SuccessClap

Best Coaching for UPSC MATHEMATICS

Test Series 2023

30th July

84/150

Topic: 03 Calculus

Date :24-07-2022

Instructions:

Time: 90 Minutes
Maximum Marks: 150
All questions are compulsory
Each question carries Equal marks
Assume suitable data if considered necessary

→ Use Laplace, Method of Variation of Parameters etc only when asked

→ Check out Euler modified formula

$$\iiiint u_1^{n_1-1} u_2^{n_2-1} u_3^{n_3-1} \cdots \quad u_1+u_2+\cdots \le 1 \qquad \frac{\Gamma(n_1)\Gamma(}{\Gamma(n_1+n_2+\cdots 1)}$$

→ Rolle Substitutions

1)Given

$$f(x) = \begin{cases} (\cos x - \sin x)^{\operatorname{cosec} x}, & -\frac{\pi}{2} < x < 0 \\ a, & x = 0 \\ \frac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}}, & 0 < x < \pi/2 \end{cases}$$

If $f(x)$ is continuous $at x = 0$, find a and b

$f(x)$ is continuous at $x=0$

$\Rightarrow f(0^-) = f(0) = f(0^+)$

$\Rightarrow \lim_{x\to 0} (\cos x - \sin x)^{\operatorname{cosec} x} = a$

$\Rightarrow \log a = \lim_{x\to 0} \operatorname{cosec} x \log(\cos x - \sin x)$

$= \lim_{x\to 0} \frac{\log(\cos x - \sin x)}{\sin x}$

$= \lim_{x\to 0} \frac{1}{\cos x - \sin x} \frac{(-\sin x - \cos x)}{\cos x}$

$\Rightarrow \log a = \frac{-1}{1\times 1} = -1$

$\Rightarrow \boxed{a = \frac{1}{e}}$

again, $\lim_{x\to 0} \frac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}} = a$

$\Rightarrow \lim_{x\to 0} \frac{e^{1/x} + e^{2/x} + e^{3/x}}{e^{-1}e^{2/x} + be^{3/x}} = \frac{1}{e}$

$\Rightarrow \lim_{x\to 0} \frac{\cancel{e^{3/x}}\left[e^{-2/x} + e^{-1/x} + 1\right]}{\cancel{e^{3/x}}\left[e^{-1}e^{-1/x} + b\right]} = \frac{1}{e}$

$\Rightarrow \frac{1}{b} = \frac{1}{e}$

$\Rightarrow \boxed{b = e}$

12

2) If λ, u, v are the roots of the equation in k,

$$\frac{x}{a+k}+\frac{y}{b+k}+\frac{z}{c+k}=1,$$

prove that

$$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,v)}=-\frac{(\mu-v)(v-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

$$\frac{x}{a+k}+\frac{y}{b+k}+\frac{z}{c+k}=1$$

$$\Rightarrow \frac{x}{a+k}+\frac{y}{b+k}=1-\frac{z}{c+k} \Rightarrow [x(b+k)+y(a+k)](c+k) = (c+k-z)(a+k)(b+k)$$

$$\Rightarrow (xb+xk+ya+yk)(c+k) = (c+k-z)(ab+bk+ak+k^2)$$

$$\Rightarrow xbc+xck+cya+cyk+xbk+xk^2+yak+yk^2$$
$$= abc+bck+ack+ck^2 + abk+bk^2+ak^2+k^3 - abz-bzk-azk-zk^2$$

$$\Rightarrow k^3+k^2[a+b+c-x-y-z]+k[ab+bc+ac-bz-az-xc-cy-xb-ya] + [abc-abz-bcx-acy]=0$$

$$\Rightarrow k^3+k^2[a+b+c-(x+y+z)]+k[ab+bc+ca-x(c+b)-y(a+c)-z(a+b)] + (abc-xbc-yac-zab)=0$$

