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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -2 PAPER 2

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Email ID		Roll No	
Phone Number		Date	11th August
Start Time :	12:00 PM	Closing Time:	3:00 PM.

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		8	5a		8	
1b		8	5b		8	
1c		8	5c			
1d		8	5d			
1e		8	5e		8	
2a			6a		18	
2b			6b		8	
2c			6c		15	
2d			6d			
3a		18	7a			
3b		13	7b			
3c		13	7c			
3d			7d			
4a		13	8a			
4b		13	8b			
4c		18	8c			
4d			8d			
Total			193/250			

Section A

1(a) Prove that a finite integral domain is a field (10)

Let R is a finite integral domain

$\Rightarrow$ R is commutative and unity exist
and R has no zero divisor

To Prove, R as field we need to Prove that every non zero element of R has its inverse in R

Now, since R is finite, let

$\Rightarrow R = \{a_1, a_2, a_3, \ldots a_n\}$. all a_i are distinct

if $\exists\, a \in R, a \neq 0$

$\Rightarrow R = \{aa_1, aa_2, aa_3 \ldots, aa_n\}$ —(1)

here all aa_i are distinct
because if $aa_i = aa_r$

$\Rightarrow a(a_i - a_r) = 0$

since no zero divisor

$\Rightarrow a_i - a_r = 0 = a_i = a_r$
which is not Possible

$\Rightarrow$ any one of set (1), ~~such~~ must be equal to unity

let

$\Rightarrow aa_i = 1$

$\Rightarrow a_i$ is inverse of a

$\therefore$ every non zero element has its inverse

$\Rightarrow$ $\boxed{\text{R is a field}}$

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1(b) Prove that if an ideal U of a ring R contains a unit of R, then $U = R$. (10)

Let a is unit in R

and given $a \in U$

$\because$ a is unit in R

$\Rightarrow \exists\ b \in R$ such that

$a \cdot b = 1$ —— (1)

Now since $a \in U$, $b \in R$

$\Rightarrow a \cdot b \in U$ as U is an ideal

$\Rightarrow 1 \in U$

Since unity of R belongs to U

$\Rightarrow$ if $x \in R$

$\Rightarrow x \cdot 1 \in U$ (U is ideal)

$\Rightarrow x \in U$

$\Rightarrow R \subseteq U$ —— (2)

Since U is ideal of R

$\Rightarrow U \subseteq R$ —— (3)

From (2) & (3)

$\boxed{U = R}$

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1(c) Show that $\int_0^{\pi/2} x^m \operatorname{cosec}^n x\,dx$ exists iff $n < (m+1)$ (10)

$$I = \int_0^{\pi/2} x^m \operatorname{cosec}^n x\,dx$$

$$I = \int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx$$

at $x \to \frac{\pi}{2}$, $\frac{x^m}{\sin^n x}$ has finite value irrespective of value of m and n

$\therefore$ $\frac{\pi}{2}$ is not point of discontinuty

$\boxed{\text{convergence at } x=0}$:→

$$\int_0^{\pi/2} \frac{x^m}{\sin^n x} = \int_0^{\pi/2} \frac{x^n}{(\sin^n x)} \cdot x^{m-n}$$

let $f(x) = \frac{x^n \cdot x^{m-n}}{(\sin^n x)}$

$g(x) = \frac{1}{x^{n-m}}$.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^n}{\sin^n x} \cdot \frac{x^{m-n}}{x^{m-n}} = 1$$

$\therefore \int_0^a f(x)dx$ & $\int_0^a g(x)\,dx$ converges & diverges together)

~~(from rule~~

Now, $\int_0^a \frac{1}{x^{n-m}}\,dx$ will converge iff

$n-m < 1$

$\Rightarrow n < m+1$

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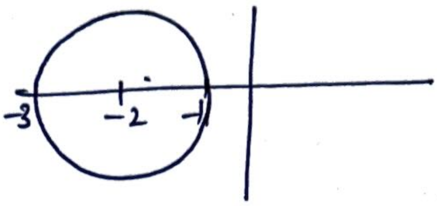
$$\therefore \int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx = \int_0^{\pi/2} x^m \operatorname{cosec}^n x \text{ will}$$

