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Partial Differential Equations

14.0 INTRODUCTION

An equation involving partial derivatives of a function w.r. to two or more independent variables is called a partial differential equation.

Examples

1. $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$

2. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$

3. $a^2\left(\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}\right)=\dfrac{\partial^2 z}{\partial t^2}$

4. $\left(\dfrac{\partial z}{\partial x}\right)^3+\dfrac{\partial^2 z}{\partial y^2}=\cos(x+y)$

Since many physical and social phenomena involve more than two independent variables, partial differential equations are the natural choice to deal with such problems. These equations arise in the study of fluid mechanics, heat transfer, electromagnetic theory and quantum mechanics.

14.1 ORDER AND DEGREE OF PARTIAL DIFFERENTIAL EQUATIONS

Definition 14.1 **Order** of a partial differential equation is the order of the highest order partial derivative occurring in the equation.

The **degree** of a partial differential equation is the degree of the highest order partial derivative occurring in the equation after the equation has been made free of radicals and fractions so far as the partial derivatives are concerned.

In example 1, order is 1 and degree is 1
In example 2, order is 2 and degree is 1
In example 4, order is 2 and degree is 1

14.2 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definition 14.2 A partial differential equation is said to be linear if the dependent variable and the partial derivatives occur in the first degree and there is no product of partial derivatives or product of derivative and dependent variable.

A partial differential equation which is not linear is said to be non-linear.
In example 1, the equation is first order, linear.
In example 2 and 3, the equations are second order, linear.
In example 4, the equation is second order, non-linear.

The partial differential equations $$\frac{\partial^2 z}{\partial x^2}+z\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=0$$

and $$\frac{\partial^2 z}{\partial x^2}+z^2=\sin(x+y)$$ are non-linear.

The partial differential equation $x^2\dfrac{\partial^2 z}{\partial x^2}+x\dfrac{\partial z}{\partial y}+y\dfrac{\partial z}{\partial y}+z=e^{x+y}$ is linear.

14.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations can be formed in two ways.

(i) By elimination of arbitrary constants from an ordinary functional relation between the variables

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the resulting partial differential equation will be of first order.

If the number of arbitrary constants to be eliminated is more than the number of independent variables, the resulting partial differential equation will be of second or higher order.

(ii) By elimination of arbitrary function or functions from an ordinary relation between the variables.

The order of the resulting partial differential equation will be equal to the number of arbitrary functions to be eliminated.

Usual Notation: If z is a function of two independent variables x and y, say $z = f(x, y)$, then we use the following notations.

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

WORKED EXAMPLES

Type 1: Formation of partial differential equation by elimination of arbitrary constants

EXAMPLE 1

Form the partial differential equation by eliminating arbitrary constants a and b from (i) $(x-a)^2 + (y-b)^2 + z^2 = c^2$ and (ii) $(x-a)^2 + (y-b)^2 + z^2 = 1$.

Solution.

(i) Given $\qquad (x-a)^2 + (y-b)^2 + z^2 = c^2 \qquad (1)$

Differentiating (1) partially w.r.to x and y,

we get, $\qquad 2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) + pz = 0 \Rightarrow x - a = -pz \qquad (2)$

and $\qquad 2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow y - b + qz = 0 \Rightarrow y - b = -qz \qquad (3)$

Substituting (2) and (3) in (1), we get

$$p^2z^2 + q^2z^2 + z^2 = c^2 \Rightarrow (p^2 + q^2 + 1)z^2 = c^2,$$

which is the required partial differential equation.

(ii) In (i) put $c = 1$

$\therefore$ The solution is $(p^2 + q^2 + 1)z^2 = 1$

EXAMPLE 2

Form the partial differential equation by eliminating a and b from $z = (x^2 + a^2)(y^2 + b^2)$.

Solution.

Given $\qquad z = (x^2 + a^2)(y^2 + b^2) \qquad (1)$

Differentiating (1) partially w.r.to x and y, we get,

$$\frac{\partial z}{\partial x} = (y^2 + b^2)\cdot 2x \quad \Rightarrow \quad p = (y^2+b^2)2x \quad \Rightarrow \quad \frac{p}{2x} = y^2 + b^2 \tag{2}$$

and $$\frac{\partial z}{\partial y} = (x^2 + a^2)2y \quad \Rightarrow \quad q = (x^2+a^2)2y \quad \Rightarrow \quad \frac{q}{2y} = x^2 + a^2 \tag{3}$$

Substituting (2) and (3) in (1), we get

$$z = \frac{q}{2y}\cdot\frac{p}{2x} \Rightarrow 4xyz = pq,$$

which is the required partial differential equation.

EXAMPLE 3

Form the partial differential equation by eliminating the arbitrary constants a and b from $\log_e(az-1) = x + ay + b$.

Solution.

Given $$\log_e(az-1) = x + ay + b \tag{1}$$

Differentiating (1) partially w.r.to x and y, we get,

$$\frac{1}{az-1}\cdot a\frac{\partial z}{\partial x} = 1 \quad \Rightarrow \quad ap = az - 1 \quad \Rightarrow \quad a(z-p) = 1 \quad \Rightarrow \quad a = \frac{1}{z-p} \tag{2}$$

and $$\frac{1}{az-1}a\cdot\frac{\partial z}{\partial y} = a \quad \Rightarrow \quad q = az - 1 \quad \Rightarrow \quad az - 1 = q$$

$\therefore$ $$\frac{z}{z-p} - 1 = q \quad \Rightarrow \quad \frac{z-z+p}{z-p} = q \quad \Rightarrow \quad \frac{p}{z-p} = q \quad \Rightarrow \quad p = q(z-p),$$

which is the required partial differential equation.

EXAMPLE 4

Find the partial differential equation of the family of spheres having their centres on the line $x = y = z$.

Solution.

Given that the centres of the spheres lie on the line $x = y = z$

$\therefore$ Centre of a sphere is (a, a, a). Let R be the radius.

So, the equation of the family of spheres is

$$(x-a)^2 + (y-a)^2 + (z-a)^2 = R^2 \tag{1}$$

Where a and R be arbitrary constants.

Differentiating (1) partially w.r.to x and y, we get,

$$2(x-a) + 2(z-a)\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad x - a + (z-a)p = 0$$

$\Rightarrow$ $$x + pz - a(p+1) = 0$$

$\Rightarrow \quad x + pz = a(p+1) \quad \Rightarrow \quad a = \dfrac{x+pz}{p+1}$ (2)

and $\quad 2(y-a) + 2(z-a)\dfrac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad y - a + (z-a)q = 0$

$\Rightarrow \quad y + qz - a(q+1) = 0 \quad \Rightarrow \quad a = \dfrac{y+qz}{q+1}$ (3)

From (2) and (3), $\dfrac{x+pz}{p+1} = \dfrac{y+qz}{q+1} \quad \Rightarrow \quad (x+pz)(q+1) = (y+qz)(p+1)$

$\Rightarrow \quad xq + x + pqz + pz = yp + y + pqz + qz$

$\Rightarrow \quad yp - pz + qz - xq = x - y \quad \Rightarrow \quad p(y-z) + q(z-x) = x - y$

which is the required partial differential equation.

EXAMPLE 5

Find the differential equation of all spheres whose centres lie on the *Z*-axis.

Solution.

Given that the centres of the spheres lie on the Z-axis.

$\therefore$ Centre is $(0, 0, c)$ and let R be the radius.

$\therefore$ Equation of the family of spheres is

$$x^2 + y^2 + (z-c)^2 = R^2 \quad (1)$$

where c and R are arbitrary constants.

Differentiating (1) partially w.r.to x and y, we get,

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad x + (z-c)p = 0 \quad \Rightarrow \quad (z-c)p = -x \quad (2)$$

and $$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad y + (z-c)q = 0 \quad \Rightarrow \quad (z-c)q = -y \quad (3)$$

$\dfrac{(2)}{(3)} \quad \Rightarrow \quad \dfrac{p}{q} = \dfrac{x}{y} \quad \Rightarrow \quad py = qx,$

which is the required partial differential equation.

EXAMPLE 6

Form a partial differential equation by eliminating the arbitrary constants *a*, *b*, *c* from $z = ax + by + cxy$.

Solution.

Given $$z = ax + by + cxy \quad (1)$$

In equation (1) the number of independent variables is two and the number of arbitrary constants is three.

$\therefore$ number of constants is greater than the number of independent variables. So, the resulting partial differential equation will be of order greater than one.

Differentiating (1) partially w.r.to x and y,

We get, $$\frac{\partial z}{\partial x} = a + cy \quad \Rightarrow \quad p = a + cy \qquad (2)$$

and $$\frac{\partial z}{\partial y} = b + cx \quad \Rightarrow \quad q = b + cx \qquad (3)$$

Using the three equations (1), (2) and (3) we cannot eliminate the three constants a, b, c. We need one more equation.
Differentiating (2) w.r.to x, we get

$$\frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial^2 z}{\partial x^2} = 0,$$

which is one partial differential equation obtained.

Note Different partial differential equations could be obtained depending upon the way, the elimination of the arbitrary constants is made. Instead of differentiating (2), if we differentiate (3) w.r.to y, then we get,

$\frac{\partial q}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y^2} = 0$, which is another partial differential equation.

If we differentiate (2) w.r.to y, then

$$\frac{\partial p}{\partial y} = c \quad \Rightarrow \quad \frac{\partial^2 z}{\partial y \partial x} = c \quad \Rightarrow s = c \qquad (4)$$

$\therefore \quad (2) \Rightarrow \quad p = a + sy \quad \Rightarrow \quad a = p - sy$

and (3) $\Rightarrow \quad q = b + sx \quad \Rightarrow \quad b = q - sx$

Substituting for a and b in (1), we get,

$$z = (p - sy)x + (q - sx)y + sxy$$

$$\Rightarrow \quad z = px - sxy + qy - sxy + sxy \quad \Rightarrow z = px + qy - sxy,$$

which is yet another partial differential equation,

Thus, three different partial differential equations could be formed. So, the resulting PDE is not unique when the number of constants is more than number of independent variables.

EXAMPLE 7

Form a partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution.

Given $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (1)$$

Differentiating (1) partially w.r.to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{x^2}{a^2} + \frac{pz}{c^2} = 0 \qquad (2)$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \quad \Rightarrow \frac{y}{b^2}+\frac{qz}{c^2}=0 \tag{3}$$

Differentiating (2) w.r.to x,

$$\frac{1}{a^2}+\frac{1}{c^2}\left[p\frac{\partial z}{\partial x}+z\frac{\partial p}{\partial x}\right]=0$$

$$\Rightarrow \quad \frac{c^2}{a^2}+p^2+z\frac{d^2z}{dx^2}=0 \quad \Rightarrow \quad \frac{c^2}{a^2}+p^2+zr=0 \tag{4}$$

[multiplying by c^2]

From (2), $$\frac{x}{a^2}=-\frac{pz}{c^2} \quad \Rightarrow \quad \frac{c^2}{a^2}=-\frac{pz}{x}$$

$\therefore$ (4) becomes, $-\dfrac{pz}{x}+p^2+zr=0 \quad \Rightarrow \quad xp^2+xzr-zp=0$

which is the required partial differential equation.

Note As in example (6), we can get different partial differential equations in this example and they are $zs+pq=0$ and $yzt+yp^2-zq=0$

EXAMPLE 8

Find a partial differential equation of all spheres of given radius.

Solution.

The equation of any sphere of given radius R is

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2 \tag{1}$$

where a,b,c are arbitrary constants.

Differentiating (1) partially w.r.to x, y we get,

$$2(x-a)+2(z-c)\frac{\partial z}{\partial x}=0 \quad \Rightarrow \quad x-a+(z-c)p=0 \tag{2}$$

and $$2(y-b)+2(z-c)\frac{\partial z}{\partial y}=0 \quad \Rightarrow \quad y-b+(z-c)q=0 \tag{3}$$

Differentiating (2) w.r.to x,

$$1+(z-c)\frac{\partial p}{\partial x}+p\cdot\frac{\partial z}{\partial x}=0 \quad \Rightarrow \quad 1+(z-c)\frac{\partial^2 z}{\partial x^2}+p^2=0$$

$$\Rightarrow \quad 1+(z-c)r+p^2=0 \qquad \left[\text{since } p=\frac{\partial z}{\partial x}\right]$$

$$\Rightarrow \quad (z-c)r=-(1+p^2) \tag{4}$$

and differentiating (3) w.r.to y,

$$1+(z-c)\frac{\partial q}{\partial y}+q\cdot\frac{\partial z}{\partial y}=0 \quad \Rightarrow \quad 1+(z-c)\frac{\partial^2 z}{\partial y^2}+q^2=0$$

$$\Rightarrow \quad 1+(z-c)t+q^2=0 \quad \Rightarrow \quad (z-c)t=-(1+q^2) \tag{5}$$

Now $\frac{(4)}{(5)} \Rightarrow \quad \frac{r}{t} = \frac{1+p^2}{1+q^2} \quad \Rightarrow \quad r(1+q^2) = t(1+p^2)$,

which is a partial differential equation.

Type 2: Formation of partial differential equation by elimination of arbitrary function(*s*)

2(A) Equation with single arbitrary function of the form $Z = f(x, y)$

The resulting partial differential equation will be of first order.

EXAMPLE 9

Eliminate the arbitrary function *f* from $z = f\left(\frac{xy}{z}\right)$ and form the partial differential equation.

Solution.

Given $$z = f\left(\frac{xy}{z}\right) \qquad (1)$$

Differentiating (1) partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right).\left[\frac{z.y - xy\frac{\partial z}{\partial x}}{z^2}\right] \quad \Rightarrow \quad p = f'\left(\frac{xy}{z}\right)\left[\frac{zy - xyp}{z^2}\right]$$

$$\Rightarrow \quad \frac{pz^2}{[zy - xyp]} = f'\left(\frac{xy}{z}\right) \qquad (2)$$

and $$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xy\frac{\partial z}{\partial y}}{z^2}\right] \quad \Rightarrow \quad q = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xyq}{z^2}\right]$$

$$\Rightarrow \quad \frac{qz^2}{zx - xyq} = f'\left(\frac{xy}{z}\right) \qquad (3)$$

From (2) and (3) we get,

$$\frac{pz^2}{zy - xyp} = \frac{qz^2}{zx - xyq} \quad \Rightarrow \quad p(zx - xyq) = q(zy - xyp)$$

$$\Rightarrow \quad pzx - xypq = qzy - xypq \quad \Rightarrow \quad pzx = qzy \quad \Rightarrow \quad px = qy,$$

which is the required partial differential equation.

EXAMPLE 10

Find the partial differential equation of eliminating *f* from $z = xy + f(x^2 + y^2 + z^2)$.

Solution.

Given $$z = xy + f(x^2 + y^2 + z^2) \qquad (1)$$

Differentiating (1) partially w.r.to x and y,

we get, $$\frac{\partial z}{\partial y} = y + f'(x^2 + y^2 + z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$\Rightarrow \qquad p = y + f'(x^2 + y^2 + z^2)(2x + 2zp)$

$\Rightarrow \qquad p - y = f'(x^2 + y^2 + z^2)\cdot(x + zp)$

$\Rightarrow \qquad \dfrac{p-y}{(x+zp)} = 2f'(x^2 + y^2 + z^2) \qquad (2)$

and $\qquad \dfrac{\partial z}{\partial y} = x + f'(x^2 + y^2 + z^2)\left(2y + 2z\dfrac{\partial z}{\partial y}\right)$

$\Rightarrow \qquad q = x + 2f'(x^2 + y^2 + z^2)(y + zq)$

$\Rightarrow \qquad q - x = 2f'(x^2 + y^2 + z^2)(y + zq)$

$\Rightarrow \qquad \dfrac{q-x}{y+zq} = 2f'(x^2 + y^2 + z^2) \qquad (3)$

From (2) and (3), we get $\dfrac{p-y}{x+zp} = \dfrac{q-x}{y+zq}$

$\Rightarrow \qquad (p-y)(y+zq) = (q-x)(x+zp)$

$\Rightarrow \qquad py + zpq - y^2 - yzp = qx + zpq - x^2 - xzp$

$\Rightarrow \qquad py - y^2 - yzp = qx - x^2 - xzp \quad \Rightarrow \quad p(y+xz) - q(x+yz) = y^2 - x^2$

which is the required partial differential equation.

EXAMPLE 11

2B. Formation of partial differential equation by eliminating arbitrary function ϕ from $\phi(u, v) = 0$, where u and v are functions of x. y. z

Method 1: $\phi(u, v) = 0$ can be rewritten as $v = f(u)$ or $u = g(v)$ where u and v are functions of x, y, z. Then proceed as in 2(A).

Method 2: Given $\phi(u, v) = 0$, where u and v are functions of x, y, z (1)

We have $d\phi = 0 \Rightarrow \dfrac{\partial \phi}{\partial u}du + \dfrac{\partial \phi}{\partial v}\cdot dv = 0$

$\therefore$ differentiating (1) partially w.r.to x, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$\Rightarrow \qquad \dfrac{\partial \phi}{\partial u}\left[\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}p\right] + \dfrac{\partial \phi}{\partial v}\left[\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}\cdot p\right] = 0 \qquad (2)$

Similarly differentiating (1) partially w.r.to y we get,

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right] = 0 \qquad (3)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix} = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}q+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}p+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}pq$$

$$-\left[\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}p+\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}q+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}pq\right]=0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} p + \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} q = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\Rightarrow \quad \frac{\partial(u, v)}{\partial(y, z)}p+\frac{\partial(u, v)}{\partial(z, x)}q=\frac{\partial(u, v)}{\partial(x, y)}$$

The partial differential equation is of the form $Pp+Qq=R$ where $P=\frac{\partial(u, v)}{\partial(y, z)}$, $Q=\frac{\partial(u, v)}{\partial(z, x)}$, and $R=\frac{\partial(u, v)}{\partial(x, y)}$

This is a partial differential equation of order 1.

EXAMPLE 12

Form the partial differential equation by eliminating the arbitrary function ϕ from $\phi\left(z^2-xy, \frac{x}{z}\right)=0$.

Solution.

Given $$\phi\left(z^2-xy, \frac{x}{z}\right)=0$$

It is of the form $\phi(u, v)=0$, where $u=z^2-xy$, $v=\frac{x}{z}$

$\therefore$ the partial differential equation obtained is of the form $Pp+Qq=R$

where $$P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)}, \quad R = \frac{\partial(u, v)}{\partial(x, y)}$$

Now $$u = z^2 - xy \quad \Rightarrow \frac{\partial u}{\partial x} = -y, \quad \frac{\partial u}{\partial y} = -x, \quad \frac{\partial u}{\partial z} = 2z$$

$$v = \frac{x}{z} \qquad \Rightarrow \frac{\partial v}{\partial x} = \frac{1}{z}, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial z} = -\frac{x}{z^2}$$

$\therefore$ $$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} -x & 2z \\ 0 & -\frac{x}{z^2} \end{vmatrix} = \frac{x^2}{z^2}$$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & -y \\ -\frac{x}{z^2} & \frac{1}{z} \end{vmatrix} = 2 - \frac{xy}{z^2}$$

and $$R = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -y & -x \\ \frac{1}{z} & 0 \end{vmatrix} = \frac{x}{z}$$

$\therefore$ the partial differential equation is $\frac{x^2}{z^2} p + \left(2 - \frac{xy}{z^2}\right) q = \frac{x}{z}$

$\Rightarrow$ $$x^2 p + (2z^2 - xy)q = xz$$

Note Assuming the given equation can be written as $z^2 - xy = f\left(\frac{x}{z}\right)$, we proceed as in type 2(A) to eliminate f. It is an exercise to the reader.

EXAMPLE 13

Form the partial differential equation by eliminating ϕ from $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$.

Solution.

Given $$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0 \tag{1}$$

It is of the form $\phi(u, v) = 0$

where $$u = x^2 + y^2 + z^2, \quad v = lx + my + nz$$

$\therefore$ the PDE is $Pp + Qq = R$

where $$P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)} \quad \text{and} \quad R = \frac{\partial(u, v)}{\partial(x, y)}$$

Now $\quad u = x^2 + y^2 + z^2 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y; \quad \frac{\partial u}{\partial z} = 2z$

$$v = lx + my + nz \quad \Rightarrow \quad \frac{\partial v}{\partial x} = l, \quad \frac{\partial v}{\partial y} = m, \quad \frac{\partial v}{\partial z} = n$$

$\therefore \quad P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ m & n \end{vmatrix} = 2ny - 2mz = 2(ny - mz)$

$$Q = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ n & l \end{vmatrix} = 2(lz - nx)$$

and $\quad R = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ l & m \end{vmatrix} = 2(mx - ly)$

$\therefore$ the partial differential equation is

$$2(ny - mz)p + 2(lz - nx)q = 2(mx - ly)$$

$\Rightarrow \quad (ny - mz)p + (lz - nx)q = mx - ly$

2(C) Formation of partial differential equation by eliminating two arbitrary functions in $z = f(x, y) + g(x, y)$

Given $\quad z = f(x, y) + g(x, y) \quad (1)$

To form the partial differential equation first we find the equations

$P = \frac{\partial z}{\partial x} \quad (2) \qquad q = \frac{\partial z}{\partial y} \quad (3) \qquad r = \frac{\partial^2 z}{\partial x^2} \quad (4)$

$s = \frac{\partial^2 z}{\partial x \partial y} \quad (5) \qquad t = \frac{\partial^2 z}{\partial y^2} \quad (6)$

From these six equations, we choose the suitable equations to eliminate f and g. The resulting partial differential equation will be a second order equation.

EXAMPLE 14

Form the partial differential equation by eliminating the arbitrary functions f and g in $z = x^2 f(y) + y^2 g(x)$.

Solution.

Given $\quad z = x^2 f(y) + y^2 g(x) \quad (1)$

$\therefore \quad P = \frac{\partial z}{\partial x} = 2xf(y) + y^2 g'(x) \quad (2)$

$$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2yg(x) \quad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g''(x) \quad (4)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = 2xf'(y) + 2yg'(x) \quad (5)$$

and $\quad t = \frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2g(x) \quad (6)$

$(2) \times x \Rightarrow \quad px = 2x^2 f(y) + xy^2 g'(x)$

$(3) \times y \Rightarrow \quad qy = x^2 yf'(y) + 2y^2 g(x)$

$\therefore \quad px + qy = 2[x^2 f(y) + y^2 g(x)] + xy[yg'(x) + xf'(y)]$

$(5) \Rightarrow \quad xf'(y) + yg'(x) = \frac{s}{2}$

$\therefore \quad px + qy = 2z + xy\frac{s}{2}$ [Using (1)]

$\Rightarrow \quad 2(px + qy) = 4z + xys$

which is the required partial differential equation.

EXAMPLE 15

Form the partial differential equation by eliminating the arbitrary functions f and g in $z = f(x^3 + 2y) + g(x^3 - 2y)$.

Solution.

Given $z = f(x^3 + 2y) + g(x^3 - 2y) \quad (1)$

$\therefore \quad p = \frac{\partial z}{\partial x} = f'(x^3 + 2y) \cdot 3x^2 + g'(x^3 - 2y) \cdot 3x^2$

$\Rightarrow \quad p = 3x^2[f'(x^3 + 2y) + g'(x^3 - 2y)] \quad (2)$

$$q = \frac{\partial z}{\partial y} = f'(x^3 + 2y) \cdot 2 + g'(x^3 - 2y)(-2)$$

$\Rightarrow \quad q = 2[f'(x^3 + 2y) - g'(x^3 - 2y)] \quad (3)$

$$r = \frac{\partial^2 z}{\partial x^2} = 3\left\{x^2[f''(x^3 + 2y) \cdot 3x^2 + g''(x^3 - 2y)3x^2] + [f'(x^3 + 2y) + g'(x^3 - 2y)] \cdot 2x\right\}$$

$\Rightarrow \quad r = 9x^4[f''(x^3+2y)+g''(x^3-2y)]+6x[f'(x^3+2y)+g'(x^3-2y)]$ (4)

$$s = \frac{\partial^2 z}{\partial x \partial y} = 2[f''(x^3+2y)\cdot 3x^2 - g''(x^3-2y)3x^2]$$

$\Rightarrow \quad s = 6x^2[f''(x^3+2y)-g''(x^3-2y)]$ (5)

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = 2[f''(x^3+2y)\cdot 2 - g''(x^3-2y)(-2)]$

$\Rightarrow \quad t = 4[f''(x^3+2y)+g''(x^3-2y)]$ (6)

(6) $\Rightarrow f''(x^3+2y)+g''(x^3-2y) = \dfrac{t}{4}$

(2) $\Rightarrow \quad \dfrac{p}{3x^2} = f'(x^3+2y)+g'(x^3-2y)$

Substituting in (4), we get $\quad r = 9x^4\cdot\dfrac{t}{4}+6x\cdot\dfrac{p}{3x^2}$

$\Rightarrow \quad r = \dfrac{9x^4 t}{4}+\dfrac{2p}{x} \quad \Rightarrow \quad 4xr = 9x^5 t + 8p,$

which is the required partial differential equation.

EXAMPLE 16

Eliminate the arbitrary functions f and g from $z = f(x+iy)+g(x-iy)$ and form a partial differential equation.

Solution.

Given $\quad z = f(x+iy)+g(x-iy)$ (1)

$\therefore \quad p = \dfrac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy)$ (2)

$$q = \frac{\partial z}{\partial y} = f'(x+iy)\cdot i + g'(x-iy)(-i)$$

$\Rightarrow \quad q = i[f'(x+iy)-g'(x-iy)]$ (3)

$r = \dfrac{\partial^2 z}{\partial x^2} = f''(x+iy)+g''(x-iy)$ (4)

$s = \dfrac{\partial^2 z}{\partial x \partial y} = i[f''(x+iy)-g''(x-iy)]$ (5)

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = i[f''(x+iy)\cdot i - g''(x-iy)(-i)]$

$\Rightarrow \quad t = i^2[f''(x+iy)+g''(x-iy)] \quad \Rightarrow \quad t = -r \quad \Rightarrow \quad r+t=0$ [using (4)]

which is the required partial differential equation.

EXAMPLE 17

Form the partial differential equation by eliminating f and ϕ from $z = xf\left(\frac{y}{x}\right) + y\phi(x)$.

Solution.

Given $$z = xf\left(\frac{y}{x}\right) + y\phi(x) \quad (1)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = xf'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$$

$$\Rightarrow \quad p = -\frac{y}{x}f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x) \quad (2)$$

$$q = \frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\frac{1}{x} + \phi(x) \quad \Rightarrow \quad q = f'\left(\frac{y}{x}\right) + \phi(x) \quad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = -y\left[\frac{1}{x}f''\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) + f'\left(\frac{y}{x}\right)\left(-\frac{1}{x^2}\right)\right] + f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + y\phi''(x)$$

$$\Rightarrow \quad r = \frac{y^2}{x^3}f''\left(\frac{y}{x}\right) + \frac{y}{x^2}f'\left(\frac{y}{x}\right) - \frac{y}{x^2}f'\left(\frac{y}{x}\right) + y\phi''(x)$$

$$\Rightarrow \quad r = \frac{y^2}{x^3}f''\left(\frac{y}{x}\right) + y\phi''(x) \quad (4)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \phi'(x) \quad \Rightarrow \quad s = -\frac{y}{x^2}f''\left(\frac{y}{x}\right) + \phi'(x) \quad (5)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right)\cdot\frac{1}{x} \quad \Rightarrow \quad tx = f''\left(\frac{y}{x}\right)$$

$$\therefore \quad s = tx\left(-\frac{y}{x^2}\right) + \phi'(x) \quad \Rightarrow \quad \phi'(x) = s + \frac{yt}{x}$$

$$(3) \Rightarrow \quad f'\left(\frac{y}{x}\right) = q - \phi(x)$$

$$(2) \Rightarrow \quad p = -\frac{y}{x}[q - \phi(x)] + f\left(\frac{y}{x}\right) + y\left[s + \frac{yt}{x}\right]$$

$$\Rightarrow \quad p = \frac{1}{x}\left[-yq + y\phi(x) + xf\left(\frac{y}{x}\right)\right] + \frac{y}{x}[xs + yt]$$

$$\Rightarrow \quad px = -yq + z + y[xs + yt] = y[xs + yt - q] + z$$

which is the required partial differential equation.

EXERCISE 14.1

I. Eliminating arbitrary constants form partial differential equation from the following:

1. $z = ax + by$
2. $z = ax + by + ab$
3. $z = ax^3 + by^3$
4. $z = (x+a)(y+b)$
5. $z = ax^n + by^n$
6. $z = (x+a)^2 + (y+b)^2$
7. $z = a^2x + ay^2 + b$
8. $z = (x^2+a)(y^2+b)$
9. $2z = (ax+y)^2 + b$
10. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
11. $z = ax - \frac{a}{a+1}y + b$
12. $4z(1+a^2) = (x+ay+b)^2$
13. $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$, where α is a given constant.
14. Find the differential equation of all planes cutting equal intercepts from the x and y axes.

II. Form partial differential equation by eliminating arbitrary function.

15. $z = f(x^2 - y^2)$
16. $z = x + y + f(xy)$
17. $z = yf\left(\frac{y}{x}\right)$
18. $z = f(x^2 + y^2 + z^2)$
19. $z = xy + f(x^2 + y^2)$
20. $z = e^{ay} f(x + by)$
21. $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$
22. $\phi(x^2 + y^2 + z^2, x + y + z) = 0$
23. $\phi(z - xy, x^2 + y^2) = 0$
24. $\phi(x + y + z, xy + z^2) = 0$

III. Form partial differential equation by eliminating the functions *f* and *g*.

25. $z = xf(x+t) + g(x+t)$
26. $z = f(x+it) + g(x-it)$, where $i = \sqrt{-1}$
27. $z = f(y+2x) + g(y-3x)$
28. $z = f(x+ay) + g(x-ay)$
29. $z = yf(x) + xg(y)$
30. $z = f(x) + e^y g(x)$

ANSWERS TO EXERCISE 14.1

1. $z = px + qy$
2. $z = px + qy + pq$
3. $3z = px + qy$
4. $z = pq$
5. $nz = px + qy$
6. $4z = p^2 + q^2$
7. $q^2 = 4py^2$
8. $4xy = pq$
9. $q^2 = px + qy$
10. $2z = px + qy$
11. $pq = p + q$
12. $z = p^2 + q^2$

13. $p^2 + q^2 = \tan^2 \alpha$
14. $p - q = 0$
15. $py + qx = 0$
16. $px - qy = x - y$
17. $z = px + qy$
18. $py - qx = 0$
19. $qx - py = x^2 - y^2$
20. $q = az + bp$
21. $z(p - q) = y - x$
22. $(y - z)p + (z - x)q = x - y$
23. $yp - xq = y^2 - x^2$
24. $(x - 2z)p + (2z - y)q = y - x$
25. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$
26. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
27. $r + s - 6t = 0$
28. $t = a^2 r$
29. $xys = xp + yq - z$
30. $t = q$

14.4 Solutions of Partial Differential Equations

Definition 14.3 A solution of a partial differential equation is a relation between the dependent and independent variables which satisfies the partial differential equation.

A solution is also known as an integral of the partial differential equation.

We have seen that a partial differential equation is formed by eliminating arbitrary constants and arbitrary functions. These relations are solutions of the partial differential equation formed. So, there are two types of solutions for partial differential equations.

1. Solution containing arbitrary constants.
2. Solution containing arbitrary functions.

Definition 14.4 Complete Integral or Complete Solution

A solution of a partial differential equation which contains as many arbitrary constants as the number of independent variables is called the **complete integral** or **complete solution**.

Definition 14.5 General Integral or General Solution

A solution of a partial differential equation which contains as many arbitrary functions as the order of the equation is called the **general solution** or **general integral**.

Note

(i) These two types of solutions can be obtained for the same partial differential equation.

For example, $z = ax + by$ is the complete integral and $z = xf\left(\frac{y}{x}\right)$ is the general integral of $z = px + qy$.

(ii) The general integral of a partial differential equation is more general than the complete integral.

For example, the complete integral $z = ax + by$ can be written as $z = x\left\{a + b\left(\frac{y}{x}\right)\right\}$. It is a particular form of the general integral $z = xf\left(\frac{y}{x}\right)$, where $f\left(\frac{y}{x}\right) = a + b\left(\frac{y}{x}\right)$

Thus, *when we solve a partial differential equation, we should find the general integral.*

However, there are partial differential equations for which it is not possible to find the general integral directly. In such cases we find the general integral from the complete integral.

Definition 14.6 Particular Integral

A solution of a partial differential equation obtained from the complete integral by giving particular values to the arbitrary constants is called a **particular integral**.

There is yet another solution called **singular integral** which does not contain any arbitrary constants and which cannot be obtained as a **particular integral**.

14.4.1 Procedure to Find General Integral and Singular Integral for a First Order Partial Differential Equation

Let $$F(x, y, z, p, q) = 0 \qquad (1)$$

be a first order partial differential equation and

let $$\phi(x, y, z, a, b) = 0 \qquad (2)$$

be the complete integral, where a and b are arbitrary constants.

1. To find the general integral of (1), put $b = f(a)$ [or $a = g(b)$] in (2) where f(or g) is an arbitrary function.

   Then (2) becomes $$\phi[(x, y, z, a, f(a))] = 0 \qquad (3)$$

   Differentiating (3) partially w.r.to a, we get

   $$\frac{\partial}{\partial a}[\phi(x, y, z, a, f(a))] = 0 \qquad (4)$$

   The eliminant of a from (3) and (4), if exists, contains the arbitrary function f, which is the general integral of (1).

   Geometrically, the envelope of the family of surfaces (3) is the general integral of (1).

2. **Singular integral**

   The complete integral is $\phi(x, y, z, a, b) = 0$

   Find $$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$$

   The eliminant of a and b from these three equations, if exists, is the singular integral of (1).

   Geometrically, singular integral is the envelope of the two parameter family of surfaces $\phi(x, y, z, a, b) = 0$

WORKED EXAMPLES

Equations solvable by direct integration

EXAMPLE 1

Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, given $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$, when $x = 0$. Show also that as $t \to \infty$, $u \to \sin x$.

Solution.

Given $$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$$

Integrating w.r.to x,

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$$

When $x = 0$, $\frac{\partial u}{\partial t} = 0 \quad \Rightarrow \quad f(t) = 0 \quad \therefore \quad \frac{\partial u}{\partial t} = e^{-t}\sin x$

Integrating w.r.to t, $$u = \frac{e^{-t}}{-1}\sin x + g(x)$$

When $t = 0, \quad u = 0 \quad \Rightarrow \quad -\sin x + g(x) = 0 \quad \Rightarrow \quad g(x) = \sin x$

$\therefore$ $$u = -e^{-t}\sin x + \sin x = \sin x[1 - e^{-t}]$$

As $t \to \infty, \quad e^{-t} \to 0, \quad \therefore \quad u = \sin x$

EXAMPLE 2

Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0, z = e^{-y}$ and $\frac{\partial z}{\partial x} = 1$.

Solution.

Given $$\frac{\partial^2 z}{\partial x^2} + z = 0$$

Integrate w.r.to x, treating z as a function of x alone, we have

$$\frac{d^2 z}{dx^2} + z = 0 \quad \Rightarrow (D^2 + 1)z = 0$$

Auxiliary equation is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$\therefore \; z = A\cos x + B\sin x$, where A and B are arbitrary constants.

Since z is a function of x and y, A and B are arbitrary functions of y, say $f(y)$ and $g(y)$.

$\therefore$ $$z = f(y)\cos x + g(y)\sin x$$

Differentiating partially w.r.to x,

$$\frac{\partial z}{\partial x} = f(y)(-\sin x) + g(y)(\cos x)$$

When $x = 0$, $\frac{\partial z}{\partial x} = 1 \quad \therefore \; g(y) = 1 \quad \therefore \quad z = f(x)\cos x + \sin x$

When $x = 0$, $z = e^{-y} \quad \therefore \; e^{-y} = f(y) \quad \therefore \quad z = e^{-y}\cos x + \sin x.$

EXAMPLE 3

Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, **when** $x = 0$, $\dfrac{\partial z}{\partial y} = -2\sin y$ **and when** y **is an odd multiple of** $\dfrac{\pi}{2}$, $z = 0$.

Solution.

Given $$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$$

Integrating w.r.to x, treating y as constant,

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$$

Given, when $x = 0$, $\dfrac{\partial z}{\partial y} = -2\sin y$.

$$\therefore \quad -2\sin y = -\cos 0 \sin y + f(y) \quad \Rightarrow \quad f(y) = -\sin y$$

$$\therefore \quad \frac{\partial z}{\partial y} = -\cos x \sin y - \sin y = -\sin y[1 + \cos x]$$

Integrating w.r.to y,

$$z = \cos y[1 + \cos x] + g(x)$$

Given, when $y = (2n+1)\dfrac{\pi}{2}$, $z = 0$

$$\therefore \quad \cos y = 0 \quad \Rightarrow g(x) = 0$$

Hence, $$z = (1 + \cos x)\cos y$$

EXAMPLE 4

Solve $\dfrac{\partial z}{\partial x} = 6x + 3y$ **and** $\dfrac{\partial z}{\partial y} = 3x - 4y$.

Solution.

Given $$\frac{\partial z}{\partial x} = 6x + 3y \quad (1) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3x - 4y \qquad (2)$$

Integrating (1) w.r.to x, we get

$$z = 6\frac{x^2}{2} + 3yx + f(y) \quad \Rightarrow \quad z = 3x^2 + 3yx + f(y) \qquad (3)$$

Differentiating (3) w.r.to y,

$$\frac{\partial z}{\partial y} = 3x + f'(y)$$

$$\Rightarrow \quad 3x - 4y = 3x + f'(y) \quad \Rightarrow \quad f'(y) = -4y$$

Integrating w.r.to y,

$$f(y) = -4\frac{y^2}{2} + C = -2y^2 + C \quad \therefore \quad z = 3x^2 + 3xy - 2y^2 + C \qquad \text{[using (3)]}$$

EXERCISE 14.2

Solve the following equations

1. $\dfrac{\partial^2 z}{\partial x^2} = xy$
2. $\dfrac{\partial^2 z}{\partial x \partial y} = 0$
3. $\dfrac{\partial^2 z}{\partial x \partial y} = 2(x+y)$
4. $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$
5. $\dfrac{\partial z}{\partial x} = 3x - y, \quad \dfrac{\partial z}{\partial y} = -x + \cos y$
6. $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0, \; z = e^y, \; \dfrac{\partial z}{\partial x} = 1$

ANSWERS TO EXERCISE 14.2

1. $z = \dfrac{x^3 y}{6} + xf(y) + g(y)$
2. $z = f(y) + g(x)$
3. $z = xy(x+y) + f(y) + g(x)$
4. $z = -\dfrac{1}{x^2}\sin xy + yf(x) + g(y)$
5. $z = \dfrac{3x^2}{2} - xy + \sin y + C$
6. $z = \sin x + e^y \cos x$

14.4.2 First Order Non-linear Partial Differential Equation of Standard Types

These standard types are Corollaries of a very general method known as Charpit's method for solving first order partial differential equation $F(x, y, z, p, q) = 0$. So, their solutions are assumed to be known.

Type 1: $F(p, q) = 0$

It is known that the complete integral is $z = ax + by + c$, where $F(a, b) = 0$

[Replacing p by a and q by b]

Solving $b = f(a)$

$\therefore$ C.I is
$$z = ax + f(a)y + c \tag{1}$$
where a and c are arbitrary constants.
To find the general integral, put $c = g(a)$, where g is arbitrary.

$\therefore$
$$z = ax + f(a)y + g(a) \tag{2}$$
Differentiating partially w.r.to a,
$$0 = x + f'(a)y + g'(a) \tag{3}$$
Elimating a from (2) and (3), we get the general solution.

To find singular integral:
Differentiating (1) w.r.to a and c partially, we get $0 = x + f'(a)y$
and $0 = 1$, which is not true.
$\therefore$ there is no singular solution for $F(p, q) = 0$

WORKED EXAMPLES

EXAMPLE 1

Solve $\sqrt{p} + \sqrt{q} = 1$.

Solution.

Given $\sqrt{p} + \sqrt{q} = 1$, which is type 1.

$\therefore$ the complete integral is $z = ax + by + c$,

where $\sqrt{a} + \sqrt{b} = 1$ [replacing p by a and q by b]

$$\Rightarrow \quad \sqrt{b} = 1 - \sqrt{a} \quad \Rightarrow \quad b = \left(1 - \sqrt{a}\right)^2$$

$\therefore$ the complete integral is $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$ (2)

where a and c are arbitrary constants.

To find the general integral; put $c = f(a)$

$$\therefore \quad z = ax + \left(1 - \sqrt{a}\right)^2 y + f(a) \qquad (2)$$

Differentiating partially w.r.to a, we get

$$0 = x + 2\left(1 - \sqrt{a}\right)\left(-\frac{1}{2\sqrt{a}}\right) y + f'(a)$$

$$\Rightarrow \quad x - \frac{\left(1 - \sqrt{a}\right)}{\sqrt{a}} y + f'(a) = 0 \qquad (3)$$

Eliminating a from (2) and (3), we get the general integral.

Differentiating (1) w.r.to c, we get $0 = 1$, which is not true.

$\therefore$ there is no singular integral.

EXAMPLE 2

Solve $p^2 - 2pq + 3q = 5$.

Solution.

Given $p^2 - 2pq + 3q = 5$, which is type1.

$\therefore$ the complete integral is $z = ax + by + c$,

where $a^2 - 2ab + 3b = 5$ [replacing p by a and q by b]

$$\Rightarrow \quad b[3 - 2a] = 5 - a^2 \quad \Rightarrow \quad b = \frac{5 - a^2}{3 - 2a}$$

$\therefore$ the complete integral is

$$z = ax + \left(\frac{5 - a^2}{3 - 2a}\right) y + c \qquad (1)$$

where a and c are arbitrary.

To find the general integral, put $c = f(a)$

$$\therefore \quad z = ax + \frac{5 - a^2}{3 - 2a} y + f(a) \qquad (2)$$

Differentiating (2) w.r.to a,

$$0 = x + \frac{(3-2a)(-2a)-(5-a^2)(-2a)}{(3-2a)^2} y + f'(a)$$

$$\Rightarrow \quad x + \frac{[-6a+4a^2+10a-2a^3]}{(3-2a)^3} y + f'(a) = 0$$

$$\Rightarrow \quad x + \frac{2a[2+2a-a^2]}{(3-2a)^3} y + f'(a) = 0 \qquad (3)$$

Eliminating a from (2) and (3), we get the general integral.
Differentiating (1) w.r.to c, we get $0 = 1$ which is not true.
$\therefore$ there is no singular integral.

Type 2: Clairaut's form

Equation of the form $z = px + qy + f(p,q)$ is called Clairaut's equation.

The complete integral is known to be $z = ax + by + f(a,b)$ (1),
replacing p by a and q by b, where a and b are arbitrary constants.
To find the general integral, put $b = \phi(a)$ in (1)

$$\therefore \quad z = ax + \phi(a)y + f[a, \phi(a)] \qquad (2)$$

Differentiating (1) w.r.to a and eliminating b, we get the general integral.

To find the singular solution

Differentiating (1) w.r.to a and b, We get

$$x + \frac{\partial f}{\partial a} = 0 \quad (3) \qquad \text{and} \qquad y + \frac{\partial f}{\partial b} = 0 \qquad (4)$$

Eliminating a and b from (1), (3), (4), we get the singular solution.
Normally, the singular integral exists for Clairaut's equation.

WORKED EXAMPLES

EXAMPLE 3

Solve $z = px + qy + p^2q^2$.

Solution.

Given equation is $\quad z = px + qy + p^2q^2$

It is Clairaut's form

$\therefore$ the complete integral is $\quad z = ax + by + a^2b^2 \qquad (1)$

To find the G.I, put $b = \phi(a)$ in (1) $\quad \therefore \quad z = ax + \phi(a)\cdot y + a^2(\phi(a))^2 \qquad (2)$

Differentiating (2) w.r.to a,

$$0 = x + \phi'(a)y + a^2 \cdot 2\phi'(a) + (\phi(a))^2 \cdot 2a \qquad (3)$$

Eliminating a from (2) and (3), we get the general Integral.

To find singular integral

Differentiating the C.I. (1) partially w.r.to a and b, we get

$$0 = x + 2ab^2 \quad \Rightarrow \quad x = -2ab^2 \tag{4}$$

and $$0 = y + 2a^2b \quad \Rightarrow \quad y = -2a^2b \tag{5}$$

$$\frac{(4)}{(5)} \Rightarrow \quad \frac{x}{y} = \frac{-2ab^2}{-2a^2b} = \frac{b}{a} \quad \Rightarrow \quad \frac{x}{b} = \frac{y}{a} = k$$

Now $$\frac{x}{b} = k \quad \Rightarrow \quad \frac{x}{k} = b \quad \text{and} \quad \frac{y}{a} = k \quad \Rightarrow \quad \frac{y}{k} = a$$

Substituting in (4), $$x = -2\frac{y}{k}\cdot\frac{x^2}{k^2} \quad \Rightarrow \quad k^3 = -2xy \quad \Rightarrow \quad k = -(2xy)^{1/3}$$

Substitution in (1), $$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4} = \frac{2xy}{k} + \frac{x^2y^2}{k^4}$$

$$\Rightarrow \quad kz = 2xy + \frac{x^2y^2}{k^3} = 2xy - \frac{x^2y^2}{2xy} = 2xy - \frac{xy}{2} = \frac{3xy}{2}$$

Cubing both sides, $$k^3z^3 = \frac{27}{8}x^3y^3 \quad \Rightarrow \quad -2xyz^3 = \frac{27}{8}x^3y^3 \qquad \text{[using (6)]}$$

$$\Rightarrow \quad -16z^3 = 27x^2y^2 \quad \Rightarrow \quad 16z^3 + 27x^2y^2 = 0$$

which is the singular integral.

EXAMPLE 4

Find the singular integral of the partial differential equation $z = px + qy + p^2 - q^2$.

Solution.

Given $$z = px + qy + p^2 - q^2$$

It is Clairaut's form.

$\therefore$ the complete integral is $z = ax + by + a^2 - b^2$ (1)

To find singular solution:

Differentiating (1) partially w.r.to a and b, we get

$$0 = x + 2a \quad \Rightarrow \quad a = -\frac{x}{2} \quad \text{and} \quad 0 = y - 2b \quad \Rightarrow \quad b = \frac{y}{2}$$

Substituting in (1), we get

$$z = -\frac{x}{2}\cdot x + \frac{y}{2}\cdot y + \left(-\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$= -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} = -\frac{x^2}{4} + \frac{y^2}{4} \quad \Rightarrow \quad 4z = y^2 - x^2,$$

which is the singular solution.

EXAMPLE 5

Find the singular solution of $z = px + qy + \sqrt{p^2 + q^2 + 1}$.

Solution.

Given $$z = px + qy + \sqrt{1 + p^2 + q^2}$$

This is clairaut's form.

So, the complete integral is $z = ax + by + \sqrt{1 + a^2 + b^2}$ (1)

where a and b are arbitrary constants.

To find singular solution:

Differentiating (1) partially w.r.to a and b, we get,

$$0 = x + \frac{1}{2\sqrt{1 + a^2 + b^2}} \cdot 2a \quad \Rightarrow x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \qquad (2)$$

and $$0 = y + \frac{1}{2\sqrt{1 + a^2 + b^2}} \cdot 2b \quad \Rightarrow y = -\frac{b}{\sqrt{1 + a^2 + b^2}} \qquad (3)$$

Now, $$x^2 + y^2 = \frac{a^2}{1 + a^2 + b^2} + \frac{b^2}{1 + a^2 + b^2} = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$\therefore$ $$1 - x^2 - y^2 = 1 - \frac{(a^2 + b^2)}{1 + a^2 + b^2} = \frac{1 + a^2 + b^2 - a^2 - b^2}{1 + a^2 + b^2} = \frac{1}{1 + a^2 + b^2}$$

$\Rightarrow$ $$1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2} \quad \Rightarrow \sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$\therefore$ (2) $\Rightarrow$ $$a = -x\sqrt{1 + a^2 + b^2} = -\frac{x}{\sqrt{1 - x^2 - y^2}}$$

and (3) $\Rightarrow$ $$b = -y\sqrt{1 + a^2 + b^2} = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

Substituting for a and b in (1), we get

$$z = -\frac{x^2}{\sqrt{1 - x^2 - y}} - \frac{y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$= \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}} = \sqrt{1 - x^2 - y^2}$$

$\Rightarrow$ $$z^2 = 1 - x^2 - y^2$$

$\Rightarrow$ $x^2 + y^2 + z^2 = 1$, which is the singular solution.

EXERCISE 14.3

Solve the following partial differential equations:

1. $p^2+q^2=4$
2. $p^2+q^2=npq$
3. $pq+p+q=0$
4. $pq=1$
5. $q+\sin p=0$

Find the singular integral of the following partial differential equations.

6. $z=px+qy-2\sqrt{pq}$
7. $z=px+qy+\sqrt{p^2+q^2}$
8. $z=px+q-4p^2q^2$
9. $(p+q)(z-px-qy)=1$
10. $(1-x)p+(2-y)q=3-z$

ANSWERS TO EXERCISE 14.3

1. C.I is $z=ax+\sqrt{4-a^2}\,y+c$
2. C.I is $z=ax+\frac{a}{2}\left(n\pm\sqrt{n^2-4}\right)y+c$
3. C.I is $z=ax-\frac{a}{a+1}y+c,\ a\neq -1$
4. C.I is $z=ax+\frac{y}{a}+c,\ a\neq 0$
5. C.I is $z=ax-y\sin a+c$
6. $xy=1$
7. $x^2+y^2=1$
8. $z^3+8x^2y^2=0$
9. $z^4=16xy$
10. $z=3$

Type 3: (a) Equation of the form $F(z, p, q)=0$

Given $F(z, p, q)=0$

$\therefore$ assume that $z=f(x+ay)$ is a solution, where a is a constant.

Put $u=x+ay$, then $z=f(u)$

$\therefore$
$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du}\cdot 1=\frac{dz}{du}$$
$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}=\frac{dz}{du}\cdot a=a\frac{dz}{du}$$

Substituting for p and q, we get an equation of the form.

$$\frac{dz}{du}=g(z,a)\quad\Rightarrow\quad\frac{dz}{g(z,a)}=du$$

Integrating both sides, $\phi(z,a)=u+c \quad\Rightarrow\quad \phi(z,a)=x+ay+c$ (1)

which is the complete integral.

To find the general integral, put $c=h(a)$ and proceed as in type 1.

Differentiating (1) partially w.r.to c, we get $0=1$, which is not true.

$\therefore$ there is no singular integral.

WORKED EXAMPLES

EXAMPLE 1

Solve $z = p^2 + q^2$.

Solution

Given $$z = p^2 + q^2 \qquad (1)$$

This is of the form $F(z, p, q) = 0$

$\therefore$ Put $u = x + ay$, then $p = \dfrac{dz}{du}, \quad q = a\dfrac{dz}{du}$

Substituting for p, q in (1), we get

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 \Rightarrow z = (1+a^2)\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \quad \sqrt{z} = \left(\sqrt{1+a^2}\right)\frac{dz}{du}$$

$$\Rightarrow \quad \frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}} \Rightarrow \left(\sqrt{1+a^2}\right)\frac{dz}{\sqrt{z}} = du$$

Integrating both sides,

$$\sqrt{1+a^2}\int z^{-1/2}dz = \int du \Rightarrow \left(\sqrt{1+a^2}\right)\frac{z^{-1/2+1}}{\frac{-1}{2}+1} = u + c, \quad c \text{ is } a \text{ constant.}$$

$$\Rightarrow \quad \left(\sqrt{1+a^2}\right)\frac{z^{1/2}}{\frac{1}{2}} = u + c \Rightarrow 2\sqrt{1+a^2}\,z^{1/2} = x + ay + c, \qquad (2)$$

which is the complete integral.
To find the general integral, put $c = h(a)$

$$\therefore \quad 2\sqrt{1+a^2}\sqrt{z} = x + ay + h(a) \qquad (3)$$

Differentiating w.r.to a partially, we get

$$2\cdot\frac{1}{2\sqrt{1+a^2}}\cdot 2a\sqrt{z} = y + h'(a) \Rightarrow \frac{2a\sqrt{z}}{\sqrt{1+a^2}} = y + h'(a) \qquad (4)$$

Eliminating a between (2) and (4), we get the general solution.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular solution.

EXAMPLE 2

Solve $9(p^2z + q^2) = 4$.

Solution.

Given $$9\cdot(p^2z + q^2) = 4 \qquad (1)$$

This is of the form $F(z, p, q) = 0$

Put $u = x + ay$, then $$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Substitution for p, q in (1), we get

$$9\left[\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \quad \Rightarrow \quad 9\left(\frac{dz}{du}\right)^2 (z + a^2) = 4$$

$$\Rightarrow \quad \frac{9}{4}(z + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow \quad \frac{3}{2}\sqrt{z + a^2}\left(\frac{dz}{du}\right) = 1 \quad \Rightarrow \quad \frac{3}{2}\sqrt{z + a^2}\, dz = du$$

Integrating both sides,

$$\frac{3}{2}\int\sqrt{z + a^2}\, dz = \int du \quad \Rightarrow \quad \frac{3}{2}\frac{(z + a^2)^{1/2+1}}{\frac{1}{2}+1} = u + c \qquad \left[\because \int [f(x)^n] f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\Rightarrow \quad \frac{3}{2}\frac{(z + a^2)^{3/2}}{\frac{3}{2}} = x + ay + c \quad \Rightarrow \quad (z + a^2)^{3/2} = x + ay + c \qquad (2)$$

which is the complete integral.
To find the general solution put $c = h(a)$ in (2)

$$\therefore \quad (z + a^2)^{3/2} = x + ay + h(a) \qquad (3)$$

Differentiating (3) w.r.to a partially and eliminating a, we get the general integral.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

EXAMPLE 3

Solve $z^2(p^2 + q^2 + 1) = 1$.

Solution.
Given $$z^2(p^2 + q^2 + 1) = 1 \qquad (1)$$
This is the form $$F(z, p, q) = 0$$

$\therefore$ Put $u = x + ay$, then $$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Substituting for p and q in (1), we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$\Rightarrow \quad z^2\left[(1 + a^2)\left(\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$\Rightarrow \quad z^2(1 + a^2)\left(\frac{dz}{du}\right)^2 = 1 - z^2$$

$$\Rightarrow \qquad z\sqrt{1+a^2}\frac{dz}{du} = \sqrt{1-z^2} \quad \Rightarrow \quad \sqrt{1+a^2}\frac{z}{\sqrt{1-z^2}}dz = du$$

Integrating, $\int \sqrt{1+a^2}\frac{z}{\sqrt{1-z^2}}dz = \int du$

$$\Rightarrow \qquad \sqrt{1+a^2}\int (1-z^2)^{1/2} \cdot z dz = u + c$$

$$\Rightarrow \qquad -\frac{\sqrt{1+a^2}}{2}\int (1-z^2)^{-1/2}(-2z)dz = u + c$$

$$\Rightarrow \qquad -\frac{\sqrt{1+a^2}}{2}\frac{(1-z^2)^{1/2}}{1/2} = x + ay + c \quad \Rightarrow \quad -\sqrt{1+a^2}\cdot\sqrt{1-z^2} = x + ay + c \qquad (2)$$

which is the complete integral.
To find the general integral, put $c = h(a)$

$$\therefore \qquad -\sqrt{1+a^2}\sqrt{1-z^2} = x + ay + h(a) \qquad (3)$$

Differentiating (3) w.r.to a and eliminate a to get the general integral.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

EXAMPLE 4

Solve $q^2 = z^2 p^2 (1 - p^2)$.

Solution.
Given $q^2 = z^2 p^2 (1 - p^2)$ (1)

This is of the form $F(z, p, q) = 0$

$\therefore$ Put $u = x + ay$, then $p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$

Substituting for p and q in (1), we get

$$a^2\left(\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1 - \left(\frac{dz}{du}\right)^2\right]$$

Since $z = f(u)$ is not a constant, $\frac{dz}{du} \neq 0$

$\therefore$ dividing by $\left(\frac{dz}{du}\right)^2$, we get

$$a^2 = z^2\left[1 - \left(\frac{dz}{du}\right)^2\right] \quad \Rightarrow \quad a^2 = z^2 - z^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \qquad z^2\left(\frac{dz}{du}\right)^2 = z^2 - a^2$$

$$\Rightarrow \qquad z\frac{dz}{du} = \sqrt{z^2 - a^2} \quad \Rightarrow \quad \frac{z}{\sqrt{z^2 - a^2}}dz = du$$

Integrating, $$\int \frac{z}{\sqrt{z^2-a^2}}\,dz = \int du$$

$\Rightarrow$ $$\frac{1}{2}\int (z^2-a^2)^{-1/2}\cdot 2z\,dz = u + c$$

$\Rightarrow$ $$\frac{1}{2}\frac{(z^2-a^2)^{-1/2+1}}{\frac{-1}{2}+1} = x + ay + c$$

$\Rightarrow$ $$\frac{1}{2}\frac{(z^2-a^2)^{1/2}}{\frac{1}{2}} = x + ay + c \quad \Rightarrow \quad (z^2-a^2)^{1/2} = x + ay + c \qquad (2)$$

which is the complete integral.
To find the general integral, put $c = h(a)$ in (2)

$\therefore$ $$\sqrt{z^2-a^2} = x + ay + h(a) \qquad (3)$$

Differentiating (3) partially w.r.to a and eliminating a, we get the general integral.
There is no singular integral, since differentiating (2) partially w.r.to a, we get $0 = 1$, which is not true.

Type 3(b): Equation of the form $F(x, p, q) = 0$
Put $q = a$, a constant and solve for $p = \phi(x, a)$
Since z is a function of x and y,
$$dz = p\,dx + q\,dy$$
$\Rightarrow$ $$dz = \phi(x, a)dx + a\,dy$$
Integrating, $$z = \int \phi(x, a)dx + a\int dy + c = f(x, a) + ay + c \qquad (1)$$
which is the complete integral.
Differentiating (1) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

WORKED EXAMPLES

EXAMPLE 5
Solve $q = px + p^2$.

Solution.
Given $$q = px + p^2 \qquad (1)$$
It is of the form $$F(x, p, q) = 0$$
Put $q = a$ in (1)

$\therefore$ $$a = px + p^2 \quad \Rightarrow \quad p^2 + px - a = 0 \quad \Rightarrow \quad p = \frac{-x \pm \sqrt{x^2+4a}}{2}$$

We know that $$dz = p\,dx + q\,dy \quad \Rightarrow \quad dz = \frac{-x \pm \sqrt{x^2+4a}}{2}dx + a\,dy$$

Integrating, $$z = \frac{1}{2}\int\left(-x \pm \sqrt{x^2+4a}\right)dx + a\int dy$$

$$\Rightarrow \qquad z = \frac{1}{2}\left[\frac{-x^2}{2} \pm \left\{\frac{x\sqrt{x^2+4a}}{2} + \frac{4a}{2}\log_e\left(x+\sqrt{x^2+4a}\right)\right\}\right] + ay + c \qquad (2)$$

which is the complete integral.
The general integral is found by putting $c = \phi(a)$ and differentiating w.r.to a partially and eliminating a.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

EXAMPLE 6

Solve $\sqrt{p} + \sqrt{q} = \sqrt{x}$.

Solution.

Given $\qquad \sqrt{p} + \sqrt{q} = \sqrt{x}$

It is of the form $\qquad F(x, p, q) = 0$

$\therefore$ Put $q = a$ then $\qquad \sqrt{p} + \sqrt{q} = \sqrt{x}$

$$\Rightarrow \qquad \sqrt{p} = \sqrt{x} - \sqrt{a} \quad \Rightarrow \quad p = x + a - 2\sqrt{a}\sqrt{x} \qquad \text{(squaring)}$$

We know that, $\qquad dz = pdx + qdy \quad \Rightarrow \quad dz = \left(x + a - 2\sqrt{a}\sqrt{x}\right)dx + ady$

Integrating $\qquad z = \int\left(x + a - 2\sqrt{a}\sqrt{x}\right)dx + a\int dy$

$$\Rightarrow \qquad z = \frac{x^2}{2} + ax - 2\sqrt{a}\,\frac{x^{3/2}}{\frac{3}{2}} + ay + c$$ [where a and c are arbitrary constants]

$$\Rightarrow \qquad z = \frac{x^2}{2} + ax - \frac{4\sqrt{a}}{3}x^{3/2} + ay + c \qquad (2)$$

which is the complete integral.
The general integral is found by putting $c = \phi(a)$ and differentiating (2) partially w.r.to a and eliminating a.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

Type 3(c): Equation of the form $F(y, p, q) = 0$

Given $\qquad F(y, p, q) = 0 \qquad (1)$

Put $p = a$ and solve for $q = \phi(y, a)$

Since z is a function of x and y,

$$dz = pdx + qdy \quad \Rightarrow \quad dz = adx + \phi(y, a)dy$$

Integrating, $\qquad z = a\int dx + \int \phi(y, a)dy$

$$\Rightarrow \qquad z = a + f(y, a) + c \qquad (2)$$

where a and c arbitrary are constants. Which is the complete integral.

The general integral is found by putting $c = \phi(a)$ and differentiating (2) partially w.r.to a and eliminating a.
Differentiating (2) partially w.r.to c, we get $0 = 1$, which is not true.
$\therefore$ there is no singular integral.

WORKED EXAMPLES

EXAMPLE 7

Solve $pq = y$.

Solution.

Given $\qquad pq = y \qquad (1)$

This of the form $\qquad F(y, p, q) = 0$

$\therefore$ Put $p = a$, then $\qquad aq = y \Rightarrow q = \frac{y}{a}$

We know that $\qquad dz = pdx + qdy = adx + \frac{y}{a}dy$

Integrating, $\qquad z = a\int dx + \frac{1}{a}\int ydy = ax + \frac{1}{a}\frac{y^2}{2} + c \qquad (2)$

where a and c are arbitrary. Which is the complete integral.
There is no singular integral.
The general integral is obtained by putting $c = \phi(a)$ in (2) and differentiating w.r.to a and eliminating a.

EXAMPLE 8

Solve $q = py + p^2$.

Solution.

Given $\qquad q = py + p^2 \qquad (1)$

This is of the form $\qquad F(y, p, q) = 0$

$\therefore$ Put $p = a$, then $\qquad q = ay + a^2$

We know that $\qquad dz = pdx + qdy = adx + (ay + a^2)dy$

Integrating, $\qquad z = ax + \left(a\frac{y^2}{2} + a^2 y\right) + c \qquad (2)$

which is the complete integral.
There is no singular integral.
The general integral is found by putting $c = \phi(a)$ in (2), differentiating (1) w.r.to a and eliminating a.

Type 4: Separable equations

A first order partial differential equation is said to be separable if it can be written as

$$f(x, p) = g(y, q)$$

Let $f(x, p) = g(y, q) = a$, where a is an arbitrary constant.

then $f(x, p) = a \quad \Rightarrow \quad p = \phi(x, a)$ and $g(y, q) = a \quad \Rightarrow \quad q = \psi(y, a)$

Since z is a function of x and y,

$$dz = pdx + qdy \quad \therefore \quad dz = \phi(x, a)dx + \psi(y, a)dy$$

Integrating, $z = F(x, a) + G(y, a) + c$, which is the complete integral.

There is no singular integral and general integral is obtained as usual.

WORKED EXAMPLES

EXAMPLE 9

Solve $p^2 y(1 + x^2) = qx^2$.

Solution.

Given $p^2 y(1+x^2) = qx^2$. It is separable type.

$$\therefore \quad \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \qquad [a \text{ is a constant}]$$

$$\therefore \quad q = ay, \quad p^2 = \frac{ax^2}{1+x^2} \quad \Rightarrow \quad p = \frac{\sqrt{a}x}{\sqrt{1+x^2}}$$

We know that

$$dz = pdx + qdy = \frac{\sqrt{a}x}{\sqrt{1+x^2}}dx + aydy$$

Integrating,

$$z = \sqrt{a}\int \frac{xdx}{\sqrt{1+x^2}} + a\int ydy$$

$$= \frac{\sqrt{a}}{2}\int (1+x^2)^{-1/2} \cdot 2xdx + a\int ydy$$

$$\Rightarrow \quad z = \frac{\sqrt{a}}{2}\frac{(1+x^2)^{1/2}}{\frac{1}{2}} + \frac{ay^2}{2} + c = \sqrt{a}\sqrt{1+x^2} + \frac{a}{2}y^2 + c \qquad (1)$$

This is the complete integral.
There is no singular integral.
The general integral is obtained by putting $c = \phi(a)$ in (1), differentiating w.r.to a and eliminating a.

EXAMPLE 10

Find the complete integral of $p + q = \sin x + \sin y$.

Solution.

Given $p + q = \sin x + \sin y$. It is separable type.

$\therefore \quad p - \sin x = \sin y - q = a$, say

$\therefore \quad p = a + \sin x, \quad$ and $\quad q = \sin y - a$

We know that $\quad dz = pdx + qdy = (a + \sin x)dx + (\sin y - a)dy$

Integrating, $\quad z = \int (a + \sin x)dx + \int (\sin y - a)dy$

$\Rightarrow \quad z = ax - \cos x - \cos y - ay + c$

This is the complete integral.

EXAMPLE 11

Solve $p^2 + q^2 = x^2 + y^2$.

Solution.

Given $\quad p^2 + q^2 = x^2 + y^2$. $\quad$ It is separable type.

$\therefore \quad p^2 - x^2 = y^2 - q^2 = a$

$\therefore \quad p^2 = x^2 + a \quad \Rightarrow \quad p = \sqrt{x^2 + a} \quad$ and $\quad q^2 = y^2 - a \quad \Rightarrow \quad q = \sqrt{y^2 - a}$

We know that $\quad dz = pdx + qdy = \sqrt{x^2 + a}\, dx + \sqrt{y^2 - a}\, dy$

Integrating, $\quad z = \int \sqrt{x^2 + a}\, dx + \int \sqrt{y^2 - a}\, dy$

$$\Rightarrow \quad z = \frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\log\left(x + \sqrt{x^2 + a}\right) + \frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2}\log\left(y - \sqrt{y^2 - a}\right) + c \qquad (1)$$

This is the complete integral.
There is no singular integral.
The general integral is found by putting $c = \phi(a)$ in (1), differentiating w.r.to a and eliminating a.

14.4.3 Equations Reducible to Standard Forms

(A) Equation of the form $\quad F(x^m p, y^n q) = 0 \qquad (1)$

and $\quad F(z, x^m p, y^n q) = 0 \qquad (2)$

Case 1: If $m \neq 1, \quad n \neq 1$, put $x^{1-m} = X, \quad y^{1-n} = Y$

$$\therefore \quad \frac{dX}{dx} = (1-m)x^{-m}, \quad \frac{dY}{dy} = (1-n)y^{-n}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{\partial z}{\partial X}(1-m)x^{-m}$$

$$\Rightarrow \quad p = (1-m)x^{-m}P, \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$\Rightarrow \quad x^m p = (1-m)P.$$

Similarly, $y^n q = (1-n)Q$, where $Q = \dfrac{\partial z}{\partial Y}$

Then (1) becomes $f(P, Q) = 0$, which is standard type (1)

and (2) becomes $f(z, P, Q) = 0$, which is standard type 3(a).

