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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -1 PAPER 1

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Phone Number		Date	
Start Time :		Closing Time:	

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		8	5a		8	
1b		8	5b		8	
1c			5c		8	
1d		8	5d		8	
1e		8	5e		5	
2a			6a			
2b			6b			
2c			6c			
2d			6d			
3a		15	7a		8	
3b		12	7b		5	
3c		12	7c		12	
3d			7d		8	
4a		12	8a			
4b			8b			
4c		12	8c			
4d			8d			
Total		165/250				

Section A

1(a) Prove the solution set W of the differential equation

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

is a subspace of vector space of all real valued functions of R. (10)

Let V is vector space of all real valued function of R.

Let W be a subset of V which contains solution of differential equation

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0. \quad \text{—} (1)$$

Now, let $\omega_1, \omega_2 \in W$ be two solution of (1)

$$\therefore \quad 2\frac{d^2\omega_1}{dx^2} \cancel{+} - 9\frac{d\omega_1}{dx} + 2\omega_1 = 0 \quad \text{—} (11)$$

$$2\frac{d^2\omega_2}{dx^2} - 9\frac{d\omega_2}{dx} + 2\omega_2 = 0 \quad \text{—} (111)$$

let $a, b \in R$

multiply (11) with a & (111) with b and add

$$2\left(a\frac{d^2\omega_1}{dx^2} + b\frac{d^2\omega_2}{dx^2}\right) - 9\left(a\frac{d\omega_1}{dx} + b\frac{d\omega_2}{dx}\right) + 2(a\omega_1 + b\omega_2) = 0$$

$$\Rightarrow 2\frac{d^2(a\omega_1 + b\omega_2)}{dx^2} - 9\frac{d}{dx}(a\omega_1 + b\omega_2) + 2(a\omega_1 + b\omega_2) = 0$$

$\Rightarrow a\omega_1 + b\omega_2$ is solution of (1)

$\Rightarrow a\omega_1 + b\omega_2 \in W$

$\therefore$ when $\omega_1 \in W$ & $W_2 \in W$

$a\omega_1 + bW_2 \in W$ for any $a, b \in R$

$\therefore$ $\boxed{\text{W is } \cancel{\text{vector}} \text{ sub space of vector space V}}$

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1(b) Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$. Prove that T maps the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ into a parallelogram.

(10)

$T(1,0) = (1,1)$

$T(0,1) = (-1,2)$

Now $T(x,y) = T(x(1,0) + y(0,1)) = xT(1,0) + yT(0,1)$

$= x(1,1) + y(-1,2)$

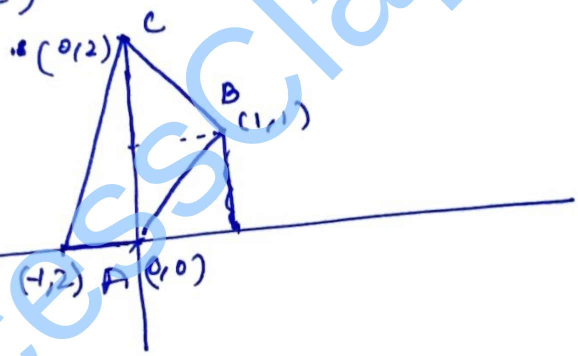
$$\boxed{T(x,y) = (x-y, x+2y)}$$

Now, $A = T(0,0) = (0,0)$

$B = T(1,0) = (1-0, 1+0) = (1,1)$

$C = T(1,1) = (1-1, 1+2) = (0,3)$

$D = T(0,1) = (-1,2)$



Now,

$AB = \sqrt{1^2+1^2} = \sqrt{2}$

$BC = \sqrt{1^2+(3-1)^2} = \sqrt{5}$

$CD = \sqrt{(0+1)^2+(1)} = \sqrt{2}$

$DA = \sqrt{1^2+4} = \sqrt{5}$

$\therefore AB = \sqrt{2} = CD$

$BC = DA = \sqrt{5}$

so opposite side are equal in size

thus ABCD is parallelogram

$\Rightarrow$ $\boxed{\text{T maps } (0,0), (1,0), (1,1), (0,1) \text{ in a parallelogram}}$

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1(c) Evaluate $\lim_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2}$ (10)

$$L = \lim_{x\to 0} \frac{(x+1)^{1/x} - e + \frac{1}{2}ex}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}\log e^{(1+x)} - e + \frac{1}{2}ex}{x^2}$$

Expand sinx, cosx $(1+x)^x$ it cannot be solved

1(d) Discuss the convergence of $\int_0^\infty \frac{x^p \sin^2 x}{1+x^2} dx$. (10)

$$\int_0^\infty \frac{x^p \sin^2 x}{1+x^2} dx = \int_0^a \frac{x^p \sin^2 x}{1+x^2} dx + \int_a^\infty \frac{x^p \sin^2 x}{1+x^2} dx$$

Check convergence at $x=0$

when $p \geq 0$ $\lim_{x\to 0} \frac{x^p \sin^2 x}{1+x^2} = 0$

$\therefore \int_0^a \frac{x^p \sin^2 x}{1+x^2} dx$ is proper integral.

when $p<0$ let $-p = q$

$$\Rightarrow \int_0^a \frac{\sin^2 x \, x^p}{1+x^2} dx = \int_0^a \frac{\sin^2 x}{x^q(1+x^2)} dx$$

$$f(x) = \frac{\sin^2 x}{x^q(1+x^2)}$$

let $g(x) = \frac{1}{x^{q-2}}$ $\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+x^2)} = 1$

$\therefore$ By comparison test

$f(x), g(x)$ converge or diverges together

~~if q $p<1$ ⇒ $p>$~~

if $q-2<1 \Rightarrow q<3$ $g(x)$ will converge

$\Rightarrow -p<3 \Rightarrow p>-3$

but $p<0$

$\therefore -3<p<0$ $\rightarrow \int_0^a f(x)\,dx$ will converge.

