \sin x] - 2C_1 \sin x + 2C_2 \cos x$$

Substituting in the equation (1), we get

$$x(-C_1\cos x - C_2\sin x) - 2C_1\sin x + 2C_2\cos x + x(C_1\cos x + C_2\sin x) = 2\cos x$$

$\Rightarrow$ $$-2C_1\sin x + 2C_2\cos x = 2\cos x$$

Equating like terms on both sides

$$-2C_1 = 0 \quad \Rightarrow C_1 = 0 \quad \text{and} \quad 2C_2 = 2 \Rightarrow C_2 = 1$$

$\therefore$ $\quad$ P.I $= x\sin x$

$\therefore$ $\quad$ the general solution is $y =$ C.F + P.I

$\Rightarrow$ $$y = A\cos x + B\sin x + x\sin x$$

EXAMPLE 3

Solve by the method of undetermined coefficients the equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = x^2 + e^x$.

Solution.

The given equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + e^x \qquad (1)$$

To find the complementary function, solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0 \quad \Rightarrow \quad (D^2 - 3D + 2)y = 0$

Auxiliary equation is

$$m^2 - 3m + 2 = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow m = 1, 2$$

$\therefore$ $\quad$ C.F $= A\,e^x + B\,e^{2x}$

To find the P.I

Since $Q(x) = x^2 + e^x$, we have to assume the P.I as

$$y = C_0x^2 + C_1x + C_2 + C_3e^x$$

Since e^x is there in the C.F, we modify by multiplying e^x by x $\qquad$ [$\because m = 1$ is a simple root]

$\therefore$ $\quad$ P.I is $y = C_0x^2 + C_1x + C_2 + C_3\,x\,e^x \qquad (2)$

Choose C_0, C_1, C_2, C_3 such that it satisfies the given equation (1)

Differentiating (2) w.r.to x, we get

$$\frac{dy}{dx} = 2C_0x + C_1 + C_3[xe^x + e^x]$$

$$\frac{d^2y}{dx^2} = 2C_0 + C_3[xe^x + e^x + e^x] = 2C_0 + C_3\,e^x(x+2)$$

Substituting in the equation (1), we get

$$2C_0 + C_3(x+2)e^x - 3\{2C_0x + C_1 + C_3(x+1)e^x\} + 2(C_0x^2 + C_1x + C_2 + C_3xe^x) = x^2 + e^x$$

$\Rightarrow$ $$2C_0 - 3C_1 + 2C_2 + [C_3x + 2C_3 - 3C_3x + 2C_3x]e^x - 6C_0x + 2C_0x^2 + 2C_1x = x^2 + e^x$$

$\Rightarrow$ $$2C_0 - 3C_1 + 2C_2 + C_3e^x + (2C_1 - 6C_0)x + 2C_0x^2 = x^2 + e^x$$

Equating like powers on both sides, we get

$$2C_0 - 3C_1 + 2C_2 = 0 \quad , \quad 2C_0 = 1 \quad \Rightarrow \quad C_0 = \frac{1}{2} \quad , \quad +C_3 = 1$$

and $2C_1 - 6C_0 = 0 \quad \Rightarrow \quad 2C_1 = 6\times\frac{1}{2} \quad \Rightarrow \quad C_1 = \frac{3}{2}$

$\therefore \quad 2\frac{1}{2} - 3\cdot\frac{3}{2} + 2C_2 = 0 \quad \Rightarrow \quad \frac{2-9}{2} + 2C_2 = 0 \quad \Rightarrow \quad 2C_2 = \frac{7}{2} \Rightarrow \quad C_2 = \frac{7}{4}$

$$\therefore \quad \text{P.I} = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4} + xe^x = \frac{1}{4}[2x^2 + 6x + 7] + xe^x$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = Ae^x + Be^{2x} + \frac{1}{4}[2x^2 + 6x + 7] + xe^x$$

EXAMPLE 4

Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$

Solution.

The given equation is

$$(D^2 - 2D)y = e^x \sin x \tag{1}$$

To find the complementary function, solve $(D^2 - 2D)y = 0$

Auxiliary equations is $m^2 - 2m = 0 \quad \Rightarrow \quad m(m-2) = 0 \quad \Rightarrow \quad m = 0, 2$

$\therefore \quad \text{C.F} = A\,e^{0x} + B\,e^{2x} = A + B\,e^{2x}$

To find the P.I

Since $Q(x) = e^x \sin x$, we have to assume the P.I as

$$y = e^x (C_1 \cos x + C_2 \sin x) \tag{2}$$

where C_1, C_2 are chosen such that it satisfies the equation (1)

Differentiating (2) w.r.to x, we get

$$\frac{dy}{dx} = e^x[-C_1 \sin x + C_2 \cos x] + [C_1 \cos x + C_2 \sin x]e^x$$

$$= e^x[(C_2 - C_1)\sin x + (C_1 + C_2)\cos x]$$

and $$\frac{d^2y}{dx^2} = e^x[(C_2 - C_1)\cos x - (C_1 + C_2)\sin x] + [(C_2 - C_1)\sin x + (C_1 + C_2)\cos x]e^x$$

$$= e^x[2C_2 \cos x - 2C_1 \sin x]$$

Substituting in the equation (1), we get

$$e^x[2C_2 \cos x - 2C_1 \sin x] - 2e^x[(C_2 - C_1)\sin x + (C_1 + C_2)\cos x] = e^x \sin x$$

$$\Rightarrow \quad e^x[(2C_2 - 2C_1 - 2C_2)\cos x \; - (2C_1 + 2C_2 - 2C_1)\sin x] = e^x \sin x$$

$$\Rightarrow \quad e^x[-2C_1 \cos x - 2C_2 \sin x] = e^x \sin x$$

$$\Rightarrow \quad -2C_1 \cos x - 2C_2 \sin x = \sin x$$

Equating like coefficients on both sides, we get

$$-2C_1 = 0 \quad \Rightarrow \quad C_1 = 0 \quad \text{and} \quad -2C_2 = 1 \quad \Rightarrow \quad C_2 = -\frac{1}{2}$$

$\therefore$ $$\text{P.I} = e^x\left(-\frac{1}{2}\sin x\right) = -\frac{e^x}{2}\sin x$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$y = A + Be^{2x} - \frac{1}{2}e^x \sin x$$

EXAMPLE 5

Solve the differential equation $y'' - 4y' + 4y = 5e^{2x}$.

Solution.

The given equation is

$$y'' - 4y' + 4y = 5e^{2x} \quad \Rightarrow \quad (D^2 - 4D + 4)y = 5e^{2x} \qquad (1)$$

To find the complementary function, solve $(D^2 - 4D + 4)y = 0$

Auxiliary equation is $m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore$ $$\text{C.F} = (A + Bx)e^{2x} = Ae^{2x} + Bx\, e^{2x}$$

To find the P.I

Since $Q(x) = 5e^{2x}$, assume P.I as $y = C_1 e^{2x}$

But e^{2x} is a term of the complementary function, so we have to modify by multiplying e^{2x} by x^2 (since 2 is the root of the auxiliary equation of multiplicity 2).

$\therefore$ $$\text{P.I is } y = Cx^2 e^{2x} \qquad (2)$$

Choose C such that it satisfies the equation (1)

Differentiating (2) w.r.to x, we get

$$y' = \frac{dy}{dx} = C[x^2 e^{2x} \cdot 2 + e^{2x} \cdot 2x] = 2C\, e^{2x}(x^2 + x)$$

and $$y'' = \frac{d^2y}{dx^2} = 2C\left\{e^{2x}(2x+1) + (x^2+x)e^{2x} \cdot 2\right\} = 2Ce^{2x}(2x^2 + 4x + 1)$$

Substituting in the equation (1), we get

$$2Ce^{2x}(2x^2 + 4x + 1) - 4 \cdot 2C\, e^{2x}(x^2 + x) + 4Cx^2 e^{2x} = 5e^{2x}$$

$\Rightarrow$ $$2Ce^{2x} + 2Ce^{2x}[2x^2 + 4x - 4x^2 - 4x + 2x^2] = 5e^{2x}$$

$\Rightarrow$ $$2Ce^{2x} = 5e^{2x} \quad \Rightarrow \quad C = \frac{5}{2}$$

$\therefore$ $$\text{P.I} = \frac{5}{2}x^2 e^{2x}$$

$\therefore$ the general solution is

$\therefore$ $$y = \text{C.F} + \text{P.I}$$

$$y = (A + Bx)e^{2x} + \frac{5}{2}x^2 e^{2x} = \left(A + Bx + \frac{5}{2}x^2\right)e^{2x}$$

EXAMPLE 6

Solve $x^2y'' - 5xy' + 8y = \log_e x$ using undetermined coefficients.

Solution.
The given equation is

$$x^2y'' - 5xy' + 8y = \log_e x \Rightarrow (x^2D^2 - 5xD + 8)y = \log_e x$$

It is Cauchy's equation
First we have to reduce it to an equation with constant coefficients.

$\therefore$ Put $x = e^z$ and $\theta = \dfrac{d}{dz}$, then $xD = \theta$ and $x^2D^2 = \theta(\theta - 1)$

$$\therefore \quad (\theta(\theta - 1) - 5\theta + 8)y = z \quad \Rightarrow \quad (\theta^2 - 6\theta + 8)y = z \qquad (1)$$

This is a second order linear differential equation with constants coefficients.
So, we use the method of undetermined coefficients to find the P.I.
To find the complementary function, solve, $(\theta^2 - 6\theta + 8)y = 0$

Auxiliary equation is $m^2 - 6m + 8 = 0 \Rightarrow (m-4)(m-2) = 0 \Rightarrow m = 2, 4$

$$\therefore \quad \text{C.F} = A\,e^{2x} + B\,e^{4x}$$

To find the P.I
Since $Q(x) = z$, assume the particular integral as

$$y = C_1 z + C_2 \qquad (2)$$

where C_1 and C_2 are chosen such that it satisfies the given equation (1)
Differentiating (2) w.r.to z, we get

$$\theta y = \frac{dy}{dz} = C_1 \quad \text{and} \quad \theta^2 y = \frac{d^2y}{dz^2} = 0$$

Substituting in the equation (1), we get

$$0 - 6C_1 + 8(C_1 z + C_2) = z \quad \Rightarrow \quad 8C_1 z - 6C_1 + 8C_2 = z$$

Equating like coefficients on both sides, we get $8C_1 = 1 \Rightarrow C_1 = \dfrac{1}{8}$

and $$8C_2 - 6C_1 = 0 \Rightarrow 8C_2 = 6C_1 \Rightarrow C_2 = \frac{6}{8}\cdot\frac{1}{8} = \frac{3}{32}$$

$$\therefore \quad \text{P.I} = \frac{1}{8}z + \frac{3}{32}$$

$\therefore$ the general solution is $y = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y = Ae^{2z} + Be^{4z} + \frac{1}{8}z + \frac{3}{32} = Ax^2 + Bx^4 + \frac{1}{8}\log_e x + \frac{3}{32}$$

EXERCISE 11.5

Solve the following equations using undetermined coefficients.

1. $y'' + 2y' - 3y = 4e^x$
2. $y'' - 4y = 12e^{4x} + 4e^{-2x}$
3. $y'' - 4y = 8x^2 - 2x$
4. $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$
5. $y'' + y = \sin x$
6. $(D^2 + 6D + 9)y = 24e^{-3x}$
7. $x^2y'' + xy' + 4y = \sin(2\log_e x)$

ANSWERS TO EXERCISE 11.5

1. $y = C_1e^{-3x} + C_2e^x + xe^x$
2. $y = C_1e^{2x} + C_2e^{-2x} + e^{4x} - xe^{-2x}$
3. $y = C_1e^{2x} + C_2e^{-2x} - 2x^2 + \frac{x}{2} - 1$
4. $y = e^x(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x) + \frac{x^2}{2} - \frac{1}{2}x + e^{-x}$
5. $y = C_1\cos x + C_2\sin x - \frac{x}{2}\cos x$
6. $y = (C_1 + C_2x)e^{-3x} + 12x^2e^{-3x}$
7. $y = C_1\cos(2\log_e x) + C_2\sin(2\log x) - \frac{1}{4}\log_e x\cdot\cos(2\log_e x)]$

SHORT ANSWERS QUESTIONS

1. **Solve $(D^2 + 1)y = 0$ given $y(0) = 0$; $y'(0) = 1$.**
2. **Solve $(D - 2)^2y = e^{2x}$.**
3. **Find the P.I of $(D^2 - 2D + 1)y = \cosh x$.**
4. **Find the particular integral of $(D - 1)^2y = \sinh 2x$.**
5. **Find the particular integral of $(D - 1)^2y = e^x\sin x$.**
6. **Find the P.I of $(D^2 + 4)y = \sin 2x$.**
7. **Find the P.I of $(D^2 - 2D + 4)y = e^x\cos x$.**
8. **Find the P.I. of $(D^2 + 1)y = xe^x$.**
9. **Find the P.I of $(D^2 + D)y = x^2 + 2x + 4$.**
10. **Find the P.I of $(D^2 + 1)y = \sin x\sin 2x$.**
11. **Find the P.I of $\frac{d^2y}{dx^2} - 4y = 3^x$.**
12. **Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.**
13. **Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$.**
14. **Solve $(x^2D^2 + xD + 1)y = 0$.**
15. **Transform the equation $(2x + 3)^2y'' - (2x + 3)y' + 2y = 6x$ into a differential equation with constant coefficients.**
16. **Transform the equation $(2x + 3)^2\frac{d^2y}{dx^2} - 2(2x + 3)\frac{dy}{dx} - 12y = 6x$ into a differential equation with constant coefficients.**
17. **Transform the equation $x^2y'' + xy' = x$ into a linear differential equation with constant coefficients.**
18. **Reduce $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x$ into a differential equation with constant coefficients.**
19. **Eliminate y from the system $\frac{dx}{dt} + 2y = -\sin t$ and $\frac{dy}{dt} - 2x = \cos t$.**
20. **Eliminate y from the simultaneous equations $\frac{dx}{dt} + y = \sin t$ and $\frac{dy}{dt} + x = \cos t$.**

Geometrical Applications

a) Orthogonal Trajectories in Casterian Coordinates

Let $F(x, y, c) = 0$ be a family of curves in a plane such that through a given point there is only one curve of the family.

A trajectory [in Latin it means cut across] of a family of curves is a curve which cuts each member of the family at a given angle. This curve is called an **isogonal trajectory**.

An **Orthogonal trajectory** is a curve which cuts each member of the family of curves at right angles

If there exists a family of curves $G(x, y, c') = 0$ such that each of its members cut orthogonally the curves of $F(x, y, c) = 0$ then the family $G(x, y, c') = 0$ is said to be orthogonal trajectories of the family $F(x, y, c) = 0$.

It is obvious $F(x, y, c) = 0$ is the orthogonal trajectories to $G(x, y, c') = 0$.

Practical examples of Orthogonal trajectories.

1. In the electric field, the paths along which current flows are orthogonal trajectories of the equipotential curves (i.e., lines of constant velocity potential) and vice versa.
2. In two dimensional heat flow the curves along which the heat flow and the isothermal curves are orthogonal trajectories.
3. Meridian and parallels on a globe are orthogonal trajectories.
4. In fluid dynamics, the stream lines and equipotential lines (i.e., lines of constant velocity) are orthogonal trajectories.
5. In geometry, the circles $x^2 + y^2 = a^2$ and the lines $y = mx$ are orthogonal trajectories.

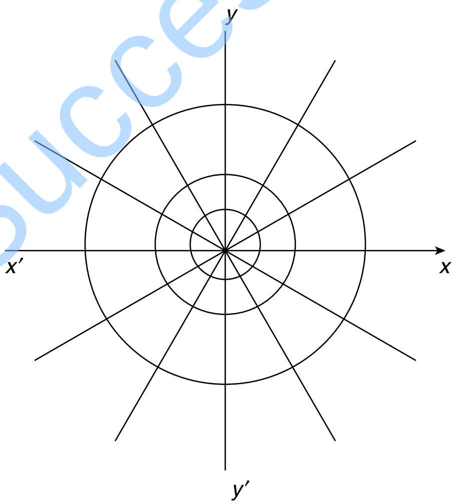


Fig. 12.2

Procedure to find the orthogonal trajectories.

Given the family of curves $F(x, y, C) = 0$ where C is an arbitrary constant. (1)

1. Form the differential equation $f\left(x, y, \frac{dy}{dx}\right) = 0$ (2)
   by eliminating C.
2. Replace $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}}$ or $-\frac{dx}{dy}$ in (2), since at a point of intersection, product of the slopes is -1.

3. The new equation is $f\left(x, y, -\frac{dx}{dy}\right) = 0$ (3)
4. Solve the equation (3), to find the orthogonal trajectories $G(x, y, C') = 0$ (4)

WORKED EXAMPLES

EXAMPLE 1

Find the orthogonal trajectories of the curve of $y^2 = 4ax$, where a is a parameter.

Solution.

The given curves are

$$y^2 = 4ax \quad (1)$$

Differentiating (1) w.r.to x, we get

$$2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad y\frac{dy}{dx} = 2a$$

Substituting the value of $2a$ in (1), we get

$$y^2 = 2 \cdot y\frac{dy}{dx}x \quad \Rightarrow \quad y = 2x\frac{dy}{dx} \quad (2)$$

To find the orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2)

$$\therefore \quad y = 2x\left(-\frac{dx}{dy}\right) \quad \Rightarrow \quad ydy = -2x\,dx$$

Integrating, $\int ydy = -2\int x\,dx$

$$\Rightarrow \quad \frac{y^2}{2} = -2\frac{x^2}{2} + C \quad \Rightarrow \quad x^2 + \frac{y^2}{2} = C \quad (3)$$

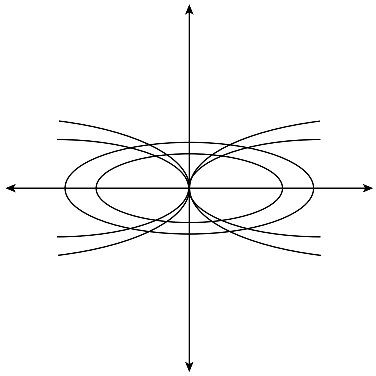


Fig. 12.3

which represents a family of ellipses.

Hence, the orthogonal trajectories the family of parabolas $y^2 = 4ax$ is the family of ellipses (3).

EXAMPLE 2

Find the orthogonal trajectories of the confocal conics $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.

Solution.

The given family is $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ (1)

Differentiating w.r. to x, we get

$$2\frac{x}{a^2}+\frac{2y}{(b^2+\lambda)}\frac{dy}{dx}=0$$

$$\Rightarrow \qquad \frac{x}{a^2}=-\frac{y}{(b^2+\lambda)}\cdot\frac{dy}{dx} \quad \Rightarrow \quad b^2+\lambda=-\frac{a^2y}{x}\frac{dy}{dx}$$

Substituting in (1), we get

$$\frac{x^2}{a^2}+\frac{y^2}{\left(-\frac{a^2y}{x}\frac{dy}{dx}\right)}=1 \quad \Rightarrow \quad \frac{x^2}{a^2}-\frac{xy}{a^2}\frac{1}{\frac{dy}{dx}}=1$$

$$\Rightarrow \qquad \frac{xy}{a^2}\cdot\frac{1}{\frac{dy}{dx}}=\frac{x^2}{a^2}-1=\frac{x^2-a^2}{a^2} \quad \Rightarrow \quad (x^2-a^2)\frac{dy}{dx}=xy \qquad (2)$$

To find the orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2)

$$\therefore \qquad (x^2-a^2)\left(-\frac{dx}{dy}\right)=xy \quad \Rightarrow \quad \frac{x^2-a^2}{x}dx=-y\,dy$$

Integrating,

$$\int\frac{x^2-a^2}{x}dx=-\int y\,dy$$

$$\Rightarrow \qquad \int\left(x-\frac{a^2}{x}\right)dx=-y\,dy$$

$$\Rightarrow \qquad \frac{x^2}{2}-a^2\log_e x=-\frac{y^2}{2}+C \quad \Rightarrow \quad x^2+y^2-2a^2\log_e x=2C$$

which is the equation of the orthogonal trajectories.

EXAMPLE 3

Find the orthogonal trajectories of semi cubical parabolas $ay^2=x^3$, where a is a parameter.

Solution.

The given family is $ay^2=x^3$ (1)

Differentiating w.r.to x, we get

$$2ay\frac{dy}{dx}=3x^2 \quad \Rightarrow \quad ay=\frac{3}{2}\frac{x^2}{\frac{dy}{dx}}$$

Substituting in (1), we get

$$\frac{3}{2}\frac{x^2}{\frac{dy}{dx}}\cdot y=x^3 \quad \Rightarrow \quad \frac{3}{2}y\frac{1}{\frac{dy}{dx}}=x \quad \Rightarrow \quad \frac{3}{2}y=x\frac{dy}{dx} \qquad (2)$$

To find the orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2)

$$\therefore \qquad \frac{3}{2}y = -x\frac{dx}{dy} \quad \Rightarrow \quad \frac{3}{2}ydy = -xdx$$

Integrating,

$$\frac{3}{2}\int ydy = -\int xdx \quad \Rightarrow \quad \frac{3}{2}\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\Rightarrow \qquad \frac{x^2}{2} + \frac{3y^2}{4} = C$$

$$\Rightarrow \qquad 2x^2 + 3y^2 = 4C \quad \Rightarrow \quad \frac{x^2}{3} + \frac{y^2}{2} = C' \qquad \text{(Dividing by 6)}$$

which is an ellipse.

$\therefore$ the orthogonal trajectories of the given family of semi-cubical parabolas $ay^2 = x^3$ is the family of ellipse $\frac{x^2}{3} + \frac{y^2}{2} = C'$.

12.1.5 (b) Orthogonal Trajectories in Polar Coordinates

Let $F(r, \theta, C) = 0$ (1) be the given family of curves, where c is an arbitrary constant. Form the differential equation

$$f\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \qquad (2)$$

by eliminating C from (1).

Let Γ' be the curve orthogonal to the curve Γ at the point P. If ϕ is the angle between the tangent at the point P to Γ and the radius Vector OP, then we know that $\tan\phi = r\frac{d\theta}{dr}$

If ϕ' is the angle between the tangent at P to Γ' and the radius vector OP, then $\phi' = \phi + 90$, since the tangent, are perpendicular.

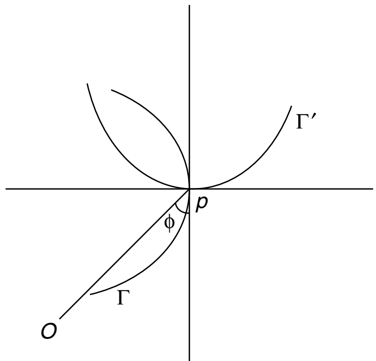


Fig. 12.4

Hence, $\tan\phi' = \tan(\phi + 90)$

$$= -\cot\phi = -\frac{1}{r\frac{d\theta}{dr}} = -\frac{1}{r}\frac{dr}{d\theta}$$

Replace $r\frac{d\theta}{dr}$ in (2) by $-\frac{1}{r\frac{d\theta}{dr}} = -\frac{1}{r}\frac{dr}{d\theta}$

This is equivalent to replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\therefore \qquad f\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$$

Solving, we get the orthogonal trajectories $G(r, \theta, C') = 0$

WORKED EXAMPLES

EXAMPLE 1

Find the orthogonal trajectories of the system of circles $r = a\cos\theta$ where a is parameter.

Solution.

The given system of circles is $r = a\cos\theta$ (1)

Differentiating w.r.to θ, we get

$$\frac{dr}{d\theta} = a\cdot(-\sin\theta) = -a\sin\theta \tag{2}$$

To eliminate a, divide (1) by (2)

$$\therefore \qquad \frac{r}{\frac{dr}{d\theta}} = -\frac{\cos\theta}{\sin\theta} = -\frac{1}{\tan\theta} \quad \Rightarrow \quad r\tan\theta = -\frac{dr}{d\theta}$$

To find the orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\therefore \qquad r\tan\theta = r^2\frac{d\theta}{dr} \quad \Rightarrow \quad \frac{dr}{r} = \frac{d\theta}{\tan\theta} = \frac{\cos\theta}{\sin\theta}d\theta$$

Integrating,

$$\Rightarrow \qquad \int\frac{dr}{r} = \int\frac{\cos\theta}{\sin\theta}d\theta$$

$$\Rightarrow \qquad \log_e r = \log_e\sin\theta + \log_e C = \log_e C\sin\theta$$

$$\Rightarrow \qquad r = C\sin\theta$$

where C is arbitrary.

$\therefore$ the orthogonal trajectory is a family of circles.

EXAMPLE 2

Find the orthogonal trajectory of cardioids $r = a(1 - \cos\theta)$, a being the parameter.

Solution.

The given family of cardioids is

$$r = a(1 - \cos\theta) \tag{1}$$

Differentiating w.r.to θ, we get

$$\frac{dr}{d\theta} = a\sin\theta \tag{2}$$

To eliminate a, divide (1) by (2)

$$\therefore \qquad \frac{r}{\frac{dr}{d\theta}} = \frac{1-\cos\theta}{\sin\theta}$$

To find the orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we get

$$\therefore \qquad \frac{r}{-r^2\dfrac{d\theta}{dr}} = \frac{1-\cos\theta}{\sin\theta} \quad \Rightarrow \quad -\frac{dr}{r} = \frac{1-\cos\theta}{\sin\theta}d\theta$$

$$\Rightarrow \qquad \frac{dr}{r} = -\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}d\theta = -\frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}d\theta$$

Integrating,

$$\int\frac{dr}{r} = -\int\frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}d\theta \quad \Rightarrow \quad \int\frac{dr}{r} = 2\int\frac{\left(-\sin\dfrac{\theta}{2}\right)\dfrac{1}{2}}{\cos\dfrac{\theta}{2}}d\theta$$

$$\Rightarrow \qquad \log_e r = 2\log_e \cos\frac{\theta}{2} + \log_e C$$

$$\Rightarrow \qquad = \log_e \cos^2\frac{\theta}{2} + \log_e C = \log_e C\cos^2\frac{\theta}{2}$$

$$\Rightarrow \qquad r = C\cos^2\frac{\theta}{2} = \frac{C}{2}(1+\cos\theta)$$

$$\Rightarrow \qquad r = C'(1+\cos\theta), \text{ where } C' = \frac{C}{2}$$

which are cardioids.
$\therefore$ the orthogonal trajectories of cardioids are again cardioids.

EXAMPLE 3

Find the orthogonal trajectories of the confocal and coaxial parabolas $r = \dfrac{2a}{1+\cos\theta}$, a being the parameter.

Solution.