Case 2: If $m = 1$, $n = 1$, then the equations are $F(xp, yp) = 0$ and $F(z, xp, yp) = 0$

Put $X = \log x$, $Y = \log y$, then $\dfrac{dX}{dx} = \dfrac{1}{x}$ and $\dfrac{dY}{dy} = \dfrac{1}{y}$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = P\frac{1}{x} \quad \Rightarrow \quad px = P$$

and
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{dY}{dy} = Q\frac{1}{y} \quad \Rightarrow \quad qy = Q$$

where
$$P = \frac{\partial z}{\partial X}, \quad Q = \frac{\partial z}{\partial Y}$$

Then (1) becomes $F(P, Q) = 0$, which is standard type (1)

and (2) becomes $F(z, P, Q) = 0$, which is standard type 3(a).

(B) Equation of the form $F(x^m z^k p, y^n z^k q) = 0$

Case 1: If $m \neq 1$, $n \neq 1$, $k \neq -1$. Put $X = x^{1-m}$, $Y = y^{1-n}$ and $Z = z^{k+1}$.

$$\therefore \frac{dX}{dx} = (1-m)x^{1-m-1} \Rightarrow \frac{dx}{dX} = \frac{x^m}{1-m} \quad \text{and} \quad \frac{\partial Z}{\partial z} = (k+1)z^k.$$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = (k+1)z^k p\cdot\frac{x^m}{1-m} \quad \Rightarrow \quad x^m\cdot z^k\cdot p = \frac{(1-m)}{k+1}P$$

Similarly, $y^n z^k q = \dfrac{(1-n)}{k+1}Q$

$\therefore$ the equation reduces to $F(P, Q) = 0$, which is standard type (1).

Case 2: If $m = 1$, $n = 1$, $k = -1$, then the equation is $F\left(\dfrac{px}{z}, \dfrac{qy}{z}\right) = 0.$

Put $X = \log_e x$, $Y = \log_e y$ and $Z = \log_e z$. $\therefore$ $\dfrac{dX}{dx} = \dfrac{1}{x}$, $\dfrac{dY}{dy} = \dfrac{1}{y}$ and $\dfrac{\partial Z}{\partial z} = \dfrac{1}{z}$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = \frac{px}{z}$$

Similarly, $Q = \dfrac{qy}{z}$

$\therefore$ the equation becomes $F(P, Q) = 0$, which is standard type (1).

(C) Equation of the form $F(z^k p, z^k q) = 0$

Case 1: If $k \neq -1$, put $Z = z^{k+1}$, then $\dfrac{\partial Z}{\partial z} = (k+1)z^k$.

$\therefore$ $$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = (k+1)z^k \cdot p \quad \Rightarrow \quad \frac{P}{k+1} = z^k p$$

Similarly, $$\frac{Q}{k+1} = z^k q$$

$\therefore$ the equation is $F(P, Q) = 0$, which is standard type 1.

Case 2: If $k = -1$, the equation is $F\left(\dfrac{p}{z}, \dfrac{q}{z}\right) = 0$

Put $Z = \log_e z$, then $P = \dfrac{p}{z}$, $Q = \dfrac{q}{z}$

$\therefore$ the equation is $F(P, Q) = 0$, which is standard type 1.

WORKED EXAMPLES

EXAMPLE 12

Solve $x^2 p^2 + y^2 q^2 = z^2$.

Solution.

Given $$x^2 p^2 + y^2 q^2 = z^2$$

$$\Rightarrow \quad \left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1 \qquad \text{(Dividing by } z^2\text{)}$$

It is the form $$F\left(\frac{px}{z}, \frac{qy}{z}\right) = 0 \qquad [m = 1, n = 1, k = -1, \text{(B) Case 2}]$$

Put $X = \log_e x$, $Y = \log_e y$, $Z = \log_e z$

$\therefore$ $$P = \frac{\partial Z}{\partial X} = \frac{px}{z}, \quad \text{and} \quad Q = \frac{\partial Z}{\partial Y} = \frac{qy}{z}$$

$\therefore$ the equation is $P^2 + Q^2 = 1$, which is standard type (1).

The complete integral is $Z = aX + bY + c$, where $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$

$\therefore$ $$\log_e z = a \log_e x + \sqrt{1 - a^2} \log_e y + C \qquad (1)$$

which is the complete integral.

There is no singular integral.

The general integral is found by putting $c = \phi(a)$ in (1), differentiating w.r.to a and eliminating a.

EXAMPLE 13

Solve $x^4 p^2 - yzq = z^2$.

Solution.

Given $$x^4 p^2 - yzq = z^2 \quad \Rightarrow \quad \frac{x^4 p^2}{z^2} - \frac{yz}{z^2} q = 1$$

$$\Rightarrow \qquad \frac{x^4p^2}{z^2} - \frac{yq}{z} = 1 \quad \Rightarrow \quad \left(\frac{x^2p}{z}\right)^2 - \left(\frac{yq}{z}\right) = 1 \qquad (1)$$

It is of the form $F(x^m z^k p, y^n z^k q) = 0$ with $m = 2, \quad k = -1, \quad n = 1$.

$\therefore$ put $\quad X = x^{1-m} = x^{-1}, \quad Y = \log_e y$ and $Z = \log_e z$

$$\therefore \quad \frac{dX}{dx} = (-1)x^{-2} \Rightarrow \frac{dX}{dx} = -x^2, \frac{dY}{dy} = \frac{1}{y} \Rightarrow \frac{dy}{dY} = y \text{ and } \frac{\partial Z}{\partial z} = \frac{1}{z}.$$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX} = \frac{1}{z} \cdot p(-x^2) = -\frac{x^2p}{z}$$

$$\text{and} \quad Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{dy}{dY} = \frac{1}{z}qy$$

Substituting for P and Q in (1) the equation is $P^2 - Q = 1$

So, the complete integral is $Z = aX + bY + c, \quad$ where $\quad a^2 - b = 1 \quad \Rightarrow \quad b = a^2 - 1$

$$\therefore \quad \log_e z = ax^{-1} + b\log_e y + c$$

$$\Rightarrow \quad \log_e z = \frac{a}{x} + (a^2 - 1)\log_e y + c, \text{ which is the complete integral.}$$

There is no singular integral.
The general integral is found by putting $c = \phi(a)$, differentiating w.r.to a and eliminating a.

EXAMPLE 14

Solve $z^2(p^2x^2 + q^2) = 1$.

Solution.

Given $\qquad z^2(p^2x^2 + q^2) = 1 \quad \Rightarrow \quad (xzp)^2 + (qz)^2 = 1 \qquad (1)$

This is of the form $\qquad F(x^m z^k p, y^n z^k q) = 0$

Here $m = 1, \quad k = 1, \quad n = 0 \neq 1$

$\therefore$ Put $\quad X = \log_e x, \qquad Y = y^{1-n} = y \quad$ and $\quad Z = z^{k+1} = z^2$

$$\therefore \quad \frac{dX}{dx} = \frac{1}{x} \Rightarrow \frac{dx}{dX} = x, \frac{dY}{dy} = 1 \Rightarrow \frac{dy}{dY} = 1 \text{ and } \frac{\partial Z}{\partial z} = 2z.$$

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX} = 2z \cdot p \cdot x \quad \Rightarrow \quad \frac{P}{2} = xzp$$

$$\text{and} \quad Q = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{dy}{dY} = 2z \cdot q \cdot 1 \quad \Rightarrow \quad \frac{Q}{2} = zq$$

Substituting in (1), we get

$$\frac{P^2}{4}+\frac{Q^2}{4}=1 \quad \Rightarrow \quad P^2+Q^2=4$$

This is of standard type (1),

$\therefore$ the complete integral is $\qquad Z=aX+bY+c$

where $\qquad a^2+b^2=4 \quad \Rightarrow \quad b=\sqrt{4-a^2}$

$$\therefore \qquad z^2=a\log_e x+\sqrt{4-a^2}\,y+c \qquad (2)$$

This is the complete integral.
There is no singular integral.
The general integral is found by putting $c=\phi(a)$ in (2), differentiating w.r.to a and eliminating a.

EXAMPLE 15

Solve $(x+pz)^2+(y+qz)^2=1$.

Solution.

Given $\qquad (x+pz)^2+(y+qz)^2=1$

Put $\qquad Z=z^{1+k}=z^2 \qquad \therefore \frac{\partial Z}{\partial z}=2z \qquad [\because k=1]$

$$\therefore \qquad P=\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=2z\cdot p \quad \Rightarrow \quad zp=\frac{P}{2}$$

$$Q=\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=2z\cdot q \quad \Rightarrow \quad zq=\frac{Q}{2}$$

$\therefore$ the equation becomes $\left(x+\frac{P}{2}\right)^2+\left(y+\frac{Q}{2}\right)^2=1$

$$\Rightarrow \qquad \left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2=a$$

$$\therefore \qquad x+\frac{P}{2}=\sqrt{a}, \quad y+\frac{Q}{2}=\sqrt{1-a}$$

$$\Rightarrow \qquad P=2\left(\sqrt{a}-x\right), \quad Q=2\left(\sqrt{1-a}-y\right)$$

We know that $\qquad dZ=Pdx+Qdy=2\left(\sqrt{a}-x\right)dx+2\left(\sqrt{1-a}-y\right)dy$

Integrating, $\qquad Z=2\int\left(\sqrt{a}-x\right)dx+2\int\left(\sqrt{1-a}-y\right)dy$

$$\Rightarrow \qquad z^2=2\left(\sqrt{a}x-\frac{x^2}{2}\right)+2\left(\sqrt{1-a}\,y-\frac{y^2}{2}\right)+c$$

$$z^2=2\sqrt{a}x-x^2+2\sqrt{1-a}\,y-y^2+c$$

There is no singular integral.
The general integral will be found by putting $c=\phi(a)$ in (2), differentiating w.r.to a and eliminating a.

EXERCISE 14.4

Solve the following partial differential equations.

1. $p(1+q)=qz$
2. $ap+bq+cz=0$
3. $z^2=1+p^2+q^2$
4. $z^2(p^2z^2+q^2)=1$
5. $pz=1+q^2$
6. $p^3+q^3=8z$
7. $p^2-q^2=z$
8. $p(1+q^2)=q(z-a)$
9. $p^2z^2+q^2=p^2q$
10. $p=2qx$
11. $\sqrt{p}+\sqrt{q}=2x$
12. $pq=y$
13. $q=py+p^2$
14. $\sqrt{p}+\sqrt{q}=x+y$
15. $p^2+q^2=x^2+y^2$
16. $p^2+q^2=x+y$
17. $pq=xy$
18. $\dfrac{x}{p}+\dfrac{y}{q}+1=0$
19. $px+qy=1$
20. $p^2x^4+y^2zq=2z^2$

    [**Hint:** $x^2z^{-1}p+y^2z^{-1}q=2;\quad m=2,\quad n=2,\quad k=-1.$ Put $X=x^{-1},\quad Y=y^{-1},\quad Z=\log_e z$]
21. $p^2+q^2=z^2(x^2+y^2)$

    $\left[\textbf{Hint:} \left(\dfrac{p}{z}\right)^2+\left(\dfrac{q}{z}\right)^2=x^2+y^2.\right.$ Here $k=-1$, Put $Z=\log_e z$ then $\left. p=\dfrac{\partial z}{\partial x}=\dfrac{p}{z},\quad Q=\dfrac{y}{z}\right]$
22. $2x^4p^2-yzq-3z^2=0$

    $\Big[$**Hint:** $2(x^2z^{-1}p)^2-(yz^{-1}q)=3.$ Here $m=2,\quad n=1,\quad k=-1.$

    Put $X=x^{-1},\quad Y=\log y,\quad Z=\log z\quad \therefore x^2z^1p=-p,\quad Q=yz^{-1}q,$

    where $P=\dfrac{\partial z}{\partial X},\quad Q=\dfrac{\partial Z}{\partial Y}\quad \therefore\quad 2P^2-Q^2=3\Big]$
23. $pz^2\sin^2 x+qz^2\cos^2 y=1$

    $\Big[$**Hint:** Put $Z=z^3$ $\qquad P=\dfrac{\partial Z}{\partial z}=\dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial x}=3z^2p;\quad Q=\dfrac{\partial Z}{\partial y}=3z^2q^2$

    $P\sin^2 x+Q\cos^2 y=3 \;\therefore\; P\sin^2 x=3-\cos^2 y=a\Big]$

ANSWERS TO EXERCISE 14.4

Complete integral is given below

1. $\log_e(az-1)=x+ay+c$
2. $\log_e z=-\dfrac{c}{a+bk}(x+ky)+c'$
3. $\log_e\left(z+\sqrt{z^2-1}\right)=\dfrac{1}{\sqrt{1+a^2}}(x+ay)+c$
4. $(z^2+a^2)^{3/2}=3(x+ay)+c$
5. $z^2-z\sqrt{z^2-4a^2}+4a^2\log_e\left(z+\sqrt{z^2-4a^2}\right)=4(x+ay)+c$
6. $3(1+a^3)^{1/3}\cdot z^{2/3}=4(x+ay)+c$
7. $2\sqrt{1-a^2}\sqrt{z}=x+ay+c$

8. $2\sqrt{bz - ab - 1} = x + ay + c$

9. $z = a\tan(x + ay + c)$

10. $z = ax^2 + ay + c$

11. $z = \frac{1}{6}\left(2x - \sqrt{a}\right)^3 + ay + c$

12. $z = ax + \frac{y^2}{2a} + c$

13. $z = ax + \frac{a^2}{2}y^2 + a^2 y + c$

14. $3z = (x + a)^3 + (y - a)^3 + c$

15. $2z = a\sin^{-1}\frac{x}{\sqrt{a}} + x\sqrt{x^2 + a} + y\sqrt{y^2 - a} - a\log_e\left(y + \sqrt{y^2 - a}\right) + c$

16. $z = \frac{2}{3}\{(x + a)^{3/2} + (y - a)^{3/2}\} + c$

17. $z = \frac{ax^2}{2} + \frac{y^2}{2a} + c$

18. $z = -\frac{a}{a+1}\cdot\frac{x^2}{2} + \frac{a}{2}y^2 + c$

19. $z(1 + a) = \log_e x + a\log_e y + c$

20. $\log_e z = \frac{a}{x} + \frac{(a^2 - 2)}{y} + c$

21. $2\log_e z = x\sqrt{x^2 + a} + a\log_e\left(x + \sqrt{x^2 + a}\right) + y\sqrt{y^2 - a} - a\log_e\left(y + \sqrt{y^2 - a}\right) + c$

22. $\log_e z = \frac{a}{x} + (2a^2 - 3)\log_e y + c$

23. $z^3 = -a\cot x + (3 - a)\tan y + c$

14.5 LAGRANGE'S LINEAR EQUATION

A partial differential equation of the form $Pp + Qq = R$, where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, is called **Lagrange's linear equation**.

We have seen already that elimination of ϕ from $\phi(u, v) = 0$, where u and v are functions x, y, z leads to Lagrange's equation.

$\therefore$ $\phi(u, v) = 0$ is the general solution of $Pp + Qq = R$, where ϕ is an arbitrary function.

The method to find the solution of $Pp + Qq = R$ is known as **Lagrange's method**.

Working Rule: To solve $Pp + Qq = R$, where P, Q, R are functions of x, y, z.

(i) Form the auxiliary equations or subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(ii) Solving the subsidiary equations, find two independent solutions $u(x, y, z) = a$ and $v(x, y, z) = b$, where a and b are arbitrary constants.

(iii) Then the required general solution is $\phi(u, v) = 0$ [or $u = f(v)$ or $v = g(u)$] where ϕ (or f or g) is an arbitrary function.

LAGRANGE'S LINEAR EQUATION

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(iii) Then the required general solution is $\phi(u, v)=0$ [or $u=f(v)$ or $v=g(u)$] where ϕ (or f or g) is an arbitrary function.

Note

1. The subsidiary equations are known as **Lagrange's subsidiary equations**.
2. The subsidiary equations can be solved by (i) the method of grouping and (ii) the method of multipliers.

   If $\frac{dx}{P} = \frac{dx}{Q} = \frac{dz}{R}$, then by the properties of ratio and proportion each ratio $= \frac{l\,dx + m\,dy + n\,dz}{l\,P + m\,Q + n\,R}$

   where l, m, n may be constants [or functions of x, y, z] and are called **Lagrange's multiplers**

   If l, m, n are found in such way that $l\,P + m\,Q + n\,R = 0$, then $l\,dx + m\,dy + n\,dz = 0$

   Integrating, we get one solution $u = a$.

   Similarly, we can find another set of independent multipliers l', m', n' or grouping method to find another solution $v = b$. Then the general solution is $\phi(u, v) = 0$

Remark

1. Since we have to find two independent solutions $u = a$ and $v = b$, it is advisable to find one solution by grouping method and the other by multiplier method or both by two independent set of multipliers.

   When both the solutions are obtained by grouping it is quite possible that they are not independent.

   For, example, if the subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$\therefore \quad \text{each ratio} = \frac{dx - dy}{-(x-y)} = \frac{dy - dz}{-(y-z)} = \frac{dz - dx}{-(z-x)} \Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)} = \frac{d(z-x)}{z-x}$$

$$\therefore \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)}$$

Integrating, $\log_e(x-y) = \log_e(y-z) + \log_e a$

$$\Rightarrow \quad \frac{x-y}{y-z} = a. \qquad \text{This is } u = a$$

and $$\frac{d(z-x)}{(z-x)} = \frac{d(y-z)}{y-z}$$

$$\Rightarrow \quad \frac{z-x}{y-z} = b. \qquad \text{This is } v = b$$

we note that u and v are not independent for

$$1 + u = 1 + \frac{x-y}{y-z} = \frac{y-z+x-y}{y-z} = -\frac{(z-x)}{y-z} = -v$$

$$\therefore \quad v = -(1+u)$$

2. **Sometimes we use the first solution to find the second solution.**

WORKED EXAMPLES

EXAMPLE 1

Solve $\frac{y^2 z}{x} p + xz\, q = y^2$.

Solution.

Given $$\frac{y^2 z}{x} p + xz\, q = y^2$$

This is Lagrange's equation $Pp + Qq = R$

Here $$P = \frac{y^2 z}{x}, \quad Q = xz, \quad R = y^2$$

The subsidiary equations are $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \Rightarrow \quad \frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Considering the first two ratios,

$$\frac{xdx}{y^2 z} = \frac{dy}{xz} \quad \Rightarrow \quad x^2 dx = y^2 dy$$

Integrating, $$\int x^2 dx = \int y^2 dy$$

$$\Rightarrow \quad \frac{x^3}{3} = \frac{y^3}{3} + c \quad \Rightarrow \quad x^3 - y^3 = 3c \quad \Rightarrow \quad x^3 - y^3 = a \qquad (1)$$

Considering the first and last ratios, $$\frac{xdx}{y^2 z} = \frac{dz}{y^2} \quad \Rightarrow \quad x\,dx = z\,dz$$

Integrating, $$\int x\,dx = \int z\,dz$$

$$\Rightarrow \quad \frac{x^2}{2} = \frac{z^2}{2} + c$$

$$x^2 = z^2 + 2c \quad \Rightarrow \quad x^2 - z^2 = 2c \quad \Rightarrow \quad x^2 - z^2 = b \qquad (2)$$

$\therefore$ the general solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$, where ϕ is arbitrary.

Note We can also write the solution as $x^3 - y^3 = f(x^2 - z^2)$ or $x^2 - z^2 = g(x^3 - y^3)$

EXAMPLE 2

Solve $x(y - z)p + y(z - x)q = z(x - y)$.

Solution.

Given $$x(y-z)p + y(z-x)q = z(x-y)$$

This is Lagrange's equation, $Pp + Qq = R$

Here $$P = x(y-z), \quad Q = y(z-x), \quad R = z(x-y)$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \qquad \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\therefore \qquad \text{each ratio} = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{0}$$

$$\therefore \qquad dx+dy+dz=0 \quad \Rightarrow d(x+y+z)=0$$

Integrating, $$x+y+z=a \qquad (1)$$

Also $$\frac{\frac{1}{x}dx}{(y-z)} = \frac{\frac{1}{y}dy}{(z-x)} = \frac{\frac{1}{z}dz}{(x-y)}$$

$$\therefore \qquad \text{each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y} \qquad \left[\text{each ratio} = \frac{\text{Sum of Nrs.}}{\text{Sum of Drs.}}\right]$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0} \quad \Rightarrow \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

Integrating, $\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$

$$\Rightarrow \qquad \log_e x+\log_e y+\log_e z=\log_e b \quad \Rightarrow \quad \log_e xyz=\log_e b \quad \Rightarrow \quad xyz=b \qquad (2)$$

$\therefore$ the general solution is $\phi(x+y+z, xyz)=0$

EXAMPLE 3

Solve $(x^2+y^2+yz)p+(x^2+y^2-zx)q=z(x+y)$.

Solution.

Given $$(x^2+y^2+yz)p+(x^2+y^2-zx)q=z(x+y)$$

This is Lagrange's equation, $Pp+Qq=R.$

Here $$P=x^2+y^2+yz, \quad Q=x^2+y^2-zx, \quad R=z(x+y)$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \qquad \frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-zx} = \frac{dz}{z(x+y)}$$

$$\text{each ratio} = \frac{dx-dy}{x^2+y^2+yz-(x^2+y^2-zx)} = \frac{dx-dy}{z(y+x)}$$

$$\therefore \qquad \frac{dx-dy}{z(y+x)} = \frac{dz}{z(x+y)} \quad \Rightarrow \quad dx-dy=dz$$

Integrating, $x - y = z + a \Rightarrow x - y - z = a$ (1)

Also each ratio $= \dfrac{x\,dx + y\,dy}{x(x^2+y^2+yz)+y(x^2+y^2-xz)}$

$$= \frac{x\,dx + y\,dy}{x(x^2+y^2)+xyz+y(x^2+y^2)-xyz} = \frac{x\,dx+y\,dy}{(x^2+y^2)(x+y)}$$

$$\therefore \quad \frac{x\,dx+y\,dy}{(x^2+y^2)(x+y)} = \frac{dz}{z(x+y)}$$

$$\Rightarrow \quad \frac{x\,dx+y\,dy}{x^2+y^2} = \frac{dz}{z}$$

$$\Rightarrow \quad \frac{1}{2}\frac{(2x\cdot dx+2y\cdot dy)}{x^2+y^2} = \frac{dz}{z} \quad \Rightarrow \quad \frac{\frac{1}{2}d(x^2+y^2)}{x^2+y^2} = \frac{dz}{z}$$

Integrating, $\dfrac{1}{2}\displaystyle\int \frac{d(x^2+y^2)}{x^2+y^2} = \int \frac{dz}{z}$

$$\Rightarrow \quad \frac{1}{2}\log_e(x^2+y^2) = \log_e z + \log c$$

$$\Rightarrow \quad \log_e(x^2+y^2) = 2\log_e c\,z = \log_e c^2 z^2$$

$$\Rightarrow \quad x^2+y^2 = c^2z^2 = b\,z^2 \Rightarrow \frac{x^2+y^2}{z^2} = b \qquad (2)$$

$\therefore$ the general solution is $\phi\left(x-y-z, \dfrac{x^2+y^2}{z^2}\right) = 0$

EXAMPLE 4

$\boldsymbol{x(y^2+z)p + y(x^2+z)q = z(x^2-y^2)}$

Solution.

Given $x(y^2+z)p + y(x^2+z)q = z(x^2-y^2)$

This is Lagrange's equation, $Pp + Qq = R.$

Here $P = x(y^2+z), \quad Q = y(x^2+z), \quad R = z(x^2-y^2)$

The subsidiary equations are $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \quad \frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Using multipliers, $x, -y, -1$, we get

$$\text{each ratio} = \frac{x\,dx - y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - dz}{0}$$

$\therefore \quad x\,dx - y\,dy - dz = 0$

Integrating, $\int x\,dx - \int y\,dy - \int dz = 0$

$$\Rightarrow \quad \frac{x^2}{2} - \frac{y^2}{2} - z = \frac{a}{2} \quad \Rightarrow \quad x^2 - y^2 - 2z = a \qquad (1)$$

Also, $$\frac{\frac{dx}{x}}{y^2+z} = \frac{\frac{dy}{y}}{x^2+z} = \frac{\frac{dz}{z}}{x^2-y^2}$$

$$\therefore \quad \text{each ratio} = \frac{\frac{dx}{x} - \frac{dy}{y}}{y^2 - x^2} = \frac{\frac{dz}{z}}{x^2-y^2}$$

$$\Rightarrow \quad \frac{dx}{x} - \frac{dy}{y} = -\frac{dz}{z} \quad \Rightarrow \quad \frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\therefore \quad \int \frac{dx}{x} - \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\Rightarrow \quad \log_e x - \log_e y + \log_e z = \log_e b \quad \Rightarrow \quad \log_e \frac{xz}{y} = \log_e b \quad \Rightarrow \frac{xz}{y} = b \qquad (2)$$

$\therefore$ the general solution is $\phi\left(x^2 - y^2 - 2z, \frac{xz}{y}\right) = 0$

EXAMPLE 5

Find the integral surface of the partial differential equation $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ and passing through the straight line $x + y = 0, z = 1$.

[**Note** Equation of a surface which satisfies the P.D.E is called an integral surface]

Solution.

Given $\quad x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

This is Lagrange's equation $\quad Pp + Qq = R.$

Here $\quad P = x(y^2+z), \quad Q = -y(x^2+z), \quad R = (x^2-y^2)z$

The subsidiary equations are $\quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \quad \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

Using multipliers $x, y, -1$, we get

$$\text{each ratio} = \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - (x^2-y^2)z}$$

$$= \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - x^2z + y^2z} = \frac{xdx + ydy - dz}{0}$$

$\therefore \qquad x\,dx + y\,dy - dz = 0$

Integrating, $\int x\,dx + \int y\,dy - \int dz = 0$

$$\Rightarrow \qquad \frac{x^2}{2} + \frac{y^2}{2} - z = \frac{a}{2} \quad \Rightarrow x^2 + y^2 - 2z = a \qquad (1)$$

Again, $$\frac{\frac{dx}{x}}{y^2 + z} = \frac{\frac{dy}{y}}{-x^2 - z} = \frac{\frac{dz}{z}}{x^2 - y^2}$$

$$\therefore \qquad \text{each ratio} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$$\Rightarrow \qquad \log_e x + \log_e y + \log_e z = \log_e b \quad \Rightarrow \log_e xyz = \log_e b \quad \Rightarrow \quad xyz = b \qquad (2)$$

$\therefore$ the general integral is $\qquad \phi(x^2 + y^2 - 2z, xyz) = 0 \qquad (3)$

From this family of surfaces, we want to find the surface passing through the line

$$x + y = 0, z = 1 \qquad (4)$$

i.e., we want to find the particular function ϕ for which the surface (3) satisfies (4). It is difficult to find the particular ϕ. So, we proceed as below.

We eliminate x, y, z using (1), (2) and (4) and get a relation between a and b, from which we find the particular surface.

From (4), $x = -y, z = 1$.Substituting in (1) and (2), we get

$$2y^2 - 2 = a \quad \text{and} \quad -y^2 = b \quad \Rightarrow y^2 = -b$$

$$\therefore \qquad -2b - 2 = a \quad \text{and} \quad a + 2b + 2 = 0 \qquad (5)$$

Now replace a and b by $x^2 + y^2 - 2z$ and xyz in (5).

$$\therefore \qquad x^2 + y^2 - 2z + 2xyz + 2 = 0,$$

which is the integral surface through the line (4)

EXAMPLE 6

Solve $\left(\frac{b-c}{a}\right)yzp + \left(\frac{c-a}{b}\right)xzq = \left(\frac{a-b}{c}\right)xy.$

Solution.

Given $$\left(\frac{b-c}{a}\right)yzp + \left(\frac{c-a}{b}\right)xzq = \left(\frac{a-b}{c}\right)xy$$

This is Lagrange's equation $Pp + Qq = R.$

Here $$P = \left(\frac{b-c}{a}\right)yz, \quad Q = \left(\frac{c-a}{b}\right)xz, \quad R = \left(\frac{a-b}{c}\right)xy$$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \quad \frac{dx}{\left(\frac{b-c}{a}\right)yz} = \frac{dy}{\left(\frac{c-a}{b}\right)xz} = \frac{dz}{\left(\frac{a-b}{c}\right)xy}$$

$$\Rightarrow \quad \frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)xz} = \frac{c\,dz}{(a-b)xy}$$

Using x, y, z as multipliers, we get

$$\text{each ratio} = \frac{ax\,dx + by\,dy + cz\,dz}{(b-c)xyz + (c-a)xyz + (a-b)xyz} = \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

$$\therefore \quad ax\,dx + by\,dy + cz\,dz = 0$$

Integrating, $a\int x\,dx + b\int y\,dy + c\int z\,dz = 0$

$$\Rightarrow \quad a\frac{x^2}{2} + b\frac{y^2}{2} + c\frac{z^2}{2} = \frac{A}{2} \quad \Rightarrow \quad ax^2 + by^2 + cz^2 = A$$

Now using ax, by, cz as multipliers, we get

$$\text{each ratio} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{a(b-c)xyz + b(c-a)xyz + c(a-b)xyz} = \frac{a^2x\,dx + b^2y\,dy + c^2z\,dz}{0}$$

$$\therefore \quad a^2x\,dx + b^2y\,dy + c^2z\,dz = 0$$

Integrating, $a^2\int x\,dx + b^2\int y\,dy + c^2\int z\,dz = 0$

$$\Rightarrow \quad a^2\cdot\frac{x^2}{2} + b^2\frac{y^2}{2} + c^2\frac{z^2}{2} = \frac{B}{2} \quad \Rightarrow \quad a^2x^2 + b^2y^2 + c^2z^2 = B$$

$\therefore$ the general integral is $\phi(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0$

EXAMPLE 7

Solve $(p-q)z = z^2 + x + y$.

Solution.

Given $(p-q)z = z^2 + x + y \Rightarrow zp - zq = z^2 + x + y$

This is Lagrange's equation, $Pp + Qq = R$.

Here $P = z, \quad Q = -z, \quad R = z^2 + x + y$

Subsidiary equations are $$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+x+y}$$

Considering the first two ratios, $$\frac{dx}{z} = \frac{dy}{-z} \Rightarrow dx = -dy$$

Integrating, $$\int dx = -\int dy \Rightarrow x = -y + a \Rightarrow x + y = a \qquad (1)$$

Here neither the grouping method nor the multiplier method can be used to find the second solution. So, we use the first solution to find the second solution.

$$\therefore \quad \frac{dx}{z} = \frac{dz}{z^2+a} \Rightarrow dx = \frac{z\,dz}{z^2+a}$$

Integrating, $$\int dx = \int \frac{z\,dz}{z^2+a} = \frac{1}{2}\int \frac{2z\,dz}{z^2+a}$$

$$\therefore \quad x = \frac{1}{2}\log_e(z^2+a) + b \Rightarrow 2x - \log_e(z^2+x+y) = b \qquad (2)$$

$\therefore$ the general solution is $\phi(x+y,\ 2x - \log_e(z^2+x+y)) = 0$

EXAMPLE 8

Solve $p - q = \log_e(x+y)$.

Solution.

Given $p - q = \log(x+y)$

This is Lagrange's equation. Here $P = 1, Q = -1, R = \log(x+y)$

The subsidiary equations are $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \quad \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log_e(x+y)}$$

$$\therefore \quad dx = -dy \Rightarrow dx + dy = 0$$

Integrating, $$\int dx + \int dy = 0 \Rightarrow x + y = a \qquad (1)$$

we use the first solution to find the second solution.

Using (1) we get, $$dx = \frac{dz}{\log_e a}$$

$\therefore$ $$\int dx = \frac{1}{\log_e a}\int dz \quad \Rightarrow \quad x = \frac{z}{\log_e a} + b \quad \Rightarrow x - \frac{z}{\log_e (x+y)} = b$$

$\therefore$ the general solution is $\phi\left(x+y,\, x - \dfrac{z}{\log_e (x+y)}\right) = 0.$

EXERCISE 14.5

Find the general integral of the following partial differential equations.

1. $px + qy = z$
2. $px^2 - qy^2 = z(x-y)$
3. $yzp + zxq = xy$
4. $x^2 p + y^2 q = z^2$
5. $x(y-z)p + y(z-x)q = z(x-y)$
6. $(y-xz)p + (yz-x)q = x^2 - y^2$
7. $(x^2 - y^2 - z^2)p + 2xy\, q = 2xz$
8. $(x-y)p + (y-x-z)q = z$
9. $(1+y)p + (1+x)q = z$
10. $(y-xz)p + (yz-x)q = (x+y)(x-y)$
11. $\left(\dfrac{y-z}{yz}\right)p + \left(\dfrac{z-x}{zx}\right)q = \dfrac{x-y}{xy}$
12. $(y^2 + z^2)p - xy\, q + xz = 0$
13. $p\tan x + q\tan y = \tan z$
14. $y^2 p - xy\, q = x(z-2y)$
15. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$
16. $(mz - ny)p + (nx - lz)q = ly - mx$
17. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$
18. $(y+z)p + (z+x)q = x+y$
19. $x(y^2 + z^2)p + y(z^2 + x^2)q = z(y^2 - x^2)$
20. $(3z - 4y)p + (4x - 2z)q = 2y - 3x$
21. $p - q = \log(x+y)$
22. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

ANSWERS TO EXERCISE 14.5

1. $\phi\left(\dfrac{x}{y}, \dfrac{x}{z}\right) = 0$
2. $\phi\left(\dfrac{1}{x} + \dfrac{1}{y}, \dfrac{x+y}{z}\right) = 0$
3. $\phi(x^2 - y^2, y^2 - z^2) = 0$
4. $\phi\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{1}{y} - \dfrac{1}{z}\right) = 0$
5. $\phi(x+y+z, xyz) = 0$
6. $\phi(xy+z, x^2+y^2+z^2) = 0$
7. $\phi\left(\dfrac{y}{z}, \dfrac{x^2+y^2+z^2}{z}\right) = 0$
8. $\phi\left[x+y+z, \dfrac{z^2}{x-y+z}\right] = 0$
9. $\phi\left(x - y + \dfrac{x^2}{2} - \dfrac{y^2}{2}, z(x-y)\right) = 0$
10. $\phi(xy+z, x^2+y^2+z^2) = 0$

11. $\phi(x+y+z, xyz)=0$

12. $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right)=0$

13. $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right)=0$

14. $\phi(x^2+y^2, x^2+yz)=0$

15. $\phi(x^2+y^2+z^2, xyz)=0$

16. $\phi(lx+my+nz, x^2+y^2+z^2)=0$

17. $\phi\left(\frac{x-y}{y-z}, xy+yz+zx\right)=0$

18. $\phi\left(\frac{x-y}{y-z}, (x+y+z)(y-z)^2\right)=0$

19. $\phi\left(\frac{x}{y+z}, x^2-y^2+z^2\right)=0$

20. $\phi(2x+3y+4z, x^2+y^2+z^2)=0$

21. $\phi\left(x+y, x-\frac{z}{\log(x+y)}\right)=0$

22. $\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0$

14.6 HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

A linear partial differential equation in which all the partial derivatives are of the same order is called a homogeneous linear partial differential equation.

We shall consider here homogeneous linear equation in two independent variables x and y.

Examples

1. $\frac{\partial^2 z}{\partial x^2}-3\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$

2. $\frac{\partial^3 z}{\partial x^3}-\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}+5\frac{\partial^3 z}{\partial y^3}=e^{x+y}$

are homogeneous linear partial differential equations of the second and third order respectively.

1. The general form of a homogeneous partial differential equation of the n[th] order with constant coefficient is

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\cdots+a_{n-1}\frac{\partial^n z}{\partial x\partial y^{n-1}}+a_n\frac{\partial^n z}{\partial y^n}=R(x, y) \quad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

Denoting $D=\frac{\partial}{\partial x}, D'=\frac{\partial}{\partial y}, D^r=\frac{\partial^r}{\partial x^r}, \quad D'^r=\frac{\partial^r}{\partial y^r}$ and $D^r D'^s=\frac{\partial^{r+s}}{\partial x^r\partial y^s}$,

the equation (1) can be written as

$$(a_0D^n+a_1D^{n-1}D'+a_2D^{n-2}D'^2+\cdots+a_nD'^n)z=R(x, y)$$

$$\Rightarrow \qquad F(D, D')z=R(x, y) \quad (2)$$

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

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Examples

1. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

2. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} + 5\frac{\partial^3 z}{\partial y^3} = e^{x+y}$

are homogeneous linear partial differential equations of the second and third order respectively.

1. The general form of a homogeneous partial differential equation of the n[th] order with constant coefficient is

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + a_n \frac{\partial^n z}{\partial y^n} = R(x, y) \quad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

Denoting $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}, D^r = \frac{\partial^r}{\partial x^r}, \quad D'^r = \frac{\partial^r}{\partial y^r}$ and $D^r D'^s = \frac{\partial^{r+s}}{\partial x^r \partial y^s}$,

the equation (1) can be written as

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = R(x, y)$$

$$\Rightarrow \quad F(D, D') z = R(x, y) \quad (2)$$

The equation $\quad F(D, D')z = 0 \quad$ (3)

is called the **reduced equation** and its general solution is called the **complementary function of (1)**. The complementary function should contain n arbitrary functions for the n^{th} order equation.

Particular integral of (1) is $\text{P.I} = \dfrac{1}{F(D, D')} R(x, y)$

which will not contain any arbitrary function
The general solution of (1) is $z = \text{C.F} + \text{P.I}$
The method of solving (1) is similar to the method of solving ordinary linear differential equation with constant coefficients.
Assuming $z = \phi(y + mx)$ is a solution of (1), where ϕ is arbitrary, we get

$$Dz = \frac{\partial z}{\partial x} = m\phi'(y + mx),$$

$$D^2 z = m^2 \phi''(y + mx), \ldots, D'z = \phi'(y + mx),$$

$$D'^2 z = \phi''(y + mx), \ldots \text{ and } D^{n-r} D'^r z = m^{n-r} \phi^{(n)}(y + mx)$$

$\therefore$ equation (1) is

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n)\phi^{(n)}(y + mx) = 0 \quad (3)$$

Since ϕ is arbitrary $\phi^{(n)}(y + mx) \neq 0$,

$(3) \Rightarrow$
$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \quad (4)$$

The equation (4) is called the **auxillary equation**.

Thus, $z = \phi(y + mx)$ is the solution of (1) if m is a root of (4).

The auxillary equation is obtained by replacing D by m and D' by 1.

14.6.1 Working Procedure to Find Complementary Function

To find the complementary function of $F(D, D')z = R(x, y)$, solve $F(D, D')z = 0$

The auxillary equation is $F(m, 1) = 0$, replacing D by m and D' by 1.

$$\Rightarrow \quad a_0 m^n + a_1 m^{n-1} + \ldots + a_n = 0$$

It has n roots $m_1, m_2, \ldots. m_n$ which are real or complex.

Case 1: If the roots are different, then

$$\text{C.F} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

Where $\phi_1, \phi_2, \ldots, \phi_n$ are n arbitrary functions.

Case 2: If two roots are equal, say, $m_1 = m_2 = m$ and others are different, then

$$\text{C.F} = \phi_1(y+mx) + x\phi_2(y+mx) + \phi_3(y+m_3x) + \ldots + \phi_n(y+m_nx)$$

If 3 roots are equal, say $m_1 = m_2 = m_3 = m$ and others are different, then

$$\text{C.F} = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx)$$
$$+\phi_4(y+m_4x) + \ldots + \phi_n(y+m_nx)$$

Note There is no separate rule for complex roots as in the case of ordinary differential equation.

14.6.2 Working Procedure to Find Particular Integral

In symbolic form $\text{P.I} = \dfrac{1}{F(D, D')} R(x, y)$.

Type 1: Let $R(x, y) = e^{ax+by}$

(a) If $F(a, b) \neq 0$, where $F(a, b)$ is got replacing D by a and D' by b in $F(D, D')$, then

$$P.I = \frac{1}{F(a, b)} e^{ax+by}$$

(b) If $F(a, b) = 0$, then $D - \dfrac{a}{b}D'$ or its power will be a factor of $F(D, D')$

We know $\dfrac{1}{D-\frac{a}{b}D'} e^{ax+by} = x\, e^{ax+by}$, $\dfrac{1}{\left(D-\frac{a}{b}D'\right)^2} e^{ax+by} = \dfrac{x^2}{2!} e^{ax+by}$,

$\dfrac{1}{\left(D-\frac{a}{b}D'\right)^3} e^{ax+by} = \dfrac{x^3}{3!} e^{ax+by}$, ...

Aliter for (b): If $F(a, b) = 0$, then multiply the numerator by x and differentiate $F(D, D')$ in the denominator w.r.to D and then replace D by a and D' by b. Even then if the denominator is 0, proceed as above again.

Type 2: Let $\quad$ **$R(x, y) = \sin(ax + by)$ or $\cos(ax + by)$**

Since $\quad D^2 \sin(ax+by) = -a^2 \sin(ax+by)$

$$DD' \sin(ax+by) = -ab \sin(ax+by)$$

$$D'^2 \sin(ax+by) = -b^2 \sin(ax+by)$$

$$F(D^2, DD', D'^2)\sin(ax+by) = F(-a^2, -ab, -b^2)\sin(ax+by)$$

$\therefore$
$$\text{P.I} = \frac{1}{F(D^2, DD', D'^2)}\sin(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2)}\sin(ax+by) \qquad \left[\text{if } F(-a^2, -ab, -b^2) \neq 0\right]$$

Similarly,
$$\text{P.I} = \frac{1}{F(D^2, DD', D'^2)}\cos(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2)}\cos(ax+by) \qquad \left[\text{if } F(-a^2, -ab, -b^2) \neq 0\right]$$

If $F(-a^2, -ab, -b^2) = 0$, then $D^2 - \frac{a^2}{b^2}D'^2$ will be factor of $F(D^2, DD', D'^2)$

As in ordinary differential equation

$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\sin(ax+by) = \frac{x}{2}\int \sin(ax+by)dx = -\frac{x}{2a}\cos(ax+by)$$

and
$$\frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax+by) = \frac{x}{2}\int \cos(ax+by)dx = \frac{x}{2a}\sin(ax+by)$$

Aliter: If $F(-a^2, -ab, -b^2) = 0$, then multiply the numerator by x and differentiate the denominator w.r.to D and then replace D^2 by $-a^2$, DD' by $-ab$ and D'^2 by $-b^2$.

Type 3: Let $R(x, y) = x^m y^n$

Then $\text{P.I} = \frac{1}{F(D, D')}x^m y^n = [F(D, D')]^{-1}x^m y^n$

If $m \geq n$, rewrite $[F(D, D')]$ by taking out the highest power of D, as $\left[1 \pm F\left(\frac{D'}{D}\right)\right]^{-1}$

and expand using binomial series in powers of $\frac{D'}{D}$.

If $m < n$, rewrite taking out the highest power of D' and expand in powers of $\frac{D}{D'}$

We have $\frac{1}{D}f(x, y) = \int f(x, y)\, dx, y$ constant and $\frac{1}{D'}f(x, y) = \int f(x, y)\, dy, x$ constant

Type 4: General rule

$R(x, y)$ may not always be of the above types. If $R(x, y)$ is any function of x, y, we can use this method.

$$\text{P.I} = \frac{1}{F(D, D')} R(x, y)$$

$F(D, D')$ can be factorised into n linear factors, in general,

$$\therefore \qquad \text{P.I} = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots (D - m_n D')} R(x, y)$$

We know that $\frac{1}{D - mD'} R(x, y) = \int R(x, c - mx)\,dx$ $\qquad [y = c - mx]$

where c is replaced by $y + mx$ after integration.

By repeated application of this rule, P.I is evaluated.

Note This general method can also be used in types 1 and 2 when the denominator become zero i.e., in the cases $F(a, b) = 0$ and $F(-a^2, -ab, -b^2) = 0$

Type 5: $R(x, y) = e^{ax+by} f(x, y)$. Exponential shifting.

$$\text{P.I} = \frac{1}{F(D, D')} e^{ax+by} f(x, y) = e^{ax+by} \frac{1}{F(D + a, D' + b)} f(x, y)$$

This can be evaluated by any of the above methods.

WORKED EXAMPLES

Type 1:

EXAMPLE 1

Solve $(D^3 + D^2 D' - DD'^2 - D'^3)z = 0$.

Solution.

Given $\qquad (D^3 + D^2 D' - DD'^2 - D'^3)z = 0$

The auxiliary equation is $m^3 + m^2 - m - 1 = 0$

$\Rightarrow \qquad m^2(m+1) - (m+1) = 0 \quad \Rightarrow \quad (m+1)(m^2 - 1) = 0 \quad \Rightarrow \quad m = -1, -1, 1$

Two equal roots.

$\therefore$ the general solution is $\qquad z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + x)$

EXAMPLE 2

Solve $(D^4 - D'^4)z = 0$.

Solution.

Given $\quad (D^4 - D'^4)z = 0$

Auxiliary equation is $m^4 - 1 = 0 \quad \Rightarrow \quad (m^2+1)(m^2-1) = 0 \quad \Rightarrow \quad m = \pm i, \pm 1$ [Roots are different]

$\therefore$ the general solution is $z = \phi_1(y+ix) + \phi_2(y-ix) + \phi_3(y+x) + \phi_4(y-x)$

EXAMPLE 3

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial^2 x \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Solution.

Given equation is $\quad (D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$

To find the C.F, solve $\quad (D^3 - 3D^2D' + 4D'^3)z = 0$

Auxiliary equation is $\quad m^3 - 3m^2 + 4 = 0$

$\Rightarrow \quad m^3 + m^2 - 4m^2 - 4m + 4m + 4 = 0$

$\Rightarrow \quad m^2(m+1) - 4m(m+1) + 4(m+1) = 0$

$\Rightarrow \quad (m+1)(m^2 - 4m + 4) = 0 \quad \Rightarrow \quad (m+1)(m-2)^2 = 0 \quad \Rightarrow \quad m = -1, 2, 2$

$\therefore \quad \text{C.F} = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x)$

$$\text{P.I} = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$$= \frac{1}{1 - 3\cdot 2 + 4\cdot 2^3} e^{x+2y} = \frac{1}{27} e^{x+2y} \quad \text{[Relpacing } D \text{ by } a = 1 \text{ and } D' \text{by } b = 2$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$= \phi_1(y-x) + \phi_2(y+2x) + x\phi_2(y+2x) + \frac{1}{27} e^{x+2y}$$

EXAMPLE 4

Solve $(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$.

Solution.

Given $\quad (D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$

To find the C.F, solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

Auxiliary equation is $m^3 - 3m + 2 = 0$. By trial 1 is a root.

1	1	0	−3	2
	0	1	1	−2
	1	1	−2	0

Other roots are given by $m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$\Rightarrow$ $m = -2, 1$

$\therefore$ the roots are $m = 1, 1, -2$

$$\therefore \quad \text{C.F} = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - 2x)$$

$$\text{P.I}_1 = \frac{1}{D^3 - 3DD'^2 + 2D'^3} e^{2x-y}$$

$$= \frac{1}{2^3 - 3\cdot 2(-1)^2 + 2\cdot(-1)^3} e^{2x-y} = \frac{1}{8-6-2} e^{2x-y} = \frac{1}{0} e^{2x-y}, \text{ case of failure.}$$

$$\therefore \quad \text{P.I}_1 = x \cdot \frac{1}{3D^2 - 3D'^2} e^{2x-y}$$ [Multiply Nr. by x and differentiate Dr. w.r.to D]

$$= x \cdot \frac{1}{3\cdot 2^2 - 3(-1)^2} e^{2x-y} = x \cdot \frac{1}{12-3} e^{2x-y} = \frac{x}{9} e^{2x-y}$$

$$\text{P.I}_2 = \frac{1}{D^3 - 3DD'^2 + 2D'^3} e^{x+y} = \frac{1}{1-3+2} e^{x+y} = \frac{1}{0} e^{x+y}, \text{ case of failure.}$$

$$\therefore \quad \text{P.I}_2 = x \cdot \frac{1}{3D^2 - 3D'^2} e^{x+y} = x \cdot \frac{x}{6D} e^{x+y} = \frac{x^2}{6} \int e^{x+y} dx = \frac{x^2}{6} e^{x+y}$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-x) + \frac{x}{9} e^{2x-y} + \frac{x^2}{6} e^{x+y}$$

Type 2:

EXAMPLE 5

Solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x + 6y)$.

Solution.

Given $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x + 6y)$

To find the C.F, solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

Auxiliary equation is $m^3 - 4m^2 + 4m = 0$

$\Rightarrow$ $m(m^2 - 4m + 4) = 0 \Rightarrow m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$

$$\text{C.F} = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

$$\text{P.I} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 6\sin(3x + 6y)$$

$$= \frac{1}{D(D^2 - 4DD' + 4D'^2)} 6\sin(3x+6y)$$

$$= \frac{6}{D} \frac{1}{[-3^2 - 4(-3\cdot 6) + 4(-6^2)]} \sin(3x+6y)$$

$$= \frac{6}{D} \frac{1}{[-9+72-144]} \sin(3x+6y)$$

$$= -\frac{6}{81} \times \frac{1}{D} \sin(3x+6y)$$

$$= -\frac{2}{27} \int \sin(3x+6y)dx = \frac{-2}{27}\left(\frac{-\cos(3x+6y)}{3}\right) = \frac{2}{81}\cos(3x+6y)$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{2}{81}\cos(3x+6y)$$

EXAMPLE 6

Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y} + 4\sin(x+y)$.

Solution.

Given $\qquad (D^3 - 2D^2D')z = e^{x+2y} + 4\sin(x+y)$

To find the C.F, solve $(D^3 - 2D^2D')z = 0$

Auxiliary equation is $\qquad m^3 - 2m^2 = 0 \quad \Rightarrow \quad m^2(m-2) = 0 \Rightarrow m = 0, 0, 2$

$\therefore \qquad \text{C.F} = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$

$$\text{P.I}_1 = \frac{1}{D^3 - 2D^2D'} e^{x+2y} = \frac{1}{1 - 2\cdot 1\cdot 2} e^{x+2y} = -\frac{1}{3} e^{x+2y}$$

$$\text{P.I}_2 = \frac{1}{D^3 - 2D^2D'} 4\sin(x+y)$$

$$= 4\cdot\frac{1}{-D - 2D(-1)}\sin(x+y) \qquad [\because DD' = -(1\cdot 1) = -1 \text{ and } D^2 = -1^2 = -1]$$

$$= 4\cdot\frac{1}{D}\sin(x+y) = 4\int \sin(x+y)dx = -4\cos(x+y)$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) - \frac{1}{3}e^{x+2y} - 4\cos(x+y)$$

EXAMPLE 7

Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$.

Solution.

Given $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

To find C.F, solve $(D^2 - DD' - 20D'^2)z = 0$

Auxiliary equation is $m^2 - m - 20 = 0$

$\Rightarrow \quad m^2 - 5m + 4m - 20 = 0$

$\Rightarrow \quad m(m-5) + 4(m-5) = 0 \quad \Rightarrow \quad (m+4)(m-5) = 0 \Rightarrow m = -4, 5$

$\therefore \quad \text{C.F} = \phi_1(y - 4x) + \phi_2(y + 5x)$

$$\text{P.I}_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} = \frac{1}{5^2 - 5\cdot 1 - 20} e^{5x+y} = \frac{1}{0} e^{5x+y}$$

Since denominator is zero, we use the alternate method in type 1.

$\therefore \quad \text{P.I}_1 = x \dfrac{1}{2D - D'} e^{5x+y}$ [Multiply Nr. by x and diff. the Dr. w.r.to D]

$$= \frac{x}{2\cdot 5 - 1} e^{5x+y} = \frac{x}{9} e^{5x+y}$$

$$\text{P.I}_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y) = \frac{1}{-4^2 - (+4) - 20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$$

Since denominator is zero, we use the alternate method.

$$\text{P.I}_2 = x \cdot \frac{1}{2D - D'} \sin(4x - y) = x \cdot \frac{2D + D'}{4D^2 - D'^2} \sin(4x - y)$$

$$= x \cdot \frac{(2D + D')\sin(4x - y)}{4(-4^2) - (-1^2)}$$

$$= \frac{x[2D\sin(4x - y) + D'\sin(4x - y)]}{-64 + 1}$$

$$= \frac{x[2\cos(4x - y)\cdot 4 + \cos(4x - y)(-1)]}{-63}$$

$$= -\frac{7x}{63}\cos(4x - y) = -\frac{x}{9}\cos(4x - y)$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad z = \phi_1(y - 4x) + \phi_2(y + 5x) + \frac{x}{9} e^{5x+y} - \frac{x}{9}\cos(4x - y)$$

EXAMPLE 8

Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$.

Solution.

Given $(D^2 - 2DD')z = \sin x \cdot \cos 2y = \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$

To find C.F, solve $(D^2 - 2DD')z = 0$

Auxiliary equation is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

$\therefore$ $$\text{C.F} = \phi_1(y) + \phi_2(y+2x)$$

$$\text{P.I}_1 = \frac{1}{D^2 - 2DD'}\left(\frac{1}{2}\sin(x+2y)\right) = \frac{1}{2[(-1^2) - 2(-1\cdot 2)]}\sin(x+2y) = \frac{1}{6}\sin(x+2y)$$

$$\text{P.I}_2 = \frac{1}{2}\cdot\frac{1}{(D^2 - 2DD')}\sin(x-2y) = \frac{1}{2[(-1^2) - 2(-1(-2))]}\sin(x-2y) = -\frac{1}{10}\sin(x-2y)$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad z = \phi_1(y) + \phi_2(y+2x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

Type 3:

EXAMPLE 9

Solve $(D^2 - 2DD')z = e^{2x} + x^3 y$.

Solution.

Given $(D^2 - 2DD')z = e^{2x} + x^3 y$

To find C.F, solve $(D^2 - 2DD')z = 0$

Auxiliary equation is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$

$\therefore$ $\text{C.F} = \phi_1(y) + \phi_2(y+2x)$

$$\text{P.I}_1 = \frac{1}{D^2 - 2DD'}e^{2x} = \frac{1}{2^2 - 2\cdot 2\cdot 0}e^{2x} = \frac{e^{2x}}{4} \qquad [\text{Here } a = 2, b = 0]$$

$$\text{P.I}_2 = \frac{1}{D^2 - 2DD'}x^3 y \quad \left[\text{Here } m = 3, n = 1, m > n \text{ and so, take out } D^2 \text{ and write as a series in } \frac{D'}{D}\right]$$

$$= \frac{1}{D^2\left(1 - 2\frac{D'}{D}\right)}x^3 y$$

$$= \frac{1}{D^2}\left(1 - 2\frac{D'}{D}\right)^{-1} x^3 y$$

$$= \frac{1}{D^2}\left(1 + 2\frac{D'}{D} + 4\frac{D'^2}{D^2} + \cdots\right)x^3y$$

$$= \frac{1}{D^2}\left(x^3y + \frac{2}{D}D'(x^3y)\right) \qquad [\because D'y = 1 \text{ and } D'^2y = 0]$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$= \frac{1}{D}\int x^3y\,dx + \frac{2}{D^2}\int x^3dx$$

$$= \frac{1}{D}\frac{x^4}{4}y + \frac{2}{D^2}\frac{x^4}{4} = \frac{y}{4}\int x^4dx + \frac{1}{2}\frac{1}{D}\int x^4dx = \frac{y}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\int\frac{x^5}{5}dx = \frac{x^5y}{20} + \frac{x^6}{60}$$

$\therefore$ the general solution is z = C.F + P.I

$$\Rightarrow \qquad z = \phi_1(y) + \phi_2(y + 2x) + \frac{x^5y}{20} + \frac{x^6}{60}$$

Note

(i) $\frac{1}{D^2}(x^3y)$ means integration of x^3y twice w.r.to x. keeping y constant and $\frac{1}{D^3}x^3$ means integration of x^3 w.r.to x thrice.

(ii) First differentiate and then integrate.

EXAMPLE 10

Solve $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$.

Solution.

Given: $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

To find the C.F solve $(D^2 + DD' - 6D'^2)z = 0$

Auxiliary equation is $m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = -3, 2$

$\therefore$ C.F $= f_1(y - 3x) + f_2(y + 2x)$

$$\text{P.I}_1 = \frac{1}{D^2 + DD' - 6D'^2}x^2y$$

Here $m = 2$, $n = 1$, $m > n$ $\therefore$ take out D^2 and proceed.

$$\therefore \quad \text{P.I}_1 = \frac{1}{D^2\left[1 + \frac{D'}{D} - \frac{6D'^2}{D^2}\right]}x^2y = \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1}x^2y$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]x^2y$$

$$= \frac{1}{D^2}\left[x^2y - \frac{D'}{D}(x^2y)\right]$$

$$= \frac{1}{D^2}(x^2y) - \frac{1}{D^3}(x^2) \qquad [\because D'(x^2y) = x^2]$$

$$= \frac{1}{D}\int x^2y\,dx - \frac{1}{D^2}\int x^2dx$$

$$= \frac{1}{D}\left(\frac{x^3}{3}y\right) - \frac{1}{D^2}\left(\frac{x^3}{3}\right)$$

$$= \frac{y}{3}\int x^3dx - \frac{1}{3D}\int x^3dx = \frac{y}{3}\cdot\frac{x^4}{4} - \frac{1}{3}\int\frac{x^4}{4}dx = \frac{yx^4}{12} - \frac{1}{12}\cdot\frac{x^5}{5} = \frac{yx^4}{12} - \frac{x^5}{60}$$

$$\text{P.I}_2 = \frac{1}{D^2 + DD' - 6D'^2}e^{3x+y} = \frac{1}{3^2 + 3\cdot 1 - 6\cdot 1}e^{3x+y} = \frac{e^{3x+y}}{6}$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad z = f_1(y-3x) + f_2(y+2x) + \frac{yx^4}{12} - \frac{x^5}{60} + \frac{e^{3x+y}}{6}$$

Type 4: General method

EXAMPLE 11

Solve $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$.

Solution.

Given $\qquad (D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$

To find the C.F solve $(D^2 + 2DD' + D'^2)z = 0$

Auxiliary equation is $\qquad m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$

$\therefore \qquad \text{C.F} = f_1(y-x) + xf_2(y-x)$

$$\text{P.I}_1 = \frac{1}{D^2 + 2DD' + D'^2}2\cos y = \frac{1}{0+1}2\cos y = 2\cos y \qquad [\because a = 0, b = 1]$$

$$\text{P.I}_2 = \frac{1}{D^2 + 2DD' + D'^2}(-x\sin y)$$

$$= \frac{-1}{(D+D')(D+D')}x\sin y$$

$$= -\frac{1}{(D+D')}\int x\sin(c+x)dx \qquad [\because y = c - mx = c + x]$$

$$= -\frac{1}{(D+D')}[x(-\cos(c+x)) + 1(\sin(c+x))] \qquad \text{[by Bernoulli's formula]}$$

$$= \frac{1}{(D+D')}[x\cos y - \sin y] \qquad [\text{replacing } c]$$

$$= \int [x\cos(c+x) - \sin(c+x)]\,dx \qquad \text{[Replacing } y \text{ by } c+x]$$

$$= [x\sin(c+x) - 1\cdot(-\cos(c+x)) + \cos(c+x)]$$

$$= [x\sin y + 2\cos y] \qquad \text{[Replacing } c \text{ after integration]}$$

$\therefore$ the general solution is z = C.F + P.I

$$\Rightarrow \qquad z = f_1(y-x) + xf_2(y-x) + x\sin y + 2\cos y$$

EXAMPLE 12

Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = \sqrt{x+3y}$.

Solution.

Given $\qquad (D^2 - 4DD' + 3D'^2)z = \sqrt{x+3y}$

To find the C.F, solve $(D^2 - 4DD' + 3D'^2)z = 0$

Auxiliary equation is $\qquad m^2 - 4m + 3 = 0 \;\Rightarrow\; (m-1)(m-3) = 0 \Rightarrow m = 1, 3$

$$\therefore \qquad \text{C.F} = f_1(y+x) + f_2(y+3x)$$

$$\text{P.I} = \frac{1}{D^2 - 4DD' + 3D'^2}\sqrt{x+3y} = \frac{1}{(D-D')(D-3D')}\sqrt{x+3y}$$

$$= \frac{1}{(D-D')}\int \sqrt{x + 3(c-3x)}\,dx \qquad [\because y = c - mx = c - 3x]$$

$$= \frac{1}{(D-D')}\int (3c-8x)^{1/2}\,dx$$

$$= \frac{1}{(D-D')}\left[\frac{(3c-8x)^{3/2}}{\frac{3}{2}(-8)}\right] \qquad \left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}\right]$$

$$= -\frac{1}{12(D-D')}[3(y+3x) - 8x]^{3/2} \qquad [\because c = y + 3x]$$

$$= -\frac{1}{12(D-D')}[3y + x]^{3/2}$$

$$= -\frac{1}{12}\int [3(c-x) + x]^{3/2}\,dx \qquad [\text{Now } y = c - x]$$

$$= -\frac{1}{12}\int (3c-2x)^{3/2}\,dx$$

$$= -\frac{1}{12}\frac{(3c-2x)^{5/2}}{\frac{5}{2}(-2)} = \frac{1}{60}[3(y+x) - 2x]^{5/2} \qquad [\because c = y + x]$$

$$= \frac{1}{60}[3y + x]^{5/2}$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad z = f_1(y+x) + f_2(y+3x) + \frac{1}{60}(x+3y)^{5/2}$$

EXAMPLE 13

Solve $(4D^2 - 4DD' + D'^2)z = 16\log_e(x+2y)$.

Solution.

Given $\quad (4D^2 - 4DD' + D'^2)z = 16\log_e(x+2y)$

To find the C.F solve $(4D^2 - 4DD' + D'^2)z = 0$

Auxiliary equation is $\quad 4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \quad \Rightarrow \quad m = \frac{1}{2}, \frac{1}{2}$

$$\therefore \quad \text{C.F} = f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I} = \frac{1}{4D^2 - 4DD' + D'^2}16\log_e(x+2y) = \frac{1}{(2D-D')(2D-D')}16\log_e(x+2y)$$

$$= \frac{16}{4}\cdot\frac{1}{\left(D-\frac{1}{2}D'\right)\left(D-\frac{1}{2}D'\right)}\log_e(x+2y)$$

$$= 4\cdot\frac{1}{\left(D-\frac{1}{2}D'\right)}\int \log_e\left[x+2\left(c-\frac{1}{2}x\right)\right]dx \left[\because y = c - \frac{1}{2}x\right]$$

$$= 4\cdot\frac{1}{\left(D-\frac{1}{2}D'\right)}\int \log_e 2c\, dx$$

$$= 4\cdot\frac{1}{D-\frac{1}{2}D'}\log_e 2c\cdot x$$

$$= 4\frac{1}{D-\frac{1}{2}D'}x\log_e(x+2y) \quad \left[\because y = c - \frac{x}{2} \Rightarrow 2c = x+2y\right]$$

$$= 4\int x\log_e\left[x+2\left(c-\frac{x}{2}\right)\right]dx$$

$$= 4\int x\log_e 2c\, dx = 4\log_e 2c\cdot\frac{x^2}{2} = 2x^2\log_e(x+2y)$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad z = f_1\left(y+\frac{1}{2}x\right) + xf_2\left(y+\frac{x}{2}\right) + 2x^2\log_e(x+2y)$$

Type 5: Exponential shifting

EXAMPLE 14

Solve $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+2y}$.

Solution.

Given $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+2y}$

To find the C.F solve $(D^2 - 2DD' + D'^2)z = 0$

Auxiliary equation is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$ $\text{C.F} = f_1(y+x) + xf_2(y+x)$

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2}x^2y^2e^{x+y} = \frac{1}{(D-D')^2}x^2y^2e^{x+y}$$

$$= e^{x+y}\frac{1}{[D+1-(D'+1)]^2}x^2y^2$$

[By shifting $D \to D+1$, $D' \to D'+1$]

$$= e^{x+y}\frac{1}{(D-D')^2}x^2y^2 \qquad \text{[Here } m = n = 2]$$

$$= e^{x+y}\frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[1 + 2\frac{D'}{D} + \frac{3D'^2}{D^2} + \cdots\right]x^2y^2$$

$$= e^{x+y}\frac{1}{D^2}\left[x^2y^2 + \frac{2}{D}D'(x^2y^2) + \frac{3}{D^2}D'^2(x^2y^2)\right]$$

$$= e^{x+y}\cdot\frac{1}{D^2}\left[x^2y^2 + \frac{2}{D}x^2\cdot 2y + \frac{3}{D^2}x^2\cdot 2\right]$$

$$= e^{x+y}\cdot\frac{1}{D^2}\left[x^2y^2 + \int 4x^2y\,dx + 6\frac{1}{D}\int x^2dx\right]$$

$$= e^{x+y}\cdot\frac{1}{D^2}\left[x^2y^2 + \frac{4x^3}{3}y + 6\int\frac{x^3}{3}dx\right]$$

$$= e^{x+y}\cdot\frac{1}{D^2}\left[x^2y^2 + \frac{4}{3}x^3y + 2\frac{x^4}{4}\right]$$

$$= e^{x+y}\frac{1}{D}\int\left(x^2y^2 + \frac{4}{3}\cdot x^3y + \frac{x^4}{2}\right)dx$$

$$= e^{x+y}\frac{1}{D}\left[\frac{x^3}{3}y^2 + \frac{4y}{3}\cdot\frac{x^4}{4} + \frac{x^5}{10}\right]$$

$$= e^{x+y}\int\left(\frac{x^3}{3}y^2 + \frac{y}{3}x^4 + \frac{x^5}{10}\right)dx = e^{x+y}\left[\frac{x^4}{12}y^2 + \frac{y}{3}\frac{x^5}{5} + \frac{x^6}{60}\right] = e^{x+y}\cdot\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad z = f_1(y+x) + xf_2(y+x) + \left(\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right)e^{x+y}$$

EXAMPLE 15

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$.

Solution.

Given $\qquad (D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$

To find the C.F solve $(D^3 + D^2D' - DD'^2 - D'^3)z = 0$

Auxiliary equation is $\qquad m^3 + m^2 - m - 1 = 0$

$$\Rightarrow \qquad m^2(m+1) - (m+1) = 0$$

$$\Rightarrow \qquad (m+1)(m^2-1) = 0 \quad \Rightarrow \quad (m+1)^2(m-1) = 0 \Rightarrow m = -1, -1, 1$$

$$\therefore \qquad \text{C.F} = f_1(y-x) + xf_2(y-x) + f_3(y+x)$$

$$\text{P.I} = \frac{1}{(D^3 + D^2D' - DD'^2 - D'^3)}e^x \cos 2y$$

$$= \frac{1}{(D+D')^2(D-D')}e^x \cos 2y$$

$$= e^x \cdot \frac{1}{(D+1+D')^2(D+1-D')}\cos 2y \qquad \text{[By shifting } D \to D+1, \text{ since } a = 1, b = 0]$$

$$= e^x \frac{1}{(D+D'+1)^2(D-D'+1)}\cos 2y$$

$$= e^x \text{R.P.}\frac{1}{(D+D'+1)^2(D-D'+1)}e^{i2y} \qquad \text{[Here } a = 0, b = 2\text{i]}$$

$$= e^x \text{R.P.}\frac{1}{(2i+1)^2(-2i+1)}e^{i2y} \qquad \text{[Replace } D \text{ by } a = 0, D' \text{ by } b = 2\text{i]}$$

$$= e^x \text{R.P.}\frac{1}{(1+2i)(1+4)}e^{i2y} \qquad [\because (1+2i)(1-2i) = 1+4]$$

$$= \frac{e^x}{5} \text{R.P} \frac{1-2i}{(1+4)} e^{i2y}$$

$$= \frac{e^x}{25} \text{R.P}(1-2i)(\cos 2y + i \sin 2y)$$

$$= \frac{e^x}{25} \text{R.P} [\cos 2y + 2\sin 2y + i(\sin 2y - 2\cos 2y)]$$

$\Rightarrow$ $$\text{P.I} = \frac{e^x}{25}[\cos 2y + 2\sin 2y]$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$

Note In the evaluation of P.I, we have used Real part of e^{i2y} similar to the one in ordinary differential equation, because the usual method is difficult for this problem.