converge iff $n < (m+1)$

1(d) Evaluate $\int_C \frac{e^{-2z}z^2}{(z-1)^3(z+2)}dz$ where C is $|z+2|=1$ using Cauchy's integral (10)

given $|z+2|=1$

$$\int_c \frac{e^{-2z} z^2}{(z-1)^3(z+2)} dz$$



Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)} dz$$

where z_0 is only singularity

$f(z)$ has no singular point in ~~f(z)~~ C

& $f(z)$ is analytic in c

here consider, $f(z) = \frac{e^{-2z}z^2}{(z-1)^3}$ and $z_0 = -2$

$\Rightarrow$ $f(z)$ is analytic in c

$$f(-2) = \frac{e^4 \cdot 2^2}{(-3)^3} = \frac{4e^4}{-27}$$

$\therefore$ $\int_c \frac{f(z)}{(z-z_0)} dz = 2\pi i \times \frac{4e^4}{-27}$

$$\therefore \boxed{\int_c \frac{e^{-2z}z^2}{(z-1)^2(z+2)} dz = \cancel{2} \frac{-8\pi i}{27} e^4}$$

✓ 8

1(e) Solve the following transportation problem by Vogel's approximation method (10)

To

From		W_1	W_2	W_3	W_4	W_5	Supply ↓
	F_1	3	4	6	8	9	20
	F_2	2	10	1	5	8	30
	F_3	7	11	20	40	3	15
	F_4	2	1	9	14	16	13
Demand →		40	6	8	18	6	

For initial Basic Feasible solution

3 (20)	4 p	6 p	8	9 p	~~20~~ (1) (5)
2 (4)	10 p	1 (8)	5 (18)	8 p	~~30~~ 22 (1) (3)
7 (9)	11 p	20 p	40 p	3 (6)	~~15~~ 9 (4)
2 (7)	1 (6)	9 p	14	16	~~13~~ 7 (1) (1/2)
~~24~~ 4 ~~40~~ 31 (0) (1)	~~6~~ (3)	~~8~~ (5)	18 (3)	~~6~~ (5)	79

Initial solution Now u-v method

					u_i
3 (20)	4 (-)	6 (-)	8 (-)	9 (-)	(3)
2 (4)	10 (-)	1 (8)	5 (18)	8 (-)	(2)
7 (9)	11 (-)	20	40	3 (6)	(7)
2 (7)	1 (6)	9 (-)	14 (-)	16 (-)	(2)
v_j: (0)	(-1)	(1)	(3)	(-4)	

as all $u_i + v_j - c_{ij}$ are negative

thus optimal has reached

$\Rightarrow$ cost $= 20 \times 3 + 2 \times 4 + 7 \times 9 + 2 \times 7 + 1 \times 8 + 5 \times 18 + 6 \times 1 + 3 \times 6$

$\boxed{\text{Cost} = 267}$

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	W1	W2	W3	W4	W5	
F1	20					20
F2	4		8	18		30
F3	9				6	15
F4	7	6				13
	40	6	8	18	6	

3(a) Let $f(x,y) = \begin{cases} x^2y^2\cos\frac{1}{x} & \text{, for all values of } y \text{ so long as } x \neq 0 \\ 0 & \text{, for } x = 0 \end{cases}$

Show that

(i) $f_{xy} = f_{yx}$ at all points (x, y).

(ii) neither f_{xy} nor f_{yx} is continuous in x at $x = 0$, if $y \neq 0$.

and (iii) both f_{xy} and f_{yx} are continuous in (x, y), together at the origin. (20)

$f(x,y) = \begin{cases} x^2y^2\cos 1/x & \text{otherwise.} \\ 0 & x=0 \end{cases}$

$f_x(x,y) = \begin{cases} 2xy^2\cos(1/x) + y^2\sin\frac{1}{x} & \text{otherwise} \\ 0 & x=0 \end{cases}$

$f_{yx}(x,y) = \begin{cases} 4xy\cos(1/x) + 2y\sin(1/x), & x\neq 0 \\ 0 & x=0 \end{cases}$ —— (1)

$f_y(x,y) = \begin{cases} 2yx^2\cos 1/x & x\neq 0 \\ 0 & x=0 \end{cases}$

$f_{xy}(x,y) = \begin{cases} 2y \times 2x\cos 1/x + 2yx^2\left(-\frac{1}{x^2}\right)\left(-\sin\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$