$f_1 = \lambda+u+v = x+y+z-a-b-c$

$f_2 = \lambda u+uv+v\lambda = \cancel{a b}\ ab+bc+ca-x(c+b)-y(a+c)-z(a+b)$

$f_3 = \lambda u v = x(bc)+y(ac)+z(ab)-abc$

$$\therefore \cancel{\frac{\partial(\lambda, u, v)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial\lambda}{\partial x} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix}}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(\lambda,u,v)} \times \frac{\partial(\lambda,u,v)}{\partial(x,y,z)} = (-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$\therefore \frac{\partial(f_1,f_2,f_3)}{\partial(\lambda,u,v)} = \begin{vmatrix} 1 & 1 & 1 \\ u+v & \lambda+v & \lambda+u \\ uv & \lambda v & \lambda u \end{vmatrix} = (\lambda+v)\lambda u - \lambda v(\lambda+u) + uv(\lambda+u) - \lambda u(u+v) + \lambda v(u+v) - (\lambda+v)uv$$

$$= \lambda^2 u - \lambda^2 v + u^2 v - u^2\lambda + v^2\lambda - v^2 u.$$

$$= -(\lambda-u)(u-v)(v-\lambda)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ -(c+b) & -(a+c) & -(a+b) \\ bc & ac & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ c+b & a+c & a+b \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ c+b & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix}$$

$$= 1(a-b)(a-c)\cdot b - (a-b)(a-c)\cdot c$$

$$= (a-b)(a-c)(b-c)$$

$$= -(a-b)(b-c)(c-a)$$

$$\therefore \frac{\partial(\lambda,\mu,\nu)}{\partial(x,y,z)} = \frac{+(a-b)(b-c)(c-a)}{-(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}$$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = \frac{-(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{(b-c)(c-a)(a-b)}}$$

12

3) If $y > 0$, show that $\int_0^\infty e^{-xy}\frac{\sin x}{x}dx = \cot^{-1} y = \pi/2 - \tan^{-1} y$

$$\int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx$$

Now, $L\left(\frac{\sin t}{t}\right) = \int_p^\infty L(\sin x)\,dx$

$$= \int_p^\infty \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}(x)\Big|_p^\infty = \frac{\pi}{2} - \tan^{-1} p.$$

$\therefore\ L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1} p$

$\Rightarrow \int_0^\infty e^{-px}\frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1} p$

$\Rightarrow$ $\boxed{\text{put } p = y}$

$$\boxed{\Rightarrow \int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1} y = \cot^{-1} y}$$

Not asked to solve by Laplace & Suppose to solve by Integration by Differentiation method

4). If $p(x)$ is polynomial and $k \in \mathbb{R}$, prove that between any two real roots of $p(x) = 0$, there is a roots of $p'(x) + kp(x) = 0$.

5) Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.

$f(x)$ is even f^n

$\Rightarrow f(x) = f(-x)$

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}.$$

$$f'(0) = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}.$$

$$2f'(0) = \lim_{h\to 0}\left[\frac{f(h)-f(0)}{h} + \frac{f(-h)-f(0)}{-h}\right]$$

$$= \lim_{h\to 0}\left[\frac{f(h)-f(-h)}{h} - \cancel{\frac{f(0)}{h}} + \cancel{\frac{f(0)}{h}}\right]$$

$$= \lim_{h\to 0} \frac{f(h)-f(h)}{h} \quad (\because f(x) \text{ is even})$$

$$= 0$$

$$\Rightarrow \boxed{f'(0) = 0}$$

12

6) If n is a positive integer, prove that $2^n\Gamma(n+1/2) = 1\cdot 3\cdot 5 \dots (2n+1)\sqrt{\pi}$.