EXAMPLE 16

Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x+3y)$.

Solution.

Given $(D^2 - D'^2)z = e^{x-y} \cdot \sin(2x+3y)$

To find the C.F solve $(D^2 - D'^2)z = 0$

Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ $$\text{C.F} = f_1(y+x) + f_2(y-x)$$

$$\text{P.I} = \frac{1}{D^2 - D'^2} e^{x-y} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{(D+1)^2 - (D'-1)^2} \sin(2x+3y)$$ [By shifting $D \to D+1$, $D' \to D'-1$, since $a = 1$, $b = -1$ in e^{x-y}]

$$= e^{x-y} \frac{1}{D^2 - D'^2 + 2(D+D')} \sin(2x+3y)$$

$$= e^{x-y} \frac{1}{-2^2 - (-3^2) + 2(D+D')} \sin(2x+3y)$$ [Replacing D^2 by -2^2, D'^2 by -3^2]

$$= e^{x-y} \frac{1}{2(D+D')+5} \sin(2x+3y)$$

$$= e^{x-y} \frac{2(D+D')-5}{4(D+D')^2 - 25} \sin(2x+3y)$$

$$= e^{x-y} \frac{[2(D+D')-5]\sin(2x+3y)}{4[D^2 + D'^2 + 2DD'] - 25}$$

$$= e^{x-y}\frac{[2(D+D')\sin(2x+3y)-5\sin(2x+3y)]}{4[-2^2-3^2+2(-6)]-25}$$

$$= e^{x-y}\frac{[2\cos(2x+3y)\cdot 2+2\cdot\cos(2x+3y)\cdot 3-5\sin(2x+3y)]}{4(-25)-25}$$

$$= \frac{e^{x-y}}{-125}[10\cos(2x+3y)-5\sin(2x+3y)]$$

$$= -\frac{e^{x-y}}{25}[2\cos(2x+3y)-\sin(2x+3y)]$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$z = f_1(y+x)+f_2(y-x)-\frac{e^{x-y}}{25}[2\cos(2x+3y)-\sin(2x+3y)]$$

EXERCISE 14.6

Solve the following partial differential equations

1. $(D^2+5DD'+6D'^2)z=0$
2. $(D^2-4DD'+4D'^2)z=0$
3. $(D^3+D^2D'-DD'^2-D'^3)z=0$
4. $(5D^2-12DD'-9D'^2)z=0$
5. $(D^2+4DD')z=e^x$
6. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+6\frac{\partial^2 z}{\partial y^2}=e^{x+y}$
7. $(D^2-2DD'+D'^2)z=8e^{x+2y}$
8. $(9D^2+6DD'+D'^2)z=(e^x+e^{-2y})^2$
9. $(D^3-3DD'^2+2D'^3)z=e^{2x-y}+e^{x+y}$
10. $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)+e^{3x+y}$
11. $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=\sin x$
12. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=\cos 2x\cdot\cos 3y$
13. $(D^2+DD'-6D'^2)z=\cos(2x+y)$
14. $(D^3+D^2D'-DD'^2-D'^3)=\cos(2x+y)$
15. $(4D^2-4DD'+D'^2)z=e^{3x-2y}+\sin x$
16. $(2D^2-5DD'+2D'^2)z=5\sin(2x+y)$
17. $(D^3-7DD'^2-6D'^3)z=\cos(x+2y)+4$

    $\left[\textbf{Hint:}\ \text{P.I}_1=\text{R.P}\frac{1}{D^3-7DD'^2-6D'^3}e^{i(x+2y)}\text{ and use type 1}\right]$

    $\text{P.I}_2=\frac{1}{D^3-7DD'^2-6D'^3}4e^{0x+0y}$, multiply by x and diff. w.r.t. to D]
18. $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=x^2+xy+y^2$
19. $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$
20. $(D-D')^2z=2e^{x+y}\cos^2\left(\frac{x+y}{2}\right)$
21. $(D^2-D'^2)z=e^{x-y}\cdot\sin(2x+3y)$

22. $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$

23. $(D^2 + DD' - 2D'^2)z = y \sin x$

24. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = y \sin x$

25. $(D^2 + 3DD' + 2D'^2)z = 12xy$

26. $(D^2 + 2DD' + D'^2)z = \sinh(x + y) + e^{x+2y}$

ANSWERS TO EXERCISE 14.6

1. $z = f_1(y - 2x) + f_2(y - 3x)$

2. $z = f_1(y + 2x) + xf_2(y + 2x)$

3. $z = f_1(y + x) + f_2(y - x) + xf_3(y - x)$

4. $z = f_1(y + 3x) + f_2\left(y - \frac{3x}{5}\right)$

5. $z = f_1(y) + f_2(y - 4x) + e^x$

6. $z = f_1(y + 2x) + f_2(y + 3x) + \frac{1}{2}e^{x+y}$

7. $z = f_1(y + x) + xf_2(y + x) + 8e^{x+2y}$

8. $z = f_1\left(y - \frac{1}{3}x\right) + xf_2\left(y - \frac{1}{3}x\right) + \frac{1}{36}e^{2x} + 2e^{x-2y} + \frac{1}{16}e^{-4y}$

9. $z = f_1(y + x) + xf_2(y + x) + f_3(y - 2x) + \frac{x}{9}e^{2x-y} + \frac{x^2}{6}e^{x+y}$

10. $z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) - \frac{1}{75}\cos(x + 2y) + \frac{x}{20}e^{3x+y}$

11. $z = f_1(y + x) + xf_2(y + x) - \sin x$

12. $z = f_1(y + x) + f_2(y - x) + \frac{1}{5}\cos 2x \cdot \cos 3y$

13. $z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$

14. $z = f_1(y + x) + f_2(y - x) + xf_3(y - x) - \frac{1}{9}\sin(2x + y)$

15. $z = f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{1}{2}x\right) + \frac{1}{64}e^{3x-2y} - \frac{1}{4}\sin x$

16. $z = f_1(y + 2x) + f_2\left(y + \frac{1}{2}x\right) - \frac{5x}{2}\cos(2x + 4)$

17. $z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{1}{75}\sin(x + 2y) + \frac{2x^3}{3}$

18. $z = f_1(y - x) + xf_2(y - x) + \frac{1}{4}(x^4 - 2x^3 y + 2x^2 y^2)$

19. $z = f_1(y - x) + f_2(y - 2x) + \frac{1}{2}x^2 y - \frac{x^3}{3}$

20. $z = f_1(y+x) + xf_2(y+x) + x^2 e^{x+y} \cos^2\left(\frac{x+y}{2}\right)$

21. $z = f_1(y+x) + f_2(y-x) + \frac{1}{25} e^{x-y} [\sin(2x+3y) - 2\cos(2x+3y)]$

22. $z = f_1(y+x) + f_2(y-x) + \frac{5x^3}{6} + \frac{3}{2}x^2 y - \frac{1}{35} e^{3x+4y}$

23. $z = f_1(y+x) + f_2(y-2x) - y\sin x - \cos x$

24. $z = f_1(y+2x) + f_2(y+3x) + 5\cos x - y\sin x$

25. $z = f_1(y-x) + f_2(y-2x) + 2x^3 y - \frac{3}{2}x^4$

26. $z = f_1(y-x) + xf_2(y-x) + \frac{1}{4}\sinh(x+y) + \frac{e^{x+2y}}{9}$

14.7 NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

Equations of the type $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} + z = e^{x+y}$

where all the partial derivatives are not of the same order is called a non-homogeneous linear equation.

More generally, in the linear equation $F(D, D')z = R(x, y)$ (1)

If $F(D, D')$ is not homogeneous, then the equation (1) is called a non homogeneous linear partial differential equation.

As in the case of homogeneous equation, the general solution is $z = \text{C.F} + \text{P.I}$

To find the C.F, solve $F(D, D')z = 0$

We factorize $F(D, D')$ into linear factors of the form $D - m\,D' - c$

The solution of $(D - m\,D' - c)z = 0$ is $z = e^{cx} f(y + mx)$

For $(D - m\,D' - c)z = 0 \Rightarrow Dz - mD'z - cz = 0$

$\Rightarrow$ $p - mq = cz$, which is Lagrange's equation.

The subsidiary equations are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$

$$\Rightarrow \quad -mx = dy \Rightarrow dy + mdx = 0$$

Integrating, $y + mx = a$ is one solution.

$$\frac{dx}{1} = \frac{dz}{cz} \Rightarrow cdx = \frac{dz}{z}$$

Integrating, $\log z = cx + \log k \Rightarrow \log\frac{z}{k} = cx$

$\Rightarrow$ $\frac{z}{k} = e^{cx} \Rightarrow \frac{z}{e^{cx}} = k$

$\therefore$ the general solution is $\frac{z}{e^{cx}} = f(y+mx) \quad \Rightarrow \quad z = e^{cx} f(y+mx)$

(i) If $F(D, D')z = (D - m_1 D' - c_1)(D - m_2 D' - c_2)\ldots(D - m_n D' - c_n)z$

Then C.F $= e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x) + \cdots + e^{c_n x} f_n(y + m_n x)$

(ii) if $(D - m D' - c)^2 z = 0$ then $z = e^{cx} f_1(y+mx) + x e^{cx} f_2(y+mx)$

i.e., for repeated factors $z = e^{cx}[f_1(y+mx) + x f_2(y+mx)]$

(iii) **If both repeated and non repeated factors occur, then a combination of case (i) and case (ii) is applied.**

Note Solution of $(D' - n D - c)z = 0$ is $z = e^{cy} f(x + ny)$

and solution of $(D' - n D - c)^2 z = 0$ is $z = e^{cy}[f_1(x+ny) + y f_2(x+ny)]$

To find P.I, the rules are the same as those for homogeneous linear partial differential equations.

WORKED EXAMPLES

EXAMPLE 1

Solve $(D^2 - DD' + D' - 1)z = 0$.

Solution.

Given $(D^2 - DD' + D' - 1)z = 0$

$\Rightarrow$ $[(D^2 - 1) - D'(D-1)]z = 0$

$\Rightarrow$ $(D-1)(D+1-D')z = 0$

$\Rightarrow$ $(D-1)(D-D'+1)z = 0$

$\Rightarrow$ $[D - 0\,D' - 1][D - D' - (-1)]z = 0$

Here $m_1 = 0, \quad c_1 = 1, \quad m_2 = 1, \quad c_2 = -1$

$\therefore$ the general solution is $z = e^{x} f_1(y) + e^{-x} f_2(y+x)$

EXAMPLE 2

Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$.

Solution.

Given $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$

To find the complementary function, solve

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = 0$$

$\Rightarrow$ $[(D+D')^2 - 2(D+D')]z = 0$

$\Rightarrow \qquad (D+D')(D+D'-2)z=0$

$\Rightarrow \qquad (D-(-1)D')(D-(-1)D'-2)z=0$

Here $m_1=-1, \quad c_1=0, \quad m_2=-1, \quad c_2=2$

$\therefore \qquad \text{C.F}=e^{0x}f_1(y-x)+e^{2x}f_2(y-x)=f_1(y-x)+e^{2x}f_2(y-x)$

$$\text{P.I}=\frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x+2y)$$

$$=\frac{1}{-1^2+2(-1\cdot 2)+(-2^2)-2D-2D'}\sin(x+2y)$$

$$=\frac{1}{-9-2D-2D'}\sin(x+2y)$$

$$=\text{I.P}\frac{-1}{2D+2D'+9}e^{i(x+2y)}$$

$$=\text{I.P}-\frac{1}{2i+2(2i)+9}e^{i(x+2y)} \qquad \text{[Replacing } D \text{ by } a=i, \text{ and } D' \text{ by } b=2i]$$

$$=-\text{I.P}\frac{1}{9+6i}e^{i(x+2y)}$$

$$=-\text{I.P}\frac{(9-6i)}{81+36}[\cos(x+2y)+i\sin(x+2y)]$$

$$=-\text{I.P}\frac{1}{117}[9\cos(x+2y)+6\sin(x+2y)+i(9\sin(x+2y)-6\cos(x+2y))]$$

$$=-\frac{1}{117}[9\sin(x+2y)-6\cos(x+2y)]$$

$$=\frac{1}{39}[2\cos(x+2y)-3\sin(x+2y)]$$

$\therefore$ the general solution is $z=\text{C.F}+\text{P.I}$

$$\Rightarrow \qquad z=f_1(y-x)+e^{2x}f_2(y-x)+\frac{1}{39}[2\cos(x+2y)-3\sin(x+2y)]$$

EXAMPLE 3

Solve $(D^2-D'^2-3D+3D')z=xy+7$.

Solution.

Given $\qquad (D^2-D'^2-3D+3D')z=xy+7$

To find the C.F, solve $(D^2-D'^2-3D+3D')z=0$

$\Rightarrow \qquad (D^2-D'^2-3(D-D'))z=0$

$\Rightarrow \qquad (D-D')(D+D'-3)z=0$

Here $m_1=1, \quad c_1=0, \quad m_2=-1, \quad c_2=3$

$\therefore \qquad \text{C.F}=e^{0x}f_1(y+x)+e^{3x}f_2(y-x)=f_1(y+x)+e^{3x}f_2(y-x)$

$$\text{P.I} = \frac{1}{D^2 - D'^2 - 3D + 3D'}(xy+7)$$

$$= \frac{1}{(D-D')(D+D'-3)}(xy+7)$$

$$= \frac{1}{-3D\left(1-\frac{D'}{D}\right)\left(1-\frac{D+D'}{3}\right)}(xy+7)$$

$$= -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)^{-1}(xy+7)$$

$$= -\frac{1}{3D}\left[1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right]\left[1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+\frac{(D+D')^3}{27}+\cdots\right](xy+7)$$

$$= -\frac{1}{3}\left[\frac{1}{D}+\frac{D'}{D^2}\right]\left[1+\frac{D}{3}+\frac{D'}{3}+\frac{D^2}{9}+\frac{2DD'}{9}+\frac{D'^2}{9}+\frac{D^3}{27}+\frac{3D^2D'}{27}\right](xy+7)$$

$$= -\frac{1}{3}\left[\frac{1}{D}+\frac{1}{3}+\frac{1D'}{3D}+\frac{D}{9}+\frac{2}{9}D'+\frac{1}{9}DD'+\frac{D'}{D^2}+\frac{D'}{3D}+\frac{D'}{9}+\frac{DD'}{27}\right](xy+7)$$

$$= -\frac{1}{3}\left[\frac{1}{D}+\frac{1}{3}+\frac{2D'}{3D}+\frac{D}{9}+\frac{1}{3}D'+\frac{4}{27}DD'+\frac{D'}{D^2}\right](xy+7)$$

$$= -\frac{1}{3}\left[\frac{1}{D}(xy+7)+\frac{1}{3}(xy+7)+\frac{2}{3D}(x)+\frac{1}{9}y+\frac{1}{3}x+\frac{4}{27}+\frac{x}{D^2}\right]$$

$$= -\frac{1}{3}\left[\int (xy+7)\,dx+\frac{1}{3}xy+\frac{7}{3}+\frac{2}{3}\int x\,dx+\frac{y}{9}+\frac{x}{3}+\frac{4}{27}+\frac{1}{D}\int x\,dx\right]$$

$$= -\frac{1}{3}\left[\frac{x^2}{2}y+7x+\frac{xy}{3}+\frac{7}{3}+\frac{2}{3}\cdot\frac{x^2}{2}+\frac{y}{9}+\frac{x}{3}+\frac{4}{27}+\int \frac{x^2}{2}dx\right]$$

$$= -\frac{1}{3}\left[\frac{x^2y}{2}+7x+\frac{xy}{3}+\frac{7}{3}+\frac{x^2}{3}+\frac{y}{9}+\frac{x}{3}+\frac{4}{27}+\frac{x^3}{6}\right]$$

$$= -\frac{1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^3}{6}+\frac{x^2}{3}+\frac{x}{3}+\frac{y}{9}+7x+\frac{67}{27}\right]$$

$\therefore$ the general solution is $z = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad z = f_1(y+x)+e^{3x}f_2(y-x)-\frac{1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^3}{6}+\frac{x^2}{3}+\frac{22x}{3}+\frac{y}{9}+\frac{67}{27}\right]$$

EXAMPLE 4

Solve $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y + ye^x$.

Solution.

Given $\qquad (2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y + ye^x$

To find the C.F, solve $(2D^2 - DD' - D'^2 + 6D + 3D')z = 0$

Now $$2D^2 - DD' - D'^2 = D^2 - DD' + D^2 - D'^2$$
$$= D(D-D') + (D+D')(D-D')$$
$$= (D-D') + (D+D+D')$$
$$= (D-D') + (2D+D')$$

$\therefore$ $$2D^2 - DD' - D'^2 + 6D + 3D' = (D - D' + l)(2D + D' + m)$$
$$= (D-D')\cdot(2D+D') + l(2D+D') + m(D-D') + lm$$
$$= 2D^2 - DD' - D'^2 + (2l+m)D + (l-m)D' + lm$$

$\therefore$ $$2l + m = 6$$

and $$l - m = 3 \quad \Rightarrow \quad 3l = 9 \quad \Rightarrow \quad l = 3 \qquad \therefore m = 0$$

$\therefore$ $$2D^2 - DD' - D'^2 + 6D + 3D' = (D - D' + 3)(2D + D')$$

$\therefore$ $$(2D^2 - DD' - D'^2 + 6D + 3D')z = 0$$

$\Rightarrow$ $$(D - D' + 3)(2D + D')z = 0$$

$\Rightarrow$ $$(D - D' - (-3))\left(D - \left(-\frac{1}{2}\right)D'\right)z = 0$$

Here $m_1 = 1, c_1 = -3, m_2 = -\frac{1}{2}, c_2 = 0$

$\therefore$ $$\text{C.F} = e^{-3x} f_1(y+x) + e^{0x} f_2\left(y - \frac{1}{2}x\right) = e^{-3x} f_1(y+x) + f_2\left(y - \frac{1}{2}x\right)$$

$$\text{P.I}_1 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y$$

$$= e^y \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x$$

$[D' \to D' + 1]$

$$= e^y \frac{1}{2D^2 - DD' - D - (D'^2 + 2D' + 1) + 6D + 3D' + 3} x$$

$$= e^y \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2} x$$

$$= \frac{e^y}{2}\left[1 + \frac{1}{2}(5D + D' + 2D' - DD' - D'^2)\right]^{-1}(x)$$

$$\text{P.I}_1 = \frac{e^y}{2}\left[1 - \frac{5}{2}D\right]x = \frac{e^y}{2}\left(x - \frac{5}{2}\right) = \frac{1}{4}(2x-5)e^y$$

$$\text{P.I}_2 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} ye^x$$

$$= e^x \frac{1}{2(D+1)^2 - (D+1)D' - D'^2 + 6(D+1) + 3D'} y$$

$$= e^x \frac{1}{8+10D+2D'+2D^2-DD'-D'^2}(y)$$

$$= \frac{e^x}{8}\left[1+\frac{1}{8}(10D+2D'+2D^2-DD'-D'^2)\right]^{-1}(y)$$

$$= \frac{e^x}{8}\left[1-\frac{1}{8}(10D+2D'\ldots)\right]y$$

$$= \frac{e^x}{8}\left[1-\frac{1}{4}D'\right]y = \frac{e^x}{8}\left[y-\frac{1}{4}\right] = \frac{1}{32}(4y-1)e^x$$

$\therefore$ the general solution is z = C.F + P.I

$$z = e^{-3x}f_1(y+x)+f_2\left(y-\frac{1}{2}x\right)+\frac{1}{4}(2x-5)e^y+\frac{1}{32}(4y-1)e^x$$

EXERCISE 14.7

Solve the following partial differential equations

1. $(D^2+DD'+D'-1)z = e^{-x}$
2. $(D^2-2DD'-3D)z = e^{x+2y}$
3. $(D^2-DD'+D'-1)z = \cos(x+2y)+e^y$
4. $(2DD'+D'^2-3D')z = 3\cos(3x-2y)$

   [**Hint:** $D'(D'+2D-3)z = 0$, Here $n_1 = 0, c_1 = 0, n_2 = -2, c_2 = 3$

   $\therefore$ C.F $= f_1(x)+e^{3y}f_2(x-2y)$]
5. $(D^2-DD'+D'-1)z = \cos^2(x+2y)$
6. $(D^2+2DD'+D'^2-2D-2D')z = e^{3x+y}+4$
7. $(D+D'-1)(D+2D'-3)z = 4+3x+6y$

ANSWERS TO EXERCISE 14.7

1. $z = e^{-x}f_1(y)+e^x f_2(y-x)-\frac{1}{2}x^{-x}$
2. $z = f_1(y)+e^{3x}f_2(y+2x)-\frac{1}{6}e^{x+2y}$
3. $z = e^x f_1(y)+e^{-x}f_2(y+x)+\frac{1}{2}\sin(x+2y)-xe^y$
4. $z = f_1(x)+e^{3y}f_2(x-2y)+\frac{3}{50}[4\cos(3x-2y)+3\sin(3x-2y)]$
5. $z = e^x f_1(y)+e^{-x}f_2(y+x)+\frac{1}{50}[4\sin(2x+4y)+3\cos(2x+4y)]-\frac{1}{2}$
6. $z = f_1(y-x)+e^{2x}f_2(y-x)+\frac{1}{8}e^{3x+y}-2x$
7. $z = e^x f_1(y-x)+e^{3x}f_2(y-2x)+x+2y+6$

Applications of Partial Differential Equations

20.0 INTRODUCTION

In Chapter-14 we have indicated that partial differential equations arise in the study of fluid mechanics, heat transfer, electromagnetic theory, quantum mechanics and other areas of physics and engineering. In fact, the areas of applications of partial differential equations is too large compared to ordinary differential equations.

The important partial differential equations that will be discussed in this chapter are the following.

1. **One-dimensional wave equation**

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

2. **One-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

3. **Steady state two-dimensional heat equation or two dimensional Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (3)$$

Generally, a partial differential equation will have many solutions. For example, the functions $u = x^2 - y^2$, $u = e^x \cos y$, $u = \log_e(x^2 + y^2)$ are different solutions of (3).

In practical problems we seek to obtain unique solution of a partial differential equation subject to certain specific conditions called boundary – value conditions. The differential equation with the boundary - value conditions is called the boundary – value problem. For instance, consider the partial differential equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$. The solution $y(x, t)$ is unique when obtained under the conditions $y(x, 0) = x^2$, $\frac{\partial y}{\partial t}(x, 0) = 5x$, called **initial conditions** and the conditions $y(0, t) = 0$, $y(l, t) = 0$, called **boundary conditions**.

The initial conditions and the boundary conditions together are known as boundary – value conditions.

The differential equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with these boundary – value conditions is known as a **boundary – value problem**.

Note When conditions are prescribed at the same point, we call them as initial conditions. Here $u(x, 0) = x^2$ and $\frac{\partial u}{\partial t}(x, 0) = 5x$ are initial conditions. When conditions are prescribed at different points, we call them as boundary conditions. Here $u(0, t) = 0$ and $u(l, t) = 0$ are boundary conditions.

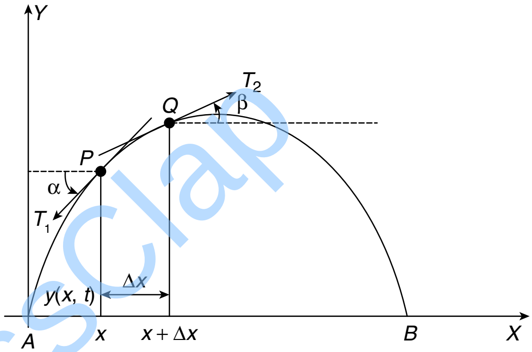
20.1 ONE DIMENSIONAL WAVE EQUATION – EQUATION OF VIBRATING STRING

Consider an elastic string which is stretched to a length l and fixed at its ends A and B.

Choose the end A as origin and AB as x-axis in the equilibrium position. The line through A and perpendicular to AB is taken as y-axis.

If the string is deflected from its original position at some instant say, $t = 0$, and released from rest, then the string vibrates transversely. That is, it vibrates at righ angles to the equilibrium position in the xy-plane. Our aim is to find the shape of the string at any instant. i.e., to find the displacement of the string $y(x, t)$ at any point x and at any time $t > 0$

In order to derive the partial differential equation satisfied by $y(x, t)$ in the simplest form, we make the following physical assumptions.



(i) The string is homogeneous. i.e., the mass of the string per unit length is constant. The string is perfectly elastic and so it does not offer any resistance to bending.

(ii) The tension T caused by stretching the string before fixing it at the ends is so large that the action of the gravitational force on the string can be neglected.

(iii) The string performs small transverse motions in a vertical plane. That is every particle of the string moves vertically so that the deflection y and the slope $\frac{\partial y}{\partial x}$ are small in absolute value, hence their higher powers may be neglected.

20.1.1 Derivation of Wave Equation

Consider the forces acting on a small portion PQ of string. Let m be the mass per unit length of the string.

$\therefore$ mass of the string PQ is $m\Delta x$. [$\because$ PQ is small, PQ is almost a straight line and so $PQ = \Delta x$]

Since the string does not offer resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1, T_2 be the tension at the end points P and Q of the element string PQ. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence, the horizontal components of the tension must be constant.

$$\therefore \quad T_1 \cos \boldsymbol{\alpha} = T_2 \cos \boldsymbol{\beta} = T, \text{ a constant} \tag{1}$$

In the vertical direction we have forces $-T_1 \sin \boldsymbol{\alpha}$ and $T_2 \sin \boldsymbol{\beta}$ of T_1 and T_2

By Newton's second law, the equation of motion in the vertical direction is

$$m\Delta x \frac{\partial^2 y}{\partial t^2} = T_2 \sin \boldsymbol{\beta} - T_1 \sin \boldsymbol{\alpha} = \frac{T}{\cos \boldsymbol{\beta}} \sin \boldsymbol{\beta} - \frac{T}{\cos \boldsymbol{\alpha}} \sin \boldsymbol{\alpha} = T(\tan \boldsymbol{\beta} - \tan \boldsymbol{\alpha}) \qquad \text{[using (1)]}$$

$$\therefore \tan \boldsymbol{\beta} - \tan \boldsymbol{\alpha} = \frac{m\Delta x}{T} \frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \frac{1}{\Delta x}(\tan \boldsymbol{\beta} - \tan \boldsymbol{\alpha}) = \frac{m}{T} \frac{\partial^2 y}{\partial t^2}$$

But $\tan \boldsymbol{\alpha}$ and $\tan \boldsymbol{\beta}$ are the slopes of the string at the points x and $x + \Delta x$

$\therefore \quad \tan \alpha = \left(\frac{\partial y}{\partial x}\right)_x$ and $\tan \beta = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \quad \therefore \quad \frac{1}{\Delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{\Delta x}\right] = \frac{m}{T}\frac{\partial^2 y}{\partial t^2}$

$$\therefore \quad \lim_{\Delta x \to 0} \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{\Delta x}}{\Delta x} = \frac{m}{T}\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{m}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{where } c^2 = \frac{T}{m}$$

Note

1. This is the partial differential equation giving the transverse vibrations of the string. It is called **the one-dimensional wave equation.**
   "One dimensional" is due to the fact that the equation involves only one space variable x.
2. The one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is involved in the study of transverse vibrations of a string, the longitudinal vibration of rods, electric oscillations in wires, the torsional oscillations of shafts, oscillation in gases and so on. This equation is the simplest of the class of **equations of the hyperbolic type**.
3. The solution $y(x, t)$ of the wave equation represents the deflection or displacement of the string at any time $t > 0$ and at any distance x from one end of the string.

$$c^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$$

   Since T and m are positive, we denote $\frac{T}{m}$ by c^2, rather than c.
4. Some times the equation is written as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

20.1.2 Solution of One-Dimensional Wave Equation by The Method of Separation of Variables (or The Fourier Method)

One-dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (1)

Since the solution $y(x, t)$ is a function of x and t, we seek a solution (not identically equal to zero) of the form $y(x, t) = X(x)\,T(t)$, where $X(x)$ is a function of x only and $T(t)$ is a function of t only.

$$\therefore \quad \frac{\partial y}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = XT''$$

$$\frac{\partial y}{\partial x} = X'T \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

$\therefore$ the equation (1) becomes $\quad XT'' = c^2 X'' T \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{c^2 T}$ (2)

Since the L.H.S is a function of x alone and R.H.S is a function of t alone and since x and t are independent variables, the equation (2) is possible if each side is a constant k.

$$\therefore \quad \frac{X''}{X} = \frac{T''}{c^2 T} = k$$

$$\Rightarrow \quad \frac{X''}{X} = k \quad \text{and} \quad \frac{T''}{c^2 T} = k$$

$$\Rightarrow \quad X'' = kX \quad \text{and} \quad T'' = kc^2 T$$

$$\Rightarrow \quad X'' - kX = 0 \quad \text{and} \quad T'' - k^2c^2T = 0 \tag{3}$$

Thus, we get two second order ordinary linear differential equations with constant coefficients. The solutions of (3) depend upon the nature of k. i.e., $k > 0$ or < 0 or 0

Case (i): If $k > 0$, let $k = \lambda^2$, $\lambda \neq 0$

Then (3) $\Rightarrow \quad X'' - \lambda^2 X = 0$

$\therefore$ auxiliary equation is $\quad m^2 - \lambda^2 = 0 \quad \Rightarrow m = \pm\lambda$

$$\therefore \quad X = Ae^{\lambda x} + Be^{-\lambda x}$$

and $\quad T'' - \lambda^2 c^2 T = 0$

$\therefore$ auxiliary equation is $\quad m^2 - \lambda^2 c^2 = 0 \quad \Rightarrow m = \pm\lambda c$

$$\therefore \quad T = Ce^{\lambda ct} + De^{-\lambda ct}$$

$\therefore$ the solution is $\quad y(x, t) = (Ae^{\lambda x} + Be^{-\lambda x})\,(Ce^{\lambda ct} + De^{-\lambda ct})$

where A, B, C, D are arbitrary constants.

Case (ii): If $k < 0$, let $k = -\lambda^2$, $\lambda \neq 0$

Then (3) $\Rightarrow \quad X'' + \lambda^2 X = 0$

$\therefore$ auxiliary equation is $\quad m^2 + \lambda^2 = 0 \quad \Rightarrow m = \pm\, i\lambda$

$$\therefore \quad X = A\cos\lambda x + B\sin\lambda x$$

Also (3) $\Rightarrow \quad T'' + \lambda^2 c^2 T = 0$

$\therefore$ auxiliary equation is $\quad m^2 + \lambda^2 c^2 = 0 \quad \Rightarrow m = \pm i\lambda c$

$$\therefore \quad T = C\cos\lambda ct + D\sin\lambda ct$$

$\therefore$ the solution is $\quad y(x, t) = (A\cos\lambda x + B\sin\lambda x)\,(C\cos\lambda ct + D\sin\lambda ct)$

where A, B, C, D are arbitrary constants.

Case (iii): If $k = 0$, then (3) $\Rightarrow X'' = 0 \quad$ and $\quad T'' = 0$

$$\Rightarrow \quad X' = A \quad \text{and} \quad T' = C$$

$$\Rightarrow \quad X = Ax + B \quad \text{and} \quad T = ct + D$$

$\therefore$ the solution is $\quad y(x, t) = (Ax + B)\,(Ct + D)$

where A, B, C, D are arbitrary constants.

Thus, there are three possible solutions of the wave equation and they are

$$y = (Ae^{\lambda x} + Be^{-\lambda x})\,(Ce^{\lambda ct} + De^{-\lambda ct}) \tag{I}$$

$$y = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \quad \text{(II)}$$

and
$$y = (Ax + B)(Ct + D) \quad \text{(III)}$$

Proper choice of the solution

Of these three solutions, we have to choose the solution which is consistent with the physical nature of the problem and the given boundary-value conditions. Since we are dealing with the vibrations of the elastic string, the displacement $y(x, t)$ of the string at any point x and at any time $t > 0$ must be periodic function of x and t. Hence, the solution (II) consisting of trigonometric functions, which are periodic functions, is the suitable solution to the one-dimensional wave equation.

The constants A, B, C, D are determined by using the boundary-value conditions of the given problem. So, in problems dealing with vibrating string, we shall assume the solution II,

$$y(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct),$$

where A, B, C, D, λ are constants, of which only 4 are independent constants to be determined. Hence, four conditions are required to solve the one dimensional wave equation.

The conditions to be satisfied by the solution $y(x, t)$ of the one-dimensional wave equation are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0$ for all $t \geq 0$

since the string is fixed at the end points, there is no displacement at the end points.

If the string is pulled up into a curve $y = f(x)$ and released (with or without a force) the conditions are (iii) $y(x, 0) = f(x)$ and (iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ or 0 for all $x \in [0, l]$

The conditions (i) and (ii) are the boundary conditions and the conditions (iii) and (iv) are the initial conditions.

The four conditions togehter are the boundary value conditions.

WORKED EXAMPLES

TYPE 1. Problems with non-zero initial displacement and zero initial velocity. i.e., the string is pulled up to the shape $y = f(x)$ and then released from rest. $f(x)$ may be given in

(a) trigonometric form (b) in algebraic form.

TYPE 1(a). Initial displacement $y(x, 0) = f(x)$ is in trigonometric form

EXAMPLE 1

A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = \alpha \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end and at time $t > 0$ is given by $y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$.

Solution.

The motion of the string is given by the partial differential equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

The solution is

$$y(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \quad (1)$$

where A, B, C, D, λ are constants to be determined.
The boundary-value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \ \forall \ t \geq 0$, which are boundary conditions.

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = f(x) = a \sin \frac{\pi x}{l}$, $0 \leq x \leq l$, which are initial conditions.

First we use the conditions with R.H.S = 0
Using condition (i), that is, when $x = 0, y = 0$ in (1), we get,

$$(A \cos 0 + B \sin 0)(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$$\Rightarrow \quad A(C \cos \lambda ct + D \sin \lambda ct) = 0 \Rightarrow A = 0, \quad \text{since } C \cos \lambda ct + D \sin \lambda ct \neq 0$$

[If $C \cos \lambda ct + D \sin \lambda$ ct = 0, then the solution $y(x, t) = 0$, which is trivial]
$\therefore$ (1) becomes $y(x, t) = B \sin \lambda x(C \cos \lambda ct + D \sin \lambda ct)$
Using condition (ii), that is, when $x = l, y = 0$, in (2), we get

$$B \sin \lambda l(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$$\Rightarrow \quad \sin \lambda l = 0, \quad \text{since } B \neq 0 \text{ and } (C \cos \lambda ct + D \sin \lambda ct) \neq 0$$

$$\Rightarrow \quad \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$$

$\therefore$ (2) becomes

$$y(x, t) = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l}\right) \quad (3)$$

Differentiating (3) partially w.r.to t, we get

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l}\left(-C \sin\left(\frac{n\pi ct}{l}\right)\cdot\left(\frac{n\pi c}{l}\right) + D \cos\left(\frac{n\pi ct}{l}\right)\cdot\left(\frac{n\pi c}{l}\right)\right)$$

Using condition (iii), that is, when $t = 0$, $\frac{\partial y}{\partial t} = 0$, we get

$$B \sin\left(\frac{n\pi x}{l}\right)\cdot\left(0 + D \cos 0 \cdot \frac{n\pi c}{l}\right) = 0 \quad \Rightarrow \quad B \sin \frac{n\pi x}{l}\cdot D\frac{n\pi c}{l} = 0 \quad \Rightarrow \quad D = 0$$

$$\therefore \quad y(x, t) = B \sin\left(\frac{n\pi x}{l}\right)\cdot C \cos\left(\frac{n\pi ct}{l}\right), \quad n = 1, 2, 3, \ldots$$

$$= BC \sin\left(\frac{n\pi x}{l}\right)\cdot \cos\left(\frac{n\pi ct}{l}\right), \quad n = 1, 2, 3, \ldots$$

Before using the R.H.S non-zero condition, we find the general solution.
The general solution is a linear combination of these solutions.
So, the general solution is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot \cos\left(\frac{n\pi ct}{l}\right) \quad (4)$$

[If $BC = k$, then the linear combination is

$$C_1 k \sin\left(\frac{\pi x}{l}\right)\cdot \cos\left(\frac{n\pi ct}{l}\right) + C_2 k \sin\left(\frac{2\pi x}{l}\right)\cdot \cos\left(\frac{2\pi ct}{l}\right) + \cdots$$

If $C_n k = B_n$, then the linear combination is as in (4)]

Using condition (iv). that is, when $t = 0$ in (4), we get $y(x, 0) = f(x) = a \sin\left(\frac{\pi x}{l}\right)$

$$\therefore \quad y(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos 0$$

$$\Rightarrow \quad a \sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \quad a \sin\left(\frac{\pi x}{l}\right) = B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots$$

Equating the like coefficients, $B_1 = a, \quad B_2 = 0, \quad B_3 = 0, \ldots$
Substituting in (4), we get

$$y(x, t) = B_1 \sin\left(\frac{\pi x}{l}\right) \cdot \cos\left(\frac{\pi ct}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) \cdot \cos\left(\frac{2\pi ct}{l}\right) + \cdots$$

$$y(x, t) = a \sin\left(\frac{\pi x}{l}\right) \cdot \cos\left(\frac{\pi ct}{l}\right)$$

Note In general, a single solution will not satisfy the initial conditions, especially the R.H.S $\neq 0$ condition. So we find the general solution for applying condition (iv). R.H.S = 0 conditions are applied before the general solution.

EXAMPLE 2

A slightly stretched string with fixed ends $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement y at any distance x from one end and at any time t.

Solution.

The displacement $y(x, t)$ of the vibrating string is given by the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

The solution is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \quad (1)$$

The boundary-value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \ \forall \ t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = f(x) = y_0 \sin^3\left(\frac{\pi x}{l}\right), \quad 0 \leq x \leq l$

Using condition (i), that is, when $x = 0$, $y = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$$\Rightarrow \quad A(C \cos \lambda ct + D \sin \lambda ct) = 0 \quad \Rightarrow \quad A = 0, \quad \text{since } C \cos \lambda ct + D \sin \lambda ct \neq 0$$

[For, if $C \cos \lambda ct + D \sin \lambda ct = 0$, then the solution $y(x, t) = 0$ for all t, which is trivial]

$\therefore$ (1) becomes $\quad y(x, t) = B \sin \lambda x (C \cos \lambda ct + D \sin \lambda ct)$ (2)

Using condition (ii), i.e., when $x = l$, $y = 0$, in (2), we get

$$B \sin \lambda l (C \cos \lambda ct + D \sin \lambda ct) = 0$$

But $B \neq 0$. $\therefore \quad \sin \lambda l = 0 \quad \Rightarrow \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$

$$\therefore \qquad y(x,t) = B \sin\left(\frac{n\pi x}{l}\right)\left(C \cos\left(\frac{n\pi ct}{l}\right) + D \sin\left(\frac{n\pi ct}{l}\right)\right) \qquad (3)$$

Differentiating w. r. to t,

$$\frac{\partial y}{\partial t} = B \sin\left(\frac{n\pi x}{l}\right)\left[-C \sin\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l} + D \cos\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l}\right]$$

Using condition (iii), i.e., when $t = 0$, $\frac{\partial y}{\partial t} = 0$, we get

$$B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D \cos 0 \cdot \frac{n\pi c}{l}\right] = 0 \quad \Rightarrow \quad B \sin\left(\frac{n\pi x}{l}\right) \cdot D \cdot \frac{n\pi c}{l} = 0 \quad \Rightarrow D = 0$$

$$\therefore \qquad y(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot C \cdot \cos\left(\frac{n\pi ct}{l}\right)$$

$$= BC \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi ct}{l}\right), n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is a linear combination of these solutions for $n = 1, 2, 3, \ldots\ldots$

The general solution is $\quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$ (4)

Using condition (iv), i.e., when $\quad t = 0, \; y = f(x) = y_0 \sin^3 \frac{\pi x}{l}$

$$\therefore \quad \text{we get} \qquad y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos 0.$$

$$\Rightarrow \qquad \frac{y_0}{4}\left[3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

[since $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

$\Rightarrow \sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$]

$$\therefore \qquad \frac{y_0}{4}\left[3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] = B_1 \sin\left(\frac{\pi x}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

Equating like coefficients, we get

$$B_1 = \frac{3y_0}{4}, \quad B_2 = 0, \quad B_3 = \frac{-y_0}{4}, \quad B_4 = 0 = B_5 = \cdots$$

(4) is $y(x,t) = B_1 \sin\left(\frac{\pi x}{l}\right)\cdot\cos\left(\frac{\pi ct}{l}\right) + B_2 \sin\left(\frac{2\pi x}{l}\right)\cdot\cos\left(\frac{2\pi ct}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right)\cdot\cos\left(\frac{3\pi ct}{l}\right) + \cdots$

$\therefore \quad y(x,t) = \frac{3y_0}{4}\sin\left(\frac{\pi x}{l}\right)\cdot\cos\left(\frac{\pi ct}{l}\right) - \frac{y_0}{4}\sin\left(\frac{3\pi x}{l}\right)\cdot\cos\left(\frac{3\pi ct}{l}\right)$

TYPE 1(b): The initial form of the string y(x, 0) = *f*(*x*) is in algebraic form.

EXAMPLE 3

A tightly stretched string of length *l* has its end fastened at *x* = 0, *x* = *l*. At *t* = 0, the string is in the form *f*(*x*) = *kx*(*l* − *x*) and then released. Find the displacement at any point on the string at a distance *x* from one end and at any time *t* > 0.

Solution.

The displacement is given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda\, ct + D\sin\lambda\, ct) \qquad (1)$$

The boundary-value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \ \forall\ t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = f(x) = kx(l - x);\ 0 \leq x \leq l$

Using condition (i), i.e., $y = 0$ when $x = 0$ in (1), we get

$$(A\cos 0 + B\sin 0)(C\cos\lambda\, ct + D\sin\lambda\, ct) = 0 \quad\Rightarrow\quad A = 0, \quad \text{since } C\cos\lambda\, ct + D\sin\lambda\, ct \neq 0$$

$\therefore$ (1) becomes $\qquad y(x, t) = B\sin\lambda x\,(C\cos\lambda\, ct + D\sin\lambda\, ct) \qquad (2)$

Using condition (ii), i.e., $y = 0$ when $x = l$ in (3), we get

$$B\sin\lambda l\,(C\cos\lambda\, ct + D\sin\lambda\, ct) = 0$$

$\Rightarrow \qquad \sin\lambda l = 0, \quad \text{since } B \neq 0 \text{ and } C\cos\lambda\, ct + D\sin\lambda\, ct \neq 0$

$\therefore \qquad \lambda l = n\pi \quad\Rightarrow\quad \lambda = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots$

$\therefore$ (2) becomes $\qquad y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi ct}{l} + D\sin\frac{n\pi ct}{l}\right) \qquad (3)$

Differentiating (3) w.r.to t, $\qquad \frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi ct}{l}\cdot\frac{n\pi c}{l} + D\cos\frac{n\pi ct}{l}\cdot\frac{n\pi c}{l}\right]$

Using condition (iii), i.e., when $t = 0$ and $\frac{\partial y}{\partial t} = 0$, we get

$$B\sin\frac{n\pi x}{l}\left[0 + D\cdot\frac{n\pi c}{l}\right] = 0 \quad\Rightarrow\quad BD\frac{n\pi x}{l}\sin\frac{n\pi x}{l} = 0 \quad\Rightarrow\quad D = 0$$

$\therefore$ (3) becomes $$y(x,t) = B\sin\frac{n\pi x}{l}\cdot C\cos\frac{n\pi ct}{l}$$

$$= BC\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l},\ n = 1, 2, 3, \ldots$$

$\therefore$ the most general solution is the linear combination of these solutions

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l} \qquad (4)$$

Using condition (iv), i.e., when $t = 0$, $y = kx(l - x)$ in (4), we get

$$kx(l-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos 0$$

$$\Rightarrow \quad kx(l-x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \qquad (5)$$

Since $f(x)$ is given in algebraic form, to find B_n we expand

$f(x) = kx(l - x)$, $0 \le x \le l$, as a half-range sine series

$$\text{Let} \quad kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (6)$$

$$\text{where} \quad b_n = \frac{2}{l}\cdot\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Compare (5) and (6), we get $B_n = b_n$

$$\text{Now} \quad b_n = \frac{2}{l}\int_0^l kx(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2k}{l}\left[(lx-x^2)\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right\} - (l-2x)\left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right\} + (-2)\left\{\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right\}\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{l}{n\pi}(lx-x^2)\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}(l-2x)\sin\left(\frac{n\pi x}{l}\right) - \frac{2l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-l}{n\pi}(l^2 - l^2)\cos n\pi + \frac{l^2}{n^2\pi^2}(l - 2l)\sin n\pi - \frac{2l^3}{n^3\pi^3}\cos n\pi - \left(0 - \frac{2l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \frac{2k}{l}\left[\frac{2l^3}{n^3\pi^3} - \frac{2l^3}{n^3\pi^3}\cos n\pi\right] = \frac{2k}{l}\cdot\frac{2l^3}{n^3\pi^3}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

$$\Rightarrow \quad b_n = \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

If n is odd then $(-1)^n = -1$ $\quad \therefore\ b_n = \dfrac{4kl^2}{n^3\pi^3}(2) \quad \Rightarrow b_n = \dfrac{8kl^2}{n^3\pi^3}$

If n is even then $(-1)^n = 1$ $\quad \therefore\ b_n = 0$

$$\therefore \qquad B_n = \frac{8kl^2}{n^3\pi^3}, \qquad n = 1, 3, 5, \ldots$$

$\therefore$ (4) becomes

$$y(x,t) = \sum_{n=1,3,5,\ldots} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

$$= \frac{8kl^2}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

EXAMPLE 4

Find the solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection $f(x) = \begin{cases} \dfrac{2kx}{l}, & 0 < x < \dfrac{l}{2} \\ \dfrac{2k}{l}(l - x), & \dfrac{l}{2} < x < l \end{cases}$ and the initial velocity is 0.

Solution.

In this problem, the wave equation is given using $u(x, t)$ [instead of $y(x, t)$]

$\therefore$ the solution of wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ is

$$u(x, t) = (A\cos \lambda x + B \sin \lambda x)(C \cos \lambda\, ct + D \sin \lambda\, ct) \qquad (1)$$

The boundary-value conditions are

(i) $u(0, t) = 0$ and (ii) $u(l, t) = 0\ \forall\ t \geq 0$

(iii) $\dfrac{\partial u}{\partial t}(x, 0) = 0$ and (iv) $u(x, 0) = f(x) = \begin{cases} \dfrac{2kx}{l}, & 0 < x < \dfrac{l}{2} \\ \dfrac{2k}{l}(l - x), & \dfrac{l}{2} < x < l \end{cases}$

Using condition (1), i.e., when $x = 0$, $u = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda\, ct + D \sin \lambda\, ct) = 0$$

$\Rightarrow$ $A(\cos \lambda\, ct + D \sin \lambda\, ct) = 0 \ \Rightarrow\ A = 0,$ since $C \cos \lambda\, ct + D \sin \lambda\, ct) \neq 0$

$\therefore$ (1) becomes $\quad u(x, t) = B \sin \lambda x\, (C \cos \lambda\, ct + D \sin \lambda\, ct) \qquad (2)$

Using condition (ii), i.e., when $x = l$, $u = 0$ in (2), we get

$$B \sin \lambda l(\cos \lambda\, ct + D \sin \lambda\, ct) = 0$$

$\Rightarrow$ $\sin \lambda l = 0,$ since $B \neq 0$, $C \cos \lambda\, ct + D \sin \lambda\, ct) \neq 0$

$\Rightarrow$ $\lambda l = n\pi \ \Rightarrow\ \lambda = \dfrac{n\pi}{l}, n = 1, 2, 3, \ldots$

$\therefore$ (2) becomes

$$u(x,t) = B \sin\left(\frac{n\pi}{l}x\right)\left(C \cos\left(\frac{n\pi ct}{l}\right) + D \sin\left(\frac{n\pi ct}{l}\right)\right)$$

Differentiating (3) w. r. to t, we get

$$\frac{\partial u}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l} + D\cos\left(\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l}\right]$$

Using condition (iii), i.e., when $t = 0$, $\frac{\partial u}{\partial t} = 0$, we get

$$B\sin\left(\frac{n\pi x}{l}\right)\left[0 + D\cdot\frac{n\pi c}{l}\right] = 0 \quad\Rightarrow\quad BD\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{n\pi c}{l} = 0,\ n = 1, 2, 3, \ldots$$

$\Rightarrow$ $D = 0$, since the other factors are $\neq 0$

$\therefore$ (3) becomes

$$u(x, t) = B\sin\frac{n\pi x}{l}\cdot C\cos\frac{n\pi ct}{l},\ n = 1, 2, 3, \ldots$$

$$= BC\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l},\ n = 1, 2, 3, \ldots$$

$\therefore$ the most general solution is the linear combination of these solutions.

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right) \qquad (4)$$

Using condition (iv), i.e., when $t = 0$, $u(x, 0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cos 0 \quad\Rightarrow\quad f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \qquad (5)$$

Since $f(x)$ is given in algebraic form, to find B_n, we express $f(x)$ as a Fourier sine series.

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (6)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Now $$b_n = \frac{2}{l}\left\{\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\left\{\int_0^{l/2}\frac{2kx}{l}\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\cdot\frac{2k}{l}\left\{\left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left\{\left[-\frac{l}{n\pi}x\cos\left(\frac{n\pi x}{l}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2}-\left[\frac{l}{n\pi}(l-x)\cos\left(\frac{n\pi x}{l}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}-0-\left[0-\left(\frac{l}{n\pi}(l-l/2)\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right.$$

$$= \frac{4k}{l^2}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{4k}{l^2}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

From (5) and (6), we get $\quad B_n = b_n$

$\therefore \quad B_n = \dfrac{8k}{n^2\pi^2}\sin\dfrac{n\pi}{2}, \quad n = 1, 2, 3, \ldots$

Substituting in (4), we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right)$$

Note We can simplify further.

If n is even, say $n = 2m$, then $\sin\dfrac{n\pi}{2} = \sin m\pi = 0$

If n is odd, say , $n = 2m + 1$, then $\sin\dfrac{n\pi}{2} = \sin(2m+1)\dfrac{\pi}{2} = \sin\left(\dfrac{\pi}{2}+m\pi\right)$

$$= \cos m\pi = (-1)^m = (-1)^{\frac{n-1}{2}}$$

$$\therefore \quad \sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \quad n = 1, 3, 5, \ldots$$

$$\therefore \quad u(x,t) = \frac{8k}{\pi^2}\sum_{n=\text{odd}}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right)$$

$$= \frac{8k}{\pi^2}\left[\sin\frac{\pi x}{l}\cdot\cos\frac{\pi ct}{l}-\frac{1}{3^2}\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi ct}{l}+\frac{1}{5^2}\sin\frac{5\pi x}{l}\cdot\cos\frac{5\pi ct}{l}-\cdots\right]$$

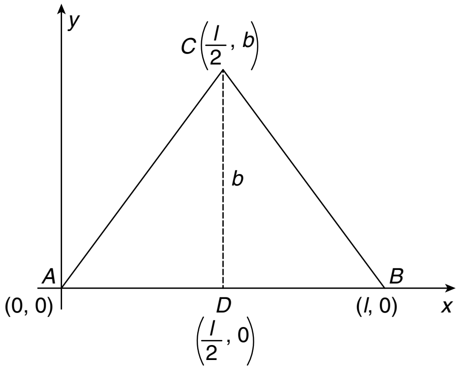
Type 1 (c): The initial form of the string $y(x, 0) = f(x)$ in algebraic form is to be found from the given problem

EXAMPLE 5

A string is tightly stretched and its ends are fastened at two points $x = 0$ and $x = l$. The mid point of the string is displaced transversely through a small distance b and the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution.

The tight string AB fixed at $x = 0$, $x = l$ is lifted to C and released from rest. So, the initial position of the string is ACB.



$$A = (0, 0),\ B = (l, 0),\ C = \left(\frac{l}{2}, b\right).$$

AC is the line joining $A(0, 0)$ and $C\left(\frac{l}{2}, b\right)$

$\therefore$ the equation of AC is $\dfrac{y-0}{b-0} = \dfrac{x-0}{\frac{l}{2}-0}$

$$\Rightarrow \qquad \frac{y}{b} = \frac{2}{l}x \quad \Rightarrow \quad y = \frac{2b}{l}x,\ 0 \le x \le \frac{l}{2}$$

BC is the line joining $\quad B = (l, 0),\ C = \left(\frac{l}{2}, b\right)$

$\therefore$ the equation of BC is $\quad \dfrac{y-b}{0-b} = \dfrac{x-\frac{l}{2}}{l-\frac{l}{2}} \quad \Rightarrow \quad \dfrac{y-b}{-b} = \dfrac{x-\frac{l}{2}}{\frac{l}{2}}$

$$\Rightarrow \qquad y - b = \frac{-2b}{l}\left(x - \frac{l}{2}\right) = b - \frac{2b}{l}x + b$$

$$\Rightarrow \qquad y = 2b - \frac{2b}{l}x = \frac{2b}{l}(l-x),\ \frac{l}{2} < x \le l$$

$$\therefore \text{ the initial position is } y(x, 0) = \begin{cases} \dfrac{2b}{l}x, & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2b}{l}(l-x), & \dfrac{l}{2} < x \le l \end{cases}$$

This is exactly the Example 4 with $k = b$

Thus, the boundary-value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0\ \forall\ t \ge 0$

(iii) $\dfrac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = \begin{cases} \dfrac{2b}{l}x \text{ if } 0 \le x \le \dfrac{1}{2} \\ \dfrac{2b}{l}(l-x) \text{ if } \dfrac{1}{2} < x \le l \end{cases}$

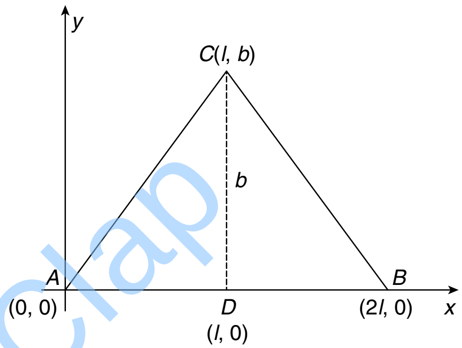
The solution of the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ is obtained as in Example 4 with b instead of k and $y(x, t)$ instead of $u(x, t)$

$$\therefore \qquad y(x, t) = \frac{8b}{\pi^2}\left[\sin\frac{\pi x}{l}\cdot\cos\frac{\pi ct}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi ct}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l}\cdot\cos\frac{5\pi ct}{l} - \cdots\right]$$

EXAMPLE 6

A tightly stretched string of length $2l$ has its ends fastened at $x = 0, x = 2l$. The midpoint of the string is taken to a height b and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the lateral displacement of a point of the string at time t from the instant of release.

Solution.
The string AB is fixed at the ends $x = 0$ and $x = 2l$.
The mid point of the string is lifted to a height b and then released from rest. So, the initial position of the string is ACB, where $A = (0, 0)$, $B = (2l, 0)$, $C = (l, b)$



Equation of AC is $y = \frac{b}{l}x, 0 \le x \le l$

Equation of BC is $\frac{y-0}{b-0} = \frac{x-2l}{l-2l}$

$$\Rightarrow \quad \frac{y}{b} = -\frac{x-2l}{l}$$

$$\Rightarrow \quad y = \frac{b}{l}(2l - x), l \le x \le 2l$$

$\therefore$ the initial position of the string is

$$y(x, 0) = f(x) = \begin{cases} \frac{b}{l}x, & 0 \le x \le l \\ \frac{b}{l}(2l - x), & l \le x \le 2l \end{cases}$$

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(2l, t) = 0 \ \forall \ t \ge 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = f(x). \ 0 \le x \le 2l$

The solution is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \quad (1)$$

where A, B, C, D, λ are constants to be determined.
Using condition (i), i,e., when $x = 0, y = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$$\Rightarrow \quad A(C \cos \lambda ct + D \sin \lambda ct) = 0 \quad \Rightarrow \quad A = 0 \quad [\because C \cos \lambda ct + D \sin \lambda ct \ne 0]$$

$$\therefore \quad y(x, t) = B \sin \lambda x \,(C \cos \lambda ct + D \sin \lambda ct) \quad (2)$$

Using condition (ii), i.e., when $x = 2l$, $y = 0$, in (2), we get

$$B \sin \lambda 2l\,(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$\Rightarrow$ $\sin 2\lambda l = 0$, since $B \neq 0$ and $C \cos \lambda ct + D \sin \lambda ct \neq 0$

$\therefore$ $2\lambda l = n\pi \Rightarrow \lambda = \dfrac{n\pi}{2l};\ n = 1, 2, 3, \ldots$

$$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{2l}\right)\left(C \cos\left(\frac{n\pi ct}{2l}\right) + D \sin\left(\frac{n\pi ct}{2l}\right)\right) \tag{3}$$

Differentiating (3) w. r. to t, we get

$$\frac{\partial y}{\partial t} = B \sin\frac{n\pi x}{2l}\left(-C \sin\left(\frac{n\pi ct}{2l}\right)\cdot\frac{n\pi c}{2l} + D \cos\left(\frac{n\pi ct}{2l}\right)\cdot\frac{n\pi c}{2l}\right)$$

When $t = 0$, $\dfrac{\partial y}{\partial t} = 0$

$$\therefore \quad B \sin\frac{n\pi x}{2l}\left(0 + D\cdot\frac{n\pi c}{2l}\right) = 0$$

$$\Rightarrow \quad BD\frac{n\pi c}{2l}\sin\left(\frac{n\pi x}{2l}\right) = 0 \Rightarrow D = 0,\ n = 1, 2, 3, \ldots$$

$$\therefore \quad y(x,t) = B \sin\frac{n\pi x}{2l}\cdot C \cos\frac{n\pi ct}{2l}$$

$$= BC \sin\frac{n\pi x}{2l}\cdot\cos\frac{n\pi ct}{2l} \text{ for } n = 1, 2, 3, \ldots$$

are all solutions.
The most general solution is the linear combination of these solutions.

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right)\cdot\cos\left(\frac{n\pi ct}{2l}\right) \tag{4}$$

Using condition (iv), i.e., when $t = 0$, $y(x, 0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right)\cdot\cos 0 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right),\ 0 \le x \le 2l \tag{5}$$

Since $f(x)$ is given in algebraic form., to find B_n, we express $f(x)$ as a Fourier half-range sine series in $(0, L)$ where $L = 2l$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right) \tag{6}$$

where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \dfrac{2}{2l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx$

Comparing (5) and (6) we find $B_n = b_n$, $\forall n = 1, 2, 3, \ldots$

and $$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^l \frac{b}{l} x \cdot \sin\left(\frac{n\pi x}{2l}\right) dx + \int_l^{2l} \frac{b}{l}(2l - x)\sin\left(\frac{n\pi x}{2l}\right) dx\right\}$$

$$= \frac{1}{l}\cdot\frac{b}{l}\left\{\left[x\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l\right.$$

$$\left. + \left[(2l - x)\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}\right\}$$

$$= \frac{b}{l^2}\left\{\left[\frac{-2l}{n\pi} x \cdot \cos\left(\frac{n\pi x}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2l}\right)\right]_0^l\right.$$

$$\left. + \left[\frac{-2l}{n\pi}(2l - x)\cos\left(\frac{n\pi x}{2l}\right) - \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2l}\right)\right]_l^{2l}\right\}$$

$$= \frac{b}{l^2}\left\{\left[\frac{-2l^2}{n\pi}\cos\left(\frac{n\pi l}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi l}{2l}\right) - 0\right]\right.$$

$$\left. + \left[0 - \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2l}2l\right) - \left(\frac{-2l}{n\pi} l\cos\left(\frac{n\pi l}{2l}\right) - \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi l}{2l}\right)\right)\right]\right\}$$

$$= \frac{b}{l^2}\left\{\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4l^2}{n^2\pi^2}\sin n\pi + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{b}{l^2}\left\{\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2},\ n = 1, 2, 3, \ldots$$

But we know that $\sin\frac{n\pi}{2} = 0$ if n is even and $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$ if n is odd

$$\therefore \qquad b_n = \frac{8b}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \text{ if } n = 1, 3, 5, \ldots$$

$$\therefore \qquad B_n = \frac{8b}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \text{ if } n = 1, 3, 5, \ldots$$

$$\therefore \qquad y(x,t) = \sum_{n=1,3,5,\ldots} \frac{8b}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \sin\left(\frac{n\pi x}{2l}\right)\cdot\cos\left(\frac{n\pi ct}{2l}\right)$$

$$= \frac{8b}{\pi^2}\sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\left(\frac{n\pi x}{2l}\right)\cdot\cos\left(\frac{n\pi ct}{2l}\right)$$

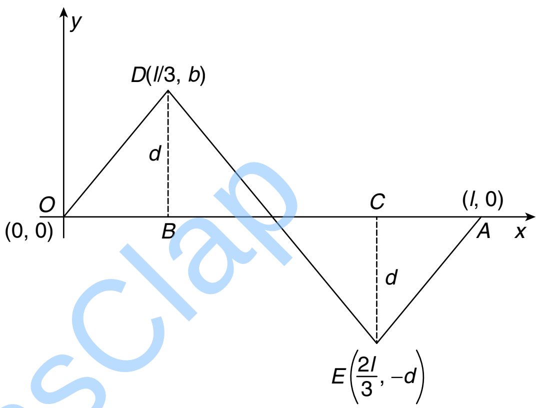
EXAMPLE 7

The points of trisection of a tightly stretched string of length l with fixed ends are pulled aside through a distance d on opposite sides of the position of equilibrium and the string is released from rest. Obtain the displacement of the string at any subsequent time and show that the midpoint of the string always remains at rest.

Solution.

Let OA be the stretched string of length l fixed at the ends $x = 0$ and $x = l$.

The points of trisection B and C of the string are pulled to a distance d in opposite directions and released from rest. So, the initial position of the string is $ODEA$.



Where $O(0, 0)$, $D\left(\frac{l}{3}, d\right)$, $E\left(\frac{2l}{3}, -d\right)$, $A(l, 0)$.

Equation of the line OD is

$$y = \frac{d}{\frac{l}{3}}x = \frac{3d}{l}x, \; 0 \le x \le \frac{l}{3}$$

Equation of DE is
$$\frac{y-d}{-d-d} = \frac{x - \frac{l}{3}}{\frac{2l}{3} - \frac{l}{3}}$$

$$\Rightarrow \quad \frac{y-d}{-2d} = \frac{x - \frac{l}{3}}{\frac{l}{3}} \quad \Rightarrow \quad y - d = -\frac{2d}{l}(3x - l)$$

$$\Rightarrow \quad y = -\frac{6d}{l}x + 3d = \frac{3d}{l}(l - 2x); \; \frac{l}{3} \le x \le \frac{2l}{3}$$

Equation of EA is
$$\frac{y+d}{0+d} = \frac{x - \frac{2l}{3}}{l - \frac{2l}{3}} \quad \Rightarrow \quad y + d = \frac{d(3x - 2l)}{l}$$

$$\Rightarrow \quad y = \frac{3d}{l}x - 2d - d = \frac{3d}{l}(x - l), \quad \frac{2l}{3} \le x \le l$$

$\therefore$ the initial shape of the string is $y(x, 0) = f(x)$

and
$$f(x) = \begin{cases} \frac{3d}{l}x, & 0 \le x \le \frac{l}{3} \\ \frac{3d}{l}(l - 2x), & \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3d}{l}(x - l), & \frac{2l}{3} \le x \le l \end{cases}$$

One-dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \ \forall\ t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x, 0) = 0$ and (iv) $y(x, 0) = f(x)$. $0 \leq x \leq l$

The solution is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct) \tag{1}$$

Using condition (i), i.e., when $x = 0$, $y = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda ct + D \sin \lambda ct) = 0$$

$\Rightarrow$ $A(C \cos \lambda ct + D \sin \lambda ct) = 0 \Rightarrow A = 0$, since $C \cos \lambda ct + D \sin \lambda ct \neq 0$

$\therefore$ $$y(x, t) = B \sin \lambda x\, (C \cos \lambda ct + D \sin \lambda ct) \tag{2}$$

Using condition (ii), i.e., when $x = l$, $y = 0$, in (2), we get

$$B \sin \lambda l\, (C \cos \lambda ct + D \sin \lambda ct) = 0$$

$\Rightarrow$ $\sin \lambda l = 0$, $[\because B \neq 0, C \cos \lambda ct + D \sin \lambda ct \neq 0]$

$\therefore$ $\lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

$\therefore$ $$y(x,t) = B \sin\left(\frac{n\pi x}{l}\right)\left[C \cos\left(\frac{n\pi ct}{l}\right) + D \sin\left(\frac{n\pi ct}{l}\right)\right] \tag{3}$$

Differentiating w. r. to t partially, we get

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l}\left[-C \sin\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l} + D \cos\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l}\right]$$

Using condition (iii), i.e., then $t = 0$, $\frac{\partial y}{\partial t} = 0$.

$\therefore$ $$B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D \cdot \frac{n\pi c}{l}\right] = 0$$

$\Rightarrow$ $BD \sin\left(\frac{n\pi x}{l}\right) \cdot \frac{n\pi c}{l} = 0 \Rightarrow D = 0$ $\left[\because B \neq 0,\ \sin \frac{n\pi x}{l} \neq 0\right]$

$\therefore$ $$y(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot C \cos\left(\frac{n\pi ct}{l}\right),\ n = 1, 2, 3, \ldots$$

$$y(x,t) = BC \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l} \text{ for } n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is the linear combination of these solutions.