$f_{xy}(x,y) = \begin{cases} 4xy\cos\frac{1}{x} + 2y\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ —— (2)

from (1) & (2)

$\boxed{f_{xy}(x,y) = f_{yx}(x,y)}$ $\forall (x,y)$

(ii) when $x=0$, & $y\neq 0$

$f_{xy}(x,y) = \lim_{x\to 0}\left(4xy\cos\frac{1}{x} + 2y\sin 1/x\right)$

$= \lim_{x\to 0}\left(2y\sin\frac{1}{x}\right)$

Since $y\neq 0$, $\Rightarrow 2y\sin\frac{1}{x}$ does not exit as $x\to 0$

$f_{xy}(x,y)$ not continuous at $x=0$, & $y\neq 0$

Similarly $f_{yx}(x,y)$ not continuous at $x=0, y\neq 0$

(ii) $f_{xy}(x,y) = \lim_{(x,y)\to(0,0)} 4xy\cos\frac{1}{x} + 2y\left(\sin\frac{1}{x}\right)$

~~Consider~~

$\cancel{f_{xy}(h,k) - f_{xy}(0,0) = 4hk\cos\left(\frac{1}{h}\right) + 2k\sin\left(\frac{1}{h}\right) - 0}$

$\cancel{f_{xy}(h,k) - f_{xy}(0,0) =}$

$$\left|f_{xy}(x,y) - f_{xy}(0,0)\right| = \left|4xy\cos\frac{1}{x} + 2y\sin\frac{1}{x}\right|$$

put $x = r\cos\theta$
$y = r\sin\theta$

$$= \left|4r^2\cos\theta\sin\theta\cos\frac{1}{r\cos\theta} + 2r\sin\theta\sin\frac{1}{r\cos\theta}\right|$$

$$= r\left|\left[4r\cos\theta\sin\theta\cos\left(\frac{1}{r\cos\theta}\right) + 2\sin\theta\sin\frac{1}{r\cos\theta}\right]\right|$$

$$\le r|4r+2| < \epsilon_1$$

when $4r^2 + 2r < \epsilon_1$

$\cancel{\Rightarrow 4r^2+2r}$

solving this let say we get $r = \alpha$.

$\boxed{\epsilon_1 > \alpha}$

$$f_{xy}(x,y) = \lim_{(x,y)\to(0,0)} xy\cos\frac{1}{x} + 2y\sin\frac{1}{x}$$

$= 0$ as both $x, y \to (0,0)$

and $f_{xy}(0,0) = 0$

$\therefore$ f_{xy} is continuous at origin

similarly f_{yx} is continuous at origin

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3(b) Using Residue theorem, Show that $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{\pi a^2}{1-a^2} (a^2 < 1)$

(15)

$$I = \int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta$$

put $z = e^{i\theta}$

$\Rightarrow \cos\theta = \frac{z^2+1}{2z}$

$\cos 2\theta = \frac{z^4+1}{2z^2}$

$$= \frac{1}{2}\int_C \frac{z^4+1 \quad dz}{2z^2(iz)\left(1-\cancel{2}a\frac{z^2+1}{\cancel{2}z}+a^2\right)}$$

$$= \frac{1}{4i}\int_C \frac{(z^4+1)\, z\, dz}{z^3(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{4i}\int_C \frac{z(z^4+1)\,dz}{z^3(-az^2+(1+a^2)z-a)}$$

$$= \frac{1}{4i}\int_C \frac{1}{-z^2} \frac{z^4+1 \quad dz}{(az^2-a^2z-z+a)}$$

$$= \frac{1}{-4i}\int_C \frac{(z^4+1)\quad dz}{z^2(az-1)(z-a)}$$

given $a^2<1$

$\Rightarrow |a|<1$

$$= -\frac{1}{4ia}\int_C \frac{(1+z^4)\,dz}{z^2(z-\frac{1}{a})(z-a)}$$

$\Rightarrow \left|\frac{1}{a}\right| > 1$

$\therefore z=0,\ z=a$ lies in C

$$f(z) = \frac{1+z^4}{z^2(z-\frac{1}{a})(z-a)}$$

residue at $z=0$

$$a_1 = \lim_{z\to 0} \frac{d}{dz} \frac{(1+z^4)}{(z-1/a)(z-a)}$$

$$= \lim_{z\to 0} \frac{(z-1/a)(z-a)(4z^3) - (z^4+1)(z-a+z-\frac{1}{a})}{(z-1/a)^2(z-a)^2}$$