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Now, for discontinuity at ∞

$$\int_a^\infty \frac{x^p \sin^2 x}{1+x^2} dx < \int_a^\infty \frac{x^p}{1+x^2} dx$$

$$\int_a^\infty \frac{x^p}{1+x^2} dx = \int_a^\infty \frac{1}{x^{2-p}(1+\frac{1}{x^2})} dx$$

will converge if

$2-p>1$

$\boxed{p<1}$

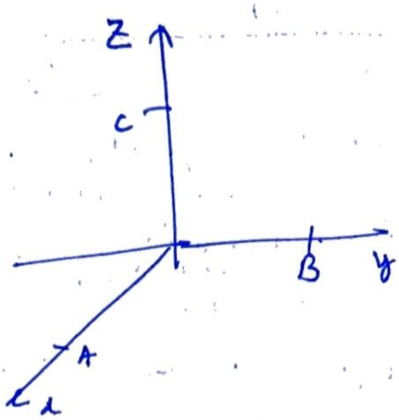
$\therefore \int_0^\infty \frac{x^p \sin^2 x}{1+x^2} dx$ will converge for $\boxed{-3<p<1}$

1(e) A variable plane is at a constant distance $3p$ from the origin and meets the axes in A, B and C. Prove that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

(10)

Let variable plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{—(i)}$$



distance from origin

$$3p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = \frac{1}{3p} \quad \text{—(ii)}$$

coordinate of A, B, C

$A \equiv (a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$

Let centroid of ΔABC is (α, β, γ)

$$\therefore \alpha = \frac{a+0+0}{3}, \quad \beta = \frac{b+0+0}{3}, \quad \gamma = \frac{c+0+0}{3}$$

$$\Rightarrow \alpha = \frac{a}{3}, \quad \beta = \frac{b}{3}, \quad \gamma = \frac{c}{3}$$

$$\Rightarrow a = 3\alpha, \quad b = 3\beta, \quad c = 3\gamma. \longrightarrow \text{(iii)}$$

put these in equation (ii)

$$\Rightarrow \sqrt{\frac{1}{3^2\alpha^2} + \frac{1}{3^2\beta^2} + \frac{1}{3^2\gamma^2}} = \frac{1}{3p}$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{p^2}$$

$\therefore$ locus of centroid is given by

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

$$\Rightarrow \boxed{x^{-2} + y^{-2} + z^{-2} = p^{-2}}$$

8

3(a) Suppose u is a unit vector in R^n, so $u^T u = 1$.
This problem is about the n by n symmetric matrix $H = I - 2uu^T$.

(a) Show directly that $H^2 = I$. Also Show H is orthogonal

(b) If one eigenvector of H is u itself. Find the corresponding eigenvalue.

(c) If v is any vector perpendicular to u, show that v is an eigenvector of H and find the eigenvalue.

With all these eigenvectors v, that eigenvalue must be repeated how many times?
Is H diagonalizable? Why or why not?

(d) If H_{ii} is defined as diagonal entry of i-row and column term of matrix H. Find the diagonal entries H_{11} and H_{ii} in terms of $u_1, \ldots, u_n$.

(e)Find the value of $H_{11} + \ldots + H_{nn}$

(20)

(a) $H = I - 2uu^T \qquad u^T \cdot u = 1$

$H^2 = (I - 2uu^T)(I - 2uu^T)$
$= I - 2uu^T - 2uu^T + 4uu^Tuu^T$
$= I - 4uu^T + 4u(u^Tu)u^T$
$= I - 4uu^T + 4uu^T$
$= I$

$\therefore \boxed{H^2 = I}$

$H \cdot H^T = (I - 2uu^T)(I - 2uu^T)^T$
$= (I - 2uu^T)(I^T - 2(u^T)^T u^T)$
$= (I - 2uu^T)(I - 2uu^T) = I$

$\therefore$ H is orthogonal

(b) u is eigen vector for λ

$\therefore Hu = u\lambda$
$\Rightarrow (I - 2uu^T)u = \lambda u$
$\Rightarrow Iu - 2u \cdot u^T \cdot u = \lambda u$
$\Rightarrow Iu - u \cdot I = \lambda u$
$\Rightarrow \lambda u = 0$
$\Rightarrow \boxed{\cancel{\lambda = 0}}$

(c) v is $\perp^r$ to u

$\Rightarrow$ ~~$u \cdot v^T = 0$~~ or $u^T \cdot v = 0$ or $\underline{v^T \cdot u = 0}$

Now $H \cdot v = (I - u \cdot u^T) v$

$= I v - u \cdot (u^T \cdot v)$

$\boxed{H v = I v .}$

$H v = v \times 1$

$\therefore$ v is eign vector of H and corresponing eign value is 1

· For these eigen vector v these will be repeated $\underline{n-1}$ times as there are n eigen vector for $n \times n$ matrix and one $\underline{\text{eigen value} = 0}$