The given family is $\quad r = \dfrac{2a}{1+\cos\theta} = \dfrac{2a}{2\cos^2\dfrac{\theta}{2}} = a\sec^2\dfrac{\theta}{2} \qquad (1)$

Differentiating w.r.to θ, we get

$$\frac{dr}{d\theta} = a\cdot 2\sec\frac{\theta}{2}\cdot\sec\frac{\theta}{2}\tan\frac{\theta}{2}\cdot\frac{1}{2} \quad \Rightarrow \quad \frac{dr}{d\theta} = a\sec^2\frac{\theta}{2}\tan\frac{\theta}{2} \qquad (2)$$

To eliminate a, divide (1) by (2)

$$\therefore \qquad \frac{r}{\dfrac{dr}{d\theta}} = \frac{a\sec^2\dfrac{\theta}{2}}{a\sec^2\dfrac{\theta}{2}\tan\dfrac{\theta}{2}} = \frac{1}{\tan\dfrac{\theta}{2}} \quad \Rightarrow \quad r\tan\frac{\theta}{2} = \frac{dr}{d\theta}$$

To find the orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we get

$$r\tan\frac{\theta}{2} = -r^2\frac{d\theta}{dr} \quad \Rightarrow \quad \frac{dr}{r} = -\frac{d\theta}{\tan\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta$$

Integrating, we get

$$\int\frac{dr}{r} = -\int\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta \quad \Rightarrow \quad \log_e r = -2\int\frac{\cos\frac{\theta}{2}\cdot\frac{1}{2}}{\sin\frac{\theta}{2}}d\theta$$

$$\Rightarrow \qquad \log_e r = -2\log_e \sin\frac{\theta}{2} + \log_e C$$

$$\Rightarrow \qquad \log_e r = -\log_e \sin^2\frac{\theta}{2} + \log_e C = \log_e\frac{C}{\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \qquad r = \frac{C}{\sin^2\frac{\theta}{2}} = \frac{2C}{1-\cos\theta}$$

which is a parabola.

∴ orthogonal trajectories are also parabolas.

EXERCISE 12.4

1. Find the orthogonal trajectories of $xy = c$, where c is a parameter.
2. Find the orthogonal trajectories of $y = ax^2$, where a is a parameter.
3. Obtain the differential equation of the family of circles touching the x-axis at the origin and hence, derive the equation of the orthogonal trajectories of these circles.
4. Find the orthogonal trajectories of a system of confocal and co-axial parabolas $y^2 = 4a(x+a)$, where a is a parameter.
5. Find the orthogonal trajectories of the system of hyperbolas $x^2 - y^2 = a^2$, a is a parameter.
6. Show that the family of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self orthogonal, where λ is the parameter.
7. Find the orthogonal trajectories of the family of curves $r = a(1+\cos\theta)$.

Laplace Transforms

19.0 INTRODUCTION

Laplace transform is a powerful tool for solving linear differential equations. Laplace transform converts a linear differential equation with initial conditions to an algebraic problem. This process of changing from operations of calculus to algebraic operations on transforms is known as **operational calculus**, which is an important area of applied mathematics. The advantage of Laplace transforms in solving initial value problems lies in the fact that the initial conditions are taken care of at the outset and the solution is directly obtained without resorting to finding the general solution and then the arbitrary constants.

The name is due to the French mathematician Pierre Simon de Laplace who used this transforms while developing the theory of probability.

Definition 19.1 Let $f(t)$ be defined for all $t \geq 0$, then the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

is defined as the **Laplace transform of** $f(t)$, if the integral exists.

This integral is a function of the parameter s.

Symbolically, we write $L[f(t)] = F(s)$

Thus, $$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s) \qquad (1)$$

***L* is called the Laplace transform operator.**

The operation is multiplication of $f(t)$ by e^{-st} and integration between 0 and ∞.

Note (1) The parameter s is a real or complex number. We shall assume s is a real number. Some times p is used instead of s.

19.1 CONDITION FOR EXISTENCE OF LAPLACE TRANSFORM

Let $f(t)$ be defined for all $t \geq 0$ such that

(i) $f(t)$ is piecewise continuous in the interval $[0, \infty)$

and (ii) $f(t)$ is of exponential order $\alpha > 0$, then the Laplace transform of $f(t)$ exists for $s > \alpha$.

Note

1. By piecewise continuity on $[0, \infty)$, we mean that the function is continuous on every finite sub interval $0 \leq t \leq a$, except possibly at a finite number of points, where they are jump discontinuities i.e., $f(x+)$, $f(x-)$ exist, but not equal.
2. $f(t)$ is of exponential order $\alpha > 0$ if $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$ and some positive constant M.

   Equivalently $\lim_{t\to\infty}\{e^{-\alpha t} f(t)\}$ is finite.

Geometrically, it means that the graph of $f(t)$, $t \geq 0$ does not grow faster than the graph of the exponential function $g(t) = Me^{\alpha t}$, $t \geq 0$

For example, t^n is of exponential order as $t \to \infty$.

$$\because \quad \lim_{t\to\infty}(e^{-\alpha t}t^n) = \lim_{t\to\infty}\frac{t^n}{e^{\alpha t}} = \lim_{t\to\infty}\frac{n!}{\alpha^n e^{\alpha t}} = 0 \quad \text{[by L'Hopital's rule]}$$

$\therefore$ t^n is of exponential order $\alpha > 0$.

Similarly, sin at, cos at, e^{at}, e^{-at} all satisfy this condition.

3. The above conditions are sufficient, but not necessary.
   For example, Laplace transform of $\frac{1}{\sqrt{t}}$ exists, but it is not continuous at $t = 0$ and hence it is not piece-wise continuous in $[0, \infty)$.
4. Generally, functions that represent physical quantities satisfy these conditions and hence we assume they have Laplace transforms.
5. When Laplace transform for a given function exists, it is unique. Conversely, two continuous functions having same Laplace transform must be equal and hence we say that inverse Laplace transform is unique. This is of practical importance because Laplace transforms are used in solving boundary value problems.

19.2 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

1. $L[c] = \frac{c}{s}$, $s > 0$ and c is a constant.

Proof

$$L[c] = \int_0^\infty e^{-st} c \, dt$$

$$= c\int_0^\infty e^{-st}\,dt = c\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{c}{s}\left[e^{-\infty} - e^0\right] = \frac{c}{s} \quad \text{if } s > 0 \qquad (\because\ t > 0)$$

In particular, $L[1] = \frac{1}{s}$, $s > 0$ and $L[0] = 0$ ■

2. $L\left[e^{at}\right] = \frac{1}{s-a}$ if $s > a$

Proof

$$L\left[e^{at}\right] = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

This limit exists if $s - a > 0$ as $t > 0$.

$$\therefore \quad L\left[e^{at}\right] = \frac{1}{-(s-a)}\left[e^{-\infty} - e^0\right] = \frac{1}{s-a} \quad \text{if } s > a$$ ■

3. Similarly $L\left[e^{-at}\right] = \frac{1}{s+a}$ if $s > -a$

4. $L\left[t^n\right] = \dfrac{n!}{s^{n+1}}$ **if** $s > 0$ **and** $n = 0, 1, 2, 3, \ldots$

Proof

$$L\left[t^n\right] = \int_0^\infty e^{-st} t^n \, dt, \quad s > 0$$

Put $\quad st = u \quad \therefore \quad dt = \dfrac{du}{s}.$

When $t = 0$, $u = 0$ and when $t = \infty$, $u = \infty$

$$\therefore \quad L\left[t^n\right] = \int_0^\infty e^{-u} \frac{u^n}{s^n} \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1} \, du = \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{if } n+1 > 0, \; s > 0$$

We know, if n is a positive integer, then $\Gamma(n+1) = n!$

$$\therefore \quad L\left[t^n\right] = \frac{n!}{s^{n+1}}$$ ■

Corollary $\quad$ $L[1] = \dfrac{1}{s}, \quad s > 0$ $\quad$ and $\quad$ $L[t] = \dfrac{1}{s^2}, \quad s > 0$

5. $L[\sin at] = \dfrac{a}{s^2 + a^2}, \quad s > 0$

Proof

$$L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt = \left[\frac{e^{-st}}{s^2 + a^2}\left(-s \sin at - \frac{d}{dt}(\sin at)\right)\right]_0^\infty$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= 0 - \frac{e^0}{s^2 + a^2}(0 - a \cos 0) = \frac{a}{s^2 + a^2} \quad \text{if } s > 0 \qquad [\because \; e^{-\infty} = 0 \;\; \text{if } s > 0]$$ ■

6. $L[\cos at] = \dfrac{s}{s^2 + a^2}, \quad s > 0$

Proof

$$L[\cos at] = \int_0^\infty e^{-st} \cos at \; dt = \left[\frac{e^{-st}}{s^2 + a^2}\left[-s \cos at - \frac{d}{dt}(\cos at)\right]\right]_0^\infty$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}[-s \cos at + a \sin at]\right]_0^\infty$$

$$= 0 - \frac{e^0}{s^2 + a^2}(-s \cos 0 + a \sin 0) = \frac{s}{s^2 + a^2} \quad \text{if } s > 0$$

In particular, $L[\sin t] = \frac{1}{s^2+1}$ and $L[\cos t] = \frac{s}{s^2+1}$

The Laplace transform of many functions can be obtained from the Laplace transform of elementary functions and some general properties just as in differentiation of functions. One such general property of Laplace transform is the linearity property.

19.3 SOME PROPERTIES OF LAPLACE TRANSFORM

Property 1 Linearity property

Let $f(t)$ and $g(t)$ be any two functions whose Laplace transform exist and a, b are any two constants then

$$L[af(t)+bg(t)] = aL[f(t)] + bL[g(t)]$$

Proof By definition,

$$L[af(t)+bg(t)] = \int_0^\infty e^{-st}[af(t)+bg(t)]\,dt$$

$$= \int_0^\infty e^{-st} af(t)\,dt + \int_0^\infty e^{-st} bg(t)\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt = aL[f(t)] + bL[g(t)]$$ ■

Note Because of this property the Laplace transform operator L is a linear operator.

Using this we can find (1) $L[\sinh at]$ and (2) $L[\cosh at]$

Proof

1. $$L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$= \frac{1}{2}\{L[e^{at}] - L[e^{-at}]\}, \qquad \text{[by linearity property]}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \qquad \text{if } s > a, s > -a$$

$$= \frac{s+a-(s-a)}{2(s-a)(s+a)} = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2} \qquad \text{if } s > |a|$$

∴ $$\mathbf{L[\sinh at] = \frac{a}{s^2-a^2}} \qquad \text{if } s > |a|$$

2. $$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\{L[e^{at}] + L[e^{-at}]\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \text{if } s > a, s > -a$$

$$= \frac{s+a+s-a}{2(s-a)(s+a)} = \frac{2s}{2(s^2-a^2)} = \frac{s}{s^2-a^2} \qquad \text{if } s > |a|$$

$$\therefore \qquad \boldsymbol{L[\cosh at] = \frac{s}{s^2-a^2}} \qquad \text{if } s > |a| \qquad \blacksquare$$

WORKED EXAMPLES

EXAMPLE 1

Find the Laplace transform of $t^{-1/2}$.

Solution.

$$\text{Let } f(t) = t^{-1/2} \quad \therefore \quad L\left[t^{-1/2}\right] = \int_0^\infty e^{-st} t^{-1/2}\, dt$$

Put $\qquad st = x \quad \therefore \quad s\,dt = dx \quad \Rightarrow \quad dt = \dfrac{dx}{s}$

When $\qquad t = 0,\ x = 0$ and when $t = \infty,\ x = \infty$

$$\therefore \qquad L\left[t^{-1/2}\right] = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^{-1/2} \cdot \frac{dx}{s}$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2}\, dx = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{1/2-1}\, dx = \frac{1}{\sqrt{s}} \Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{s}} \qquad \left[\because \Gamma(1/2) = \sqrt{\pi}\right]$$

EXAMPLE 2

Find the Laplace transforms of (i) sin 2*t* sin3*t*.

Solution.

(i) Let $\qquad f(t) = \sin 2t \sin 3t$

$$= \frac{1}{2}\left[\cos(3t-2t) - \cos(3t+2t)\right] = \frac{1}{2}\left[\cos t - \cos 5t\right]$$

$$\therefore \qquad L[f(t)] = \frac{1}{2}\left[L(\cos t) - L(\cos 5t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{s}{2}\frac{\left[s^2+25-s^2-1\right]}{(s^2+1)(s^2+25)} = \frac{s}{2}\left[\frac{24}{(s^2+1)(s^2+25)}\right]$$

$$\Rightarrow \quad L[\sin 2t \sin 3t] = \frac{12s}{(s^2+1)(s^2+25)}$$

We shall now list some important Laplace transform pairs $f(t)$ and $F(s)$ for ready reference to do problems.

	Function $f(t)$	**Laplace transform $F(s) = L[f(t)]$**
(1)	$t^n, n = 0, 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}, s > 0, \; n = 0, 1, 2, 3, \ldots$
(2)	$t^\alpha, \alpha > 0$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \; \alpha > 0, s > 0$
(3)	e^{at}	$\frac{1}{s-a}, s > a$
(4)	$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
(5)	$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
(6)	$\sinh at$	$\frac{a}{s^2-a^2}, s > \lvert a \rvert$
(7)	$\cosh at$	$\frac{s}{s^2-a^2}, s > \lvert a \rvert$
(8)	$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}, s > 0$

EXAMPLE 3

Find the Laplace transforms of (i) $\cos^2 2t$ (ii) $\sin^3 2t$.

Solution.

(i) Let $$f(t) = \cos^2 2t = \frac{1+\cos 4t}{2}$$

$\therefore$

$$L[f(t)] = \frac{1}{2}[L(1) + L(\cos 4t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4^2}\right] = \frac{1}{2}\left[\frac{s^2+16+s^2}{s(s^2+16)}\right] = \frac{1}{2}\left[\frac{2s^2+16}{s(s^2+16)}\right] = \frac{s^2+8}{s(s^2+16)}$$

(ii) We know $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \;\Rightarrow\; \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

$$\therefore \quad L[\sin^3 2t] = L\left[\frac{1}{4}(3\sin 2t - \sin 6t)\right]$$

$$= \frac{3}{4} L[\sin 2t] - \frac{1}{4} L[\sin 6t]$$

$$= \frac{3}{4} \cdot \frac{2}{s^2+2^2} - \frac{1}{4} \cdot \frac{6}{s^2+6^2}$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right] = \frac{3}{2}\left[\frac{32}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

Property 2 First shifting property (or the *s*-shifting)

If $L[f(t)] = F(s)$ then (i) $L\left[e^{at} f(t)\right] = F(s-a)$ if $s - a > \alpha$

(ii) $L\left[e^{-at} f(t)\right] = F(s+a)$ if $s + a > \alpha$

Proof Given $L[f(t)] = F(s)$

$$\therefore \quad L\left[e^{at} f(t)\right] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a) \quad (\text{if } s-a>\alpha)$$

Similarly, $L\left[e^{-at} f(t)\right] = F(s+a)$ if $s+a>\alpha$ ■

Note The above properties are called shifting properties because the multiplication of $f(t)$ by e^{at} and e^{-at} shifts the argument s to $s-a$, $s+a$ respectively.

The results can be rewritten as $L\left[e^{at} f(t)\right] = [F(s)]_{s\to s-a} = L[f(t)]_{s\to s-a}$

and $L\left[e^{-at} f(t)\right] = [F(s)]_{s\to s+a} = L[f(t)]_{s\to s-a}$

'$s \to s+a$' means s is replaced by $s+a$ or s changed to $s+a$

Property 3 Change of scale property

$$\textbf{If } L[f(t)] = F(s), \textbf{ then } L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a>0$$

Proof Given $L[f(t)] = F(s)$ (where $s > \alpha$)

Now $$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

Put $at = u \quad \therefore \quad a\,dt = du \quad \Rightarrow \quad dt = \frac{1}{a} du, \quad a>0$

When $t=0, u=0$ and when $t=\infty, u=\infty$

$$\therefore \qquad L[f(at)] = \int_0^\infty e^{\frac{-su}{a}} \cdot f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)\,du = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad \blacksquare$$

Note $\quad L\left[e^{at} f(bt)\right] = \frac{1}{b}F\left(\frac{s-a}{b}\right); \quad L\left[e^{-at} f(bt)\right] = \frac{1}{b}F\left(\frac{s+a}{b}\right)$

WORKED EXAMPLES

EXAMPLE 4

Find the Laplace transform of the following functions:

(i) $e^{at}t^n,\ n \in N$, **(ii)** $e^{at}\sin bt$, **(iii)** $e^{at}\sinh bt$.

Solution.

(i) $L\left[e^{at}t^n\right] = \left[L(t^n)\right]_{s\to s-a}$, by shifting formula

$$= \left[\frac{n!}{s^{n+1}}\right]_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}$$

(ii) $$L\left[e^{at}\sin bt\right] = \left[L(\sin bt)\right]_{s\to s-a} = \left[\frac{b}{s^2+b^2}\right]_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

(iii) $$L\left[e^{at}\sinh bt\right] = \left[L(\sinh bt)\right]_{s\to s-a} = \left[\frac{b}{s^2-b^2}\right]_{s\to s-a} = \frac{b}{(s-a)^2-b^2}$$

EXAMPLE 5

Find $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$.

Solution.

$$\cos\sqrt{t} = 1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - \frac{(\sqrt{t})^6}{6!} + \cdots$$

$$\therefore \qquad \frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left[1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots\right] = t^{-1/2} - \frac{1}{2!}t^{1/2} + \frac{1}{4!}t^{3/2} - \frac{t^{5/2}}{6!} + \cdots$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L[t^{-1/2}] - \frac{1}{2!}L[t^{1/2}] + \frac{1}{4!}L[t^{3/2}] - \frac{1}{6!}L[t^{5/2}] + \cdots$$

$$= \sqrt{\frac{\pi}{s}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \cdots$$

$$= \sqrt{\frac{\pi}{s}} - \frac{1/2\,\Gamma(1/2)}{2!\,s^{3/2}} + \frac{1}{4!}\frac{3/2\cdot 1/2\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\cdot 3/2\cdot 1/2\cdot\Gamma(1/2)}{s^{7/2}}$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{1!}\frac{1}{4s} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3} + \cdots\right] = \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

EXAMPLE 6

Find the Laplace transform of cosh *at* cos *at*.

Solution.

$$L[\cosh at \cos at] = L\left[\frac{e^{at}+e^{-at}}{2}\cdot \cos at\right]$$

$$= \frac{1}{2}L[e^{at}\cos at] + \frac{1}{2}L[e^{-at}\cos at]$$

$$= \frac{1}{2}[L[\cos at]]_{s\to s-a} + \frac{1}{2}[L[\cos at]]_{s\to s+a}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+a^2}\right]_{s\to s-a} + \frac{1}{2}\left[\frac{s}{s^2+a^2}\right]_{s\to s+a}$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{(s-a)[(s+a)^2+a^2]+(s+a)[(s-a)^2+a^2]}{2[(s-a)^2+a^2][(s+a)^2+a^2]}$$

$$= \frac{(s-a)[(s+a)[s+a+s-a]+a^2[s+a+s-a]]}{2(s^2-2as+a^2+a^2)(s^2+2as+a^2+a^2)}$$

$$= \frac{2s(s^2-a^2)+2sa^2}{2(s^2+2a^2-2as)(s^2+2a^2+2as)}$$

$$= \frac{2s^3-2a^2s+2a^2s}{2[(s^2+2a^2)^2-4a^2s^2]}$$

$$= \frac{2s^3}{2[s^4+4a^2s^2+4a^4-4a^2s^2]} = \frac{s^3}{s^4+4a^4}$$

EXERCISE 19.1

Find the Laplace transform of the following functions:

1. $\sin 6t + 5e^{-3t} + \cos 3t + 2$
2. $\cos^3 3t$
3. $\sin 2t \cos 3t$
4. $\sin 5t \cdot \sin 3t$
5. $\sinh^2 2t$
6. $(t^2+1)^2$
7. $(\sin t - \cos t)^2$
8. $\cosh(5t+2)$
9. $\sin 3t \cos^2 t$
10. $e^{-2t}\sin 4t$
11. $t^3 e^{-3t}$
12. $e^{-t}\sin^2 t$
13. $e^{-t}(3\sinh 2t - 5\cosh 2t)$
14. $(t+2)^2 e^t$
15. $e^{-3t}\cos^2 t$
16. $(1+te^{-t})3$
17. $e^{4t}\sin 2t \cos t$
18. $e^{3t}\sin 2t . \sin t$
19. $(2e^t + e^{-2t})^2 \cdot t^2$
20. $t^{1/2}$
21. $e^{-5t}t^7$
22. $e^{at}\cos bt$
23. $e^{at}\cosh bt$
24. $e^t t^{-1/2}$
25. $\cosh at \sin bt$
26. $e^{-3t}(2\cos 5t - 3\sin 5t)$
27. $(1+te^{-t})^3$

28. $e^t \sin^3 2t$ 29. $\dfrac{(\sqrt{t}-1)^2}{\sqrt{t}}$ 30. $\sin\sqrt{t}$

31. $\dfrac{e^{at}}{\sqrt{t}}$.

ANSWERS TO EXERCISE 19.1

1. $\dfrac{6}{s^2+36}+\dfrac{5}{s+3}+\dfrac{s}{s^2+9}+\dfrac{2}{s}$ 2. $\dfrac{s}{4}\left\{\dfrac{1}{s^2+81}+\dfrac{3}{s^2+9}\right\}$ 3. $\dfrac{1}{2}\left\{\dfrac{5}{s^2+25}-\dfrac{1}{s^2+1}\right\}$

4. $\dfrac{s}{2}\left\{\dfrac{1}{s^2+4}-\dfrac{1}{s^2+64}\right\}$ 5. $\dfrac{1}{2}\left\{\dfrac{s}{s^2-16}-\dfrac{1}{s}\right\}$ 6. $\dfrac{24}{s^5}+\dfrac{4}{s^3}+\dfrac{1}{s}$

7. $\dfrac{1}{s}-\dfrac{2}{s^2+4}$ 8. $\dfrac{1}{2}\left[\dfrac{e^2}{s-5}+\dfrac{e^{-2}}{s+5}\right]$ 9. $\dfrac{3}{2}\left[\dfrac{1}{s^2-9}+\dfrac{s^2-13}{s^4-10s^2+169}\right]$

10. $\dfrac{4}{s^2+4s+20}$ 11. $\dfrac{6}{(s+3)^4}$ 12. $\dfrac{2}{(s+1)(s^2+2s+5)}$

13. $\dfrac{1-5s}{s^2+2s-3}$ 14. $\dfrac{2(2s^2-2s+1)}{(s-1)^3}$ 15. $\dfrac{1}{2}\left[\dfrac{1}{s+3}+\dfrac{s+3}{(s+3)^2+4}\right]$

16. $\dfrac{1}{s}+\dfrac{3}{(s+1)^2}+\dfrac{6}{(s+2)^3}+\dfrac{6}{(s+3)^4}$ 17. $\dfrac{1}{2}\left[\dfrac{3}{(s-4)^2+9}+\dfrac{1}{(s-4)^2+1}\right]$

18. $\dfrac{1}{2}\left[\dfrac{s-3}{(s-3)^2+1}-\dfrac{s-3}{(s-3)^2+9}\right]$ 19. $\dfrac{8}{(s-2)^3}+\dfrac{2}{(s+4)^3}+\dfrac{8}{(s+1)^3}$

20. $\dfrac{\sqrt{\pi}}{2s^{3/2}}$ 21. $\dfrac{7!}{(s+5)^8}$ 22. $\dfrac{s-a}{(s-a)^2+b^2}$

23. $\dfrac{s-a}{(s-a)^2-b^2}$ 24. $\dfrac{\sqrt{\pi}}{\sqrt{s-1}}$ if $s-1>0$

25. $\dfrac{b}{2}\left\{\dfrac{1}{(s-a)^2+b^2}+\dfrac{1}{(s+a)^2+b^2}\right\}$ 26. $\dfrac{2s-9}{(s+3)^2+25}$

27. $\dfrac{1}{s}+\dfrac{3}{(s+1)^2}+\dfrac{6}{(s+2)^3}+\dfrac{6}{(s+3)^4}$ 28. $\dfrac{48}{[(s-1)^2+4][(s-1)^2+36]}$

29. $\dfrac{\sqrt{\pi}(1+2s)-4\sqrt{s}}{2s^{3/2}}$ 30. $\dfrac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$ 31. $\sqrt{\dfrac{\pi}{s-a}}$, $s-a>0$

19.4 DIFFERENTIATION AND INTEGRATION OF TRANSFORMS

Theorem 19.1 Differentiation of transform (or multiplication by t^n)

If $L[f(t)] = F(s)$ **then** $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s)), \quad n=1,2,3,\ldots$

Proof By definition, $$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating w.r.to s under the integral sign by Leibnitz's theorem,

we get $$\frac{dF(s)}{ds} = \int_0^\infty -te^{-st} f(t)\, dt \tag{1}$$

$$= -\int_0^\infty e^{-st} tf(t)\, dt = -L[tf(t)]$$

∴ $$L[tf(t)] = -\frac{dF(s)}{ds}$$

Differentiating (1) w.r.to s, we get

$$\frac{d^2(F(s))}{ds^2} = -\int_0^\infty -te^{-st}(tf(t))\, dt$$

$$= \int_0^\infty e^{-st} t^2 f(t)\, dt = (-1)^2 \int_0^\infty e^{-st} t^2 f(t)\, dt = (-1)^2 L[t^2 f(t)]$$

∴ $$L[t^2 f(t)] = (-1)^2 \frac{d^2 F(s)}{ds^2}$$

Proceeding in this way, we get

$$L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}, \quad n = 1,2,3,\ldots$$ ■

Theorem 19.2 Integration of transform (or Division by t)

If $L[f(t)] = F(s)$, **then** $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$

Proof

By definition, $$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Integrating both sides w.r.to s from s to ∞, we get

$$\int_s^\infty F(s) ds = \int_s^\infty \int_0^\infty e^{-st} f(t) dt$$

Since t is independent of s, changing the order of integration, we have

$$\int_s^\infty F(s)ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{-t}[e^{-\infty} - e^{-st}]dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left[\frac{f(t)}{t}\right] \qquad [\because t > 0; s > 0; e^{-\infty} = 0]$$

$$\therefore \qquad \boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds} \qquad ■$$

Note

1. In theorem 19.1 multiplication of $f(t)$ by t results in differentiation of $L[f(t)] = F(s)$ w.r. to s. Multiplication by t^2 results in differentiation of $L[f(t)]$ twice and so on, with proper sign.
2. In theorem 19.2 division of $f(t)$ by t results in integrating $F(s)$ from s to ∞. Division by t^2 will result in integration w.r.to s twice

$$\therefore \qquad L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty\int_s^\infty F(s)\,ds\,ds$$

Proceeding like this, we get

$$L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty\int_s^\infty \cdots \int_s^\infty F(s)(ds)^n, \text{ where } n \text{ is a positive integer.}$$

WORKED EXAMPLES

EXAMPLE 1

Find $L[t\cos^3 t]$.