$$= \frac{1\times 4\times 0 - (1)(-a-\frac{1}{a})}{\frac{1}{a^2}\times a^2} = a+\frac{1}{a} = \frac{a^2+1}{a}$$

Residue at $z=a$

$$a_2 = \lim_{z\to a} \frac{1+z^4}{z^2\left(z-\frac{1}{a}\right)}$$

$$= \lim_{z\to a} \frac{1+a^4}{a^2\left(a-\frac{1}{a}\right)} = \frac{(1+a^4)}{a(a^2-1)}$$

$$I = \frac{-1}{4ia}\int_C \frac{(1+z^4)\,dz}{z^2(z-a)(z-1/a)} = \frac{-1}{4\cancel{i}a} \times 2\pi \cancel{i}\left[\frac{1+a^4}{a(a^2-1)} + \frac{a^2+1}{a}\right]$$

$$= \frac{-\pi}{2a^2}\left[\frac{\cancel{1}+a^4+a^4-\cancel{1}}{a^2-1}\right]$$

$$= \frac{\pi}{2a^2} \times \frac{2a^4}{1-a^2}$$

$$\boxed{I = \frac{\pi a^2}{1-a^2}}$$

$$\therefore \boxed{\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a}\,d\theta = \frac{\pi a^2}{1-a^2}}$$

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3(c) Solve the following L.P.P. using Big. M. method.

$$\text{Min } Z = x_1 + x_2$$
$$\text{S.T. } 2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

(15)

$\min z = x_1 + x_2$

$\Rightarrow \text{Max } z^* = -x_1 - x_2$

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

$\xrightarrow{\text{standard form}}$

$\max z^* = -x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

Simplex table :-

	$C_j \rightarrow$	-1	-1	0	0	$-M$	$-M$		
		x_1	x_2	S_1	S_2	A_1	A_2	b	θ
$-M$	A_1	2	1	-1	0	1	0	4	4
$-M$	A_2	1	7	0	-1	0	1	7	$7/7 = 1 \rightarrow$
	Z_j	$-3M$	$-8M$	M	M	$-M$	$-M$	$-11M$	
	$C_j - Z_j$	$3M-1$	$\boxed{8M-1}$	$-M$	$-M$	0	0		
$-M$	A_1	$\boxed{\frac{13}{7}}$	0	-1	$\frac{1}{7}$	1		3	$\frac{21}{13} \rightarrow$
-1	x_2	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	0		1	7
	Z_j	$\frac{-13M}{7} - \frac{1}{7}$	-1	M	$-\frac{M}{7} + \frac{1}{7}$	$-M$		$-3M-1$	
	$C_j - Z_j$	$\boxed{\frac{13M}{7} - \frac{6}{7}}$	0	$-M$	$\frac{M}{7} - \frac{1}{7}$	0			
-1	x_1	1	0	$-\frac{7}{13}$	$\frac{1}{13}$			$\frac{21}{13}$	
-1	x_2	0	1	$\frac{1}{13}$	$-\frac{2}{13}$			$\frac{10}{13}$	
	Z_j	-1	-1	$\frac{6}{13}$	$\frac{1}{13}$			$-\frac{31}{13}$	
	$C_j - Z_j$	0	0	$-\frac{6}{13}$	$-\frac{1}{13}$				

optimality reached

$\boxed{\text{with } x_1 = \frac{21}{13},\ x_2 = \frac{10}{13}}$

$\therefore \max z^* = \frac{-31}{13}$

$\boxed{\therefore \min z = \frac{31}{13}}$

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4(a) Show that the sequence $\{f_n\}$, here $f_n(x) = \frac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing zero. (15)