· H is non diagonalizable because matix of eigen vector is singular

(d) ~~H~~ $u = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$ $u^T = \{u_1, u_2, u_3 \ldots u_n\}$

$$u u^T = \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 & \ldots & u_1u_n \\ u_2u_1 & u_2u_2 & \ldots & \ldots & u_2u_n \\ \vdots & & & & \\ u_nu_1 & \ldots & \ldots & & u_nu_n \end{bmatrix}$$

$$I - 2uu^T = \begin{bmatrix} 1-2u_1u_1 & -2u_1u_2 & \ldots & -2u_1u_n \\ -2u_2u_1 & 1-2u_2u_2 & \ldots & -2u_2u_n \\ \vdots & & & \\ -2u_nu_1 & -2u_nu_2 & & 1-2u_nu_n \end{bmatrix}$$

$\therefore$ $\boxed{H_{ii} = 1 - 2u_iu_i}$

$\boxed{H_{11} = 1 - 2u_1u_1}$

(e) $H_{11} + H_{22} + \ldots H_{nn} = n - 2(u_1^2 + u_2^2 + \ldots u_n^2)$

$= \underline{n - 2 \times 1} = \underline{\underline{n-2}}.$

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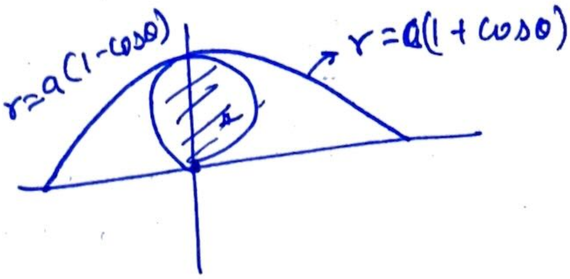
3(b) Show that the area of the region included between the cardioids $r = a(1+\cos\theta), r = a(1-\cos\theta)$ is $\frac{a^2}{2}(3\pi - 8)$. (15)

area included

$$= 2\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\cdot dr\, d\theta.$$



$$= 2\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{a(1-\cos\theta)} d\theta.$$

$$= 2\int_0^{\pi/2} \frac{a^2}{2}(1-\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi/2}(1+\underline{\cos^2\theta} - 2\cos\theta)\, d\theta$$

$$= a^2\int_0^{\pi/2}\left(1+ \frac{\cos 2\theta + 1}{2} - 2\cos\theta\right) d\theta$$

$$= a^2\left[\theta + \frac{\sin 2\theta}{2} + \frac{1}{2}\theta - 2\sin\theta\right]_0^{\pi/2}$$

$$= a^2\left[\frac{3}{2}\times\frac{\pi}{2} + 0 + 0 - 2(1-0)\right]$$

$$= a^2\left[\frac{3\pi}{4} - 2\right]$$

$$\therefore \boxed{\text{area} = a^2\left[\frac{3\pi}{4} - 2\right]}$$

12

3(c) Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0$$

lie on the same sphere and find its equation. Also find the value 'a' for which $x+y+z=a\sqrt{3}$ touches the sphere.

(15)

Sphare passes through

$(x^2+y^2+z^2-2x+3y+4z-5)=0$ & $5y+6z+1=0$ is

$\Rightarrow x^2+y^2+z^2-2x+3y+4z-5+k_1(5y+6z+1)=0$

$\Rightarrow x^2+y^2+z^2+x(-2\cancel{+5k_1})+y(3+5k_1)+z(4+6k_1)-5+k_1=0$ —①

Sphare passes through

$x^2+y^2+z^2-3x-4y+5z-6=0$ & $x+2y-7z=0$ is

$x^2+y^2+z^2-3x-4y+5z-6+k_2(x+2y-7z)=0$

$\Rightarrow x^2+y^2+z^2+x(-3+k_2)+y(-4+2k_2)+z(5-7k_2)-6=0$ —②

eqn ① & ② will be same if

$-2\cancel{+5k_1}=-3+k_2$ —ⓐ

& $5k_1+3=(-4+2k_2)$ —ⓑ

$4+6k_1=5-7k_2$ —ⓒ

$-5+k_1=-6$ —ⓓ

From ⓐ & ⓓ, a, b

$\Rightarrow k_2=1,\ k_1=-1$

this satisfy eqn ⓑ & ⓒ

so, $k_1=-1,\ k_2=1$ satisfies all a, b, c, d

$\therefore$ eqn ① & ② are same for $k_1=-k_2=-1$

$\therefore$ required sphare

$x^2+y^2+z^2+(-2x)+(-2y)+(-2z)-6=0$ —③

Now centre $(1,1,1)$

radius $=(\sqrt{1+1+1+6}=3)$

if $x+y+z=a\sqrt{3}$ touch ③

then $\dfrac{|1+1+1-a\sqrt{3}|}{\sqrt{3}}=3$

4

$|3 - a\sqrt{3}| = 3\sqrt{3}$

$\Rightarrow 3 - a\sqrt{3} = 3\sqrt{3}$ or $3 - a\sqrt{3} = -3\sqrt{3}$

$\Rightarrow a = \frac{3 - 3\sqrt{3}}{\sqrt{3}}$ or $a = \frac{3 + 3\sqrt{3}}{\sqrt{3}}$

$\boxed{\Rightarrow a = (\sqrt{3} - 3) \text{ or } a = \sqrt{3} + 3}$

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4(a) Find the equations to the generating lines of the hyperboloid $x^2/4 + y^2/9 - z^2/16 = 1$ which pass through the points $(2,3,-4)$ and $(2,-1,4/3)$.