$\therefore$ $$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l} \tag{4}$$

Using condition (iv), i.e., when $t = 0$, $y = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot \cos 0 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad (5)$$

Since $f(x)$ is given in algebraic form., to find B_n, we express $f(x)$ as a Fourier sine series in $0 \le x \le l$

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad (6)$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Comparing (5) and (6), we find $\quad B_n = b_n,\ \forall n = 1, 2, 3, \ldots$

Now,
$$b_n = \frac{2}{l}\left\{\int_0^{l/3} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/3}^{2l/3} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{2l/3}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\left\{\int_0^{l/3}\frac{3d}{l}x\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/3}^{2l/3}\frac{3d}{l}(l-2x)\sin\left(\frac{n\pi x}{l}\right)dx\right.$$
$$\left. + \int_{2l/3}^{l}\frac{3d}{l}(x-l)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\cdot\frac{3d}{l}\left\{\left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/3}\right.$$
$$+\left[(l-2x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (-2)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/3}^{2l/3}$$
$$\left.+\left[(x-l)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right]_{2l/3}^{l}\right\}$$

$$= \frac{6d}{l^2}\left\{\left[\frac{-l}{n\pi}x\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/3}\right.$$
$$-\left[\frac{l}{n\pi}(l-2x)\cos\left(\frac{n\pi x}{l}\right) + \frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_{l/3}^{2l/3}$$
$$\left.+\left[\frac{-l}{n\pi}(x-l)\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_{2l/3}^{l}\right\}$$

$$= \frac{6d}{l^2}\left\{\left[\frac{-l}{n\pi}\cdot\frac{l}{3}\cdot\cos\left(\frac{n\pi}{l}\cdot\frac{l}{3}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{l}{3}\right)-0\right]\right.$$

$$-\left[\frac{l}{n\pi}\left(l-\frac{4l}{3}\right)\cos\left(\frac{n\pi}{l}\cdot\frac{2l}{3}\right)+\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{2l}{3}\right)\right]$$

$$-\left\{\frac{l}{n\pi}\left(l-\frac{2l}{3}\right)\cdot\cos\left(\frac{n\pi}{l}\cdot\frac{l}{3}\right)+\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{l}{3}\right)\right\}$$

$$\left.+\left[0-\left\{\frac{-l}{n\pi}\left(\frac{2l}{3}-l\right)\cos\left(\frac{n\pi}{l}\cdot\frac{l}{3}\right)+2\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{l}\cdot\frac{2l}{3}\right)\right\}\right]\right\}$$

$$= \frac{6d}{l^2}\left\{\frac{-l^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)+\frac{l^2}{3n\pi}\cos\left(\frac{2n\pi}{3}\right)\right.$$

$$-\frac{2l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)+\frac{l^2}{3n\pi}\cos\left(\frac{n\pi}{3}\right)+\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)$$

$$\left.-\frac{l^2}{3n\pi}\cos\left(\frac{2n\pi}{3}\right)-\frac{l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\right\}$$

$$= \frac{6d}{l^2}\left\{\frac{3l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)-\frac{3l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\right\}$$

$$= \frac{18d}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{3}\right)-\sin\left(\frac{2n\pi}{3}\right)\right\}$$

$$= \frac{18d}{n^2\pi^2}\left\{\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right\}$$

$$= \frac{18d}{n^2\pi^2}\left\{\sin\frac{n\pi}{3}-\left(\sin n\pi\cdot\cos\frac{n\pi}{3}-\cos n\pi\cdot\sin\frac{n\pi}{3}\right)\right\}$$

$$= \frac{18d}{n^2\pi^2}\left\{\sin\frac{n\pi}{3}+\cos n\pi\cdot\sin\frac{n\pi}{3}\right\}$$

$$\Rightarrow \qquad b_n = \frac{18d}{n^2\pi^2}\left[1+(-1)^n\right]\sin\frac{n\pi}{3}$$

If n is odd, $(-1)^n = -1 \;\therefore \quad b_n = 0$

If n is even, $(-1)^n = 1 \;\therefore \quad b_n = \frac{18d}{n^2\pi^2}\cdot 2\sin\frac{n\pi}{3}, \quad n = 2, 4, 6, \ldots$

$$\Rightarrow \qquad b_n = \frac{36d}{n^2\pi^2}\cdot\sin\frac{n\pi}{3}, \quad n = 2, 4, 6, \ldots$$

$$\therefore \qquad B_n = \frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3}, \quad n = 2, 4, 6, \ldots$$

$$\therefore \qquad y(x,t) = \sum_{n=2,4,6,\ldots} \frac{36d}{n^2\pi^2} \sin\frac{n\pi}{3} \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi ct}{l}\right)$$

$$= \frac{36d}{\pi^2} \sum_{n=2,4,6,\ldots} \frac{1}{n^2} \sin\frac{n\pi}{3} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

This is the displacement of the string at any time t.

Mid point of the string is $x = \frac{l}{2}$.

When $x = \frac{l}{2}$, the displacement is given by $y\left(\frac{l}{2}, t\right)$.

But, then $\sin\frac{n\pi x}{l} = \sin\frac{n\pi}{l} \cdot \frac{l}{2} = \sin\frac{n\pi}{2}$

Since n is even, say $n = 2m$, $\sin\frac{n\pi}{2} = \sin m\pi = 0$

$$\therefore \qquad y(x, t) = 0, \text{ when } x = \frac{l}{2} \text{ and for any } t.$$

So, the midpoint of the string is not displaced at all.
That is the mid point is at rest.

Type 2. Zero initial displacement and non-zero initial velocity.
That is the string in equilibrium position is set vibrating with an initial velocity $\frac{\partial y}{\partial t}(x, 0) = g(x)$, $g(x)$ may be in trigonometric form or in algebraic form.

WORKED EXAMPLES

Type 2(a): Initial velocity $\frac{\partial y}{\partial t}(x, 0) = g(x)$ is in trigonometric form

EXAMPLE 1

A tightly stretched string with fixed end points $x = 0$ and $x = 50$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each point a velocity $v_0 \sin\frac{\pi x}{50} \cdot \cos\frac{2\pi x}{50}$, then find the displacement of any point of the string at any subsequent time.

Solution.
The displacement $y(x, t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(50, t) = 0 \ \forall\ t \geq 0$
(iii) $y(x, 0) = 0$, since there is no initial displacement

and (iv) $\frac{\partial y}{\partial t}(x, 0) = v_0 \sin\frac{\pi x}{50} \cdot \cos\frac{3\pi x}{50}$ $0 \leq x \leq 50$

The solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct) \quad (1)$$

Using condition (i), i.e., when $x = 0$, $y = 0$ in (1), we get

$$(A\cos 0 + B\sin 0)(C\cos\lambda ct + D\sin\lambda ct) = 0$$

$$\Rightarrow \quad A(C\cos\lambda ct + D\sin\lambda ct) = 0 \quad \Rightarrow \quad A = 0, \qquad \text{since } A(C\cos\lambda ct + D\sin\lambda ct) \neq 0$$

$$\therefore \quad y(x,t) = B\sin\lambda x(C\cos\lambda ct + D\sin\lambda ct) \quad (2)$$

Using condition (ii), i.e., when $x = 50$, $y = 0$, in (2), we get

$$B\sin 50\lambda(C\cos\lambda ct + D\sin\lambda ct) = 0$$

$$\sin 50\lambda = 0, \quad \text{since } B \neq 0 \text{ and } C\cos\lambda ct + D\sin\lambda ct \neq 0$$

$$\therefore \quad 50\lambda = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{50},\ n = 1, 2, 3, \ldots$$

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{50}\left(C\cos\frac{n\pi ct}{50} + D\sin\frac{n\pi ct}{50}\right) \quad (3)$$

Using condition (iii), i.e., when $t = 0$, $y = 0$, in (3), we get

$$\therefore \quad B\sin\frac{n\pi x}{50}(C\cos 0 + 0) = 0 \quad \Rightarrow \quad BC\sin\frac{n\pi x}{50} = 0$$

$$\Rightarrow \quad C = 0 \qquad \text{since } B \neq 0;\ \sin\frac{n\pi x}{50} \neq 0$$

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{50}\cdot D\sin\frac{n\pi ct}{50}$$

$$= BD\sin\frac{n\pi x}{50}\cdot\sin\frac{n\pi ct}{50}, \quad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is a linear combination of these solutions.

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{50}\cdot\sin\frac{n\pi ct}{50} \quad (4)$$

Differentiating w.r.to t,
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{50}\cdot\cos\frac{n\pi ct}{50}\cdot\frac{n\pi c}{50}$$

Using condition (iv), i.e., when $t = 0$, $\frac{\partial y}{\partial t} = v_0\sin\frac{\pi x}{50}\cdot\cos\frac{2\pi x}{50}$

$$\therefore \quad v_0\sin\frac{\pi x}{50}\cdot\cos\frac{2\pi x}{50} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{50}\cdot\cos 0\cdot\frac{n\pi c}{50}$$

$$\Rightarrow \quad \frac{v_0}{2}\left[\sin\frac{3\pi x}{50} - \sin\frac{\pi x}{50}\right] = \sum_{n=1}^{\infty} B_n\cdot\frac{n\pi c}{50}\cdot\sin\frac{n\pi x}{50}$$

$$\frac{v_0}{2}\sin\frac{3\pi x}{50} - \frac{v_0}{2}\sin\frac{\pi x}{50} = B_1\cdot\frac{\pi c}{50}\cdot\sin\frac{\pi x}{50} + B_2\cdot\frac{2\pi c}{50}\cdot\sin\frac{2\pi x}{50} + B_3\cdot\frac{3\pi c}{50}\cdot\sin\frac{3\pi x}{50} + \cdots$$

Equating like coefficients, we get

$$\frac{\pi c}{50}B_1 = -\frac{v_0}{2} \quad \Rightarrow \quad B_1 = -\frac{25v_0}{\pi c}, \quad B_2 = 0,$$

$$B_3 \cdot \frac{3\pi c}{50} = \frac{v_0}{2} \quad \Rightarrow \quad B_3 = \frac{25v_0}{3\pi c}, \quad B_4 = 0, \quad B_5 = 0 \ldots$$

$$\therefore \quad y(x, t) = B_1 \sin\frac{\pi x}{50}\cdot\sin\frac{\pi ct}{50} + B_2 \sin\frac{2\pi x}{50}\cdot\sin\frac{2\pi ct}{50} + B_3 \sin\frac{3\pi x}{50}\cdot\sin\frac{3\pi ct}{50} + \cdots$$

$$\Rightarrow \quad y(x,t) = -\frac{25v_0}{\pi c}\sin\frac{\pi x}{50}\cdot\sin\frac{\pi ct}{50} + \frac{25v_0}{3\pi c}\sin\frac{3\pi x}{50}\cdot\sin\frac{3\pi ct}{50}$$

$$= \frac{25v_0}{3\pi c}\left[-3\sin\frac{\pi x}{50}\cdot\sin\frac{\pi ct}{50} + \sin\frac{3\pi x}{50}\cdot\sin\frac{3\pi ct}{50}\right]$$

Type 2(b): Initial velocity $\frac{\partial y}{\partial t}(x, 0) = g(x)$ is in algebraic form

EXAMPLE 2

A tightly stretched string with end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. It is set vibrating giving each point a velocity $\lambda x(l - x)$, then show that

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1,3,5,\ldots}\frac{1}{n^4}\sin\frac{n\pi x}{l}\cdot\sin\frac{n\pi at}{l}.$$

Solution.
The displacement $y(x, t)$ is given by the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

[$\because$ a is in the answer instead of c]

The boundary value conditions are
(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \;\forall\; t \geq 0$
(iii) $y(x, 0) = 0$, since there is no initial displacement

and (iv) $\frac{\partial y}{\partial t}(x, 0) = g(x) = \lambda x(l - x),\ 0 \leq x \leq l$

The solution is

$$y(x, t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{1}$$

Using condition (i), i.e., when $x = 0$, $y = 0$ in (1), we get

$$(A\cos 0 + B\sin 0)(C\cos\lambda at + D\sin\lambda at) = 0$$

$$\Rightarrow \quad A(C\cos\lambda at + D\sin\lambda at) = 0 \quad \Rightarrow \quad A = 0 \qquad \text{since } C\cos\lambda at + D\sin\lambda at \neq 0$$

$$\therefore \quad y(x, t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \tag{2}$$

Using condition (ii), i.e., when $x = l$, $y = 0$, in (2), we get

$$B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$$

$\Rightarrow \quad \sin \lambda l = 0, \quad$ since $B \neq 0$, $C \cos \lambda at + D \sin \lambda at \neq 0$

$\therefore \quad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$

$$\therefore \quad y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right) \tag{3}$$

Using condition (iii), i.e., when $t = 0$, $y = 0$ in (3), we get

$B \sin \frac{n\pi x}{l} (C \cos 0 + D \cdot 0) = 0 \quad \Rightarrow \quad BC \sin \frac{n\pi x}{l} = 0 \quad \Rightarrow \quad C = 0, \quad$ since $B \neq 0$; $\sin \frac{n\pi x}{l} \neq 0$

$$\therefore \quad y(x,t) = B \sin \frac{n\pi x}{l} \cdot D \sin \frac{n\pi at}{l}$$

$$= BD \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi at}{l}, \quad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is the linear combination of these solutions.

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi at}{l} \tag{4}$$

Differentiating w. r. to t, $\quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l} \cdot \frac{n\pi a}{l}$

Using condition (iv), i.e., when $t = 0$, $\frac{\partial y}{\partial t} = g(x) = \lambda x(l - x)$.

$$\therefore \quad \lambda x(l - x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \cos 0 \cdot \frac{n\pi a}{l}$$

$$\Rightarrow \quad \lambda x(l - x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \cdot \sin \frac{n\pi x}{l} \tag{5}$$

Since the initial velocity is in algebraic form, to find B_n, express $g(x) = \lambda x(l - x)$ as a Fourier sine series in $0 < x < l$

$$\therefore \quad g(x) = \lambda x(l - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{6}$$

where $\quad b_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$

Comparing (5) and (6), we see $B_n \frac{n\pi a}{l} = b_n$, $n = 1, 2, \ldots$

Now $\quad b_n = \frac{2}{l} \int_0^l \lambda(lx - x^2) \sin\left(\frac{n\pi x}{l} \right) dx$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\cdot\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-l}{n\pi}(lx - x^2)\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}(l-2x)\sin\left(\frac{n\pi x}{l}\right) - 2\frac{l^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[0 + \frac{l^2}{n^2\pi^2}(l-2l)\sin n\pi - 2\frac{l^3}{n^3\pi^3}\cos n\pi - \left(0 - \frac{2l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \frac{2\lambda}{l}\left[-\frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right] = \frac{2\lambda}{l}\cdot\frac{2l^3}{n^3\pi^3}[1-\cos n\pi] = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

If n is even, $(-1)^n = 1$ $\quad\therefore\quad b_n = 0$

If n is odd, $(-1)^n = -1$ $\quad\therefore\quad b_n = \frac{4\lambda l^2}{n^3\pi^3}(2) = \frac{8\lambda l^2}{n^3\pi^3},\quad n = 1, 3, 5, \ldots$

$$\therefore \qquad B_n \cdot \frac{n\pi a}{l} = \frac{8\lambda l^2}{n^3\pi^3}, \qquad n = 1, 3, 5, \ldots$$

$$\Rightarrow \qquad B_n = \frac{8\lambda l^3}{n^4\pi^4 a}, \qquad n = 1, 3, 5, \ldots$$

Substituting in (4), we get

$$y(x,t) = \sum_{n=1,3,5,\ldots}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi at}{l}\right)$$

$$= \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1,3,5,\ldots}\frac{1}{n^4}\sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi at}{l}\right)$$

EXAMPLE 3

A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities $v = \begin{cases} \frac{cx}{l} & \text{in } 0 < x < l \\ \frac{c}{l}(2l - x) & \text{in } l < x < 2l \end{cases}$ x being the distance from an end point. Find the displacement of the string.

Solution.

The displacement $y(x, t)$ is given by the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

[$\because$ c is used in the hypothesis, we are taking a in the P.D equation]

The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(2l, t) = 0\ \forall\ t \ge 0$

(iii) $y(x, 0) = 0\ \forall\ x \in (0, 2l)$, since there is no initial displacement

and (iv) $\frac{\partial y}{\partial t}(x, 0) = v$, $0 \le x \le 2l$

The solution of the P.D.E is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda at + D \sin \lambda at) \quad (1)$$

where A, B, C, D, λ are constants to be determined.

Using condition (i), i.e., when $x = 0, y = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda at + D \sin \lambda at) = 0$$

$$\Rightarrow \quad A(C \cos \lambda at + D \sin \lambda at) = 0 \quad \Rightarrow \quad A = 0, \quad \text{since } C \cos \lambda at + D \sin \lambda at \neq 0$$

$$\therefore \quad y(x, t) = B \sin \lambda x\,(C \cos \lambda at + D \sin \lambda at) \quad (2)$$

Using condition (ii), i.e., when $x = 2l, y = 0$, we get

$$B \sin 2\lambda l\,(C \cos \lambda at + D \sin \lambda at) = 0$$

$$\Rightarrow \quad \sin 2\lambda l = 0, \quad \text{since } B \neq 0,\ C \cos \lambda at + D \sin \lambda at \neq 0$$

$$\therefore \quad 2\lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{2l}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{2l}\right)\left(C \cos\left(\frac{n\pi at}{2l}\right) + D \sin\left(\frac{n\pi at}{2l}\right)\right) \quad (3)$$

Using condition (iii), i.e., when $t = 0, y = 0$ in (3), we get

$$B \sin\frac{n\pi x}{2l}(C\cos 0 + D \sin 0) = 0 \quad \Rightarrow \quad BC \sin\frac{n\pi x}{2l} = 0$$

$$\Rightarrow \quad C = 0 \quad \left[\because B \neq 0; \sin\frac{n\pi x}{2l} \neq 0\right]$$

$$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{2l}\right) \cdot D \sin\left(\frac{n\pi at}{2l}\right)$$

$$= BD \sin\left(\frac{n\pi x}{2l}\right) \cdot \sin\left(\frac{n\pi at}{2l}\right), \quad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is the linear combination of these solutions.

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right) \cdot \sin\left(\frac{n\pi at}{2l}\right) \quad (4)$$

Differentiating w. r. to t,

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right) \cdot \cos\left(\frac{n\pi at}{2l}\right) \cdot \frac{n\pi a}{2l}$$

Using condition (iv), i.e., when $t = 0$, $\frac{\partial y}{\partial t} = v$.

$$\therefore \quad v = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right) \cdot \cos 0 \cdot \left(\frac{n\pi a}{2l}\right)$$

$$v = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{2l}\right) \cdot \sin\left(\frac{n\pi x}{2l}\right) \quad (5)$$

Since v is the given algebraic form

$$v = \begin{cases} \dfrac{cx}{l} & \text{if } 0 < x < l \\ \dfrac{c}{l}(2l - x) & \text{if } l < x < 2l \end{cases}$$

to find B_n, we express v as a Fourier sine series in (0, L), where $L = 2l$.

$$\therefore \quad v = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \quad v = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right) \tag{6}$$

where $$b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx$$

Comparing (5) and (6), we get $B_n \dfrac{n\pi a}{2l} = b_n, \qquad n = 1, 2, 3, \ldots$

Now $$b_n = \frac{2}{2l}\left\{\int_0^l f(x)\sin\left(\frac{n\pi x}{2l}\right)dx + \int_l^{2l} f(x)\sin\left(\frac{n\pi x}{2l}\right)dx\right\}$$

$$= \frac{1}{l}\left\{\int_0^l \frac{cx}{l}\sin\left(\frac{n\pi x}{2l}\right)dx + \int_l^{2l} \frac{c}{l}(2l - x)\sin\left(\frac{n\pi x}{2l}\right)dx\right\}$$

$$= \frac{1}{l}\left\{\frac{c}{l}\left[x\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l\right.$$
$$\left. + \frac{c}{l}\left[(2l - x)\left(\frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}\right\}$$

$$= \frac{c}{l^2}\left\{\left[\frac{-2l}{n\pi}\cdot x\cdot\cos\left(\frac{n\pi x}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{2l}\right)\right]_0^l\right.$$
$$\left. - \left[\frac{2l}{n\pi}\cdot(2l - x)\cos\left(\frac{n\pi x}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi x}{2l}\right]_l^{2l}\right\}$$

$$= \frac{c}{l^2}\left\{\left[\frac{-2l^2}{n\pi}\cdot\cos\left(\frac{n\pi\cdot l}{2l}\right) + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi l}{2l}\right) - 0\right]\right.$$
$$\left. - \left[0 + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2l}\cdot 2l\right) - \left(\frac{2l^2}{n\pi}\cos\left(\frac{n\pi}{2l}\cdot l\right) + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi l}{2l}\right)\right)\right]\right\}$$

$$= \frac{c}{l^2}\left\{\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$\Rightarrow \qquad b_n = \frac{c}{l^2}\left\{\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}$$

If n is even, then $\sin\frac{n\pi}{2} = 0$ and if n is odd, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$

$$\therefore \qquad b_n = \frac{8c}{n^2\pi^2}(-1)^{\frac{n-1}{2}}, \quad n = 1, 3, 5, \ldots$$

$$\therefore \qquad B_n \cdot \frac{n\pi a}{2l} = \frac{8c}{n^2\pi^2}(-1)^{\frac{n-1}{2}}$$

$$\Rightarrow \qquad B_n = \frac{16lc(-1)^{\frac{n-1}{2}}}{n^3\pi^3 a}, \quad n = 1, 3, 5, \ldots$$

Substituting in (4), we get

$$y(x,t) = \sum_{n=1,3,5,\ldots} \frac{16lc(-1)^{\frac{n-1}{2}}}{n^3\pi^3 a}\sin\left(\frac{n\pi x}{2l}\right)\cdot\sin\left(\frac{n\pi at}{2l}\right)$$

$$= \frac{16lc}{a\pi^3}\sum_{n=1,3,5,\ldots}\frac{(-1)^{\frac{n-1}{2}}}{n^3}\sin\frac{n\pi x}{2l}\cdot\sin\frac{n\pi at}{2l}$$

EXAMPLE 4

If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that $v = \begin{cases} cx & \text{for } 0 < x \le \frac{l}{2} \\ c(l-x) & \text{for } \frac{l}{2} < x \le l \end{cases}$

Determine the displacement $y(x, t)$ at any time t.
Show that the displacement is given by

$$y(x,t) = \frac{4l^2c}{\pi^3 a}\left[\sin\frac{\pi x}{l}\cdot\sin\frac{\pi at}{l} - \frac{1}{3^3}\sin\frac{3\pi x}{l}\cdot\sin\frac{3\pi at}{l} + \cdots\right].$$

Solution.
The displacement $y(x, t)$ is given by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

[a is used here, since c is in the hypothesis]
The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0 \ \forall\ t \ge 0$
(iii) $y(x, 0) = 0 \ \forall\ x \in (0, l)$, since there is no initial displacement

and (iv) $\frac{\partial y}{\partial t}(x, 0) = v \quad 0 \le x \le l$

Solution is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda at + D \sin \lambda at) \tag{1}$$

where A, B, C, D, λ are constants to be determined.

Using condition (i), i.e., when $x = 0, y = 0$ in (1), we get

$$(A \cos 0 + B \sin 0)(C \cos \lambda at + D \sin \lambda at) = 0$$

$\Rightarrow \quad A(C \cos \lambda at + D \sin \lambda at) = 0 \quad \Rightarrow \quad A = 0, \quad$ since $C \cos \lambda at + D \sin \lambda at \neq 0$

$$\therefore \quad y(x, t) = B \sin \lambda x \,(C \cos \lambda at + D \sin \lambda at) \tag{2}$$

Using condition (ii), i.e., when $x = l, y = 0$ in (2), we get

$$B \sin \lambda l \,(C \cos \lambda at + D \sin \lambda at) = 0$$

$\Rightarrow \quad \sin \lambda l = 0, \quad$ since $B \neq 0$, $C \cos \lambda at + D \sin \lambda at \neq 0$

$$\therefore \quad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}, \; n = 1, 2, 3, \ldots$$

$$\therefore \quad y(x,t) = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l}\right) \tag{3}$$

Using condition (iii), i.e., when $t = 0, y = 0$ in (3), we get

$$\therefore \quad B \sin\left(\frac{n\pi x}{l}\right)(C \cos 0 + D \sin 0) = 0$$

$\Rightarrow \quad B \sin \frac{n\pi x}{l} \cdot C = 0 \quad \Rightarrow \quad C = 0 \qquad$ since $B \sin \frac{n\pi x}{l} \neq 0$

$$\therefore \quad y(x,t) = B \sin\left(\frac{n\pi x}{l}\right) \cdot D \sin\left(\frac{n\pi at}{l}\right)$$

$$= BD \sin\left(\frac{n\pi x}{l}\right) \cdot \sin\left(\frac{n\pi at}{l}\right), \; n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is the linear combination of these solutions.

So the general solution is

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \sin\left(\frac{n\pi at}{l}\right) \tag{4}$$

Differentiating (4) w. r. to t,

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi at}{l}\right) \cdot \frac{n\pi a}{l}$$

Using condition (iv), i.e., when $t = 0$, $\frac{\partial y}{\partial t} = v$.

$$\therefore \quad v = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos 0 \cdot \frac{n\pi a}{l}$$

$$v = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{l} \cdot \sin\left(\frac{n\pi x}{l}\right) \tag{5}$$

Since $$v = \begin{cases} cx, & 0 < x \le \dfrac{l}{2} \\ c(l-x), & \dfrac{l}{2} < x \le l \end{cases}$$

is the given algebraic form, to find B_n, we express v as a Fourier sine series.

Let $$v = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \ldots \qquad (6)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Comparing (5) and (6), we have $B_n \dfrac{n\pi a}{l} = b_n$, $n = 1, 2, 3, \ldots$

Now $$b_n = \frac{2}{l}\left\{\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{2}{l}\left\{\int_0^{l/2} cx\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} c(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2c}{l}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2c}{l}\left\{\left[-\frac{l}{n\pi}x\cdot\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2}\right.$$

$$\left. - \left[\frac{l}{n\pi}(l-x)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}\right\}$$

$$= \frac{2c}{l}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 - \left[0 - \left(\frac{l}{n\pi}\cdot\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{2c}{l}\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2c}{l}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4cl}{n^2\pi^2}\cdot\sin\frac{n\pi}{2}$$

$$b_n = \frac{4cl}{n^2\pi^2}(-1)^{\frac{n-1}{2}}, \qquad n = 1, 3, 5, \ldots$$

$\left[\text{If } n \text{ is even } \sin\dfrac{n\pi}{2} = 0\right.$ and n is odd, $\left.\sin\dfrac{n\pi}{2} = (-1)^{\frac{n-1}{2}}\right]$

$$\therefore\quad B_n\cdot\frac{n\pi a}{l} = \frac{4cl}{n^2\pi^2}(-1)^{\frac{n-1}{2}}$$

$$\Rightarrow \qquad B_n = \frac{4cl^2(-1)^{\frac{n-1}{2}}}{an^3\pi^3},\ n = 1, 3, 5, \ldots$$

Substituting in (4),

$$y(x,t) = \sum_{n=1,3,5,\ldots} \frac{4cl^2}{an^3\pi^3}(-1)^{\frac{n-1}{2}} \sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi at}{l}\right)$$

$$= \frac{4cl^2}{a\pi^3} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi at}{l}\right)$$

$$= \frac{4cl^2}{a\pi^3}\left[\sin\left(\frac{\pi x}{l}\right)\cdot\sin\left(\frac{\pi at}{l}\right) - \frac{1}{3^3}\sin\left(\frac{3\pi x}{l}\right)\cdot\sin\left(\frac{3\pi at}{l}\right) + \frac{1}{5^3}\cdot\sin\left(\frac{5\pi x}{l}\right)\cdot\sin\left(\frac{5\pi at}{l}\right) - \cdots\right]$$

EXAMPLE 5

A uniform string of length l is struck in such a way that an initial velocity v_0 is imparted to the position of the string between $\frac{l}{4}$ and $\frac{3l}{4}$, while the string is in its equilibrium position. Find the displacement of the string at any time.

Solution.

The string of length l is fixed at the ends $x = 0$ and $x = l$. The part BC of the string OA is given a constant velocity v_0 and so the string vibrates. The displacement $y(x, t)$ at any time is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

The boundary value conditions are

(i) $y(0, t) = 0$ and (ii) $y(l, t) = 0\ \forall\ t \geq 0$

(iii) $y(x, 0) = 0$, since the string is in equilibrium position and so initially there is no displacement.

$$\text{(iv)}\quad \frac{\partial y}{\partial t}(x,0) = g(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{l}{4} \\ v_0 & \text{if } \frac{l}{4} x \leq \frac{3l}{4} \\ 0 & \text{if } \frac{3l}{4} < x \leq l \end{cases}$$

The solution is $y(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda\, ct + D \sin \lambda\, ct)$.
Proceeding as in the earlier problems, using conditions (i), (ii), (iii) we get the general solution

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi ct}{l}\right) \qquad (1)$$

Differentiating w. r. to t, partially we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l}$$

Using condition (iv), i.e., $\frac{\partial y}{\partial t} = g(x)$ when $t = 0$, we get

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot\cos 0\cdot\left(\frac{n\pi c}{l}\right)$$

$\Rightarrow$
$$g(x) = \sum_{n=1}^{\infty} B_n \cdot\left(\frac{n\pi c}{l}\right)\cdot\sin\left(\frac{n\pi x}{l}\right) \qquad (2)$$

Since $g(x) = \begin{cases} 0, & \text{if } 0 < x \le \frac{l}{4} \\ v_0 & \text{if } \frac{l}{4} x \le \frac{3l}{4} \\ 0 & \text{if } \frac{3l}{4} < x \le l \end{cases}$

is the given algebraic form, to find B_n, we express $g(x)$ as a Fourier sine series in $0 < x < l$.

Then
$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (3)$$

where
$$b_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Comparing (2) and (3), we find $B_n \frac{n\pi c}{l} = b_n$

Now
$$b_n = \frac{2}{l}\left\{\int_{l/4}^{3l/4} v_0 \sin\left(\frac{n\pi x}{l}\right)dx\right\}, \text{ since } g(x) = 0 \text{ otherwise}$$

$$= \frac{2v_0}{l}\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]_{l/4}^{3l/4}$$

$$= \frac{2v_0}{l}\left(-\frac{l}{n\pi}\right)\left[\cos\left(\frac{n\pi}{l}\cdot\frac{3l}{4}\right) - \cos\left(\frac{n\pi}{l}\cdot\frac{l}{4}\right)\right]$$

$$= -\frac{2v_0}{n\pi}\left[\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{4}\right]$$

$\Rightarrow$
$$= -\frac{2v_0}{n\pi}\left[\cos\left(n\pi - \frac{n\pi}{4}\right) - \cos\frac{n\pi}{4}\right]$$

$$= -\frac{2v_0}{n\pi}\left[\cos n\pi \cdot \cos\frac{n\pi}{4} + \sin n\pi \cdot \sin\frac{n\pi}{4} - \cos\frac{n\pi}{4}\right]$$

$$= -\frac{2v_0}{n\pi}\left[(-1)^n \cdot \cos\frac{n\pi}{4} - \cos\frac{n\pi}{4}\right]$$

$$b_n = \frac{2v_0}{n\pi}\left[1-(-1)^n\right]\cos\frac{n\pi}{4}$$

If n is even, $(-1)^n = 1 \quad \therefore \quad b_n = 0$

If n is odd, $(-1)^n = -1 \quad \therefore \quad b_n = \frac{2v_0}{n\pi}\cdot 2\cdot\cos\frac{n\pi}{4} = \frac{4v_0}{n\pi}\cos\frac{n\pi}{4},\ n = 1, 3, 5, \ldots$

$$\therefore \quad B_n \cdot \frac{n\pi c}{l} = \frac{4v_0}{n\pi}\cos\frac{n\pi}{4} \quad \Rightarrow \quad B_n = \frac{4lv_0}{n^2\pi^2 c}\cos\frac{n\pi}{4},\ n = 1, 3, 5, \ldots$$

$\therefore$ the displacement at any time is

$$y(x,t) = \sum_{n=1,3,5,\ldots} \frac{4lv_0}{n^2\pi^2 c}\cos\frac{n\pi}{4}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi ct}{l}\right)$$

$$= \frac{4lv_0}{\pi^2\cdot c}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}\cos\frac{n\pi}{4}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi ct}{l}\right)$$

EXERCISE 20.1

1. A string is stretched and the ends are fixed at the points $x = 0$ and $x = l$. The string is initially displaced to the form $y = 2\sin\left(\frac{3\pi x}{l}\right)\cdot\cos\left(\frac{2\pi x}{l}\right)$ and then released. Find the displacement $y(x, t)$.
2. A tightly stretched string with fixed ends $x = 0$ and $x = l$ is initially in the position $y = k\left[\sin\frac{\pi x}{l} - \sin\frac{2\pi x}{l}\right]$. If it is released from rest, find the displacement at any time t and at any distance x from one end.
3. Solve the boundary-values problem $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$ subject to the conditions $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = 0$ and $\frac{\partial y}{\partial t}(x, 0) = 3\sin 2\pi x - 2\sin 5\pi x$.
4. A string is stretched and the end are fixed at the points $x = 0$ and $x = l$. Motion is started by displacing the string in the form of the curve $y = 2\sin\left(\frac{2\pi x}{l}\right) + 3\sin\left(\frac{3\pi x}{l}\right)$ and then releasing it from rest in this position. Find the displacement $y(x, t)$ at any time t.
5. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially displaced in a sinusoidal arc of height y_0 and then released from rest. Find the displacement y at any distance x from one end and at time t.

   $\left[\textbf{Hint:} \text{ Sinusoidal arc of height } y_0 \text{ is } y = y_0 \sin\frac{\pi x}{l}\right]$

6. A tightly stretched string has its ends fixed at $x = 0$ and $x = l$. Initially the string is in the form $y = kx^2(l - x)$, where k is a constant, and then released from rest. Find the displacement at any point x and any time $t > 0$.
7. A uniform string with ends fixed at $x = 0$ and $x = l$ is lifted to a small height d at the point $x = b$ and released from rest. Find the transverse displacement of any point of the string at any time.
8. A string is stretched and fixed at the points $x = 0$ and $x = 60$ and the point of the string are given initial velocity

$$v = \begin{cases} \dfrac{kx}{30} & \text{if } 0 < x < 30 \\ \dfrac{k}{30}(60 - x) & \text{if } 30 < x < 60 \end{cases}$$

where x is the distance from the end $x = 0$. Find the displacement of the string any time t.
9. A tightly stretched string with fixed end points $x = 0$, $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$, find the displacement.
10. A taut string of length 20 cm fastened at both ends, is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by $v = \begin{cases} x & \text{in } 0 \le x \le 10 \\ 20 - x & \text{in } 10 \le x \le 20 \end{cases}$ x being the distance from one end. Determine the displacement at any subsequent time.
11. An elastic string is stretched between two fixed points at a distance π apart. In its initial position the string is in the shape of the curve $f(x) = k(\sin x - \sin^3 x)$. Obtain $y(x, t)$, the vertical displacement if y satisfies the equation $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$.
12. A taut string of length l has its ends $x = 0$, $x = l$ fixed. The point $x = \dfrac{l}{3}$ is drawn aside a small distance h, the displacement $y(x, t)$ satisfies $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$. Determine $y(x, t)$ at any time t.
13. A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3\left(\dfrac{\pi x}{l}\right)$. Find the displacement $y(x, t)$.

ANSWERS TO EXERCISE 20.1

1. $y(x,t) = \sin\left(\dfrac{\pi x}{l}\right) \cdot \cos\left(\dfrac{\pi ct}{l}\right) + \sin\left(\dfrac{5\pi x}{l}\right) \cdot \cos\left(\dfrac{5\pi ct}{l}\right)$

2. $y(x,t) = k\left[\sin\left(\dfrac{\pi x}{l}\right) \cdot \cos\left(\dfrac{\pi ct}{l}\right) - \sin\left(\dfrac{2\pi x}{l}\right) \cdot \cos\left(\dfrac{2\pi ct}{l}\right)\right]$

3. $y(x,t) = \dfrac{1}{5\pi}\left[15\sin(2\pi x) \cdot \sin(4\pi t) - \sin 5\pi x \cdot \sin 10\pi t\right]$

4. $y(x,t) = 2\sin\frac{2\pi x}{l}\cdot\cos\frac{2\pi ct}{l} + 3\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi ct}{l}$

5. $y(x,t) = y_0\sin\frac{\pi x}{l}\cdot\cos\frac{\pi ct}{l}$

6. $y(x,t) = -\frac{4l^3}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1+2(-1)^n}{n^3}\right]\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l}$

7. $y(x,t) = \frac{2dl^2}{\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

8. $y(x,t) = \frac{480k}{c\pi^3}\sum_{n=1,3,5,\ldots}\frac{(-1)^{\frac{n-1}{2}}}{n^3}\sin\frac{n\pi x}{60}\cdot\sin\frac{n\pi ct}{60}$

9. $y(x,t) = \frac{24l^3}{c\pi^4}\sum_{n=1,3,5,\ldots}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$

10. $y(x,t) = \frac{1600}{c\pi^3}\left[\sin\frac{\pi x}{20}\sin\frac{\pi ct}{20} - \frac{1}{3^3}\sin\frac{3\pi x}{20}\cdot\sin\frac{3\pi ct}{20} + \ldots\right]$

11. $y(x,t) = \frac{k}{4}[\sin x\cos t + \sin 3x\cos 3t]$

12. $y(x,t) = \sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3}\cdot\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l} = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi at}{l}\right)$

13. $y(x,t) = \frac{lv_0}{12\pi c}\left[9\sin\frac{\pi x}{l}\cdot\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\cdot\sin\frac{3\pi ct}{l}\right]$

20.1.3 Classification of Partial Differential Equation of Second Order

In the field of wave propagation such as heat conduction, vibrations, elasticity, boundary layer theory and so on, second order partial differential equations occur. Their nature is important in the discussions.

The general form of a second-order partial differential equation in two independent variables x and y is

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0 \tag{1}$$

where F represents the first order part.

This equation is linear in second order terms.

If the first order part F is linear, then P.D. equation is linear.

If F is non-linear, then the P.D.E is called a quasi-linear differential equation.

The P.D.E is called elliptic if $B^2 - 4AC < 0$

Parabolic if $B^2 - 4AC = 0$

and hyperbolic if $B^2 - 4AC > 0$

1. The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$$

Independent variables are x and t.
Here $A = c^2$, $B = 0$, $C = -1$ $\quad \therefore \quad B^2 - 4AC = 0 - 4 \times c^2(-1) = 4c^2 > 0$
So, **it is hyperbolic**.

2. One-dimensional heat-flow equation $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ **is parabolic**.
For, here $A = a^2$, $B = 0$, $C = 0$ and $\quad B^2 - 4AC = 0 - 4a^2 \cdot 0 = 0$

3. Two-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ **is elliptic**.
For, here $A = 1$, $B = 0$, $C = 1$ and $\quad B^2 - 4AC = 0 - 4 \cdot 1 \cdot 1 = -4 < 0$

Note For an elliptic equation boundary conditions are prescribed in a closed region, whereas for parabolic and hyperbolic equations boundary conditions and initial conditions are prescribed in an open ended region.

WORKED EXAMPLES

EXAMPLE 1

Classify the partial differential equation $(1 + x^2)u_{xx} + (5 + 2x^2)u_{xy} + (4 + x^2)u_{yy} = \sin(x + y)$.

Solution.
Given $(1+x^2)u_{xx} + (5+2x^2)u_{xy} + (4+x^2)u_{yy} = \sin(x+y)$.
Here $A = 1 + x^2$, $\quad B = 5 + 2x^2$, $\quad C = 4 + x^2$

$$\therefore \quad B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2)$$
$$= 25 + 20x^2 + 4x^4 - 4(4 + 5x^2 + x^4) = 9 > 0 \ \forall\, x \in R$$

$\therefore$ the equation is hyperbolic $\forall\, x \in R$.

EXAMPLE 2

Classify the partial differential equation
$(1 - x^2)u_{xx} - 2xy\ u_{xy} + (1 - y^2)u_{yy} + x\ u_x + 3x^2y\ u_y - 2u = 0$.

Solution.
Given $(1-x^2)u_{xx} - 2xy\ u_{xy} + (1-y^2)u_{yy} + x\ u_x + 3x^2y\ u_y - 2u = 0$
Here $A = 1 - x^2$, $\quad B = -2xy$, $\quad C = 1 - y^2$

$$\therefore \quad B^2 - 4AC = 4x^2y^2 - 4(1 - x^2)(1 - y^2)$$
$$= 4\{x^2y^2 - (1 - x^2 - y^2 + x^2y^2)\} = 4\{x^2 + y^2 - 1\}$$

(i) The equation is parabolic if $B^2 - 4AC = 0$
i.e., if $\quad x^2 + y^2 - 1 = 0$
So, the equation is parabolic for points on the circle $x^2 + y^2 = 1$.

(ii) The equation is elliptic if $\quad B^2 - 4AC < 0$
i.e., if $\quad x^2 + y^2 - 1 < 0 \quad \Rightarrow x^2 + y^2 < 1$
So, the equation is elliptic inside the circle $x^2 + y^2 = 1$.

(iii) The equation is hyperbolic if $B^2 - 4AC > 0$
i.e., if $\quad x^2 + y^2 - 1 > 0 \quad \Rightarrow x^2 + y^2 > 1$
So, the circle is hyperbolic outside the circle $x^2 + y^2 = 1$.

EXAMPLE 3

Classify the partial differential equation $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = e^{x+y}$.

Solution.

Given $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = e^{x+y}$

Here $A = 1, \quad B = 4, \quad C = x^2 + 4y^2$

$$\therefore \qquad B^2 - 4AC = 16 - 4 \cdot 1 \cdot (x^2 + 4y^2) = -16\left\{\frac{x^2}{4} + y^2 - 1\right\}$$

(i) The equation is parabolic if $B^2 - 4AC = 0$

i.e., if $$\frac{x^2}{4} + y^2 - 1 = 0 \quad \Rightarrow \quad \frac{x^2}{4} + y^2 = 1$$

So, the equation is parabolic for the points on the ellipse $\frac{x^2}{4} + y^2 = 1$.

(ii) The equation is elliptic if $B^2 - 4AC < 0$.

i.e., if $$\frac{x^2}{4} + y^2 - 1 < 0 \quad \Rightarrow \quad \frac{x^2}{4} + y^2 < 1$$

So, the equation is elliptic at points inside the ellipse $\frac{x^2}{4} + y^2 = 1$.

(iii) The equation is hyperbolic if $B^2 - 4AC > 0$.

i.e., if $$\frac{x^2}{4} + y^2 - 1 > 0 \quad \Rightarrow \quad \frac{x^2}{4} + y^2 > 1$$

So, the equation is hyperbolic at points outside the ellipse $\frac{x^2}{4} + y^2 = 1$.

EXERCISE 20.2

Classify the following partial differential equations.

1. $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$
2. $u_{xx} - y^4 u_{yy} - 2y^3 u_y = 0$
3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$
4. $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$
5. $y^2 u_{xx} + u_{yy} + u_x^2 + u_y^2 + 5u = 0$

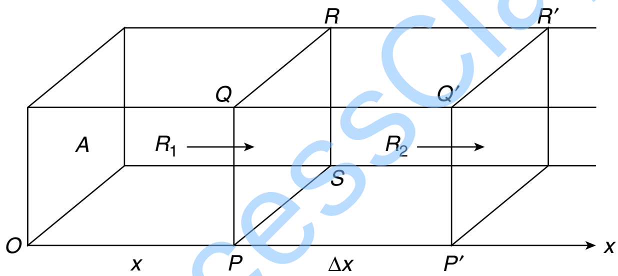
ANSWERS TO EXERCISE 20.2

1. Parabolic for all points (x, y)
2. Hyperbolic for all points $y \neq 0$ and parabolic for points on $y = 0$
3. Elliptic for all points (x, y)
4. Hyperbolic for all points (x, y)
5. Elliptic for all points $y \neq 0$ and parabolic for points on $y = 0$

ONE-DIMENSIONAL EQUATION OF HEAT CONDUCTION (IN A ROD)

1. We shall now consider the flow of heat and the consequent variation of temperature with position and time in conducting materials.
   In the derivation of the one-dimensional heat equation, we use the following empirical laws.
   (i) Heat flows from a higher to lower temperature.
   (ii) The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and the temperature change. The constant of proportionality is known as the specific heat (c) of the conducting material.
   (iii) **Fourier law of heat conduction:** The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. This constant of proportionality is called the thermal conductivity (k) of the material.

20.2.1 Derivation of Heat Equation



Consider a long thin bar (or wire or rod) of constant cross sectional area A and homogeneous conducting material. Let ρ be the density of the material, c be the specific heat and k be the thermal conductivity of the material. We assume that the surface of the bar is insulated so that the heat flow is along parallel lines which are perpendicular to the area A.

Choose one end of the bar as origin and the direction of heat flow as +ve x-axis.

Let $u(x, t)$ be the temperature at a distance x from 0. If Δu be the temperature change in the slab of thickness Δx of the bar, and time change Δt

Then the quantity of heat in this slab

= (specific heat) × (mass of the element slab) × (change in temperature) = $c(A\rho\Delta x)\,\Delta u$

Hence, the rate of change (i.e., increase) of heat in the slab at time t is

$$= c(A\rho\Delta x)\cdot\lim_{\Delta t\to 0}\frac{\Delta u}{\Delta t} = c(A\rho\Delta x)\frac{\partial u}{\partial t}$$

Let R_1 be the rate of inflow of heat at x in the slab and R_2 be the rate of out flow of heat at $x+\Delta x$

Then $$c(A\rho\Delta x)\frac{\partial u}{\partial t} = R_1 - R_2 \tag{1}$$

where $R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x$ and $R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$

The negative sign is due to the fact that heat flows from higher to lower.

i.e., $\frac{\partial u}{\partial t}$ is negative and R_1 and R_2 are positive.

$\therefore$ rate of increase of heat at time t is

$$R_1 - R_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad (2)$$

From (1) and (2) we get,

$$c(A\rho\Delta x)\frac{\partial u}{\partial t} = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\Delta x}$$

As $\Delta x \to 0$, $\quad \frac{\partial u}{\partial t} = \frac{k}{c\rho}\cdot\frac{\partial^2 u}{\partial x^2}$, where $\frac{k}{c\rho}$ is a positive constant.

It is called the diffusivity of the material of the bar. Put $\frac{k}{c\rho} = \alpha^2$

$\therefore$ the heat equation is $\quad \frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$

Note

1. It is called one-dimensional because there is only one space variable x.
2. The one dimensional heat equation is also known as **one dimensional diffusion equation**.

20.2.2 Solution of Heat Equation by Variable Separable Method

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ (1)

To solve, we use the method of separation of variables.
Let $u(x, t) = X(x)\, T(t)$ be a solution.

Then $\quad \frac{\partial u}{\partial t} = XT'$, and $\frac{\partial^2 u}{\partial x^2} = X''T$

Substituting in (1), we get $\quad XT' = \alpha^2 X''T \Rightarrow \frac{T'}{\alpha^2 T} = \frac{X''}{X}$

Since x and t are independent variables, LHS is a function of t alone and RHS is a function of x alone. This is possible if each side is a constant k.

$$\therefore \quad \frac{T'}{\alpha^2 T} = \frac{X''}{X} = k$$

$$\therefore \quad T' = k\alpha^2 T \Rightarrow T' - k\alpha^2 T = 0 \quad (2)$$

$$\text{and} \quad X'' = kX \Rightarrow X'' - kX = 0 \quad (3)$$

(2) and (3) are ordinary differential equations.

Case (i): Let $k < 0$, say $k = -\lambda^2$, $\lambda \neq 0$

$\therefore \quad T' + \lambda^2 \alpha^2 T = 0 \quad \Rightarrow \quad \frac{T'}{T} = -\lambda^2 \alpha^2$

$\Rightarrow \quad \int \frac{T'}{T} dt = -\lambda^2 \alpha^2 \int dt$

$\Rightarrow \quad \log T = -\lambda^2 \alpha^2 t + \log C$

$\Rightarrow \quad \log \frac{T}{C} = -\lambda^2 \alpha^2 t \quad \Rightarrow \quad \frac{T}{C} = e^{-\lambda^2 \alpha^2 t} \quad \Rightarrow \quad T = Ce^{-\lambda^2 \alpha^2 t}$

where C is an arbitrary constant.

(3) $\Rightarrow \quad X'' + \lambda^2 X = 0 \quad \Rightarrow \quad \frac{d^2 X}{dx^2} + \lambda^2 X = 0$

Auxiliary equation is $\quad m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m = \pm i\lambda$

$\therefore \quad X = A_1 \cos \lambda x + B_1 \sin \lambda x$

Hence, $\quad u(x, t) = (A_1 \cos \lambda x + B_1 \sin \lambda x) \cdot Ce^{-\lambda^2 \alpha^2 t}$

$\Rightarrow \quad u(x, t) = (A \cos \lambda x + B \sin \lambda x) e^{-\lambda^2 \alpha^2 t}$ (I)

where $A = A_1 C$ and $B = B_1 C$

Case (ii): Let $k > 0$ i.e., $k = \lambda^2$, $\lambda \neq 0$

Then $\quad T' - \lambda^2 \alpha^2 T = 0 \quad \Rightarrow \quad \frac{T'}{T} = \lambda^2 \alpha^2$

$\therefore \quad \int \frac{T'}{T} dt = \int \lambda^2 \alpha^2 \, dt$

$\Rightarrow \quad \log_e T = \lambda^2 \alpha^2 t + \log_e C$

$\Rightarrow \quad \log_e \frac{T}{C} = \lambda^2 \alpha^2 t \quad \Rightarrow \quad \frac{T}{C} = e^{\lambda^2 \alpha^2 t} \quad \Rightarrow \quad T = C\, e^{\lambda^2 \alpha^2 t}$

where C is an arbitrary constant.

and (3) $\Rightarrow \quad X'' - \lambda^2 X = 0$

Auxiliary equation is $\quad m^2 - \lambda^2 = 0 \quad \Rightarrow \quad m = \pm\lambda$

$\therefore \quad X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$

$\therefore \quad u(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) C e^{\lambda^2 \alpha^2 t}$

$\Rightarrow \quad u(x, t) = (A e^{\lambda x} + B e^{-\lambda x}) e^{\lambda^2 \alpha^2 t}$ (II)

where $A = A_1 C$; and $B = B_1 C$

Case (iii): Let $k = 0$ then $\quad X'' = 0 \quad$ and $\quad T' = 0$

$\Rightarrow \quad X = C_1 x + C_2 \quad$ and $\quad T = C_3$

$\therefore \quad u(x, t) = (C_1 x + C_2) C_3$

$\Rightarrow \quad u(x, t) = \mathrm{A}x + B$ (III)

where $A = C_1 C_3$ and $B = C_2 C_3$

Proper choice of the solution

Of the three possible solutions, we choose the solution which is consistent with the physical nature of the problem and the given boundary-value conditions. Since $u(x, t)$ represent the temperature at any time t and at a distance x from one end of the rod, the temperature cannot be increasing as t is increasing. So, as t increases, u must decrease hence the suitable solution for unsteady state conditions (or transient) is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \quad \text{(I)}$$

A, B, λ are independent constants to be determined. Hence, three conditions are required to solve the one-dimensional heat equation in transient state.

In steady state conditions, the temperature at any point is independent of time (i.e., it does not change with time). Hence, the suitable solution for steady state heat flow is

$$u(x,t) = Ax + B \quad \text{(III)}$$

In problems, we will use these solutions directly depending upon the hypothesis temperature distribution is transient or steady state.

TYPE 1. Problems with zero boundary values
That is the temperatures at the ends of the rod are kept at zero

The boundary-values conditions are

(i) $u(0, t) = 0$ and (ii) $u(l, t) = 0 \ \forall\ t \geq 0$, which are boundary conditions
(iii) $u(x, 0) = f(x) \ \forall\ x \in (0, l)$ is the initial condition.

$f(x)$ may be in trigonometric form or algebraic form.

WORKED EXAMPLES

TYPE 1(a): $u(x, 0) = f(x)$ is in trigonometric form

EXAMPLE 1

A uniform rod of length l through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\sin^3\frac{\pi x}{l}$, $0 < x < l$, find the temperature distribution in the bar at any time t.

Solution.

The temperature distribution in the bar is given by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

The boundary-value conditions are

(i) $u(0, t) = 0$ and (ii) $u(l, t) = 0 \ \forall\ t \geq 0$, (iii) $u(x, 0) = k\sin^3\frac{\pi x}{l}$, $0 < x < l$

The suitable solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \quad (1)$$

where A, B, λ are constants to be determined.

Using condition (i), i.e., when $x = 0$, $u = 0$, in (1), we get

$$(A\cos 0 + B\sin 0)e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad A e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad A = 0 \qquad \text{since } e^{-\alpha^2\lambda^2 t} \neq 0$$

$$\therefore \qquad u(x,t) = B\sin\lambda x \cdot e^{-\alpha^2\lambda^2 t} \qquad (2)$$

Using condition (ii), i.e., when $x = l$, $u = 0$, in (2), we get

$$B\sin\lambda l \cdot e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad \sin\lambda l = 0, \quad \text{since } B \neq 0,\ e^{-\alpha^2\lambda^2 t} \neq 0$$

$$\Rightarrow \qquad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots$$

$$\therefore \qquad u(x,t) = B\sin\left(\frac{n\pi}{l}x\right)\cdot e^{-\alpha^2\cdot\frac{n^2\pi^2}{l^2}t},\ n = 1, 2, 3, \ldots \qquad (3)$$

Before using the non-zero condition, we have to find the general solution.
For each value of n, (3) is a solution. So their linear combination is also a solution.
$\therefore$ the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot e^{\frac{-\alpha^2 n^2\pi^2}{l^2}t} \qquad (4)$$

Using condition (iii), i.e., when $t = 0$, $u = k\sin^3\left(\frac{\pi x}{l}\right)$, in (4), we get

$$k\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\cdot e^0$$

$$\Rightarrow \qquad \frac{k}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \qquad \frac{3k}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{k}{4}\sin\left(\frac{3\pi x}{l}\right) = B_1\sin\left(\frac{\pi x}{l}\right) + B_2\sin\left(\frac{2\pi x}{l}\right) + B_3\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

Equating like coefficients, we get

$$B_1 = \frac{3k}{4}, \qquad B_2 = 0, \qquad B_3 = \frac{-k}{4}, \qquad B_4 = 0 = B_5 = B_6 = \cdots$$

$\therefore$ the general solution is
$$u(x,t) = B_1\sin\left(\frac{\pi x}{l}\right)\cdot e^{\frac{-\alpha^2\pi^2}{l^2}t} + \cdots$$

$$u(x,t) = \frac{3k}{4}\sin\left(\frac{\pi x}{l}\right)\cdot e^{\frac{-\alpha^2\pi^2}{l^2}t} - \frac{k}{4}\sin\left(\frac{3\pi x}{l}\right)\cdot e^{\frac{-3\alpha^2\pi^2 t}{l^2}}.$$

TYPE 1(b): $u(x, 0) = f(x)$ is an algebraic function

EXAMPLE 2

Heat flows through a uniform bar of length l which has its sides insulated and the temperature at the ends kept at zero. If the initial temperature at the interior points of the bar is given by $k(lx - x^2)$, $0 < x < l$, find the temperature distribution in the bar at time t.

Solution.

The temperature distribution in the bar is given by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary-value conditions are

(i) $u(0, t) = 0$ (ii) $u(l, t) = 0 \; \forall \; t \geq 0$

(iii) $u(x, 0) = f(x) = k(lx - x^2), 0 < x < l$

The solution is $$u(x, t) = (A \cos \lambda x + B \sin \lambda x) e^{-\alpha^2 \lambda^2 t} \tag{1}$$

where A, B, λ are constants to be determined.

Using condition (i), i.e., when $x = 0$, $u = 0$, in (1) we get

$$(A \cos 0 + B \sin 0) e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A = 0$$

$\therefore$ $$u(x, t) = B \sin \lambda x \cdot e^{-\alpha^2 \lambda^2 t} \tag{2}$$

Using condition (ii), i.e., when $x = l$, $u = 0$, in (2), we get

$$B \sin \lambda l \cdot e^{-\alpha^2 \lambda^2 t} = 0$$

But $B \neq 0$, $e^{-\lambda^2 \alpha^2 t} \neq 0 \quad \Rightarrow \quad \sin \lambda l = 0 \quad \Rightarrow \quad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}, n = 1, 2, 3, \ldots$

$\Rightarrow$ $$u(x, t) = B \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2 \pi^2 \alpha^2 t}{l^2}}, n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is the linear combination of these solutions.

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2 \pi^2 \alpha^2 t}{l^2}} \tag{3}$$

Using condition (iii), i.e., when $t = 0$, $u = k(lx - x^2)$, we get

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^0$$

$\Rightarrow$ $$k(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \tag{4}$$

Since $u(x, 0) = f(x) = k(lx - x^2)$ is an algebraic function, to find B_n we express $f(x)$ as a Fourier sine series in $0 < x < l$.

$\therefore$ $$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{5}$$

where $$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

Comparing (4) and (5), we get $B_n = b_n$, $n = 1, 2, 3, \ldots$

Now $$b_n = \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{l}{n\pi}(lx - x^2)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[0 - \frac{2l^3}{n^3\pi^3}\cos n\pi - \left(0 - \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2k}{l}\left[\frac{2l^3}{n^3\pi^3} - \frac{2l^3}{n^3\pi^3}\cos n\pi\right]$$

$\Rightarrow$ $$b_n = \frac{2k}{l}\cdot\frac{2l^3}{n^3\pi^3}(1-\cos n\pi) = \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]$$

If n is even, $(-1)^n = 1$ $\quad\therefore\quad$ $b_n = 0$

If n is odd, $(-1)^n = -1$ $\quad\therefore\quad$ $b_n = \frac{4kl^2}{n^3\pi^3}(2) = \frac{8kl^2}{n^3\pi^3}$, $n = 1, 3, 5, \ldots$

$\therefore$ $$B_n = \frac{8kl^2}{n^3\pi^3},\ n = 1, 3, 5, \ldots$$

Substituting in (3), we get

$$u(x,t) = \sum_{n=1,3,5,\ldots}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}} = \frac{8l^2k}{\pi^3}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}}$$

EXAMPLE 3

Find the solution of the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions.

(i) $u(0, t) = 0$, (ii) $u(l, t) = 0$ for $t > 0$

and (iii) $$u(x, 0) = \begin{cases} x, & 0 \le x < \frac{l}{2} \\ l - x, & \frac{l}{2} < x < l \end{cases}$$

Solution.

The solution of $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \qquad (1)$$

Using condition (i), i.e., when $x = 0$, $u = 0$ in (1), we get

$$(A\cos 0 + B\sin 0)e^{-\alpha^2\lambda^2 t} = 0 \quad\Rightarrow\quad Ae^{-\alpha^2\lambda^2 t} = 0 \quad\Rightarrow\quad A = 0$$

$\therefore$ $$u(x,t) = B\sin\lambda x\cdot e^{-\alpha^2\lambda^2 t} \qquad (2)$$

Using condition (ii), i.e., when $x = l$, $u = 0$ in (2), we get

$$B \sin \lambda l \, e^{-\alpha^2\lambda^2 t} = 0$$

$$\Rightarrow \quad \sin \lambda l = 0 \quad \Rightarrow \quad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad u(x,t) = B \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2\pi^2}{l^2}\alpha^2 t},\ n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}} \tag{3}$$

Using condition (iii), i.e., when $t = 0$, $u = 0$ in (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^0 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \tag{4}$$

Since $u(x, 0)$ is an algebraic function, to find B_n we express $u(x, 0)$ as a Fourier sine series.

$$\therefore \quad u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{5}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

Comparing (4) and (5), we get $B_n = b_n$, $n = 1, 2, 3, \ldots$

Now
$$b_n = \frac{2}{l}\left\{\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2}{l}\left\{\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2}{l}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left\{\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2} - \left[\frac{l}{n\pi}(l-x)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left\{\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0\right] - \left[0 - \left(\frac{l}{n\pi}\cdot\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{2}{l}\left\{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{l}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

If n is even, then $\sin \frac{n\pi}{2} = 0$ and if n is odd $\sin \frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$

$$\therefore \quad b_n = \frac{4l}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \text{ if } n = 1, 3, 5, \ldots \quad \therefore \quad B_n = \frac{4l}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \quad \text{if } n = 1, 3, 5, \ldots$$

Substituting in (3), we get

$$u(x,t) = \sum_{n=1,3,5,\ldots} \frac{4l}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}} = \frac{4l}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin \frac{n\pi x}{l} \cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}}$$

TYPE 1(c): Intial temperatue $u(x, 0) = f(x)$ is to be found from the given problem.

EXAMPLE 4

Find the temperature distribution of a homogenous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature at the centre of the bar is k and falls uniformly to 0 at the ends.

Solution.

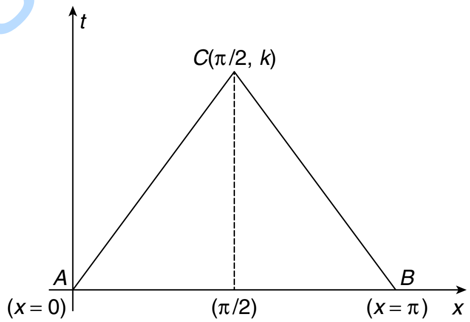
The temperature distribution of the bar is given by the one-dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The boundary conditions are given by

(i) $u(0,t) = 0$ and (ii) $u(\pi, t) = 0 \quad \forall \quad t \geq 0$

The initial temperature is to be found out from the temperature graph ACB

where $A(0, 0)$, $B(\pi, 0)$, $C\left(\frac{\pi}{2}, k\right)$



Equation of AC is $\frac{y-0}{0-k} = \frac{x-0}{0-\frac{\pi}{2}} \Rightarrow y = \frac{2k}{\pi}x, \quad 0 \leq x \leq \frac{\pi}{2}$

Equation of BC is $\frac{y-0}{k-0} = \frac{x-\pi}{\frac{\pi}{2}-\pi} \Rightarrow \frac{y}{k} = -\frac{2}{\pi}(x-\pi) \Rightarrow y = \frac{2k}{\pi}(\pi - x), \quad \frac{\pi}{2} < x \leq \pi$

$\therefore$ the initial condition is

$$u(x,0) = \begin{cases} \frac{2k}{\pi}x, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), & \frac{\pi}{2} < x \leq \pi \end{cases}$$

The suitable solution is

$$u(x,t) = (A \cos \pi x + B \sin \pi x)e^{-\alpha^2\lambda^2 t} \tag{1}$$

Using condition (i), i.e., when $x = 0$, $u = 0$ in (1), we get

$$(A\cos 0 + B\sin 0)e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad A e^{e^{-\alpha^2\lambda^2 t}} = 0 \quad \Rightarrow \quad A = 0$$

$$\therefore \qquad u(x,t) = B\sin \lambda x\, e^{-\alpha^2\lambda^2 t} \qquad (2)$$

Using condition (ii), i.e., when $x = \pi$, $u = 0$ in (2), we get

$$B\sin \pi\lambda\, e^{-\alpha^2\lambda^2 t} = 0$$

$$\Rightarrow \qquad \sin \pi\lambda = 0 \qquad [\because B \neq 0, e^{-\alpha^2\lambda^2 t} \neq 0]$$

$$\Rightarrow \qquad \pi\lambda = n\pi \quad \Rightarrow \quad \lambda = n, \qquad n = 1, 2, 3, \ldots$$

$$\therefore \qquad u(x,t) = B\sin nx \cdot e^{-\alpha^2 n^2 t}, \qquad n = 1, 2, 3, \ldots \qquad (3)$$

The general solution is the linear combination of these solutions.

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} B_n \sin nx\; e^{-\alpha^2 n^2 t} \qquad (4)$$

Using initial condition (iii), i.e., when $t = 0$, we get

$$u(x,0) = \begin{cases} \dfrac{2k}{\pi}x, & 0 \le x \le \dfrac{\pi}{2} \\ \dfrac{2k}{\pi}(\pi - x), & \dfrac{\pi}{2} < x \le \pi \end{cases}$$

$$\therefore \qquad u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx \cdot e^0 = \sum_{n=1}^{\infty} B_n \sin nx \qquad (5)$$

Since $u(x, 0)$ is an algebraic function, to find B_n we express $u(x, 0)$ as a Fourier sine series

$$\therefore \qquad u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad (6)$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Comparing (5) and (6), we find $B_n = b_n$, $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{2k}{\pi}x\sin nx\, dx + \int_{\pi/2}^{\pi} \frac{2k}{\pi}(\pi - x)\sin nx\, dx\right\}$$

$$= \frac{2}{\pi}\cdot\frac{2k}{\pi}\left\{\int_0^{\pi/2} x\sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right\}$$

$$= \frac{4k}{\pi^2}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{4k}{\pi^2}\left\{\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[-\frac{1}{n}(\pi - x)\cos nx - \left(\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{4k}{\pi^2}\left\{\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - 0 + \left[0 - \left(\frac{-1}{n}\frac{\pi}{2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{4k}{\pi^2}\left\{\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right\}$$

$$b_n = \frac{4k}{\pi^2}\cdot\frac{2}{n^2}\sin\frac{n\pi}{2} = \frac{8k}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

If n is even, $\sin\frac{n\pi}{2} = 0 \qquad \therefore\ b_n = 0$

If n is odd, $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}} \qquad \therefore\ b_n = \frac{8k}{\pi^2 n^2}(-1)^{\frac{n-1}{2}},\ n = 1, 3, 5, \ldots$

$$\therefore \qquad B_n = \frac{8k}{\pi^2 n^2}(-1)^{\frac{n-1}{2}}, \quad n = 1, 3, 5, \ldots$$

Substituting in (4), we get

$$u(x, t) = \sum_{n = \text{odd}} \frac{8k}{\pi^2 n^2}(-1)^{\frac{n-1}{2}} \sin nx \cdot e^{-\alpha^2 n^2 t}$$

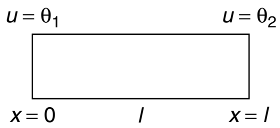
$$u(x, t) = \frac{8k}{\pi^2}\sum_{n = 1, 3, 5, \ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin nx \cdot e^{-\alpha^2 n^2 t}$$

TYPE 2. Non-zero temperature at the end points of the bar in steady state and zero temperature in unsteady state

In steady state the temperature $u(x, t)$ is a function of x alone, as it is independent of t.

$$\therefore \qquad u(x, t) = u(x)$$

$$\therefore \qquad \frac{\partial u}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 0$$



$$\therefore \qquad \frac{\partial^2 u}{\partial x^2} = 0 \quad \Rightarrow \quad u = Ax + B$$

When $x = 0$, $u = \theta_1 \ \Rightarrow\ \theta_1 = B$. When $x = l$, $u = \theta_2$.

$$\therefore \qquad \theta_2 = Al + \theta_1 \quad \Rightarrow \quad A = \frac{(\theta_2 - \theta_1)}{l} \qquad \therefore \quad u = \frac{(\theta_2 - \theta_1)}{l}x + \theta_1$$

When the state changes from steady to unsteady, the temperature at the ends are reduced to zero.

WORKED EXAMPLES

EXAMPLE 1

A rod, 30 cm, long has its ends *A* and *B* kept at 20°C and 80°C respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function *u*(*x*, *t*), taking *x* = 0 at *A*.

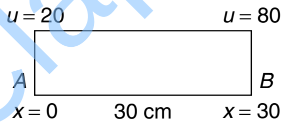
Solution.