Solution.

Let $\qquad f(t) = \cos^3 t$

$\therefore$ $tf(t)$ is the multiplication of $f(t)$ by t.

We know $\cos 3t = 4\cos^3 t - 3\cos t \quad \Rightarrow \quad \cos^3 t = \frac{1}{4}[\cos 3t + 3\cos t]$

$$\therefore \qquad F(s) = L[f(t)] = L[\cos^3 t] = \frac{1}{4}[L[\cos 3t] + 3L[\cos t]] = \frac{1}{4}\left[\frac{s}{s^2+9} + 3\frac{s}{s^2+1}\right]$$

$$\therefore \quad L[t\cos^3 t] = -\frac{d}{ds}F(s) = -\frac{d}{ds}\left[\frac{1}{4}\left(\frac{s}{s^2+9} + \frac{3s}{s^2+1}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{(s^2+9)\cdot 1 - s\cdot 2s}{(s^2+9)^2} + \frac{3\{(s^2+1)\cdot 1 - s\cdot 2s\}}{(s^2+1)^2}\right]$$

$$= -\frac{1}{4}\left[\frac{(9-s^2)}{(s^2+9)^2} + \frac{3(1-s^2)}{(s^2+1)^2}\right] = \frac{1}{4}\left[\frac{(s^2-9)}{(s^2+9)^2} + \frac{3(s^2-1)}{(s^2+1)^2}\right]$$

EXAMPLE 2

Find $L[te^{-t}\sin t]$.

Solution.

Let $f(t) = e^{-t}\sin t$

$$\therefore \quad F(s) = L[f(t)] = L\left[e^{-t}\sin t\right] = L[\sin t]_{s\to s+1} = \left[\frac{1}{s^2+1}\right]_{s\to s+1} = \frac{1}{(s+1)^2+1}$$

Now $te^{-t}\sin t = tf(t)$ is the multiplication of $f(t)$ by t.

$$\therefore \quad L[tf(t)] = -\frac{dF(s)}{ds} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -\left[\frac{-1}{\left\{(s+1)^2+1\right\}^2}\right]2(s+1) \qquad \left[\because \frac{d}{dx}\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)\cdot 2x}{(x^2+a^2)^2}\right]$$

$$= \frac{2(s+1)}{\left\{(s+1)^2+1\right\}^2}$$

EXAMPLE 3

Find $L[t^2e^{-3t}\sin 2t]$.

Solution.

Let $f(t) = \sin 2t$

$$\therefore \quad L[t^2 f(t)] = L[t^2\sin 2t] = (-1)^2\frac{d^2F(s)}{ds^2},$$

where

$$F(s) = L[\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore \quad L[t^2 f(t)] = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = 2\frac{d}{ds}\left[\frac{(-1)\cdot 2s}{(s^2+4)^2}\right]$$

$$= -4\frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$$

$$= -4\frac{[(s^2+4)^2\cdot 1 - s\cdot 2(s^2+4)\cdot 2s]}{(s^2+4)^4} = -4\frac{[s^2+4-4s^2]}{(s^2+4)^3} = +4\frac{[3s^2-4]}{(s^2+4)^3}$$

$$\therefore \quad L[t^2\sin 2t] = \frac{4[3s^2-4]}{(s^2+4)^3}$$

$$\therefore \quad L[t^2e^{-3t}\sin 2t] = L[t^2\sin 2t]_{s\to s+3}$$

$$= \left[\frac{4(3s^2-4)}{(s^2+4)^3}\right]_{s\to s+3} = \frac{4\cdot[3(s+3)^2-4]}{[(s+3)^2+4]^3} = \frac{4[3s^2+18s+23]}{(s^2+6s+13)^3}$$

EXAMPLE 4

Find $L\left[\dfrac{1-\cos 2t}{t}\right]$.

Solution.

Let $f(t) = 1 - \cos 2t$

$\therefore$ $F(s) = L[f(t)] = L[1-\cos 2t] = \dfrac{1}{s} - \dfrac{s}{s^2+4}$

$\therefore$ $L\left[\dfrac{1-\cos 2t}{t}\right] = L\left[\dfrac{f(t)}{t}\right]$ [division of $f(t)$ by t]

$$= \int_s^\infty F(s)\,ds \quad \text{[By theorem 19.2, page 19.11]}$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s\,ds}{s^2+4}$$

$$= [\log_e s]_s^\infty - \frac{1}{2}\int_s^\infty \frac{2s\,ds}{s^2+4}$$

$$= \left[\log_e s - \frac{1}{2}\log_e (s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}[2\log_e s - \log_e (s^2+4)]_s^\infty$$

$$= \frac{1}{2}[\log_e s^2 - \log_e (s^2+4)]_s^\infty$$

$$= \frac{1}{2}\left[\log_e \left[\frac{s^2}{s^2+4}\right]\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e \frac{1}{1+\frac{4}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e 1 - \log_e \frac{1}{1+\frac{4}{s^2}}\right] \quad \left[\text{since } \frac{4}{s^2} \to 0 \text{ as } s \to \infty\right]$$

$$= \frac{1}{2}\left[0 - \log_e \frac{s^2}{s^2+4}\right] = \frac{1}{2}\log_e \frac{s^2+4}{s^2} \quad [\because \ \log_e 1 = 0]$$

EXAMPLE 5

Find the Laplace transform of $\frac{1-e^t}{t}$.

Solution.

Let $$f(t) = 1 - e^t$$

$\therefore$ $$F(s) = L[f(t)] = L[1-e^t] = L[1] - L[e^t] = \frac{1}{s} - \frac{1}{s-1}$$

$\therefore$ $$L\left[\frac{1-e^t}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds \qquad \text{[division of } f(t) \text{ by } t]$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$$

$$= [\log s - \log(s-1)]_s^\infty$$

$$= \left[\log \frac{s}{s-1}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{1-\frac{1}{s}} = 0 - \log \frac{1}{1-\frac{1}{s}} = -\log \frac{s}{s-1} = \log\left(\frac{s-1}{s}\right)$$

EXAMPLE 6

Find the Laplace transform of $\frac{\sin at}{t}$.

Solution.

Since $\lim_{t\to 0} \frac{\sin at}{t} = \lim_{t\to 0} \frac{a \sin at}{at} = a$, $\frac{\sin at}{t}$ is continuous $\forall\, t \geq 0$

$\therefore$ Laplace transform of $\frac{\sin at}{t}$ exists.

Let $$f(t) = \sin at$$

$\therefore$ $$F(s) = L[f(t)] = L[\sin at] = \frac{a}{s^2 + a^2}$$

$\therefore$ $$L\left[\frac{\sin at}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \qquad \text{[division of } f(t) \text{ by } t]$$

$$= \int_s^\infty \frac{a\,ds}{s^2 + a^2}$$

$$= a \cdot \frac{1}{a}\left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Note $L\left[\frac{\cos at}{t}\right]$ does not exist because the function $\frac{\cos at}{t}$ is discontinuous at $t = 0$.

EXAMPLE 7

Find the Laplace transform of $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution.

Let $$f(t) = e^{-at} - e^{-bt}$$

$$\therefore \quad F(s) = L[f(t)] = L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\therefore \quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \qquad \text{[division by } t\text{]}$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \left[\log_e(s+a) - \log_e(s+b)\right]_s^\infty$$

$$= \left[\log_e \frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[\log_e \left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]\right]_s^\infty = \log_e 1 - \log_e \left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right] = -\log_e \frac{s+a}{s+b} = \log_e \frac{s+b}{s+a}$$

EXAMPLE 8

Find $L\left[\dfrac{e^{-3t}\sin 2t}{t}\right]$.

Solution.

Let $$f(t) = e^{-3t} \sin 2t$$

$$\therefore \quad F(s) = L[f(t)] = L[e^{-3t}\sin 2t] = [L[\sin 2t]]_{s \to s+3} = \frac{2}{(s+3)^2+4}$$

$$\therefore \quad L\left[\frac{e^{-3t}\sin t}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \qquad \text{[division of } f(t) \text{ by } t\text{]}$$

$$= \int_s^\infty \frac{2}{(s+3)^2+4}\,ds$$

$$= 2\left[\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s+3}{2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right) = \cot^{-1}\left(\frac{s+3}{2}\right)$$

EXAMPLE 9

Find $L\left[\dfrac{1-\cos t}{t^2}\right]$.

Solution.

Let $\quad f(t) = 1 - \cos t$

$$\therefore \quad F(s) = L[1-\cos t] = L[1] - L[\cos t] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\therefore \quad L\left[\frac{1-\cos t}{t^2}\right] = L\left[\frac{f(t)}{t^2}\right] \qquad \text{[dividing } f(t) \text{ by } t^2\text{]}$$

$$= \int_s^\infty \int_s^\infty F(s)ds \cdot ds \qquad \text{[by theorem 19.2]}$$

Now $\displaystyle\int_s^\infty F(s)ds = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds = \int_s^\infty \frac{ds}{s} - \frac{1}{2}\int \frac{2s\,ds}{s^2+1}$

$$= \left[\log_e s - \frac{1}{2}\log_e(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log_e s - \log_e(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e\left(\frac{1}{\left(1+\frac{1}{s^2}\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left\{\log_e 1 - \log_e\left(\frac{1}{1+\frac{1}{s^2}}\right)\right\} = -\frac{1}{2}\log_e\left(\frac{s^2}{s^2+1}\right) = \frac{1}{2}\log_e\frac{s^2+1}{s^2}$$

$$\therefore \quad L\left[\frac{1-\cos t}{t^2}\right] = \int_s^\infty \frac{1}{2}\log_e\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty [\log_e(s^2+1) - \log_e s^2]\,ds$$

$$= \frac{1}{2}\int_s^\infty [\log_e(s^2+1) - 2\log_e s]\,ds$$

$$= \frac{1}{2}\int_s^\infty \log_e(s^2+1)\,ds - \int_s^\infty \log_e s\,ds = \frac{1}{2}I_1 - I_2$$

where $\displaystyle I_1 = \int_s^\infty \log_e(s^2+1)\,ds = \left[s\log_e(s^2+1)\right]_s^\infty - \int_s^\infty \frac{1}{s^2+1}2s\cdot s\,ds$

$$= \left[s \log_e (s^2+1) \right]_s^\infty - 2\int_s^\infty \frac{s^2+1-1}{s^2+1} ds$$

$$= \left[s \log_e (s^2+1) \right]_s^\infty - 2\int_s^\infty \left[1 - \frac{1}{(s^2+1)} \right] ds$$

$$= \left[s \log_e (s^2+1) \right]_s^\infty - 2\left[s - \tan^{-1} s \right]_s^\infty$$

and $I_2 = \int_s^\infty \log_e s \, ds = [s \log_e s]_s^\infty - \int_s^\infty s \cdot \frac{1}{s} ds = [s \log_e s]_s^\infty - [s]_s^\infty$

$$\therefore \quad L\left[\frac{1-\cos t}{t^2}\right] = \frac{1}{2}\left\{ [s \log_e (s^2+1)]_s^\infty - 2[s - \tan^{-1} s]_s^\infty \right\} - \left\{ [s \log_e s]_s^\infty - [s]_s^\infty \right\}$$

$$= \left[\frac{s}{2} \log_e (s^2+1) - s + \tan^{-1} s - s \log_e s + s \right]_s^\infty$$

$$= \left[s \log_e \sqrt{s^2+1} + \tan^{-1} s - s \log_e s \right]_s^\infty$$

$$= \left[s \log_e \sqrt{\frac{s^2+1}{s^2}} + \tan^{-1} s \right]_s^\infty$$

$$= \lim_{s\to\infty} s \log_e \sqrt{\frac{s^2+1}{s^2}} + \tan^{-1} \infty - s \log_e \sqrt{\frac{s^2+1}{s^2}} - \tan^{-1} s$$

$$= \lim_{s\to\infty} \frac{s}{2} \log_e \left(1 + \frac{1}{s^2} \right) + \frac{\pi}{2} - s \log_e \left(\sqrt{\frac{s^2+1}{s^2}} \right) - \tan^{-1} s$$

$$= \lim_{s\to\infty} \frac{s}{2} \left[\frac{1}{s^2} - \frac{1}{2s^4} + \frac{1}{3s^6} - \cdots \right] + \frac{\pi}{2} - \tan^{-1} s - s \log_e \left(\frac{\sqrt{s^2+1}}{s} \right)$$

$$= \lim_{s\to\infty} \frac{1}{2} \left(\frac{1}{s} - \frac{1}{2s^3} + \frac{1}{3s^5} - \cdots \right) + \cot^{-1} s - s \log_e \left(\frac{\sqrt{s^2+1}}{s} \right)$$

$$= 0 + \cot^{-1} s - s \log_e \left(\frac{\sqrt{s^2+1}}{s} \right) = \cot^{-1} s - s \log_e \frac{\sqrt{s^2+1}}{s}$$

EXAMPLE 10

Find the Laplace transform of $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$.

Solution.

Given $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$

$\therefore \quad L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$

$$= \int_0^4 e^{-st} t\,dt + \int_4^\infty e^{-st}\, 5\,dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} - 1.\frac{e^{-st}}{(-s)^2} \right]_0^4 + 5\left[\frac{e^{-st}}{-s} \right]_4^\infty$$

$$= -\left[\frac{4e^{-4s}}{s} + \frac{e^{-4s}}{s^2} - \left(0 + \frac{e^0}{s^2} \right) \right] - \frac{5}{s}[e^{-\infty} - e^{-4s}]$$

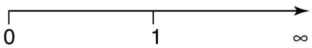
$$= \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} = \frac{e^{-4s}}{s} + \frac{1}{s^2}(1 - e^{-4s})$$

EXAMPLE 11

Find the Laplace transform of $f(t) = |t-1| + |t+1|, \ t \geq 0$.

Solution.

Given $\quad f(t) = |t-1| + |t+1|, \ t \geq 0$



We know $\quad |x| = x \quad$ if $x \geq 0$ and $|x| = -x \quad$ if $x < 0$

If $0 \leq t < 1$, then $t + 1 > 0$ and $t - 1 < 0 \quad \therefore \ |t + 1| = t + 1$ and $|t - 1| = -(t - 1)$

$\therefore \quad f(t) = -(t-1) + t + 1 = 2 \quad$ if $0 \leq t < 1$

If $t \geq 1, \ t+1 > 0, \ t-1 \geq 0 \quad \therefore \ |t+1| = t+1, \ |t-1| = t-1$

$\therefore \quad f(t) = t + 1 + t - 1 = 2t \quad$ if $t \geq 1$

Thus $\quad f(t) = \begin{cases} 2 & \text{if } 0 \leq t < 1 \\ 2t & \text{if } t \geq 1 \end{cases}$

$\therefore \quad L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$

$$= \int_0^1 e^{-st}\, 2\,dt + \int_1^\infty e^{-st}\, 2t\,dt$$

$$= 2\left[\frac{e^{-st}}{-s} \right]_0^1 + 2\left[\frac{te^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{(-s)^2} \right]_1^\infty$$

$$= -\frac{2}{s}[e^{-s} - e^0] - 2\left[0 - \left(\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \right) \right]$$

$$= -\frac{2}{s}(e^{-s} - 1) + 2e^{-s}\left(\frac{1}{s} + \frac{1}{s^2} \right) = \frac{2}{s} + \frac{2e^{-s}}{s^2} = \frac{2}{s^2}[s + e^{-s}]$$

EXERCISE 19.2

Find the Laplace Transform of the following functions.

1. $te^{-t}\sin 2t$
2. $t\cosh 3t$
3. $te^{at}\sin at$
4. $t^2\cos^2 2t$
5. $(t+2)\cos 3t$
6. $te^{-2t}\sinh t$
7. $te^{-t}\cosh t$
8. $t^2e^{3t}\sinh t$
9. $\dfrac{\cos 2t-\cos 3t}{t}$
10. $\dfrac{e^{-3t}-e^{-4t}}{t}$
11. $\dfrac{1-e^{-t}}{t}$
12. $\dfrac{\cos 4t\sin 2t}{t}$
13. $\dfrac{\sin^2 t}{t}$
14. $\dfrac{1-e^{-at}}{t}$
15. $f(t)=\begin{cases}\sin t, 0<t<\pi\\ 0, \quad t\geq \pi\end{cases}$
16. $f(t)=\begin{cases}e^t & \text{if } 0<t<1\\ 0 & \text{if } t\geq 1\end{cases}$
17. $f(t)=\begin{cases}\cos t & \text{if } 0<t<\pi\\ \sin t & \text{if } \quad t\geq \pi\end{cases}$
18. $f(t)=\begin{cases}\dfrac{t}{c} & \text{if } 0<t<c\\ 1, & \text{if } \quad t\geq c\end{cases}$
19. $t\cos 3t$.
20. $te^{-4t}\sin 3t$.
21. $t^2e^{-t}\cos t$.
22. $t^2\sin at$.
23. $t\sin 3t\cos 2t$.
24. $\dfrac{\cos at-\cos bt}{t}$.
25. $\dfrac{e^{at}-\cos bt}{t}$.

ANSWERS TO EXERCISE 19.2

1. $\dfrac{4(s+1)}{(s^2+2s+5)^2}$
2. $\dfrac{s^2+9}{(s^2-9)^2}$
3. $\dfrac{2a(s-a)}{(s^2-2as+a^2)^2}$
4. $\dfrac{1}{s^3}+\dfrac{s(s^2-48)}{(s^2+16)^3}$
5. $\dfrac{s^2-9}{(s^2+9)^2}+\dfrac{2s}{s^2+9}$
6. $\dfrac{2(s+2)}{(s^2+4s+3)^2}$
7. $\dfrac{s^2+2s+2}{s^4+4s^3+4s^2}$
8. $\dfrac{1}{(s-2)^3}+\dfrac{1}{(s-4)^3}$
9. $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$
10. $\log\left(\dfrac{s+4}{s+3}\right)$
11. $\log\left(\dfrac{s+1}{s}\right)$
12. $\dfrac{1}{2}\left[\tan^{-1}\dfrac{s}{2}-\tan^{-1}\dfrac{s}{6}\right]$
13. $\dfrac{1}{2}\log\dfrac{\sqrt{s^2+4}}{s^2}$
14. $\log\left(\dfrac{s+a}{s}\right)$
15. $\dfrac{(1+e^{-\pi s})}{s^2+1}$
16. $\dfrac{1-e\cdot e^{-s}}{s-1}$
17. $\dfrac{1}{s^2+1}[s+(s-1)e^{-\pi s}]$
18. $-\dfrac{1}{s^2c}(e^{-sc}-1)$
19. $\dfrac{s^2-9}{(s^2+9)^2}$
20. $\dfrac{6(s+4)}{[(s+4)^2+9]^2}$
21. $\dfrac{2(s+1)[s^2+2s-2]}{[(s+1)^2+1]^3}$
22. $\dfrac{2a[3s^2-a^2]}{(s^2+a^2)^3}$
23. $\dfrac{5s}{(s^2+25)^2}+\dfrac{s}{(s^2+1)^2}$
24. $\dfrac{1}{2}\log_e\left[\dfrac{s^2+b^2}{s^2+a^2}\right]$
25. $\dfrac{1}{2}\log_e\dfrac{s^2+b^2}{(s-a)^2}$

19.5 LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Theorem 19.3 Laplace transform of derivative

Let $f(t)$ be continuous for all $t \geq 0$ and $f'(t)$ be piecewise continuous on every finite interval $0 \leq t \leq a$ in $[0, \infty)$. If $f(t)$ and $f'(t)$ are of exponential order as $t \to \infty$, then

$$L[f'(t)] = s\,L[f(t)] - f(0).$$

Proof By definition

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt \qquad \text{[using integration by parts]}$$

$$= \lim_{t\to\infty} e^{-st} f(t) - e^0 f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

Since $f(t)$ is of exponential order, $\lim_{t\to\infty} e^{-st} f(t) = 0$, if $s > \alpha$

$$\therefore \quad L[f'(t)] = s\int_0^\infty e^{-st} f(t)\,dt - f(0) = s\,L[f(t)] - f(0)$$

$$\therefore \quad L[f'(t)] = s\,L[f(t)] - f(0)$$ ■

Note Laplace transform of $f'(t)$ results in multiplication of $L\,[f(t)]$ by s

Corollary $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

We have, $L[f'(t)] = s\,L[f(t)] - f(0)$

$$\therefore \quad L[f''(t)] = s\,L\,[f'(t)] - f'(0)$$

$$= s\{s\,L[f(t)] - f(0)\} - f'(0) = s^2\,L[f(t)] - s\,f(0) - f'(0)$$

$$\therefore \quad L[f''(t)] = s^2\,L[f(t)] - s\,f(0) - f'(0)$$

Similarly, we can prove

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

and in general

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)$$

Theorem 19.4 Laplace transform of integral

If $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq a$ in $[0, \infty)$ and $f(t)$ is of exponential order $\alpha > 0$ and if $L[f(t)] = F(s)$, then

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s} \qquad \text{for } s > \alpha$$

i.e., $$L\left[\int_0^t f(u)du\right] = \frac{L[f(t)]}{s} \quad \text{for } s > \alpha$$

Proof Let $$g(t) = \int_0^t f(u)\,du$$

Then $$g'(t) = f(t) \text{ and } g(0) = \int_0^0 f(u)\,du = 0$$

$$\therefore \quad L[f(t)] = L[g'(t)]$$

$$= s\,L[g(t)] - g(0) = s\,L[g(t)] = s\,L\left[\int_0^t f(u)\,du\right]$$

$$\therefore \quad \frac{1}{s}L[f(t)] = L\left[\int_0^t f(u)du\right] \Rightarrow \boldsymbol{L\left[\int_0^t f(u)du\right] = \frac{L[f(t)]}{s}}$$ ■

Note

1. The Laplace transform of integral of $f(t)$ results in division of $L[f(t)]$ by s.
2. Replacing the dummy variable u by t, the above result is usually written as $L\left[\int_0^t f(t)\,dt\right] = \frac{L[f(t)]}{s}$

   Similarly $L\left[\int_0^t\int_0^t f(t)\,dt\,dt\right] = \frac{L[f(t)]}{s^2}$

3. $$\int_0^t f(t)\,dt = \int_0^a f(t)\,dt + \int_a^t f(t)\,dt$$

$$\therefore \quad \int_a^t f(t)\,dt = \int_0^t f(t)\,dt - \int_0^a f(t)\,dt$$

$$\therefore \quad L\left[\int_a^t f(t)\,dt\right] = L\left[\int_0^t f(t)\,dt\right] - L\left[\int_0^a f(t)\,dt\right]$$

But $\int_0^a f(t)\,dt$ is a constant, say K.

$$\therefore \quad L\left[\int_0^a f(t)\,dt\right] = L[K] = K\,L[1] = \frac{K}{s}$$

$$\therefore \quad L\left[\int_0^a f(t)\,dt\right] = \frac{1}{s}\int_0^a f(t)\,dt$$

$$\therefore \quad L\left[\int_a^t f(t)\,dt\right] = \frac{L[f(t)]}{s} - \frac{1}{s}\int_0^a f(t)\,dt$$

WORKED EXAMPLES

EXAMPLE 1

Find the Laplace transform of $\int_0^t te^{-t}\sin t\,dt$.

Solution.

$$L\left[\int_0^t te^{-t}\sin t\,dt\right]=\frac{L(te^{-t}\sin t)}{s} \qquad \text{[by theorem 19.4]} \qquad (1)$$

Let $f(t)=e^{-t}\sin t$

$$\therefore \quad F(s)=L[f(t)]=L[e^{-t}\sin t]=L[\sin t]_{s\to s+1}=\frac{1}{(s+1)^2+1}$$

$$\therefore \quad L[te^{-t}\sin t]=L[tf(t)]=-\frac{dF(s)}{ds}=-\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=-\left[\frac{(-1)\cdot 2(s+1)}{[(s+1)^2+1]^2}\right]=\frac{2(s+1)}{[(s+1)^2+1]^2}=\frac{2(s+1)}{[s^2+2s+2]^2}$$

Substituting in (1) we get,

$$L\left[\int_0^t te^{-t}\sin t\,dt\right]=\frac{1}{s}\cdot\frac{2(s+1)}{(s^2+2s+2)^2}$$

EXAMPLE 2

Find $L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right]$.

Solution.

$$L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right]=\frac{1}{s}L\left[\frac{e^{-t}\sin t}{t}\right] \qquad \text{[by theorem 19.4]}$$

But $$L\left[\frac{e^{-t}\sin t}{t}\right]=\int_s^\infty L[e^{-t}\sin t]ds \qquad \text{[by theorem 19.2]}$$

$$=\int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$=[\tan^{-1}(s+1)]_s^\infty=\tan^{-1}\infty-\tan^{-1}(s+1)=\frac{\pi}{2}-\tan^{-1}(s+1)=\cot^{-1}(s+1)$$

$$\therefore \quad L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right]=\frac{\cot^{-1}(s+1)}{s}$$

EXAMPLE 3

Find $L\left[\int_0^t \frac{\sin x}{x}dx + te^{-t}\cos^2 2t\right]$.

Solution.

$$L\left[\int_0^t \frac{\sin x}{x}dx + te^{-t}\cos^2 2t\right] = L\left[\int_0^t \frac{\sin x}{x}dx\right] + L[te^{-t}\cos^2 2t] \quad (1)$$

[By linearity property]

Now $$L\left[\int_0^t \frac{\sin x}{x}dx\right] = L\left[\int_0^t \frac{\sin t}{t}dt\right]$$

$$= \frac{1}{s}L\left[\frac{\sin t}{t}\right] = \frac{\cot^{-1}s}{s}$$ [example 6, page 19.15]

$$L[te^{-t}\cos^2 2t] = L[e^{-t}t\cos^2 2t]$$

$$= L[t\cos^2 2t]_{s\to s+1} = \left[-\frac{d}{ds}L(\cos^2 2t)\right]_{s\to s+1}$$

Let $$f(t) = \cos^2 2t = \frac{1+\cos 4t}{2}$$

$$\therefore \quad L[f(t)] = L\left[\frac{1+\cos 4t}{2}\right] = \frac{1}{2}[L[1]+L[\cos 4t]] = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+16}\right]$$

$$\therefore \quad L[t\cos^2 2t] = L[tf(t)]$$

$$\therefore \quad L[t\cos^2 2t] = -\frac{d}{ds}L[f(t)]$$

$$= -\frac{d}{ds}\left[\frac{1}{2}\left\{\frac{1}{s}+\frac{s}{s^2+16}\right\}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+16)\cdot 1 - s\cdot 2s}{(s^2+16)^2}\right] = \frac{1}{2}\left[\frac{1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right]$$

$$\therefore \quad L[te^{-t}\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{1}{(s+1)^2}+\frac{(s+1)^2-16}{[(s+1)^2+16]^2}\right] = \frac{1}{2}\left[\frac{1}{(s+1)^2}+\frac{s^2+2s-15}{(s^2+2s+17)^2}\right]$$

Substituting in (1), we get

$$L\left[\int_0^t \frac{\sin x}{x}dx + te^{-t}\cos^2 2t\right] = \frac{\cot^{-1}s}{s} + \frac{1}{2}\left[\frac{1}{(s+1)^2}+\frac{s^2+2s-15}{(s^2+2s+17)^2}\right].$$

19.5.1 Evaluation of Improper Integrals Using Laplace Transform

For the definition of Improper integrals, refer Chapter 7.