(15)

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1 - \frac{y^2}{9}$$

∴ generating lines are given by system

system ①

$$\left(\frac{x}{2} - \frac{z}{4}\right) = \lambda\left(1 - \frac{y}{3}\right)$$

$$\frac{x}{2} + \frac{z}{4} = \frac{1}{\lambda}\left(1 + \frac{y}{3}\right)$$

system ②

$$\frac{x}{2} - \frac{z}{4} = \mu\left(1 + \frac{y}{3}\right)$$

$$\frac{x}{2} + \frac{z}{4} = \frac{1}{\mu}\left(1 - y/3\right)$$

For $(2,3,-4)$

system ①

$$(1 + 1) = \lambda(1 - 1)$$

$\cancel{\lambda = 2}$

$$2 = \lambda \times 0$$

$$\Rightarrow \lambda = \infty$$

system ②

$$\frac{2}{2} + \frac{4}{4} = \mu\left(1 + \frac{3}{3}\right)$$

$$2 = \mu \times 2.$$

$$\Rightarrow \mu = 1$$

∴ Generating lines for $(2,3,-4)$ is given by

$$\frac{x}{2} - \frac{y}{4} = \frac{1}{0}\left(1 - \frac{y}{3}\right)$$

$$\Rightarrow 1 - \frac{y}{3} = 0 \Rightarrow y = 3$$

and

$$\frac{x}{2} + \frac{z}{4} = 0$$

$$\Rightarrow \boxed{\frac{x}{2} + \frac{z}{4} = 0 \text{ and } y = 3} \rightarrow ①.$$

and

$$\frac{x}{2} - \frac{z}{4} = \left(1 + \frac{y}{3}\right)$$

$$\boxed{\begin{aligned} &\Rightarrow \frac{x}{2} - \frac{y}{3} - \frac{z}{4} = 1. \\ &\frac{x}{2} + \frac{z}{4} + \frac{y}{6} = \frac{1}{2} \end{aligned}} \rightarrow ②$$

eqⁿ ① & ② give generating line for $(2,-3,4)$

Now again for $(2,-1,4/3)$

system ①

$$\frac{x}{2} - \frac{z}{4} = \lambda\left(1 - \frac{y}{3}\right)$$

$$\Rightarrow 1 - \frac{1}{3} = \lambda\left(1 + \frac{1}{3}\right)$$

$$\Rightarrow \frac{2}{3} = \lambda \frac{4}{3} \Rightarrow \boxed{\lambda = \frac{1}{2}}$$

system ②

$$\frac{x}{2} - \frac{z}{4} = \mu\left(1 + \frac{y}{3}\right)$$

$$\Rightarrow 1 - \frac{1}{3} = \mu\left(1 - \frac{1}{3}\right)$$

$$\Rightarrow \boxed{\mu = 1}$$

Generating lines

system ①

$\frac{x}{2} - \frac{z}{4} = \frac{1}{2}\left(1 + \frac{y}{3}\right)$ and

$\frac{x}{2} + \frac{z}{4} = 2\left(1 + \frac{y}{3}\right)$

system ②

$\frac{x}{2} - \frac{z}{4} = 1\left(1 + \frac{y}{3}\right)$ and

$\frac{x}{2} + \frac{z}{4} = 1\left(1 - \frac{y}{3}\right)$

$\Rightarrow \boxed{\frac{x}{2} + \frac{y}{6} - \frac{z}{4} = 1 \text{ and } \frac{x}{2} - \frac{2y}{3} + \frac{z}{4} = 2}$ and $\boxed{\frac{x}{2} - \frac{y}{3} - \frac{z}{3} = 1 \text{ and } \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1}$

is generating lines for $(2, -1, 4/3)$

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4(b) Reduce the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to a diagonal form and interpret the result in terms of quadratic form (20)

diagonal form is not diagonalization

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$$

$$\Rightarrow (6-\lambda)\left[\lambda^2 - 6\lambda + 9 - 1\right] + 4(2\lambda - 4) = 0$$

$$\Rightarrow -(\lambda-6)\left[\lambda^2 - 6\lambda + 8\right] + 8(\lambda-2) = 0$$

$$\Rightarrow -(\lambda-6)(\lambda-2)(\lambda-4) + 8(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)\left((\lambda-6)(\lambda-4) - 8\right) = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2 - 10\lambda + 24 - 8) = 0 \Rightarrow (\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\lambda = 2, \lambda = 8, \underline{\lambda = 2}$$

when $\underline{\lambda = 2}$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x - y + z = 0$

$\Rightarrow$ when, $y = k_1, z = k_2$

$2x = y - z = k_1 - k_2$

$x = \dfrac{k_1 - k_2}{2}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} - \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ eign vector for 2 is $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x - 2y + 2z = 0 \Rightarrow x + y - z = 0$

$-3y - 3z = 0 \qquad y + z = 0$

let $x = k$

$y + z = -k$

$y + z = 0$

$\Rightarrow y = -\frac{k}{2}$

$\boxed{z = \frac{k}{2}}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$

$\therefore$ $D = P^{-1} A P$

$$\boxed{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 1 & 1 & -1/2 \\ 1 & 1 & 1/2 \end{bmatrix}^{-1} A \begin{bmatrix} 1/2 & -1/2 & 1 \\ 1 & 1 & -1/2 \\ 1 & 1 & 1/2 \end{bmatrix}}$$