Temperature function $u(x, t)$ is given by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Initially steady state conditions prevails with $u = 20$ at $x = 0$ and $u = 80$ at $x = 30$.
In steady state $u(x, t)$ is independent of t and so is a function of x alone:

$$\therefore \qquad u = \frac{\theta_2 - \theta_1}{l} x + \theta_1$$



Here $\theta_1 = 20$, $\theta_2 = 80$, $l = 30$

$$\therefore \qquad u = \frac{80 - 20}{30} x + 20$$

$$\Rightarrow \qquad u = 2x + 20 \qquad (2)$$

When the temperature at the ends are changed to 0°C, the heat flow or the temperature distribution in the bar will not be in steady state and so will depend on time. So, the temperature distribution $u(x, t)$ is given by (1).

$$\therefore \qquad u(x, t) = (A \cos \lambda x + B \sin \lambda x)\, e^{-\alpha^2 \lambda^2 t}$$

The new boundary conditions are (i) $u(0, t) = 0$ and (ii) $u(30, t) = 0 \;\forall\; t \geq 0$
The initial distribution of temperature is given by (2)

$\therefore$ (iii) $u(x, 0) = 2x + 20,\; 0 < x < 30$

Using condition (i), i.e., when $x = 20$, $u = 0$, we get

$$(A \cos 0 + B \sin 0) e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A = 0$$

$$\therefore \qquad u(x, t) = B \sin (\lambda x) \cdot e^{-\alpha^2 \lambda^2 t}$$

Using condition (ii), i.e., when $x = 30$, $u = 0$, we get

$$B \sin (30\lambda) e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad \sin 30\lambda = 0 \qquad \left[\because B e^{-\alpha^2 \lambda^2 t} \neq 0\right]$$

$$\Rightarrow \qquad 30\lambda = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{30}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \qquad u(x,t) = B \sin \frac{n\pi x}{30} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{900}}, \quad n = 1, 2, 3, \ldots$$

The general solution is the linear combination of these solutions.

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{900}} \tag{3}$$

Using condition (iii), i.e., when $t = 0$, $u = 2x + 20$

$$\therefore \qquad u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} \cdot e^0 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}$$

Since $u(x, 0) = 2x + 20$ is algebraic, to find B_n, we express $u(x, 0)$ as a Fourier sine series in (0, 30)

$$\therefore \qquad u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30} \tag{5}$$

where $\qquad b_n = \frac{2}{30} \int_0^{30} f(x) \sin \frac{n\pi x}{30} dx \qquad$ and $f(x) = u(x, 0)$

Comparing (4) and (5), we get $B_n = b_n, \qquad n = 1, 2, 3, \ldots$

Now $\qquad b_n = \frac{1}{15} \int_0^{30} (2x+20) \sin \frac{n\pi x}{30} dx$

$$= \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - 2\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{900}}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left[\frac{-30}{n\pi}(2x+20)\cos\frac{n\pi x}{30} + \frac{1800}{n^2\pi^2}\sin\frac{n\pi x}{30}\right]_0^{30}$$

$$= \frac{1}{15}\left[\frac{-30}{n\pi} \times 80 \cos n\pi + 0 - \left(\frac{-600}{n\pi}\right)\cos 0\right]$$

$$= \frac{1}{15}\left[\frac{-2400(-1)^n + 600}{n\pi}\right]$$

$$\Rightarrow \qquad b_n = \frac{600}{15}\left[\frac{1 - 4(-1)^n}{n\pi}\right] = \frac{40}{n\pi}[1 - 4(-1)^n]$$

$$\therefore \qquad B_n = \frac{40}{n\pi}[1 - 4(-1)^n], \qquad n = 1, 2, 3, \ldots$$

Substituting in (3), we get the solution.

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}[1 - 4(-1)^n] \sin \frac{n\pi x}{30} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{900}}$$

$$\Rightarrow \qquad u(x,t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{[1 - 4(-1)^n]}{n} \sin \frac{n\pi x}{30} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{900}}.$$

EXAMPLE 2

A rod of length l has its ends A and B kept 0°C and 100°C until steady state conditions prevail. If the temperature at B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A and at any time t.

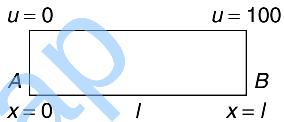
Solution.

The temperature distribution is given by the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Initially, steady state conditions prevail with the conditions $u = 0$ at $x = 0$ and $u = 100$ at $x = l$

In steady state, u is independent of time t.



$$\therefore \qquad u = \frac{\theta_2 - \theta_1}{l} x + \theta_1$$

Here $\theta_1 = 0$, $\theta_2 = 100$.

$$\therefore \qquad u = \frac{100 - 0}{l} x + 0$$

$$\Rightarrow \qquad u(x) = \frac{100}{l} x, \quad 0 \le x \le l \qquad (2)$$

If the temperature at B is reduced to 0°C, then the temperature distribution changes from steady state to unsteady state. So, the temperature distribution $u(x, t)$ is given by (1)

$$\therefore \qquad u(x,t) = (A \cos \lambda x + B \sin \lambda x) e^{-\alpha^2 \lambda^2 t} \qquad (3)$$

The new boundary-value conditions are

(i) $u(0,t) = 0$ (ii) $u(l,t) = 0 \quad \forall \quad t \ge 0$ and (iii) $u(x,0) = \frac{100}{l} x, \quad x \in (0, l)$

Using condition (i), i.e., when $x = 0$, $u = 0$, in (3), we get

$$(A \cos 0 + B \sin 0) e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A = 0. \qquad [\therefore\ e^{-\alpha^2 \lambda^2 t} \ne 0]$$

$$\therefore \qquad u(x,t) = B \sin \lambda x \cdot e^{-\alpha^2 \lambda^2 t} \qquad (4)$$

Using condition (ii), i.e., when $x = l$, $u = 0$, in (4), we get

$$B \sin \lambda l \cdot e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad \sin \lambda l = 0 \qquad [\because B e^{-\alpha^2 \lambda^2 t} \ne 0]$$

$$\Rightarrow \qquad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}, \qquad n = 1, 2, 3, \ldots$$

$$\therefore \qquad u(x,t) = B \sin \frac{n\pi x}{l} \cdot e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}, \qquad n = 1, 2, 3, \ldots$$

The general solution is the linear combination of these solutions.

$$\therefore \text{ the general solution is} \qquad u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \qquad (5)$$

Using condition (iii), i.e., when $t = 0$, $u(x, 0) = \dfrac{100}{l}x$, we get

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^0 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

Since $u(x, 0) = \dfrac{100}{l}x$ is an algebraic function, to find B_n, we express $u(x, 0)$ as a Fourier sine series in $(0, l)$

$$\therefore \qquad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{7}$$

where $\qquad b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$ and $f(x) = u(x, 0)$

Comparing (6) and (7), we get $B_n = b_n$, $\qquad n = 1, 2, 3, \ldots$

Now $\qquad b_n = \dfrac{2}{l}\displaystyle\int_0^l \frac{100}{l} x \sin \frac{n\pi x}{l} dx$

$$= \frac{200}{l^2}\left[x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 1 \cdot \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l}{n\pi} x \cdot \cos \frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi} \cos n\pi + \frac{l^2}{n^2\pi^2} \sin n\pi - (0) \right]$$

$$b_n = \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n \right] = \frac{200}{n\pi}(-1)^{n+1}$$

$$\therefore \qquad B_n = \frac{200}{n\pi}(-1)^{n+1}, \quad n = 1, 2, 3, \ldots$$

Substituting in (5), we get,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{l} \cdot e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}$$

$$\Rightarrow \qquad u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}.$$

Type 3. Non-zero temperatures at the ends of the bar, both in steady state and unsteady state. In type 1 and type 2, the temperatures at the ends in unsteady (or transient) state are kept at 0°C. So, the constants in the suitable solution could be obtained easily. In this type the temperatures at the ends are non-zero in the unsteady state and so the computations of constants cannot be done as before. We split the required solution $u(x, t)$ into two parts as $u(x,t) = u_1(x) + u_2(x,t)$, where $u_1(x)$ is the steady state solution. (i.e., the solution corresponding to the end points which do not change with time) and $u_2(x, t)$ is the unsteady state solution (i.e., the solution corresponding to the interior points of the bar which vary with time t), in order to get zero boundary conditions.

WORKED EXAMPLES

EXAMPLE 1

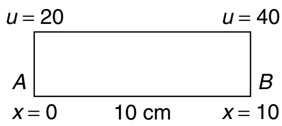
A bar, 10 cm long, with insulated sides, has its ends A and B kept at 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly raised to 50°C and at the same instant at B is lowered to 10°C. Find the subsequent temperature at any point of the bar at any time.

Solution.

The temperature at any point is given by the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

In steady state, the temperature is



$$u = \frac{\theta_2 - \theta_1}{l} x + \theta_1$$

$$= \frac{40 - 20}{10} x + 20$$

$$\Rightarrow \qquad u(x) = 2x + 20 \qquad (2)$$

Then suddenly the temperature at A is increased to 50°C and that at B is decreased to 10°C. So, the temperature distribution in the bar is changed from steady state to unsteady state.

Then the temperature $u(x, t)$ satisfies (1)

The boundary-value conditions of the unsteady state are

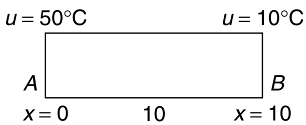
(i) $u(0,t) = 50$ and (ii) $u(10,t) = 10 \quad \forall\, t \geq 0$ and (iii) $u(x,0) = 2x + 20, \qquad 0 \leq x \leq 10$

As the temperature at the ends are not equal to 0°C, we split the solution $u(x, t)$ of (1) into two parts in order to get zero boundary conditions

$$u(x,t) = u_1(x) + u_2(x,t) \qquad (3)$$

where $u_1(x)$ is steady state solution of (1)

and $u_2(x, t)$ is the transient solution of (1)



Since $u_1(x)$ is steady state solution, $\quad u_1 = \dfrac{10 - 50}{10} x + 50$

$$\therefore \qquad u_1(x) = -4x + 50, \qquad 0 < x \leq 10$$

Since u_2 is transient solution of (1), $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$.

$\therefore$ we have $$u_2(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \quad (4)$$

But $$u_2(x,t) = u(x,t) - u_1(x)$$

$\therefore$ the boundary-value conditions of $u_2(x, t)$ are

(iv) $u_2(0,t) = u(0,t) - u_1(0) = 50 - 50 = 0,$ [Using (i)]

(v) $u_2(10,t) = u(10,t) - u_1(10) = 10 - 10 = 0,$ [Using (ii)]

and (vi) $u_2(x,0) = u(x,0) - u_1(x) = 2x + 20 - (-4x + 50) = 6x - 30$

Thus, $u_2(x, t)$ satisfies the heat equation (1) and the boundary-value conditions (iv), (v) and (vi)
Using condition (iv), i.e., when $x = 0$, $u_2 = 0$, in (4), we get

$$(A\cos 0 + B\sin 0)e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad Ae^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad A = 0$$

$\therefore$ (4) is $$u_2(x,t) = B\sin\lambda x \cdot e^{-\alpha^2\lambda^2 t} \quad (5)$$

Using condition (v), i.e., when $x = 10$, $u_2 = 0$ in (5), we get

$$B\sin 10\lambda \cdot e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow \sin 10\lambda = 0, \qquad \text{since } Be^{-\alpha^2\lambda^2 t} \neq 0$$

$$\Rightarrow \quad 10\lambda = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{10}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \quad u_2(x,t) = B\sin\frac{n\pi x}{10} \cdot e^{-\alpha^2 \frac{n^2\pi^2}{100} t}, \qquad n = 1, 2, 3, \ldots$$

$\therefore$ the most general solution is the linear combination of these solutions.

$$\therefore \quad u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \cdot e^{\frac{-\alpha^2 n^2\pi^2 t}{100}} \quad (6)$$

Using condition (vi), i.e., when $t = 0$, $u_2 = 6x - 30$, in (6), we get

$$u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \cdot e^0 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \quad (7)$$

Since $u_2(x,0) = 6x - 30$ is an algebraic function, to find B_n, we express $u_2(x, 0)$ as Fourier sine series in (0, 10).

$$\therefore \quad u_2(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \quad (8)$$

where $$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx \text{ and } f(x) = 6x - 30$$

Comparing (7) and (8), we find $B_n = b_n \quad \forall\, n = 1, 2, 3, \ldots.$

Now $$b_n = \frac{1}{5}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[-\frac{10}{n\pi}(6x-30)\cos\frac{n\pi x}{10} + \frac{600}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5}\left[\frac{-10}{n\pi}\cdot 30\cdot\cos n\pi + \frac{600}{n^2\pi^2}\sin n\pi - \left(\frac{-10}{n\pi}(-30)\right)\cos 0\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}(-1)^n - \frac{300}{n\pi}\right]$$

$\Rightarrow$ $$b_n = \frac{1}{5}\left(\frac{-300}{n\pi}\right)[1+(-1)^n] = -\frac{60}{n\pi}[1+(-1)^n]$$

If n is odd, $(-1)^n = -1$. $\therefore\ b_n = 0$

If n is even, $(-1)^n = 1$. $\therefore\ b_n = -\frac{60}{n\pi}(2) = -\frac{120}{n\pi}$, $n = 2, 4, 6, \ldots$

$\therefore$ $$B_n = -\frac{120}{n\pi}, \quad n = 2, 4, 6, \ldots$$

Substituting in (6), we get

$$u_2(x,t) = \sum_{n=2,4,\ldots} \frac{-120}{n\pi}\sin\frac{n\pi x}{10}e^{\frac{-\alpha^2 n^2\pi^2 t}{100}} = \frac{-120}{\pi}\sum_{n=2,4,\ldots}\frac{1}{n}\sin\frac{n\pi x}{10}\cdot e^{-\alpha^2\frac{n^2\pi^2}{100}t}$$

$\therefore$ the required temperature is $u(x,t) = u_1(x) + u_2(x,t)$

$\Rightarrow$ $$u(x,t) = -4x + 50 - \frac{120}{\pi}\sum_{n=2,4,\ldots}\frac{1}{n}\sin\frac{n\pi x}{10}\cdot e^{-\alpha^2\frac{n^2\pi^2}{100}t}.$$

EXAMPLE 2

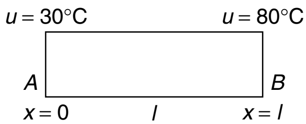
The ends *A* and *B* of a rod *l* cm long have their temperatures kept at 30°C and 80°C, until steady state conditions prevail. The temperature of the end *B* is suddenly reduced to 60°C and that of *A* increased to 40°C. Find the temperature distribution of the rod after time *t*.

Solution.

The temperature at any point is given by the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

In steady state, the temperature is



$$u = \frac{\theta_2 - \theta_1}{l} x + \theta_1$$

$$\Rightarrow \qquad u(x) = \frac{80-30}{l} x + 30 = \frac{50}{l} x + 30 \qquad (2)$$

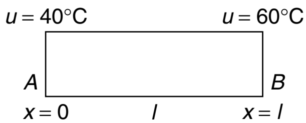
Suddenly the temperatures at A and B are changed to 40°C and 60°C. So, the temperature distribution in the bar is changed from steady state to unsteady state. The temperature $u(x, t)$ is given by (1)
The boundary-value conditions of the unsteady state are

(i) $u(0,t) = 40$, (ii) $u(l,t) = 60 \quad \forall \quad t \geq 0$ and (iii) $u(x, 0) = \frac{50}{l} x + 30, \ 0 \leq x \leq l$

As the temperature at the ends are not 0°C, we split the solution $u(x, t)$ of (1) into two parts in order to get zero boundary conditions.

$$u(x,t) = u_1(x) + u_2(x,t) \qquad (3)$$

where $u_1(x)$ is the steady state solution of (1)
and $u_2(x, t)$ is the transient state solution of (1)
Since u_1 is the steady state solution,



$$u_1(x) = \frac{\theta_2 - \theta_1}{l} x + \theta_1$$

$$= \frac{60-40}{l} x + 40 = \frac{20}{l} x + 40, \ 0 \leq x \leq l$$

Since u_2 is the unsteady state solution of (1), $\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2}$

$$\therefore \qquad u_2(x,t) = (A \cos \lambda x + B \sin \lambda x) e^{-\alpha^2 \lambda^2 t} \qquad (4)$$

But $\qquad u_2(x,t) = u(x,t) - u_1(x)$

$\therefore$ the boundary-value conditions of $u_2(x, t)$ are

(iv) $u_2(0,t) = u(0,t) - u_1(0) = 40 - 40 = 0$ [Using (i)]

(v) $u_2(l,t) = u(l,t) - u_1(l) \quad = 60 - 60 = 0 \quad \forall \quad t \geq 0$ [Using (ii)]

and (vi) $u_2(x,0) = u(x,0) - u_1(x) = \frac{50}{l} x + 30 - \left(\frac{20}{l} x + 40\right) = \frac{30}{l} x - 10, \qquad 0 < x < l$

Thus, $u_2(x, t)$ satisfies the equation (1) and the conditions (iv), (v) and (vi)
Using condition (iv), i.e., when $x = 0$, $u_2 = 0$, in (4), we get

$$(A \cos 0 + B \sin 0) e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad A = 0$$

$$\therefore \qquad u_2(x,t) = B \sin \lambda x \cdot e^{-\alpha^2 \lambda^2 t} \qquad (5)$$

Using condition (v), i.e., when $x = l$, $u_2 = 0$, in (5), we get

$$B \sin \lambda l \cdot e^{-\alpha^2 \lambda^2 t} = 0 \quad \Rightarrow \quad \sin \lambda l = 0 \qquad [\because B e^{-\alpha^2 \lambda^2 t} \neq 0]$$

$\Rightarrow \qquad \lambda l = n\pi \quad \Rightarrow \quad \lambda = \dfrac{n\pi}{l}, \qquad n = 1, 2, 3, \ldots$

$\therefore \qquad u_2(x,t) = B \sin \dfrac{n\pi x}{l} \cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}, \qquad n = 1, 2, 3, \ldots$

So, the general solution is

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \qquad (6)$$

Using condition (vi), i.e., when $t = 0$, $u_2 = 0$, in (6) we get

$$u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^0$$

$$\Rightarrow \qquad u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \qquad (7)$$

Since $u_2(x,0) = \dfrac{30x}{l} - 10$ is algebraic, we express $u_2(x, 0)$ as a Fourier sine series in $0 < x < l$.

$$\therefore \qquad u_2(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad (8)$$

where $\qquad b_n = \dfrac{2}{l}\displaystyle\int_0^l \left(\dfrac{30}{l}x - 10\right) \sin \dfrac{n\pi x}{l}\, dx$

Comparing (7) and (8), we find $B_n = b_n \quad \forall\, n$

Now $\qquad b_n = \dfrac{2}{l}\displaystyle\int_0^l \left(\dfrac{30}{l}x - 10\right) \sin \dfrac{n\pi x}{l}\, dx$

$$\Rightarrow \qquad b_n = \frac{2}{l}\left[\left(\frac{30x}{l} - 10\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \frac{30}{l}\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi}\left(\frac{30x}{l} - 10\right)\cos \frac{n\pi x}{l} + \frac{30}{l} \cdot \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi}(30 - 10) \cos n\pi + \frac{30\, l}{n^2\pi^2} \sin n\pi - \left(10 \frac{l}{n\pi} \cdot \cos 0\right)\right]$$

$$= \frac{2}{l}\left[-\frac{20l}{n\pi} \cos n\pi - \frac{10l}{n\pi}\right] = \frac{2}{l}\left(-\frac{10l}{n\pi}\right)[1 + 2\cos n\pi]$$

$$\Rightarrow \qquad b_n = -\frac{20}{n\pi}\left[1 + 2(-1)^n\right], \qquad n = 1, 2, 3, \ldots$$

Since $B_n = b_n$, $\qquad B_n = -\dfrac{20}{n\pi}[1+2(-1)^n] \quad n = 1, 2, 3, \ldots$

Substituting in (6),

$$\therefore \qquad u_2(x,t) = \sum_{n=1}^{\infty} \frac{-20}{n\pi}[1+2(-1)^n]\sin\frac{n\pi x}{l}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

$$= -\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{[1+2(-1)^n]}{n}\sin\frac{n\pi x}{l}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

$\therefore$ the required temperature is

$$u(x,t) = u_1(x) + u_2(x,t)$$

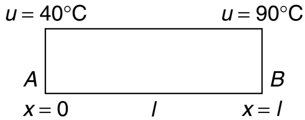
$$\Rightarrow \qquad u(x,t) = \frac{20}{l}x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{[1+2(-1)^n]}{n}\sin\frac{n\pi}{l}x\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

EXAMPLE 3

The ends *A* and *B* of a rod *l* cm long have the temperatures 40°C and 90°C, until steady state prevail. The temperatures at *A* is suddenly raised to 90°C and at the same time that at *B* is reduced to 40°C. Find the temperature distribution in the rod after time *t*. Also show that the temperature at the mid point of the rod remains unaltered for all time, regardless of the material of the rod.

Solution.
The temperature distribution is given by the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$



In steady state, the temperature is

$$u(x) = \frac{\theta_2 - \theta_1}{l}x + \theta_1$$

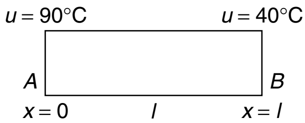
$$\Rightarrow \qquad = \frac{90-40}{l}x + 40 = \frac{50}{l}x + 40 \qquad (2)$$

When the temperatures are changed at the ends, steady state is changed to unsteady state, the temperature $u(x, t)$ is given by (1).
The boundary-value conditions of the unsteady state are

(i) $u(0,t) = 90$ (ii) $u(l,t) = 40 \quad \forall \quad t \geq 0$ and (iii) $u(x,0) = \dfrac{50}{l} + 40, \qquad 0 < x < l$

Since the temperature at the ends are non-zero, we split the temperature function $u(x, t)$ into two parts in order to get zero boundary conditions

$$u(x,t) = u_1(x) + u_2(x,t) \qquad (3)$$



where $u_1(x)$ is the steady state solution of (1)
and $u_2(x, t)$ is the transient state solution of (1)
In steady state, the solution is

$$u_1(x) = \frac{40-90}{l}x + 90$$

$$= -\frac{50}{l}x + 90\,, \quad 0 < x < l$$

Since u_2 is a solution of (1), we have

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2}$$

$\therefore$ $$u_2(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \quad (4)$$

But $$u_2(x,t) = u(x,t) - u_1(x)$$

$\therefore$ the corresponding boundary-value conditions of u_2 are

(iv) $u_2(0,t) = u(0,t) - u_1(0) = 90 - 90 = 0$ [Using (i)]

(v) $u_2(l,t) = u(l,t) - u_1(l) = 40 - 40 = 0$ [Using (ii)]

and (vi) $u_2(x,0) = u(x,0) - u_1(x) = \frac{50}{l}x + 40 - \left(\frac{-50}{l}x + 90\right) = \frac{100}{l}x - 50,\qquad 0 < x < l$

Using condition (iv), i.e., when $x = 0$, $u_2 = 0$, in (4) we get

$$(A\cos 0 + B\sin 0)e^{-\alpha^2\lambda^2 t} = 0 \;\Rightarrow\; Ae^{-\alpha^2\lambda^2 t} = 0 \;\Rightarrow\; A = 0$$

$\therefore$ $$u_2(x,t) = B\sin\lambda x \cdot e^{-\alpha^2\lambda^2 t} \quad (5)$$

Using condition (v), i.e., when $x = l$, $u_2 = 0$, in (5), we get

$$B\sin\lambda l\cdot e^{-\alpha^2\lambda^2 t} = 0 \;\Rightarrow\; \sin\lambda l = 0 \qquad [\text{since } Be^{-\alpha^2\lambda^2 t} \neq 0]$$

$\Rightarrow$ $$\lambda l = n\pi \;\Rightarrow\; \lambda = \frac{n\pi}{l}, \qquad n = 1, 2, 3, \ldots$$

$\therefore$ $$u_2(x,t) = B\sin\frac{n\pi x}{l}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}, \qquad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t} \quad (6)$$

Using condition (vi), i.e., when $t = 0$, $u_2 = \frac{100x}{l} - 50$ in (6), we get

$$u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad (7)$$

Since $u_2(x, 0)$ is an algebraic function, to find B_n, we express $u_2(x, 0)$ as a Fourier sine series in $0 < x < l$.

$$\therefore \qquad u_2(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{8}$$

where $$b_n = \frac{2}{l}\int_0^l \left(\frac{100x}{l} - 50\right)\sin \frac{n\pi x}{l}\,dx$$

Comparing (7) and (8), we get $B_n = b_n \quad \forall\, n$

Now $$b_n = \frac{2}{l}\left[\left(\frac{100x}{l} - 50\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \frac{100}{l}\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l}{n\pi}\left(\frac{100x}{l} - 50\right)\cos \frac{n\pi x}{l} + \frac{100l}{n^2\pi^2}\sin \frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l}{n\pi}(100-50)\cos n\pi + \frac{100l}{n^2\pi^2}\sin n\pi - \left(\frac{-l}{n\pi}(-50)\right)\right]$$

$$= \frac{2}{l}\left[\frac{-50l}{n\pi}\cos n\pi - \frac{50l}{n\pi}\right] = \frac{2}{l}\left(\frac{-50l}{n\pi}\right)[1+\cos n\pi]$$

$$\Rightarrow \qquad b_n = -\frac{100}{n\pi}\left[1+(-1)^n\right]$$

If n is even, then $b_n = -\frac{100}{n\pi}(2) = -\frac{200}{n\pi}$ and if n is odd, then $b_n = 0$

Since $B_n = b_n$, $\quad B_n = -\frac{200}{n\pi}, \quad n = 2, 4, 6, \ldots$

Substituting in (6), we get

$$u_2(x,t) = \sum_{n=2,4,6,\ldots} -\frac{200}{n\pi}\sin \frac{n\pi x}{l}\cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t}$$

$$\Rightarrow \qquad u_2(x,t) = -\frac{200}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{n}\sin \frac{n\pi x}{l}\cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t}$$

$$\therefore \qquad u(x,t) = u_1(x) + u_2(x,t)$$

$$\Rightarrow \qquad u(x,t) = -\frac{50x}{l} + 90 - \frac{200}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{n}\sin \frac{n\pi x}{l}\cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t}$$

The midpoint of the rod is $x = \frac{l}{2}$.

When $x = \frac{l}{2}$, the temperature is

$$u\left(\frac{l}{2}, t\right) = -\frac{50}{l}\cdot\frac{l}{2} + 90 - \frac{200}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{n}\sin\frac{n\pi}{2}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

$$= -25 + 90 - \frac{200}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{n}\sin\frac{n\pi}{2}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$$

Since n is even, $\sin\frac{n\pi}{2} = 0$, $n = 2, 4, 6, \ldots$

$\therefore$ $u\left(\frac{l}{2}, t\right) = 65$, which is constant.

Hence, the temperature is unaltered at all times at the mid-point of the rod.

EXERCISE 20.3

1. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \cdot \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$ and $u(x, 0) = x$.
2. Solve $\frac{\partial \theta}{\partial t} = \alpha^2 \cdot \frac{\partial^2 \theta}{\partial x^2}$ given that (i) θ is finite as $t \to \infty$.
   (ii) $\theta = 0$ when $x = 0$ and $x = \pi$ for all values of t.
   (iii) $\theta = x$ from $x = 0$ to $x = \pi$ when $t = 0$.
   [**Hint**: (i) $\theta(x, t) = A(\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ (ii) $\theta(0, t) = 0, \theta(\pi, t) = 0\ \forall\ t \geq 0$
   (iii) $\theta(x, 0) = x, 0 < x < \pi$]
3. A rod of length l is heated so that its ends A and B are kept at 0°C. If initially the temperature is given by $u = \frac{cx(l-x)}{l^2}$, find the temperature at time t.
4. A uniform bar of length 10 cm through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $3\sin\frac{\pi x}{5} + 2\sin\frac{2\pi x}{5}$, find the temperature at time t.
5. A rod of length 100 cm has its ends A and B are kept at 0°C and 100°C until steady state conditions prevail. If the temperature at B is reduced to 0°C and kept so while that of A is maintained find the temperature $u(x, t)$ at a distance x from A and at time t.
6. A homogeneous rod conducting material of length 100 cm has its ends kept at zero temperature and initially the temperature is $u(x, 0) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100 - x, & 50 \leq x \leq 100 \end{cases}$
   Find the temperature $u(x, t)$ at any time t.
7. A bar of 10 cm long has its ends A and B kept at 50°C and 100°C until steady state conditions prevail. The temperature at A is then suddenly raised to 90°C and at the same instant that at B is

reduced to 60°C and the temperatures are maintained thereafter. Find the temperature distribution of the bar at any time t and at a distance x from A.

8. Two ends A and B of a rod of length 20 cm have the temperatures at 30°C and 80°C respectively, until the steady-state conditions prevail. Then the temperatures at the ends A and B are changed to 40°C and 60°C respectively. Find $u(x, t)$.
9. A rod of length l has its ends A and B kept at 0°C and 100°C respectively until steady state conditions prevail. The temperatures of the ends are changed to 25°C and 75°C respectively. Find the temperature distribution in the rod at time t.
10. A bar of length 10 cm, has its ends A and B kept at 50°C and 100°C until steady state conditions prevail. The temperature at A is then suddenly raised to 90°C and that at B is lowered to 60°C and the temperatures are maintained thereafter. Find the subsequent temperature at any time t.
11. A uniform bar of length 10 cm through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $2\sin\frac{\pi x}{5}\cos\frac{2\pi x}{5}$, find the temperature distribution of the rod.
12. Find the temperature distribution of a homogenous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature at the centre of the bar is k and falls uniformly to 0 at the ends.
13. A rod of length 20 cm has its ends A and B kept at 30°C and 90°C respectively until steady state conditions prevail. If the temperature at each end is then suddenly reduced to 0°C and maintained so, find the temperature $u(x, t)$ at a distance x from A at time t.
14. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If the temperatures at A is suddenly raised to 20°C and that at B is reduced to 80°C, find the temperature at any time.

ANSWERS TO EXERCISE 20.3

1. $u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cdot e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}$

2. $\theta(x,t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\cdot e^{-n^2\alpha^2 t}$

3. $u(x,t) = \frac{8c}{\pi^3}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\sin\frac{n\pi x}{l}\cdot e^{\frac{-n^2\pi^2\alpha^2}{l^2}t}$

4. $u(x,t) = 3\sin\frac{\pi x}{5}\cdot e^{\frac{-4\pi^2\alpha^2}{100}t} + 2\sin\frac{2\pi x}{5}\cdot e^{\frac{-16\pi^2\alpha^2 t}{100}}$

5. $u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{100}\cdot e^{\frac{-n^2\pi^2\alpha^2}{100^2}t}$

6. $u(x,t) = \frac{400}{\pi^2}\sum_{n=1,3,5,\ldots}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin\frac{n\pi x}{100}\cdot e^{\frac{-n^2\pi^2\alpha^2}{100^2}t}$

7. $u(x,t) = 90 - 3x - \frac{160}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{n}\sin\frac{n\pi x}{10}\cdot e^{\frac{-n^2\pi^2\alpha^2}{100}t}$

8. $u(x,t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{[1+2(-1)^n]}{n}\sin\frac{n\pi}{20}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{400}t}$

9. $u(x,t)=\frac{50}{l}+25+\frac{50}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[3-(-1)^n]\sin\frac{n\pi x}{l}\cdot e^{\frac{-n^2\pi^2\alpha^2 t}{l^2}}$

10. $u(x,t)=-3x+90-\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}\cdot e^{\frac{-\alpha^2 n^2\pi^2 t}{25}}$

11. $u(x,t)=-\sin\frac{\pi x}{5}\cdot e^{\frac{-2^2\pi^2\alpha^2 t}{100}}+\sin\frac{3\pi x}{5}\cdot e^{\frac{-6\pi^2\alpha^2 t}{100}}$

12. $u(x,t)=\frac{8k}{\pi^2}\sum_{n=1,3,5,\dots}\frac{(-1)^{\frac{n-1}{2}}}{n^2}\sin nx\cdot e^{-\alpha^2 n^2 t}$

13. $u(x,t)=\frac{60}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-3(-1)^n}{n}\right]\sin\frac{n\pi x}{20}\cdot e^{\frac{-\alpha^2 n^2\pi^2 t}{400}}$

14. $u(x,t)=\frac{60}{l}x+20-\frac{80}{\pi}\sum_{n=2,4,6,\dots}\frac{1}{n}\sin\frac{n\pi x}{l}\cdot e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$

TYPE 4. Bars with both ends thermally insulated

When both the ends of the bar are insulated, no heat can flow through the ends. So, the corresponding boundary conditions are $\frac{\partial u}{\partial x}(0,t)=0$ and $\frac{\partial u}{\partial x}(l,t)=0$ for all t.

Note $\frac{\partial u}{\partial x}(0,t)$ is the temperature gradient at $x=0$ and $\frac{\partial u}{\partial x}(l,t)$ is the temperature gradient at $x=l$.

WORKED EXAMPLES

EXAMPLE 1

Find the solution of the one-dimensional diffusion equation satisfying the boundary conditions

(i) u is bounded as $t\to\infty$, (ii) $\left(\frac{\partial u}{\partial x}\right)_{x=0}=0$ and (iii) $\left(\frac{\partial u}{\partial x}\right)_{x=a}=0\ \forall\ t$,

(iv) $u(x,0)=x(a-x),\ 0<x<a$.

Solution.

One dimensional heat equation is

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Since u is finite as $t\to\infty$, the suitable solution is

$$u(x,t)=(A\cos\lambda x+B\sin\lambda x)\cdot e^{-\alpha^2\lambda^2 t} \tag{2}$$

Differentiating partially w. r. to x, we get

$$\frac{\partial u}{\partial x}=(-A\lambda\sin\lambda x+B\lambda\cos\lambda x)\cdot e^{-\alpha^2\lambda^2 t}$$

Using condition (ii), i.e., when $x = 0, \dfrac{\partial u}{\partial x} = 0$, we get

$$(-A\lambda \sin 0 + B\lambda \cos 0)e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad B\lambda e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad B = 0 \qquad [\because \lambda e^{-\alpha^2\lambda^2 t} \neq 0]$$

Using condition (iii), i.e., when $x = a, \dfrac{\partial u}{\partial x} = 0$, we get

$$(-A\lambda \sin \lambda a + B\lambda \cos \lambda a)e^{-\alpha^2\lambda^2 t} = 0 \quad \Rightarrow \quad -A\lambda \sin \lambda a \cdot e^{-\alpha^2\lambda^2 t} = 0$$

But $A \neq 0$ [for, since $B = 0$, if A is also 0, then $u(x, t) = 0$, which is trivial]

$$\therefore \qquad \sin \lambda a = 0 \quad \Rightarrow \quad \lambda a = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{a}, \qquad n = 0, 1, 2, 3, \ldots$$

$$\therefore \qquad u(x,t) = A\cos\frac{n\pi}{a}x \cdot e^{-\alpha^2\frac{n^2\pi^2}{a^2}t}, \qquad n = 0, 1, 2, 3, \ldots$$

[Here $n = 0$ is also possible, where as in the earlier types $n = 0$ is not possible as $\sin\dfrac{n\pi x}{l}$ is a factor]

Before using the non zero condition, we write the general solution.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} e^{-\alpha^2\frac{n^2\pi^2 t}{a^2}} \qquad (3)$$

Using condition (iv), i.e., when $t = 0, u = x(a - x), 0 < x < a$ in (3), we get

$$\therefore \qquad u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} \cdot e^{0}$$

$$\Rightarrow \qquad u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a} \qquad (4)$$

Since $u(x, 0) = x(a - x)$, is algebraic, to find A_0, A_n we express it as a Fourier cosine series.

$$\therefore \qquad f(x) = x(a-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \qquad (5)$$

where $\qquad a_0 = \dfrac{2}{a}\displaystyle\int_0^a f(x)\,dx \qquad$ and $\qquad a_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$

Comparing (4) and (5), $A_0 = \dfrac{a_0}{2}$ and $A_n = a_n \ \forall\ n \geq 1$

$$a_0 = \frac{2}{a}\int_0^a (ax - x^2)\,dx = \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{2}{a}\left[\frac{a^3}{2} - \frac{a^3}{3}\right] = \frac{a^2}{3} \quad \Rightarrow \quad A_0 = \frac{a^2}{6}$$

$$a_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[(ax - x^2)\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}} - (a - 2x)\left(\frac{-\cos\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\frac{a}{n\pi}(ax - x^2)\sin\frac{n\pi x}{a} + \frac{a^2}{n^2\pi^2}(a - 2x)\cos\frac{n\pi x}{a} + \frac{2a^3}{n^3\pi^3}\sin\frac{n\pi x}{a}\right]_0^a$$

$$= \frac{2}{a}\left[0 + \frac{a^2}{n^2\pi^2}(-a)\cos n\pi x + 0 - \left(0 + \frac{a^3}{n^2\pi^2}\cdot\cos 0\right)\right]$$

$$= \frac{2}{a}\left[\frac{-a^3}{n^2\pi^2}\cdot\cos n\pi - \frac{a^3}{n^2\pi^2}\right]$$

$$\Rightarrow \qquad a_n = -\frac{2a^2}{n^2\pi^2}[\cos n\pi + 1] = -\frac{2a^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

If n is odd, then $(-1)^n = -1$. $\quad \therefore \quad a_n = 0$

If n is even, then $(-1)^n = 1$. $\quad \therefore \quad a_n = -\frac{2a^2}{n^2\pi^2}\cdot 2 = -\frac{4a^2}{n^2\pi^2}$, $n = 2, 4, 6, \ldots$

Since $A_n = a_n$, $\quad n \ge 1$, $\quad A_n = -\frac{4a^2}{n^2\pi^2}$, $\quad n = 2, 4, 6, \ldots$

Substituting in (3), we get

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{a^2}t}$$

$$\Rightarrow \qquad u(x,t) = \frac{a^2}{6} + \sum_{n=2,4,6,\ldots} -\frac{4a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{a^2}t} = \frac{a^2}{6} - \frac{4a^2}{\pi^2}\sum_{n=2,4,6,\ldots}\frac{1}{n^2}\cos\frac{n\pi x}{a}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{a^2}t}.$$

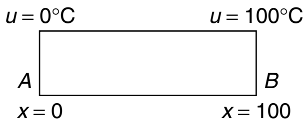
EXAMPLE 2

A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are the suddenly insulated and kept so. Find the temperature distribution.

Solution.

The temperature distribution is given by

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$



In steady state the temperature distribution of the bar is

$$u(x) = \frac{100 - 0}{100}x + 0$$

$$= x, \quad 0 \le x \le 100$$

Suddenly, the ends are insulated so the temperature distribution changes to unsteady state, which is given by (1).

The temperature is $u(x,t) = (A\cos\lambda x + B\sin\lambda x)\cdot e^{-\alpha^2\lambda^2 t}$ (2)

Since the ends are insulated, the boundary conditions are

(i) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$, (ii) $\left(\frac{\partial u}{\partial x}\right)_{x=100} = 0$ for $t \geq 0$

and (iii) the initial distribution is $u(x, 0) = x$, $0 < x < 100$

Differentiating (2) w. r. to x, we get

$$\frac{\partial u}{\partial x} = (-A\lambda\sin\lambda x + B\lambda\cos\lambda x)\, e^{-\alpha^2\lambda^2 t}$$

Using condition (i), i.e., when $x = 0$, $\frac{\partial u}{\partial x} = 0$, we get

$$(-A\lambda\sin 0 + B\lambda\cos 0)e^{-\alpha^2\lambda^2 t} = 0 \;\Rightarrow\; B\lambda e^{-\alpha^2\lambda^2 t} = 0 \;\Rightarrow\; B = 0 \qquad [\because \lambda e^{-\alpha^2\lambda^2 t} \neq 0]$$

$$\therefore \quad \frac{\partial u}{\partial x} = -A\lambda\sin\lambda x\cdot e^{-\alpha^2\lambda^2 t}$$

Using condition (ii), i.e., when $x = 100$, $\frac{\partial u}{\partial x} = 0$, we get $-A\lambda\sin 100\lambda\cdot e^{-\alpha^2\lambda^2 t} = 0$

But $A \neq 0$. $\therefore$ $\sin 100\lambda = 0 \;\Rightarrow\; 100\lambda = n\pi \;\Rightarrow\; \lambda = \frac{n\pi}{100}$, $n = 0, 1, 2, 3, \ldots$

$\therefore$ (2) is $u(x,t) = A\cos\left(\frac{n\pi x}{100}\right)e^{-\frac{\alpha^2 n^2\pi^2}{100^2}t}$, $n = 0, 1, 2, 3, \ldots$

$\therefore$ the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{100}\cdot e^{-\alpha^2\frac{n^2\pi^2 t}{100^2}} \qquad (3)$$

Using condition (iii), i.e., when $t = 0$, $u(x, 0) = x$, we get

$$u(x,0) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{100}\cdot e^0 = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{100}$$

$$\Rightarrow \quad x = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{100} \qquad (4)$$

Since $u(x, 0) = x$, is algebraic, to find A_0 and A_n, we express $f(x) = x$ as a Fourier cosine series.

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{100} \qquad (5)$$

Comparing (4) and (5), we find $A_0 = \frac{a_0}{2}$ and $A_n = a_n$, $n \geq 1$

where $$a_0 = \frac{2}{100}\int_0^{100} f(x)\,dx \text{ and } a_n = \frac{2}{100}\int_0^{100} f(x)\cos\frac{n\pi x}{100}\,dx$$

$\therefore \qquad a_0 = \frac{1}{50}\int_0^{100} x\,dx = \frac{1}{50}\left[\frac{x^2}{2}\right]_0^{100} = \frac{1}{100}\cdot 100^2 = 100$

$\therefore \qquad A_0 = \frac{100}{2} = 50$

and $\qquad a_n = \frac{1}{50}\int_0^{100} x\cos\frac{n\pi x}{100}\,dx$

$$= \frac{1}{50}\left[x\cdot\frac{\sin\frac{n\pi x}{100}}{\frac{n\pi}{100}} - 1\left(\frac{-\cos\frac{n\pi x}{100}}{\frac{n^2\pi^2}{100^2}}\right)\right]_0^{100}$$

$$= \frac{1}{50}\left[\frac{100}{n\pi}\cdot x\sin\frac{n\pi x}{100} + \frac{100^2}{n^2\pi^2}\cos\frac{n\pi x}{100}\right]_0^{100}$$

$$= \frac{1}{50}\left[\frac{100}{n\pi}\cdot 100\cdot\sin n\pi + \frac{100^2}{n^2\pi^2}\cos n\pi - \left(0 + \frac{100^2}{n^2\pi^2}\right)\right]$$

$$= \frac{1}{50}\left[\frac{100^2}{n^2\pi^2}\cos n\pi - \frac{100^2}{n^2\pi^2}\right] = \frac{1}{50}\cdot\frac{100^2}{n^2\pi^2}[\cos n\pi - 1]$$

$\Rightarrow \qquad a_n = \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right],\ n = 1, 2, 3, \ldots$

If n is even, then $(-1)^n = 1$. $\qquad \therefore\ a_n = 0$

If n is odd, then $(-1)^n = -1$. $\qquad \therefore\ a_n = \frac{200}{n^2\pi^2}(-2) = -\frac{400}{n^2\pi^2},\ n = 1, 3, 5, \ldots$

$\therefore \qquad A_n = -\frac{400}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$

Now (3) is $\qquad u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{100}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{100^2}t}$

$\Rightarrow \qquad u(x,t) = 50 + \sum_{n=1,3,5,\ldots}\frac{-400}{n^2\pi^2}\cos\frac{n\pi x}{100}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{100^2}t}$

$\Rightarrow \qquad u(x,t) = 50 + \frac{-400}{\pi^2}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}\cos\frac{n\pi x}{100}\cdot e^{\frac{-\alpha^2 n^2\pi^2}{100^2}t}.$

EXERCISE 20.4

1. The temperature at one end of a bar, 50 cm long and with insulated sides, is kept at 0°C and the other end is kept at 100°C until steady state conditions prevail. The two ends are then suddenly insulated, so that temperature gradient is zero at each end thereafter. Find the temperature distribution.

2. An insulated metal rod of length 100 cm has one end A kept at 0°C and the other end B at 100°C until steady state condition prevail. At time $t = 0$, the end B is suddenly insulated while the temperature at A is maintained at 0°C. Find the temperature at any point of the rod and at any time. [**Hint:** It is insulated at one end. The boundary conditions are (i) $u(0, t) = 0$ and (ii) $\dfrac{\partial u}{\partial t}(100, t) = 0, \forall\, t \geq 0$

Initial condition is (iii) $u(x, 0) = x$, $0 < x < 100$]

ANSWERS TO EXERCISE 20.4

1. $u(x,t) = 50 - \dfrac{400}{\pi^2} \displaystyle\sum_{n=1,3,5,\ldots} \dfrac{1}{n^2} \cos\dfrac{n\pi x}{50} \cdot e^{\frac{-\alpha^2 n^2 \pi^2}{2500^2} t}$

2. $u(x,t) = \dfrac{800}{\pi^2} \displaystyle\sum_{n=1,3,5,\ldots} \dfrac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\dfrac{n\pi x}{200} e^{-n^2 \frac{\alpha^2 \pi^2}{100^2} t}$

20.3 TWO DIMENSIONAL HEAT EQUATION IN STEADY STATE

Consider the flow of heat in a metal plate of uniform thickness h, density ρ, specific heat c and thermal conductivity k. Then the temperature distribution in the plate is given by

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $\alpha^2 = \dfrac{k}{\rho c}$ is called diffusivity of the material of the plate.

It is called the two dimensional heat equation because there are two space variables x, y.

In steady state, u is independent of t and so $\dfrac{\partial u}{\partial t} = 0$

∴ the two dimensional heat equation in the steady state is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

This is called the **Laplace's equation in two dimension**.

The solution $u(x, y)$ of the Laplace's equation (1) in a rectangular region can be obtained by the method of separation of variables.

A rectangular thin plate, with its two faces insulated is considered so that the heat flow is purely two-dimensional. The boundary conditions are prescribed on the four edges of the plate. The steady state heat flow in such a plate is obtained by solving Laplace's equation in two-dimension.

20.3.1 Solution of Two Dimensional Heat Equation

Two-dimensional steady state heat equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Since u is function of x and y, let the solution of (1) be

$$u(x, y) = X(x)Y(y) \quad (2)$$

$$\therefore \quad \frac{\partial u}{\partial x} = X'Y \text{ and } \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial u}{\partial y} = XY' \text{ and } \frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting in (1), we get $X''Y + XY'' = 0 \Rightarrow X''Y = -XY'' \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y}$

LHS is a function of x alone and R.H.S is a function of y alone and x and y are independent variables.
$\therefore$ the above equation is true if each side is a constant k

$$\therefore \quad \frac{X''}{X} = -\frac{Y''}{Y} = k \Rightarrow \frac{X''}{X} = k \quad \text{and} \quad \frac{-Y''}{Y} = k$$

$$\Rightarrow \quad X'' - kX = 0 \quad (3) \quad \text{and} \quad Y'' + kY = 0 \quad (4)$$

Case (i): Let $k > 0$, say $k = \lambda^2$, $\lambda \neq 0$ $\therefore$ $X'' - \lambda^2 X = 0$

Auxiliary equation is $m^2 - \lambda^2 = 0 \Rightarrow m = \pm\lambda$

$\therefore$ $X = Ae^{\lambda x} + Be^{-\lambda x}$

and $Y'' + \lambda^2 Y = 0$

Auxiliary equation is $m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$

$\therefore$ $Y = C\cos\lambda y + D\sin\lambda y$

$\therefore$ the solution is $u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$ (I)

where A, B, C, D are constants.

Case (ii): Let $k < 0$, say $k = -\lambda^2$, $\lambda \neq 0$ $\therefore$ (3) $\Rightarrow$ $X'' + \lambda^2 X = 0$

Auxiliary equation is $m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$

$\therefore$ $X = A\cos\lambda x + B\sin\lambda x$

(4) $\Rightarrow$ $Y'' - \lambda^2 y = 0$

Auxiliary equation is $m^2 - \lambda^2 = 0 \Rightarrow m = \pm\lambda$

$\therefore$ $Y = Ce^{\lambda y} + De^{-\lambda y}$

$\therefore$ $u(x, y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ (II)

where A, B, C, D are constants.

Case (iii): Let $k = 0$ $\therefore$ $X'' = 0$ and $Y'' = 0$

$\Rightarrow$ $X = Ax + B$ and $Y = Cy + D$

$\therefore$ $u(x, y) = (Ax + B)(Cy + D)$ (III)

where A, B, C, D are constants.

Proper choice of solution:
Of the three solutions (I), (II), (III), we have to choose that solution which is consistent with the nature of the problem and given boundary-value conditions.

We consider rectangles or squares whose sides are parallel to the coordinate axes.

If three of the boundary values are zero and the fourth one is non-zero, then (I) or (II) is a suitable solution.

WORKED EXAMPLES

TYPE 1. Finite plates with only one non-zero boundary condition

EXAMPLE 1

A square plate is bounded by the lines $x = 0, y = 0, x = 20$, and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x), 0 < x < 20$, while other three edges are kept at 0°C. Find the steady state temperature in the plate.

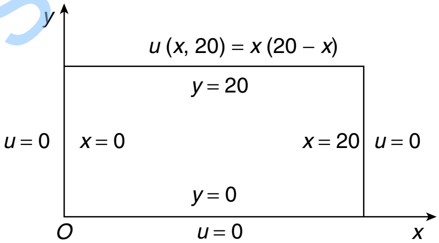
Solution.
The steady state temperature in the plate is given by the two dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The boundary-conditions are

$$\left.\begin{array}{ll}\text{(i)} & u(0, y) = 0 \\ \text{(ii)} & u(20, y) = 0\end{array}\right\} 0 \le y \le 20$$

$$\left.\begin{array}{ll}\text{(iii)} & u(x, 0) = 0 \\ \text{(iv)} & u(x, 20) = x(20 - x)\end{array}\right\} 0 \le x \le 20,$$



Since $u(x, 20) \neq 0$, the appropriate solution of (1) is the solution involving trigonometric function in x.

$$\therefore \quad u(x, y) = (A \cos \lambda x + B \sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \tag{1}$$

Using condition (i), i.e., when $x = 0, u = 0$, in (1), we get

$$(A \cos 0 + B \sin 0)(Ce^{\lambda y} + De^{-\lambda y}) = 0 \quad \Rightarrow \quad A(Ce^{\lambda y} + De^{-\lambda y}) = 0 \quad \Rightarrow \quad A = 0$$

$$\therefore \quad u(x, y) = B \sin \lambda x (Ce^{\lambda y} + De^{-\lambda y}) \tag{2}$$

Using condition (ii), i.e., when $x = 20, u = 0$, in (2), we get

$$\Rightarrow \quad B \sin 20\lambda (Ce^{\lambda y} + De^{-\lambda y}) = 0 \quad \Rightarrow \quad \sin 20\lambda = 0 \qquad [\because B \neq 0, Ce^{\lambda y} + De^{-\lambda y} \neq 0]$$

$$\Rightarrow \quad 20\lambda = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{20}, n = 1, 2, 3, \ldots$$

Using condition (iii), i.e., when $y = 0, u = 0$, in (2), we get

$$B \sin \lambda x \cdot (C + D) = 0 \quad \Rightarrow \quad C + D = 0 \quad \Rightarrow \quad D = -C \qquad [\because B \sin \lambda x \neq 0]$$

$$\therefore \quad u(x, y) = B \sin \lambda x \cdot (Ce^{\lambda y} - Ce^{-\lambda y})$$

$$= BC \sin \lambda x \cdot (e^{\lambda y} - e^{-\lambda y})$$
$$= BC \sin \lambda x \cdot 2 \sinh \lambda y$$
$$= (2BC) \sin \lambda x \cdot \sinh \lambda y = (2BC) \sin \frac{n\pi x}{20} \cdot \sinh \frac{n\pi y}{20}, \; n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{20} \cdot \sinh \frac{n\pi y}{20} \qquad (3)$$

Using condition (iv), i.e., when $y = 20$, $u = x(20 - x) = f(x)$, say, we get

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{20} \cdot \sinh n\pi \quad \Rightarrow f(x) = \sum_{n=1}^{\infty} A_n \cdot \sinh n\pi \cdot \sin \frac{n\pi x}{20} \qquad (4)$$

Since $f(x)$ is an algebraic expression, to find A_n, we express $f(x)$ as a Fourier sine series

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \qquad (5)$$

where $$b_n = \frac{2}{20} \int_0^{20} f(x) \sin \frac{n\pi x}{20} \, dx$$

Comparing (4) and (5), we get $A_n \sinh n\pi = b_n \; \forall \; n \geq 1$

$$\Rightarrow \qquad A_n = \frac{b_n}{\sinh n\pi}, \quad n \geq 1$$

Now $$b_n = \frac{1}{10} \int_0^{20} x(20 - x) \sin \frac{n\pi x}{20} \, dx$$

$$= \frac{1}{10} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20} \, dx$$

$$= \frac{1}{10} \left[(20x - x^2) \left(\frac{-\cos \frac{n\pi x}{20}}{\frac{n\pi}{20}} \right) - (20 - 2x) \left(\frac{-\sin \frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{20}}{\frac{n^3\pi^3}{20^3}} \right) \right]_0^{20}$$

$$= \frac{1}{10} \left[\frac{-20}{n\pi} (20x - x^2) \cdot \cos \frac{n\pi x}{20} + \frac{20^2}{n^2\pi^2} (20 - 2x) \sin \frac{n\pi x}{20} - 2 \cdot \frac{20^3}{n^3\pi^3} \cos \frac{n\pi x}{20} \right]_0^{20}$$

$$= \frac{1}{10} \left[\frac{-20}{n\pi} \cdot 0 + \frac{20^2}{n^2\pi^2} (-20) \sin n\pi - 2 \cdot \frac{20^3}{n^3\pi^3} \cos n\pi - \left(0 - 2 \cdot \frac{20^3}{n^3\pi^3} \cos 0 \right) \right]$$

$$= \frac{1}{10} \left[-2 \cdot \frac{20^3}{n^3\pi^3} \cos n\pi + 2 \cdot \frac{20^3}{n^3\pi^3} \right]$$

$$\Rightarrow \qquad = \frac{1}{10} \cdot 2 \cdot \frac{20^3}{n^3\pi^3} [1 - \cos n\pi] = \frac{4 \cdot 20^2}{n^3\pi^3} [1 - (-1)^n]$$

If n is even, then $(-1)^n = 1$. $\quad \therefore \; b_n = 0$

If n is odd, then $(-1)^n = -1$. $\quad \therefore \; b_n = \frac{4 \cdot 20^2}{n^3\pi^3}[2] = 8 \cdot \frac{20^2}{n^3\pi^3}, \quad n = 1, 3, 5, \ldots$

But $\quad A_n = \frac{b_n}{\sinh n\pi} = \frac{1}{\sinh n\pi} \cdot 8 \frac{20^2}{n^3\pi^3}, \; = \frac{3200}{\pi^3} \cdot \frac{1}{n^3 \sinh n\pi}, \quad n = 1, 3, 5, \ldots$

Substituting in (3), we get

$$u(x, t) = \sum_{n=1,3,5,\ldots} \frac{3200}{\pi^3 \cdot n^3 \sinh n\pi} \sin \frac{n\pi x}{20} \cdot \sinh \frac{n\pi y}{20}$$

$$= \frac{3200}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{\sin \frac{n\pi x}{20} \cdot \sinh \frac{n\pi y}{20}}{n^3 \sinh n\pi}$$

TYPE 2. Finite plate with two non-zero boundary conditions

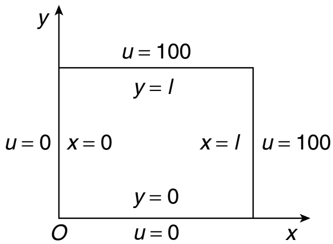
EXAMPLE 2

Find the steady state temperature at any point of a square plate if two adjacent edges are kept at 0°C and the others at 100°C.

Solution.
The steady state temperature at any point of the plate is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Let the side of the square be l
Then the boundary-conditions are



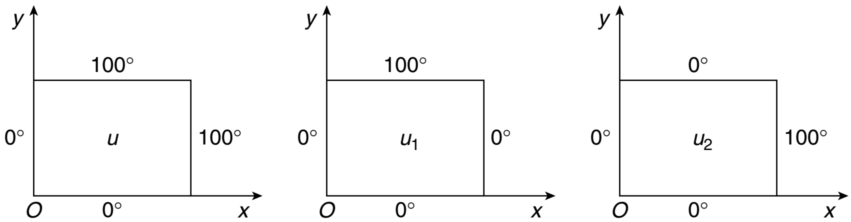
$$\left.\begin{array}{l}\text{(i) } u(x, 0) = 0 \\ \text{(ii) } u(x, l) = 100\end{array}\right\} 0 < x < l \qquad \left.\begin{array}{l}\text{(iii) } u(0, y) = 0 \\ \text{(iv) } u(l, y) = 100\end{array}\right\} 0 < y < l,$$

Note that two boundary conditions are non-zero.
As two adjacent edges have non-zero temperature,
we split $u(x, y)$ into two parts $u_1(x, y)$ and $u_2(x, y)$,

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where $u_1(x, y)$ and $u_2(x, y)$
satisfy (1) and the following boundary conditions.

(i) $u_1(0, y) = 0$ $\qquad$ (v) $u_2(0, y) = 0$
(ii) $u_1(l, y) = 0$ $\qquad$ (vi) $u_2(x, 0) = 0$
(iii) $u_1(x, 0) = 0$ $\qquad$ (vii) $u_2(x, l) = 0$
(iv) $u_1(x, l) = 100$ $\qquad$ (viii) $u_2(l, y) = 100$



To find $u_1(x, y)$:

Since u_1 satisfies (1) and with one non-zero condition in x, the appropriate solution contains trigonometric function in x.

$$\therefore \quad u_1(x, y) = (A \cos \lambda x + B \sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \quad (2)$$

Using condition (i), i.e., when $x = 0$, $u_1 = 0$, in (2), we get

$$(A \cos 0 + B \sin 0)(Ce^{\lambda y} + De^{-\lambda y}) = 0 \Rightarrow A(Ce^{\lambda y} + De^{-\lambda y}) = 0 \Rightarrow A = 0$$

$$\therefore \quad u(x, y) = B \sin \lambda x(Ce^{\lambda y} + De^{-\lambda y}) \quad (3)$$

Using condition (ii), i.e., when $x = l$, $u_1 = 0$, in (3), we get

$$\Rightarrow \quad B \sin \lambda l(Ce^{\lambda y} + De^{-\lambda y}) = 0 \Rightarrow \sin \lambda l = 0 \quad [\because B \neq 0,\ Ce^{\lambda y} + De^{-\lambda y} \neq 0]$$

$$\Rightarrow \quad \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots$$

Using condition (iii), i.e., when $y = 0$, $u_1 = 0$, in (3), we get

$$B \sin \lambda x \cdot (C + D) = 0 \Rightarrow C + D = 0 \Rightarrow D = -C$$

$$\therefore \quad u_1(x, y) = B \sin \lambda x(Ce^{\lambda y} - Ce^{-\lambda y})$$

$$= BC \sin \lambda x(e^{\lambda y} - e^{-\lambda y}) = BC \sin \lambda x \cdot 2 \sinh \lambda y$$

$$u_1(x, y) = (2BC) \sin\left(\frac{n\pi x}{l}\right) \cdot \sinh\left(\frac{n\pi y}{l}\right), \quad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \sinh\left(\frac{n\pi y}{l}\right) \quad (4)$$

Using condition (iv), i.e., when $y = l$, $u_1 = 100$ in (4), we get

$$\therefore \quad u_1(x, l) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \sinh n\pi$$

$$\Rightarrow \quad 100 = \sum_{n=1}^{\infty} B_n \sinh n\pi \cdot \sin\left(\frac{n\pi x}{l}\right) \quad (5)$$

Let $f(x) = 100$. To find B_n, express $f(x) = 100$ as a Fourier sine series in $0 < x < l$.

$$\therefore \qquad 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (6)$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Comparing (5) and (6), we find $B_n \sin\text{h}\, n\pi = b_n \Rightarrow B_n = \dfrac{b_n}{\sinh n\pi}$

Now
$$b_n = \frac{2}{l}\int_0^l 100\sin\frac{n\pi x}{l}dx$$
$$= \frac{200}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l = -\frac{200}{n\pi}[\cos n\pi - \cos 0] = \frac{200}{n\pi}[1-(-1)^n]$$

If n is even, then $(-1)^n = 1$. $\quad \therefore \quad b_n = 0$

If n is odd, then $(-1)^n = -1$. $\quad \therefore \quad b_n = \dfrac{200}{n\pi}\cdot 2 = \dfrac{400}{n\pi}$

$$\therefore \qquad B_n = \frac{400}{n\pi\sinh n\pi},\ n = 1, 3, 5, \ldots$$

Substituting in (4), we get

$$u_1(x, y) = \sum_{n=1,3,5,\ldots} \frac{400}{n\pi\sinh n\pi}\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot\sinh\left(\frac{n\pi y}{l}\right)$$

We observe that in the boundary conditions of u_1 and u_2, the roles of x and y are interchanged.
$\therefore$ to obtain $u_2(x, y)$ interchange the roles of x and y in the R.H.S of $u_1(x, y)$.

$$u_2(x, y) = \sum_{n=1,3,5,\ldots} \frac{400}{n\pi\sinh n\pi}\cdot\sin\left(\frac{n\pi y}{l}\right)\cdot\sinh\left(\frac{n\pi x}{l}\right)$$

$$\therefore \qquad u(x, y) = u_1(x, y) + u_2(x, y)$$

$$\Rightarrow \qquad u(x, y) = \frac{400}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n\pi\sinh n\pi}\left\{\sin\left(\frac{n\pi x}{l}\right)\cdot\sinh\left(\frac{n\pi y}{l}\right) + \sin\left(\frac{n\pi y}{l}\right)\cdot\sinh\left(\frac{n\pi x}{l}\right)\right\}.$$

TYPE 3. Infinite plates with only one non-zero boundary condition

EXAMPLE 3

A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $y = 0$ is given by

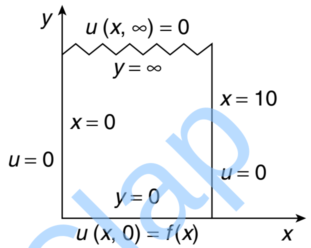
$$u = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10 - x), & 5 \le x \le 10 \end{cases}$$

and all the other three edges are kept at 0°C. Find the steady state temperature at any point in the plate.

Solution.
The steady state temperature $u(x,y)$ in a plate is given by the two-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

The boundary conditions in the problem are



(i) $u(0,y)=0$

(ii) $u(10,y)=0$

(iii) $u(x,\infty)=0$

(iv) $u(x,0)=f(x)=\begin{cases}20x\ , & 0\le x\le 5\\ 20(10-x), & 5\le x\le 10\end{cases}$

The short edge is $y=0$, the x-axis, so the long edge is parallel to y-axis.
Since $u(x,0)\neq 0$, the appropriate solution is the one with trigonometric function in x.

$$\therefore \qquad u(x,y)=(A\cos\lambda x+B\sin\lambda x)(Ce^{\lambda y}+De^{-\lambda y}) \tag{2}$$

Using condition (i), i.e., when $x=0$, $u=0$, in (2), we get

$$\therefore \qquad (A\cos 0+B\sin 0)(Ce^{\lambda y}+De^{-\lambda y})=0 \Rightarrow A(Ce^{\lambda y}+De^{-\lambda y})=0 \Rightarrow A=0$$

$$\therefore \qquad u(x,y)=B\sin\lambda x(Ce^{\lambda y}+De^{-\lambda y}) \tag{3}$$

Using condition (ii), i.e., when $x=10$, $u=0$, in (3), we get

$$B\sin 10\lambda(Ce^{\lambda y}+De^{-\lambda y})=0 \Rightarrow \sin 10\lambda=0 \qquad [\because B(Ce^{\lambda y}+De^{-\lambda y})\neq 0]$$

$$\Rightarrow \qquad 10\lambda=n\pi \Rightarrow \lambda=\frac{n\pi}{10},\ n=1,2,3,\ldots$$

Using condition (iii), i.e., $y\to\infty$, $u=0$, in (3), we get $C=0$.

For, if $C\neq 0$, then $e^{\lambda y}\to\infty$ as $y\to\infty$ $\quad\therefore\quad u\to\infty$, which contradicts the hypothesis.

$$\therefore \qquad u(x,y)=B\sin\lambda x\cdot De^{-\lambda y}=BD\sin\lambda x\cdot e^{-\lambda y}=BD\sin\frac{n\pi}{10}x\cdot e^{\frac{-n\pi}{10}y} \qquad n=1,2,3,\ldots$$

$\therefore$ the most general solution is

$$u(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{10}\cdot e^{\frac{-nx}{10}y} \tag{4}$$

Using condition (iv), i.e., when $y = 0,\ u = f(x)$, in (4), we get

$$\therefore \qquad u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10} \cdot e^0 \quad \Rightarrow \quad f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10} \qquad (5)$$

Since $f(x)$ is in algebraic form, to find A_n we express $f(x)$ as a Fourier sine series in $0 \le x \le 10$

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \qquad (6)$$

where $b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx = \frac{1}{5}\left\{\int_0^5 f(x)\sin\frac{n\pi x}{10}dx + \int_5^{10} f(x)\sin\frac{n\pi x}{10}dx\right\}$

$$= \frac{1}{5}\left\{\int_0^5 20x\cdot\sin\frac{n\pi x}{10}dx + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx\right\}$$

$$= \frac{1}{5}\cdot 20\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5\right.$$

$$\left. + \left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right]_5^{10}\right\}$$

$$= 4\left\{\left[-\frac{10}{n\pi}x\cdot\cos\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^5\right.$$

$$\left. + \left[-\frac{10}{n\pi}(10-x)\cos\frac{n\pi x}{10} - \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_5^{10}\right\}$$

$$= 4\left\{\left[-\frac{10}{n\pi}\cdot 5\cdot\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} - 0\right]\right.$$

$$\left. - \left[\frac{10}{n\pi}\cdot 0 + \frac{100}{n^2\pi^2}\sin n\pi - \left(\frac{10}{n\pi}\cdot 5\cdot\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\cdot\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$= 4\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = 4\left\{\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

If n is even, then $\quad \sin\frac{n\pi}{2} = 0. \qquad \therefore \quad b_n = 0$

If n is odd, then $\quad \sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}. \qquad \therefore \quad b_n = \frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}},\ n = 1, 3, 5, \ldots$

Comparing (5) and (6), we get $\quad A_n = b_n \quad \forall \; n \geq 1$

$$\therefore \qquad A_n = \frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}}, \; n = 1, 3, 5, \ldots$$

Substituting in (4), we get the solution

$$u(x,y) = \sum_{n=1,3,5,\ldots} \frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \sin\frac{n\pi x}{10} \cdot e^{\frac{-nxy}{10}} = \frac{800}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\frac{n\pi x}{10} \cdot e^{\frac{-n\pi y}{10}}$$

EXAMPLE 4

A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by

$$u = \begin{cases} 10y, & 0 \leq y \leq 10 \\ 10(20 - y), & 10 \leq y \leq 20 \end{cases}$$

and the two long-edges as well as the other short edge are kept at 0°C, find the steady state temperature distribution in the plate.