WORKED EXAMPLES

EXAMPLE 4

Evaluate $\int_0^\infty e^{-2t} t \sin 3t \, dt$.

Solution.

Let $$I = \int_0^\infty e^{-2t} t \sin 3t \, dt.$$

If $$f(t) = t \sin 3t,$$

then $$I = \int_0^\infty e^{-2t} f(t) dt = [F(s)]_{s=2} = F(2),$$ [by definition of Laplace transform]

and $$F(s) = L[f(t)] = L[t \sin 3t]$$

$$= -\frac{d}{ds} L[\sin 3t] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\left[\frac{3(-1)}{(s^2+9)^2} \cdot 2s\right] = \frac{6s}{(s^2+9)^2}$$

$\therefore$ $$F(2) = \frac{6 \times 2}{(2^2+9)^2} = \frac{12}{169} \quad \therefore \quad I = F(2) = \frac{12}{169}$$

EXAMPLE 5

Evaluate $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt$.

Solution.

Let $$I = \int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt$$

If $f(t) = \frac{\sin^2 t}{t}$, then $$I = \int_0^\infty e^{-t} f(t) dt = [F(s)]_{s=1} = F(1),$$ [by definition of L. T]

But $$F(s) = L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L[\sin^2 t] ds$$

$$= \int_s^\infty L\left[\frac{1-\cos 2t}{2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left[\log_e s - \frac{1}{2}\log_e (s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[2\log_e s - \log_e(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log_e s^2 - \log_e(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log_e\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log_e \frac{1}{\left(1+\frac{4}{s^2}\right)}\right]_s^\infty = \frac{1}{4}\left[\log_e 1 - \log_e \frac{1}{1+\frac{4}{s^2}}\right] = -\frac{1}{4}\log_e \frac{s^2}{s^2+4}$$

$$\Rightarrow \quad F(s) = \frac{1}{4}\log_e \frac{s^2+4}{s^2} \quad \text{and} \quad F(1) = \frac{1}{4}\log_e(1+4) = \frac{1}{4}\log_e 5$$

$$\therefore \quad I = F(1) = \frac{\log_e 5}{4}$$

EXAMPLE 6

Evaluate $\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right)dt.$

Solution.

Let $$I = \int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right)dt$$

If $f(t) = \frac{1-\cos t}{t}$, then $I = \int_0^\infty e^{-t} f(t)\,dt = [F(s)]_{s=1} = F(1)$

But $$F(s) = L[f(t)] = L\left[\frac{1-\cos t}{t}\right]$$

$$= \int_s^\infty L[1-\cos t]\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]ds$$

$$= \left[\log_e s - \frac{1}{2}\log_e(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log_e\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]_s^\infty = \frac{1}{2}\left[\log_e 1 - \log_e\left(\frac{1}{1+\frac{1}{s^2}}\right)\right] = -\frac{1}{2}\log_e \frac{s^2}{s^2+1}$$

$\Rightarrow \qquad F(s) = \frac{1}{2}\log_e \frac{s^2+1}{s^2} \quad \text{and} \quad F(1) = \frac{1}{2}\log_e\left(\frac{1+1}{1}\right) = \frac{1}{2}\log_e 2$

$\therefore \qquad I = F(1) = \frac{1}{2}\log_e 2$

EXAMPLE 7

Given $L[\sin\sqrt{t}\,] = \frac{\sqrt{\pi}e^{-1/4s}}{2s^{3/2}}$, **show that** $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}e^{-1/4s}$.

Solution.

Given $\qquad L[\sin\sqrt{t}\,] = \frac{\sqrt{\pi}e^{-1/4s}}{2s^{3/2}}$

If $\qquad f(t) = \sin\sqrt{t}$

then $f(0) = 0$ and $f'(t) = \cos\sqrt{t}\cdot\frac{1}{2\sqrt{t}} = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$

We know $\qquad L[f'(t)] = s\,L\,[f(t)] - f(0)$

$\Rightarrow \qquad L\left[\frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = s\,L\,[\sin\sqrt{t}\,] - 0$

$\Rightarrow \qquad \frac{1}{2}L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = s\cdot\frac{\sqrt{\pi}e^{-1/4s}}{2s^{3/2}} \Rightarrow L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{s^{1/2}}e^{-1/4s} = \sqrt{\frac{\pi}{s}}e^{-1/4s}$

19.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS AND OTHER SPECIAL TYPE OF FUNCTIONS

Mathematical representation of physical quantities whose values repeat periodically give rise to periodic functions.

Definition 19.2 A real function $f(t)$ is said to be periodic if there exists a positive constant T such that $f(t+T) = f(t)$ for all values of t. The smallest such T is called the period of the function or fundamental period.

For example: $\sin(t+2\pi) = \sin t, \quad \sin(t+4\pi) = \sin t, \quad \sin(t+6\pi) = \sin t \quad \forall t \in \mathbb{R}$

$\therefore$ 2π is the smallest number which satisfies $\sin(t+T) = \sin t \quad \forall t$.

So, the period of $\sin t$ is 2π.

Note It is clear that $\qquad f(t+T) = f(t)\,\forall t$

$\Rightarrow \qquad f(t+2T) = f(t+T+T) = f(t+T) = f(t)$ and so on.

In general, $f(t+n\text{T}) = f(t) \quad \forall t$, where n is an integer positive or negative.

The advantage of periodic function is that it is enough we study it in an interval of length T, because copies of it only will be available in all other intervals of length T. So, the Laplace transform of $f(t)$ will be expressed as an integral of $e^{-st}f(t)$ over the interval $(0, T)$ of length T.

Theorem 19.5 Laplace transform of periodic function

If $f(t)$ is a periodic function with period T, then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Proof Since $f(t)$ is periodic with period T, we have

$$f(t) = f(t+T) = f(t+2T) = \cdots = f(t+nT) \text{ for all } t$$

By definition,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Put $t = u + T$ in the second integral

$t = u + 2T$ in the third integral and so on.

Then $dt = du$

In the second integral, when $t = T$, $u = 0$ and when $t = 2T$, $u = T$

In the third integral, when $t = 2T$, $u = 0$ and when $t = 3T$, $u = T$ and so on

$$\therefore \quad L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du + e^{-3sT}\int_0^T e^{-su} f(u)\,du + \cdots$$

$$= [1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots]\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

[$1 + e^{-sT} + e^{-2sT} + \cdots$ is a infinite geometric series with C.R e^{-sT}] ■

Note If $f(x)$ is periodic with period T, then $f(ax+b)$ is periodic with period $\dfrac{T}{a}$ if $a > 0$.

WORKED EXAMPLES

EXAMPLE 1

Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t \le \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases} \quad \text{with period } \frac{2\pi}{\omega}.$$

Solution.

Here $f(t)$ is given in the interval $\left(0, \dfrac{2\pi}{\omega}\right)$ and $f(t)$ is periodic with period $T = \dfrac{2\pi}{\omega}$ because $\sin t$ is periodic with period 2π.

$$\therefore \qquad L[f(t)] = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}} \qquad \text{[by theorem 19.5]}$$

$$= \frac{\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt}{1 - e^{\frac{-2\pi s}{\omega}}}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} \cdot 0\, dt \right]$$

$$= \frac{\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt}{1 - e^{\frac{-2\pi s}{\omega}}}$$

$$= \frac{\left\{ \dfrac{e^{-st}}{s^2 + \omega^2} [-s \sin \omega t - \omega \cos \omega t] \right\}_0^{\frac{\pi}{\omega}}}{1 - e^{\frac{-2\pi s}{\omega}}}$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{\frac{-\pi s}{\omega}}}{s^2 + \omega^2} \left\{ -s \sin\left(\omega \frac{\pi}{\omega}\right) - \omega \cos\left(\omega \frac{\pi}{\omega}\right) \right\} - \frac{e^0}{s^2 + \omega^2} \{ -s \sin 0 - \omega \cos 0 \} \right]$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega e^{-\frac{s\pi}{\omega}}}{s^2 + \omega^2} + \frac{\omega}{s^2 + \omega^2} \right]$$

$$= \frac{\omega\left[1+e^{-\frac{s\pi}{\omega}}\right]}{(s^2+\omega^2)\left[1-e^{-\frac{s\pi}{\omega}}\right]\left[1+e^{-\frac{s\pi}{\omega}}\right]} = \frac{\omega}{(s^2+\omega^2)[1-e^{-\frac{s\pi}{\omega}}]} = \frac{\omega}{(s^2+\omega^2)\left(1-e^{\frac{-s\pi}{\omega}}\right)}$$

$$\left[\because \quad e^{\frac{-2s\pi}{\omega}} = \left(e^{\frac{-\pi s}{\omega}}\right)^2 \text{ and } a^2-b^2=(a-b)(a+b)\right]$$

Note This function is called the half-sine wave rectifier function of period $\dfrac{2\pi}{\omega}$.

EXAMPLE 2

Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases} \quad \text{and} \quad f(t+2a) = f(t).$$

Solution.

Given $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$ and $f(t)$ is of period $2a$.

$\therefore$ by theorem 19.5,

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} = \frac{\int_0^{2a} e^{-st} f(t)\,dt}{1-e^{-2as}}$$

$$= \frac{\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt}{1-e^{-2as}}$$

$$= \frac{\int_0^a e^{-st}\, t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt}{1-e^{-2as}}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{te^{-st}}{-s} - 1\cdot\frac{e^{-st}}{(-s)^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - \frac{(-1)\cdot e^{-st}}{(-s)^2}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[-\left(\frac{a\cdot e^{-as}}{s} + \frac{e^{-as}}{s^2}\right) + \left(0 + \frac{e^0}{s^2}\right)\right]\right.$$

$$\left. + \left[\left(0 + \frac{e^{-2as}}{s^2}\right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{2e^{-sa}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\frac{(1-e^{-as})^2}{s^2} = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{1}{s^2}\tan h\frac{as}{2}.$$

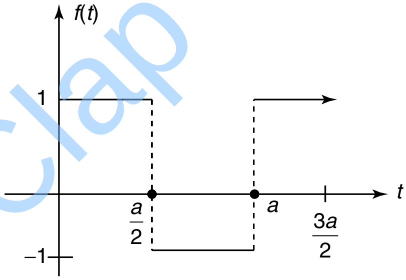
Note This function is called the triangular wave function of period $2a$.

EXAMPLE 3

Find the Laplace transform of the square-wave function (or Meoander function) of period a defined as $f(t) = \begin{cases} 1 & \text{when } 0 < t \le \frac{a}{2} \\ -1 & \text{when } \frac{a}{2} < t < a \end{cases}$.

Solution.

Given $$f(t) = \begin{cases} 1, & \text{if } 0 < t < \frac{a}{2} \\ -1, & \text{if } \frac{a}{2} < t < a \end{cases}$$



$\therefore$ $f(t)$ is of period $T = a$

By theorem 19.5

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} = \frac{\int_0^{\frac{a}{2}} e^{-st} f(t)\,dt + \int_{\frac{a}{2}}^{a} e^{-st} f(t)\,dt}{1-e^{-sa}}$$

$$= \frac{\int_0^{\frac{a}{2}} e^{-st}\cdot 1\,dt + \int_{\frac{a}{2}}^{a} e^{-st}(-1)\,dt}{1-e^{-sa}}$$

$$= \frac{1}{[1-e^{-sa}]}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{a}\right\}$$

$$= \frac{1}{[1-e^{-sa}]}\left[\frac{-e^{\frac{-sa}{2}}}{s} + \frac{e^0}{s} + \frac{e^{-sa}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{s(1-e^{-sa})}[1 - 2e^{\frac{-sa}{2}} + e^{-sa}]$$

$$= \frac{\left[1-e^{\frac{-sa}{2}}\right]^2}{s\left[1-e^{\frac{-sa}{2}}\right]\left[1+e^{\frac{-sa}{2}}\right]} = \frac{1-e^{-\frac{sa}{2}}}{s[1+e^{-\frac{sa}{2}}]} = \frac{1}{s}\tan h\frac{sa}{4}$$

Theorem 19.6 Initial value theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$,

then
$$\lim_{t\to 0+} f(t) = \lim_{s\to} s\,F(s).$$

Proof We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$\Rightarrow \quad \int_0^\infty e^{-st} f'(t)\,dt = s\,F(s) - f(0)$$

$$\therefore \quad \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to\infty} s\,F(s) - f(0)$$

$$\int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\,dt = \lim_{s\to\infty} s\,F(s) - f(0)$$

Since t is independent of s, $\lim_{s\to\infty} e^{-st} = 0 \quad \therefore \quad \int_0^\infty \lim_{s\to\infty} e^{-st} f'(t)\,dt = 0$.

$$\therefore \quad f(0) = \lim_{s\to\infty} s\,F(s)$$

$$\Rightarrow \quad \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$ [Since $f(t)$ is continuous on the right at 0, $\lim_{t\to 0+} f(t) = f(0)$] ■

Theorem 19.7 Final value theorem

If the Laplace transform of $f(t)$ and $f'(t)$ exist and $F(s) = L[f(t)]$, then
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s).$$

Proof We know that $L\,[f'(t)] = s\,L\,[f(t)] - f(0)$

$$\Rightarrow \quad \int_0^\infty e^{-st} f'(t)dt = s\,F\,(s) - f(0)$$

$$\therefore \quad \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to 0} (s\,F(s) - f(0))$$

$$\Rightarrow \quad \int_0^\infty \lim_{s\to 0} e^{-st} f'(t)\,dt = \lim_{s\to 0} s\,F(s) - f(0)$$

$$\Rightarrow \quad \int_0^\infty f'(t)\,dt = \lim_{s\to 0} s\,F(s) - f(0)$$

$$\Rightarrow \quad [f(t)]_0^\infty = \lim_{s\to 0} s\,F(s) - f(0)$$

$$\Rightarrow \quad \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} s\,F(s) - f(0)$$

$$\Rightarrow \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s) \qquad \blacksquare$$

Note Since $f(t)$ is defined for all $t \in [0, \infty)$, $f(0)$ is the first value or initial value of $f(t)$ and $\lim_{t\to\infty} f(t)$ is the final value of $f(t)$.

WORKED EXAMPLES

EXAMPLE 4

Verify initial value theorem for the function $f(t) = e^{-t} \sin t$.

Solution.

Initial value theorem is $\quad \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

Here $\quad f(t) = e^{-t} \sin t$

$$\therefore \quad F(s) = L[f(t)] = L[e^{-t} \sin t] = \frac{1}{(s+1)^2 + 1}$$

Now $\quad \text{L.H.S.} = \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-t} \sin t = e^0 \cdot 0 = 0$

and $\quad \text{R.H.S.} = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{(s+1)^2 + 1} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{s\to\infty} \frac{1}{2(s+1)} = 0 \qquad \text{[by L'Hopital's rule]}$$

$\therefore \quad$ L.H.S = R.H.S.

Hence the theorem is verified.

EXAMPLE 5

Verify final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

Solution.

Final value theorem is $\quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

Given $\quad f(t) = 1 + e^{-t}(\sin t + \cos t)$

$$\therefore \quad \lim_{t\to\infty} f(t) = \lim_{t\to\infty} [1 + e^{-t}(\sin t + \cos t)] = 1 + 0 = 1$$

$$\begin{aligned} F(s) &= L[f(t)] \\ &= L[1 + e^{-t}(\sin t + \cos t)] \\ &= L[1] + L[e^{-t} \sin t] + L[e^{-t} \cos t] \\ &= \frac{1}{s} + \frac{1}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2 + 1} \end{aligned}$$

$\therefore$
$$\lim_{s\to 0} s\,F(s) = \lim_{s\to 0} s\left[\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}\right]$$
$$= \lim_{s\to 0}\left[1+\frac{s}{(s+1)^2+1}+\frac{s^2+s}{(s+1)^2+1}\right] = 1$$

$\therefore$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

Hence the theorem is verified.

EXAMPLE 6

Verify the initial and final value theorems for $f(t) = e^{-t}(t+2)^2$.

Solution.

(1) Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

Here
$$f(t) = e^{-t}(t+2)^2$$

$\therefore$
$$F(s) = L[f(t)] = L[e^{-t}(t^2+4t+4)]$$
$$= L[e^{-t}t^2] + 4L[e^{-t}t] + 4L\,[e^{-t}]$$
$$= \frac{2}{(s+1)^3} + 4\cdot\frac{1}{(s+1)^2} + \frac{4}{s+1}$$

Now
$$\text{L.H.S.} = \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-t}(t+2)^2 = e^0(0+2)^2 = 4$$

and
$$\text{R.H.S.} = \lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3}+\frac{4s}{(s+1)^2}+\frac{4s}{s+1}\right]$$

But
$$\lim_{s\to\infty}\frac{2s}{(s+1)^3} = \lim_{s\to\infty}\frac{2}{3(s+1)^2} = 0 \qquad \text{[By L'Hopital's rule]}$$

$$\lim_{s\to\infty}\frac{4s}{(s+1)^2} = \lim_{s\to\infty}\frac{4}{2(s+1)} = 0 \qquad \text{[By L'Hopital's rule]}$$

$$\lim_{s\to\infty}\frac{4s}{s+1} = \lim_{s\to\infty}\frac{4}{1+\frac{1}{s}} = 4$$

Hence,
$$\text{R.H.S.} = \lim_{s\to\infty} s\,F(s) = 0+0+4 = 4$$

$\therefore$
$$\text{L.H.S.} = \text{R.H.S.}$$

Hence, the initial value theorem is verified.

(2) Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$.

Now L.H.S. $= \lim_{t\to\infty} e^{-t}(t+2)^2 = \lim_{t\to\infty} \frac{(t+2)^2}{e^t}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{t\to\infty} \frac{2(t+2)}{e^t}$ [By L'Hopital's rule]

$= \lim_{t\to\infty} \frac{2}{e^t} = 0$ [By L'Hopital's rule]

and R.H.S. $= \lim_{s\to 0} s\,F(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{(s+1)}\right] = 0$

$\therefore$ L.H.S. = R.H.S.

Hence, final value theorem is verified.

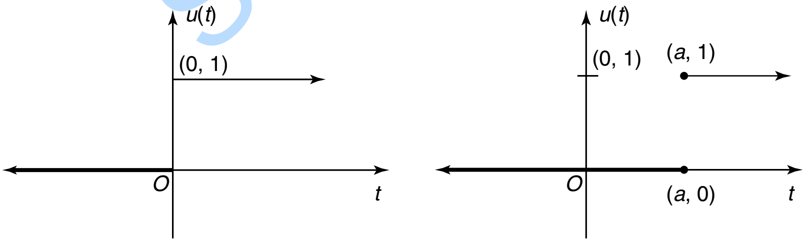
Definition 19.3 Unit step function or Heaviside function

The unit step function u is defined by $u(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$

This function has jump discontinuity at $t = 0$

More generally, if a is any positive number, then $u(t-a)$ is the unit step function shifted a units to the right and is defined as

$$u(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a \end{cases}$$



$u(t-a)$ is also denoted by $u_a(t)$

Note The unit step function may be thought of as a flat signal of magnitude 1 or a flat switching function. It is also called **Heaviside function in honour of the British electrical engineer Oliver Heaviside**.

19.6.1 Laplace Transform of Unit Step Function

By definition, $$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st}\,dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}[e^{-\infty} - e^{-as}] = \frac{e^{-as}}{s} \quad \text{if } s > 0$$

$\therefore$ $$\mathbf{L[u(t-a)] = \frac{e^{-as}}{s} \quad \text{if } s > 0}$$

Theorem 19.8 Second shifting property (or *t*-shifting)

If $L[f(t)] = F(s)$, then $L[f(t-a)u(t-a)] = e^{-as}F(s)$

Proof The unit step function is

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

$\therefore$ $$f(t-a)u(t-a) = \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t \geq a \end{cases}$$

$\therefore$ $$L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t - a = x \quad \therefore \quad dt = dx$

When $t = a$, $x = 0$ and when $t = \infty$, $x = \infty$

$\therefore$ $$L[f(t-a)u(t-a)] = \int_0^\infty e^{-s(x+a)} f(x)\,dx = e^{-sa}\int_0^\infty e^{-sx} f(x)\,dx = e^{-sa}F(s)$$

$\therefore$ $$L[f(t-a)u(t-a)] = e^{-sa}L[f(t)]$$ ■

Note The above property can also be stated without using the name unit step function as below.

If $\quad L[f(t)] = F(s)$ and $G(t) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a \end{cases}$

then $\quad L[G(t)] = e^{-as} F(s) = e^{-as} L[f(t)]$

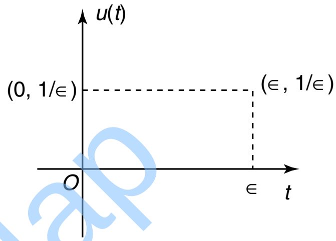
19.6.2 Unit Impulse Function

Many engineering applications involve the concept of an impulse, which may be considered as a very large force acting for a very short time.

For example: the force due to a hammer blow.

We can model an impulse as below in terms of unit step function.

For any positive number $\in$, the impulse function $\delta_\in$ is defined by



$$\delta_\in(t) = \begin{cases} \frac{1}{\in}[u(t) - u(t-\in)], & 0 \leq t < \in \\ 0, & \text{otherwise} \end{cases}$$

This is a pulse of magnitude $\frac{1}{\in}$ and duration $\in$.

Note that the area of the graph representing the pulse is $\frac{1}{\in} . \in = 1$, which remains a constant 1 as $\in \to 0$.

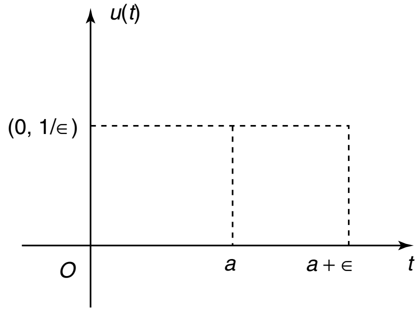
19.6.3 Dirac-delta Function

The function $\delta_\in(t)$ obtained by taking $\in \to 0$ is called the **Dirac delta function** and denoted by $\delta(t)$.

Thus $$\delta(t) = \lim_{\in \to 0+} \delta_\in(t)$$

In other words $$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$

$$\left[\in \to 0 \Rightarrow \frac{1}{\in} \to \infty\right]$$



The shifted delta function is $\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$

19.6.4 Laplace Transform of Delta Function

We have $$\delta_\in(t-a) = \frac{1}{\in}[u(t-a) - u(t-a-\in)] \quad \text{if } a \leq t \leq a+\in$$

Then $$L[\delta_\in(t-a)] = \frac{1}{\in} L[u(t-a) - u(t-a-\in)]$$

$$= \frac{1}{\in}[L[u(t-a)] - L[t-(a+\in)]]$$

$$= \frac{1}{\in}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\in)s}}{s}\right] = \frac{e^{-as}(1-e^{-\in s})}{\in s}$$

As $\in \to 0$, $$L[\boldsymbol{\delta}(t-a)] = e^{-as} \lim_{\in \to 0+}\left[\frac{1-e^{-\in s}}{\in s}\right]$$

i.e. $$L[\boldsymbol{\delta}(t-a)] = e^{-as} \qquad \left[\because \lim_{x\to 0}\frac{1-e^{-x}}{x}=1\right]$$

In particular if $a = 0$, then we get $L[\boldsymbol{\delta}(t)] = e^0 = 1$

An important property of Dirac-delta function is $\int_0 f(t)\delta(t-a)\,dt = f(a)$

Note Paul Dirac (1902–84) was a British physicist who was awarded the noble prize in 1933 for his work in quantum mechanics.

In mechanical problems delta function is used to represent an impulse. In electrical systems it is used to represent the application of a very large voltage for a short time or the sudden discharge of energy contained in a capacitor. This function is applied to study the behaviour of circuits subjected to transients like high input voltage. Sometimes the system may break down. So, before a circuit is built, transients are modeled by the delta function and their effects on the circuit are studied to set safety standards for the system to be fabricated.

WORKED EXAMPLES

EXAMPLE 7

Find $L[(t-1)^2u(t-1)]$.

Solution.

Here $$f(t) = t^2$$

$\therefore$ $$f(t-1) = (t-1)^2 \quad \text{and} \quad a = 1$$

$\therefore$ $$L[(t-1)^2u(t-1)] = e^{-as}L[t^2] = e^{-s}\cdot\frac{2}{s^3} = \frac{2e^{-s}}{s^3} \qquad \text{[By theorem 19.8]}$$

EXAMPLE 8

Find $L[e^{-4t}u(t-1)]$.

Solution.

$$L[e^{-4t}u(t-1)] = L[e^{-4(t-1+1)}u(t-1)] = e^{-4}L[e^{-4(t-1)}u(t-1)]$$

Here $$f(t) = e^{-4t} \text{ and } a = 1$$

$\therefore$ $$L[e^{-4(t-1)}u(t-1)] = e^{-s}L[e^{-4t}] = e^{-s}\frac{1}{s+4} \qquad \text{[By theorem 19.8]}$$

$\therefore$ $$L[e^{-4t}u(t-1)] = e^{-4}\cdot e^{-s}\cdot\frac{1}{s+4} = \frac{e^{-(4+s)}}{s+4}$$

EXAMPLE 9

Find $L\left[(\sin t)u\left(t-\frac{\pi}{2}\right)\right]$.

Solution.

$$L\left[(\sin t)u\left(t-\frac{\pi}{2}\right)\right]=L\left[\sin\left\{t-\frac{\pi}{2}+\frac{\pi}{2}\right\}\left\{u\left(t-\frac{\pi}{2}\right)\right\}\right]$$

$$=L\left[\cos\left(t-\frac{\pi}{2}\right)\left\{u\left(t-\frac{\pi}{2}\right)\right\}\right]$$

$$=e^{-\frac{\pi}{2}s}L[\cos t] \qquad \left[\because\ a=\frac{\pi}{2}\right]\text{[Using theorem 19.8]}$$

$$=e^{-\frac{\pi s}{2}}\cdot\frac{s}{s^2+1}=\frac{se^{-\frac{\pi s}{2}}}{s^2+1}$$

EXAMPLE 10

Find the Laplace transform of $f(t)=\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$.