4(c)Derive Legendre Duplication Formula

$$\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n), n>0.$$

(15)

we know

$$B(n,n)=2\int_0^{\pi/2}\sin^{2n-1}\theta\ \cos^{2n-1}\theta\, d\theta$$

$$\Rightarrow \frac{\Gamma n\,\Gamma n}{\Gamma 2n} = 2\int_0^{\pi/2}\sin^{2n-1}\theta\ \cos^{2n-1}\theta\, d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2}\sin^{2n-1}2\theta\, d\theta$$

put $2\theta=t$

$$= \frac{\cancel{2}}{2^{2n-1}}\int_0^{\pi}\sin^{2n-1}t\ \frac{dt}{\cancel{2}}$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2}\sin^{2n-1}t\, dt$$

$$\frac{(\Gamma n)^2}{\Gamma 2n} = \frac{1}{2^{2n-1}}\ \frac{\Gamma n\ \Gamma\frac{1}{2}}{\Gamma n+\frac{1}{2}}$$

$$\Rightarrow \frac{\Gamma n}{\Gamma 2n} = \frac{1}{2^{2n-1}}\times\frac{\sqrt{\pi}}{\Gamma n+\frac{1}{2}}.$$

$$\therefore \boxed{\Gamma n\ \Gamma n+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2n-1}}\ \Gamma 2n}$$

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SECTION B

5(a) Find the family of curves whose tangents form the angle of $\pi/4$ with the hyperbola xy = c .

(10)

$xy = c$

differntsale with respect to x

$\Rightarrow y + x\frac{dy}{dx} = 0$

put $\frac{dy}{dx} = p$

$\Rightarrow y + xp = 0$. — (1)

Now, For Family of curve at 45°

Put p as $\frac{1+p}{1-p}$

$\Rightarrow$ Differential equation of the required family

$\Rightarrow y + x\left(\frac{1+p}{1-p}\right) = 0$

$\Rightarrow y(1-p) + x(1+p) = 0$

$\Rightarrow p(x-y) + x + y = 0$.

$\Rightarrow \frac{dy}{dx} = \frac{y+x}{y-x}$

$\Rightarrow -ydx + xdy + xdx + ydy = 0$

$\Rightarrow xdy + ydx + xdx - ydy = 0$

$\Rightarrow d(xy) + \frac{1}{2}d(x^2 - y^2) = 0$

~~dy~~ integrate

$\Rightarrow xy + \frac{1}{2}(x^2 - y^2) = C_2$

$\Rightarrow \boxed{2xy + (x^2 - y^2) = C_3}$

is required family

8

5(b) If y_1 and y_2 be solutions of the equation $\frac{dy}{dx} + P(x)y = Q(x)$ and $y_2 = y_1 z$, then show $z = 1 + ae^{-\int\left(\frac{Q}{y_1}\right)dx}$ where a is arbitrary constant. (10)

$$\frac{dy}{dx} + P(x)y = Q(x)$$

if $y_2 = y_1 z$ is solution

then

$$\frac{dy_2}{dx} + P(x)y_2 = Q(x)$$

$$\Rightarrow \frac{d}{dx}(y_1 z) + y_1 z P(x) = Q(x)$$

$$\Rightarrow z\frac{dy_1}{dx} + y_1\frac{dz}{dx} + y_1 z P(x) = Q(x)$$

$$\Rightarrow z\left(\frac{dy_1}{dx} + y_1 P(x)\right) + y_1\frac{dz}{dx} = Q(x)$$

$$\Rightarrow z(+Q(x)) + y_1\frac{dz}{dx} = Q(x) \qquad \text{as}\left(\frac{dy_1}{dx} + P(x)y_1 = Q(x)\right)$$

$$\Rightarrow \frac{dz}{dx} + \frac{+z(Q(x))}{y_1} = \frac{Q(x)}{y_1}$$

$$\therefore \cancel{z = IF = e^{\int\frac{Q}{y_1}dx}}$$

$$\therefore \cancel{z \cdot e^{\int Q/y_1 dx} =}$$

$$\Rightarrow \frac{dz}{dx} = (1-z)\frac{Q}{y_1}$$

$$\Rightarrow \frac{dz}{1-z} = dx\,\frac{Q}{y_1}$$

$$\Rightarrow \frac{dz}{z-1} = -\frac{Q}{y_1}dx$$

Integrate

$$\Rightarrow \log(z-1) = \int\frac{-Q}{y_1}dx + \log a.$$

$$\Rightarrow \log(z-1) = \log e^{\int-\frac{Q}{y_1}dx} + \log a$$

$$\Rightarrow z - 1 = a \cdot e^{\int -Q/y_1\, dx}$$

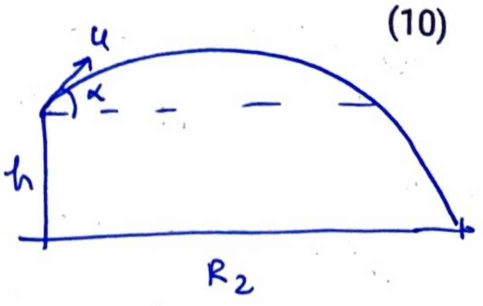
$$\boxed{\therefore z = 1 + ae^{\int -Q/y_1\, dx}}$$

8

5(c) A gun is fixed from the sea level out to sea. It is then mounted on a battery h meters higher up and fired at the same elevation α. Show that the range is increased by $(1/2)\left\{\left(1+2gh/u^2\sin^2\alpha\right)^{1/2}-1\right\}$ of itself, u being the velocity of projection.