Solution.

The steady state temperature $u(x,y)$ in a plate is given by the two-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

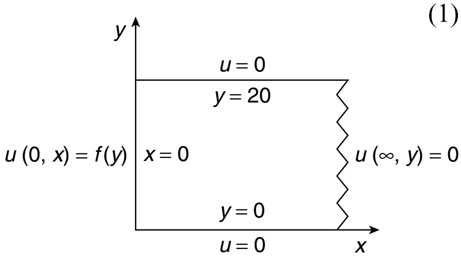
The boundary conditions in the problem are

(i) $u(x,0) = 0$

(ii) $u(x,20) = 0$

(iii) $u(\infty, y) = 0$

(iv) $u(0,y) = f(y) = \begin{cases} 10y & , \; 0 \leq y \leq 10 \\ 10(20-y), & 10 \leq y \leq 20 \end{cases}$



The short edge is $x = 0$, the y-axis and so, the long edge is parallel to the x-axis.
Since $u(0,y) \neq 0$, the appropriate solution to (1) is the one with trigonometric function in y.

$$\therefore \qquad u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y) \tag{2}$$

Using condition (i), i.e., when $y = 0$, $u = 0$, in (2), we get

$$(Ae^{\lambda x} + Be^{-\lambda x})(C\cos 0 + D\sin 0) = 0 \quad \Rightarrow \quad (Ae^{\lambda x} + Be^{-\lambda x})C = 0 \quad \Rightarrow \quad C = 0$$

$$\therefore \qquad u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})D\sin\lambda y \tag{3}$$

Using condition (ii), i.e., when $y = 20$, $u = 0$, in (3), we get

$$(Ae^{\lambda x} + Be^{-\lambda x})D\sin 20\lambda = 0 \quad \Rightarrow \quad \sin 20\lambda = 0 \qquad [\because \; D \neq 0, \; Ae^{\lambda x} + Be^{-\lambda x} \neq 0]$$

$$\Rightarrow \qquad 20\lambda = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{20}, \; n = 1, 2, 3, \ldots$$

Using condition (iii), i.e., when $x \to \infty,\ u \to 0$, in (3), we get

$A = 0$, for if $A \neq 0$, then $e^{\lambda x} \to \infty$ and so $u \to \infty$, which contradicts the hypothesis $u = 0$.

$$\therefore \qquad u(x,y) = Be^{-\lambda x} D \sin \lambda y = BDe^{-\lambda x} \sin \lambda y = BDe^{\frac{-n\pi x}{20}} \cdot \sin \frac{n\pi y}{20}, \quad n = 1, 2, 3, \ldots$$

$\therefore$ the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} A_n e^{\frac{-n\pi x}{20}} \cdot \sin \frac{n\pi y}{20} \qquad (4)$$

Using condition (iv), i.e., when $x = 0,\ u = f(y)$, in (4) we get

$$f(y) = \sum_{n=1}^{\infty} A_n e^0 \cdot \sin \frac{n\pi y}{20} \Rightarrow f(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{20} \qquad (5)$$

Since $f(y) = \begin{cases} 10y &, \ 0 \le y \le 10 \\ 10(20-y), & 10 \le y \le 20 \end{cases}$ is algebraic,

to find A_n, express $f(y)$ as a Fourier sine series in $0 \le y \le 20$.

$$\therefore \qquad f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{20} \qquad (6)$$

where $\qquad b_n = \frac{2}{20} \int_0^{20} f(y) \sin \frac{n\pi y}{20} dy$

Comparing (5) and (6), we get $A_n = b_n,\ n \ge 1$

$$\text{Now } b_n = \frac{1}{10} \left\{ \int_0^{10} 10y \sin \frac{n\pi y}{20} dy + \int_{10}^{20} 10(20-y) \sin \frac{n\pi y}{20} dy \right\}$$

$$= \frac{10}{10} \left\{ \left[y \cdot \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - 1 \cdot \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right) \right]_0^{10} + \left[(20-y) \left(\frac{-\cos \frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (-1) \left(\frac{-\sin \frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}} \right) \right]_{10}^{20} \right\}$$

$$= \left[-\frac{20}{n\pi} \cdot y \cdot \cos \frac{n\pi y}{20} + \frac{20^2}{n^2\pi^2} \sin \frac{n\pi y}{20} \right]_0^{10} - \left[\frac{20}{n\pi} (20-y) \cdot \cos \frac{n\pi y}{20} + \frac{20^2}{n^2\pi^2} \cdot \sin \frac{n\pi y}{20} \right]_{10}^{20}$$

$$= \frac{-20}{n\pi} 10 \cdot \cos \frac{n\pi}{2} + \frac{20^2}{n^2\pi^2} \sin \frac{n\pi}{2} - 0 - \left[0 + \frac{20^2}{n^2\pi^2} \sin n\pi - \left(\frac{20}{n\pi} \cdot 10 \cos \frac{n\pi}{2} + \frac{20^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \right]$$

$$= -\frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{200}{n\pi} \cos \frac{n\pi}{2} + \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$b_n = \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2}$$

If n is even, then $\sin\frac{n\pi}{2}=0$. $\therefore\quad b_n=0$

If n is odd, then $\sin\frac{n\pi}{2}=(-1)^{\frac{n-1}{2}}$. $\therefore\quad b_n=\frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}},\ n=1,3,5,\ldots$

$$\therefore\qquad A_n=\frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}},\quad n=1,3,5,\ldots$$

Substituting in (4), we get

$$u(x,y)=\sum_{n=1,3,5,\ldots}\frac{800}{n^2\pi^2}(-1)^{\frac{n-1}{2}}\cdot e^{-\frac{n\pi x}{20}}\cdot\sin\frac{n\pi y}{20}$$

$$\Rightarrow\qquad u(x,y)=\frac{800}{\pi^2}\sum_{n=1,3,5,\ldots}\frac{(-1)^{\frac{n-1}{2}}}{n^2}e^{-\frac{n\pi x}{20}}\cdot\sin\frac{n\pi y}{20}$$

EXAMPLE 5

A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x=0$ and $x=a$ and the short edge at infinity are kept at temperature 0°C, while the other short edge $y=0$ is kept at temperature $u_0\sin^3\frac{\pi x}{a}$, find the steady state temperature at any point (x,y) of the plate.

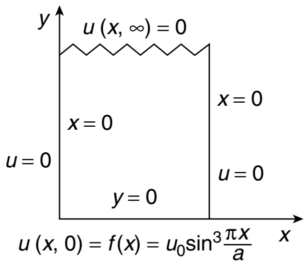
Solution.
The steady state temperature $u(x,y)$ at any point in the plate is given by the two dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

The boundary conditions are

(i) $u(0,y)=0$
(ii) $u(a,y)=0$ $\Big\}\ 0\le y<\infty$

(iii) $u(x,\infty)=0$
(iv) $u(x,0)=f(x)=u_0\sin^3\frac{\pi x}{a}$ $\Big\}\ 0\le x\le a,$



Since $u(x,0)\neq 0$, the appropriate solution of (1) is the one with trigonometric functions in x.

$$\therefore\qquad u(x,y)=(A\cos\lambda x+B\sin\lambda x)(Ce^{\lambda y}+De^{-\lambda y}) \tag{2}$$

Using condition (i), i.e., when $x=0,\ u=0$, in (2), we get

$$(A\cos 0+B\sin 0)(Ce^{\lambda y}+De^{-\lambda y})=0\quad\Rightarrow\quad A(Ce^{\lambda y}+De^{-\lambda y})=0\quad\Rightarrow\quad A=0$$

$$\therefore\qquad u(x,y)=B\sin\lambda x(Ce^{\lambda y}+De^{-\lambda y}) \tag{3}$$

Using condition (ii), i.e., when $x = a,\ u = 0$, in (3), we get

$$B\sin\lambda a(Ce^{\lambda y} + De^{-\lambda y}) = 0 \quad \Rightarrow \quad \sin\lambda a = 0, \qquad [\because B(Ce^{\lambda y} + De^{-\lambda y}) \neq 0]$$

$$\Rightarrow \qquad \lambda a = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{a},\ n = 1, 2, 3, \ldots$$

Using condition (iii), i.e., when $y \to \infty$, $u \to 0$, we find $C = 0$.

For if $C \neq 0$, as $y \to \infty$, $e^{\lambda y} \to \infty$ and so $u \to \infty$, which contradicts the hypothesis $u = 0$

$\therefore$ (3) is $\quad u(x,y) = B\sin\lambda x D\cdot e^{-\lambda y} = BD\sin\lambda x\cdot e^{-\lambda y} = BD\sin\dfrac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}},\quad n = 1, 2, 3, \ldots$

$\therefore$ the general solution is $\quad u(x,y) = \displaystyle\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} \qquad (4)$

Using condition (iv), i.e., when $y = 0$, $u = f(x) = u_0\sin^3\dfrac{\pi x}{a}$, we get

$$u_0\sin^3\frac{\pi x}{a} = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{a}\cdot e^0 = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{a}$$

$$\Rightarrow \qquad \frac{u_0}{4}\left[3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\right] = A_1\sin\frac{\pi x}{a} + A_2\sin\frac{2\pi x}{a} + A_3\sin\frac{3\pi x}{a} + \cdots$$

Equating like coefficients we get,

$$A_1 = \frac{3u_0}{4}, \qquad A_2 = 0, \qquad A_3 = -\frac{u_0}{4}, \qquad A_4 = 0 = A_5 = A_6 = \ldots$$

$$\therefore \quad u(x,y) = A_1\sin\frac{\pi x}{a}\cdot e^{-\frac{\pi}{a}y} + A_2\sin\frac{2\pi x}{a}\cdot e^{\frac{-2\pi y}{a}} + A_3\sin\frac{3\pi x}{a}\cdot e^{\frac{-3\pi y}{a}} + A_4\sin\frac{4\pi x}{a}\cdot e^{\frac{-4\pi y}{a}} + \cdots$$

$$\Rightarrow u(x,y) = \frac{3u_0}{4}\sin\frac{\pi x}{a}\cdot e^{\frac{-\pi y}{a}} - \frac{u_0}{4}\sin\frac{3\pi x}{a}\cdot e^{\frac{-3\pi y}{a}}$$

EXAMPLE 6

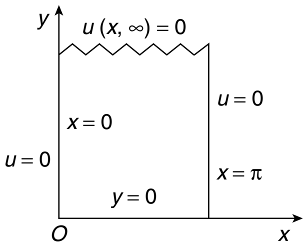
An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.

Solution.
The steady state temperature $u(x,y)$ at any point is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

The boundary conditions are

$$\left.\begin{array}{l}\text{(i) } u(0,y)=0\\ \text{(ii) } u(\pi,y)=0\end{array}\right\}\ 0\le y<\infty \qquad \left.\begin{array}{l}\text{(iii) } u(x,\infty)=0\\ \text{(iv) } u(x,0)=u_0\end{array}\right\}\ 0<x<\pi$$



Since $u(x,0)\neq 0$, the appropriate solution of (1) is the solution with trigonometric function of x.

$$\therefore \qquad u(x,y)=(A\cos\lambda x+B\sin\lambda x)(Ce^{\lambda y}+De^{-\lambda y}) \qquad (2)$$

Using condition (i), i.e., when $x=0,\ u=0$, in (2), we get

$$(A\cos 0+B\sin 0)(Ce^{\lambda y}+De^{-\lambda y})=0 \quad\Rightarrow\quad A(Ce^{\lambda y}+De^{-\lambda y})=0 \quad\Rightarrow\quad A=0$$

$$\therefore \qquad u(x,y)=B\sin\lambda x(Ce^{\lambda y}+De^{-\lambda y}) \qquad (3)$$

Using condition (ii), i.e., when $x=\pi,\ u=0$, in (3)

$$B\sin\pi\lambda(Ce^{\lambda y}+De^{-\lambda y})=0 \quad\Rightarrow\quad \sin\pi\lambda=0, \qquad [\because B(Ce^{\lambda y}+De^{-\lambda y})\neq 0]$$

$$\Rightarrow \qquad \pi\cdot\lambda=n\pi \quad\Rightarrow\quad \lambda=n;\ n=1,2,3,\ldots$$

Using condition (iii), i.e., when $y\to\infty,\ u=0$, in (3), we get $C=0$.

For if $C\neq 0$, then $u\to\infty$, as $y\to\infty$ which contradicts $u=0$

$$\therefore \qquad u(x,y)=B\sin\lambda x\cdot De^{-\lambda y}=BD\sin\lambda x\cdot e^{-\lambda y}=BD\sin nx\cdot e^{-ny}, \quad n=1,2,3,\ldots$$

$\therefore$ the most general solution is

$$u(x,y)=\sum_{n=1}^{\infty}A_n\sin nx\cdot e^{-ny} \qquad (4)$$

Using condition (iv), i.e., when $y=0,\ u=u_0$, in (4), we get

$$\Rightarrow \qquad u_0=\sum_{n=1}^{\infty}A_n\sin nx\cdot e^0=\sum_{n=1}^{\infty}A_n\sin nx \qquad (5)$$

To find A_n, we express u_0 as a Fourier sine series

$$\therefore \qquad u_0=\sum_{n=1}^{\infty}b_n\sin nx \qquad (6)$$

where
$$b_n=\frac{2}{\pi}\cdot\int_0^{\pi}u_0\sin nx\,dx=\frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$=\frac{-2u_0}{n\pi}[\cos n\pi-\cos 0]=-\frac{2u_0}{n\pi}[(-1)^n-1]$$

When n is even, $(-1)^n=1$, then $b_n=0$

When n is odd, $(-1)^n=-1$, then $b_n=\dfrac{4u_0}{n\pi}$, $n=1,3,5,\ldots$

Comparing (5) and (6), we find $A_n = b_n,\ n > 1 \quad \therefore \quad A_n = \frac{4u_0}{n\pi},\ n = 1, 3, 5, \ldots$

Substituting in (4), we have

$$u(x,y) = \sum_{n=1,3,5,\ldots} \frac{4u_0}{n\pi} \sin nx \cdot e^{-ny} = \frac{4u_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin nx \cdot e^{-ny}.$$

EXERCISE 20.5

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin \frac{n\pi x}{l}$.
2. The function $v(x,y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for the points in the rectangle $x = 0,\ x = a,\ y = b$, it satisfies the conditions $v(0,y) = v(a,y) = v(x,b) = 0$ and $v(x,0) = x(a-x),\ 0 < x < a$. Show that $v(x,y)$ is given by

$$v(x,y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \cdot \sin \frac{(2n+1)\pi x}{a} \cdot \frac{\sinh\left\{\frac{(2n+1)\pi}{a}(b-y)\right\}}{\sinh(2n+1)\frac{\pi b}{a}}.$$

3. A square plate of length 20 cm has its faces insulated and its edges along $x = 0,\ x = 20,\ y = 0,\ y = 20$. If the temperature along the edge $x = 20$ is given by $u = \begin{cases} \frac{T}{10} y, & 0 \le y \le 10 \\ \frac{T}{10}(20-y), & 10 \le y \le 20 \end{cases}$
While the other three edges are kept at 0°C, find the steady state temperature distribution in the plate.
4. A rectangular plate is bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b$ and the edge temperatures are $u(0,y) = 0,\ u(a,y) = 0,\ u(x,b) = 0,\ u(x,0) = 5\sin\frac{4\pi x}{a} + 3\sin\frac{3\pi x}{a}$.
Find the temperature distribution.
5. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions
$u(0,y) = u(\pi,y) = u(x,\pi) = 0$ and $u(x,0) = \sin^2 x$.
6. Find the steady-state temperature at any point of a rectangular plate of sides a and b insulated on the lateral surface and satisfying $u(0,y) = 0,\ u(a,y) = 0, u(x,b) = 0$ and $u(x0,) = x(a-x)$.
7. An infinitely long-plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth of this edge $x = 0$ is π, this end is maintained at temperature as $u = k(\pi y - y^2)$ at all points while the other edges are at zero temperature. Determine the temperature $u(x,y)$ at any point of the plate in the steady state if u satisfies the Laplace equation.

8. A rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0,y)=0$, $v(a,y)=0$, $v(x,\infty)=0$, and $v(x,0)=kx$.

   Show that the steady state temperature within the plate is $v(x,y)=\frac{2ak}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}e^{\frac{-n\pi y}{a}}\cdot\sin\frac{n\pi x}{a}$.

9. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x,0)=100\sin\frac{\pi x}{8}, 0<x<8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plate is given by $u(x,y)=100e^{\frac{-\pi y}{8}}\cdot\sin\frac{\pi x}{8}$.

10. An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature , while the other short edge $x = 0$ is kept at temperature given by $u=\begin{cases}20y & , 0\le y\le 5\\ 20(10-y), & 5\le y\le 10\end{cases}$.

    Find the steady state temperature distribution in the plate.

11. A rectangular plate is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$ and the edge temperature are $u(0, y) = 0$, $u(x, b) = 0$, $u(a, y) = 0$, $u(x, 0) = 5\sin\frac{5\pi x}{a} + 3\frac{\sin 3\pi x}{a}$. Find the steady state temperature distribution at any point of the plate.

12. Find the steady sate temperature distribution in a square plate of side π with the boundary conditions $u(0, y) = u(\pi, y) = u(x, \pi) = 0$; $u(x, \mathbf{0}) = \sin^2 x$.

13. A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = b$. Its surfaces are insulated and the temperatures, along two adjacent edges are kept at 100°C and the other two at 0°C. Find the steady state temperature at any point of the plate.

14. A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered as an infinite plate. If the temperature along short edge $y = 0$ is $u(x,0)=8\sin\frac{\pi x}{10}$, $0<x<10$, while two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at 0°C. Find the steady-state temperature at any point of the plate.

ANSWERS TO EXERCISE 20.5

1. $u(x,y)=\dfrac{\sin\frac{n\pi x}{l}\cdot\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}}$

2. $u(x,y)=\dfrac{8T}{\pi^2}\sum_{n=1,3,5,\ldots}\dfrac{(-1)^{\frac{n-1}{2}}}{n^2}\cdot\dfrac{\sinh\frac{n\pi x}{20}\cdot\sin\frac{n\pi y}{20}}{\sinh n\pi}$

4. $u(x,y)=5\sin\frac{4\pi x}{a}\sinh\frac{4\pi}{a}(b-y)\operatorname{cosec}\frac{4\pi b}{a}+3\sin\frac{3\pi x}{a}\cdot\sinh\frac{3\pi}{a}(b-y)\cdot\operatorname{cosec}\frac{3\pi b}{a}$

5. $u(x,y) = \frac{8}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin nx \cdot \sinh n(y-\pi)}{n(n^2-4)\sinh n\pi}$

6. $u(x,y) = \frac{-8a^2}{\pi^3} \sum \frac{1}{n^3 \sinh \frac{n\pi b}{a}} \cdot \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi}{a}(b-y)$

7. $u(x,y) = \frac{8k}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^3} \sin ny \cdot e^{-nx}$

10. $u(x,y) = \frac{800}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2}(-1)^{\frac{n-1}{2}} \cdot e^{\frac{-n\pi x}{10}} \cdot \sin \frac{n\pi y}{10}$

11. $u(x,y) = -\frac{3}{\sin\left(\frac{3\pi b}{a}\right)} \sin \frac{3\pi x}{a} \cdot \sinh \frac{3\pi}{a}(y-b) - \frac{5}{\sin\left(\frac{5\pi b}{a}\right)} \sin \frac{5\pi x}{a} \cdot \sinh \frac{5\pi}{a}(y-b)$

12. $u(x,y) = \frac{8}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin nx \cdot \sinh n(y-\pi)}{n(n^2-4)\sinh n\pi}$

13. $u(x,y) = \frac{400}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \left\{ \frac{\sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi y}{a}}{\sin \frac{n\pi b}{a}} + \frac{\sin \frac{n\pi y}{b} \cdot \sinh \frac{n\pi x}{b}}{\sin \frac{n\pi a}{b}} \right\}$

14. $u(x,y) = 8 \sin \frac{\pi x}{10} \cdot e^{\frac{-\pi y}{10}}$

SHORT ANSWER QUESTIONS

1. **In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, what does c^2 stand for?**
2. **What are all the solutions of one dimensional wave equation?**
3. **Write down the partial differential equation governing the transverse vibrations of an elastic string.**
4. **State the suitable solution to the one dimensional wave equation.**
5. **How many boundary-value conditions are required to solve the one-dimensional wave equation?**
6. **Write down the initial conditions when a taut string of length $2l$ is fastened on both ends. The midpoint of the string is taken to a height b and released from the rest in that position.**
7. **In the diffusion equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, what does α^2 stand for?**
8. **State the three possible solutions of the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.**
9. **In steady state conditions derive the solution of one dimensional heat flow equation.**

10. **What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation?**

11. **A rod 30 cm long has its ends A and B kept to 20°C and 80°C respectively untill steady state conditions prevail. Find the steady state temperature in the rod.**

12. **The ends A and B of a rod of length 10 cm long have their temperature kept 20°C and 70°C. Find the steady state temperature distribution on the rod.**

13. **An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively and the rod is in steady state condition. Find the temperature at any point in terms of its distance from one end.**

14. **State the initial and boundary conditions of one dimensional heat equation.**

15. **What is meant by two dimensional heat flow?**

16. **Write down the two dimensional heat flow equation in steady state.**
(OR)
Write down the Cartesian form of Laplace equation in two dimension.

17. **Write down the possible solutions of the two dimensional heat equation in steady state.**

18. **Write any two solutions of the Laplace equation $u_{xx} + u_{yy} = 0$ involving exponential terms in x or y.**

19. **An infinitely long plane uniform plate is bounded by two parallel edges and an edge at right angles to them. The breadth is p. This edge $u(x, 0)$ is maintained at a temperature 608C at all points and the other edges are at zero temperature. Formulate the boundary value problem to determine the steady state temperature.**

20. **Given the boundary conditions on a square plate how will you identify the proper solution?**

21. **An insulated rod of length 60 cm has its ends at A and B maintained at 20°C and 80°C respectively. Find the steady state solution of the rod.**

22. **A plate is bounded by the lines $x = 0$, $y = 0$, $x = l$ and $y = l$. Its faces are insulated. The edge coinciding with x-axis is kept at 100°C. The edge coinciding with y-axis is kept at 50°C. The other two edges are kept at 0°C. Write the boundary condition that are needed for solving two dimensional heat flow equation.**

OBJECTIVE TYPE QUESTIONS

A. Fill up the blanks

1. The one-dimensional wave equation is ___________.
2. The boundary value conditions for the transverse vibrations of a string of length l with fixed ends and initial displacement $y = f(x)$ is ___________.
3. The general solution of a string of a length l whose ends points are fixed and which stands from rest is __________.
4. The number of boundary conditions required to solve one-dimensional wave equation is __________.
5. In one-dimensional heat equation, $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, α^2 stands for _________.
6. A rod of length 60 cm has its ends A and B insulated and the temperature is maintained at 20°C and 80°C. Then the steady state temperature at any point of the rod is _________.

7. The variable separable solution of the one-dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ with separated expressions equal $-\lambda^2$ is ________.

8. An infinitely long metal plate is in the form of an area enclosed between the lines $y = 0$ and $y = 10$ for positive values of x. The temperature is zero along the edges $y = 0$ and $y = 10$ and the edge at infinity. If the edge $x = 0$ is kept at temperature $u = 4K \sin^3 \frac{\pi y}{10}$. Then the boundary value conditions for $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ are ________.

9. The suitable solution for the heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is ________.

10. For the boundary value problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with

$$\left.\begin{aligned}&\text{(i) } u(0, y) = 0\\ &\text{(ii) } u(10, y) = 0\end{aligned}\right\} 0 < y < \infty \qquad \left.\begin{aligned}&\text{(iii) } u(x, \infty) = 0\\ &\text{(iv) } u(x, 0) = \sin x\end{aligned}\right\} 0 < x < 10$$

the suitable solution is ________.

B. Choose the correct answer

1. The nature of the partial differential equation $x^2 u_{xx} + 2xy u_{xy} + (1 + y^2) u_{yy} - 2u_x = 0$ is
   (a) elliptic if $x \neq 0$ (b) hyperbolic if $x \neq 0$
2. The nature of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is
   (a) elliptic for all (x, t) (b) hyperbolic for all (x, t)
   (c) parabolic for all (x, t) (d) None of these
3. The suitable solution to one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is
   (a) $y = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda ct} + De^{-\lambda ct})$
   (b) $y = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct)$
   (c) $y = (Ax + B)(Cx + D)$
   (d) None of these
4. The minimum number of boundary value conditions required for unique solution of $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is
   (a) 2 (b) 3 (c) 4 (d) 5
5. A hot rod 40 cm long with insulated sides has its ends A and B kept at 20°C and 60°C. Then the steady state temperature at a point 15 cm from end A is
   (a) 25°C (b) 27.5°C (c) 28.5°C (d) None of these
6. A hot rod of length 20 cm where one end is kept at 30° and the other end is kept at 70° is maintained until steady state conditions prevail. Then the steady state temperature is given by
   (a) $u(x) = x + 30,\ 0 \le x \le 20$ (b) $u(x) = 3x + 30,\ 0 \le x \le 20$
   (c) $u(x) = 2x + 30,\ 0 \le x \le 20$ (d) None of these
7. An insulated rod of length 60 cm has its ends at A and B maintained at 20°C and 80°C, respectively. Then the steady state temperature of the rod in $0 \le x \le 60$ is
   (a) $u(x) = x + 30$ (b) $u(x) = x + 20$ (c) $u(x) = 3x + 20$ (d) None of these
8. The ends A and B of a hot rod 40 cm long have their temperature kept at 0°C and 80°C, respectively, until steady state conditions prevail. Then the steady state temperature of the rod in $0 \le x \le 40$ is
   (a) $u(x) = 2x$ (b) $u(x) = x + 2$ (c) $u(x) = x$ (d) $u(x) = 2x + 2$

9. The number of boundary conditions required to obtain unique solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is

(a) 2 (b) 3 (c) 4 (d) 5

10. Any solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is

(a) $u(x, y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ (b) $u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda y} + De^{-\lambda y})$

(c) $u(x, y) = (Ax + B)(Ce^{\lambda y} + De^{-\lambda y})$ (d) None of these

ANSWERS

A. Fill up the blanks

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

2. The boundary conditions of $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ are

(i) $y(0, t) = 0$ (ii) $y(l,t) = 0 \, \forall t \geq 0$ for fixed ends

(iii) $\frac{\partial y}{\partial t(x,0)} = 0$ (iv) $y(x,0) = f(x), 0 < x < l$

3. $y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}$

4. 4 boundary value conditions consisting of 2 initial conditions and 2 boundary conditions.

5. $\alpha^2 = \frac{k}{pc} = \frac{\text{thermal conductivity}}{\text{density} \times \text{specific heat}}$

6. $u(x) = x + 20$

7. $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\lambda^2\alpha^2 t}$

8. $\left.\begin{array}{l}\text{(i) } u(x,0) = 0 \\ \text{(ii) } u(x,10) = 0\end{array}\right\} 0 \leq x < \infty$ (iii) $u(\infty, y) = 0$

(iv) $u(0, y) = 4K\sin^3\frac{\pi y}{10}, 0 \leq y \leq 10$

9. $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ or $u(x,y) = (A\cos\lambda y + B\sin\lambda y)(Ce^{\lambda x} + De^{-\lambda x})$

10. $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$

B. Choose the correct answer

1. (a) 2. (b) 3. (b) 4. (c) 5. (b) 6. (c) 7. (b) 8. (a) 9. (c) 10. (a)

Unit - 5

Chapter 11

Reduction of Second Order Partial Differential Equation into Canonical Forms
(Non-linear Equations of Second Order)

11.1 Laplace Transformation (Canonical Forms)

[Avadh 2001, Delhi Maths (Hons.) 2004]

Consider the linear partial differential equation of second order of the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ...(1)$$

in which R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high order as necessary.

This equation can be reduced to a more simple form by a suitable change of the independent variables. The simpler forms which result in this way are called **canonical forms** of the equation (1).

Let the independent variables x and y be changed to u and v by means of the transformations

$$u = u(x, y) \text{ and } v = v(x, y) \qquad ...(2)$$

Now, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} \equiv \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)\right]\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}$$

$$+\left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)\right]\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\left(\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u^2}+\frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v\,\partial u}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}$$

$$+\left(\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v^2}\right)\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2\cdot\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\cdot\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

Similarly,

$$t=\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

and

$$s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x\,\partial y}$$

$$=\left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)\right]\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x\,\partial y}$$

$$+\left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)\right]\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x\,\partial y}$$

$$=\left(\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u^2}+\frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v\,\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\,\partial y}$$

$$+\left(\frac{\partial u}{\partial x}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial v}{\partial x}\frac{\partial^2 z}{\partial v^2}\right)\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x\,\partial y}$$

$$= \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\cdot\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x\,\partial y}$$

Putting these values of p, q, r, s and t in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u\,\partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \qquad ...(3)$$

where,
$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \qquad ...(4)$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} \qquad ...(5)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S.\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad ...(6)$$

and $F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is the function of $u, v, z, \frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Now, we determine u and v so that the equation (3) takes the simplest possible form.

The equation (3) reduces to the simplest integrable form when the discriminant. $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad ...(7)$$

is either positive, negative or zero everywhere.

The three cases are discussed separately as follows.

Case I : If $S^2 - 4RT > 0$ (*i.e.*, if equation (1) is hyperbolic)

In this case the roots λ_1, λ_2 (say) of the equation (7) are real and distinct. If we choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda_1\frac{\partial u}{\partial y} \qquad ...(8)$$

and
$$\frac{\partial v}{\partial x} = \lambda_2\frac{\partial v}{\partial y} \qquad ...(9)$$

then
$$A = (R\lambda_1^2 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$\because \quad R\lambda_1{}^2 + S\lambda_1 + T = 0$ as λ_1 is a root of equation (7).

Similarly, $C = 0$

Now equation (8), can be written as

$$\frac{\partial u}{\partial x} - \lambda_1 \frac{\partial u}{\partial y} = 0$$

which is Lagrange's form and its auxiliary equations, are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

Which gives $du = 0$ $\therefore$ $u = c_1$ (constant)

and $$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{...(10)}$$

Let $f_1(x, y) = c_2$ (constant) be the solution of equation (10).

$\therefore$ the solution of (8), can be taken as

$$u = f_1(x, y) \quad \text{...(11)}$$

which is a suitable choice for u.

Similarly, the solution of (9) *i.e.,* $f_2(x, y) = c_2$, can be taken as

$$v = f_2(x, y) \quad \text{...(12)}$$

is a suitable choice for v.

Now it can be shown that

$$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \quad \text{...(13)}$$

Since $A = 0$, $C = 0$ when u and v are chosen such that (8) and (9) are satisfied, therefore from (13), we get

$$B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

$\because$ $S^2 - 4RT > 0$, $\therefore$ $B^2 > 0$

$\because$ $A = 0 = C$, $\therefore$ equation (3) reduces to

$$2B\frac{\partial^2 z}{\partial u\, \partial v} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0$$

Since $B^2 > 0$, *i.e.,* $B \neq 0$, so we can divide this equation by B.

$\therefore$ dividing by $2B$, it reduces to

$$\frac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u,\ v,\ z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Hence, *if we make the substitutions of u and v defined by* (11) *and* (12), *the given equation* (1) *reduces to the form*

$$\frac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u,\ v,\ z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad ...(14)$$

which is the **canonical form of given equation** (1) and is simpler to solve, than the given equation (1).

Case II : If $S^2 - 4RT = 0$ (*i.e.,* if equation (1) is parabolic)

In this case the roots of equation (7) *i.e.,* $R\lambda^2 + S\lambda + T = 0$, are real and equal

i.e., $\lambda_2 = \lambda_1$

Here we choose u as in case I, such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \text{ which gives } u = f(x,\ y) \qquad ...(15)$$

Also we take v to be any function of x and y, which is independent of u.

$\therefore$ As in case I, $A = 0$. Also from (13), $B^2 = 0 \;\because\; S^2 = 4RT$.

i.e., $B = 0$

Here C cannot be zero, otherwise v would be a function of u.

Putting $A = 0$, $B = 0$ in (3) and dividing by $C \neq 0$, equation (3) becomes

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Hence, if we make the substitution of u as defined by (15) and v any function of x and y, the given equation (1) in this case reduces to the form

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad ...(16)$$

which is another **canonical form of given equation** (1) and is simpler to solve than the given equation.

Case III : If $S^2 - 4RT < 0$. (*i.e.,* if given equation (1) is elliptic)

In this case the roots of equation (7) *i.e.,* $R\lambda^2 + S\lambda + T = 0$, are complex conjugates.

Proceeding as in case I, here the equation (1) will reduce to the same canonical form [equation (14)] as in case I but here the variables u and v are not real but are infact the complex conjugates.

To find the real canonical form.

Let $\quad u = \alpha + i\beta \quad$ and $\quad v = \alpha - i\beta$

$\therefore \quad \alpha = \frac{1}{2}(u+v) \quad$ and $\quad \beta = \frac{1}{2}i(v-u).$

Now we transform the independent variables u and v to α and β with the help of these relations.

We have, $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u\,\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right).$$

Substituting in (14), the **canonical form** of equation (1), in this case is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \qquad ...(17)$$

Summary : Here we list the canonical forms of the second order partial differential equation (1), obtained in different cases, in the following table.

Type of the Equation	Canonical Form
Hyperbolic $S^2 - 4RT > 0$	$\frac{\partial^2 z}{\partial u\,\partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$
Parabolic $S^2 - 4RT = 0$	$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$
Elliptic $S^2 - 4RT < 0$	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$ Here $u = \alpha + i\beta,\ v = \alpha - i\beta$

11.2 Working Method of Reducing a Hyperbolic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ...(1)$$

be *hyperbolic i.e.,* $S^2 - 4RT > 0$.

To reduce (1) to canonical form, proceed as follows.

Step 1 : Write λ-quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad ...(2)$$

whose roots will be real and distinct $\quad \because \quad S^2 - 4RT > 0$

Solve (2) and let λ_1, and λ_2 be its two distinct real roots.

Step 2 : Write the corresponding characteristic equations

i.e., $\quad \dfrac{dy}{dx} + \lambda_1 = 0 \quad$ and $\quad \dfrac{dy}{dx} + \lambda_2 = 0$

Solve these equations and let

$f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ be the solutions.

where c_1 and c_2 are the arbitrary constants which are known as the characteristic curves or simply the characteristics of the equation (1).

Step 3 : Let u and v be two functions of x and y, such that

$$u = f_1(x, y) \text{ and } v = f_2(x, y) \qquad ...(3)$$

Step 4 : Using the relations (3), find p, q, r, s and t as needed for (1) in terms of u and v as in § 11.1

Step 5 : Substituting these values of p, q, r, s, t obtained in step (4) in (1), simplify to get the *canonical form* of (1), which will be of the form $\dfrac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u. v. z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}\right)$.

Solved Examples

Example 1: ***Reduce the following equation to canonical form.***

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

[Agra 2005; Meerut 2009 (B.P.), 11 (Sem. I), 12 (Sem. I); Delhi Maths (Hons.) 2002, 06; Himachal 2005; Kurukshetra 2004; Ravishankar 2004; I.A.S. 2008]

Solution: The given equation can be written as

$$r - x^2 t = 0 \qquad \ldots(1)$$

Comparing (1) with the equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = 1,\ S = 0,\ T = -x^2 \quad \therefore \quad S^2 - 4RT = 4x^2 > 0$

and so (1) is a hyperbolic equation

$\therefore$ λ – quadratic equation, $R\lambda^2 + S\lambda + T = 0$, becomes

$\lambda^2 - x^2 = 0$, giving $\lambda = \pm x$

Let $\lambda_1 = x$ and $\lambda_2 = -x$. (Real and distinct roots)

$\therefore$ The characteristic equations $\dfrac{dy}{dx} + \lambda_1 = 0$ and $\dfrac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} + x = 0 \quad \text{and} \quad \frac{dy}{dx} - x = 0$$

Integrating them, we get the following characteristics of (1)

$$y + \frac{1}{2}x^2 = c_1 \quad \text{and} \quad y - \frac{1}{2}x^2 = c_2$$

$\therefore$ To change the independent variable x, y, to u, v, in the given equation (1), we take

$$\therefore \quad u = y + \frac{1}{2}x^2 \quad \text{and} \quad v = y - \frac{1}{2}x^2$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = x.\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x.\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

$$= x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$\therefore$ Substituting the values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}-x^2\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)=0$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4x^2}.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

which is the required *canonical form* of the given equation.

Example 2: ***Reduce the equation***

$$(y-1)r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$$

to canonical form and hence solve it. **[Rohilkhand 2001; Delhi 2008]**

Solution: Comparing the given equation with the equation

$Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have

$R=y-1,\ S=-(y^2-1),\ T=y(y-1)$

$\therefore\ S^2-4RT=(y-1)^4>0 \Rightarrow$ Given equation is hyperbolic.

$\therefore$ The quadratic equation $R\lambda^2+S\lambda+T=0$, becomes

$$(y-1)\lambda^2-(y^2-1)\lambda+y(y-1)=0$$

or $$\lambda^2-(y+1)\lambda+y=0$$

or $$(\lambda-1)(\lambda-y)=0, \qquad \text{giving } \lambda=1, y$$

i.e., $\lambda_1=1$ and $\lambda_2=y$ (Real and distinct roots)

$\therefore$ The characteristic equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ becomes

$$\frac{dy}{dx}+1=0 \qquad \text{and} \qquad \frac{dy}{dx}+y=0$$

Integrating these equations, we get the following characteristic curves (or characteristics) of the given equation.

$$x+y=c_1 \qquad \text{and} \qquad ye^x=c_2$$

$\therefore$ To change the independent variables x, y to u, v, in the given equation, we take

$$u=x+y \text{ and } v=ye^x.$$

$\therefore$
$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)+v\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\,\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\,\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}$$

Substituting the values of p, q, r, s, t in the given equation, we get

$$(y-1)\left(\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial v^2}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v}\right)$$

$$-(y^2-1)\left[\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\,\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}\right]$$

$$+y(y-1)\left[\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}\right]+(v-e^x)\frac{\partial z}{\partial v}=2\,ye^{2x}(1-y)^3$$

or $$(1-y)^3 e^x\frac{\partial^2 z}{\partial u\,\partial v}=2\,ye^{2x}(1-y)^3$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=2\,ye^x \quad \text{or} \quad \frac{\partial^2 z}{\partial u\,\partial v}=2v \qquad ...(1)$$

which is the required *canonical form*.

To Find the Solution : Integrating (1) partially, w.r.t. v, we have

$$\frac{\partial z}{\partial u}=v^2+f(u). \qquad ...(2)$$

where $f(u)$ is an arbitrary function of u.

Now integrating (3) partially w.r.t. u, we have

$$z=uv^2+\phi(u)+\psi(v) \quad \text{where} \quad \phi(u)=\int f(u)du.$$

Where $\psi(v)$ is an arbitrary function of v.

or $$z=(x+y)\,y^2e^{2x}+\phi(x+y)+\psi(y\,e^x)$$

which is the solution of the given equation.

Example 3: ***Reduce the equation*** $yr+(x+y)s+xt=0$ ***to canonical form and hence find its general solution.*** **[Delhi Maths (Hons.) 2007]**

Solution: Given, $yr+(x+y)s+x\,t=0$ ...(1)

Comparing (1) with the equation

$$Rr+Ss+Tt+f(x,y,z,p,q)=0, \text{ we have}$$

$$R=y,\ S=x+y,\ T=x,$$

$$\therefore \quad S^2-4RT=(x+y)^2-4xy=(x-y)^2>0 \text{ if } x\neq y$$

which shows that (1) is hyperbolic equation if $x \neq y$.

$\therefore$ λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y\lambda^2 + (x+y)\lambda + x = 0 \quad \text{or} \quad (y\lambda + x)(\lambda + 1) = 0$$

$$\therefore \quad \lambda = -x/y,\ -1$$

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes,

$$\frac{dy}{dx} - \frac{x}{y} = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 = 0$$

Solving $y^2/2 - x^2/2 = c_1$ and $y - x = c_2$

where c_1 and c_2 are arbitrary constants are the characteristics of (1)

To change the independent variables x, y to u, v, in (1)

we choose $u = y^2/2 - x^2/2$ and $v = y - x$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

and
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\left(x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right)$$

$$= -1\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right).\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right).\frac{\partial v}{\partial x}\right] - \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= -\frac{\partial z}{\partial u} - x\left[-x\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u\,\partial v}\right] - \left[-x\frac{\partial^2 z}{\partial u\,\partial v} - \frac{\partial^2 z}{\partial v^2}\right]$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right).\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}\right]$$

$$= y\left[-x\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial u\,\partial v}\right]+\left[-x\frac{\partial^2 z}{\partial u\,\partial v}-\frac{\partial^2 z}{\partial v^2}\right]=-\left[xy\frac{\partial^2 z}{\partial u^2}+(x+y)\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right]$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=1.\frac{\partial z}{\partial u}+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial z}{\partial u}+y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial z}{\partial u}+y\left[y\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v\,\partial u}\right]+\left[y\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right]$$

$$=y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}$$

Substituting the values of r, s and t in (1), we get

$$y\left[x^2\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}\right]-(x+y)\left[xy\frac{\partial^2 z}{\partial u^2}+(x+y)\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right]$$
$$+x\left[y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}\right]=0$$

or $$[4xy-(x+y)^2]\frac{\partial^2 z}{\partial u\,\partial v}-y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial u}=0$$

or $$-(y-x)^2\frac{\partial^2 z}{\partial u\,\partial v}-(y-x)\frac{\partial z}{\partial u}=0$$ or $$(y-x)\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial z}{\partial u}=0 \text{ as } y\neq x.$$

or $$v\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial z}{\partial u}=0 \qquad ...(2)$$

which is the required *canonical form*.

To Find the Solution : Multiplying both sides of (2) by u, we get

$$uv\frac{\partial^2 z}{\partial u\,\partial v}+u\frac{\partial z}{\partial u}=0 \quad \text{or} \quad (uvDD'+uD)z=0 \qquad ...(3)$$

where $D\equiv\dfrac{\partial}{\partial u}$ and $D'\equiv\dfrac{\partial}{\partial v}$.

To reduce (2) to linear differential equation with constant coefficients, let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$.

$$\therefore \quad u\frac{\partial}{\partial u} \equiv \frac{\partial}{\partial X} = D_1 \text{ (say)} \quad i.e., \quad uD \equiv D_1, \; v\frac{\partial}{\partial v} \equiv \frac{\partial}{\partial Y} \equiv D_1' \text{ (say)}$$

$$\text{and} \quad uv\frac{\partial^2}{\partial u\,\partial v} = \frac{\partial^2}{\partial X\,\partial Y} \equiv D_1 D_1' \quad i.e., \quad uvDD' = D_1 D_1'$$

Substituting in (2), it reduces to

$$(D_1 D_1' + D_1)z = 0 \qquad \text{or} \qquad D_1(D_1' + 1)\,z = 0$$

whose general solution is

$$z = \phi_1(Y) + e^{-Y}\phi_2(X) = \phi_1(\log v) + v^{-1}\phi_2(\log u)$$

$$\text{or} \qquad z = \psi_1(v) + v^{-1}\,\psi_2(u)$$

$$\text{or} \qquad z = \psi_1(y - x) + (y - x)^{-1}\psi_2(y^2 - x^2)$$

where ψ_1 and ψ_2 are arbitrary functions.

Example 4: ***Reduce the following equation to canonical form and hence solve it.***

$$\boldsymbol{y(x + y)(r - s) - xp - yq - z = 0.}$$ **[Meerut 2012 (O)]**

Solution: Comparing the given equation with the equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = y(x + y),\; S = -y(x + y),\; T = 0$

$S^2 - 4RT = y^2(x + y)^2 > 0 \;\Rightarrow$ Given equation is hyperbolic

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y(x + y)\lambda^2 - y(x + y)\lambda + 0 = 0$$

$$\text{or} \qquad \lambda(\lambda - 1) = 0 \qquad \therefore \qquad \lambda = 0, 1$$

$\therefore$ Let $\lambda_1 = 0$ and $\lambda_2 = 1$

$\therefore$ The characteristic equations $\dfrac{dy}{dx} + \lambda_1 = 0$ and $\dfrac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} + 1 = 0 \quad \text{or} \quad \frac{dy}{dx} = 0 \text{ and } dy + dx = 0$$

Integrating, $y = c_1$ and $y + x = c_2$, which are the characteristics of the given equation.

$\therefore$ To change the independent variables x and y to u and v in the given equation, we take $u = y$ and $v = x + y$

$\therefore$ $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial v}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$\therefore$ $$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$-y(x+y)\cdot\frac{\partial^2 z}{\partial u\,\partial v} - x\frac{\partial z}{\partial v} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0$$

or $$-y(x+y)\cdot\frac{\partial^2 z}{\partial u\,\partial v} - y\frac{\partial z}{\partial u} - (x+y)\frac{\partial z}{\partial v} - z = 0$$

or $$uv\frac{\partial^2 z}{\partial u\,\partial v} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0$$

or $$\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0 \qquad \text{...(1)}$$

which is the *canonical form* of the given equation.

To Find the Solution : Equation (1), can be written as

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{1}{u}z\right) + \frac{1}{v}\left(\frac{\partial z}{\partial u} + \frac{1}{u}z\right) = 0 \qquad \text{...(2)}$$

or $\dfrac{\partial t}{\partial v} + \dfrac{1}{v}t = 0,$ Putting $\dfrac{\partial z}{\partial u} + \dfrac{1}{u}z = t$

which is L.D.E. in t with v as independent variable, $\therefore$ I.F. $= e^{\int\left(\frac{1}{v}\right)dv} = e^{\log v} = v$

$$\therefore \quad v.t = \int 0.dv + f(u) \quad \text{or} \quad v\left(\frac{\partial z}{\partial u} + \frac{1}{u}z\right) = f(u)$$

$$\text{or} \quad \frac{\partial z}{\partial u} + \frac{1}{u}z = \frac{1}{v}f(u)$$

which is L.D.E. in z, with u as independent variable (treating v constant)

$$\therefore \quad \text{I.F.} = e^{\int\left(\frac{1}{u}\right)du} = e^{\log u} = u$$

$$\therefore \quad u.z = \int u.\frac{1}{v}f(u)\,du + \psi(v) = \frac{1}{v}\phi(u) + \psi(v), \text{ where } \phi(u) = \int u\,f(u)\,du$$

$$\text{or} \quad z = \frac{1}{uv}\phi(u) + \frac{1}{u}\psi(v)$$

$$\text{or} \quad z = \frac{1}{y(x+y)}\phi(y) + \frac{1}{y}\psi(x+y)$$

where ϕ and ψ are arbitrary functions.

which is the required solution.

Note : If we write equation (1) as

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{1}{v}z\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{1}{v}z\right) = 0$$

and proceed as above, then the solution of the equation will be

$$z = \frac{1}{y(x+y)}\phi(x+y) + \frac{1}{(x+y)}\psi(y).$$

Example 5: ***Reduce the equation***

$$(n-1)^2\frac{\partial^2 z}{\partial x^2} - y^{2n}\frac{\partial^2 z}{\partial y^2} = ny^{2n-1}\frac{\partial z}{\partial y}$$

to canonical form, and find its general solution.

[Delhi Maths (Hons.) 2000, 01, 05; Himachal 2004]

Solution: The given equation can be written as

$$(n-1)^2 r - y^{2n}t - ny^{2n-1}q = 0 \qquad \ldots(1)$$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = (n-1)^2,\ S = 0,\ T = -y^{2n}.$$

$\therefore\ S^2 - 4RT = 4(n-1)^2 y^{2n} > 0 \Rightarrow$ (1) is a hyperbolic equation.

$\therefore$ The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$(n-1)^2\lambda^2 - y^{2n} = 0 \quad \text{or} \quad \lambda^2 = \frac{1\, y^{2n}}{(n-1)^2}$$

Giving $\lambda = \pm \dfrac{1}{(n-1)} y^n$ *i.e.,* $\lambda_1 = \dfrac{1}{(n-1)} y^n$, $\lambda_2 = \dfrac{-1}{(n-1)} y^n$ (Two real distinct roots)

$\therefore$ The characterise equations $\dfrac{dy}{dx} + \lambda_1 = 0$ and $\dfrac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + \frac{1}{(n-1)} y^n = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{1}{(n-1)} y^n = 0$$

or $\quad (n-1)\, y^{-n} dy + dx = 0 \quad$ and $\quad (n-1)\, y^{-n} dy - dx = 0$

integrating, we have

$$x - y^{-n+1} = c_1 \quad \text{and} \quad x + y^{-n+1} = c_2$$

which are the characteristic curves of (1).

$\therefore$ To change the independent variables x, y to u, v in the given equation (1), we take

$$u = x - y^{-n+1} \quad \text{and} \quad v = x + y^{-n+1}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} + \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = (n-1)\, y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

and

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[(n-1)\, y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= -n(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)\, y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)\, y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2\, y^{-2n}\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right].$$

Substituting these values in (1) and simplifying, we have

$$\frac{\partial^2 z}{\partial u\,\partial v} = 0 \qquad ...(2)$$

which is the required, *canonical form*.

To Find the Solution : Integrating, (2) w.r.t. v, we have

$$\frac{\partial z}{\partial u} = f(u), \text{ where } f(u) \text{ is an arbitrary function of } u.$$

Again integrating partially w.r.t. u, we have

$$z = \phi(u) + \psi(v) \quad \text{where } \phi(u) = \int f(u)\, du$$

where $\phi(u)$ and $\psi(v)$ is arbitrary functions.

Hence the required solution is

$$z = \phi(x - y^{-n+1}) + \psi(x + y^{-n+1}).$$

Example 6: ***Reduce the following equation to canonical form and hence solve***

$$x(xy-1)r - (x^2y^2 - 1)s + y(xy-1)t + (x-1)p + (y-1)q = 0.$$

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = x(xy-1),\ S = -(x^2y^2-1),\ T = y(xy-1)$$

$$\therefore \quad S^2 - 4RT = (xy-1)^4 > 0 \Rightarrow \text{The given equation is hyperbolic.}$$

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x(xy-1)\lambda^2 - (x^2y^2-1)\lambda + y(xy-1) = 0$$

or $\quad (xy-1)(x\lambda - 1)(\lambda - y) = 0 \quad$ or $\quad \lambda = 1/x, y$

Let $\quad \lambda_1 = 1/x$ and $\lambda_2 = y$

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} + \frac{1}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + y = 0$$

or $$dy + \frac{dx}{x} = 0 \quad \text{and} \quad \frac{dy}{y} + dx = 0$$

Integrating $y + \log x = \log c_1$ and $\log y + x = \log c_2$

$$\therefore \quad x e^y = c_1 \quad \text{and} \quad y e^x = c_2$$

Which are the characterise curves of the given equation.

$\therefore$ To change the independent variables x and y to u and v in the given equation, we take $u = x e^y$ and $v = y e^x$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = e^y \frac{\partial z}{\partial u} + ye^x \frac{\partial z}{\partial v} = e^y \frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\,e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} = u\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(e^y \frac{\partial z}{\partial u} + y\,e^x \frac{\partial z}{\partial v}\right) = e^y \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + ye^x \frac{\partial z}{\partial v} + ye^x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right).\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + ye^x\frac{\partial z}{\partial v} + ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y\left[\frac{\partial^2 z}{\partial u^2}e^y + \frac{\partial^2 z}{\partial u\,\partial v}ye^x\right] + ye^x\frac{\partial z}{\partial v} + ye^x\left[\frac{\partial^2 z}{\partial u\,\partial v}e^y + \frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$= e^{2y}\frac{\partial^2 z}{\partial u^2} + 2\,ye^{x+y}\frac{\partial^2 z}{\partial u\,\partial v} + ye^x\frac{\partial z}{\partial v} + y^2e^{2x}\frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\,e^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$= e^y\frac{\partial z}{\partial u} + x\,e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial z}{\partial v} + e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y\frac{\partial z}{\partial u} + x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$+ e^x\frac{\partial z}{\partial v} + e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y\frac{\partial z}{\partial u} + x\,e^y\left[\frac{\partial^2 z}{\partial u^2}e^y + \frac{\partial^2 z}{\partial v\,\partial u}ye^x\right] + e^x\frac{\partial z}{\partial v} + e^x\left[\frac{\partial^2 z}{\partial u\,\partial v}e^y + \frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$= x\,e^{2y}\frac{\partial^2 z}{\partial u^2} + (1+xy)\,e^{x+y}\frac{\partial^2 z}{\partial u\,\partial v} + e^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v} + ye^{2x}\frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\,e^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$= x\,e^y\frac{\partial z}{\partial u} + x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x\,e^y\frac{\partial z}{\partial u} + x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\,e^y\frac{\partial z}{\partial u} + x\,e^y\left[\frac{\partial^2 z}{\partial u^2}x\,e^y + \frac{\partial^2 z}{\partial v\,\partial u}e^x\right] + e^x\left[\frac{\partial^2 z}{\partial v\,\partial u}x\,e^y + \frac{\partial^2 z}{\partial v^2}e^x\right]$$

$$= x^2 e^{2y} \frac{\partial^2 z}{\partial u^2} + 2x e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2} + x e^y \frac{\partial z}{\partial u}$$

Substituting the values of p, q, r, s, t in the given equation, we have,

$$x(xy-1)\left[e^{2y}\frac{\partial^2 z}{\partial u^2} + 2y e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + y e^x \frac{\partial z}{\partial v} + y^2 e^{2x}\frac{\partial^2 z}{\partial v^2}\right]$$

$$-(x^2y^2-1)\left[x e^{2y}\frac{\partial^2 z}{\partial u^2} + (xy+1)e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} + y e^{2x}\frac{\partial^2 z}{\partial v^2}\right]$$

$$+y(xy-1)\left[x^2 e^{2y}\frac{\partial^2 z}{\partial u^2} + 2x e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + e^{2x}\frac{\partial^2 z}{\partial v^2} + x e^y \frac{\partial z}{\partial u}\right]$$

$$+(x-1)\left[e^y \frac{\partial z}{\partial u} + y e^x \frac{\partial z}{\partial v}\right] + (y-1)\left[x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right] = 0$$

or $$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad ...(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t., u, we get $\frac{\partial z}{\partial v} = f(v)$

Again integrating partially w.r.t., v, we get

$$z = \phi(v) + \psi(u), \quad \text{where} \quad \phi(v) = \int f(v)\, dv$$

or $$z = \phi(y e^x) + \psi(x e^y)$$

where ϕ and ψ are arbitrary functions.

Example 7: ***Reduce the equation***

$$x(y-x)r - (y^2 - x^2)s + y(y-x)t + (y+x)(p-q) = 2x + 2y + 2$$

to canonical form and hence solve.

Solution: Comparing the given equation with the equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = x(y-x)$, $S = -(y^2 - x^2)$, $T = y(y-x)$

$\therefore$ $$S^2 - 4RT = (y^2 - x^2)^2 - 4xy(y-x)^2 = (y-x)^2[(y+x)^2 - 4xy] = (y-x)^4 > 0$$

$\Rightarrow$ given equation is hyperbolic.

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x(y-x)\lambda^2-(y^2-x^2)\lambda+y(y-x)=0$$

or $\quad x\lambda^2-(y+x)\lambda+y=0$ or $(x\lambda-y)(\lambda-1)=0 \quad \therefore \quad \lambda=y/x, 1$

$\therefore$ Let $\quad \lambda_1=y/x \quad$ and $\quad \lambda_2=1$

$\therefore$ The characteristic equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$, becomes

$$\frac{dy}{dx}+\frac{y}{x}=0 \quad \text{and} \quad \frac{dy}{dx}+1=0$$

or $$\frac{dy}{y}+\frac{dx}{x}=0 \quad \text{and} \quad dy+dx=0$$

Integrating $\log y+\log x=\log c_1$ *i.e.*, $xy=c_1$ and $y+x=c_2$

Which are the characterises of the given equation

$\therefore$ To change the independent variables x and y to u and v, in the given equation, we take

$$u=xy \quad \text{and} \quad v=x+y$$

$\therefore$ $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\frac{\partial z}{\partial v}$$

$$=y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=y\left[\frac{\partial^2 z}{\partial u^2}\cdot y+\frac{\partial^2 z}{\partial v\,\partial u}.1\right]+\left[\frac{\partial^2 z}{\partial v\,\partial u}y+\frac{\partial^2 z}{\partial v^2}.1\right]$$

$$=y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial z}{\partial u}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}+\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2} y + \frac{\partial^2 z}{\partial v\,\partial u}.1\right] + \frac{\partial z}{\partial u} + \left[\frac{\partial^2 z}{\partial u\,\partial v} y + \frac{\partial^2 z}{\partial v^2}.1\right]$$

$$= x\,y\frac{\partial^2 z}{\partial u^2} + (x+y)\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2} x + \frac{\partial^2 z}{\partial v\,\partial u}.1\right] + \left[\frac{\partial^2 z}{\partial u\,\partial v}.x + \frac{\partial^2 z}{\partial v^2}.1\right] = x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$x(y-x)\left[y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right]$$

$$-(y^2-x^2)\left[x\,y\frac{\partial^2 z}{\partial u^2} + (x+y)\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}\right]$$

$$+y(y-x)\left[x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right]$$

$$+(y+x)\left[y\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial u}\right] = 2x + 2y + 2$$

or $$-(y-x)^3\frac{\partial^2 z}{\partial u\,\partial v} = 2x + 2y + 2$$

or $$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{2(x+y)+2}{[(x+y)^2 - 4xy]^{3/2}}$$

or $$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{2(v+1)}{(v^2-4u)^{3/2}} \qquad ...(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t. 'u', (treating v constant), we get

$$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2-4u)}} + f(v)$$

or $$\frac{\partial z}{\partial v} = \frac{v}{\sqrt{(v^2 - 4u)}} + \frac{1}{\sqrt{(v^2 - 4u)}} + f(v)$$

Integrating partially w.r.t. 'v', (treating u constant)

$$z = \sqrt{(v^2 - 4u)} + \log [v + \sqrt{(v^2 - 4u)}] + \phi (v) + \psi (u), \text{where, } \phi (v) = \int f(v) dv$$

or $$z = \sqrt{\{(x+y)^2 - 4xy\}} + \log [x + y + \sqrt{\{(x+y)^2 - 4xy\}}] + \phi (x+y) + \psi (xy)$$

or $$z = (x - y) + \log (2x) + \phi (x + y) + \psi (xy)$$

where ϕ and ψ are arbitrary functions.

which is the solution of the given equation.

Example 8: ***Reduce the following equation to canonical form and hence solve it***

$$xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2).$$ **[Delhi Maths (Hons.) 2006]**

Solution: Comparing the given equation with the equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = xy,\ S = -(x^2 - y^2),\ T = -xy. \quad \therefore \quad S^2 - 4RT = (x^2 + y^2)^2 > 0$

$\Rightarrow$ Given equation is hyperbolic.

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$xy\lambda^2 - (x^2 - y^2)\lambda - xy = 0$$

or $$(x\lambda + y)(y\lambda - x) = 0 \qquad \text{or} \qquad \lambda = -y/x,\ x/y$$

Let $\lambda_1 = -y/x$ and $\lambda_2 = x/y$

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} - \frac{y}{x} = 0 \qquad \text{and} \qquad \frac{dy}{dx} + \frac{x}{y} = 0$$

or $$\frac{dy}{y} - \frac{dx}{x} = 0 \qquad \text{and} \quad y\,dy + x\,dx = 0$$

Integrating $\log y - \log x = \log c_1$ and $y^2 + x^2 = c_2$

$$\therefore \qquad y/x = c_1 \qquad \text{and} \qquad y^2 + x^2 = c_2$$

Which are the characteristics of the given equation.

$\therefore$

To change the independent variables x and y to u and v, in the given equation we take $u = y/x$ and $v = x^2 + y^2$.

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\left(-\frac{y}{x^2}\right) + \frac{\partial z}{\partial v}.2x$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{1}{x}\right) + \frac{\partial z}{\partial v}2y$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{2y}{x^3}\right)\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial v} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{2y}{x^3}\right)\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right).\frac{\partial v}{\partial x}\right] + 2\frac{\partial z}{\partial v} + 2x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \left(\frac{2y}{x^3}\right)\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial^2 z}{\partial u^2}.\left(-\frac{y}{x^2}\right) + \frac{\partial^2 z}{\partial v\,\partial u}2x\right] + 2\frac{\partial z}{\partial v} + 2x\left[\frac{\partial^2 z}{\partial u\,\partial v}.\left(-\frac{y}{x^2}\right) + \frac{\partial^2 z}{\partial v^2}2x\right]$$

$$= \left(\frac{y^2}{x^4}\right)\frac{\partial^2 z}{\partial u^2} - \frac{4y}{x}\frac{\partial^2 z}{\partial u\,\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)$$

$$= \left(-\frac{1}{x^2}\right)\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + 2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left(-\frac{1}{x^2}\right)\frac{\partial z}{\partial u} + \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right).\frac{\partial v}{\partial x}\right] + 2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left[\frac{\partial^2 z}{\partial u^2}.\left(-\frac{y}{x^2}\right) + \frac{\partial^2 z}{\partial v\,\partial u}2x\right] + 2y\left[\frac{\partial^2 z}{\partial u\,\partial v}.\left(-\frac{y}{x^2}\right) + \frac{\partial^2 z}{\partial v^2}.2x\right]$$

$$= -\frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} + \left(2 - \frac{2y^2}{x^2}\right)\frac{\partial^2 z}{\partial u\,\partial v} + 4xy\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u}$$

and
$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)$$

$$= \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial v} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial z}{\partial u}\frac{\partial v}{\partial y}\right] + 2\frac{\partial z}{\partial v} + 2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= \frac{1}{x}\left[\frac{\partial^2 z}{\partial u^2}.\frac{1}{x} + \frac{\partial^2 z}{\partial v\,\partial u}.2y\right] + 2\frac{\partial z}{\partial v} + 2y\left[\frac{\partial^2 z}{\partial u\,\partial v}.\frac{1}{x} + \frac{\partial^2 z}{\partial v^2}.2y\right]$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{4y}{x}\frac{\partial^2 z}{\partial u\,\partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$xy\left[\frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} - \frac{4y}{x}\frac{\partial^2 z}{\partial u\,\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}\right]$$

$$-(x^2 - y^2)\left[-\frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} + 2\left(1 - \frac{y^2}{x^2}\right)\frac{\partial^2 z}{\partial u\,\partial v} + 4xy\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u}\right]$$

$$-xy\left[\frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + 4\frac{y}{x}\frac{\partial^2 z}{\partial u\,\partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v}\right] + \left[-\frac{y}{x^2}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right]y$$

$$-\left[\frac{1}{x}\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right]x = 2(x^2 - y^2)$$

or
$$\frac{-2(x^2 + y^2)^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v} = 2(x^2 - y^2)$$

or
$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{(y^2 - x^2)x^2}{(x^2 + y^2)^2} = \frac{(y/x)^2 - 1}{(1 + y^2/x^2)}$$

or
$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{u^2 - 1}{(1 + u^2)^2} \qquad ...(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t. 'u' (treating v constant), we get

$$\frac{\partial z}{\partial v} = \int \frac{u^2 - 1}{(u^2 + 1)^2}\,du + f(v) = \int \frac{(1 - 1/u^2)}{(u + 1/u)^2}\,du + f(v)$$

$$= \int \frac{dt}{t^2} + f(v) = -\frac{1}{t} + f(v) = -\frac{u}{u^2 + 1} + f(v), \quad \text{Putting } u + \frac{1}{u} = t$$

Integrating again partially w.r.t. 'v' (treating u constant), we get

$$z = \left(-\frac{u}{u^2+1}\right)v + \int f(v)\, dv + \psi(u)$$

$$= -\frac{uv}{u^2+1} + \phi(v) + \psi(u) \quad \text{where} \quad \phi(v) = \int f(v)\, dv$$

or $\quad z = -x\,y + \phi(x^2 + y^2) + \psi(y/x)$, where ϕ and ψ are arbitrary functions.

11.3 Working Method of Reducing a Parabolic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ...(1)$$

be parabolic *i.e.*, $\quad S^2 - 4RT = 0.$

To reduce (1) to canonical form, proceed as follows.

Step 1 : Write λ quadratic equation $R\lambda^2 + S\lambda + T = 0$ ...(2)

whose roots will be real and equal as $S^2 - 4RT = 0$

Solve (2) and let λ_1, λ_1 be two equal roots.

Step 2 : Write the corresponding *characteristic equation*

i.e., $\quad \frac{dy}{dx} + \lambda_1 = 0$. Solve this equation and let $f_1(x, y) = c_1$ be its solution i.e., the characteristic curve (or characteristics) of (1).

where c_1 is an arbitrary constant.

Step 3 : Let u and v be two arbitrary functions of x and y.

such that $\quad u = f_1(x, y)$. Choose $v = f_2(x, y)$ ...(3)

such that, the chosen arbitrary function $f_2(x, y)$ is independent of $f_1(x, y)$.

For this verify that the Wronskian

$$W(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

Step 4 : Using the relations (3), find p, q, r, s, t (as needed for (1)) in terms of u and v as in § 11.1.

Step 5 : Substituting these values of p, q, r, s, t (obtained in step 4) in (1), simplify to get the *canonical form* of (1), which will be of the form $\dfrac{\partial^2 z}{\partial u\,\partial v} = \phi\left(u, v, z\dfrac{\partial u}{\partial u}, \dfrac{\partial v}{\partial v}\right)$.

Solved Examples

Example 1: ***Reduce the equation***

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form and hence solve it. **[Meerut 2006 (B.P.) 2010 (Sem. I), 13 (Sem. I); Delhi Maths (Hons.) 2000, 04, 06, 08; Himachal 2001, 05; Rajasthan 2003; Jabalpur 2004]**

Solution: The given equation can be written as

$$r + 2s + t = 0 \qquad \text{...(1)}$$

Comparing this equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \text{, we have}$$

$$R = 1,\ S = 2 \text{ and } T = 1. \quad \therefore\ S^2 - 4RT = 0 \Rightarrow (1) \text{ is a parabolic equation}$$

$\therefore$ The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 + 2\lambda + 1 = 0 \quad \text{or} \quad (\lambda + 1)^2 = 0.$$

Giving $\lambda = -1$, *i.e.* $\lambda_1 = \lambda_2 = -1$. (Equal roots)

$\therefore$ The characterise equation $\dfrac{dy}{dx} + \lambda = 0$, becomes $\dfrac{dy}{dx} - 1 = 0$

Integrating, we have $x - y = c$ (constant), which is the characteristics of (1).

$\therefore$ To change the independent variables x, y to u, v we take $u = x - y$.

Also we have to take v as some function of x and y independent of u,

$\therefore$ Let $v = x + y$.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \therefore \qquad \frac{\partial}{\partial y} \equiv -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

and $$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting in (1), we have $\frac{\partial^2 z}{\partial v^2} = 0$ ...(2)

which is required *canonical form.*

To Find the Solution : Integrating (2) partially w.r.t. v, we have

$$\frac{\partial z}{\partial v} = \phi(u)$$

where $\phi(u)$ is some arbitrary function of u.

Again integrating partially w.r.t. v, we have

$$z = v\,\phi(u) + \psi(u)$$

where $\psi(u)$ is another arbitrary function of u.