Solution.

We shall rewrite $f(t)$ in terms of unit-step function.

$$f(t)=(t-1)[u(t-1)-u(t-2)]+(3-t)[u(t-2)-u(t-3)]$$

$$[\because u(t-1)=1,\ u(t-2)=0 \text{ in } 1<t<2;\ u(t-2)=1,\ u(t-3)=0 \text{ in } 2<t<3]$$

$$\therefore \quad f(t)=(t-1)u(t-1)+[3-t-t+1]u(t-2)+(t-3)u(t-3)$$

$$=(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)$$

$$\therefore \quad L[f(t)]=L[(t-1)u(t-1)]-2L[(t-2)u(t-2)]+L[(t-3)u(t-3)]$$

$$=e^{-s}L[t]-2e^{-2s}L[t]+e^{-3s}L[t]$$

$$=e^{-s}\cdot\frac{1}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}=\frac{e^{-s}}{s^2}[1-2e^{-s}+e^{-2s}]=\frac{e^{-s}(1-e^{-s})^2}{s^2}$$

EXERCISE 19.3

Evaluate the following integrals using Laplace transform.

1. $\int_0^\infty \frac{e^{-t}-e^{-3t}dt}{t}$
2. $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$
3. $\int_0^\infty e^{3t}t^3\cos t\,dt$
4. $\int_0^\infty t^3e^{-t}\sin t\,dt$
5. $\int_0^\infty e^{-2t}t\sin 3t\cdot dt$
6. $\int_0^\infty te^{-t}\sin t\,dt$
7. $\int_0^\infty e^{-3t}\,t\sin t\,dt$
8. $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt$
9. $\int_0^\infty \frac{\cos at-\cos bt}{t}dt$
10. $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}dt$

Verify initial and final value theorems for the following functions.

11. $f(t)=1-e^{-at}$
12. $f(t)=t^2e^{-3t}$
13. $f(t)=(2t-3)^2$
14. $f(t)=a\,e^{-bt}$

15. Find the Laplace transform of the square-wave function $f(t)$ given by $f(t)=\begin{cases} K, & 0\le t\le a \\ -K, & a\le t\le 2a \end{cases}$ and $f(t+2a)=f(t)$.
16. Find the Laplace transform of the full-sine wave rectifier function $f(t)=|\sin\omega t|, \quad t\ge 0$.
17. Find the Laplace transform of the periodic saw-tooth wave function given by $f(t)=\frac{\alpha t}{\omega}, 0<t<\omega$ and $f(t+\omega)=f(t)$.
18. Find the Laplace transform of the periodic function defined by the triangular wave $f(t)=\begin{cases} \frac{t}{a}, & 0\le t\le a \\ \frac{2a-t}{a}, & a\le t\le 2a \end{cases}$ and $f(t+2a)=f(t)$.
19. Find the Laplace transform of the periodic function $f(t)=\begin{cases} \cos t, & 0<t<\pi \\ 0, & \pi<t<2\pi \end{cases}$, $f(t+2\pi)=f(t)$.
20. Find the Laplace transform of $f(t)=\begin{cases} t & \text{for } 0<t<\pi \\ \pi-t & \text{for } \pi<t<2\pi \end{cases}$ and $f(t+2\pi)=f(t)$.
21. Find $L\left[e^{-2t}\int_0^t\left(\frac{1-\cos u}{u}\right)du\right]$.
22. Find $L[e^{-t}\int_0^t t^2\cos t\,dt]$.
23. Find $L\left[\int_0^t \frac{e^{-t}\sin t}{t}dt\right]$.
24. Find $L[e^{-t}\int_0^t t\cos t\,dt]]$.
25. Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}dt$.
26. Find $L\left[\frac{\sin at}{t}\right]$ and hence prove that $\int_0^{\frac{\pi}{2}} \frac{\sin t}{t}dt=\frac{\pi}{2}$.
27. Using Laplace transform of derivative formula, prove that $L[t\cos at]=\frac{s^2-a^2}{(s^2+a^2)^2}$.
28. If $L[t\sin\omega t]=\frac{2\omega s}{(s^2+\omega^2)^2}$ evaluate $L[\omega t\cos\omega t+\sin\omega t]$.
29. Find the Laplace transform of the saw-toothed wave function of period T given by $f(t)=\frac{t}{T},\ 0<t<T$.
30. If $F(s)=\frac{3s^2+5s+2}{s^3+4s^2+2s}$, then find $f(0)$ and $f(\infty)$.

ANSWERS TO EXERCISE 19.3

1. $\log_e 3$ 2. $\log_e \frac{2}{3}$ 3. $\frac{168}{10^4}$ 4. 0 5. $\frac{12}{169}$

6. $\frac{1}{2}$ 7. $\frac{3}{50}$ 8. $\log_e 2$ 9. $\log_e \frac{b}{a}$ 10. $\frac{\pi}{3}$

15. $\frac{k}{s}\tanh\left(\frac{as}{2}\right)$ 16. $\frac{\omega}{s^2+\omega^2}\coth\left(\frac{\pi s}{2\omega}\right)$ 17. $\frac{\alpha}{\omega s^2}-\frac{\alpha e^{-\omega s}}{s(1-e^{-\omega s})}$ 18. $\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$

19. $\frac{s}{(s^2+1)(1-e^{-\pi s})}$ 20. $\frac{1}{s^2}\tanh\frac{s\pi}{2}-\frac{\pi e^{-s\pi}}{s(1+e^{-s\pi})}$ 21. $\frac{1}{2(s+2)}\log\left(\frac{s^2+4s+5}{s^2+4s+4}\right)$

22. $\frac{2(s^2+2s-2)}{(s^2+2s+2)^3}$ 23. $\frac{1}{s}\cot^{-1}(s+1)$ 24. $\frac{s(s+2)}{(s+1)(s^2+2s+2)^2}$ 25. $\frac{\cot^{-1}(s+1)}{s+1}$

27. $\frac{(s^2-a^2)}{(s^2+a^2)^2}$ 28. $\frac{2\omega s^2}{(s^2+\omega^2)^2}$ 29. $\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-sT})}$ 30. 3 and 1

19.7 INVERSE LAPLACE TRANSFORMS

Definition 19.4 If the Laplace transform of a function $f(t)$ is $F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is written as $\boldsymbol{L^{-1}[F(s)] = f(t)}$

i.e., if $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$

L^{-1} is called the inverse Laplace transform operator.

For example: $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$, since $L[e^{at}] = \frac{1}{s-a}$

Linearity Property

If $$L[f(t)] = F(s) \text{ and } L[G(t)] = G(s)$$

then $$L^{-1}[aF(s)+bG(s)] = a\,L^{-1}[F(s)] + b\,L^{-1}[G(s)]$$

First shifting property

If $$L[f(t)] = F(s),\ L[e^{-at}f(t)] = F(s+a) \quad \text{and} \quad L[e^{at}f(t)] = F(s-a)$$

then (i) $L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$ (ii) $L^{-1}[F(s-a)] = e^{at}L^{-1}[F(s)]$

INVERSE LAPLACE TRANSFORMS

Definition 19.4 If the Laplace transform of a function $f(t)$ is $F(s)$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is written as $\boldsymbol{L^{-1}[F(s)] = f(t)}$

i.e., if $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$

L^{-1} is called the inverse Laplace transform operator.

For example: $L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$, since $L[e^{at}] = \dfrac{1}{s-a}$

Linearity Property

If $$L[f(t)] = F(s) \text{ and } L[G(t)] = G(s)$$

then $$L^{-1}[aF(s) + bG(s)] = a\,L^{-1}[F(s)] + b\,L^{-1}[G(s)]$$

First shifting property

If $L[f(t)] = F(s)$, $L[e^{-at} f(t)] = F(s+a)$ and $L[e^{at} f(t)] = F(s-a)$

then (i) $L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$ (ii) $L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$

For example: $L^{-1}\left[\frac{s+2}{(s+4)^2}\right]=L^{-1}\left[\frac{s+4-2}{(s+4)^2}\right]=e^{-4t}L^{-1}\left[\frac{s-2}{s^2}\right]=e^{-4t}L^{-1}\left[\frac{1}{s}-\frac{2}{s^2}\right]=e^{-4t}[1-2t]$

We have seen while finding Laplace transforms of most of the functions $F(s)$ are rational algebraic functions. Hence, to find the inverse Laplace transforms, the most important technique is to split $F(s)$ into partial fractions and using the table given below, we get $L^{-1}[F(s)]$.

Many problems given under the other types can also be done by partial fraction method.

The following table is important in finding the inverse Laplace transform.

S.No.	$L[f(t)]=F(s)$	$L^{-1}[F(s)]=f(t)$
1.	$L[1]=\frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right]=1$
2.	$L[t]=\frac{1}{s^2},\ s>0$	$L^{-1}\left[\frac{1}{s^2}\right]=t$
3.	$L[t^n]=\frac{n!}{s^{n+1}},\ n=1,2,3,\ldots$	$L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!},\ n=1,2,3,\ldots$
4.	$L[t^\alpha]=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, α is a real number >-1	$L^{-1}\left[\frac{1}{s^{\alpha+1}}\right]=\frac{t^\alpha}{\Gamma(\alpha+1)}$,
5.	$L[e^{at}]=\frac{1}{s-a},\ s>a$	$L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
6.	$L[e^{-at}]=\frac{1}{s+a},\ s>-a$	$L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$
7.	$L[\sin at]=\frac{a}{s^2+a^2},\ s>0$	$L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$
8.	$L[\cos at]=\frac{s}{s^2+a^2},\ s>0$	$L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$
9.	$L[\sinh at]=\frac{a}{s^2-a^2},\ s>\|a\|$	$L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$
10.	$L[\cosh at]=\frac{s}{s^2-a^2},\ s>\|a\|$	$L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$

19.7.1 Type 1 – Direct and Shifting Methods

WORKED EXAMPLES

EXAMPLE 1

Find $L^{-1}\left[\dfrac{3s+5}{s^2+9}\right]$.

Solution.

$$L^{-1}\left[\frac{3s+5}{s^2+9}\right]=3L^{-1}\left[\frac{s}{s^2+9}\right]+5L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$=3\cos 3t+5\frac{\sin 3t}{3}=\frac{9\cos 3t+5\sin 3t}{3}$$

EXAMPLE 2

Find $L^{-1}\left[\dfrac{s^2-3s+4}{s^3}\right]$.

Solution.

$$L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]=L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s^2}\right]+4L^{-1}\left[\frac{1}{s^3}\right]=1-3t+2t^2$$

EXAMPLE 3

Find $L^{-1}\left[\dfrac{s+3}{s^2-4s+13}\right]$.

Solution.

$$L^{-1}\left[\frac{s+3}{s^2-4s+13}\right]=L^{-1}\left[\frac{s+3}{(s-2)^2-4+13}\right]$$

$$=L^{-1}\left[\frac{s-2+5}{(s-2)^2+9}\right]$$

$$=e^{2t}L^{-1}\left[\frac{s+5}{s^2+9}\right] \qquad \text{[by shifting property]}$$

$$=e^{2t}\left\{L^{-1}\left[\frac{s}{s^2+9}\right]+5L^{-1}\left[\frac{1}{s^2+9}\right]\right\}$$

$$=e^{2t}\left[\cos 3t+5.\frac{\sin 3t}{3}\right]=\frac{e^{2t}}{3}[3\cos 3t+5\sin 3t]$$

EXAMPLE 4

Find $L^{-1}\left[\dfrac{s}{(s+6)^3}\right]$.

Solution.

$$L^{-1}\left[\frac{s}{(s+6)^3}\right]=L^{-1}\left[\frac{s+6-6}{(s+6)^3}\right]$$

$$= e^{-6t} L^{-1}\left[\frac{s-6}{s^3}\right] = e^{-6t}\left\{L^{-1}\left[\frac{1}{s^2}\right] - 6L^{-1}\left[\frac{1}{s^3}\right]\right\}$$

$$= e^{-6t}\left\{t - 6\frac{t^2}{2!}\right\} = e^{-6t}[t-3t^2] = te^{-6t}(1-3t)$$

19.7.2 Type 2 – Partial Fraction Method

WORKED EXAMPLES

EXAMPLE 5

Find $L^{-1}\left[\dfrac{s+2}{s(s+4)(s+9)}\right]$.

Solution.

Given $$F(s) = \frac{s+2}{s(s+4)(s+9)}$$

Splitting into partial fractions, we get

$$\frac{s+2}{s(s+4)(s+9)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+9}$$

$\Rightarrow$ $$s+2 = A(s+4)(s+9) + Bs(s+9) + Cs(s+4)$$

Put $s = 0$. $\therefore$ $A(0+4)(0+9) = 2 \Rightarrow 36A = 2 \Rightarrow A = \dfrac{1}{18}$

Put $s = -4$. $\therefore$ $B(-4)(-4+9) = -4+2 \Rightarrow -20B = -2 \Rightarrow B = \dfrac{1}{10}$

Put $s = -9$. $\therefore$ $C(-9)(-9+4) = -9+2 \Rightarrow 45C = -7 \Rightarrow C = -\dfrac{7}{45}$

$\therefore$ $$F(s) = \frac{1}{18}\cdot\frac{1}{s} + \frac{1}{10}\cdot\frac{1}{s+4} - \frac{7}{45}\cdot\frac{1}{s+9}$$

$\therefore$ $$L^{-1}[F(s)] = \frac{1}{18}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{10}L^{-1}\left[\frac{1}{s+4}\right] - \frac{7}{45}L^{-1}\left[\frac{1}{s+9}\right]$$

$$= \frac{1}{18} + \frac{e^{-4t}}{10} - \frac{7e^{-9t}}{45}$$

EXAMPLE 6

Find $L^{-1}\left[\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right]$.

Solution.

Given $$F(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$$

For s^2+2s+5, $b^2-4ac = 4-20<0$ So, it cannot be factorised into real factors.

Splitting into partial fractions, we get

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$\Rightarrow$ $$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$

Put $s=1$. $\therefore$ $A(1+2+5) = 5+3 \Rightarrow 8A = 8 \Rightarrow A = 1$

Put $s=0$. $\therefore$ $5A - C = 3 \Rightarrow C = -3+5A = -3+5 = 2.$

Equating, the coefficients of s^2 on both sides, we get $A+B=0 \Rightarrow B = -A = -1$

$$\therefore \quad F(s) = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5} = \frac{1}{s-1} - \frac{s-2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s-2}{(s+1)^2+4} = \frac{1}{s-1} - \frac{s+1-3}{(s+1)^2+4}$$

$$\therefore \quad L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s+1-3}{(s+1)^2+4}\right]$$

$$= e^t - e^{-t} L^{-1}\left\{\frac{s-3}{s^2+4}\right\}$$

$$= e^t - e^{-t}\left\{L^{-1}\left[\frac{s}{s^2+4}\right] - 3L^{-1}\left[\frac{1}{s^2+4}\right]\right\}$$

$$= e^t - e^{-t}\left[\cos 2t - \frac{3\sin 2t}{2}\right] = e^t - \frac{e^{-t}}{2}[2\cos 2t - 3\sin 2t]$$

EXAMPLE 7

Find $L^{-1}\left[\dfrac{4s+5}{(s-1)^2(s+2)}\right]$.

Solution.

Given $$F(s) = \frac{4s+5}{(s-1)^2(s+2)}$$

Splitting into partial fractions, we get

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$\Rightarrow$ $$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

Put $s = 1$. $\therefore$ $B(1+2) = 4+5 \Rightarrow B = \frac{9}{3} = 3$

Put $s = -2$. $\therefore$ $C(-2-1)^2 = -8+5 \Rightarrow 9C = -3 \Rightarrow C = -\frac{1}{3}$

Equating coefficients of s^2, we get $A + C = 0 \Rightarrow A = -C = \frac{1}{3}$

$\therefore$
$$F(s) = \frac{1}{3}\cdot\frac{1}{(s-1)} + 3\cdot\frac{1}{(s-1)^2} - \frac{1}{3}\cdot\frac{1}{(s+2)}$$

$$L^{-1}[F(s)] = \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{3}e^t + 3e^t L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^t + 3e^t\cdot t - \frac{1}{3}e^{-2t} = \frac{e^t}{3}(1+9t) - \frac{e^{-2t}}{3}$$

EXAMPLE 8

Find $L^{-1}\left[\frac{s^2+16}{(s^2+1)(s^2+4)}\right]$.

Solution.

Given
$$F(s) = \frac{s^2+16}{(s^2+1)(s^2+4)}$$

Since there is no odd powers of s, we can regard $F(s)$ as a function of s^2 and write the special partial fraction treating s^2 as x.

$\therefore$
$$\frac{x+16}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

$\Rightarrow$
$$x+16 = A(x+4) + B(x+1)$$

Put $x = -1$. $\therefore$ $A(-1+4) = -1+16 \Rightarrow 3A = 15 \Rightarrow A = 5$

Put $x = -4$. $\therefore$ $B(-4+1) = -4+16 \Rightarrow -3B = 12 \Rightarrow B = -4$

$\therefore$
$$\frac{x+16}{(x+1)(x+4)} = \frac{5}{x+1} + \frac{-4}{x+4}$$

$\Rightarrow$
$$\frac{s^2+16}{(s^2+1)(s^2+4)} = \frac{5}{s^2+1} - \frac{4}{s^2+4}$$

$\Rightarrow$
$$F(s) = \frac{5}{s^2+1} - \frac{4}{s^2+4}$$

$\therefore$
$$L^{-1}[F(s)] = 5L^{-1}\left[\frac{1}{s^2+1}\right] - 4L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 5\sin t - 4\cdot\frac{1}{2}\sin 2t = 5\sin t - 2\sin 2t$$

EXAMPLE 9

Find $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$.

Solution.

Given
$$\frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2)^2 - 2s^2.2a^2} \qquad [\because\ a^2+b^2=(a+b)^2-2ab]$$
$$= \frac{s}{(s^2+2a^2)^2-(2sa)^2}$$
$$= \frac{s}{(s^2+2a^2+2sa)(s^2+2a^2-2sa)}$$
$$= \frac{s}{(s^2+2sa+2a^2)(s^2-2sa+2a^2)}$$

Splitting into partial fractions, we get
$$\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$
$$\therefore \qquad s = (As+B)[s^2-2as+2a^2]+(Cs+D)[s^2+2as+2a^2]$$

Equating coefficients of s^3, s^2, s and constant terms on both sides, we get
$$A+C=0 \quad \Rightarrow \quad C=-A \qquad (1)$$
$$-2aA+B+2aC+D=0 \quad \Rightarrow \quad 2a(C-A)+B+D=0 \qquad (2)$$
$$2a^2A-2aB+2a^2C+2aD=1$$
$$\Rightarrow \quad 2a^2(A+C)+2a(D-B)=1 \quad \Rightarrow \quad 2a(D-B)=1 \qquad \text{[using (1)]} \quad (3)$$
and
$$2a^2B+2a^2D=0 \quad \Rightarrow \quad B+D=0 \quad \Rightarrow \quad D=-B \qquad (4)$$
$$\therefore \quad 2a[-B-B]=1 \quad \Rightarrow \quad -4aB=1 \quad \Rightarrow \quad B=-\frac{1}{4a} \qquad [\because\ D=-B]$$
$$\therefore \quad D=\frac{1}{4a}$$

Substituting $C=-A$ and $B+D=0$ in (2), we get
$$2a(-A-A)=0 \quad \Rightarrow \quad -4A=0 \quad \Rightarrow \quad A=0 \quad \therefore \quad C=0$$
$$\therefore \quad \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} = -\frac{1}{4a}\left[\frac{1}{s^2+2as+2a^2}\right]+\frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2}\right]$$

$$\Rightarrow \qquad F(s) = -\frac{1}{4a}\left[\frac{1}{s^2+2as+2a^2}\right] + \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2}\right]$$

$$\therefore \qquad L^{-1}[F(s)] = -\frac{1}{4a}L^{-1}\left[\frac{1}{s^2+2as+2a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{s^2-2as+2a^2}\right]$$

$$= -\frac{1}{4a}L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]$$

$$= -\frac{e^{-at}}{4a}L^{-1}\left[\frac{1}{s^2+a^2}\right] + \frac{e^{at}}{4a}L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= -\frac{e^{-at}}{4a}\cdot\frac{\sin at}{a} + \frac{e^{at}}{4a}\cdot\frac{\sin at}{a} = \frac{\sin at}{4a^2}[e^{at} - e^{-at}] = \frac{\sin at \sinh at}{2a^2}$$

19.7.3 Type 3 – 1. Multiplication by s and 2. Division by s

1. If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then

$$L^{-1}[s\,F(s)] = f'(t) = \frac{d}{dt}\left\{L^{-1}[F(s)]\right\}$$

In general, $$L^{-1}[s^n F(s)] = f^{(n)}(t), \text{ if } f(0) = 0, f'(0) = 0, \cdots, f^{(n-1)}(0)$$

2. If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

Similarly, $$L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t\int_0^t f(t)\,dt\,dt$$

3. We know that $$L[tf(t)] = -\frac{d}{ds}[F(s)] = F'(s)$$

$$\therefore \qquad L^{-1}[F'(s)] = -tf(t) = -tL^{-1}[F(s)]$$

WORKED EXAMPLES

EXAMPLE 10

Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$.

Solution.

Let $$F'(s) = \frac{s}{(s^2+a^2)^2}, \text{ then } \quad L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

Integrating (1) w.r.to s, we get

$$F(s) = \int \frac{s}{(s^2+a^2)^2} ds = \frac{1}{2}\int (s^2+a^2)^{-2} \cdot 2s\, ds$$

$$= \frac{1}{2} \cdot \frac{(s^2+a^2)^{-2+1}}{(-2+1)} = -\frac{1}{2(s^2+a^2)} \qquad \left[\because \int [f(x)]^n f'(x) dx = \frac{[f(x)^{n+1}]}{n+1}\right]$$

$$\therefore \quad L^{-1}[F'(s)] = -tL^{-1}\left[-\frac{1}{2(s^2+a^2)}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{t}{2} \cdot \frac{\sin at}{a} = \frac{t \sin at}{2a}$$

EXAMPLE 11

Find $L^{-1}\left[\dfrac{1}{s(s+2)^3}\right]$.

Solution.

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = L^{-1}\left[\frac{F(s)}{s}\right], \qquad \text{where } F(s) = \frac{1}{(s+2)^3}$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t L^{-1}[F(s)]dt \qquad \text{by formula (2)}$$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right]dt$$

$$= \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^3}\right]dt$$

$$= \int_0^t e^{-2t} . \frac{t^2}{2!} dt$$

$$= \frac{1}{2}\int_0^t t^2 e^{-2t}\, dt$$

$$= \frac{1}{2}\left[t^2 \frac{e^{-2t}}{-2} - 2t . \frac{e^{-2t}}{(-2)^2} + 2 . \frac{e^{-2t}}{(-2)^3}\right]_0^t$$

$$= \frac{1}{2}\left[-\frac{t^2 e^{-2t}}{2} - \frac{t}{2} e^{-2t} - \frac{1}{4} e^{-2t} - \left(-\frac{e^0}{4}\right)\right]$$

$$= \frac{1}{8}\left[-e^{-2t}(2t^2+2t+1)+1\right] = \frac{1}{8}\left[1 - e^{-2t}(2t^2+2t+1)\right]$$

Note This problem can also be done by partial fraction method.

EXAMPLE 12

Find $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right]$.

Solution.

Let $F'(s)=\dfrac{s+2}{(s^2+4s+5)^2}$. Then $F(s)=\displaystyle\int\dfrac{s+2}{(s^2+4s+5)^2}ds$

Put $x=s^2+4s+5$ $\therefore$ $dx=2s+4=2(s+2)ds \Rightarrow \dfrac{1}{2}dx=(s+2)ds$

$$\therefore \quad F(s)=\int\frac{dx}{2x^2}=\frac{1}{2}\int x^{-2}dx$$

$$=\frac{1}{2}\left[\frac{x^{-2+1}}{-2+1}\right]=-\frac{1}{2x}=-\frac{1}{2(s^2+4s+5)}$$

But $L^{-1}[F'(s)]=-t\,L^{-1}[F(s)]$ [Formula 3, page 19.48]

$$=-t\,L^{-1}\left[-\frac{1}{2(s^2+4s+5)}\right]$$

$$=\frac{t}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right]=\frac{t}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]=\frac{t}{2}e^{-2t}\sin t$$

$$\therefore \quad L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]=\frac{t}{2}e^{-2t}\sin t$$

19.7.4 Type 4 – Inverse Laplace Transform of Logarithmic and Trigonometric Functions

WORKED EXAMPLES

EXAMPLE 13

Find $L^{-1}\left[\log_e\left(\dfrac{s+1}{s-1}\right)\right]$.

Solution.

Let $F(s)=\log_e\dfrac{s+1}{s-1}=\log_e(s+1)-\log_e(s-1)$

$$\therefore \quad F'(s)=\frac{1}{s+1}-\frac{1}{s-1}$$

But we know that $L[tf(t)]=-F'(s)=-\dfrac{1}{s+1}+\dfrac{1}{s-1}=\dfrac{1}{s-1}-\dfrac{1}{s+1}$

$$\therefore \quad tf(t)=L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s+1}\right]=e^{t}-e^{-t}$$

$\therefore \qquad f(t) = \dfrac{e^t - e^{-t}}{t} = \dfrac{2\sinh t}{t}$

$\Rightarrow \qquad L^{-1}\left[\log_e\left(\dfrac{s+1}{s-1}\right)\right] = \dfrac{2}{t}\sinh t.$

EXAMPLE 14

Find $L^{-1}\left[\log_e \dfrac{s(s+1)}{s^2+1}\right]$.

Solution.

Let $\quad F(s) = \log_e \dfrac{s(s+1)}{s^2+1} = \log_e s + \log_e(s+1) - \log_e(s^2+1)$

$\therefore \qquad F'(s) = \dfrac{1}{s} + \dfrac{1}{s+1} - \dfrac{1}{s^2+1}2s$

But $\quad L[tf(t)] = -F'(s) = -\dfrac{1}{s} - \dfrac{1}{s+1} + \dfrac{2s}{s^2+1} = \dfrac{2s}{s^2+1} - \dfrac{1}{s} - \dfrac{1}{s+1}$

$\therefore \qquad tf(t) = 2L^{-1}\left[\dfrac{s}{s^2+1}\right] - L^{-1}\left[\dfrac{1}{s}\right] - L^{-1}\left[\dfrac{1}{s+1}\right] = 2\cdot\cos t - 1 - e^{-t}$

$\therefore \qquad f(t) = \dfrac{2\cos t - (1+e^{-t})}{t}$

$\therefore \qquad L^{-1}\left[\log_e \dfrac{s(s+1)}{s^2+1}\right] = \dfrac{2\cos t - (1+e^{-t})}{t}$

EXAMPLE 15

Find $L^{-1}\left[\tan^{-1}\dfrac{2}{s}\right]$.