(10)

Let R_1 be initial range with velocity u and elevation α



$\Rightarrow R_1 = \frac{u^2\sin 2\alpha}{g}$ — (1)

Now for battery h,

equation of motion in x direction for any point (x, y)

$\Rightarrow x = u\cos\alpha\, t$ — (2)

$y = u\sin\alpha\, t - \frac{1}{2}gt^2$ — (3)

now for Range R_2, $y = -h$.

$\therefore -h = u\sin\alpha\, t - \frac{1}{2}gt^2$ & $R_2 = u\cos\alpha \cdot t$

$\Rightarrow gt^2 - 2u\sin\alpha\, t - 2h = 0$

$\Rightarrow t = \frac{u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}}{g}$

Put in $R_2 = u\cos\alpha\, t$

$\Rightarrow R_2 = \frac{u\cos\alpha}{g}\left(u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh}\right)$

Now $R_2 - R_1 = \frac{u^2\cos\alpha\sin\alpha}{g} + \frac{u\cos\alpha}{g}\sqrt{u^2\sin^2\alpha + 2gh} - \frac{u^2\sin 2\alpha}{g}$

$= \frac{u\cos\alpha \cdot u\sin\alpha}{g}\sqrt{1 + \frac{2gh}{u^2\sin^2\alpha}} - \frac{u^2\sin 2\alpha}{2g}$

$= \frac{u^2\sin 2\alpha}{2g}\left[\sqrt{1 + \frac{2gh}{u^2\sin^2\alpha}} - 1\right]$

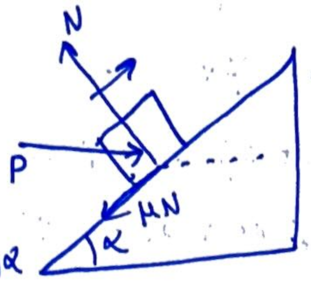
$\therefore$ increase in Range $R_2 - R_1$ is of itself ~~is~~ given by $\frac{R_2 - R_1}{R} = \cancel{u^2\sin\alpha}\ \frac{1}{2}\left[\sqrt{1 + \frac{2gh}{u^2\sin^2\alpha}} - 1\right]$

8

5(d) Find the least force, required to drag a heavy body up a rough plane of inclination α, when the force acts horizontally.

(10)

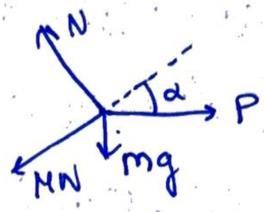
Along Plane balancing the force



$\Rightarrow P\cos\alpha = \mu N + mg\sin\alpha$ — (I)

& $\perp^r$ to Plane

$P\sin\alpha + mg\cos\alpha = N$ — (II)



put (II) in (I)

$\Rightarrow P\cos\alpha = \mu(P\sin\alpha + mg\cos\alpha) - mg\sin\alpha$

$\Rightarrow P\cos\alpha - \mu P\sin\alpha = \mu mg\cos\alpha - mg\sin\alpha$

$= mg(\mu\cos\alpha - \sin\alpha)$

$\therefore \boxed{P = \dfrac{mg(\mu\cos\alpha - \sin\alpha)}{(\cos\alpha - \mu\sin\alpha)}}$

is required min force P

where m is mass of body

and μ is friction cofficient

8

5(e) Find the equations of the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. (10)

$2xz^2 - 3xy - 4x = 7$ —(1)

$f = 2xz^2 - 3xy - 4x - 7 = 0$ —(2)

gradient of f (∇f) Provide direction ratio of normal at any point.

$\therefore \nabla f = \frac{\partial}{\partial x} f \hat{i} + \frac{\partial}{\partial y} f \hat{j} + \frac{\partial f}{\partial z} \hat{k}$

$\Rightarrow \nabla f = (2z^2 - 3y - 4)\hat{i} + (-3x)\hat{j} + 2x \times 2z \hat{k}$

$\Rightarrow \nabla f = (2z^2 - 3y^2 - 4)\hat{i} - 3x\hat{j} + 4xz\hat{k}$

∇f at $(1,-1,2)$ is

$(\nabla f)_{(1,-1,2)} = (2\times 4 - 3\times 1 - 4)\hat{i} - 3(1)\hat{j} + 4(1\times 2)\hat{k}$

$= \hat{i} - 3\hat{j} + 8\hat{k}$

$\therefore$ direction ratio of normal $(1, -3, 8)$

$\therefore$ equation of tangent plane is

$(x-1)1 + (y+1)(-3) + (z-2)8 = 0$

$\Rightarrow x - 3y + 8z - 1 - 3 - 16 = 0$

$\Rightarrow \boxed{x - 3y + 8z - 20 = 0}$

is required equation of tangent plane at $(1,-1,2)$

equation of normal at $(1,-1,2)$

$$\boxed{\frac{x-1}{1} = \frac{y+1}{-3} = \frac{z-2}{8}}$$

5

7(a) Using Laplace Solve $(D^2+2)x - Dy = 1,\ Dx + (D^2+2)y = 0$, if $x = Dx = y = Dy = 0$, when $t = 0$. (10)

$(D^2+2)x - Dy = 1$ — (I)

$Dx + (D^2+2)y = 0$ — (II)

let $L(x) = X$

$L(y) = Y$

$\therefore$ taking lapalance of (I)

$L\{D^2x\} + L(2x) - L(Dy) = L(1)$

$\Rightarrow p^2X - px(0) - x'(0) + 2X - Yp + y(0) = \frac{1}{p}$

$\Rightarrow p^2X + 2X - Yp = \frac{1}{p}$ — (a)

taking laplace of (II)

$L(D(x)) + L(D^2+2)y = 0$.