Hence, the solution is

$$z = (x + y)\,\phi(x - y) + \psi(x - y).$$

Example 2: ***Reduce the equation***

$$y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canonical form, and hence solve it. **[Meerut 2006, 07, 09 (BP); Delhi Maths (Hons.) 2001, 05; GNDU Amritsar 2005; Nagpur 2005]**

Solution: The given equation can be written as

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x}p - \frac{x^2}{y}q = 0 \qquad \text{...(1)}$$

Comparing with, $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = y^2,\ S = -2xy,\ T = x^2 \quad \therefore \quad S^2 - 4RT = 0 \Rightarrow (1) \text{ is a parabolic equation.}$$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0 \qquad \text{or} \quad (y\lambda - x)^2 = 0$$

Giving $\quad \lambda = \dfrac{x}{y}, \dfrac{x}{y} \quad$ *i.e.,* $\quad \lambda_1 = \lambda_2 = \dfrac{x}{y}$ (equal roots)

$\therefore$ The characterise equation $\dfrac{dy}{dx} + \lambda = 0$, becomes $\dfrac{dy}{dx} + \dfrac{x}{y} = 0$

or $\quad y\,dy + x\,dx = 0$. Integrating $x^2 + y^2 = c$ (constant), which is the characteristics of (1).

$\therefore$ To change the independent variables x, y to u, v in the given equation (1), we take

$$u = x^2 + y^2$$

also we have to take v as some function of x and y independent of u,

$\therefore$ Let $\quad v = x^2 - y^2$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}.2x + \frac{\partial z}{\partial v}.2x = 2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$$

$$= 2\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= 2\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + 2x.\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}\right]$$

$$= 2\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + 4x^2\left[\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right]$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{2y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = 2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= 2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right).\frac{\partial v}{\partial x}\right] = 4xy\left[\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right]$$

and $$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[2y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= 2\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= 2\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right).\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= 2\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 4y^2\left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right]$$

Substituting in (1) and simplifying, we have

$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad ...(2)$$

which is the required *canonical form.*

To Find the Solution : Integrating (2) partially w.r.t. v, $\frac{\partial z}{\partial v} = \phi(u)$

Again integrating w.r.t. v, we have

$$z = v\,\phi(u) + \psi(u)$$

where $\phi(u)$, $\psi(u)$ are arbitrary functions of u.

$\therefore$ The solution is

$$z = (x^2 - y^2)\,\phi(x^2 + y^2) + \psi(x^2 + y^2)$$

which is the required solution.

11.4 Working Method of Reducing Elliptic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ...(1)$$

be elliptic *i.e.,* $S^2 - 4RT < 0$.

Step 1 : Write quadratic equation $R\lambda^2 + S\lambda + T = 0$ ...(2)

Whose roots will be complex conjugates as $S^2 - 4RT < 0$. Solve (2) and let λ_1, λ_2 be complex conjugates.

Step 2 : Write the corresponding characteristic equations

i.e., $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$.

Solving these equations we shall get the solutions of the form

$$f_1(x, y) + i\, f_2(x, y) = c_1 \quad \text{and} \quad f_1(x, y) - i\, f_2(x, y) = c_2$$

where c_1 and c_2 are arbitrary constants.

Step 3 : Let u and v be two arbitrary functions of x and y such that

$$u = f_1(x, y) + i\, f_2(x, y) \text{ and } v = f_1(x, y) - i\, f_2(x, y) \qquad ...(3)$$

Take α and β be two new real independent variables such that $u = \alpha + i\beta$ and $v = \alpha - i\beta$

$$\Rightarrow \qquad \alpha = f_1(x, y) \quad \text{and } \beta = f_2(x, y). \qquad ...(4)$$

Step 4 : Using the relations (4), find p, q, r, s, t in terms of α and β (as needed for (1)) as in § 11.1.

Step 5 : Substituting these values of p, q, r, s, t (obtained in step 4) in (1), simplify to get the canonical form of (1), which will be of the form

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \text{ where } u = \alpha + i\beta,\ v = \alpha - i\beta.$$

Solved Examples

Example 1: ***Reduce*** $x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2\ (x > 0)$ ***to canonical form.***

[Delhi Maths (Hons.) 2007]

Solution: Given differential equation can be written as

$$x\,r + t - x^2 = 0\,, \quad (x > 0) \qquad ...(1)$$

Comparing (1) with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = x,\ S = 0,\ T = 1. \quad \therefore \quad S^2 - 4RT = -4x < 0 \quad \because \quad x > 0$$

which shows that (1) is an elliptic equation

λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x\lambda^2 + 1 = 0 \quad \text{or} \quad \lambda = i/\sqrt{x},\ -i/\sqrt{x}.$$

$\therefore$ The corresponding characteristic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0, \text{ becomes}$$

$$\frac{dy}{dx} + \frac{i}{\sqrt{x}} = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{i}{\sqrt{x}} = 0$$

Solving, we get $y + 2i\sqrt{x} = c_1$ and $y - 2i\sqrt{x} = c_2$

where c_1 and c_2 are arbitrary constants. Let u and v be two arbitrary functions of x and y such that $u = y + 2i\sqrt{x} = \alpha + i\beta$ and $v = y - 2i\sqrt{x} = \alpha - i\beta$

$$\Rightarrow \qquad \alpha = y \quad \text{and} \quad \beta = 2\sqrt{x} \qquad \text{...(2)}$$

which are new independent variables.

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta} \qquad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \qquad \text{...(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right) = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\cdot\left[\frac{\partial^2 z}{\partial \alpha\,\partial \beta}\cdot 0 + \frac{\partial^2 z}{\partial \beta^2}\cdot x^{-1/2}\right] = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1}\frac{\partial^2 z}{\partial \beta^2}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2} \quad \because \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha} \text{ from (4)}$$

Substituting the values of r and t in (1), we get

$$x\cdot\left[-\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1}\frac{\partial^2 z}{\partial \beta^2}\right] + \frac{\partial^2 z}{\partial \alpha^2} - x^2 = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} - \frac{1}{2}x^{-1/2}\frac{\partial z}{\partial \beta} - x^2 = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \left(\frac{1}{\beta}\right)\frac{\partial z}{\partial \beta} + \frac{\beta^4}{16}$$

which is the required canonical form.

Example 2: ***Reduce the following equation to canonical form***

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0.$$ **[Delhi Maths (Hons.) 2006, 08]**

Solution: The given equation can be written as

$$r + x^2 t = 0 \quad \text{...(1)}$$

Comparing it with

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = 1,\ S = 0,\ T = x^2 \quad \therefore\ S^2 - 4RT = -4x^2 < 0 \Rightarrow$ (1) is an elliptic equation

The quadratic equation, $R\lambda^2 + S\lambda + T = 0$, becomes

$\lambda^2 + x^2 = 0,$ giving $\lambda = \pm ix$

i.e., $\lambda_1 = ix,\ \lambda_2 = -ix$ (Complex roots)

$\therefore$ The characteristic equations, $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + ix = 0 \quad \text{and} \quad \frac{dy}{dx} - ix = 0$$

Integrating them, we have

$$y + \frac{i}{2}x^2 = \text{constant and } y - \frac{i}{2}x^2 = \text{constant}$$

$\therefore$ To change the independent variables x, y to u, v we take

$$u = y + \frac{i}{2}x^2 = \alpha + i\beta \text{ (say)}$$

and $$v = y - \frac{i}{2}x^2 = \alpha - i\beta$$

so that $\alpha = y$ and $\beta = \frac{1}{2}x^2.$

Now we transform the independent variables x and y to α and β. With the help of these relations

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}.\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}.\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}.\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \Rightarrow \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x.\frac{\partial z}{\partial \beta}\right) = 1.\frac{\partial z}{\partial \beta} + x.\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right).\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}.$$

Substituting in (1), we have

$$\left(\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}\right) + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2}\frac{\partial z}{\partial \beta}$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \alpha}$$

which is the required *canonical form*.

Example 3: ***Reduce the equation***

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + 5\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} - 3z = 0 \qquad \textit{to canonical form.}$$

Solution: The given equation can be written as

$$r + 2s + 5t + p - 2q - 3z = 0 \qquad \ldots(1)$$

Comparing the given equation (1) with the equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = 1, S = 2, T = 5 \quad \therefore \quad S^2 - 4RT = -16 < 0 \Rightarrow (1)$ is an elliptic equation.

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 + 2\lambda + 5 = 0 \qquad \therefore \qquad \lambda = -1 \pm 2i$$

∴ Let $\lambda_1 = -1 + 2i,\ \lambda_2 = -1 - 2i$ (Complex roots)

∴ The characterise equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} - 1 + 2i = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 - 2i = 0$$

Integrating $y - x + 2x\,i = c_1$ and $y - x - 2x\,i = c_2$

∴ To change the independent variables x and y in the given equation, we take

$$u = y - x + 2x\,i = \alpha + i\,\beta \quad \text{and} \quad v = y - x - 2x\,i = \alpha - i\,\beta$$

so that $\alpha = y - x$ and $\beta = 2x$.

Now we transform the independent variables x and y to α and β such that

$$\alpha = y - x \quad \text{and} \quad \beta = 2x.$$

∴
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = -\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \Rightarrow \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial}{\partial \alpha}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}$$

$$= \left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \alpha\,\partial \beta}\right)(-1) + \left(-\frac{\partial^2 z}{\partial \beta\,\partial \alpha} + 2\frac{\partial^2 z}{\partial \beta^2}\right).2$$

$$= \frac{\partial^2 z}{\partial \alpha^2} - 4\frac{\partial^2 z}{\partial \alpha\,\partial \beta} + 4\frac{\partial^2 z}{\partial \beta^2}$$

$$s = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \beta}{\partial x} = -\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \alpha\,\partial \beta}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$\left(\frac{\partial^2 z}{\partial \alpha^2}-4\frac{\partial^2 z}{\partial \alpha\,\partial \beta}+4\frac{\partial^2 z}{\partial \beta^2}\right)+2\left(-\frac{\partial^2 z}{\partial \alpha^2}+2\frac{\partial^2 z}{\partial \alpha\,\partial \beta}\right)+5\frac{\partial^2 z}{\partial \alpha^2}$$
$$+\left(-\frac{\partial z}{\partial \alpha}+2\frac{\partial z}{\partial \beta}\right)-2\frac{\partial z}{\partial \alpha}-3z=0$$

or $$4\frac{\partial^2 z}{\partial \alpha^2}+4\frac{\partial^2 z}{\partial \beta^2}-3\frac{\partial z}{\partial \alpha}+2\frac{\partial z}{\partial \beta}-3z=0$$

or $$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 \beta}{\partial \beta^2}=\frac{3}{4}\frac{\partial z}{\partial \alpha}-\frac{1}{2}\frac{\partial z}{\partial \beta}+\frac{3}{4}z$$

which is the required *canonical form.*

Exercise

Numerical Questions

Reduce the following equations to canonical form

1. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0.$ **[Delhi Maths (Hons.) 2002; Sagar 2004]**
2. $\frac{\partial^2 z}{\partial x^2}+4\frac{\partial^2 z}{\partial x\,\partial y}-5\frac{\partial^2 z}{\partial y^2}+6\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}-9z=0.$
3. $6\frac{\partial^2 z}{\partial x\,\partial y}+3\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}=0.$
4. $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\,\partial y}-8\frac{\partial^2 z}{\partial y^2}+9\frac{\partial z}{\partial x}=0.$
5. $y^2\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x^2}=0.$ **[Delhi Maths (Hons.) 2005]**
6. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\,\partial y}+6\frac{\partial^2 z}{\partial y^2}=0.$
7. $\frac{\partial^2 z}{\partial x^2}-6\frac{\partial^2 z}{\partial x\,\partial y}+9\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}-z=0.$
8. $\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\,\partial y}+13\frac{\partial^2 z}{\partial y^2}-9\frac{\partial z}{\partial y}=0.$
9. $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+5\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}-2\frac{\partial z}{\partial y}-3z=0.$
10. $t-s+p-q(1+1/x)+z/x=0.$ **[Delhi Maths (Hons.) 2004]**

Reduce the following equations to canonical form and hence solve.

11. $x^2 r - y^2 t + px - qy = x^2.$ **[Kurukshetra 2003]**

12. $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0.$

13. $x^2 r - xys + px + qy - z = 2x^2 y.$

14. $x^2 r - 2xys + y^2 t - xp + 3yq = 8y/x.$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. The equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 8\dfrac{\partial^2 z}{\partial y^2} + 9\dfrac{\partial z}{\partial x} = 0$ is

   (a) Elliptic (b) Hyperbolic
   (c) Parabolic (d) None of these

2. The characteristic equations of $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 8\dfrac{\partial^2 z}{\partial y^2} + 9\dfrac{\partial z}{\partial x} = 0$ are :

   (a) $\dfrac{dy}{dx} - 2 = 0,\ \dfrac{dy}{dx} + 4 = 0$ (b) $\dfrac{dy}{dx} + 2 = 0,\ \dfrac{dy}{dx} - 4 = 0$
   (c) $\dfrac{dy}{dx} - 2x = 0,\ \dfrac{dy}{dx} + 4x = 0$ (d) None of these

3. The characteristic equations of $y^2 r - x^2 t = 0$ are :

   (a) $\dfrac{dy}{dx} + \dfrac{x}{y} = 0,\ \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ (b) $\dfrac{dy}{dx} + \dfrac{y}{x} = 0,\ \dfrac{dy}{dx} - \dfrac{y}{x} = 0$
   (c) $\dfrac{dy}{dx} + y = 0,\ \dfrac{dy}{dx} + x = 0$ (d) None of these

4. The characteristics of $y^2 r - x^2 t = 0$ are :

   (a) $x^2 + y^2 = c_1,\ x^2 + 2y^2 = c_2$ (b) $x^2 + y^2 = c_1,\ x^2 - y^2 = c_2$
   (c) $2x^2 + y^2 = c_1,\ x^2 + y^2 = c_2$ (d) None of these

5. The characteristic equations of $4r + 5s + t + 2p + q - 3 = 0$ are :

   (a) $\dfrac{dy}{dx} - 1 = 0,\ \dfrac{dy}{dx} + \dfrac{1}{4} = 0$ (b) $\dfrac{dy}{dx} + 1 = 0,\ \dfrac{dy}{dx} + \dfrac{1}{4} = 0$
   (c) $\dfrac{dy}{dx} - 1 = 0,\ \dfrac{dy}{dx} - \dfrac{1}{4} = 0$ (d) None of these

6. The characteristics of equations of $4r+5s+t+p+q-2=0$ are :

(a) $y-x=c_1,\ 4y-x=c_2$ (b) $y+x=c_1,\ 4y-x=c_2$

(c) $y+x=c_1,\ 4y+x=c_2$ (d) None of these

7. The number of characterise equations of

$$\frac{\partial^2 z}{\partial x^2}-6\frac{\partial^2 z}{\partial x\partial y}+9\frac{\partial^2 z}{\partial y^2}+\frac{2\partial z}{\partial x}+3\frac{\partial z}{\partial y}-z=0 \text{ is :}$$

(a) 3 (b) 2

(c) 1 (d) 0

8. The characteristic of $\frac{\partial^2 z}{\partial x^2}=(1+y)^2\frac{\partial^2 z}{\partial y^2}$ are :

(a) $\log(1+y)+x=c_1;\ \log(x+y)-x=c_2$

(b) $\log(1-y)+x=c_1,\ \log(1+y)-x=c_2$

(c) $\log(1-y)-x=c_1,\ \log(1+y)+y=c_2$

(d) None of these

9. If $y-\cos x+x=c_1$ is one of the characteristics of the equation $r-(2\sin x)s-(\cos^2 x)t-(\cos x)q=0$, then the other characteristics is

(a) $y+\cos x+x=c_2$ (b) $y-\cos x-x=c_2$

(c) $y-\cos x+x=c_2$ (d) None of these

10. The characteristic equations of equation $xys-x^2r-px-qy+z=-2xy^2$ are :

(a) $\frac{dy}{dx}-\frac{y}{x}=0,\ \frac{dy}{dx}+\frac{y}{x}=0$ (b) $\frac{dy}{dx}+\frac{y}{x}=0,\ \frac{dy}{dx}=0$

(c) $\frac{dy}{dx}-\frac{y}{x}=0,\ \frac{dy}{dx}+x=0$ (d) None of these

Fill in the Blank(s)

Fill in the blanks "..........." so that the following statements are complete and correct.

1. The characteristic equations of $x^2r-y^2t=0$ are $\frac{dy}{dx}+$...........$=0$ and

$\frac{dy}{dx}+$...........$=0$

2. The characteristics of $x^2r-y^2t=0$ is/are .... and ....

3. The characteristics of

$$\frac{\partial^2 z}{\partial x^2}-6\frac{\partial^2 z}{\partial x\partial y}+9\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}-3=0 \text{ is/are ....}$$

4. The characteristic equation of $x^2 r - 2xys + y^2 t - xp + 3yq = 8y/x$ is/are ....

5. The characteristic equation and characteristics of the equation $r + 2xs + x^2 t = 0$ are .... and .... respectively.

6. To reduce the equation $r - 2s + t + p - q = e^x(2y - 3) - e^y$ to canonical form, choose $u =$ .... and $v =$ ....

Answers

Numerical Questions

1.	$\frac{\partial^2 z}{\partial u \partial v} = 0$	**2.**	$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\frac{\partial z}{\partial u} - \frac{3}{4}\frac{\partial z}{\partial v} - \frac{1}{4}z$
3.	$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{6}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$	**4.**	$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} - \frac{1}{2}\frac{\partial z}{\partial v}$
5.	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}$	**6.**	$\frac{\partial^2 z}{\partial u \partial v} = 0$
7.	$\frac{\partial^2 z}{\partial v^2} = -\frac{\partial z}{\partial u} - \frac{1}{3}\frac{\partial z}{\partial v} + \frac{1}{9}u$	**8.**	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha}$
9.	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3}{4}\frac{\partial z}{\partial \alpha} - \frac{1}{2}\frac{\partial z}{\partial \beta} + \frac{3}{4}z$	**10.**	$\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} + \frac{1}{v}\frac{\partial z}{\partial u} - \frac{z}{u} = 0$
11.	$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}, z = (1/4)x^2 + \phi(xy) + \psi(x/y)$	**12.**	$\frac{\partial^2 z}{\partial u \partial v} = 0, \quad z = \phi(xy) + \phi(xe^y)$
13.	$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} - \frac{2u}{v^2} - \frac{z}{uv}, \quad z = x^2 y + xy\,\phi(y) + y\,\psi(xy)$		
14.	$\frac{\partial^2 z}{\partial v^2} = \frac{2}{v} - \frac{2}{v}\frac{\partial z}{\partial v}, \quad z = \frac{y}{x} + x^2\,\phi(xy) + \psi(xy)$		

Objective Type Questions

Multiple Choice Questions

1.	(b)	**2.**	(a)	**3.**	(a)	**4.**	(b)
5.	(a)	**6.**	(a)	**7.**	(c)	**8.**	(a)
9.	(b)	**10.**	(b)				

Fill in the Blank(s)

1.	$y/x, -y/x$	**2.**	$xy = c_1, \ y/x = c_2$
3.	$y + 3x = c$	**4.**	$xdy + ydx = 0$
5.	$\frac{dy}{dx} - x = 0, \ y - x^2/2 = c_1$	**6.**	$x + y, \ y$

❍❍❍

14.5 Laplace Equation in Plane Polar Coordinates

[Meerut 2008 (B.P.)]

Laplace equation in two dimensions is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

Proceeding as in {13.8 on page 632 in plane polar coordinates (r, θ), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Hence, the Laplace equation (1) in plane polar coordinates (r, θ) is transformed to

$$\boldsymbol{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

14.6 Solution of Laplace Equation in Plane Polar Coordinates by Separation of Variables

The Laplace equation in plane polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(A)}$$

Let the solution of (A) which is a function of r and θ be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad \text{...(1)}$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F\frac{d^2 R}{d r^2} + \frac{F}{r}\frac{dR}{d r} + \frac{R}{r^2}\frac{d^2 F}{d\theta^2} = 0$$

or $$\frac{1}{R}\left(r^2 \frac{d^2 R}{d r^2} + r\frac{dR}{d r}\right) = -\frac{1}{F}\frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \qquad \text{...(2)}$$

In (2) the two sides are functions of different independent variables, so the two will be equal to each other if each is equal to the same constant, say λ.

Thus, from (2), we get the following two ordinary differential equations

$$\frac{d^2F}{d\theta^2} + \lambda F = 0 \qquad ...(3)$$

and $$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0 \qquad ...(4)$$

The equation (4) is a homogeneous differential equation with variable coefficients.

Now let $r = e^z$ so that $\dfrac{dr}{dz} = e^z = r$

$\therefore$ $$\frac{dR}{dr} = \frac{dR}{dz}\cdot\frac{dz}{dr} = \frac{dR}{dz}\frac{1}{r} \quad \text{or} \quad r\frac{dR}{dr} = \frac{dR}{dz} \quad \therefore\ r\frac{d}{dr} \equiv \frac{d}{dz} \equiv D_1 \text{ (say)}$$

i.e., $$r\frac{dR}{dr} = \frac{dR}{dz} = D_1 R$$

and $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} \quad \text{or} \quad r^2\frac{d^2R}{dr^2} = r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - r\frac{dR}{dr}$$

or $$r^2\frac{d^2R}{dr^2} = \left(r\frac{d}{dr} - 1\right)\left(r\frac{dR}{dr}\right) = (D_1 - 1)D_1 R$$

Putting in (4), we get

$$(D_1 - 1)D_1 R + D_1 R - \lambda R = 0 \quad \text{or} \quad (D_1^2 - \lambda)R = 0 \qquad ...(5)$$

Now there are the following three cases :

Case I : *If* $\lambda = 0$, then the equations (3) and (5), reduce to

$$\frac{d^2F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = 0 \ \ i.e., \ \frac{d^2R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1\theta + A_2 \quad \text{and} \quad R = A_3 z + A_4 = A_3 \log r + A_4.$$

$\therefore$ $$u(r, \theta) = FR = (A_1\theta + A_2)(A_3 \log r + A_4) \qquad ...(6)$$

Case II : *If* $\lambda = k^2 > 0$, then the equations (3) and (5), reduce to

$$\frac{d^2F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad (D_1^2 - k^2)R = 0$$

Whose general solutions are

$$F = B_1 \cos k\theta + B_2 \sin k\theta$$

and $$R = B_3 e^{kz} + B_4 e^{-kz} = B_3 r^k + B_4 r^{-k}$$

$$\therefore \quad u(r, \theta) = FR = (B_1 \cos k\theta + B_2 \sin k\theta)(B_3 r^k + B_4 r^{-k}) \qquad ...(7)$$

Case III : *If* $\lambda = -k^2 < 0$, then the equations (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} - k^2 F = 0 \quad \text{and} \quad (D_1^2 + k^2) R = 0$$

Whose general solutions are

$$F = C_1 e^{k\theta} + C_2 e^{-k\theta}$$

and $$R = C_3 \cos kz + C_4 \sin kz = C_3 \cos(k \log r) + C_4 \sin(k \log r)$$

$$\therefore \quad u(r, \theta) = FR = (C_1 e^{k\theta} + C_2 e^{-k\theta})[C_3 \cos(k \log r) + C_4 \sin(k \log r)] \qquad ...(8)$$

Thus, the solutions of the given equation (A) are given by (6), (7) and (8) for different values of λ. Out of these solutions the most suitable solution of (A) is chosen which depends upon the physical nature of the problem under consideration and the given boundary conditions.

Solved Examples

Example 1: ***A thin semi-circular plate of radius a which is insulated on both the sides has its boundary diameter kept at 0°C and its temperature along the semi-circular boundary is*** $f(\theta)$***. Find the temperature distribution in the plate in the steady state.*** **[Meerut 2011 (B.P.)]**

Solution: In steady state, the temperature distribution in the plate is governed by the Laplace equation (in plane polar coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(A)$$

Here it is given that the temperature $u(r, \theta)$ is zero along the bounding diameter AOB *i.e.,* $u(r, \theta)$ is zero along $OA(\theta = 0)$ and $OB(\theta = \pi)$.

Also temperature $u(r, \theta)$ along the semi-circular boundary $ACB(r = a)$ is $f(\theta)$.

Thus, the boundary conditions of the problem are

$$u(r, 0) = 0 \qquad 0 \le r \le a, \qquad ...(B_1)$$

$$u(r, \pi) = 0 \qquad 0 \le r \le a, \qquad ...(B_2)$$

and $$u(a, \theta) = f(\theta) \qquad 0 < \theta < \pi \qquad ...(B_3)$$

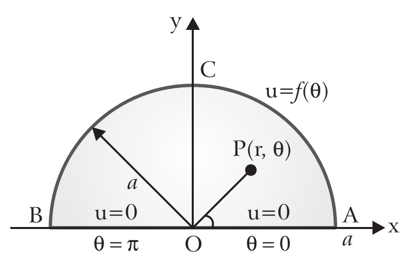


Fig. 14.5

Let the solution of equation (A) be given by

$$u(r, \theta) = R(r)\, F(\theta) = RF \text{ (say)} \qquad ...(1)$$

where $R(r)$ is a function of r alone and $F(\theta)$ a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F\frac{d^2R}{dr^2} + \frac{F}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2R}{d\theta^2} = 0$$

or
$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{F}\frac{d^2F}{d\theta^2} = \lambda \text{ (say)} \qquad ...(2)$$

In (2), two sides are functions of different artificial variables, so they will be equal to each other if each is equal to the same constant say λ.

Thus, from (2) we get the following two ordinary differential equations

$$\frac{d^2F}{d\theta^2} + \lambda F = 0 \qquad ...(3)$$

and
$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0 \qquad ...(4)$$

(4) is a homogeneous differential equation.

Let $r = e^z$, so that $r\dfrac{dR}{dr} = D_1R$, $r^2\dfrac{d^2R}{dr^2} = (D_1 - 1)D_1R$, where $D_1 \equiv r\dfrac{d}{dr} \equiv \dfrac{d}{dz}$

Putting in (4), it reduces to $(D_1 - 1)D_1R + D_1R - \lambda R = 0$

or
$$(D_1^2 - \lambda)R = 0 \qquad ...(5)$$

Now there are three cases as follows :

Case I : *If* $\lambda = 0$, then the equations in (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2R = \frac{d^2R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1\theta + A_2 \quad \text{and} \quad R = A_3z + A_4 = A_3 \log r + A_4$$

$\therefore \quad u(r, \theta) = FR = (A_1\theta + A_2)(A_3 \log r + A_4)$

$\therefore$ The boundary conditions (B_1) and (B_2) *i.e.*, $u(r, 0) = 0$ and $u(r, \pi) = 0$

$\Rightarrow \quad A_2(A_3 \log r + A_4) = 0 \quad$ and $\quad (A_1\pi + A_2)(A_3 \log r + A_4) = 0$

$\Rightarrow \quad A_2 = 0 \quad \text{and} \quad A_1\pi + A_2 = 0 \quad \because \; A_3 \log_r + A_4 \neq 0$

$\Rightarrow \quad A_1 = 0 \quad \text{and} \quad A_2 = 0$

$\therefore$ $u(r, \theta) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0$

Case II : *If* $\lambda = -k^2 < 0$, then the equations in (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} - k^2 F = 0 \quad \text{and} \quad (D_1^2 + k^2) R = 0$$

Whose general solutions are $F = B_1 e^{k\theta} + B_2 e^{-k\theta}$

and $\quad R = B_3 \cos kz + B_4 \sin kz = B_3 \cos(k \log r) + B_4 \sin(k \log r)$

$$\therefore \quad u = FR = (B_1 e^{k\theta} + B_2 e^{-k\theta})\,[B_3 \cos(k \log r) + B_4 \sin(k \log r)] \quad ...(6)$$

Since on physical grounds of the problem, we must have $u(r, \theta) = u(r, \theta + 2\pi)$, which is possible if $u(r, \theta)$ involve trigonometric functions of θ.

Since here $u(r, \theta)$ in (6) does not contain trigonometrical functions of θ, so we reject the case when $\lambda = -k^2 < 0$.

Case III : *If* $\lambda = k^2 > 0$, then from (3) and (5) we get the following two ordinary differential equations.

$$\frac{d^2F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2R}{dz^2} - k^2 R = 0$$

Whose general solutions are

$$F = A \cos k\theta + B \sin k\theta \quad \text{and} \quad R = C e^{kz} + D e^{-kz} = C r^k + D r^{-k}$$

$$\therefore \quad u(r, \theta) = FR = (A \cos k\theta + B \sin k\theta)(Cr^k + Dr^{-k}) \quad ...(7)$$

Now from (7), the boundary condition (B_1) *i.e.,* $u(r, 0) = 0$

$\Rightarrow \quad A(Cr^k + Dr^{-k}) = 0 \Rightarrow A = 0, \quad \because \; C r^k + D r^{-k} \neq 0$

$\therefore$ from (7), we get $\quad u(r, \theta) = B \sin k\theta\,(Cr^k + Dr^{-k}) \quad ...(8)$

Since when $r \to 0$, $u(r, \theta) \to 0$, but in (8) when $r \to \theta, r^{-k} \to \infty$ as $k > 0$, so in (8), we must take $D = 0$. Taking $D = 0$ in (8), we get

$$u(r, \theta) = (B \sin k\theta)(C r^k) \quad ...(9)$$

$\therefore$ from (9) and the boundary condition (B_2) *i.e.*, $u(r, \pi) = 0$

$\Rightarrow$ $(B \sin k\pi)(C\, r^k) = 0$

$\Rightarrow$ $\sin k\pi = 0,$ $\quad\therefore\quad B \neq 0$ and $C\, r^k \neq 0$

$\Rightarrow$ $k\pi = n\pi$ $\quad$ *i.e.*, $k = n$, where n is any integer.

$\therefore$ From (9), for integral values of n, the solutions of (A) satisfying the conditions (B_1) and (B_2) are given by

$$u_n(r, \theta) = (E_n\, r^n) \sin n\theta.$$

Where $E_n = BC$ is the new arbitrary constant.

Taking $n = 1, 2, 3, \ldots$, more general solution is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} E_n\, r^n \sin n\theta \qquad \ldots(10)$$

Which also satisfy conditions (B_1) and (B_2).

Now from (10), the boundary condition (B_3) *i.e.*, $u(a, \theta) = f(\theta)$

$$\Rightarrow \quad f(\theta) = \sum_{n=1}^{\infty} E_n\, a^n \sin n\theta \qquad \ldots(11)$$

In (11), right hand side can be considered as the Fourier sine series of function $f(\theta)$ on its left hand side

$$\therefore \quad a^n E_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta$$

or
$$E_n = \frac{2}{\pi\, a^n} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta \qquad \ldots(12)$$

Hence, the required temperature distribution, in steady state, in the plate is given by (10), where E_n is given by (12).

Example 2: ***The boundary diameter of a semi-circular plate of radius 10 cm is kept at 0°C and its temperature along the semi-circular boundary is given by***

$$u(10, \theta) = f(\theta) = \begin{cases} 50\,\theta, & \text{for } 0 \le \theta \le \pi/2 \\ 50\,(\pi - \theta), & \text{for } \pi/2 \le \theta \le \pi \end{cases}$$

Find the steady state temperature $u(r, \theta)$ is the plate.

Solution: The steady state temperature in the semi-circular plate is governed by the Laplace equation (in plane polar coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(A)}$$

(Refer to fig. 14.5 in Ex. 1)

As in Ex. 1, the boundary conditions of the problem are

$$u(r, 0) = 0 \qquad 0 \le r \le 10 \qquad \text{...(B}_1\text{)}$$

$$u(r, \pi) = 0 \qquad 0 \le r \le 10 \qquad \text{...(B}_2\text{)}$$

and $$u(10, \theta) = f(\theta) \qquad 0 < \theta < \pi \qquad \text{...(B}_3\text{)}$$

Let the solution of equation (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad \text{...(1)}$$

Where R is a function of r alone and F a function of θ alone.

Now proceeding similarly as in Ex. 1, the required solution is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta \qquad \text{...(2)}$$

where $$E_n = \frac{2}{\pi(10^n)} \int_0^{\pi} f(\theta) \sin n\theta\, d\theta$$

or $$E_n = \frac{2}{\pi . 10^n}\left[\int_0^{\pi/2} 50\theta . \sin n\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta) \sin n\theta\, d\theta\right]$$

$$= \frac{2}{\pi . 10^n}\left[\left\{50\,\theta . \left(-\frac{1}{n}\cos n\theta\right) - 50\left(-\frac{1}{n^2}\sin n\theta\right)\right\}_0^{\pi/2} + \left\{50(\pi - \theta).\left(-\frac{1}{n}\cos n\theta\right) - (-50)\left(-\frac{1}{n^2}\sin n\theta\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{100}{\pi . 10^n}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{200}{\pi n^2 . 10^n}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & \text{if } n = 2m \text{ (even)}, \quad m = 1, 2, \ldots \\ \dfrac{200(-1)^{m-1}}{\pi(2m-1)^2 . 10^{(2m-1)}} & \text{if } n = 2m - 1 \text{ (odd)}, \quad m = 1, 2, \ldots \end{cases}$$

$\because$ When $n = 2m-1$, $\sin \dfrac{n\pi}{2} = \sin (2m-1)\dfrac{\pi}{2} = \sin (m\pi - \pi/2)$

$$= \sin m\pi \cos \frac{\pi}{2} - \cos m\pi \sin \frac{\pi}{2} = -(1)^m = (-1)^{m+1} = (-1)^{m-1}$$

Hence, putting in (2) the required solution is

$$u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{10}\right)^{2m-1} (-1)^{m-1} . \frac{\sin (2m-1)\theta}{(2m-1)^2}$$

Example 3: ***Solve the differential equation***

$\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0$ ***(Laplace equation in plane polar coordinates) in the region*** $0 \le r \le a$, $0 \le \theta \le 2\pi$ ***and satisfying the boundary conditions.***

(i) ***u remains finite as*** $r \to 0$ ***and***

(ii) $u = \sum_n C_n \cos n\theta$, ***when*** $r = a$.

Solution: The given differential equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(A)}$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad \text{...(1)}$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F\frac{d^2R}{dr^2} + \frac{F}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2F}{d\theta^2} = 0$$

or
$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{F}\frac{d^2F}{d\theta^2} \qquad \text{...(2)}$$

The two sides of (1) are functions of different independent variables, so they will be equal to each other if each is equal to the same constant. Since there is trigonometric function $\cos n\theta$ in the given conditions, so the solution must involve trigonometric function of θ. Hence, we take each side of (2) equal to the constant $n^2 > 0$ where n is an integer.

$\therefore$ From (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + n^2 F = 0 \qquad ...(3)$$

and
$$r^2 \frac{d^2 R}{d r^2} + r \frac{dR}{dr} - n^2 R = 0 \qquad ...(4)$$

Equation (4) is homogeneous equation, so we substitute $r = e^z$ so that

$$\frac{dR}{dr} = D_1 z \text{ and } r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 z, \text{ where } D_1 \equiv r\frac{dR}{dr} \equiv \frac{d}{dz}$$

∴ (4) reduces to

$$\{(D_1 - 1) D_1 + D_1 - n^2\} R = 0 \quad \text{ro} \quad (D_1^2 - n^2) R = 0 \qquad ...(5)$$

The general solutions of (3) and (5) are

$$F = A_n \cos n\theta + B_n \sin n\theta \quad \text{and} \quad R = C_n e^{nz} + D_n e^{-nz} = C_n r^n + D_n r^{-n} \qquad ...(6)$$

∴ from (1) the solutions of (A) for integral values of n are given by

$$u_n(r, \theta) = FR = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \qquad ...(7)$$

Taking $n = 1, 2, 3, \ldots$, the more general solution of (A) is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{n} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \qquad ...(8)$$

According to given condition (i) when $r \to 0$, u is finite.

But in (7), when $r \to 0$, $r^{-n} \to \infty$. ∴ We must take $D_n = 0$

∴
$$u(r, \theta) = \sum_{r=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n)$$

or
$$u(r, \theta) = \sum_{r=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta)\, r^n \qquad ...(9)$$

where $E_n = A_n C_n$ and $F_n = B_n C_n$ are the new arbitrary constants.

According to given condition (ii), $u = \sum C_n \cos n\theta$ when $r = a$

i.e.,
$$u(a, \theta) = \sum C_n \cos n\theta$$

$\therefore$ from (9), we get $\sum C_n \cos n\theta = \sum (E_n \cos n\theta + F_n \sin n\theta)\, a^n$

$\therefore$ $F_n = 0$ and $E_n . a^n = C_n$ i.e., $E_n = C^n / a^n$

Putting in (9) the required solution is

$$u(r, \theta) = \sum_{r=1}^{n} C_n (r/a)^n \cos n\theta$$

Example 4: ***Find the steady state temperature in a circular plate of radius a whose circular edge r = a is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface.***

Solution: The steady state temperature in a circular plate with insulated surface is governed by Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{...(A)}$$

Here we are required to solve (A) subject to the following boundary condition

$$u(a, \theta) = f(\theta), \quad 0 \le \theta \le 2\pi \quad \text{...(B)}$$

Obviously $u(r,\theta)$ the solution of (A) must be periodic in θ and finite when $r \to 0$.

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \quad \text{...(1)}$$

Differentiating (1) and substituting in (A), we get

$$F r^2 \frac{d^2 R}{dr^2} + F r \frac{dR}{dr} + R \frac{d^2 F}{d\theta^2} = 0$$

or $$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr}\right) = -\frac{1}{F}\frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \text{...(2)}$$

Two sides of (2) are functions of different artificial variables, so the two sides can be equal to each other if each is equal to the same constant λ (say).

Thus, (2), reduce to the following two ordinary differential equations.

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \text{...(3)}$$

and $$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \text{...(4)}$$

Equation (4) is a homogeneous equation with variable coefficients. To solve (4), let $r = e^z$, so that $r\dfrac{dR}{dr} = D_1 z$ and $r^2\dfrac{d^2R}{dr^2} = (D_1 - 1)\, D_1 z$, where $D_1 \equiv r\dfrac{d}{dr} \equiv \dfrac{d}{dz}$

Putting in (4), it reduces to

$$[(D_1 - 1)D_1 + D_1 - \lambda]\, R = 0 \qquad \text{or} \qquad (D_1^2 - \lambda)R = 0 \qquad \text{...(5)}$$

Now there are following cases :

Case I : *If* $\lambda = 0$, then equation (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} = 0 \qquad \text{and} \qquad D_1^2 R = \frac{d^2R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1\theta + A_2 \quad \text{and} \quad R = A_3 z + A_4 = A_3 \log r + A_4$$

$\therefore \quad u(r, \theta) = FR = (A_1\theta + A_2)(A_3 \log r + A_4)$

Since u must be periodic and finite when $r \to 0$

$\therefore$ we take $A_1 = 0$ and $A_3 = 0$, $\because$ $\log r \to \infty$ as $r \to 0$

$\therefore \quad u(r, \theta) = A_2 A_4 = A$ (constant) ...(6)

Case II : *If* $\lambda = k^2 > 0$, then equations (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} + k^2 F = 0 \qquad \text{and} \qquad (D_1^2 - k^2)R = 0$$

Whose general solutions are

$$F = A_k \cos k\theta + B_k \sin k\theta \;\; \text{and} \;\; R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}$$

$\therefore \quad u(r, \theta) = FR = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k + D_k r^{-k})$ ...(7)

$\because \quad u(r, \theta)$ is periodic with period 2π, so we must take $k = n$

where $n = 1, 2, 3, \ldots,$ (an integer)

(Note that we cannot take $\lambda = -k^2 < 0$, since in this case $u(r, \theta)$ will not involve trigonometric function, of θ while $u(r, \theta)$ must involve trigonometric function of θ as it is periodic function of θ of period 2π.)

$\therefore$ For $n = 1, 2, 3, \ldots,$ (an integer), from (7) the solutions of (A) are given by

$$u_n(r, \theta) = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \qquad \text{...(8)}$$

From (6) and (8) more general solution of (A) can be taken as

$$u(r, \theta) = A + \sum_{n=1}^{\infty} u_n(r, \theta)$$

$$= A + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \qquad ...(9)$$

Since $u(r, \theta)$ is finite when $r \to 0$, but when $r \to 0$, $r^{-n} \to \infty$, so in (9), we must take $D_n = 0$. Putting in (9), we get

$$\therefore \qquad u(r, \theta) = A + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta)\, r^n \qquad ...(10)$$

Where $E_n = A_n C_n$ and $F_n = B_n D_n$ are new arbitrary constants.

Using conditions(B), *i.e.*, $u(a, \theta) = f(\theta)$, from (10), we get

$$f(\theta) = A + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta)\, a^n \qquad ...(11)$$

Multiplying both sides of (11) by $\cos n\theta$ and integrating between the limits $\theta = 0$ to 2π, we get

$$\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = A \int_0^{2\pi} \cos n\theta \, d\theta + a^n E_n \int_0^{2\pi} \cos^2 n\theta \, d\theta + a^n F_n \int_0^{2\pi} \sin n\theta \cos n\theta \, d\theta$$

$$= \left[A \frac{\sin n\theta}{n}\right]_0^{2\pi} + \frac{1}{2} a^n E_n \int_0^{2\pi} (1 + \cos 2n\theta)\, d\theta + \frac{1}{2} a^n F_n \int_0^{2\pi} \sin 2n\theta \, d\theta$$

$$= 0 + \frac{1}{2} a^n E_n \left[\theta + \frac{1}{2n} \sin 2n\theta\right]_0^{2\pi} + \frac{1}{2} a^n F_n \left[-\frac{1}{2n} \cos 2n\theta\right]_0^{2\pi}$$

$$= \frac{1}{2} a^n E_n 2\pi = \pi a^n E_n$$

$$\therefore \qquad E_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \qquad ...(12)$$

Similarly, multiplying both sides of (11) by $\sin n\theta$ and integrating between the limits $\theta = 0$ to 2π, we get

$$F_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \qquad ...(13)$$

Integrating both sides of (11) between the limits, $\theta = 0$ to 2π, we get

$$\int_0^{2\pi} f(\theta)\, d\theta = \int_0^{2\pi} A\, d\theta + \sum_{n=1}^{\infty} [E_n \int_0^{2\pi} \cos n\theta \, d\theta + F_n \int_0^{2\pi} \sin n\theta \, d\theta]$$

$$= A 2\pi + 0$$

$$\therefore \qquad A = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{2} E_0 \qquad ...(14)$$

From (10), $$u(r, \theta) = \frac{1}{2} E_0 + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta)\, r^n \qquad ...(15)$$

Hence, the required solution is given by (15) where E_0, E_n are given by (12) and F_n by (13).

Example 5: ***Find the steady state temperature in a circular plate of radius a which has half of its circumference at 0°C and the other half at temperature u_0°C.***

Solution: The steady state temperature in a circular plate is governed by Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(A)$$

Here we are required to solve (A) subject to the boundary condition

$$u(a, \theta) = f(\theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$$

Obviously $u(r, \theta)$, the solution of (A), must be periodic in θ and finite when $r \to 0$.

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad ...(1)$$

Now proceeding similarly as in Ex. 4, we get

$$u(r, \theta) = \frac{1}{2} E_0 + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \qquad ...(2)$$

where $$E_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$= \frac{1}{\pi a^n} \left[\int_0^{\pi} f(\theta) \cos n\theta \, d\theta + \int_{\pi}^{2\pi} f(\theta) \cos n\theta \, d\theta \right]$$

$$= \frac{1}{\pi a^n} \left[\int_0^{\pi} u_0 \cos n\theta \, d\theta + 0 \right] = \frac{u_0}{\pi a^n} \left(\frac{1}{n} \sin n\theta \right)_0^{\pi} = 0$$

$$E_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \, d\theta = \frac{1}{\pi} \left[\int_0^{\pi} f(\theta) \, d\theta + \int_{\pi}^{2\pi} f(\theta) \, d\theta \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} u_0 \, d\theta + 0 \right] = \frac{1}{\pi} (\pi u_0) = u_0$$

and $$F_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

$$= \frac{1}{\pi a^n} \left[\int_0^{\pi} f(\theta) \sin n\theta \, d\theta + \int_{\pi}^{2\pi} f(\theta) \sin n\theta \, d\theta \right]$$

$$= \frac{1}{\pi a^n} \left[\int_0^{\pi} u_0 \sin n\theta \, d\theta + 0 \right]$$

$$= \frac{u_0}{\pi a^n} \left[-\frac{1}{n} \cos n\theta \right]_0^{\pi} = \frac{u_0}{\pi n a^n} (1 - \cos n\theta) = \frac{u_0}{\pi n a^n} [1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{if } n = 2m \text{ (even)}, \quad m = 1, 2, \ldots \\ \dfrac{2u_0}{\pi (2m-1) a^{(2m-1)}} & \text{if } n = 2m-1 \text{ (odd)}, \quad m = 1, 2, \ldots \end{cases}$$

Hence, from (2), the required solution is

$$u(r, \theta) = \frac{1}{2} u_0 + \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{2m-1} \cdot \frac{\sin (2m-1)\theta}{2m-1}$$

Example 6: ***A circular sector is determined by $0 \le r \le a, 0 \le \theta \le \alpha$. The temperature is kept at 0°C along the straight edges and at $f(\theta)$ along the curved edge. Find the steady state temperature at any point of the sector with its surfaces insulated.***

Solution: Let OAB be a thin circular sector of radius a and $\angle AOB = \alpha$. The surfaces of the sector are insulated. The steady state temperature $u(r, \theta)$ in the sector is governed by the Laplace equation (in place polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(A)}$$

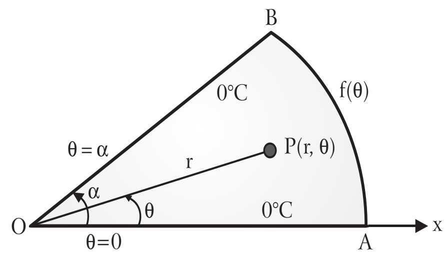


Fig. 14.6

Here the boundary conditions of the problem are

$$u(r, 0) = 0,\ u(r, \alpha) = 0, \quad 0 < r < a \qquad \text{...(B}_1\text{)}$$

and
$$u(a, \theta) = f(\theta),\ 0 < \theta < \alpha \qquad \text{...(B}_2\text{)}$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad \text{...(1)}$$

Where R is a function of r alone and F function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F r^2 \frac{d^2 R}{dr^2} + F r \frac{dR}{dr} + R\frac{d^2 F}{d\theta^2} = 0$$

or
$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{F}\frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \qquad \text{...(2)}$$

Since each side of (2) is a function of different independent variables, so the two will be equal to each other if each is equal to the same constant say λ.

Thus, from (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \qquad \text{...(3)}$$

and
$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0 \qquad \text{...(4)}$$

Equation (4) is a homogeneous equation with variable coefficients. To solve (4), let $r = e^z$, so that

$$r\frac{dR}{dr} = D_1 z \quad \text{and} \quad r^2 \frac{d^2R}{dr^2} = (D_1 - 1)D_1 z \quad \text{where } D_1 \equiv r\frac{d}{dr} \equiv \frac{d}{dz}$$

Putting in (4), it reduces to

$$\{(D_1 - 1)D_1 + D_1 - \lambda\}\ R = 0 \quad \text{or} \quad (D_1^2 - \lambda)R = 0 \qquad \text{...(5)}$$

Now there are following cases :

Case I : *If* $\lambda = 0$, then (3) and (5), reduce to

$$\frac{d^2F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2\ R = \frac{d^2R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1\theta + A_2,\ \ R = A_3 z + A_4 = A_3 \log r + A_4$$

$\therefore$ $u(r, \theta) = FR = (A_1\theta + A_2)(A_3 \log r + A_4)$

$\therefore$ Condition (B_1) *i.e.,* $u(r, 0) = 0$ and $u(r, \alpha) = 0$

$\Rightarrow$ $A_2(A_3 \log r + A_4) = 0$ and $(A_1\alpha + A_2)(A_3 \log r + A_4) = 0$

$\Rightarrow$ $A_2 = 0$ and $A_1\alpha + A_2 = 0$, $\because$ $A_3 \log r + A_4 \neq 0$

otherwise we get $u(r, \theta) = 0$, which is inadmissible.

$\Rightarrow$ $A_1 = 0$ and $A_2 = 0$

$\Rightarrow$ $u(r, \theta) = 0$, which is inadmissible.

So we reject the case $\lambda = 0$.

Case II : *If* $\lambda = k^2 > 0$, then (3) and (5), reduce to

$$\frac{d^2F}{d\theta^2} + k^2F = 0 \quad \text{and} \quad (D_1^2 - k^2)R = 0$$

Whose general solutions are

$$F = A_k \cos k\theta + B_k \sin k\theta \text{ and } R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}$$

$\therefore$ $u(r, \theta) = F.R = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k + D_k r^{-k})$ ...(6)

Note that we cannot take $\lambda = -k^2 < 0$. Since in his case $u(r, \theta)$ will not involve trigonometric function of θ while $u(r, \theta)$ must involve trigonometric function of θ as it is periodic function of period 2π.

Since $u(r, \theta)$ is finite when $r \to 0$, but when $r \to 0$, $r^{-k} \to \infty$ $\because$ $k > 0$,

So in (6) we must take $D_k = 0$

$\therefore$ from (6) $u(r, \theta) = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k)$ ...(7)

Now from (7), the boundary condition (B_1) *i.e.*, $u(r, 0) = 0$

$\Rightarrow$ $A_k (C_k . r^k) = 0 \Rightarrow A_k = 0,$

$\because$ $r^k = 0$ or $C_k = 0$ will lead to trivial solution $u(r, \theta) = 0$ which is inadmissible.

$\therefore$ from (7), we have

$$u(r, \theta) = (B_k \sin k\theta)(C_k . r^k) \quad ...(8)$$

Now from (8), the boundary condition (B_1) *i.e.*, $u(r, \alpha) = 0$

$\Rightarrow$ $B_k C_k \sin k\alpha . r^k = 0$

$\Rightarrow$ $\sin k\alpha = 0.$ $\because$ $B_k C_k \neq 0$ and $r^k \neq 0$

$\Rightarrow$ $k\alpha = n\pi$ *i.e.*, $k = n\pi/\alpha$, n is an integer.

$\therefore$ From (8), for integral values of n, the solutions of (A), satisfying conditions (B_1) are given by

$$u_n(r, \theta) = F_n \sin(n\pi\theta/\alpha) . r^{n\pi/\alpha}$$

Taking $F_n = B_k C_k$ is the new arbitrary constant.

Now taking $n = 1, 2, 3, \ldots$, the more general solution of (A) is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} F_n \sin(n\pi\theta/\alpha) . r^{n\pi/\alpha} \quad ...(9)$$

Which also satisfy the boundary conditions (B_1).

Now from (9), the boundary condition (B_2) *i.e.*, $u(a, \theta) = f(\theta)$

$$\Rightarrow \quad f(\theta) = \sum_{n=1}^{\infty} F_n \sin(n\pi\theta/\alpha) . a^{n\pi/\alpha} \quad ...(10)$$

The right hand side of (10) can be considered as the Fourier series of the function $f(\theta)$ on its left hand side for $0 < \theta < \alpha$

$$\therefore \qquad a^{n\pi/\alpha} . F_n = \frac{2}{\alpha} \int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta$$

or
$$F_n = \frac{2}{\alpha . a^{n\pi/\alpha}} \int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta \qquad ...(11)$$

Hence, the required temperature in the sector is given by (9) where F_n is given by (11).

Note : If $\alpha = \pi$, then the thin circular sector reduces to a semi-circular plate and the result (9) and (11), reduce to the results (10) and (12) of Ex. 1 on page 703.

Example 7: ***Find the steady state temperature at any point of a circular sector; determined by $0 \le r \le a, 0 \le \theta \le \pi/2$, if the temperature is maintained at 0°C along the straight edge and $50(\pi\theta - 2\theta^2)$ along the curved edge. Also find the steady state temperature at $(a/2, \pi/4)$.***

Solution: The steady state temperature $u(r, \theta)$ in the sector will be governed by the Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(A)$$

Here the boundary conditions of the problem are

$$u(r, 0) = 0, \qquad u(r, \pi/2) = 0, \qquad 0 < r < a \qquad ...(B_1)$$

and
$$u(a, \theta) = f(\theta) = 50(\pi\theta - 2\theta^2) \qquad ...(B_2)$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad ...(1)$$

Where R is a function of r alone and F a function of θ alone.

Now proceeding similarly as in Ex. 6 page 741, we get

$$u(r, \theta) = \sum_{n=1}^{\infty} F_n \sin(2n\theta) . r^{2n} \quad \text{Here } \alpha = \pi/2 \qquad ...(2)$$

where
$$F_n = \frac{2}{(\pi/2)\, a^{2n}} \int_0^{\pi/2} f(\theta) \sin 2n\theta\, d\theta,$$

$$= \frac{4}{\pi\, a^{2n}} \int_0^{\pi/2} 50\,(\pi\theta - 2\theta^2) \sin 2n\theta\, d\theta$$

$$= \frac{200}{\pi a^{2n}}\left[(\pi\theta - 2\theta^2).\left(-\frac{\cos 2n\theta}{2n}\right) - (\pi - 4\theta)\left(-\frac{\sin 2n\theta}{4n^2}\right) + (-4)\left(\frac{\cos 2n\theta}{8n^3}\right)\right]_0^{\pi/2}$$

$$= \frac{200}{\pi a^{2n}}\left[\frac{4}{8n^3}(1-\cos n\pi)\right] = \frac{100}{\pi n^3 a^{2n}}[1-(-1)^n]$$

$$= \begin{cases} 0, & \text{when } n = 2m \text{ (even)}, \quad m = 1, 2, \ldots \\ \dfrac{200}{\pi (2m-1)^3 \; a^{2n}}, & \text{when } n = 2m-1 \text{ (odd)}, \quad m = 1, 2, \ldots \end{cases}$$

Hence, from (2), the steady state temperature at any point (r, θ) in the sector is given by

$$u(r, \theta) = \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2(2m-1)}\frac{\sin 2(2m-1)\theta}{(2m-1)^3} \qquad \ldots(3)$$

2nd Part : Putting $r = a/2$ and $\theta = \pi/4$ in (3), the temperature at $(a/2, \pi/4)$ is given by

$$u(a/2, \pi/4) = \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{2(2m-1)}\frac{\sin\{(2m-1)\pi/2\}}{(2m-1)^3}$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^{4m-2}(2m-1)^3}$$

$$\because \quad \sin\{(2m-1)\pi/2\} = \sin(m\pi - \pi/2) = (-1)^{m+1}.$$

Example 8: ***u is a function of r and θ satisfying the equation***

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a, r = b, \theta = 0, \theta = \pi/2$. ***Its value along the boundary*** $r = a$ ***is*** $\theta(\pi/2 - \theta)$ ***and along the other boundary is zero. Prove that***

$$u(r, \theta) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}}\frac{\sin(4m-2)\theta}{(2m-1)^3}.$$

Solution: The function $u(r, \theta)$ satisfy the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \ldots(A)$$

The boundary conditions of the problem are

$$u(r, 0) = 0, u(r, \pi/2) = 0, \quad a < r < b \qquad \ldots(B_1)$$

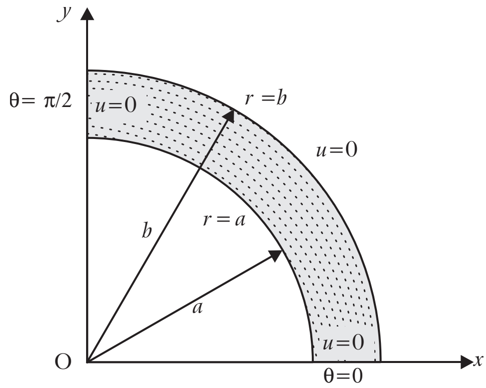


Fig. 14.7

$$u(b, \theta) = 0, 0 < \theta < \pi/2 \qquad ...(B_2)$$

and $$u(a, \theta) = \theta(\pi/2 - \theta), 0 < \theta < \pi/2 \qquad ...(B_3)$$

Here we are required to find the solution of (A), satisfying the conditions (B_1), (B_2) and (B_3).

Let the solution of (A) be given by

$$u(r, \theta) = R(r)\ F(\theta) = RF \text{ (say)} \qquad ...(1)$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F\frac{d^2R}{dr^2} + \frac{F}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2F}{d\theta^2} = 0$$

or $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{F}\frac{d^2F}{d\theta^2} = \lambda \text{ (say)} \qquad ...(2)$$

Now the two sides of (2) are functions of different independent variables, so the two functions will be equal to each other if each is equal to the same constant λ (say).

Thus, from (2), we get the following two ordinary differential equations.

$$\frac{d^2F}{d\theta^2} + \lambda F = 0 \qquad ...(3)$$

and $$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0 \qquad ...(4)$$

The equation (4) is a homogeneous differential equation with variable coefficients. To solve it let $r = e^z$, so that

$$r\frac{dR}{dr} = D_1 z \text{ and } r^2\frac{d^2R}{dr^2} = (D_1 - 1)D_1 z, \text{ where } D_1 \equiv r\frac{d}{dr} \equiv \frac{d}{dz}$$

Putting in (4), it reduces to

$$[(D_1 - 1)D_1 + D_1 - \lambda]R = 0 \quad \text{or} \quad (D_1^2 - \lambda)R = 0 \qquad ...(5)$$

Now there are following cases :

Case I : *If* $\lambda = 0$, then (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = \frac{d^2R}{dz^2} = 0$$

Whose general solutions are

$$F = A\theta + B \quad \text{and} \quad R = Cz + D = C\log r + D$$

$$\therefore \quad u(r, \theta) = FR = (A\theta + B)(C\log r + D) \qquad ...(6)$$

From (6), conditions (B_1) *i.e.,* $u(r, 0) = 0$ and $u(r, \pi/2) = 0$

$\Rightarrow$ $B(C\log r + D) = 0$ and $(A.\pi/2 + B)(C\log r + D) = 0$

$\Rightarrow$ $B = 0$ and $A\pi/2 + B = 0$,

$\because$ $C\log r + D = 0$ will lead to, trivial solution $u(r, \theta) = 0$, which is inadmissible.

$\Rightarrow$ $A = 0$ and $B = 0$

$\Rightarrow$ $u(r, \theta) = 0$, which is inadmissible.

Hence, we reject the case $\lambda = 0$.

Case II : If $\lambda = k^2 > 0$, then (3) and (5) reduce to

$$\frac{d^2F}{d\theta^2} + k^2F = 0 \quad \text{and} \quad (D_1^2 - k^2)R = 0$$

Whose general solutions are

$$F = A_k\cos k\theta + B_k\sin k\theta \text{ and } R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}.$$

$$\therefore \quad u(r, \theta) = F.R = (A_k\cos k\theta + B_k\sin k\theta)(C_k r^k + D_k r^{-k}) \qquad ...(7)$$

Note that $\lambda \neq -k^2 < 0$. Since in the problem $u(r, \theta)$ is a periodic function of period 2π, i.e., $u(r, \theta) = u(r, \theta + 2\pi)$, so $u(r, \theta)$ must involve trigonometric function of θ while if we take $\lambda = -k^2 < 0$, then $u(r, \theta)$ will not involve trigonometric function of θ.

Now condition in (B_1), *i.e.*, $u(r, 0) = 0$ and (7)

$\Rightarrow \quad A_k(C_k r^k + D_k r^{-k}) = 0 \Rightarrow A_k = 0 \qquad \because C_k r^k + D_k r^{-k} \neq 0$

Putting $A_k = 0$, in (7), we get

$$u(r, \theta) = (C_k r^k + D_k r^{-k}) B_k \sin k\theta \qquad ...(8)$$

Now condition in (B_1) *i.e.*, $u(r, \pi/2) = 0$ and (8)

$\Rightarrow \quad (C_k r^k + D_k r^{-k}) B_k \sin(k\pi/2) = 0$

$\Rightarrow \quad \sin(k\pi/2) = 0, \qquad \because C_k r^k + D_k r^{-k} \neq 0, B_k \neq 0$

$\Rightarrow \quad k\pi/2 = n\pi$, where n is integer.

$\Rightarrow \quad k = 2n.$

$\therefore$ from (8)

$$u_n(r, \theta) = (E_n r^{2n} + F_n r^{-2n}) \sin 2n\theta$$

Where $E_n = C_k B_k$ and $F_n = D_k B_k$ are the new arbitrary constants are solutions of (A) satisfying conditions (B_1) for integral values of n.

Taking $n = 1, 2, 3, \ldots$, the more general solution of (A) can be written as

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} (E_n r^{2n} + F_n r^{-2n}) \sin 2n\theta \qquad ...(9)$$

Which also satisfy the conditions (B_1).

Now condition (B_2) *i.e.*, $u(b, \theta) = 0$ and (9)

$\Rightarrow \quad (E_n b^{2n} + F_n b^{-2n}) \sin 2n\theta = 0$

$\Rightarrow \quad E_n b^{2n} + F_n b^{-2n} = 0, \qquad \because \sin 2n\theta \neq 0$

$\Rightarrow \quad F_n = -E_n b^{4n}$

Putting in (9), we get

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n (r^{2n} - b^{4n} r^{-2n}) \sin 2n\theta \qquad ...(10)$$

Now condition (B_3) *i.e.,* $u(a, \theta) = \theta(\pi/2 - \theta)$ and (10)

$$\Rightarrow \qquad \theta(\pi/2 - \theta) = \sum_{n=1}^{\infty} E_n(a^{2n} - b^{4n}\, a^{-2n}) \sin 2n\theta \qquad \text{...(11)}$$

The right hand side of (11) can be considered as Fourier series of $\theta(\pi/2 - \theta)$ the function on its left hand side in half range $0 \le \theta \le \pi/2$

$$\therefore \quad E_n(a^{2n} - b^{4n}\, a^{-2n}) = \frac{2}{(\pi/2)} \int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right) \sin 2n\theta \, d\theta$$

$$= \frac{4}{\pi}\left[\theta\left(\frac{\pi}{2} - \theta\right)\left(-\frac{\cos 2n\theta}{2n}\right) - \left(\frac{\pi}{2} - 2\theta\right)\left(-\frac{\sin 2n\theta}{4n^2}\right) + (-2)\left(\frac{\cos 2n\theta}{8n^3}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{\pi n^3}[1 - \cos n\pi] = \frac{1}{\pi n^3}[1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{when } n = 2m \text{ (even)}, \quad m = 1, 2, \ldots \\ \dfrac{2}{\pi(2m-1)^3} & \text{when } n = 2m-1 \text{ (odd)}, \quad m = 1, 2, \ldots \end{cases}$$

$$\therefore \quad E_n = \begin{cases} 0, & \text{when } n = 2m, \quad m = 1, 2, \ldots \\ \dfrac{2}{\pi(2m-1)^3\{a^{2(2m-1)} - b^{4(2m-1)}\, a^{-2(2m-1)}\}} & \text{when } n = 2m-1, \quad m = 1, 2, \ldots \end{cases}$$

Hence, from (10), we have

$$u(r, \theta) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\left\{r^{2(2m-1)} - b^{4(2m-1)} r^{-2(2m-1)}\right\} \sin 2(2m-1)\theta}{(2m-1)^3 \left\{a^{2(2m-1)} - b^{4(2m-1)} a^{-2(2m-1)}\right\}}$$

$$= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}} \cdot \frac{\sin(4m-2)\theta}{(2m-1)^3}$$

Exercise 14.2

1. Obtain steady state temperature distribution in a semi-circular plate of radius a, insulated on both faces, with its curved boundary kept at a constant temperature u_0 and its bounding diameter kept at temperature zero. **[Meerut 2011 (B.P.)]**

2. A thin semi-circular plate of radius a, insulated on both faces, has its boundary diameter kept at 0°C and its temperature along the semi-circular boundary is 100°C. If $u(r, \theta)$ is the steady state temperature, in the plate, then prove that

$$u(r, \theta) = \frac{400}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}$$

Also find $u(a/4, \pi/2)$.

3. Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at 100°C.

4. Find the steady state temperature at the points in the sector given by $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate if the temperature is maintained at 0°C along the side edges and at a constant temperature u_0 °C along the curved edge.

Answers 14.2

1.	$u(r, \theta) = \frac{440}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}$
2.	$u(a/4, \pi/2) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)} \left(\frac{1}{4}\right)^{2m-1}$
3.	$u(r, \theta) = 50 + \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{(2m-1)}$
4.	$u(r, \theta) = \frac{440}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{4(2m-1)} \frac{\sin 4(2m-1)\theta}{(2m-1)}$

14.7 Solution of Laplace's Equation in Rectangular Cartesian Coordinates (x, y, z) by the Method of Separation of Variables

Three dimensional Laplace's equation in cartesian coordinates is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad \text{...(A)}$$

Let the solution of (A) be of the form

$$u(x, y, z) = F(x)\ Y(y)\ Z(z) = FYZ \text{ (say)} \quad \text{...(1)}$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$YZ\frac{d^2F}{dx^2}+FZ\frac{d^2Y}{dy^2}+FY\frac{d^2Z}{dz^2}=0$$

or

$$\frac{1}{F}\frac{d^2F}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2} \quad \text{...(2)}$$

All the three terms in (2) are functions of different independent variables, therefore (2) will be satisfied if each of the three terms is constant. So let

$$\frac{1}{F}\frac{d^2F}{dx^2}=\lambda_1,\ \frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda_2 \quad \text{and} \quad -\frac{1}{Z}\frac{d^2Z}{dz^2}=\lambda$$

where $\lambda_1+\lambda_2=\lambda$

Thus, we get the following three ordinary differential equations

$$\frac{d^2F}{dx^2}-\lambda_1 F=0,\ \frac{d^2Y}{dy^2}-\lambda_2 Y=0 \quad \text{and} \quad \frac{d^2Z}{dz^2}+\lambda Z=0 \quad \text{...(3)}$$

where $\lambda_1+\lambda_2=\lambda$

Now there are the following three cases :

Case I : When $\lambda_1=0=\lambda_2=\lambda$, then the equations given in (3) reduce to

$$\frac{d^2F}{dx^2}=0,\ \frac{d^2Y}{dy^2}=0 \quad \text{and} \quad \frac{d^2Z}{dz^2}=0$$

Whose general solutions are

$$F = Ax + B,\ Y = Cy + D \text{ and } Z = Ez + F$$

$\therefore$ from (1) solution of (A) is given by

$$u(x, y) = FYZ = (Ax + B)(Cy + D)(Ez + F) \qquad ...(4)$$

Case II : When $\lambda_1 = k_1^2 > 0$, $\lambda_2 = k_2^2 > 0$ and $\lambda = k^2 > 0$ s.t., $k_1^2 + k_2^2 = k^2$, then the equations in (3) reduce to

$$\frac{d^2F}{dx^2} - k_1^2 F = 0,\quad \frac{d^2Y}{dy^2} - k_2^2 Y = 0 \quad\text{and}\quad \frac{d^2Z}{dz^2} + k^2 Z = 0$$

Whose general solutions are

$$F = A_{k_1} e^{xk_1} + B_{k_1} e^{-xk_1} \qquad \text{or} \qquad F = A'_{k_1} e^{\pm xk_1}$$

$$Y = C_{k_2} e^{yk_2} + D_{k_2} e^{-yk_2} \qquad \text{or} \qquad Y = B'_{k_2} e^{\pm yk_2}$$

and $$Z = E_k \cos kz + F_k \sin kz \qquad \text{or} \qquad Z = C'_k e^{\pm ikz}$$

$\therefore$ from (1), the most general solution of (A) is given by

$$u(x, y, z) = \sum_{k_1, k_2} (A_{k_1} e^{xk_1} + B_{k_1} e^{-xk_1})(C_{k_2} e^{yk_2} + D_{k_2} e^{-yk_2}).$$

$$(E_k \cos kz + F_k \sin kz) \qquad ...(5)$$

where $$k^2 = k_1^2 + k_2^2$$

The solution can also be taken as

$$u(x, y, z) = \sum_{k_1, k_2} H_{k_1 k_2} e^{\pm xk_1 \pm yk_2 \pm ikz} \qquad ...(6)$$

where $$k^2 = k_1^2 + k_2^2$$

and $H_{k_1 k_2} = A'_{k_1} B'_{k_2} C'_k$ is the new arbitrary constant.