Solution.

Let $\quad F(s) = \tan^{-1}\dfrac{2}{s} \quad \therefore \quad F'(s) = \dfrac{1}{1+\left(\dfrac{2}{s}\right)^2}\left(-\dfrac{2}{s^2}\right) = -\dfrac{2}{s^2+4}$

But $\quad L[tf(t)] = -F'(s) = \dfrac{2}{s^2+4}$

$\therefore \qquad tf(t) = L^{-1}\left[\dfrac{2}{s^2+4}\right] = \sin 2t \quad \Rightarrow \quad f(t) = \dfrac{1}{t}\sin 2t$

$\therefore \qquad L^{-1}\left[\tan^{-1}\dfrac{2}{s}\right] = \dfrac{\sin 2t}{t}$

EXAMPLE 16

Find $L^{-1}\left[s\log_e \dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s\right]$.

Solution.

Let $$F(s)=s\log_e\frac{s}{\sqrt{s^2+1}}+\cot^{-1}s=s\left[\log_e s-\frac{1}{2}\log_e(s^2+1)\right]+\cot^{-1}s$$

$\therefore$ $$F'(s)=s\left[\frac{1}{s}-\frac{1}{2(s^2+1)}\cdot 2s\right]+\left[\log_e s-\frac{1}{2}\log_e(s^2+1)\right]\cdot 1+\frac{-1}{1+s^2}$$

$$=1-\frac{s^2}{s^2+1}-\frac{1}{s^2+1}+\log_e s-\frac{1}{2}\log_e(s^2+1)$$

$$=1-\frac{s^2+1}{s^2+1}+\log_e s-\frac{1}{2}\log_e(s^2+1)=\log_e s-\frac{1}{2}\log_e(s^2+1)$$

But $$L[tf(t)]=-F'(s)=\frac{1}{2}\log_e(s^2+1)-\log_e s$$

$$tf(t)=L^{-1}\left[\frac{1}{2}\log_e(s^2+1)-\log_e s\right]$$

$\therefore$ $$f(t)=\frac{1}{t}\left\{L^{-1}\left[\frac{1}{2}\log_e(s^2+1)-\log_e s\right]\right\}$$

Let $$G(s)=\frac{1}{2}\log_e(s^2+1)-\log_e s \quad \therefore \quad G'(s)=\frac{1}{2}\cdot\frac{2s}{s^2+1}-\frac{1}{s}=\frac{s}{s^2+1}-\frac{1}{s}$$

But $$L[tg(t)]=-G'(s)=\frac{1}{s}-\frac{s}{s^2+1}$$

$\therefore$ $$tg(t)=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{s^2+1}\right]$$

$\Rightarrow$ $$tg(t)=1-\cos t=2\sin^2\frac{t}{2} \quad \Rightarrow \quad g(t)=\frac{2}{t}\sin^2\frac{t}{2}$$

$\therefore$ $$f(t)=\frac{1}{t}\cdot\frac{2}{t}\sin^2\frac{t}{2}=\frac{2}{t^2}\sin^2\frac{t}{2}$$

$$\therefore L^{-1}\left[s\log_e\frac{s}{\sqrt{s^2+1}}+\cot^{-1}s\right]=\frac{2}{t^2}\sin^2\frac{t}{2}$$

EXAMPLE 17

Find the inverse Laplace transform of $\cot^{-1}\left[\dfrac{2}{s+1}\right]$.

Solution.

Let $$F(s) = \cot^{-1}\left(\frac{2}{s+1}\right)$$

$\therefore$ $$F'(s) = -\frac{1}{1+\frac{4}{(s+1)^2}}\left[-\frac{2}{(s+1)^2}\right] = \frac{2}{(s+1)^2+4}$$

But $$L[tf(t)] = -F'(s) = -\frac{2}{(s+1)^2+4}$$

$\therefore$ $$tf(t) = -2L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= -2e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right] = -2e^{-t}.\frac{\sin 2t}{2} = -e^{-t}\sin 2t$$

$\therefore$ $$f(t) = -\frac{e^{-t}\sin 2t}{t}$$

$\therefore$ $$L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right] = -\frac{e^{-t}\sin 2t}{t}$$

EXERCISE 19.4

Type I Find the inverse Laplace transform of the following:

1. $\frac{3s-2}{s^2+1}$
2. $\frac{s}{(s+2)^2}$
3. $\frac{3s^2-4s+6}{s^4}$
4. $\frac{2s+1}{s^2+s}$
5. $\frac{s}{s^2+4s+8}$
6. $\frac{5s+3}{s^2+2s+5}$
7. $\frac{s}{s^2+4s+5}$
8. $\frac{3s+7}{s^2-2s-3}$
9. $\frac{3s-2}{s^2-4s+20}$
10. $\frac{2s^2+5s+2}{(s-2)^4}$
11. $\frac{4s+15}{16s^2-25}$
12. $\frac{s^3-3s^2+8s-6}{(s^2-2s+2)^2}$
13. $\frac{3s+2}{s^2-4}$
14. $\frac{s}{a^2s^2+b^2}$
15. $\frac{2s-3}{s^2+4s+13}$
16. $\frac{s}{s^2-4s+5}$
17. $\frac{1}{(s-3)^5}$

Type II Find the inverse Laplace transform of the following functions by partial fraction method.

18. $\frac{s-1}{s^2+3s+2}$
19. $\frac{s+2}{s(s-1)(s-4)}$
20. $\frac{1-s}{(s+1)(s^2+4s+13)}$

21. $\dfrac{s+5}{(s+1)(s^2+1)}$
22. $\dfrac{1}{s^2(s^2+4)}$
23. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$
24. $\dfrac{s}{(s^2+1)(s^2+4)}$
25. $\dfrac{s^2+s-2}{s(s+3)(s-2)}$
26. $\dfrac{7s-11}{(s+1)(s-2)^2}$
27. $\dfrac{s}{s^4+s^2+1}$ **[Hint:** $s^4+s^2+1=(s^2+1)^2-s^2=[(s^2+1)+s][(s^2+1)-s]$]
28. $\dfrac{s}{(s+1)(s-2)^2}$
29. $\dfrac{s}{(s+1)^2(s^2+1)}$
30. $\dfrac{1}{s^2(s^2+8)}$
31. $\dfrac{1}{(s+a)(s+b)}$
32. $\dfrac{s}{(s-4)(s+5)}$
33. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$
34. $\dfrac{s+9}{(s+2)(s^2+3)}$
35. $\dfrac{2s+1}{(s+2)^2(s-1)^2}$
36. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

Type III Find the inverse Laplace transform by multiplication by *s* and division by *s* types.

37. $\dfrac{s^2}{(s-2)^2}$
38. $\dfrac{s^2}{(s-1)^4}$
39. $\dfrac{s}{(s+2)^4}$
40. $\dfrac{1}{s(s+2)^3}$
41. $\dfrac{1}{s(s^2-2s+5)}$
42. $\dfrac{s^2}{(s^2+a^2)^2}$
43. $\dfrac{1}{(s^2+a^2)^2}$

Type IV Find the inverse Laplace transform of the following Logarithmic and trigonometric functions.

44. $\log\left(1+\dfrac{\omega^2}{s^2}\right)$
45. $\log\left(\dfrac{s+2}{s+4}\right)$
46. $\log\left(\dfrac{s^2+a^2}{s^2-b^2}\right)$
47. $\log\left(\dfrac{s-a}{s^2+a^2}\right)$
48. $\tan^{-1}\left(\dfrac{2}{s+1}\right)$
49. $\cot^{-1}\left(\dfrac{s+a}{b}\right)$
50. $\log_e\left(\dfrac{1+s}{s^2}\right)$
51. $s\log_e\dfrac{s+1}{s-1}+2$
52. $\tan^{-1}\left(\dfrac{a}{s}\right)+\cot^{-1}\left(\dfrac{s}{b}\right)$

ANSWERS TO EXERCISE 19.4

1. $3\cos t-2\sin t$
2. $e^{-2t}(1-2t)$
3. $3t-2t^2+t^3$
4. $1+e^{-t}$
5. $e^{-2t}(\cos 2t-\sin 2t)$
6. $e^{-t}(5\cos 2t-\sin 2t)$
7. $e^{-2t}(\cos t-2\sin t)$
8. $4e^{3t}-e^{-t}$
9. $e^{2t}(3\cos 4t+\sin 4t)$

10. $\frac{e^{2t}}{16}(12t+39t^2+20t^3)$
11. $\frac{1}{4}\cos\frac{5t}{4}+\frac{3}{4}\sinh\frac{5t}{4}$
12. $e^{-t}(\cos t+2t\sin t)$
13. $3\cosh 2t+\sinh 2t$
14. $\frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$
15. $\frac{e^{-2t}}{3}[6\cos 3t-7\sin 3t]$
16. $e^{2t}\{\cos t+2\sin t\}$
17. $\frac{t^4e^{3t}}{24}$
18. $-2e^{-t}+3e^{-2t}$
19. $\frac{1}{2}-e^t+\frac{1}{2}e^{4t}$
20. $\frac{1}{5}(e^{-t}-e^{-2t}\cos 3t-2e^{-2t}\sin 3t)$
21. $2e^t+3\sin t-2\cos t$
22. $\frac{1}{8}(2t-\sin 2t)$
23. $\frac{1}{a^2-b^2}[a\sin at-b\sin bt]$
24. $\frac{1}{3}(\cos t-\cos 2t)$
25. $\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$
26. $-2e^{-t}+2e^{2t}+te^{2t}$
27. $\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t.\sinh\frac{t}{2}$
28. $\frac{1}{4}[e^t-e^{-t}-2te^t]$
29. $\frac{1}{2}(\sin t-te^{-t})$
30. $\frac{1}{81}\left(t-\frac{\sin 9t}{9}\right)$
31. $\frac{1}{b-a}[e^{-at}-e^{-bt}]$
32. $\frac{1}{9}[4e^{4t}+5e^{-5t}]$
33. $\frac{e^t}{2}[1-2e^t+5e^{2t}]$
34. $e^{-2t}-\cos\sqrt{3}t+\frac{3\sin\sqrt{3}t}{\sqrt{3}}$
35. $\frac{t}{3}[e^t-e^{-2t}]$
36. $\frac{1}{b^2-a^2}[\cos at-\cos bt]$
37. $4e^{2t}(1+2t)$
38. $e^t\left(t+t^2+\frac{t^3}{6}\right)$
39. $\frac{1}{6}e^{-2t}(3t^2-2t^3)$
40. $\frac{1}{8}[1-e^{-2t}(2t^2+2t+1)]$
41. $\frac{1}{10}[e^t\sin 2t-2e^t\cos 2t+2]$
42. $\frac{1}{2a}[t\cdot a\cos at+\sin at]$
43. $\frac{1}{2a^3}[\sin at-at\cos at]$
44. $\frac{2}{t}(1-\cos\omega t)$
45. $\frac{1}{t}(e^{-4t}-e^{-2t})$
46. $\frac{2}{t}(\cosh bt-\cos at)$
47. $\frac{1}{t}(2\cos at-e^{at})$
48. $\frac{e^{-t}\sin 2t}{t}$
49. $\frac{1}{t}e^{-at}\sin bt$
50. $\frac{1}{t}(2-e^{-t})$
51. $\frac{2}{t^2}[t\cosh t-\sinh t]$
52. $\frac{1}{t}(\sin at+\sin bt)$

19.7.5 Type 5 – Method of Convolution

Definition 19.5 Let $f(t)$ and $g(t)$ be two functions defined for all $t \geq 0$. The convolution of $f(t)$ and $g(t)$ is defined as the integral

$$\int_0^t f(u)g(t-u)du.$$

It is denoted by **$f(t) * g(t)$ or $(f * g)(t)$**

$$\therefore \quad f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$$

Note

$$f(t)*g(t)=\int_0^t f(t-u)g(t-(t-u))\,du \qquad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^t f(t-u)g(u)du=\int_0^t g(u)f(t-u)du=g(t)*f(t)$$

$\therefore$ the operation of the convolution or convolution product is commutative.

Theorem 19.9 Convolution theorem

If $L[f(t)]=F(s)$ and $L[g(t)]=G(s)$,

then $L[f(t)*g(t)]=L[f(t)]L[g(t)]=F(s).G(s)$

Equivalently, $L^{-1}[F(s).G(s)]=f(t)*g(t)=L^{-1}[F(s)]*L^{-1}[G(s)]$

Proof We have $f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$

$$\therefore \quad L[f(t)*g(t)]=\int_0^\infty e^{-st}[f(t)*g(t)]dt=\int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right]dt$$

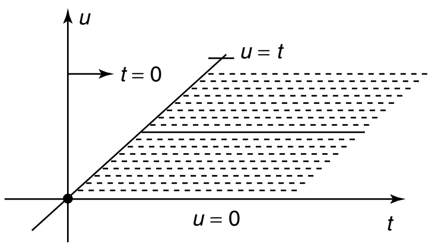
$$\Rightarrow \quad L[f(t)*g(t)]=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt \qquad (1)$$

The region of integration of this double integral is bounded by the lines $u=0$, $u=t$, $t=0$ and $t=\infty$ as in figure.

Changing the order of integration, we take a strip parallel to t-axis.

t varies from u to ∞ and u varies from 0 to ∞

$$\therefore \quad L[f(t)*g(t)]=\int_0^\infty f(u)\left[\int_u^\infty e^{-st}g(t-u)dt\right]du$$



Put $v=t-u$ in the inner integral $\therefore\ dv=dt$

When $t=u,\ v=0$ and when $t=\infty,\ v=\infty$

$$\therefore \quad \int_u^\infty e^{-st}g(t-u)dt=\int_0^\infty e^{-s(v+u)}g(v)dv$$

$$=e^{-su}\int_0^\infty e^{-sv}g(v)dv$$

$\therefore$ $$L[f(t)*g(t)] = \int_0^\infty f(u)\left\{e^{-su}\int_0^\infty e^{-sv} g(u)\,dv\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv$$

$$= L[f(t)]\cdot L[g(t)] = F(s)\cdot G(s)$$

$\therefore$ $$\boldsymbol{L^{-1}[F(s)G(s)] = f(t)*g(t) \; = L^{-1}[F(s)]*L^{-1}[G(s)]}$$

It can be written as $L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)\,du$ ■

WORKED EXAMPLES

EXAMPLE 1

Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.

Solution.

We can write $$\frac{s}{(s^2+a^2)^2} = \frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}$$

Here $$F(s) = \frac{s}{s^2+a^2} \quad \text{and} \quad G(s) = \frac{1}{s^2+a^2}$$

$\therefore$ $$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$ [by convolution theorem]

$$= \cos at * \frac{1}{a}\sin at$$

$$= \int_0^t \cos au.\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t 2\sin(at-au)\cos au\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin(at-au+au)+\sin(at-au-au)\}\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin at + \sin(at - 2au)\}\,du$$

$$= \frac{1}{2a}\left[\sin at \cdot u - \frac{\cos(at-2au)}{-2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \left(0 + \frac{\cos at}{2a}\right)\right] = \frac{1}{2a}t\sin at$$

EXAMPLE 2

Apply the convolution theorem to find $L^{-1}\left[\dfrac{1}{s(s^2-a^2)}\right]$.

Solution.

We can write $\quad \dfrac{1}{s(s^2-a^2)} = \dfrac{1}{s}\cdot\dfrac{1}{(s^2-a^2)}$

$\therefore \quad F(s) = \dfrac{1}{s} \quad$ and $\quad G(s) = \dfrac{1}{s^2-a^2}$

$\therefore \quad L^{-1}\left[\dfrac{1}{s(s^2-a^2)}\right] = L^{-1}\left(\dfrac{1}{s}\right) * L^{-1}\left(\dfrac{1}{s^2-a^2}\right)$ [by convolution theorem]

$$= 1 * \frac{1}{a}\sinh at = \frac{1}{a}\int_0^t \sinh au \cdot 1\,du$$

Here $\quad f(t) = \sinh at,\ g(t) = 1 \quad \therefore\ f(u)\,g(t-u) = f(u)\cdot 1 = \sinh au$

$\therefore\ L^{-1}\left[\dfrac{1}{s(s^2-a^2)}\right] = \dfrac{1}{a}\left[\dfrac{\cosh au}{a}\right]_0^t = \dfrac{1}{a^2}[\cosh at - \cosh 0] = \dfrac{1}{a^2}[\cosh at - 1]$ $\quad [\because\ \cosh 0 = 1]$

EXAMPLE 3

Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem.

Solution.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au . \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos au\cos(bt - bu)\,du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\}+\cos((a+b)u-bt)]du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b}+\frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2(a-b)}[\sin\{(a-b)t+bt\}-\sin\{(a-b).0+bt\}]$$

$$+\frac{1}{2(a+b)}[\sin\{(a+b)t-bt\}-\sin(0-bt)]$$

$$= \frac{1}{2(a-b)}(\sin at-\sin bt)+\frac{1}{2(a+b)}(\sin at+\sin bt)$$

$$= \frac{1}{2}\left[\frac{a+b+a-b}{(a+b)(a-b)}\sin at+\frac{a-b-(a+b)}{(a+b)(a-b)}\sin bt\right]$$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at-\frac{2b}{a^2-b^2}\sin bt\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

EXAMPLE 4

Using convolution theorem find the inverse Laplace transform of $\frac{4}{(s^2+2s+5)^2}$.

Solution.

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]=L^{-1}\left[\frac{2}{(s^2+2s+5)}\cdot\frac{2}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{2}{(s^2+2s+5)}\right]*L^{-1}\left[\frac{2}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{2}{(s+1)^2+4}\right]*L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right]*e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= e^{-t}\sin 2t * e^{-t}\sin 2t$$

$$= \int_0^t e^{-u}\sin 2u\, e^{-(t-u)}\sin 2(t-u)du$$

$$= \int_0^t e^{-t}\sin 2u\sin(2t-2u)du$$

$$= \frac{e^{-t}}{2}\int_0^t [\cos(4u-2t)-\cos 2t]du$$

$$= \frac{e^{-t}}{2}\left[\frac{\sin(4u-2t)}{4} - \cos 2t \cdot u\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[\frac{1}{4}[\sin 2t - \sin(0-2t)] - \cos 2t[t-0]\right]$$

$$= \frac{e^{-t}}{2}\left[\frac{\sin 2t + \sin 2t}{4} - t\cos 2t\right]$$

$$= \frac{e^{-t}}{2}\left[\frac{2\sin 2t}{4} - t\cos 2t\right] = \frac{e^{-t}}{4}[\sin 2t - 2t\cos 2t]$$

EXAMPLE 5

Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$ using convolution theorem.

Solution.

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{(s+1)}\cdot\frac{1}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} * e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t} * e^{-t}\sin t$$

$$= \int_0^t e^{-u}\sin u\, e^{-(t-u)}du \qquad [\text{Here } f(t) = e^{-t}\sin t, g(t) = e^{-t}]$$

$$= e^{-t}\int_0^t \sin u\, du = e^{-t}[-\cos u]_0^t = -e^{-t}[\cos t - \cos 0] = e^{-t}[1-\cos t]$$

EXERCISE 19.5

Using convolution theorem evaluate the inverse Laplace transform of the following functions.

1. $\frac{1}{(s+a)(s+b)}$
2. $\frac{1}{s(s^2+4)}$
3. $\frac{s^2}{(s^2+4)^2}$
4. $\frac{1}{s^2(s^2+9)}$
5. $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$
6. $\frac{s}{(s^2+4)^3}$

7. $\dfrac{1}{s(s^2+4)^2}$ 8. $\dfrac{2}{(s+1)(s^2+4)}$ 9. $\dfrac{1}{s^2(s+1)^2}$

10. $\dfrac{s}{s^4+4}$ 11. $\dfrac{1}{(s+1)(s+2)}$ 12. $\dfrac{1}{(s+1)(s^2+1)}$

ANSWERS TO EXERCISE 19.5

1. $\dfrac{1}{b-a}[e^{-at}-e^{-bt}]$ 2. $\dfrac{1}{4}[1-\cos 2t]$ 3. $\dfrac{1}{4}[\sin t+2t\cos 2t]$

4. $\dfrac{1}{27}[3t-\sin 3t]$ 5. $\dfrac{1}{5}[e^{-t}(\sin t-3\cos t)+\sin t+3\cos t]$

6. $\dfrac{t}{64}[\sin 2t-2t\cos 2t]$ 7. $\dfrac{1}{16}[1-\cos 2t-t\sin 2t]$ 8. $\dfrac{1}{5}[2e^{-t}+\sin 2t-2\cos 2t]$

9. $t-2+e^{-t}(t+2)$ 10. $\dfrac{\sin t\sinh t}{2}$ 11. $e^{-t}(1-e^{-t})$

12. $\dfrac{1}{2}[\sin t-\cos t+e^{-t}]$

19.7.6 Type 6: Inverse Laplace Transform as Contour Integral

Let $f(t)$ be a piecewise continuous function for all $t \geq 0$ and is of exponential order $\alpha > 0$. Suppose $L\,[f(t)] = F(s)$, where s is complex and Re $s > \alpha$. Then $F(s)$ is analytic in the domain Re $s > \alpha$.

If $F(s) \to 0$ as $s \to \infty$, then

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,F(s)\,ds,\; c > \alpha \tag{1}$$

where α is large enough so that all the finite number of singularities of $F(s)$ lie in the part of the Re $s < \alpha$.

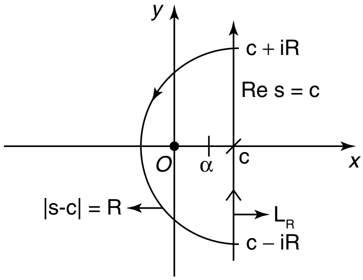
The integral (1) is called the **complex inversion formula** or **Bromwhich's integral formula,** which gives the inverse Laplace transform of the given function $F(s)$.

The complex line integral (1) is evaluated by using the residue theorem, choosing a contour C consisting of the line L_R from $c-iR$ to $c+iR$ and the semicircle of radius R and centre at $s = c$ lying on the left of the line L_R as in figure. The radius R is taken so large such that C encloses all the singularities of the function $e^{st}F(s)$.

As $R \to \infty$, we get

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,F(s)\,ds$$

$$= \sum_{k=1}^{n} \text{Res}\;[e^{st}\,F(s)]_{s=s_k},\; t > 0$$



Working Rule

Given $F(s)$, where $L[f(t)] = F(s)$ and $F(s) \to 0$ as $s \to \infty$, then

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)\,ds$$

$$= \text{Sum of the residues of } e^{st}F(s) \text{ at the poles of } F(s).$$

WORKED EXAMPLES

EXAMPLE 1

Evaluate $L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right]$ by method of residues.

Solution.

Given $F(s) = \frac{1}{(s-1)(s^2+1)}$ Clearly as $s \to \infty$, $F(s) \to 0$

By contour integral, $L^{-1}[F(s)] =$ Sum of the residues of $e^{st}F(s)$ at the poles of $F(s)$.

The poles of $F(s)$ are given by $(s-1)(s^2+1) = 0 \quad \Rightarrow \quad s = 1$ and $s = \pm i$,

which are simple poles.

$$R(1) = \lim_{s\to 1}(s-1)e^{st}F(s) = \lim_{s\to 1}(s-1)e^{st}\frac{1}{(s-1)(s^2+1)}$$

$$= \lim_{s\to 1}\frac{e^{st}}{s^2+1} = \frac{e^t}{1+1} = \frac{e^t}{2}$$

$$R(i) = \lim_{s\to i}(s-i)e^{st}F(s) = \lim_{s\to i}(s-i)e^{st}\frac{1}{(s-1)(s+i)(s-i)}$$

$$= \lim_{s\to i}\frac{e^{st}}{(s-1)(s+i)} = \frac{e^{it}}{(i-1)(i+i)}$$

$$= \frac{e^{it}}{2(-1-i)} = -\frac{e^{it}}{2(1+i)} = -\frac{e^{it}(1-i)}{4}$$

Changing i to $-i$, $\quad R(-i) = -\frac{e^{-it}(1+i)}{4}$

$\therefore \quad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] =$ Sum of the residues of $e^{st}F(s)$ at the poles of $F(s)$

$$= \frac{e^t}{2} - \frac{e^{it}(1-i)}{4} - \frac{e^{-it}(1+i)}{4}$$

$$= \frac{e^t}{2} - \frac{1}{4}[e^{it} + e^{-it}] + \frac{i}{4}[e^{it} - e^{-it}]$$

$$= \frac{e^t}{2} - \frac{2\cos t}{4} + \frac{i}{4}2i\sin t$$

$$= \frac{e^t}{2} - \frac{\cos t}{2} - \frac{\sin t}{2} = \frac{1}{2}[e^t - \cos t - \sin t]$$

EXAMPLE 2

Evaluate the inverse Laplace transform of $\dfrac{1}{s^2(s^2-a^2)}$ by the method of residues.

Solution.

Given $\quad F(s)=\dfrac{1}{s^2(s^2-a^2)}$ Clearly as $s\to\infty,\ F(s)\to 0$

By contour integral,

$$L^{-1}[F(s)] = \text{Sum of the residues of } e^{st}F(s) \text{ at the poles of } F(s)$$

The poles of $F(s)$ are $s=0$, $s=\pm a$, where $s=0$ is a pole of order 2 and $s=a,\ -a$ are simple poles.

$$\therefore \quad R(0)=\frac{1}{(2-1)!}\lim_{s\to 0}\frac{d}{ds}[s^2e^{st}F(s)] = \lim_{s\to 0}\frac{d}{ds}\left[s^2\frac{e^{st}}{s^2(s^2-a^2)}\right]$$

$$=\lim_{s\to 0}\frac{d}{ds}\left[\frac{e^{st}}{(s^2-a^2)}\right]$$

$$=\lim_{s\to 0}\frac{(s^2-a^2)e^{st}\cdot t-e^{st}\cdot 2s}{(s^2-a^2)^2}$$

$$=\frac{(-a^2)e^0\cdot t-0\times e^0}{(0-a^2)^2}=\frac{-a^2t}{a^4}=-\frac{t}{a^2}$$

$$R(a)=\lim_{s\to a}[(s-a)e^{st}F(s)]$$

$$=\lim_{s\to a}(s-a)\frac{e^{st}}{s^2(s^2-a^2)}=\frac{e^{st}}{s^2(s+a)}=\frac{e^{at}}{a^2\cdot 2a}=\frac{e^{at}}{2a^3}$$

Changing a to $-a$, $\quad R(-a)=\dfrac{e^{-at}}{2(-a)^3}=-\dfrac{e^{-at}}{2a^3}$.

$$\therefore \quad L^{-1}\left[\frac{1}{s^2(s^2-a^2)}\right] = \text{Sum of the residues of } e^{st}F(s) \text{ at the poles of } F(s)$$

$$=-\frac{t}{a^2}+\frac{e^{at}}{2a^3}-\frac{e^{-at}}{2a^3}$$

$$=-\frac{t}{a^2}+\frac{1}{2a^3}(e^{at}\cdot -e^{-at})=-\frac{t}{a^2}+\frac{\sinh at}{a^3}=\frac{1}{a^3}[\sinh at-at]$$

EXAMPLE 3

Find the inverse Laplace transform of $\dfrac{1}{(s^2+1)^2}$, by the method of residues.