$\Rightarrow pX - x(0) + p^2Y - pY(0) - y'(0) + 2L(y) = 0$

$\Rightarrow pX + p^2Y + 2Y = 0$ — (b)

$\Rightarrow (p^2+2)Y = -pX$

$\Rightarrow Y = \frac{-pX}{(p^2+2)}$

$\therefore p^2X + 2X - p\left(\frac{-pX}{p^2+2}\right) = \frac{1}{p}$

$\Rightarrow X\left(p^2+2+\frac{p^2}{p^2+2}\right) = \frac{1}{p}$

$\Rightarrow X \times \frac{(p^2+2)^2+p^2}{(p^2+2)} = \frac{1}{p}$

$\Rightarrow X = \frac{(p^2+2)}{(p^4+5p^2+4)p} = \frac{p^2+2}{(p^2+4)(p^2+1)p}$

$\cancel{X = \frac{1}{2p} - \frac{p}{6(p^2+4)} - \frac{p}{3(p^2+1)}}$

$$x = \frac{p^2+2}{p(p^2+1)(p^2+4)}$$

$$\frac{p^2+2}{p(p^2+1)(p^2+4)} = \frac{A}{p} + \frac{Bp+C}{p^2+1} + \frac{Dp+E}{p^2+4}$$

$$p^2+2 = A(p^2+1)(p^2+4) + (Bp+C)(p)(p^2+4) + (Dp+E)p(p^2+1)$$

Put $p=0$

$$\Rightarrow 2 = A(1)*4 \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow p^2+2 = A(p^4+5p^2+4) + p(Bp^3+Cp^2+4Bp+C4) + (Dp^3+Ep^2+Dp+E)p$$

$$\Rightarrow A+B+D = 0 \Rightarrow B+D = -\frac{1}{2}$$

$$C+E=0$$

$$4C+E=0 \Rightarrow \underline{E=C=0}$$

$$1 = 5A+4B+D \Rightarrow 4B+D = 1-5A = 1-\frac{5}{2} = -\frac{3}{2}$$

$$4B+D = -\frac{3}{2}$$

$$B+D = -\frac{1}{2}$$

$$3B = -\frac{3}{2}+\frac{1}{2} = -\frac{2}{2} = -1$$

$$\boxed{B = -\frac{1}{3}}$$

$$D = -\frac{1}{6}.$$

$$\therefore x = \frac{1}{2p} - \frac{1}{6(p^2+1)} - \frac{1}{3(p^2+4)}$$

$$\boxed{x = \frac{t}{2} - \frac{1}{6}\sin(t) - \frac{1}{6}\sin 2t}$$

$$y = \frac{-1}{(p^2+4)(p^2+1)} = \frac{1}{3}\left(\frac{1}{p^2+4} - \frac{1}{p^2+1}\right)$$

$$y(t) = \frac{1}{3}\left(\frac{\sin 2t}{2} - \sin t\right)$$

$$\boxed{y(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\sin t}$$

8

7(b) A particle is moving with central acceleration $\mu(r^5 - c^4 r)$ being projected from an apse at a distance c with velocity $c^3(2\mu/3)^{1/2}$, show that its path is the curve $x^4 + y^4 = c^4$ (15)

at apse $r = c$, $V = c^3\left(\frac{2\mu}{3}\right)^{1/2}$.

central acceleration $= A = \mu(r^5 - c^4 r)$

$u = \frac{1}{r} \Rightarrow A = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)$

. we know,

$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{A}{u^2} = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)\frac{1}{u^2}$

$\Rightarrow h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)$

integrate after multiplying with $2\frac{du}{d\theta}$

$\Rightarrow h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\int\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)du + B = V^2$

$\Rightarrow h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[\frac{1}{u^6(-6)} - \frac{c^4}{(-2)u^2}\right] + B = V^2$

$\Rightarrow h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[\frac{1}{-6u^6} + \frac{c^4}{2u^2}\right] + B = V^2$ —— (1)

at $u = \frac{1}{c}$, $V = c^3\left(\frac{2\mu}{3}\right)^{1/2}$, $\left(\frac{du}{d\theta}\right) = 0$

$\Rightarrow h^2\left[\frac{1}{c^2} + 0\right] = 2\mu\left[\frac{c^6}{-6} + \frac{c^4 \times c^2}{2}\right] + B = c^6\left(\frac{2\mu}{3}\right)$

$\frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \frac{2}{6}$

$\Rightarrow \frac{h^2}{c^2} = 2\mu\left[c^6 \times \frac{2}{6}\right] + B = c^6\left(\frac{2\mu}{3}\right)$