$\{f_n\} \Rightarrow$

$f_n(x) = \frac{nx}{n^2x^2+1}$

$f(x) = \lim_{n\to\infty} \frac{nx}{n^2x^2+1} = 0$

let $|f_n(x) - f(x)| = \left|\frac{nx}{n^2x^2+1}\right|$

let $\phi(x) = \frac{nx}{n^2x^2+1}$

$\phi'(x) = \frac{(n^2x^2+1)n - nx(2n^2x)}{(n^2x^2+1)^2} = \frac{n - n^3x^2}{(n^2x^2+1)^2}$

$\Rightarrow$ $\phi'(x) = 0$ when $n - n^3x^2 = 0$

$\Rightarrow \boxed{x = \frac{1}{n}}$

$\phi''(x) = \frac{(n^2x^2+1)^2[-2xn^3] - (n-n^3x^2)(2)(1+n^2x^2)(2n^2x)}{(n^2x^2+1)^4}$

at $(x = \frac{1}{n})$

$\phi''(x) = \frac{4(-2n^2) - 0}{(2)^4} < 0$

$\therefore$ $\phi(x)$ is maximum at $x = \frac{1}{n} \in$ interval as it contains zero

$\therefore \phi(x)_{max} = \frac{n \times \frac{1}{n}}{1 + n^2 \times \frac{1}{n^2}} = \frac{1}{2}$

$\therefore |f_n(x) - f(x)| < \frac{1}{2}$

let $M_n = \frac{1}{2}$

here $|f_n(x) - f(x)| < M$

but when $n \to 0$ $M_n \neq 0$

$\therefore$ From M-test

$\{f_n(x)\}$ is not uniformly convergent on any interval containing zero

✓ 13

4(b) If R is a ring, prove that $\frac{R[x]}{(x)} \approx R$, (x) is the ideal generated by x. (15)

Consider

a mapping

$\phi : R[x] \longrightarrow R$

$\phi(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = a_0$

Check for homomorphism.

Let $f(x) = a_0 + a_1 x + \dots + a_n x^n$

$g(x) = b_0 + b_1 x + \dots + b_m x^m$

$\phi(f(x) + g(x)) = a_0 + b_0$

$\phi[f(x) + g(x)] = \phi(f(x)) + \phi(g(x))$ —— (1)

$f(x) \cdot g(x) = a_0 b_0 + (a_1 b_0 + b_0 a_1) x + \dots + a_n b_m x^{n+m}$

$\phi[f(x) \cdot g(x)] = a_0 b_0$

$\phi[f(x) \cdot g(x)] = \phi[f(x)] \cdot \phi[g(x)]$ —— (2)

from (1) & (2)

ϕ is homomorphism.

∴ from fundamental theorm of homomorphism

$R \cong \frac{R[x]}{\text{Kernel } \phi}$ —— (3)

Now, for kernel ϕ

$\phi(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = 0$

$\Rightarrow a_0 = 0$

∴ $a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n \in \text{kern } \phi$

$\{a_1 x + a_2 x^2 + \dots + a_n x^n\} = \langle x \rangle$ where $\{a_1, a_2, \dots a_n\} \in R$.

and $\langle x \rangle$ is ideal of R.

∴ kernel of $\phi = \langle x \rangle$

∴ From (3)

$\cancel{\frac{R[x]}{\text{kernel } \phi}}$ $\boxed{\frac{R[x]}{\langle x \rangle} \cong R}$

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4(c) Solve the following assignment problem for minimum optimal cost.

To City

From City	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

(20)

Add dummy city with zero cost

12	10	15	22	18	8
10	18	25	15	16	12
11	10	3	8	5	9
6	14	10	13	13	12
8	12	11	7	13	10
0	0	0	0	0	0

Apply row operation => Subtracting with minimum in the row

4	2	7	14	10	0
0	8	15	5	6	2
8	7	0	5	2	6
0	8	4	7	7	6
1	5	4	0	6	3
0	0	0	0	0	0

only 5 lines require instead of 6

Apply column operation will not change any thing.
minimum uncoverd is 2

4	0	7	14	8	0
0	6	15	5	4	2
8	5	0	5	0	6
0	6	4	7	5	6
1	3	4	0	4	3
0	0	1	0	0	0

Again min is 2 substraction stage

	1	2	3	4	5	6
A	6	$\boxed{0}$	7	14	8	0
B	0	4	13	3	2	$\boxed{0}$
C	10	5	$\boxed{0}$	5	0	6
D	$\boxed{0}$	4	2	5	3	4
E	2	3	4	$\boxed{0}$	4	3
F	2	0	1	1	$\boxed{0}$	1