Solution.

Given $\quad F(s)=\dfrac{1}{(s^2+1)^2}$ Clearly as $s\to\infty,\ F(s)\to 0$

By contour integral $L^{-1}[F(s)] =$ Sum of the residues of $e^{st} F(s)$ at the poles of $F(s)$.

The poles of $F(s)$ are $s = \pm i$, which are poles of order 2.

$$\therefore \qquad R(i) = \frac{1}{(2-1)!} \lim_{s \to i} \frac{d}{ds}[(s-i)^2 e^{st} F(s)]$$

$$= \lim_{s \to i} \frac{d}{ds}\left[\frac{(s-i)^2 e^{st}}{(s+i)^2 (s-i)^2}\right]$$

$$= \lim_{s \to i} \frac{d}{ds}\left[\frac{e^{st}}{(s+i)^2}\right]$$

$$= \lim_{s \to i} \frac{(s+i)^2 \cdot e^{st} \cdot t - e^{st} 2(s+i) \cdot 1}{(s+i)^4}$$

$$= \lim_{s \to i} \frac{(s+i)t\, e^{st} - 2e^{st}}{(s+i)^3}$$

$$= \frac{(i+i)t\, e^{it} - 2e^{it}}{(i+i)^3} = \frac{2i\, t\, e^{it} - 2e^{it}}{-8i} = -\frac{1}{4}[te^{it} + ie^{it}]$$

Changing i to $-i$, $\quad R(-i) = -\frac{1}{4}[te^{-it} - ie^{-it}]$

$$\therefore \qquad L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = -\frac{1}{4}[te^{it} + ie^{it}] - \frac{1}{4}[te^{-it} - ie^{-it}]$$

$$= -\frac{1}{4}\{t(e^{it} + e^{-it}) + i[e^{it} - e^{-it}]\}$$

$$= -\frac{1}{4}[t\ 2\cos t + i\ 2i \sin t] = -\frac{1}{2}[t\cos t - \sin t] = \frac{1}{2}[\sin t - t\cos t]$$

EXERCISE 19.6

I. Evaluate the Laplace transform of the following functions by using the method of residues:

1. $\frac{1}{(s-2)(s^2+1)}$
2. $\frac{1}{(s-1)^2(s^2+1)}$
3. $\frac{2s+3}{(s-2)(s+1)^2}$
4. $\frac{1}{(s+1)^3(s-2)^2}$
5. $\frac{s^2}{(s^2+4)^2}$
6. $\frac{2}{(s+1)(s^2+1)}$
7. $\frac{5}{s^2(s+5)^2}$
8. $\frac{s}{(s^2+1)(s^2+4)}$
9. $\frac{2s}{2s^2+1}$
10. $\frac{1}{(s+1)(s-2)^2}$
11. $\frac{2s-2}{(s+1)(s^2+2s+5)}$

ANSWERS TO EXERCISE 19.6

1. $\frac{1}{5}[e^{-2t} - 2\sin t - \cos t]$
2. $\frac{1}{2}[e^{t}(t-1) + \cos t]$
3. $\frac{7}{9}e^{2t} - \frac{1}{9}(7+3t)e^{-t}$
4. $\frac{e^{2t}}{27}(t-1) + \frac{e^{-t}}{54}[3t^2 + 4t + 2]$
5. $\frac{1}{4}[\sin 2t + 2\sin t]$
6. $e^{-t} + \cos t + 3\sin t$
7. $\frac{t}{5}[e^{-5t} + 1] + \frac{2}{25}[e^{-5t} - 1]$
8. $\cos t - \cos 2t$
9. $\cos\left(\frac{t}{\sqrt{2}}\right)$
10. $\frac{1}{9}[e^{-t} + e^{2t}(3t-1)]$
11. $e^{-t}[\cos 2t + \sin 2t - 1]$

19.8 APPLICATION OF LAPLACE TRANSFORM TO THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Given the linear differential equation with constant coefficients, we apply Laplace transform L on both sides and get the solution of the differential equation.

For this we apply the formulae

$$L\left[\frac{dy}{dt}\right] = L[y'] = s\,L[y] - y(0)$$

$$L\left[\frac{d^2y}{dt^2}\right] = L[y''] = s^2\,L[y] - sy(0) - y'(0)$$

and $$L\left[\frac{d^3y}{dt^3}\right] = L[y'''] = s^3\,L[y] - s^2y(0) - sy'(0) - y''(0)$$

Then the equation is reduced to an algebraic equation in $L[y]$ and s, incorporating the initial conditions.

We group the terms and obtain $L[y] = F(s)$

$$\therefore \qquad y = L^{-1}[F(s)] = f(t)$$

which is the required solution.

19.8.1 First Order Linear Differential Equations with Constant Coefficients

Refer Chapter 10 for basic concepts.

WORKED EXAMPLES

EXAMPLE 1

Solve the equation $\frac{dx}{dt} + x = \sin\omega t$, $x(0) = 2$ by using Laplace transforms.

Solution.

The given equation is $$\frac{dx}{dt} + x = \sin\omega t \qquad (1)$$

and when $t = 0$, $x = 2$.

19.8 APPLICATION OF LAPLACE TRANSFORM TO THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Given the linear differential equation with constant coefficients, we apply Laplace transform L on both sides and get the solution of the differential equation.

For this we apply the formulae

$$L\left[\frac{dy}{dt}\right] = L[y'] = s\,L[y] - y(0)$$

$$L\left[\frac{d^2y}{dt^2}\right] = L[y''] = s^2 L[y] - sy(0) - y'(0)$$

and
$$L\left[\frac{d^3y}{dt^3}\right] = L[y'''] = s^3 L[y] - s^2 y(0) - sy'(0) - y''(0)$$

Then the equation is reduced to an algebraic equation in $L[y]$ and s, incorporating the initial conditions.

We group the terms and obtain $L[y] = F(s)$

$$\therefore \qquad y = L^{-1}[F(s)] = f(t)$$

which is the required solution.

19.8.1 First Order Linear Differential Equations with Constant Coefficients

Refer Chapter 10 for basic concepts.

WORKED EXAMPLES

EXAMPLE 1

Solve the equation $\frac{dx}{dt} + x = \sin \omega t$, $x(0) = 2$ by using Laplace transforms.

Solution.

The given equation is $$\frac{dx}{dt} + x = \sin \omega t \qquad (1)$$

and when $t = 0$, $x = 2$.

Taking Laplace transform on both sides of equation (1), we get

$$L\left[\frac{dx}{dt}\right]+L[x]=L[\sin\omega t]$$

$$\Rightarrow \quad sL[x]-x(0)+L(x)=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow \quad (s+1)L[x]-2=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow \quad (s+1)L[x]=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow \quad L[x]=\frac{\omega}{(s^2+\omega^2)(s+1)}+\frac{2}{(s+1)}$$

$$\therefore \quad x=L^{-1}\left[\frac{\omega}{(s^2+\omega^2)(s+1)}\right]+L^{-1}\left[\frac{2}{(s+1)}\right]=L^{-1}\left[\frac{\omega}{(s^2+\omega^2)(s+1)}\right]+2e^{-t}$$

Using partial fractions,

Let $\quad \dfrac{1}{(s^2+\omega^2)(s+1)}=\dfrac{A}{s+1}+\dfrac{Bs+C}{s^2+\omega^2}$

$$1=A(s^2+\omega^2)+(Bs+C)(s+1)$$

Putting $\quad s=-1, \quad 1=A(1+\omega^2)\Rightarrow A=\dfrac{1}{1+\omega^2}$

Equating coefficients of s^2 and s on both sides, we get

$$A+B=0 \quad \Rightarrow B=-A=-\frac{1}{1+\omega^2}$$

and $\quad B+C=0 \quad \Rightarrow C=-B=\dfrac{1}{1+\omega^2}$

$$\therefore \quad \frac{1}{(s+1)(s^2+\omega^2)}=\frac{1}{(1+\omega^2)}\frac{1}{(s+1)}+\frac{-\frac{1}{1+\omega^2}s+\frac{1}{1+\omega^2}}{s^2+\omega^2}$$

$$=\frac{1}{(1+\omega^2)}\cdot\frac{1}{(s+1)}+\frac{1}{1+\omega^2}\left[-\frac{s}{s^2+\omega^2}+\frac{1}{s^2+\omega^2}\right]$$

$$\therefore \quad L^{-1}\left[\frac{\omega}{(s+1)(s^2+\omega^2)}\right]=\frac{\omega}{(1+\omega^2)}L^{-1}\left[\frac{1}{s+1}\right]-\frac{\omega}{1+\omega^2}L^{-1}\left[\frac{s}{s^2+\omega^2}\right]+\frac{1}{1+\omega^2}L^{-1}\left[\frac{\omega}{s^2+\omega^2}\right]$$

$$=\frac{\omega}{1+\omega^2}e^{-t}-\frac{\omega}{1+\omega^2}\cos\omega t+\frac{1}{1+\omega^2}\sin\omega t$$

$$=\frac{\omega}{1+\omega^2}e^{-t}+\frac{1}{1+\omega^2}[\sin\omega t-\omega\cos\omega t]$$

∴ the solution is

$$x = 2e^{-t} + \frac{\omega}{1+\omega^2}e^{-t} + \frac{1}{1+\omega^2}[\sin\omega t - \omega\cos\omega t]$$

$$\Rightarrow \quad x = \left[2 + \frac{\omega}{1+\omega^2}\right]e^{-t} + \frac{1}{1+\omega^2}[\sin\omega t - \omega\cos\omega t]$$

EXAMPLE 2

Using Laplace transform, solve $\frac{dy}{dt} - y = 1 - 2t$, given that $y = -1$ when $t = 0$.

Solution.

The given equation is $\frac{dy}{dt} - y = 1 - 2t$ (1)

and when $t = 0$, $y = -1$.

Taking Laplace transform on both sides of equation (1), we get

$$L\left[\frac{dy}{dt}\right] - L[y] = L[1-2t]$$

$$\Rightarrow \quad sL[y] - y[0] - L[y] = L[1] - 2L[t]$$

$$\Rightarrow \quad (s-1)L[y] + 1 = \frac{1}{s} - \frac{2}{s^2}$$

$$(s-1)L[y] = \frac{1}{s} - \frac{2}{s^2} - 1$$

$$\Rightarrow \quad L[y] = \frac{1}{s(s-1)} - \frac{2}{s^2(s-1)} - \frac{1}{s-1} = \frac{s-2}{s^2(s-1)} - \frac{1}{s-1}$$

$$\therefore \quad y = L^{-1}\left[\frac{s-2}{s^2(s-1)}\right] - L^{-1}\left[\frac{1}{s-1}\right] = L^{-1}\left[\frac{s-2}{s^2(s-1)}\right] - e^t$$

Using partial fractions,

Let $$\frac{s-2}{(s-1)s^2} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}$$

$$\Rightarrow \quad s - 2 = As^2 + Bs(s-1) + C(s-1)$$

Putting $s = 0$, we get $-2 = -C \quad \Rightarrow C = 2$

Putting $s = 1$, we get $1 - 2 = A \quad \Rightarrow A = -1$

Equating coefficients of s^2 on both sides, we get

$$A + B = 0 \quad \Rightarrow B = -A = 1$$

$$\therefore \quad \frac{s-2}{(s-1)s^2} = -\frac{1}{s-1} + \frac{1}{s} + \frac{2}{s^2}$$

$\therefore$
$$L^{-1}\left[\frac{s-2}{(s-1)s^2}\right] = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right] = -e^t + 1 + 2t$$

$\therefore$ the solution is
$$y = -e^t + 1 + 2t - e^t = 1 + 2t - 2e^t$$

19.8.2 Ordinary Second and Higher Order Linear Differential Equations with Constant Coefficients

Refer Chapter 11 for the basic concepts.

WORKED EXAMPLES

EXAMPLE 1

Solve, using Laplace transform $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = e^{-t}$, given $y(0) = 1$ and $y'(0) = 0$.

Solution.

The given equations is $\qquad y'' - 4y' + 3y = e^{-t} \qquad (1)$

and $y(0) = 1, y'(0) = 0$

Taking Laplace transform on both sides of equation (1), we get
$$L[y''] - 4L[y'] + 3L[y] = L[e^{-t}]$$

$$\Rightarrow s^2L[y] - sy(0) - y'(0) - 4\{sL[y] - y(0)\} + 3L[y] = \frac{1}{s+1}$$

$$\Rightarrow \quad [s^2 - 4s + 3]L[y] = s + \frac{1}{s+1} - 4 \qquad [\text{Since } y(0) = 1\ y'(0) = 0]$$

$$= \frac{s(s+1) + 1 - 4(s+1)}{s+1}$$

$$= \frac{s^2 + s + 1 - 4s - 4}{s+1} = \frac{s^2 - 3s - 3}{s+1}$$

$$\therefore \quad L[y] = \frac{s^2 - 3s - 3}{(s^2 - 4s + 3)(s+1)} = \frac{s^2 - 3s - 3}{(s-3)(s-1)(s+1)}$$

$$\therefore \quad y = L^{-1}\left[\frac{s^2 - 3s - 3}{(s-3)(s-1)(s+1)}\right]$$

Let
$$\frac{s^2 - 3s - 3}{(s-3)(s-1)(s+1)} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$\Rightarrow \quad s^2 - 3s - 3 = A(s-1)(s+1) + B(s+1)(s-3) + C(s-3)(s-1)$$

Put $s = 3$, then $\quad A(3-1)(3+1) = 9 - 9 - 3 \quad \Rightarrow \quad 8A = -3 \quad \Rightarrow \quad A = -\dfrac{3}{8}$

Put $s=1$, then $B(1+1)(1-3)=1-3-3 \quad \Rightarrow -4B=-5 \quad \Rightarrow \quad B=\frac{5}{4}$

Put $s=-1$, then $C(-1-3)(-1-1)=1+3-3 \quad \Rightarrow \quad 8C=1 \quad \Rightarrow \quad C=\frac{1}{8}$

$$\therefore \quad \frac{s^2-3s-3}{(s-3)(s-1)(s+1)}=-\frac{3}{8}\frac{1}{(s-3)}+\frac{5}{4}\cdot\frac{1}{(s-1)}+\frac{1}{8}\frac{1}{(s+1)}$$

$$\therefore \quad L^{-1}\left[\frac{s^2-3s-3}{(s-3)(s-1)(s+1)}\right]=L^{-1}\left[-\frac{3}{8}\cdot\frac{1}{(s-3)}+\frac{5}{4}\cdot\frac{1}{(s-1)}+\frac{1}{8}\frac{1}{(s+1)}\right]$$

$$\Rightarrow \quad y=-\frac{3}{8}L^{-1}\left[\frac{1}{s-3}\right]+\frac{5}{4}L^{-1}\left[\frac{1}{s-1}\right]+\frac{1}{8}L^{-1}\left[\frac{1}{s+1}\right]$$

$$=-\frac{3}{8}e^{3t}+\frac{5}{4}e^{t}+\frac{1}{8}e^{-t}=\frac{1}{8}[e^{-t}-3e^{3t}+10e^{t}]$$

EXAMPLE 2

Using Laplace transform, solve $\frac{d^2y}{dt^2}+\frac{dy}{dt}=t^2+2t$ given that $y=4$ and $y'=-2$ when $t=0$.

Solution.

The given equation is $y''+y'=t^2+2t$

and $y=4$, $y'=-2$ when $t=0$.

Taking Laplace transform on both sides of equation (1), we get

$$L[y'']+L[y']=L[t^2]+2L[t]$$

$$\Rightarrow \quad s^2L[y]-sy(0)-y'(0)+sL[y]-y(0)=\frac{2!}{s^3}+2\cdot\frac{1}{s^2}$$

$$\Rightarrow \quad (s^2+s)\,L[y]-4s+2-4=\frac{2(s+1)}{s^3}$$

$$\Rightarrow \quad s(s+1)\,L[y]=4s+2+\frac{2(s+1)}{s^3}=2(s+1)+2s+\frac{2(s+1)}{s^3}$$

$$\therefore \quad L[y]=\frac{2}{s}+\frac{2}{s+1}+\frac{2}{s^4}$$

$$\therefore \quad y=2L^{-1}\left[\frac{1}{s}\right]+2L^{-1}\left[\frac{1}{s+1}\right]+2L^{-1}\left[\frac{1}{s^4}\right]$$

$$=2\cdot 1+2\cdot e^{-t}+2\frac{t^3}{3!}=2(1+e^{-t})+\frac{t^3}{3}=\frac{6(1+e^{-t})+t^3}{3}$$

EXAMPLE 3

Solve $\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=3\cos 3t-11\sin 3t$ with $y(0)=0$ and $\frac{dy}{dt}=6$ at $t=0$ using Laplace transforms.

Solution.

The given equation is $y''+y'-2y=3\cos 3t-11\sin 3t$ (1)

and $y=0,\ y'=6$ at $t=0$

Taking Laplace transform on both sides, we get

$$L[y'']+L[y']-2L[y]=3L[\cos 3t]-11L[\sin 3t]$$

$$\Rightarrow \quad s^2L[y]-sy(0)-y'(0)+sL[y]-y(0)-2L[y]=3\frac{s}{s^2+9}-11\cdot\frac{3}{s^2+9}$$

$$\Rightarrow \quad (s^2+s-2)\,L[y]-s\cdot 0-6-0=\frac{3(s-11)}{s^2+9}$$

$$\Rightarrow \quad (s+2)(s-1)\,L[y]=\frac{3(s-11)}{(s^2+9)}+6$$

$$\Rightarrow \quad L[y]=\frac{3(s-11)+6(s^2+9)}{(s-1)(s+2)(s^2+9)}=\frac{6s^2+3s+21}{(s-1)(s+2)(s^2+9)}$$

$$\therefore \quad y=L^{-1}\left[\frac{6s^2+3s+21}{(s-1)(s+2)(s^2+9)}\right]$$

Let $$\frac{6s^2+3s+21}{(s-1)(s+2)(s^2+9)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{Cs+D}{s^2+9}$$

$$\Rightarrow \quad 6s^2+3s+21=A(s+2)(s^2+9)+B(s-1)(s^2+9)+(Cs+D)(s-1)(s+2)$$

Put $s=1$. $\therefore$ $A(1+2)(1+9)=6+3+21 \Rightarrow 30A=30 \Rightarrow A=1$

Put $s=-2$ $\therefore$ $B(-2-1)(4+9)=24-6+21 \Rightarrow -39B=39 \Rightarrow B=-1$

Equating coefficients of s^3, $A+B+C=0 \Rightarrow 1-1+C=0 \Rightarrow C=0$

Equating the constant terms, $18A-9B-2D=21 \Rightarrow 18+9-2D=21 \Rightarrow -2D=-6 \Rightarrow D=3$

$$\therefore \quad y=L^{-1}\left[\frac{1}{s-1}-\frac{1}{s+2}+\frac{3}{s^2+9}\right]$$

$$=L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+3L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$=e^t-e^{-2t}+3\frac{\sin 3t}{3}=e^t-e^{-2t}+\sin 3t$$

EXAMPLE 4

Solve the differential equation $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$, where $y = 1, \frac{dy}{dt} = 2, \frac{d^2y}{dt^2} = 2$ at $t = 0$.

Solution.

The given equation is

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0 \tag{1}$$

and when $t = 0,\ y = 1,\ \frac{dy}{dt} = 2$ and $\frac{d^2y}{dt^2} = 2$

That is, when $t = 0,\ y = 1,\ y'(0) = 2$ and $y''(0) = 2$.

Taking Laplace transform on both sides of equation (1), we get

$$L\left[\frac{d^3y}{dt^3}\right] + 2\left[\frac{d^2y}{dt^2}\right] - L\left[\frac{dy}{dt}\right] - 2L[y] = 0$$

$$\Rightarrow \quad s^3L[y] - s^2y(0) - sy'(0) - y''(0) + 2[s^2L[y] - sy(0) - y'(0)] - [sL[y] - y(0)] - 2L[y] = 0$$

$$\Rightarrow \quad [s^3 + 2s^2 - s - 2]L[y] - s^2 - 2s - 2 - 2s - 4 + 1 = 0$$

$$\Rightarrow \quad [s^2(s+2) - (s+2)]L[y] - (s^2 + 4s + 5) = 0$$

$$\Rightarrow \quad [(s^2 - 1)(s+2)]L[y] = s^2 + 4s + 5$$

$$\Rightarrow \quad L[y] = \frac{s^2 + 4s + 5}{(s^2 - 1)(s+2)} = \frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)}$$

$$\therefore \quad y = L^{-1}\left[\frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)}\right]$$

Using partial fractions,

let $$\frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow \quad s^2 + 4s + 5 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

Putting $s = 1$, we get $\quad 1 + 4 + 5 = A(1+1)(1+2) \quad \Rightarrow \quad 6A = 10 \quad \Rightarrow A = \frac{5}{3}$

Putting $s = -1$, we get $\quad 1 - 4 + 5 = B(-1-1)(-1+2) \quad \Rightarrow -2B = 2 \quad \Rightarrow B = -1$

Putting $s = -2$, we get $\quad 4 - 8 + 5 = C(-2-1)(-2+1) \quad \Rightarrow \quad 3C = 1 \quad \Rightarrow C = \frac{1}{3}$

$$\therefore \quad \frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)} = \frac{5}{3}\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{3}\frac{1}{s+2}$$

$$\therefore \quad L^{-1}\left[\frac{s^2+4s+5}{(s-1)(s+1)(s+2)}\right] = \frac{5}{3}L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right]$$

$$\Rightarrow \quad y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

$\therefore$ the solution is

$$y = \frac{5}{3}e^{t} + \frac{1}{3}e^{-2t} - e^{-t} = \frac{1}{3}[5e^{t} + e^{-2t}] - e^{-t}$$

19.8.3 Ordinary Second Order Differential Equations with Variable Coefficients

Refer Chapter11, Section 11.4, page 11.44

We know that

$$L[tf(t)] = -\frac{d}{ds}\{L[f(t)]\}$$

$$L[tf'(t)] = -\frac{d}{ds}\{L[f'(t)]\}$$

$$L[tf''(t)] = -\frac{d}{ds}\{L[f''(t)]\}$$

Using these formulae, we can solve second order differential equations with variable coefficients.

WORKED EXAMPLES

EXAMPLE 1

Solve the differential equation $t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + ty = \cos t$, given that $y(0) = 1$.

Solution.

The given solution is $$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = \cos t \qquad (1)$$

and when $t = 0$, $y = 1$

Taking Laplace transform on both sides of equation (1), we get

$$L\left[t\frac{d^2y}{dt^2}\right] + 2L\left[\frac{dy}{dt}\right] + L[ty] = L[\cos t]$$

$$\Rightarrow \quad -\frac{d}{ds}\{L[y'']\} + 2L[y'] - \frac{d}{ds}\{L[y]\} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad -\frac{d}{ds}[s^2L[y] - sy(0) - y'(0)] + 2[sL[y] - y(0)] - \frac{d}{ds}\{L[y]\} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad -\frac{d}{ds}\left[s^2L[y] - s - y'(0)\right] + 2sL[y] - 2 - \frac{d}{ds}\{L[y]\} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad -\left\{s^2\frac{d}{ds}\{L[y]\} + 2sL[y]\right\} + 1 + 2s\{L[y]\} - 2 - \frac{d}{ds}\{L[y]\} = \frac{s}{s^2+1} \qquad \left[\begin{array}{l}\therefore\ y'(0) \text{ is a constant} \\ \dfrac{d}{ds}(y'(0)) = 0\end{array}\right]$$

$\Rightarrow \quad -(s^2+1)\dfrac{d}{ds}\{L[y]\} = \dfrac{s}{s^2+1}+1$

$\Rightarrow \quad \dfrac{d}{ds}\{L[y]\} = -\dfrac{s}{(s^2+1)^2} - \dfrac{1}{s^2+1}$

$\Rightarrow \quad L^{-1}\dfrac{d}{ds}\{L[y]\} = -L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right] - L^{-1}\left[\dfrac{1}{s^2+1}\right]$

We know $\quad L^{-1}[F'(s)] = -t f(t)$, where $L[f(t)] = F(s)$. Here $y = f(t)$

$\therefore \quad -ty = -L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right] - \sin t \quad \Rightarrow \quad ty = L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right] + \sin t$

To find $L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right]$

We know $\quad L[t] = \dfrac{1}{s^2} \quad \therefore \quad L[t\,e^{it}] = \dfrac{1}{(s-i)^2}$

$\Rightarrow \quad L[t(\cos t + i\sin t)] = \dfrac{(s+i)^2}{(s-i)^2(s+i)^2}$

$\Rightarrow \quad L[t\cos t] + iL[t\sin t] = \dfrac{s^2-1+2is}{(s^2+1)^2} = \dfrac{s^2-1}{(s^2+1)^2} + i\dfrac{2s}{(s^2+1)^2}$

Equating imaginary parts, we get

$$L[t\sin t] = \frac{2s}{(s^2+1)^2} \quad \Rightarrow \quad t\sin t = 2L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$$

$\therefore \quad L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right] = \dfrac{1}{2}t\sin t$

$\therefore \quad t\,y = \dfrac{1}{2}t\sin t + \sin t$

$\Rightarrow \quad y = \dfrac{1}{2}\sin t + \dfrac{\sin t}{t} = \left[\dfrac{1}{2}+\dfrac{1}{t}\right]\sin t$

EXAMPLE 2

Solve the differential equation $y'' + 2t\,y' - 4y = 1$, $y(0) = y'(0) = 0$.

Solution.