$\Rightarrow \frac{h^2}{c^2} = \frac{2}{3}\mu(c^6) + B = c^6\,\frac{2\mu}{3}$

$\Rightarrow B = 0$, $h^2 = \frac{2}{3}\mu c^8 \Rightarrow \frac{\mu}{h^2} = \frac{3}{2}c^{-8}$

Now again from (1)

$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \frac{\cancel{2}\mu}{\cancel{2}}\left(\frac{c^4}{u^2} - \frac{1}{3u^6}\right)$

$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{3}{2}\frac{1}{c^8}\left(\frac{c^4}{u^2} - \frac{1}{3u^6}\right)$

$\left(\frac{du}{d\theta}\right)^2 = \frac{3}{2}\left(\frac{1}{c^4u^2} - \frac{1}{3u^6c^8}\right) - u^2$

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$$\left(\frac{du}{d\theta}\right)^2 = \frac{3}{2}\left(\frac{1}{c^4u^2} - \frac{1}{3c^8u^6}\right) - u^2$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{3}{2}\left(\frac{r^2}{c^4} - \frac{r^6}{3c^8}\right) - \frac{1}{r^2}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{3}{2}\left(\frac{r^6}{c^4} - \frac{r^{10}}{3c^8}\right) - r^2$$

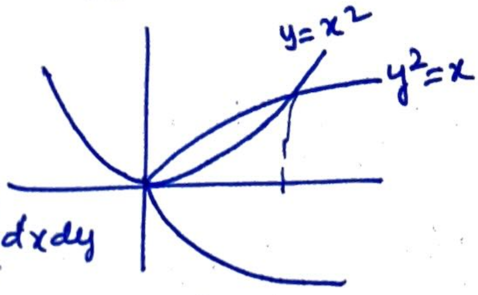
$$= \frac{9c^4r^6 - 3r^{10} - 6c^8r^2}{6c^8}$$

$$6c^8\left(\frac{dr}{d\theta}\right)^2 =$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{c^8u^6}\left[\frac{3}{2}\right.$$

7(c) Using Green's theorem evaluate $\int_C [(x^2 - y^2)dx + 2xydy]$, where C is the closed curve of the region bounded by $y^2 = x$ and $x^2 = y$. (15)

$$\int_C (x^2-y^2)\,dx + 2xy\,dy$$



$$= \iint_A \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right] dx\,dy$$

$$= \iint_A (2y+2y)\,dx\,dy$$

$$= \iint_A 4y\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 4y\,dy\,dx$$

$$= \int_0^1 2y^2\Big|_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (2x - 2x^4)\,dx$$

$$= x^2 - \frac{2x^5}{5}\Big|_0^1$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

$$\therefore \boxed{\int_C (x^2-y^2)\,dx + 2xy\,dy = \frac{3}{5}}$$

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7(d) The cosine integral is denoted by $C_i(t)$ and is defined as

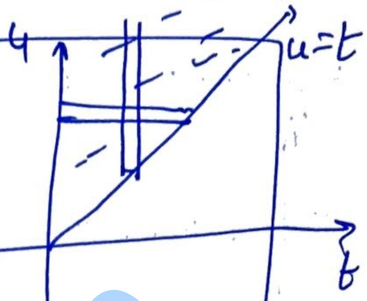
$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$. prove that $L\{C_i(t)\} = (1/2s) \times \log(s^2+1)$

(10)

$$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$$

$$L(C_i(t)) = \int_0^\infty \left(\int_t^\infty \frac{\cos u}{u} du \right) e^{-st} dt$$

$$= \int_{t=0}^{\infty} \int_{u=t}^{\infty} \frac{\cos u}{u} \cdot e^{-st} du\, dt$$



Now change order of integration.

$$= \int_{u=0}^{\infty} \int_{t=0}^{u} \frac{\cos u}{u} e^{-st} dt\, du$$

$$= \int_0^\infty \frac{e^{-st}}{-s}\Big|_0^u \frac{\cos u}{u} du$$

$$= \frac{1}{-s} \int_0^\infty \left[e^{-su} \frac{\cos u}{u} - \frac{\cos u}{u} \right] du$$

$$= \frac{1}{s} \int_0^\infty \left[\frac{\cos u}{u} - e^{-su} \frac{\cos u}{u} \right] du$$

$$C_i(t) = \int_t^\infty \frac{\cos u}{u} du$$

$$F(t) = \int_t^\infty \frac{\cos u}{u} du$$

$$F'(t) = -\frac{\cos u}{u}\Big|_t = -\frac{\cos t}{t}$$

$$\Rightarrow t F'(t) = -\cos t$$

$$\Rightarrow L\{t F'(t)\} = \frac{-s}{1+s^2}$$

$$\Rightarrow \cancel{+\frac{d}{ds}\left[L(F(t)) \cdot s - F(0) \right]} = \cancel{\frac{+s}{1+s^2}}$$

$$\Rightarrow +\frac{d}{ds} L[F'(t)] = +\frac{s}{1+s^2}$$

$$\Rightarrow \frac{d}{ds}\left[s\, L\{F(t)\} - F(0) \right] = \frac{s}{1+s^2}$$

$$\Rightarrow \frac{d}{ds} \cdot s f(s) = \frac{s}{1+s^2}$$

$$s f(s) = \frac{1}{2} \log(1+s^2)$$

$$f(s) = \frac{1}{2s} \log(1+s^2) \Rightarrow L\left(\int_t^\infty \frac{\cos u}{u} du \right) = \frac{1}{2s} \log(1+s^2)$$

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