6 lines needed
Optimal reached

∴ min cost =

A − 2 ⇒ 10
B − 6 ⇒ 12
C − 3 ⇒ 3
D − 1 ⇒ 6
E − 4 ⇒ 7

$\boxed{38}$

min cost = 38

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Section B

5(a) Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy.$ (10)

Auxilary equation of homogenous part

$(D^2 - DD' - 2D'^2)z = 0$

$m^2 - m - 2 = 0$

$\Rightarrow m^2 - 2m + m - 2 = 0 \Rightarrow (m-2)m + 1(m-2) = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m = 2,\ m = -1$

$\therefore z_{CF} = \phi_1(y+2x) + \phi_2(y-x)$ —— (1)

Now, $z_P = \dfrac{1}{(D-2D')(D+D')}\left[(2x^2+xy-y^2)\sin xy - \cos xy\right]$

$= \dfrac{1}{(D-2D')(D+D')}\left[(2x-y)(x+y)\sin xy - \cos xy\right]$

$\dfrac{1}{D+D'}(2x-y)(x+y)\sin xy - \cos xy$ $\quad y - x = c$, $y = c+x$.

$= \int (x-c)(c+2x)\sin(x(c+x)) - \cos(x(c+x))\, dx.$

$= (x-c)\dfrac{\cos(cx+x^2)}{(-)} - \int 1 \times -\cos(cx+x^2)\ \dfrac{}{dx} - \int \cos(cx+x^2)\,dx$

$= (c-x)\cos((c+x)\cdot x)$

$= (y-x-x)\cos(yx)$

$= (y-2x)\cos(xy)$

$\dfrac{1}{D-2D'}(y-2x)(\cos xy)$ $\quad$ put $y+2x = c$

$\Rightarrow y = c-2x.$

$\Rightarrow \int (c-2x-2x)\cos(x(c-2x))\,dx$

$= \int (c-4x)\cos(cx-2x^2)\,dx.$

$= \sin(cx-2x^2)$

$= \sin xy$

$\therefore z_p = \sin xy$

$\therefore z = z_{CF} + z_p$

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + \sin xy}$$

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5(b) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$. What is the order of this partial differential equation? (10)

Let $u = x+y+z$

$V = x^2+y^2-z^2$ $\quad p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$

$\therefore \phi(u,V) = 0$

$\Rightarrow \frac{\partial \phi}{\partial x} = 0 = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial V}\cdot\frac{\partial V}{\partial x} \cancel{=0}$

$\Rightarrow \frac{\partial \phi}{\partial u}(1+p) + \frac{\partial \phi}{\partial V}(2x-2zp) = 0$ — (1)

Again

$\frac{\partial \phi}{\partial y} = 0$

$\Rightarrow \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial V}\cdot\frac{\partial V}{\partial y} = 0$

$\Rightarrow \frac{\partial \phi}{\partial u}(1+q) + \frac{\partial \phi}{\partial V}(2y-2zq) = 0$ — (2)

from (1) & (2)

$$\begin{vmatrix} 1+p & 2(x-zp) \\ 1+q & 2(y-zq) \end{vmatrix} = 0$$

$\Rightarrow (1+p)(y-zq) - (1+q)(x-zp) = 0$

$\Rightarrow (1+p)(y-zq) = (1+q)(x-zp) = 0$ is required partial differential equation

order of P.D.E is one

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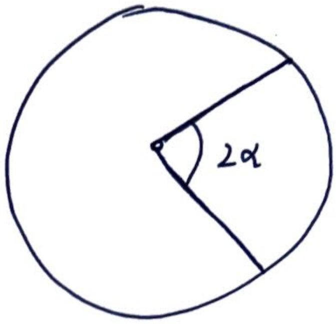
5(c) Find the value of the integral using Gauss Quadrature formula for n=4

$$\int_0^1 xdx$$

(10)