The given equation is

$$y'' + 2ty' - 4y = 1 \tag{1}$$

and when $t = 0, y = 0$ and $y' = 0$

Taking Laplace transforms on both sides of equation (1), we get

$$L[y''] + 2L[t\,y'] - 4L[y] = L[1]$$

We know $$L[t\,y'] = -\frac{d}{ds}\{L[y']\} = -\frac{d}{ds}\{sL[y] - y(0)\}$$

$$\therefore \quad s^2 L[y] - s\,y(0) - y'(0) - 2\frac{d}{ds}[(sL[y] - y(0))] - 4L[y] = \frac{1}{s}$$

$$\Rightarrow \quad s^2 L[y] - 2\left[s\frac{d}{ds}(L[y]) + L[y]\right] - 4L[y] = \frac{1}{s}$$

$$\Rightarrow \quad -2s\frac{d}{ds}(L[y]) - (6 - s^2)L[y] = \frac{1}{s}$$

$$\Rightarrow \quad \frac{d}{ds}(L[y]) + \frac{6 - s^2}{2s}L[y] = -\frac{1}{2s^2}$$

This is linear equation in $L[y]$, where

$$P = \frac{6 - s^2}{2s} = \frac{3}{s} - \frac{s}{2}, \qquad Q = -\frac{1}{2s^2}.$$

$\therefore$ the solution is

$$L[y]e^{\int P\,ds} = \int Q\,e^{\int P\,ds}\,ds + c$$

Now, $$\int P\,ds = \int\left(\frac{3}{s} - \frac{s}{2}\right)ds = 3\log_e s - \frac{s^2}{4} = \log_e s^3 - \frac{s^2}{4}$$

$$\therefore \quad e^{\int P\,ds} = e^{\log_e s^3 - \frac{s^2}{4}} = s^3 e^{-\frac{s^2}{4}}$$

and $$\int Q\,e^{\int P\,ds}\,ds = \int -\frac{1}{2s^2}s^3 e^{-\frac{s^2}{4}}\,ds = -\frac{1}{2}\int e^{-\frac{s^2}{4}}\,s\,ds$$

Put $\frac{s^2}{4} = t \Rightarrow \frac{2s}{4}ds = dt \quad \Rightarrow \quad s\,ds = 2\,dt$

$$\therefore \quad \int Q\,e^{\int P\,ds}\,ds = -\frac{1}{2}\int e^{-t}\,2\,dt = -\left[\frac{e^{-t}}{-1}\right] = e^{-t} = e^{-\frac{s^2}{4}}$$

$$\therefore \quad L[y]s^3 e^{-\frac{s^2}{4}} = e^{-\frac{s^2}{4}} + c \quad \Rightarrow \quad L[y]s^3 = 1 + c\,e^{\frac{s^2}{4}}$$

$$\Rightarrow \quad L[y] = \frac{1}{s^3} + \frac{c}{s^3}e^{\frac{s^2}{4}}$$

All the initial conditions are used.

$\therefore$ there is no initial condition to determine c.
Since we must have

$$\lim_{s\to\infty} L[y] = \lim_{s\to\infty} F(s) = 0, \text{ we choose } c = 0.$$

$$\therefore \quad L[y] = \frac{1}{s^3} \quad \Rightarrow \quad y = L^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2}t^2, \quad t \geq 0$$

$\therefore$ the solution is $\quad y = \frac{1}{2}t^2, \quad t \geq 0$

EXERCISE 19.7

I. Solve the following first order differential equations

1. $\frac{dy}{dt} + 2y = e^{2t}$, given $y = 0$, when $t = 0$.
2. $\frac{dx}{dt} + x = e^t$, given $x(0) = 1$.
3. $\frac{dy}{dx} - y = e^x, y(0) = -1$.
4. $y'(t) - 4y(t) = t, y(0) = -1$.
5. $\frac{dy}{dt} + 2y = \sin t, y(0) = -1$.

II. Solve the following second and higher order differential equations:

1. $y'' + 2y' - 3y = 3$ given $y(0) = 4$, $y'(0) = -7$.
2. $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$.
3. $y'' - 3y' + 2y = 4t + e^{3t}$, when $y(0) = 1$ and $y'(0) = -1$.
4. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t, y = \frac{dy}{dt} = 0$ when $t = 0$.
5. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t$, when $x(0) = 1$, $x'(0) = 1$.
6. $y'' + y = t, y(0) = 1, y'(0) = -2$.
7. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 10\sin t$, given $y = 0 = \frac{dy}{dt}$ when $t = 0$.
8. $\frac{d^2x}{dt^2} + x = t\cos 2t, x(0) = 0, x'(0) = 0$.
9. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 2e^{-3t}, y(0) = 1, y'(0) = -2$.
10. $y'' - 3y' + 2y = e^{-t}$, given $y(0) = 1, y'(0) = 0$.
11. $y'' - 2y' + y = e^t$, given $y(0) = 2, y'(0) = 1$.

12. $2y'' - 5y' + 2y = 3\sin t$, given $y(0) = 1, y'(0) = 0$.

13. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 8y = 0$, given $y(0) = 3, y'(0) = 6$.

14. $(D^2 + 4D + 4)y = e^{-t}$ given that $y(0) = 0, y'(0) = 0$.

15. $(D^2 + 9)y = \cos 2t$ given that $y(0) = 1, y\left(\dfrac{\pi}{2}\right) = -1$.

16. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$ with $x = 2, \dfrac{dx}{dt} = -1$ at $t = 0$.

17. $y'' + 5y' + 6y = 2$ gives $y'(0) = 0$ and $y(0) = 0$.

18. $y'' - 3y' + 2y = 4$ given that $y(0) = 2, y'(0) = 3$.

19. $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = e^{-t}$ with $y(0) = 1$ and $y'(0) = 0$.

20. $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 0$, given $y(0) = 1, y'(0) = 2$ and $y''(0) = 2$.

21. $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 0$, given $y(0) = y'(0) = 0$ and $y''(0) = 6$.

22. $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} = 2\cos t, y(0) = 3, y'(0) = 2, y''(0) = 1$.

23. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = 16e^{3t}$, given $y(0) = 0, y'(0) = 4, y''(0) = 6$.

24. $\dfrac{d^4y}{dt^4} + 2\dfrac{d^2y}{dt^2} + y = \sin t$, given $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

III. Solve the following second order differential equations with variable coefficients:

25. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1, y'(0) = 2$.

26. $y'' + 2ty' - y = t$, when $y(0) = 0, y'(0) = 1$.

27. $ty'' + 2y' + ty = \cos t$, given $y(0) = 1$.

ANSWERS TO EXERCISE 19.7

I.

1. $y = \dfrac{1}{3}[e^{2t} - e^{-t}]$

2. $y = \dfrac{e^t + e^{-t}}{2}$ or $\cosh t$

3. $y = e^x(x - 1)$

4. $y = -\dfrac{1}{4}t - \dfrac{5}{4}e^{4t}$

5. $y = \dfrac{1}{2}e^{-t} - \dfrac{1}{2}\cos t + \dfrac{3}{2}\sin t$

II.

1. $y = \frac{1}{4}(15e^{t} + 5e^{-t} - 4)$
2. $y = \frac{1}{4}(7e^{-t} - 3e^{-3t} - 2te^{-t})$
3. $y = 2t + 3 + \frac{1}{2}(e^{3t} - e^{t}) - 2e^{2t}$
4. $y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$
5. $x = \frac{1}{3}\cdot e^{-t}(\sin t + \sin 2t)$
6. $y = t - \sin t + \cos t$
7. $y = \frac{1}{2}(5e^{-t} - e^{-3t}) + \sin t - 2\cos t$
8. $x = \frac{1}{9}(4\sin 2t - 5\sin t - 3t\cos 2t)$
9. $y = e^{-3t}(1 + t + t^{2})$
10. $y = \frac{1}{6}(e^{-t} + 9e^{t} - 4e^{2t})$
11. $y = \frac{1}{2}\cdot e^{t}(t-2)^{2}$
12. $y = \frac{1}{5}(3\sin t + e^{2t} - 4e^{t/2})$
13. $2e^{4t} + e^{-2t}$
14. $e^{-t} - (1+t)e^{-2t}$
15. $\frac{1}{5}[4(\cos 3t + \sin 3t) + \cos 2t]$
16. $\frac{e^{t}}{2}[t^{2} - 6t + 4]$
17. $\frac{1}{3}[1 - 3e^{-2t} + 2e^{-3t}]$
18. $2 - 3(e^{t} - e^{2t})$
19. $\frac{3}{2}\cdot e^{t} - \frac{2}{3}\cdot e^{2t} + \frac{1}{6}\cdot e^{-t}$.
20. $y = \frac{1}{3}[5e^{t} + 2e^{-2t}] - e^{-t}$
21. $y = e^{t} - 3e^{-t} + 2e^{-2t}$
22. $y = \frac{7}{2}e^{t} + \frac{e^{-t}}{2} - \cos t$
23. $y = \frac{1}{4}[e^{2t} + e^{-2t} + 2t]$
24. $y = \frac{1}{8}[(3 - t^{2})\sin t - 3t\cos t]$

III.

25. $y = e^{2t}$
26. $y = t$
27. $y = \frac{1}{2}\left[1 + \frac{2}{t}\right]\sin t$

19.8.4 Simultaneous Differential Equations

Refer Chapter 11, Section 11.3, page 11.34.

WORKED EXAMPLES

EXAMPLE 1

Solve by using Laplace transform $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ given that $x = 2$ and $y = 0$, when $t = 0$.

Solution.

The given equations are

$$\frac{dx}{dt} + y = \sin t \qquad (1) \qquad \text{and} \qquad \frac{dy}{dt} + x = \cos t \qquad (2)$$

When $t = 0$, $x = 2$ and $y = 0$

Taking Laplace transform on both sides of equations (1) and (2), we get

$$L\left[\frac{dx}{dt}\right] + L[y] = L[\sin t]$$

$$\Rightarrow \quad sL[x] - x(0) + L[y] = \frac{1}{s^2+1}$$

$$\Rightarrow \quad sL[x] - 2 + L[y] = \frac{1}{s^2+1}$$

$$\Rightarrow \quad sL[x] + L[y] = 2 + \frac{1}{s^2+1} \quad (3)$$

and $$L\left[\frac{dy}{dt}\right] + L[x] = L[\cos t]$$

$$\Rightarrow \quad sL[y] - y(0) + L[x] = \frac{s}{s^2+1}$$

$$\Rightarrow \quad L[x] + sL[y] = \frac{s}{s^2+1} \quad (4)$$

$$(3) \times s \Rightarrow \quad s^2 L[x] + sL[y] = 2s + \frac{s}{s^2+1} \quad (5)$$

$$(5) - (4) \quad (s^2 - 1)L[x] = 2s + \frac{s}{s^2+1} - \frac{s}{s^2+1} = 2s$$

$$\Rightarrow \quad L[x] = \frac{2s}{s^2-1}$$

$$\therefore \quad x = L^{-1}\left[\frac{2s}{s^2-1}\right] = 2\cosh t = e^t + e^{-t}$$

$$(1) \Rightarrow \quad y = \sin t - \frac{dx}{dt}$$

$$= \sin t - \frac{d}{dt}(e^t + e^{-t}) = \sin t - [e^t - e^{-t}] = \sin t - e^t + e^{-t}$$

$\therefore$ the solution is

$$x = e^t + e^{-t} \quad \text{and} \quad y = \sin t - e^t + e^{-t}$$

EXAMPLE 2

The coordinates (x, y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin 2t$ and $\frac{dx}{dt} - 2y = \cos 2t$, $(t > 0)$.

If at $t = 0$, $x = 1$ and $y = 0$, then show that by Laplace transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Solution.
The given equations are

$$\frac{dx}{dt} - 2y = \cos 2t \qquad (1) \qquad \text{and} \qquad \frac{dy}{dt} + 2x = \sin 2t \qquad (2)$$

Given, when $t = 0$, $x = 1$ and $y = 0$.
Taking Laplace transform on both sides of equations (1) and (2), we get

$$L\left[\frac{dx}{dt}\right] - 2L[y] = L[\cos 2t]$$

$$\Rightarrow \qquad sL[x] - x(0) - 2L[y] = \frac{s}{s^2+4}$$

$$\Rightarrow \qquad sL[x] - 1 - 2L[y] = \frac{s}{s^2+4}$$

$$\Rightarrow \qquad sL[x] - 2L[y] = \frac{s}{s^2+4} + 1 \qquad (3)$$

and

$$L\left[\frac{dy}{dt}\right] + 2L[x] = L[\sin 2t]$$

$$\Rightarrow \qquad sL[y] - y(0) + 2L[x] = \frac{2}{s^2+4}$$

$$\Rightarrow \qquad 2L[x] + sL[y] = \frac{2}{s^2+4} \qquad (4)$$

$$(3) \times 2 \Rightarrow \qquad 2sL[x] - 4L[y] = \frac{2s}{s^2+4} + 2 \qquad (5)$$

$$(4) \times s \Rightarrow \qquad 2sL[x] + s^2L[y] = \frac{2s}{s^2+4} \qquad (6)$$

$$(6) - (5) \Rightarrow \qquad (s^2+4)L[y] = -2$$

$$\Rightarrow \qquad L[y] = -\frac{2}{s^2+4} \quad \Rightarrow \quad y = -L^{-1}\left[\frac{2}{s^2+4}\right] \quad \Rightarrow \quad y = -\sin 2t$$

From equation (2),

$$2x = \sin 2t - \frac{dy}{dt} = \sin 2t - \frac{d}{dt}(-\sin 2t) = \sin 2t + 2\cos 2t$$

$$\Rightarrow \qquad x = \frac{1}{2}[\sin 2t + 2\cos 2t]$$

$\therefore$ the general solution is

$$x = \frac{1}{2}(\sin 2t + 2\cos 2t)$$

and

$$y = -\sin 2t$$

This is the parametric equations of the curve.

Eliminating t, we get the Cartesian equation.

$$y = -\sin 2t \Rightarrow \sin 2t = -y$$

$\therefore$ $$x = \frac{1}{2}[-y - 2\cos 2t] \Rightarrow x + \frac{1}{2}y = -\cos 2t \Rightarrow \cos 2t = -\frac{2x+y}{2}$$

and $$\sin 2t = -y$$

But $$\cos^2 2t + \sin^2 2t = 1 \Rightarrow \frac{(2x+y)^2}{4} + y^2 = 1$$

$$\Rightarrow 4x^2 + 4xy + y^2 + 4y^2 = 4 \Rightarrow 4x^2 + 4xy + 5y^2 = 4 \quad (7)$$

$\therefore$ the particle moves along the curve given by equation (7).

EXAMPLE 3

Solve the differential equations $\frac{dx}{dt} + \frac{dy}{dt} = t$ and $\frac{d^2x}{dt^2} - y = e^{-t}$, given $x = 3$, $\frac{dx}{dt} = -2$, $y = 0$ when $t = 0$.

Solution.
The given equations are

$$\frac{dx}{dt} + \frac{dy}{dt} = t \quad (1) \quad \text{and} \quad \frac{d^2x}{dt^2} - y = e^{-t} \quad (2)$$

Given when $t = 0$, $x(0) = 3$, $x'(0) = -2$ and $y(0) = 0$

Taking Laplace transform on both sides of equations (1) and (2), we get

$$L\left[\frac{dx}{dt}\right] + L\left[\frac{dy}{dt}\right] = L[t]$$

$$\Rightarrow \quad sL[x] - x(0) + L[y] - y(0) = \frac{1}{s^2}$$

$$\Rightarrow \quad sL[x] - 3 + sL[y] - 0 = \frac{1}{s^2}$$

$$\Rightarrow \quad s(L[x] + L[y]) = \frac{1}{s^2} + 3 \Rightarrow L[x] + L[y] = \frac{1}{s^3} + \frac{3}{s} \quad (3)$$

and $$L\left[\frac{d^2x}{dt^2}\right] - L[y] = L[e^{-t}]$$

$$\Rightarrow \quad s^2L[x] - sx(0) - x'(0) - L[y] = \frac{1}{s+1}$$

$$\Rightarrow \quad s^2L[x] - 3s + 2 - L[y] = \frac{1}{s+1} \Rightarrow s^2L[x] - L[y] = \frac{1}{s+1} + 3s - 2 \quad (4)$$

(3) + (4) $\Rightarrow$ $$(s^2+1)L[x] = \frac{1}{s^3} + \frac{3}{s} + \frac{1}{s+1} + 3s - 2$$

$$\Rightarrow \qquad L[x] = \frac{1}{s^3(s^2+1)} + \frac{3}{s(s^2+1)} + \frac{1}{(s+1)(s^2+1)} + \frac{3s}{s^2+1} - \frac{2}{s^2+1}$$

$$\Rightarrow \qquad x = L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] + 3L^{-1}\left[\frac{1}{s(s^2+1)}\right] + L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] + 3L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{s^2+1}\right]$$

Now $$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t\int_0^t\int_0^t L^{-1}\left[\frac{1}{s^2+1}\right]dt\,dt\,dt$$

$$= \int_0^t\int_0^t\int_0^t \sin t\,dt\,dt\,dt$$

$$= \int_0^t\int_0^t [-\cos t]_0^t dt\,dt$$

$$= \int_0^t\int_0^t -[\cos t - \cos 0]\,dt\,dt$$

$$= \int_0^t\int_0^t (1-\cos t)\,dt\,dt$$

$$= \int_0^t [t - \sin t]_0^t\,dt$$

$$= \int_0^t [t - \sin t]\,dt$$

$$= \left[\frac{t^2}{2} - (-\cos t)\right]_0^t = \frac{t^2}{2} - 0 + [\cos t - \cos 0] = \frac{t^2}{2} + \cos t - 1$$

and $$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+1}\right]dt$$

$$= \int_0^t \sin t\,dt = [-\cos t]_0^t = -(\cos t - \cos 0) = 1 - \cos t$$

To find $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$

Using partial functions,

Let $$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s+1)$$

Putting $s=-1$, $$1 = 2A \Rightarrow A = \frac{1}{2}$$

Equating the coefficients of s^2 and constant terms on both sides, we get

$$A+B=0 \quad \Rightarrow \quad B=-A \quad =-\frac{1}{2}$$

and $$A+C=1 \quad \Rightarrow \quad C=1-A=1-\frac{1}{2}=\frac{1}{2}$$

$\therefore$ $$\frac{1}{(s+1)(s^2+1)}=\frac{1}{2}\frac{1}{(s+1)}+\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

$\therefore$ $$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

$$L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t \quad \text{and} \quad L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t$$

$\therefore$ $$x=\frac{t^2}{2}+\cos t-1+3(1-\cos t)+\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t+3\cos t-2\sin t$$

$\Rightarrow$ $$x=\frac{t^2}{2}+\frac{1}{2}e^{-t}+\frac{1}{2}\cos t-\frac{3}{2}\sin t+2 \qquad (5)$$

From equation (2), $$y=\frac{d^2x}{dt^2}-e^{-t}$$

Differentiating Eq. (5) with respect to t, we get

$$\frac{dx}{dt}=t-\frac{1}{2}e^{-t}-\frac{1}{2}\sin t-\frac{3}{2}\cos t$$

$$\frac{d^2x}{dt^2}=1+\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

$\therefore$ $$y=1+\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t-e^{-t}$$

$$=1-\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

$\therefore$ the solution is

$$x=\frac{t^2}{2}+\frac{1}{2}e^{-t}+\frac{1}{2}\cos t-\frac{3}{2}\sin t+2$$

and $$y=1-\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{3}{2}\sin t$$

19.8.5 Integral–Differential Equation

Differential equations involving integrals is known as integral–differential equation.

EXAMPLES:

1. $\dfrac{dy}{dt} + 3y + 2\int_0^t y\,dt = t$

2. $\dfrac{dx}{dt} = t + \int_0^t x(t-u)\cos u\,du$

WORKED EXAMPLES

EXAMPLE 1

Solve the integral–differential equation $\dfrac{dx}{dt} = t + \int_0^t x(t-u)\cos u\,du, x(0) = 2.$

Solution.

The given equation is

$$\frac{dx}{dt} = t + \int_0^t x(t-u)\cos u\,du \quad \Rightarrow \quad x'(t) = t + \int_0^t x(t-u)\cos u\,du \qquad (1)$$

and $x(0) = 2$.

In this equation, the integral is a convolution integral.

We know, $$f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$$

$$\therefore \qquad \int_0^t x(t-u)\cdot\cos u\,du = \cos t * x(t)$$

$\therefore$ the equation (1) becomes

$$x'(t) = t + \cos t * x(t) \qquad (2)$$

Applying Laplace transform on both sides of equation (2), we get

$$L[x'(t)] = L[t] + L[\cos t]\cdot L[x(t)]$$

$$\Rightarrow \qquad sL[x(t)] - x(0) = \frac{1}{s^2} + \frac{s}{s^2+1}L[x(t)]$$

$$\Rightarrow \qquad \left[s - \frac{s}{s^2+1}\right]L[x(t)] - 2 = \frac{1}{s^2}$$

$$\Rightarrow \qquad \left[\frac{s(s^2+1)-s}{s^2+1}\right]L[x(t)] = \frac{1}{s^2} + 2$$

$$\Rightarrow \qquad \frac{s^3}{s^2+1}L[x(t)] = \frac{1}{s^2} + 2$$

$$\Rightarrow \qquad L[x(t)] = \frac{s^2+1}{s^3}\left[\frac{1}{s^2} + 2\right]$$

$$= \frac{s^2+1}{s^5}[1+2s^2]$$

$$= \frac{s^2+2s^4+1+2s^2}{s^5} = \frac{3s^2+2s^4+1}{s^5} = \frac{3}{s^3}+\frac{2}{s}+\frac{1}{s^5}$$

$\therefore$ $$x(t) = L^{-1}\left[\frac{3}{s^3}\right]+L^{-1}\left[\frac{2}{s}\right]+L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 3\frac{1}{2}t^2+2\cdot 1+\frac{1}{4!}t^4$$

$\Rightarrow$ $$x(t) = \frac{3}{2}t^2+\frac{t^4}{24}+2$$

which is the solution.

EXAMPLE 2

Solve the integral–differential equation $y' + 3y + 2\int_0^t y\,dt = t, \quad y(0) = 0.$

Solution.

The given equation is

$$y'+3y+2\int_0^t y\,dt = t \tag{1}$$

and when $t=0$, $y=0$

Applying Laplace transform on both sides, we get

$$L[y']+3L[y]+2L\left[\int_0^t y\,dt\right] = L[t]$$

$\Rightarrow$ $$sL[y]-y'(0)+3L[y]+2\cdot\frac{1}{s}L[y] = \frac{1}{s^2}$$

$\Rightarrow$ $$sL[y]-0+3L[y]+\frac{2}{s}L[y] = \frac{1}{s^2}$$

$\Rightarrow$ $$\left[s+3+\frac{2}{s}\right]L[y] = \frac{1}{s^2}$$

$\Rightarrow$ $$\left[\frac{s^2+3s+2}{s}\right]L[y] = \frac{1}{s^2}$$

$\Rightarrow$ $$L[y] = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)}$$

$\therefore$ $$y = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$$

Using partial fractions,

Let $$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$\Rightarrow$ $$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Putting $s = 0$, then $1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

Putting $s = -1$, then $1 = B(-1)(1) \quad \Rightarrow \quad B = -1$

Putting $s = -2$, then $1 = C(-2)(-1) \quad \Rightarrow \quad C = \frac{1}{2}$

$\therefore$ $$y = L^{-1}\left[\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t}$$

$$= \frac{1}{2}[1 - 2e^{-t} + e^{-2t}] = \frac{1}{2}(1 - e^{-t})^2$$

EXERCISE 19.8

I. Solve the following simultaneous equations:

1. $\frac{dx}{dt} + 2y = 5e^t$, $\frac{dy}{dt} - 2x = 5e^t$, given $x = -1, y = 3$ when $t = 0$
2. $\frac{dx}{dt} + y = \sin t + 1$, $\frac{dy}{dt} + x = \cos t$, given that $x = 1$ and $y = 2$ at $t = 0$
3. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given $x = 2, y = 0$ at $t = 0$
4. $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$, given $x = y = 0$ when $t = 0$

II. Solve the following integral–differential equations:

1. $y'(t) - 4y(t) + 3\int_0^t y(t)\,dt = t,\ y(0) = 1$
2. $y'(t) - y(t) = 6\int_0^t y(t)\,dt + \sin t,\ y(0) = 2$
3. $y(t) = 1 + \int_0^t \sin(t-u)\,y(u)\,du$
4. $y(t) = e^{-t} + \int_0^t \sin(t-u)\,y(u)\,du$

ANSWERS TO EXERCISE 19.8

I.

1. $x = -e^t,\ y = 3e^t$
2. $x = e^{-t},\ y = 1 + \sin t + e^{-t}$
3. $x = e^{-t} + e^t,\ y = e^{-t} - e^t + \sin t$
4. $x = \dfrac{1}{27}[1 + 3t - (1 + 6t)e^{-3t}],\ y = \dfrac{2}{27}[2 - 3t - e^{-3t}(3t + 2)]$

II.

1. $y(t) = \dfrac{1}{3} - e^{-t} + \dfrac{5}{3}e^{3t}$
2. $y(t) = -\dfrac{7}{50}\cot - \dfrac{1}{50}\sin t + \dfrac{63}{50}e^{3t} + \dfrac{22}{50}e^{-2t}$
3. $y(t) = 1 + \dfrac{t^2}{2}$
4. $y(t) = -1 + t + 2e^{-t}$

SHORT ANSWER QUESTIONS

1. **State the conditions under which Laplace transform of $f(t)$ exists.**
2. **What is Laplace transform of $t^{3/2}$?**
3. **Find the Laplace transform of $e^t t^{1/2}$.**
4. **Find the Laplace transform of $\dfrac{t}{e^t}$.**
5. **Find L [$t \cosh 3t$].**
6. **Find $L[t^2 \sin at]$.**
7. **Find $L[\sin at - at \cos at]$.**
8. **Find L [$f(t)$], where $f(t) = \begin{cases} t, & 0 < t < 4 \\ 5, & t > 4 \end{cases}$**
9. **State and prove final value theorem.**
10. **Find L [$\cos(3t - 1)$].**
11. **Find the Laplace transform of $f(t) = \cos^2 3t$.**
12. **State the initial value theorem and final value theorem for Laplace transforms.**
13. **Give examples of two functions for which Laplace transform do not exist.**
14. **Evaluate $\int_0^\infty e^{-t} \cos 2t\ dt$.**
15. **Evaluate $\int_0^\infty e^{-2t} \sin 3t\ dt$.**
16. **If $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}\dfrac{s}{a}$, then find $\int_0^\infty \dfrac{\sin at}{t} dt$.**
17. **Find $L^{-1}\left[\dfrac{1}{s^2 + 4s + 4}\right]$.**
18. **Find the inverse Laplace theorem of $\dfrac{1}{(s+1)^4}$.**
19. **Verify intial value thorem for $1 + e^{-t}(\sin t + \cos t)$.**
20. **Find $L^{-1}\left[\dfrac{s-3}{(s-3)^2 + 4}\right]$.**

OBJECTIVE TYPE QUESTIONS

A. Fill up the blanks

1. Find $L[t^2 e^{-3t}]$ = __________
2. The Laplace transform of $e^{-2t} \sin 5t$ is __________
3. $L\left[\dfrac{t}{e^t}\right]$ = __________
4. $L[\sin(2t + 3)]$ = __________
5. The Laplace transform of $t\, e^{-t} \sin t$ is __________.