Case III : When $\lambda_1 = -k_1^2 < 0$, $\lambda_2 = -k_2^2 < 0$ and $\lambda = -k^2 < 0$ s.t. $k^2 = k_1^2 + k_2^2$, then the equations in (3) reduce to

$$\frac{d^2F}{dx^2} + k_1^2 F = 0,\quad \frac{d^2Y}{dy^2} + k_2^2 Y = 0 \qquad\text{and}\qquad \frac{d^2Z}{dz^2} - k^2 Z = 0$$

Whose general solutions are

$$F = A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x \qquad \text{or} \qquad F = A'_{k_1} e^{\pm ik_1 x}$$

$$Y = C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \qquad \text{or} \qquad Y = B'_{k_2} e^{\pm ik_2 y}$$

and $$Z = E_k e^{kz} + F_k e^{-kz} \quad \text{or} \quad Z = C'_k e^{\pm kz}$$

∴ from (1), the most general solution of (A) is given by

$$u(x, y, z) = \sum_{k_1, k_2} (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y).$$

$$(E_k e^{kz} + F_k e^{-kz}) \quad ...(7)$$

where $$k^2 = k_1^2 + k_2^2$$

The solution can also be taken as

$$u(x, y, z) = \sum_{k_1, k_2} H_{k_1 k_2} e^{\pm ik_1 x \pm ik_2 y \pm kz}, \text{ where } k^2 = k_1^2 + k_2^2$$

and $H_{k_1 k_2} = A'_{k_1} . B'_{k_2} . C'_k$ is the new arbitrary constant.

14.8 Laplace Equation in Cylindrical Coordinates

Laplace equation in cartesian coordinates (x, y, z) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ...(1)$$

Proceeding as in {12.9 on page 568 in cylindrical coordinates (ρ, ϕ, z), we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Hence, the Laplace equation (1) in cylindrical coordinates (ρ, ϕ, z) is transformed to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

14.9 Solution of Laplace's Equation in Cylindrical Coordinates by the Method of Separation of Variables

[Meerut 2001, 02 (B.P.)]

The Laplace's equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ...(A)$$

Let the solution of (A) be of the form

$$u(\rho, \phi, z) = R(\rho)\ F(\phi)\ Z(z) = RFZ \text{ (say)} \qquad ...(1)$$

Where R is a function of ρ alone, F a function of ϕ alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$FZ\frac{d^2R}{d\rho^2} + \frac{FZ}{\rho}\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\frac{d^2F}{d\phi^2} + RF\frac{d^2Z}{dz^2} = 0$$

or $$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 F}\frac{d^2F}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 \text{ (say)} \qquad ...(2)$$

Here in (2), right hand side is a function of independent variable z alone while left hand side is a function of ρ and ϕ. So the two sides will be equal to each other if each is equal to the same constant say $-k^2$. Here constant $-k^2$ is chosen, because the solutions obtained from the resulting equations are useful for practical problems.

Thus, from (2) we get the following equations

$$\frac{d^2Z}{dz^2} - k^2 F = 0 \qquad ...(3)$$

and $$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{\rho^2 F}\frac{d^2F}{d\phi^2} = -k^2$$

or $$\frac{\rho^2}{F}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + k^2\rho^2 = -\frac{1}{F}\frac{d^2F}{d\phi^2} = m^2 \text{ (say)} \qquad ...(4)$$

Here in (4) right hand side is a function of independent variable ϕ only while left hand side is a function of independent variable ρ only, so the two will be equal to each other if each is equal to the same constant. On physical grounds of the problem the solution must involve the trigonometrical functions of ϕ, so we choose this constant equal to m^2 (say). Also the physical condition of the problem *i.e.*, $u(\rho, \phi) = u(\rho, \phi + 2\pi)$ will be satisfied only if we suppose m to be an integer.

Thus, from (4), we get the following equations

$$\frac{d^2F}{d\phi^2} + m^2 F = 0 \qquad ...(5)$$

and $$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + k^2\rho^2 = m^2$$

or $$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (k^2\rho^2 - m^2)R = 0 \qquad ...(6)$$

Now let $k\rho = x$, so that $\dfrac{d\rho}{dx} = \dfrac{1}{k}$

and $$\frac{dR}{d\rho} = \frac{dR}{dx} \cdot \frac{dx}{d\rho} = k\frac{dR}{dx} \quad \Rightarrow \quad \frac{d}{d\rho} \equiv k\frac{d}{dx}$$

and $$\frac{d}{d\rho}\left(\frac{dR}{d\rho}\right) = k\frac{d}{dx}\left(k\frac{dR}{dx}\right)$$

$\therefore$ $$\frac{d^2R}{d\rho^2} = k^2\frac{d^2R}{dx^2}$$

Putting in (6), it reduces to

$$\frac{x^2}{k^2}k^2\frac{d^2R}{dx^2} + \frac{x}{k}.k\frac{dR}{dx} + (x^2 - m^2)R = 0$$

or $$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right)R = 0 \qquad ...(7)$$

Now general solution of (3) is

or $$\left.\begin{aligned} Z &= A_k e^{kz} + B_k e^{-kz} \\ Z &= A'_k e^{\pm kz} \end{aligned}\right\} \qquad ...(8)$$

General solution of (5) is

or $$\left.\begin{aligned} F &= C_m \cos m\phi + D_m \sin m\phi \\ F &= C'_m e^{\pm im\phi} \end{aligned}\right\} \qquad ...(9)$$

Equation (7) is Bessel's equation, so its general solution is

$$R = E_{km} J_m(x) + F_{km} J_{-m}(x) = E_{km} J_m(k\rho) + F_{km} J_{-m}(k\rho),$$

when m is not an integer

and $$R = E_{km} J_m(k\rho) + F_{km} Y_m(k\rho), \text{ when } m \text{ is an integer.}$$

Hence, from (1), the most general solution of (A) is given by

$$u(\rho, \theta, z) = \sum_{k,m} (A_k e^{kz} + B_k e^{-kz})(C_m \cos m\phi + D_m \sin m\phi)$$

$$[E_{km} J_m(k\rho) + F_{km} J_{-m}(k\rho)]$$

when m is not an integer.

and $$u(\rho, \phi, z) = \sum_{k,m} (A_k e^{kz} + B_k e^{-kz})(C_m \cos m\phi + D_n \sin m\phi).$$

$[E_{km} J_m(k\rho) + F_{km} Y_m(k\rho)]$ when m is an integer.

Note : 1. The solution can also be written as

$$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} [G_{km} J_m(k\rho) + H_{km} J_{-m}(k\rho)]$$

when m is not an integer.

And $$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} [G_{km} J_m(k\rho) + H_{km} Y_m(k\rho)]$$

when m is an integer.

Where $G_{km} = A'_k C'_m . E_{km}$ and $H_{km} = A'_k C'_m F_{km}$ are the new arbitrary constants.

2. The solution can also be written as

$$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} R(\rho)$$

when $R(\rho)$ is the solution of Bessel's equation (7).

14.10 Laplace Equation in Spherical Coordinates

Laplace equation in cartesian coordinates (x, y, z) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

Proceeding as in {12.11 on page 573 in spherical polar coordinates (r, θ, ϕ), we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

Hence, the Laplace equation (1), in spherical coordinates (r, θ, ϕ) is transformed to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

14.11 Solution of Laplace's Equation in Spherical Coordinates by the Method of Separation of Variables

The Laplace's equation in spherical coordinates is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=0 \quad \text{...(A)}$$

Obviously solution of (A) is a function of r, θ and ϕ.

∴ Let the solution $u(r,\theta,\phi)$ of (A) be given by

$$u(r,\theta,\phi)=R(r)\ F_1(\theta)\ F_2(\phi)=R\ F_1\ F_2 \text{ (say)} \quad \text{...(1)}$$

Where R is a function of r alone, F_1 a function of θ alone and F_2 a function of ϕ alone.

Differentiating (1) and substituting in (A), we get

$$F_1 F_2\frac{d^2R}{dr^2}+\frac{2F_1F_2}{r}\frac{dR}{dr}+\frac{RF_2}{r^2}\frac{d^2F_1}{d\theta^2}+\frac{RF_2\cot\theta}{r^2}\frac{dF_1}{d\theta}+\frac{RF_1}{r^2\sin^2\theta}\frac{d^2F_2}{d\phi^2}=0$$

or
$$\left[\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{r^2F_1}\left(\frac{d^2F_1}{d\theta^2}+\cot\theta\frac{dF_1}{d\theta}\right)\right]r^2\sin^2\theta$$
$$=-\frac{1}{F_2}\frac{d^2F_2}{d\phi^2}=m^2 \text{ (say)} \quad \text{...(2)}$$

In (2), right hand side is a function of independent variable ϕ alone while left hand side is independent of ϕ. So the two sides will be equal only if both are equal to the same constant. We choose this constant equal to m^2, as in many physical problems we require the solution which involve trigonometric functions of θ.

Thus (2), gives

$$\frac{d^2F_2}{d\phi^2}+m^2F_2=0 \quad \text{...(3)}$$

and
$$\left[\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{r^2F_1}\left(\frac{d^2F_1}{d\theta^2}+\cot\theta\frac{dF_1}{d\theta}\right)\right]r^2\sin^2\theta=m^2$$

or
$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)=\frac{m^2}{\sin^2\theta}-\frac{1}{F_1}\left(\frac{d^2F_1}{d\theta^2}+\cot\theta\frac{dF_1}{d\theta}\right)=n(n+1)\text{ say} \quad \text{...(4)}$$

In (4) two sides are functions of different artificial variable, so they will be equal to each other if each is equal to the same constant say $n(n+1)$. (Note)

Thus, from (4) we get the following two differential equations

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)=n(n+1)$$

or $$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0 \quad ...(5)$$

and $$\frac{m^2}{\sin^2\theta}-\frac{1}{F_1}\left(\frac{d^2F_1}{d\theta^2}+\cot\theta\frac{dF_1}{d\theta}\right)=n(n+1)$$

or $$\frac{d^2F_1}{d\theta^2}+\cot\theta\frac{dF_1}{d\theta}+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)F_1=0 \quad ...(6)$$

Equation (5) is a homogeneous equation with variable coefficient. So let $r=e^z$ so that $r\frac{dR}{dr}=D_1z$ and $r^2\frac{d^2R}{dr^2}=(D_1-1)D_1z$

where $$r\frac{d}{dr}\equiv\frac{d}{dz}=D_1$$

Putting in (5), we get

$$[(D_1-1)D_1+2D_1-n(n+1)]R=0$$

or $$(D_1^2+D_1-n^2-n)R=0$$

or $$[(D_1-n)(D_1+n)+(D_1-n)]R=0$$

or $$(D_1-n)(D_1+n+1)R=0 \quad ...(7)$$

Now to solve (6), let $\mu=\cos\theta$, so that $\frac{dF_1}{d\theta}=\frac{dF_1}{d\mu}\frac{d\mu}{d\theta}=-\sin\theta\frac{dF_1}{d\mu}$

and $$\frac{d^2F_1}{d\theta^2}=\frac{d}{d\theta}\left(-\sin\theta\frac{dF_1}{d\mu}\right)=-\cos\theta\frac{dF_1}{d\mu}-\sin\theta\frac{d^2F_1}{d\mu^2}\cdot\frac{d\mu}{d\theta}$$

$$=-\cos\theta\frac{dF_1}{d\mu}+\sin^2\theta\frac{d^2F_1}{d\mu^2}=-\cos\theta\frac{dF_1}{d\mu}+(1-\cos^2\theta)\frac{d^2F_1}{d\mu^2}$$

Putting in (6), we get

$$-\cos\theta\frac{dF_1}{d\mu}+(1-\cos^2\theta)\frac{d^2F_1}{d\mu^2}+\cot\theta(-\sin\theta)\frac{dF_1}{d\mu}+\left[n(n+1)-\frac{m^2}{1-\cos^2\theta}\right]F_1=0$$

or $$(1-\cos^2\theta)\frac{d^2F_1}{d\mu^2} - 2\cos\theta\frac{dF_1}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\cos^2\theta}\right]F_1 = 0$$

or $$(1-\mu^2)\frac{d^2F_1}{d\mu^2} - 2\mu\frac{dF_1}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]F_1 = 0 \qquad \text{...(8)}$$

The general solution of (3) is

$$\left.\begin{aligned} F &= A_m \cos m\phi + B_m \sin m\phi \\ \text{or} \quad F &= A'_m e^{\pm im\phi} \end{aligned}\right\} \qquad \text{...(9)}$$

General solution of (7) is

$$R = C_n e^{nz} + D_n e^{-(n+1)z} = C_n r^n + D_n / r^{n+1} \qquad \text{...(10)}$$

Equation (8) is Associated Legendre equation, whose solution is

$$F_1 = E_{mn} P_n^m(\mu) + F_{mn} Q_n^m(\mu)$$

or $$F_1 = E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta) \qquad \text{...(11)}$$

From (1), using (9), (10) and (11), the solution of (A) for various values of m and n are given by

$$u_{mn}(r, \theta, \phi) = (A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n / r^{n+1}).$$
$$[E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta)]$$

A more general solution of (A) is given by

$$u(r, \theta, \phi) = \sum_m \sum_n u_{mn}(r, \theta, \phi)$$
$$= \sum_m \sum_n [(A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n / r^{n+1}).$$
$$\{E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta)\}] \qquad \text{...(12)}$$

Which can also be taken as

$$u(r, \theta, \phi) = \sum_m \sum_n A'_m e^{\pm im\phi}(C_n r^n + D_n / r^{n+1}).$$
$$[E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta)]$$
$$= \sum_m \sum_n [(A_{mn} r^n + B_{mn} / r^{n+1})$$
$$[E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta)] e^{\pm im\phi} \qquad \text{...(13)}$$

Note : Along the polar axis (initial line) $\theta = 0$, $\cos\theta = 1 \Rightarrow Q_n^m(\cos\theta) \to \infty$

$\therefore$ If the solution $u(r, \theta, \phi)$ remains finite along the polar axis $\theta = 0$, then we must take $F_{mn} = 0$ in (12). In this case the solution is given by

$$u(r, \theta, \phi) = \sum_m \sum_n (A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n / r^{n+1}).P_n^m(\cos\theta)$$

which can also be taken as

$$u(r, \theta, \phi) = \sum_m \sum_n (C_n r^n + D_n / r^{n+1}).e^{\pm im\phi} P_n^m(\cos\theta)$$

Solved Examples

Example 1: ***Find the potential $\phi(x, y, z)$ in the parallelopiped $0 \le x \le a, 0 \le y \le b$, $0 \le z \le c$, satisfying the conditions :***

(i) $\phi = 0$ ***on*** $x = 0,\ x = a,\ y = 0,\ y = b$ ***and*** $z = 0$

(ii) $\phi = f(x, y)$ ***on*** $z = c,\ 0 \le x \le a,\ 0 \le y \le b.$ **[Meerut 2010 (Sem. I)]**

Solution : The potential $\phi(x, y, z)$ in the parallelopiped satisfies the Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad ...(A)$$

Here we are required to find the solution of (A), satisfying the following conditions.

$$\phi(0, y, z) = 0, \qquad \phi(a, y, z) = 0 \qquad ...(B_1)$$

$$\phi(x, 0, z) = 0, \qquad \phi(x, b, z) = 0 \qquad ...(B_2)$$

and $$\phi(x, y, 0) = 0, \qquad \phi(x, y, c) = f(x, y) \qquad ...(B_3)$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x)\ Y(y)\ Z(z) \qquad ...(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$YZ\frac{d^2F}{dx^2} + FZ\frac{d^2Y}{dy^2} + FY\frac{d^2Z}{dz^2} = 0$$

or $$\frac{1}{F}\frac{d^2F}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda \text{ (say)} \qquad ...(2)$$

Here in (2) each term is function of different independent variable. Thus (2), will be true only if each term is equal to a constant. So let

if $$-\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda, \ \frac{1}{F}\frac{d^2F}{dx^2} = \lambda_1 \text{ and } \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda_2 \text{ s.t. } \lambda = \lambda_1 + \lambda_2.$$

Thus, we get the following three ordinary differential equations

$$\frac{d^2F}{dx^2} - \lambda_1 F = 0, \ \frac{d^2Y}{dy^2} - \lambda_2 Y = 0 \ \text{ and } \ \frac{d^2Z}{dz^2} + \lambda Z = 0 \qquad ...(3)$$

where $\lambda = \lambda_1 + \lambda_2$

Now the following three cases arise.

Case I : *If* $\lambda = 0 = \lambda_1 = \lambda_2$, then the equations in (3) reduce to

$$\frac{d^2F}{dx^2} = 0, \ \frac{d^2Y}{dy^2} = 0 \ \text{ and } \ \frac{d^2Z}{dz^2} = 0$$

Whose general solutions are

$$F = Ax + B, \ Y = Cx + D \text{ and } Z = Ez + F$$

$\therefore$ $$\phi(x, y, z) = (Ax + B)(Cx + D)(Ez + F)$$

Now the conditions (B_1) *i.e.,* $\phi(0, y, z) = 0$ and $\phi(a, y, z) = 0$

$\Rightarrow$ $0 = B(Cx + D)(Ez + F)$ and $(Aa + B)(Cx + D)(Ez + F)$

$\Rightarrow$ $B = 0$ and $Aa + B, \because \ Cx + D = 0$ and $Ez + F = 0$,

will lead to $\phi(x, y, z) = 0$, which is trivial solution and is inadmissible.

$\Rightarrow$ $A = 0$ and $B = 0$

$\therefore$ $\phi(x, y, z) = 0$. Which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0 = \lambda_1 = \lambda_2$.

Case II : *If* $\lambda = k^2 > 0, \ \lambda_1 = k_1^2 > 0$ *and* $\lambda_2 = k_2^2 > 0$ s.t., $\lambda = \lambda_1 + \lambda_2$ *i.e.,* $k^2 = k_1^2 + k_2^2$,

then the equations in (3) reduce to

$$\frac{d^2F}{dx^2} - k_1^2 F = 0, \frac{d^2Y}{dy^2} - k_2^2 Y = 0 \quad \text{and} \quad \frac{d^2Z}{dz^2} + k^2 Z = 0$$

Whose general solutions are

$$F = Ae^{k_1 x} + Be^{-k_1 x},\ Y = Ce^{k_2 y} + De^{-k_2 y} \text{ and } Z = E\cos kz + F\sin kz$$

$\therefore$ $$\phi(x, y, z) = (A\,e^{k_1 x} + B\,e^{-k_1 x})(Ce^{k_2 y} + De^{-k_2 y})(E\cos kz + F\sin kz)$$

$\therefore$ The conditions (B_1) *i.e.*, $\phi(0, y, z) = 0$ and $\phi(a, y, z) = 0$

$\Rightarrow$ $$0 = (A + B)(C\,e^{k_2 y} + D\,e^{-k_2 y})(E\cos kz + F\sin kz)$$

and $$0 = (A\,e^{k_1 a} + B\,e^{-k_2 a})(C\,e^{k_2 y} + D\,e^{-k_2 y})(E\cos kz + F\sin kz)$$

$\Rightarrow$ $$A + B = 0 \text{ and } A\,e^{k_1 a} + B\,e^{-k_1 a} = 0,$$

$$\because \quad C\,e^{k_2 y} + D\,e^{-k_2 y} \neq 0 \text{ and } E\cos kz + F\sin kz \neq 0$$

$\Rightarrow$ $A = 0$ and $B = 0$

$\because$ $\phi(x, y, z) = 0$, which is trivial solution and is inadmissible.

So we also reject the case when $\lambda = k^2$, $\lambda_1 = k_1^2$ and $\lambda_2 = k_2^2$.

Case III : *If* $\lambda = -k^2 < 0$, $\lambda_1 = -k_1^2 < 0$, $\lambda_2 = -k_2^2 < 0$, $s.t. k^2 = k_1^2 + k_2^2$, then the equations in (3) reduce to

$$\frac{d^2F}{dx^2} + k_1^2\,F = 0,\ \frac{d^2Y}{dy^2} + k_2^2\,Y = 0 \quad \text{and} \quad \frac{d^2Z}{dz^2} - k^2\,Z = 0$$

Whose general solutions are

$$F = A_{k_1}\cos k_1 x + B_{k_1}\sin k_1 x,\quad Y = C_{k_2}\cos k_2 y + D_{k_2}\sin k_2 y$$

and $$Z = E_k\,e^{kz} + F_k\,e^{-kz}$$

$\therefore$ $$\phi(x, y, z) = (A_{k_1}\cos k_1 x + B_{k_1}\sin k_1 x)(C_{k_2}\cos k_2 y + D_{k_2}\sin k_2 y).(E_k\,e^{kz} + F_k\,e^{-kz}) \quad \ldots(4)$$

$\therefore$ Conditions (B_1) *i.e.*, $\phi(0, y, z) = 0$

$\Rightarrow$ $$A_{k_1}(C_{k_2}\cos k_2 y + D_{k_2}\sin k_2 y)(E_k\,e^{kz} + F_k\,e^{-kz}) = 0$$

$\Rightarrow$ $A_{k_1} = 0,$

$\because$ $C_{k_2}\cos k_2 y + D_{k_2}\sin k_2 y \neq 0$ and $E_k\,e^{kz} + F_k\,e^{-kz} \neq 0$

And $\phi(a, y, z) = 0$

$\Rightarrow \quad (0 + B_{k_1} \sin k_1 a)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y)(E_k e^{kz} + F_k e^{-kz}) = 0$

$\Rightarrow \quad \sin k_1 a = 0,$

$\because \quad B_{k_1} \neq 0, C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \neq 0, E_k e^{kz} + F_k e^{-kz} \neq 0$

$\Rightarrow \quad k_1 a = m\pi$, where m is an integer.

i.e., $\quad k_1 = m\pi/a$

Similarly, the conditions (B_2) *i.e.,* $\phi(x, 0, y) = 0$ and $\phi(x, b, y) = 0$

$\Rightarrow \quad C_{k_2} = 0$ and $\quad k_2 = n\pi/b$, where n is an integer.

$\therefore \quad k^2 = k_1^2 + k_2^2 \Rightarrow k^2 = k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2.$

Thus, for integral values of m and n, the solutions of (A) from (1) satisfying the conditions (B_1) and (B_2) are given by

$$\phi_{mn}(x, y, z) = \left(B_{k_2} \sin \frac{m\pi x}{a}\right)\left(D_{k_2} \sin \frac{n\pi y}{b}\right)(E_{k_{mn}} e^{k_{mn} z} + F_{k_{mn}} e^{-k_{mn} z})$$

$$= \sin \frac{m\pi x}{a} . \sin \frac{n\pi y}{b} (G_{mn} e^{k_{mn} z} + H_{mn} e^{-k_{mn} z})$$

Where $G_{mn} = B_{k_1} D_{k_2} E_{k_{mn}}$ and $H_{mn} = B_{k_1} D_{k_2} F_{k_{mn}}$ are the new arbitrary constants.

Taking $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$, the more general solution of (A) is given by

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(x, y, z)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} . \sin \frac{n\pi y}{b} (G_{mn} e^{k_{mn} z} + H_{mn} e^{-k_{mn} z}) \quad \ldots(5)$$

Which also satisfy the conditions (B_1) and (B_2).

$\therefore$ the conditions (B_3) *i.e.,* $\phi(x, y, 0) = 0$

$$\Rightarrow \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (G_{mn} + H_{mn}) = 0$$

$\Rightarrow \quad G_{mn} + H_{mn} = 0 \;\Rightarrow\; H_{mn} = -G_{mn}$

$\therefore$ from (5), we get

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} G_{mn}(e^{k_{mn} z} - e^{-k_{mn} z})$$

or
$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh(k_{mn} z) \qquad \text{...(6)}$$

Where $I_{mn} = 2\,G_{mn}$ is the new arbitrary constant. Finally from (6), the condition (B_3) *i.e.,* $\phi(x, y, c) = f(x, y)$.

$$\Rightarrow \quad f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sinh(c\,k_{mn}) \sin \frac{m\pi x}{a} . \sin \frac{n\pi y}{b} \qquad \text{...(7)}$$

Here in (7) right hand side can be considered as the double Fourier sine series of the function $f(x, y)$ on its left hand side.

$$\therefore \quad I_{mn} \sinh(c\,k_{mn}) = \frac{2}{a} . \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$$

or
$$I_{mn} = \frac{4}{ab \sinh(c\,k_{mn})} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \qquad \text{...(8)}$$

where $\quad k_{mn}^2 = (m^2/a^2 + n^2/b^2)\,\pi^2$

Hence, the required solution of (A) is given by (6) where I_{mn} is given by (8).

Example 2: ***Find the potential ϕ in the parallelopiped, $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$, satisfying the conditions :***

(i) $\phi = 0$ ***on*** $x = 0,\ x = a,\ y = 0,\ y = b$ ***and*** $z = 0$

(ii) $\phi = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$ ***on*** $z = c, 0 \le x \le a, 0 \le y \le b.$ **[Meerut 2009 (B.P.)]**

Solution: The potential $\phi(x, y, z)$ in the parallelopiped satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{...(A)}$$

Here we are required to find the solution of (A) satisfying the following conditions

$$\phi(0, y, z) = 0, \qquad \phi(a, y, z) = 0 \qquad \ldots(B_1)$$

$$\phi(x, 0, z) = 0, \qquad \phi(x, b, z) = 0 \qquad \ldots(B_2)$$

and $$\phi(x, y, 0) = 0, \qquad \phi(x, y, c) = f(x, y) = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad \ldots(B_3)$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x)\, Y(y)\, Z(z) \qquad \ldots(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone. Now proceeding as in previous Ex. 1, the required solution is given by

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \sinh(k_{mn} z) \qquad \ldots(2)$$

where $$k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2$$

and $$I_{mn} = \frac{4}{ab \sinh(c\, k_{mn})} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4}{ab \sinh(c\, k_{mn})} \int_{x=0}^{a} \int_{y=0}^{b} A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4A}{ab \sinh(c\, k_{mn})} \left[\int_{x=0}^{a} \sin\frac{\pi x}{a} \sin\frac{m\pi x}{a}\, dx. \int_{y=0}^{b} \sin\frac{\pi y}{b} \sin\frac{n\pi y}{b}\, dy \right]$$

Now $$I_1 = \int_{x=0}^{a} \sin\frac{\pi x}{a} \sin\frac{m\pi x}{a}\, dx = 0 \qquad \text{if } m \neq 1$$

and when $$m = 1,\ I_1 = \int_{x=0}^{a} \sin^2\frac{\pi x}{a}\, dx = \frac{1}{2} \int_{0}^{\pi} \left(1 - \cos\frac{2\pi x}{a}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{a}{2\pi} \sin\frac{2\pi x}{a}\right]_0^a = \frac{1}{2} a$$

Similarly, $$I_2 = \int_{y=0}^{a} \sin\frac{\pi y}{b} \sin\frac{n\pi y}{b}\, dy = 0, \text{ where } n \neq 1$$

and for $n = 1$, $$I_2 = \frac{1}{2} b$$

$\therefore$ for $m = 1, n = 1, k_{mn}^2 = k_{11}^2 = (1/a^2 + 1/b^2)\pi^2$

and $$I_{11} = \frac{4A}{ab \sinh(c\, k_{11})} \cdot \frac{a}{2} \cdot \frac{b}{2} = \frac{A}{\sinh(c\, k_{11})}$$

and $I_{mn} = 0$ if $m \neq 1$, and $n \neq 1$.

$\therefore$ From (2), the required solution is

$$\phi(x, y, z) = I_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sinh(k_{11} z)$$

$$= \frac{A}{\sinh(c\, k_{11})} \sin\frac{\pi x}{a} . \sin\frac{\pi y}{b} \sinh(k_{11} z)$$

where $$k_{11}{}^2 = (1/a^2 + 1/b^2)\pi^2$$

Example 3: ***Find the potential ϕ in the parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$, satisfying the conditions***

(i) $\phi = 0$ ***on*** $x = 0, y = 0, z = 0; x = a, y = b$

(ii) $\phi = A$ ***on*** $z = c, 0 \le x \le a, 0 \le y \le b.$

Solution: The potential $\phi(x, y, z)$ in the parallelopiped satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{...(A)}$$

Here we are required to find the solution of (A) satisfying the following conditions

$$\phi(0, y, z) = 0, \qquad \phi(a, y, z) = 0 \qquad \text{...(B}_1\text{)}$$

$$\phi(x, 0, z) = 0, \qquad \phi(x, b, z) = 0 \qquad \text{...(B}_2\text{)}$$

and $$\phi(x, y, 0) = 0, \qquad \phi(x, y, c) = f(x, y) = A \qquad \text{...(B}_3\text{)}$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x)\, Y(y)\, Z(z) \qquad \text{...(1)}$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone. Now proceeding as in Ex. 1 on page 760, the required solution is given by

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh(k_{mn} z) \qquad \text{...(2)}$$

where $k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2$

and $$I_{mn} = \frac{4}{ab \sinh(c\, k_{mn})} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4}{ab \sinh(c\, k_{mn})} \int_{x=0}^{a} \int_{y=0}^{b} A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{4A}{ab \sinh(c\, k_{mn})} \left[-\frac{a}{m\pi} \cos \frac{m\pi x}{a} \right]_{x=0}^{a} \left[-\frac{b}{n\pi} \cos \frac{n\pi y}{b} \right]_{y=0}^{b}$$

$$= \frac{4A}{mn\pi^2 \sinh(c\, k_{mn})} [1-(-1)^m].[1-(-1)^n]$$

We have

$$1-(-1)^m = \begin{cases} 0, & \text{when } m = 2r, \quad \text{even}, \quad r = 1, 2, \dots \\ 2, & \text{when } m = 2r-1, \quad \text{odd}, \quad r = 1, 2, 3, \dots \end{cases}$$

And $$1-(-1)^n = \begin{cases} 0, & \text{when } n = 2s, \quad \text{even}, \quad s = 1, 2, \dots \\ 2, & \text{when } m = 2s-1, \quad \text{odd}, \quad s = 1, 2, 3, \dots \end{cases}$$

∴ $I_{mn} = 0$ if one or both of m and n are even.

∴ Taking $m = 2r-1$ and $n = 2s-1, r, s = 1, 2, \dots$, from (2) the required solution of (A) satisfying the given conditions is given by

$$\phi(x, y, z) = \frac{16A}{\pi^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sinh(c\, k_{rs})}{(2r-1)(2s-1)} \sin \frac{(2r-1)\pi x}{a} \sin \frac{(2s-1)\pi y}{b}$$

where $$k_{rs}^2 = \left(\frac{(2r-1)^2}{a^2} + \frac{(2s-1)^2}{b^2} \right) \pi^2$$

Example 4: *Show that the solutions of Laplace's equation $\nabla^2 u = 0$ in cylindrical coordinates satisfying the conditions :*

(i) $u \to 0$ *as* $z \to \infty$ *and*

(ii) u *is finite as* $\rho \to 0$ *are of the form*

$$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm i n \phi}.$$

Solution: The Laplace's equation in cylindrical coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(A)}$$

Here we are required to find the solution of (A), satisfying the following conditions

$$u \to 0 \text{ as } z \to \infty \qquad \text{...}(B_1)$$

and $$u \text{ is finite as } \rho \to 0. \qquad \text{...}(B_2)$$

Let the solution of (A) be of the form

$$u(\rho, \phi, z) = R(\rho)\ F(\phi)\ Z(z) = RFZ \text{ (say)} \qquad \text{...(1)}$$

Where R is a function of ρ alone, F a function of ϕ alone and Z a function of z alone.

Now proceeding similarly as in 14.9, taking m^2 for k^2 and n^2 for m^2, the solution of (A) is given by

$$u(\rho, \phi, z) = \sum_m \sum_n (A_m e^{mz} + B_m e^{-mz}) e^{\pm in\phi}.[E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho) \qquad \text{...(2)}$$

Since according to condition (B_1) $u \to 0$ as $z \to \infty$, so in (2) we must take $A_m = 0$.

$\because$ as $z \to \infty, e^{mz} \to \infty$.

Also according to the condition (B_2) u is finite as $\rho \to 0$, so in (2), we must take $F_{mn} = 0$.

$\because$ as $\rho \to 0, Y_n(m\rho) \to \infty$.

Putting $A_m = 0$ and $F_{mn} = 0$ in (2), the required solution is

$$u(\rho, \phi, z) = \sum_m \sum_n G_{mn} e^{-mz}.e^{\pm in\phi}.J_n(m\rho) = \sum_m \sum_n G_{mn} J_n(m\rho).e^{-mz \pm in\phi}.$$

Where $G_{mn} = B_m.E_{mn}$ is a new arbitrary constants.

Example 5: ***Obtain the axially symmetrical solution of the three dimensional Laplace's equation.***

Solution: Laplace's equation in cylindrical coordinates (ρ, φ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad ...(1)$$

If u is symmetrical about the axis of z, then $\frac{\partial^2 u}{\partial \phi^2}=0$

∴ (1) reduces to
$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{\partial^2 u}{\partial z^2}=0 \qquad ...(2)$$

Let the solution of (2) be given by

$$u(\rho, z)=R(\rho)\, Z(z)=RZ \text{ (say)} \qquad ...(3)$$

Differentiating (3) and substituting in (2), we get

$$Z\frac{d^2 R}{d\rho^2}+\frac{Z}{\rho}\frac{dR}{d\rho}+R\frac{d^2 Z}{dz^2}=0$$

or
$$\frac{1}{R}\left(\frac{d^2 R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)=-\frac{1}{Z}\frac{d^2 Z}{dz^2}=-m^2 \text{ (say)} \qquad ...(4)$$

Since right hand side in (4) is a function of independent variable z alone while left hand side is a function of different independent variable ρ. So the two sides will be equal to each other if each is equal to the same constant say $-m^2$.

Thus, from (4) we get the following two ordinary differential equations

$$\frac{d^2 Z}{dz^2}-m^2 Z=0 \qquad ...(5)$$

and
$$\frac{d^2 R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+m^2 R=0 \qquad ...(6)$$

General solution of (5) is

$$Z=A_m e^{mz}+B_m e^{-mz} \quad \text{or} \quad Z=A_m e^{\pm mz} \qquad ...(7)$$

Equation (6) is Bessel's equation of zeroth order whose general solution is

$$R=C_m J_0(m\rho)+D_m Y_0(m\rho) \qquad ...(8)$$

Where J_0 and Y_0 are Bessel's functions of zeroth order of first and second kind respectively.

$\therefore$ from (3), the solutions of (2), for different values of m are given by

$$u_m(\rho, z) = (A_m e^{mz} + B_m e^{-mz})[C_m J_0(m\rho) + D_m Y_0(m\rho)]$$

Hence, more general solution of (2) is given by

$$u(\rho, z) = \sum_m u_m(\rho, z) = \sum_m (A_m e^{mz} + B_m e^{-mz})[C_m J_0(mp) + D_m Y_0(mp)] \quad ...(9)$$

This, solution can also be written as

$$u(\rho, z) = \sum_m A_m e^{\pm mz}.[C_m J_0(m\rho) + D_m Y_0(m\rho)]$$

$$= \sum_m e^{\pm mz}[E_m J_0(m\rho) + F_m Y_0(m\rho)]$$

Where $E_m = A_m C_m$ and $F_m = A_m D_m$ are the new arbitrary constants.

Note : In (9) when $\rho \to 0$, $Y_0(m\rho) \to \infty$. Thus for the solution $u(\rho, z)$ given by (9), to remain finite along the line $\rho = 0$ at the origin, we must take $D_m = 0$.

Hence, the solution of the Laplace's equation which is symmetrical about z-axis and remains finite on the line $\rho = 0$ is given by [putting $D_m = 0$ in (9)]

$$u(\rho, z) = \sum_m (G_m e^{mz} + H_m e^{-mz}).J_0(m\rho)$$

where $G_m = A_m.C_m$ and $H_m = B_m C_m$ are the new arbitrary constants.

This solution can also be written as

$$u(\rho, z) = \sum_m G_m e^{\pm mz}.J_0(m\rho).$$

Exercise 14.3

1. Find the potential ϕ in the cuboid $0 \le x \le a, 0 \le y \le a, 0 \le z \le a$, satisfying the conditions.

   (i) $\phi = 0$, on $x = 0 = y = z,\ \ x = a = y$

   (ii) $\phi = f(x, y)$ on $z = c, 0 \le x \le a,\ \ 0 \le y \le a$.

2. Find the potential ϕ in the parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, satisfying the conditions.

   (i) $\phi = 0$ on $x = 0 = y = z, x = a$ and $y = b$

   (ii) $\phi = A \sin \dfrac{\pi x}{a} \sin \dfrac{2\pi y}{b}$ on $z = c$, $0 \le x \le a$, $0 \le y \le b$.

3. Find the potential ϕ in the cuboid $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$, satisfying the conditions.

   (i) $\phi = 0$, on $x = 0 = y = z,\ \ x = a = y$

   (ii) $\phi = A$, on $z = c$, $0 \le x \le a$, $0 \le y \le a$.

4. Obtain the general solution of Laplace's equation in cylindrical coordinates which remain finite on the axis of z and is symmetrical about it.

   [**Hint** : See note of Ex. 5 on page 769].

5. Obtain the solution of Laplace's equation in cylindrical coordinates which is symmetrical about the z-axis and tends to zero as $\rho \to 0$ and as $z \to \infty$.

   [**Hint** : Take $G_m = 0$ in Q.N. 4.].

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. When the temperature is in steady state, the heat equation becomes :

   (a) Wave equation (b) Laplace equation

   (c) Bessel's equation (d) None of these

2. Laplace's equation is also called :

   (a) Harmonic equation (b) Wave equation

   (c) Legendre equation (d) None of these

3. Consider the steady state temperature $u(x, y)$ in a rectangular plate of length a and width b, the sides of which are kept at temperature zero, the lower end is kept at temperature $f(x)$ and the upper edge is insulated. Then $u(x, y)$ satisfy the equation :

(a) $\frac{\partial^2 u}{\partial x^2} = 0$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

(d) None of these

4. The boundary conditions of the boundary value problem given in Q.N. 3. are :

(a) $u(0, y) = 0 = u(a, y) = u_y(x, b),\ u(x, 0) = f(x)$

(b) $u(0,\ y,) = 0 = u(a,\ y),\ u(x,\ 0) = 0 = u(x,\ b)$

(c) $u(0,\ y) = 0 = u(a, y),\ u(x,\ 0) = f(x) = u(x,\ b)$

(d) None of these

5. Let $u(x,\ y)$ be steady state temperature function in a square plate $0 \le x < \pi$, $0 \le y < \pi$, the sides of which are kept at temperature zero, the lower edge is kept at temperature $f(x)$ and upper edge at temperature v_0. Then $u(x, y)$ satisfy the equation :

(a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(c) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$

(d) None of these

6. The boundary conditions of the problem in Q.N. 5 are :

(a) $u(0,\ y) = 1 = u(\pi,\ y),\ u(x,\ 0) = v_0 = u(x,\ \pi)$

(b) $u(0,\ y) = v_0 = u(\pi, y),\ u(x,\ 0) = f(x) = u(x,\ \pi)$

(c) $u(0,\ y) = 0 = u(\pi,\ y),\ u(x,\ 0) = f(x),\ u(x,\ \pi) = v_0$

(d) None of these

7. The Laplace equation in polar coordinates in given by

(a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$

(b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

(c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$

(d) None of these

8. The boundary diameter of a semi-circular plate of radius a is kept at 0°C and its temperature along the semi-circular boundary is given by $f(\theta)$. Then the steady state temperature function $u(r,\ \theta)$ satisfy :

(a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

(d) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

9. The boundary conditions of the problem in Q.N. 8. are given by :

(a) $u(r, 0) = 0, -a \le r \le a, u(a,\theta) = f(\theta), 0 < \theta < 2\pi$

(b) $u(r, 0) = 0 = u(r, \pi), 0 \le r \le a, u(a, \theta) = f(\theta), 0 < \theta < \pi$

(c) $u(r, 0) = 0 = u(r, \pi), 0 \le r \le a, u(a, 0) = f(\theta)$

(d) None of these

10. $u(r,\theta)$ is the steady state temperature distribution in a circular plate of radius 10 cm, which has half of its circumference at 0°C and the other half at temperature v_0^2 C. Then the boundary conditions of the problem is/are :

(a) $u(10, \theta) = 0, 0 < \theta < 2\pi$

(b) $u(10,\theta) = v_0, 0 < \theta < 2\pi$

(c) $u(10,\theta) = v_0, 0 < \theta < \pi, u(10,\theta) = 0, \pi < \theta < 2\pi$

(d) None of these

Fill in the Blank

Fill in the blanks "............." so that the following statements are complete and correct.

1. The Laplace's equation is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} =$ ....

2. The Laplace's equation in cylindrical coordinates is given by $\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \ldots\ldots\ldots\ldots + \frac{\partial^2 u}{\partial z^2} = 0.$

3. The Laplace's equation in spherical coordinates is given by $\frac{\partial^2 u}{\partial r^2} + \ldots\ldots\ldots\ldots + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0.$

4. The potential ϕ in the parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$, satisfy the equation ....

5. The surfaces of a sector determined by $0 \le r \le a, 0 \le \theta \le \pi/4$ are insulated. The temperature is kept at 0°C along the straight edges and at $f(\theta)$ along the current edge. Then the boundary conditions of the problem are $u(r, 0) = 0 = u(r, \pi/4)$, $0 < r < a$ and ....

Answers

Exercise 14.3

1.	$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \sinh(k_{mn} z)$ where $k_{mn}^2 = (m^2 + n^2)\pi^2/a^2$ and $I_{mn} = \frac{4}{a^2 \sinh(c\, k_{mn})} \int_{x=0}^{a} \int_{y=0}^{a} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}\, dx\, dy$
2.	$\phi(x, y, z) = A \operatorname{cosech}(c\, k_{12}) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \sinh(k_{12} z)$ where $k^2{}_{12} = (1/a^2 + 4/b^2)\pi^2$.
3.	$\phi(x, y, z) = \frac{16A}{\pi^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sinh(c\, k_{rs})}{(2r-1)(2s-1)} \sin \frac{(2r-1)\pi x}{a} \sin \frac{(2s-1)\pi y}{a}$ where $k_{rs}^2 = [(2r-1)^2 + (2s-1)^2]\pi^2/a^2$.
4.	$u(\rho, z) = \sum_{m} (G_m e^{mz} + H_m e^{-mz}) J_0(m\rho)$ or $u(\rho, z) = \sum_{m} G_m e^{\pm mz} J_0(m\rho)$.
5.	$u(\rho, z) = \sum_{m} H_m e^{-mz} J_0(m\rho)$.

Objective Type Questions

Multiple Choice Questions

1.	(b)	2.	(a)	3.	(c)	4.	(a)
5.	(b)	6.	(c)	7.	(b)	8.	(a)
9.	(b)	10.	(c)				

Fill in the Blank

1.	0	2.	$\frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2}$
3.	$\frac{2}{r} \frac{\partial u}{\partial r}$	4.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$
5.	$u(a, \theta) = f(\theta),\ 0 < \theta < \pi/4$		

OOO

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0 \quad \textit{i.e.,} \quad p_1+p_3p=0 \quad \text{and} \quad p_2+p_3q=0$$

so that $p=-(p_1/p_3)$ and $q=-(p_2/p_3)$ ... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1$, $p_2=\partial u/\partial y=\partial u/\partial x_2$, $p_3=\partial u/\partial z=\partial u/\partial x_3$, $p=\partial z/\partial x$, $q=\partial z/\partial y$

by taking $x=x_1$, $y=x_2$ and $z=x_3$ ... (4)

Using (3) and (4), (1) reduces to $p_1^2/p_3^2+p_2^2/p_3^2=k^2$ or $p_1^2+p_2^2=k^2p_3^2$

Let $f(x_1,x_2,x_3,p_1,p_2,p_3)=p_1^2+p_2^2-k^2p_3^2=0$ ... (5)

Now, the Jacobi auxilliary equations are given by

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0}=\frac{dx_1}{-2p_1}=\frac{dp_2}{0}=\frac{dx_2}{-2p_2}=\frac{dp_3}{0}=\frac{dx_3}{2k^2p_3}, \text{ using (5)}$$

From the first and third fractions of (5), $dp_1=0$ and $dp_2=0$

Integrating, $p_1=a_1$ and $p_2=a_2$, a_1 and a_2 being arbitrary constants

With $p_1=a_1$ and $p_2=a_2$, (5) gives $p_3=(a_1^2+a_2^2)^{1/2}/k$

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1dx_1+a_2dx_2+\{(a_1^2+a_2^2)^{1/2}/k\}dx_3$$

Integrating, $u=a_1x+a_2x_2+\{(a_1^2+a_2^2)^{1/2}/k\}x_3+a_3$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$a_1x+x_2+\{(a_1^2+1)^{1/2}/k\}x_3+a_3=0,$$

which is the complete integral of (1) containing two arbitrary constants a_1 and a_3.

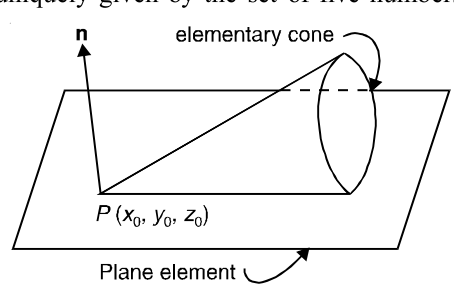
Ex. 3. *Solve the following partial differential equations by Jacobi's method:*

(*i*) $p=(z+qy)^2$ (*ii*) $(p^2+q^2)x=pz$ (*iii*) $xpq+yq^2=1$ **[Nagpur 2005]**

Hint. Proceed as in the above solved Ex. 1

3.23 Cauchy's method of characteristics for solving non-linear partial differential equation $f(x,y,z,\ \partial z/\partial x,\partial z/\partial y)=0$ *i.e.,* $f(x,y,z,p,q)=0$ ... (1)

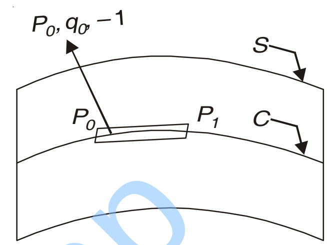
We know that the plane passing through the point $P(x_0,y_0z_0)$ with its normal parallel to the direction **n** whose direction ratios are $p_0,q_0,-1$ is uniquely given by the set of five numbers $D(x_0,y_0,z_0,p_0,q_0)$ and conversely any such set of five numbers defines a plane in three dimensional space. In view of this fact a set of five numbers $D(x,y,z,p,q)$ is known as a *plane element* of a three dimensional space. As a special case a plane element (x_0,y_0,z_0,p_0,q_0) whose components satisfy (1) is known as an *integral element* of (1) at P. Solving (1) for q, suppose we get



$$q = F(x, y, z, p).$$

which gives a value of q corresponding to known values of x, y, z and p. Then, keeping x_0, y_0 and z_0 fixed and varying p, we shall arrive at a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies, we get a set of plane elements all of which pass through the point P. Hence the above mentioned set of plane elements envelop a cone with vertex P. The cone thus obtained is known as the *elementary cone of* (1) at the point P.

Consider a surface S with equation $\qquad z = g(x, y) \qquad$ ... (2)



If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous in a certain region R of the xy-plane, then the tangent plane at each point of S determines a plane element of the form $\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\}$ which will be referred as *the tangent element* of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$.

Consider a curve C with parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad \text{t being the parameter.} \qquad \text{... (3)}$$

Then curve C lies on (2) provided $\qquad z(t) = g\{x(t), y(t)\} \qquad$ ... (4)

holds good for all values of t in the appropriate interval I. Let P_0 be a point on curve C corresponding to $t = t_0$. Now, the direction ratios of the tangent line $P_0 P_1$ are $x'(t_0), y'(t_0), z'(t_0)$ where $x'(t_0), y'(t_0)\ z'(t_0)$ denote the values of dx/dt, dy/dt, dz/dt respectively at $t = t_0$

This direction will be perpendicular to direction of normal $\boldsymbol{n}$ (with direction ratios $p_0, q_0, -1$)

if $\qquad p_0 x'(t_0) + q_0\ y'(t_0) + (-1)\ z'(t_0) = 0 \qquad$ or $\qquad z'(t_0) = p_0\ x'(t_0) + q_0\ y'(t_0)$

It follows that any set $\qquad \{x(t), y(t), z(t), p(t), q(t)\} \qquad$ ...(5)

of five real functions satisfying the condition that $\qquad z'(t) = p(t)\ x'(t) + q(t)\ y'(t) \qquad$ ... (6)

defines a strip at the point (x, y, z) of the curve C. When such a strip is also an integral element of (1), then the strip under consideration is known as an *integral strip* of (1). In other words, the set of functions (5) is known as an integral strip of (1) provided these satisty (6) and the following additional condition

$$f\{x(t), y(t), z(t), p(t), q(t)\} = 0, \quad \text{for all } t \text{ in I.}$$

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a *characteristic strip*.

Derivation of the equations determining a *characteristic strip*

Clearly, the point $(x + dx,\ y + dy,\ z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = p\,dx + q\,dy \qquad \text{... (7)}$$

where p, q satisfy (1). Differentiation (7) w.r.t. 'p', we get

$$0 = dx + (dq/dp)\,dy \qquad \text{... (8)}$$

Again, differentiating (1) partially w.r.t. 'p', we have

$$\partial f/\partial p + (\partial f/\partial q)\,(dq/dp) = 0 \qquad \textit{i.e.,} \qquad f_p + f_q (dq/dp) = 0 \qquad \text{... (9)}$$

Here, $\qquad \partial f/\partial p = f_p \qquad$ and $\qquad \partial f/\partial q = f_q$

Solving (7), (8) and (9) for the ratios of dy, dz to dx, we get

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q}. \qquad \text{... (10)}$$

Hence along a characteristic strip $x'(t)$, $y'(t)$, $z'(t)$ will be proportional to f_p, f_q, $p f_p + q f_q$ respectively. If the parameter t be selected satisfying the relations

$$x'(t) = f_p \qquad \text{and} \qquad y'(t) = f_q.$$

then, we have $$z'(t) = p f_p + q f_q$$

Since along a characteristic strip p is a function of t, hence

$p'(t) = (\partial p/\partial x)\,(dx/dt) + (\partial p/\partial y)\,(dy/dt) = (\partial p/\partial x)\,(\partial f/\partial p) + (\partial p/\partial y)\,(\partial f/\partial q)$, using (11)

Thus, $$p'(t) = (\partial p/\partial x)\,(\partial f/\partial p) + (\partial q/\partial x)\,(\partial f/\partial q) \qquad \text{... (12)}$$

$$\left[\because \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}\right]$$

Now, differentiating (1) partially w.r.t. 'x', gives $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

or $$f_x + p\, f_z + p'(t) = 0, \text{ using (12)}$$

Hence on a characteristic strip, $$p'(t) = -f_x - p\, f_z \qquad \text{... (13)}$$

Similarly, we have $$q'(t) = -f_y - q\, f_z \qquad \text{... (14)}$$

Here $f_x = \partial f/\partial x$, $\quad f_y = \partial f/\partial y$, $\quad f_z = \partial f/\partial z$

From (11), (13) and (14), we get the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t) = f_p, \quad y'(t) = f_q, \quad z'(t) = p\, f_p + q\, f_q, \quad p'(t) = -f_x - p\, f_z \quad \text{and} \quad q'(t) = -f_y - q\, f_z \qquad \text{...(15)}$$

The above equations are called the *characteristic equations* of (1). In view of a well known result if the functions which are involved in (15) satisfy a Lipschitz condition, there exists a unique solution of (15) for given set of initial values of the variables. It follows that the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_o of t.

Working rule for solving Cauchy's problem. **[Meerut 2005]**

Suppose we wish to find the integral surface of (1) which passes through a given curve with parametric equation $x = f_1(\lambda)$, $y = f_2(\lambda)$, $z = f_3(\lambda)$, λ being the parameter ... (16)

then in the solution $$x = x\,(p_0, q_0, x_0, y_0, t_0, t) \text{ etc.} \qquad \text{... (17)}$$

of the characteristic equations (15), we shall assume that

$$x_0 = f_1(\lambda), \qquad y_0 = f_2(\lambda), \qquad z_0 = f_3(\lambda)$$

are the initial values of x, y, z respectively. Then the corresponding initial values of p_0, q_0 can be obtained by the following relations

$$f_3'(\lambda) = p_0\, f_1'(\lambda) + q_0\, f_3'(\lambda) \qquad \text{and} \qquad f\{f_1(\lambda),\ f_2(\lambda),\ f_3(\lambda),\ p_0,\ q_0\} = 0$$

When the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted in (17), we shall be able to express x, y, z involving the two parameters t and λ of the form

$$x = \phi_1 (t, \lambda), \qquad y = \phi_2 (t, \lambda) \qquad \text{and} \qquad z = \phi_3(t, \lambda) \qquad \text{... (18)}$$

which are known as characteristics of (1)

Finally, by eliminating λ and t from (18), we arrive at a relation of the form $G(x,y,z) = 0$, which is the required equation of the integral surface of (1) passing through the given curve (16).

3.24 Some Theorems:

Theorems 1. *A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.*

Proof. Using geometrical considerations and Art 3.23, complete the proof yourself.

Theorem II. *Along every characteristic strip of the partial differential equation $f(x, y, z, p, q) = 0$ the function $f(x, y, z, p, q)$ is a constant.*

Proof. Along a characteristic strip, we have

$$\frac{d}{dt} f\{x(t),\ y(t),\ z(t),\ p(t), q(t)\} = f_x\ x'(t) + f_y\ y'(t) + f_z\ z'(t) + f_p\ p'(t) + f_q\ q'(t)$$

$$= f_x f_p + f_y f_q + f_z\ (p\ f_p + q\ f_q) - f_p(f_x + p\,f_z) - f_q(f_y + q\,f_z)$$

$= 0$, using the characteristic equation (15) of Art. 3.23

showing that $f(x, y, z, p, q) = K$, a constant along the strip.

Corollary to theorem II. *If a characteristic strip contains at least one integral element of $f(x,y,z,p,q) = 0$ it is an integral strip of the equation $f(x,y,z,\ \partial z/\partial x,\ \partial z/\partial y) = 0$*

Proof. Left as an exercise.

3.25 SOLVED EXAMPLES BASED ON ART. 3.23

Ex. 1. *Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0$, $y^2 = z$.* **[Meerut 2005; I.A.S. 1999]**

Sol. Given equation is $\qquad pq = z \qquad$ ... (1)

We are to find its integral surface which passes through the given parabola given by

$$x = 0, \qquad \text{and} \qquad y^2 = z \qquad \text{... (2)}$$

Re-writing (2) in parametric form, we have

$$x = 0, \qquad y = \lambda, \qquad z = \lambda^2, \qquad \lambda \text{ being a parameter} \qquad \text{... (3)}$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = 0, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = \lambda^2 \qquad \text{... (4A)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$$p_0 q_0 = z_0, \qquad \text{or} \qquad p_0\ q_0 = \lambda^2, \text{ by (4A)} \qquad \text{... (5)}$$

Also, we have $\qquad z_0'(\lambda) = p_0\ x_0'(\lambda) + q_0\ y_0'(\lambda)$

so that $\qquad 2\lambda = p_0 \times 0 + q_0 \times 1 \qquad$ or $\qquad q_0 = 2\lambda$, by (4A) $\qquad$ ... (6)

Solving (5) and (6), $\qquad p_0 = \lambda/2 \qquad$ and $\qquad q_0 = 2\lambda \qquad$ ... (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = 0, \quad y_0 = \lambda, \quad z_0 = \lambda^2, \quad p_0 = \lambda/2, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Re-writing (1), let $$f(x, y, z, p, q) = pq - z = 0 \qquad \text{... (8)}$$

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = p \qquad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \qquad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q \qquad \text{... (13)}$$

From (9) and (13), $(dx/dt) - (dq/dt) = 0$, so that $x - q = C_1$, ... (14)

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$x_0 - q_0 = C_1$ or $0 - 2\lambda = C_1$ or $C_1 = -2\lambda$. Then (14) becomes

$x - q = -2\lambda$ or $x = q - 2\lambda$, ... (15)

From (10) and (12), $(dy/dt) - (dp/dt) = 0$ so that $y - p = C_2$, ... (16)

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2$ or $\lambda - (\lambda/2) = C_2$ or $C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2$ or $y = p + (\lambda/2)$ ... (17)

From (12), $(1/p)\,dp = dt$ so that $\log p - \log C_3 = t$ or $p = C_3\, e^t$ ... (18)

Using initial values (7), (18) gives $p_0 = C_3 e^0$ or $\lambda/2 = C_3$

Hence (18) reduces to $$p = (\lambda/2) \times e^t \qquad \text{... (19)}$$

From (13), $(1/q)\; dq = dt$ so that $\log q - \log C_4 = t$ or $q = C_4\, e^t$ ... (20)

Using initial values (7), (20) gives $q_0 = C_4\, e^0$ or $2\lambda = C_4$

Hence (20) reduces to $$q = 2\lambda\, e^t \qquad \text{... (21)}$$

Using (21), (15) becomes $x = 2\lambda\, e^t - 2\lambda$ or $x = 2\lambda\,(e^t - 1)$ ... (22)

Using (19), (17) becomes $y = (\lambda/2)\, e^t + \lambda/2$ or $y = (\lambda/2) \times (e^t + 1)$ ... (23)

Substituting values of p and q from (19) and (21) in (11), we get

$dz/dt = 2\{(\lambda/2) \times e^t\} \times \{2\lambda\, e^t\}$ or $dz = 2\lambda^2 e^{2t}\, dt$.

Integrating, $z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant .... (24)

Using initial values (7), (24) gives $z_0 = \lambda^2 e^0 + C_5$ or $\lambda^2 = \lambda^2 + C_5$ or $C_5 = 0$

Then, (24) gives $z = \lambda^2 e^{2t}$ or $z = \lambda^2 (e^t)^2$ ... (25)

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$e^t = (x+4y)/(4y-x)$ and $\lambda = (4y-x)/4$

Substituting these values of e^t and λ in (25), we have

$z = \{(4y-x)^2/16\} \times \{(x+4y)/(4y-x)\}^2$ or $16z = (4y+x)^2$,

which is the required integral surface of (1) passing through (2).

Ex. 2. *Find the solution of the equation* $z = (p^2+q^2)/2 + (p-x)(q-y)$ *which passes through the x-axis.* **[Himachal 1996; 2004; I.A.S. 2002]**

Sol. Given equation is $z = (p^2+q^2)/2 + (p-x)(q-y)$ ... (1)

We are to find its integral surface which passes through x-axis which is given by equations

$y = 0$ and $z = 0$ ... (2)

Re-writing (2) in parametric form, $x = \lambda$, $y = 0$, $z = 0$, λ being the parameter ... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = \lambda$, $y_0 = y_0(\lambda) = 0$, $z = z_0(\lambda) = 0$ ... (4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$z_0 = (p_0^2 + q_0^2)/2 + (p_0 - x_0)(q_0 - x_0)$ or $0 = (p_0^2+q_0^2)/2 + q_0(p_0 - \lambda)$, by (4A)

or $p_0^2 + q_0^2 + 2q_0 p_0 - 2q_0\lambda = 0$ ... (5)

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $0 = p_0 \times 1 + q_0 \times 0$ or $p_0 = 0$, by (4A) ... (6)

Solving (5) and (6), $p_0 = 0$ and $q_0 = 2\lambda$ ... (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 = \lambda$, $y_0 = 0$, $z_0 = 0$, $p_0 = 0$, $q_0 = 2\lambda$ when $t = t_0 = 0$ ... (7)

Let $f(x,y,z,p,q) = (p^2+q^2)/2 + pq - py - qx + xy - z = 0$ ... (8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = p + q - y \quad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = q + p - x \quad \text{... (10)}$$

$$dz/dt = p\,(\partial f/\partial p) + q(\partial f/\partial q) = p(p+q-y) + q(q+p-x), \quad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y \quad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q\,(\partial f/\partial z) = p + q - x \quad \text{... (13)}$$

From (9) and (12), $(dx/dt) - (dp/dt) = 0$ so that $x - p = C_1$ ... (14)

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $x - p = \lambda$ or $x = p + \lambda$ ... (15)

From (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = C_2$, ... (16)

where C_2 is an arbitrary constant.

Using initial conditions (7), (16) gives $0 - 2\lambda = C_2$ or $C_2 = -2\lambda$.

Hence (16) reduces to $y - q = -2\lambda$ or $y = q - 2\lambda$ ... (17)

$$\therefore \quad \frac{d(p+q-x)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p+q-y+p+q-x-(p+q-y), \text{ using (9), (12) and (13)}$$

or $\dfrac{d(p+q-x)}{dt} = p+q-x$ or $\dfrac{d(p+q-x)}{p+q-x} = dt$.

Integrating, $\log(p+q-x) - \log C_3 = t$ or $p+q-x = C_3\, e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = C_3$ or $C_3 = \lambda$.

Hence (18) reduces to $p+q-x = \lambda e^t$ ... (19)

$$\text{Now,} \quad \frac{d(p+q-y)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p+q-y+p+q-x-(q+p-x), \text{ by (10), (12) and (13)}$$

or $\dfrac{d(p+q-y)}{dt} = p+q-y$ or $\dfrac{d(p+q-y)}{b+q-y} = dt$.

Integrating, $\log(p+q-y) - \log C_4 = t$ or $p+q-y - C_4\, e^t$ ... (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0 + 2\lambda - 0 = C_4$ or $C_4 = 2\lambda$.

Hence (20) reduces to $p+q-y = 2\lambda\, e^t$ ... (21)

From (9) and (21), $dx/dt = 2\lambda\, e^t$ so that $x = 2\lambda\, e^t + C_5$ ... (22)

where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda = 2\lambda + C_5$ or $C_5 = -\lambda$.

Hence (22) reduces to $x = 2\lambda\, e^t - \lambda$ or $x = \lambda\,(2e^t - 1)$ ... (23)

From (10) and (19), $dy/dt = \lambda\, e^t$ so that $y = \lambda\, e^t + C_6$ ... (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0 = \lambda + C_6$ or $C_6 = -\lambda$.

Hence (24) reduces to $y = \lambda e^t - \lambda$ or $y = \lambda\,(e^t - 1)$ ... (25)

Substituting value of y from (17) in (12), we get

$dp/dt = p + q - (q - 2\lambda)$ or $(dp/dt) - p = 2\lambda$, ... (26)

which is a linear equation whose integrating factor = $e^{\int (-1)dt} = e^{-t}$ and solution is

$p\, e^{-t} = \int (2\lambda)\, e^{-t} dt + C_7 = -2\lambda\, e^{-t} + C_3$ or $p = -2\lambda + C_3\, e^t$ ... (27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7$ or $C_7 = 2\lambda$.

Hence (27) reduces to $p = -2\lambda + 2\lambda\, e^t$ or $p = 2\lambda\,(e^t - 1)$ ... (28)

Substituting value of x from (15) in (13), we get

$$dq/dt = p+q-(p+\lambda) \quad \text{or} \quad dq/dt - q = -\lambda, \qquad \text{... (29)}$$

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$$q\,e^{-t} = \int(-\lambda)\,e^{-t}dt + C_8 = \lambda e^{-t} + C_8 \quad \text{or} \quad q = \lambda + C_8\,e^{t} \qquad \text{... (30)}$$

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.

Hence (30) reduces to $\quad q = \lambda + \lambda\,e^t \quad$ or $\quad q = \lambda\,(1+e^t)$ ... (31)

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda\,e^t) + q(\lambda\,e^t) = 2\lambda\,(e^t-1)\,(2\lambda\,e^t) + \lambda(1+e^t)\,(\lambda\,e^t)$$

[on putting values of p and q with help of (28) and (31)]

or $\quad dz/dt = 5\lambda^2e^{2t} - 3\lambda^2e^t \quad$ or $\quad dz = (5\lambda^2e^{2t} - 3\lambda^2e^t)dt$.

Integrating, $$z = (5/2)\times\lambda^2e^{2t} - 3\lambda^2e^t + C_9 \qquad \text{... (32)}$$

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives $0 = (5/2)\times\ \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to

$$z = (5/2)\times\lambda^2(e^{2t}-1) - 3\lambda^2(e^t-1) \qquad \text{... (33)}$$

Solving (23) and (25) for λ and e^t, $\quad \lambda = x-2y \quad$ and $\quad e^t = (x-y)/(x-2y)$ ... (34)

Eliminating λ and e^t from (33) and (34), we have

$(p^2+q^2)x = pz$

$$z = \frac{5}{2}(x-2y)^2\left\{\left(\frac{x-y}{x-2y}\right)^2 - 1\right\} - 3(x-2y)^2\left(\frac{x-y}{x-2y} - 1\right)$$

or $$z = (5/2)\times\{(x-y)^2 - (x-2y)^2\} - 3\ \{(x-2y)\,(x-y) - (x-2y)^2\}$$

or $\quad z = (y/2)\times(4x-3y)$, on simplification.

Ex. 3. *Determine the characteristics of the equation* $z = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4z + x^2 = 0,\ y = 0$. **[Himachal 2000, 05]**

Sol. Do yourself, the required characteristics are $x = 2\lambda(2-e^{-t})$, $y = 2\sqrt{2}\lambda\,(e^{-t}-1)$, $z = -\lambda^2e^{-2t}$, λ being parameter. Solution is $4z + (x+y\sqrt{2})^2 = 0$.

Ex. 4. *Determine the characteristics of the equation* $p^2 + q^2 = 4z$ *and find the solution of this equation which reduces to* $z = x^2 + 1$ *when* $y = 0$.

Miscellaneous Problem on Chapter 3

1. Show that the envelope of the family of surfaces touch each member of the family at all points of its characteristics. **[Meerut 2008]**

2. Find a complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the surface solution which passes through the curve $x = 0,\ z^2 = 4y$. **[Meerut 2007]**

3. Solve $p^2y + p^2yx^2 = qx^2$ **[Pune 2010]**

Ans. Complete integral is $\quad z = a\,(1+x^2)^{1/2} + (a^2y^2)/2 + b$.

4. Given thta $(x-a)^2 + (y-b)^2 + z^2 = 1$ is complete integral of $z^2\,(1+p^2+q^2) = 1$. Find its singular integral. **[Pune 2010]**

Hint. Use definition on page 3.1. **Ans.** $z^2 = 1$