To prove $\boxed{2^n \Gamma\left(n+\frac{1}{2}\right) = 1\cdot 3\cdot 5 \cdots (2n+1)\sqrt{\pi}}$

We know $B(n,n) = \frac{\Gamma n \Gamma n}{\Gamma 2n}$

$$2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma n \Gamma n}{\Gamma 2n}$$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta = \frac{\Gamma n \Gamma n}{\Gamma 2n}$$

$2\theta \to t$

$$\Rightarrow \frac{1}{2^{2n-1}}\int_0^{\pi} (\sin t)^{2n-1} dt = \frac{\Gamma n \Gamma n}{\Gamma 2n}$$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\pi/2} (\sin t)^{2n-1} dt = \frac{(\Gamma n)^2}{\Gamma 2n}$$

$$\Rightarrow \frac{1}{2^{2n-1}} \frac{\cancel{\Gamma n}\, \Gamma \tfrac{1}{2}}{\Gamma n+\tfrac{1}{2}} = \frac{(\Gamma n)^{\cancel{2}}}{\Gamma 2n}$$

$$\Rightarrow 2^{2n-1}\Gamma n+\tfrac{1}{2} = \sqrt{\pi}\,\frac{\Gamma 2n}{\Gamma n}$$

$$\Rightarrow 2^{2n-1}\Gamma n+\tfrac{1}{2} = \sqrt{\pi}\,\frac{(2n-1)!}{(n-1)!}$$

$$\Rightarrow 2^{2n-1}\Gamma n+\tfrac{1}{2} = \sqrt{\pi}\,\frac{(2n-1)!}{(n-1)!}$$

$$\Rightarrow 1\cdot 3\cdot 5 \cdots (2n+1) = \frac{1\cdot 2\cdot 3\cdot 4 \cdots (2n+1)}{2\cdot 4\cdot 6 \cdots 2n} = \frac{(2n+1)!}{2^n n!}$$

$$= \frac{(2n-1)!(2n)(2n+1)}{2^n \cdot \cancel{n}(n-1)!}$$

$$= \frac{(2n+1)}{2^{n-1}}\frac{(2n-1)!}{(n-1)!}$$

$$\therefore 2^{2n-1}\Gamma n+\tfrac{1}{2} = \sqrt{\pi}\,\frac{(1\cdot 3\cdot 5 \cdots 2n+1)\,2^{n-1}}{(2n+1)}$$

$$\Rightarrow \boxed{2^n \Gamma n+\tfrac{1}{2} = \sqrt{\pi}\,(1\cdot 3\cdot 5 \cdots 2n-1)}$$

12

7) Find all the asymptotes of the curve $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$.

$$(x^2-y^2)(x+2y+1)+(x+y+1)=0$$

$$\boxed{x^2-y^2 = \frac{-(x+y+1)}{x+2y+1}}$$

Asymptots $x^2-y^2 = \lim_{y=x\to\infty} \frac{-(x+y+1)}{x+2y+1} = \lim_{y\to 0} \frac{-(2y+1)}{3y+1} = \frac{-2}{3}$

$$(x-y)(x+y)(x+2y) + x^2-y^2 + (x+y+1) = 0.$$

$$x+y + \frac{x^2-y^2+(x+y+1)}{(x-y)(x+2y)} = 0$$

Asymptots ~~x=-y~~ for $x+y$

$$\Rightarrow x+y+\lim_{y\to -x\to\infty} \frac{x^2-y^2+(x+y+1)}{(x-y)(x+2y)} = 0$$

$$\Rightarrow x+y+\lim_{y\to\infty} \frac{y+y+1}{-2y(3y)} = 0$$

$$\boxed{\Rightarrow x+y=0}$$

$$x-y + \frac{x^2-y^2+(x+y+1)}{(x+y)(x+2y)} = 0$$

Asymptots for $x=y$

$$x-y+\lim_{x\to y\to\infty} \frac{(x^2-y^2)(x+y+1)}{(x+y)(x+2y)} = 0$$

$$\Rightarrow x-y+\lim_{x\to 0} \frac{2x+1}{2x\cdot(3x)} = 0$$

$$\Rightarrow x-y+0=0$$

$$\Rightarrow \boxed{x-y=0}$$

Asymptot for $x+2y$

$$(x+2y)+\lim_{\substack{x\to -2y\\ \to\infty}} \frac{x^2-y^2+(x+y+1)}{(x^2-y^2)} = 0$$

$$\Rightarrow (x+2y)+\lim_{-2y\to\infty} \frac{4y^2-y^2+(-y+1)}{4y^2-y^2} = 0$$

$$\Rightarrow \boxed{x+2y+1=0}$$

12

8) Show that $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}=\frac{\pi}{6}$.

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\sqrt{4-(r/n)^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{4-x^2}}$$ (limit of sum as integral)

$$=\sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

$$\therefore \boxed{\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}=\frac{\pi}{6}}$$

12

9) If $z = x^m f(y/x) + x^n g(x/y)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$

Let $u = x^m f(y/x)$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu$ — (I)

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)u$ — (II)

Similarly $v = x^n g(x/y)$

$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\cdot v$ — (III)

and $x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$ — (IV)

adding equation (I), (II), (III), (IV)

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\cdot y^2 + x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2}$

$+ x\frac{\partial v}{\partial x} + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} + y\frac{\partial u}{\partial y} = nv + n(n-1)v + mu + m(m-1)u$

$= n^2 v + m^2 u$

$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial^2 z}{\partial y^2}\cdot y^2 + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$= m^2 u + n^2 v$

$= m^2 u + mnu + n^2 v + mnv - (mnu + mnv)$

$= (m+n)(u+v) - mn(u+v)$

$= (m+n)z - mnz$

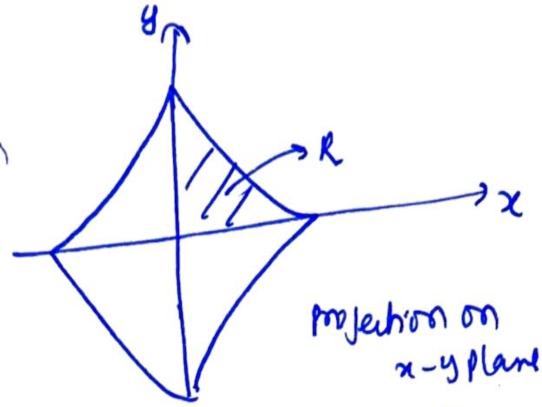
$\therefore \boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)}$

12

10) Find the volume in the first octant determined by the surface $x^n + y^n + z^n = a^n, (n > 0)$

$x^n+y^n+z^n=a^n$.

~~Put~~



$\Rightarrow x^n+y^n=a^n$

$V = \iiint_0 dx\,dy\,dz$

$= \iint_R (a^n-x^n-y^n)^{1/n}\,dx\,dy$

$x = \cancel{a\cos} \; ar\cos^2\theta$

$y = ar\sin^2\theta$.

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} a\cos^2\theta & a\sin^2\theta \\ -ar\sin 2\theta & ar\sin 2\theta \end{vmatrix}$

$= a^2 r\sin 2\theta\left[\cos^2\theta + \sin^2\theta\right]$

$= a^2 r\sin 2\theta$

$\therefore \int_0^a\int_0^{\pi/2} \left[a^n - a^n r^n(\sin^2\theta+\cos^2\theta)^n\right]^{1/n} a^2 r\sin 2\theta\, d\theta$

$\Rightarrow \int_0^a\int_0^{\pi/2} a\left[1-r^n\right]^{1/n} a^2 r\sin 2\theta\, d\theta\, dr$.

$= a^3 \int_0^a \left|\frac{-\cos 2\theta}{2}\right|_0^{\pi/2} r(1-r^n)^{1/n}\,dr$.

$= a^3 \int_0^a r(1-r^n)^{1/n}\,dr$

Use Soley method to solve Quickly

$\frac{x^n}{a^n} = u_1$, $\frac{y^n}{a^n} = u_2$

$u_1+u_2+u_3 \le 1$

$V = \frac{a^3}{n^3} \iiint u_1^{1/n-1} u_2^{1/n-1} u_3^{1/n-1}$

$\frac{\left(\Gamma(1/n)\right)^3}{\Gamma\left(\frac{3}{n}+1\right)}$