Gauss quadrature formula

5(d) From a uniform sphere of radius a, spherical sector of vertical angle 2α is removed. Show that the moment of inertia of the remainder of mass M about the axis of symmetry is $\frac{1}{5}Ma^2(1+\cos\alpha)(2-\cos\alpha)$. (10)



5(e) If velocity distributon of an incompressible fluid at point (x, y, z) is gives by $\left\{\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{(kz^2-r^2)}{r^5}\right\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential. (10)

$$u = \frac{3xz}{r^5}, \quad v = \frac{3yz}{r^5}, \quad w = \frac{k(z^2-r^2)}{r^5}$$

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} + 3xz\left(\frac{\partial}{\partial r}r^5\right) = \frac{3z}{r^5} + 3xz\left(\frac{-5}{r^6}\right)\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2z}{r^7} \quad \text{——(1)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{3yz}{r^5}\right) = \frac{3z}{r^5} - \frac{3yz(5)}{r^6}\left(\frac{y}{r}\right)$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2z}{r^7} \quad \text{——(2)}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(\frac{kz^2}{r^5} - \frac{1}{r^3}\right) = \frac{k\,2z}{r^5} + kz^2\cdot\frac{(-5)}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\left(\frac{z}{r}\right)$$

$$\frac{\partial w}{\partial z} = \frac{2kz}{r^5} + \frac{3z}{r^5} - \frac{5kz^2z}{r^7} \quad \text{——(3)}$$

∴ for possible flow, continuity eqution should be satisfied

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{3z}{r^5} - \frac{15x^2z}{r^7} + \frac{3z}{r^5} - \frac{15y^2z}{r^7} + \frac{(2k+3)z}{r^5} - \frac{5kz^2z}{r^7} = 0$$

$$\Rightarrow \frac{z}{r^5}(9+2k) - \frac{15zx^2+15y^2z+5z^2kz}{r^7} = 0$$

$$\Rightarrow (9+2k)r^2 = 15(x^2+y^2)+5kz^2$$

$$\Rightarrow 9+2k = 15 \;\&\; 9+2k = 5k$$

$$\Rightarrow \boxed{k=3} \qquad \Rightarrow \boxed{k=3}$$

$k=3$ satisfies

∴ for $k=3$ it is possible fluid flow

<u>velocity potential,</u>

ϕ,

$u = -\frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial \phi}{\partial x} = -\frac{3xz}{r^5}$ —— (a)

$\frac{\partial \phi}{\partial y} = -\frac{3yz}{r^5}$ —— (b)

$\frac{\partial \phi}{\partial z} = \frac{r^2 - 3z^2}{r^5}$ —— (c)

from (a)

$\phi = \int \frac{-3xz}{(x^2+y^2+z^2)^{5/2}} dx$

$= -\frac{\cancel{3}z}{\cancel{2}} \frac{(x^2+y^2+z^2)^{-5/2+1}}{-\cancel{3/2}}$

$\boxed{\phi = \frac{z}{(r^2)^{3/2}}} = \frac{z}{r^3}$

from (b) similarly

$\boxed{\phi = \frac{z}{(r^2)^{3/2}} = \frac{z}{r^3}}$

from (c)

$\cancel{\phi = \frac{x^2+y^2+z^2-3z^2}{r}}$

if $\phi = \frac{z}{r^3} \Rightarrow \frac{\partial \phi}{\partial z} = \frac{r^3 - z \cdot 3r^2 \cdot \frac{z}{r}}{r^6}$

$= \frac{r^2 - 3z^2}{r^5}$ which satisfy (c)

$\therefore$ $\boxed{\text{velocity potential} = \phi = \frac{z}{r^3}}$

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6(a) Write down an algorithm for Simpson's $\frac{1}{3}$ rule. Hence, compute $\int_0^1 x^2(1-x)dx$ correct up to three decimal places with step size $h = 0.1$ and compare the result with its exact value (20)

Simpson's $\frac{1}{3}$rd rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 \cdots + y_{n-2})\right]$$

Algorithm for this

step 1: start

step 2: read $y = f(x)$, a, b, h.

step 3: ~~$x_0 = a$, $x_n = x_n = x_0$~~ Calculate $n = \frac{b-a}{h}$

step 4: if $n \pmod 2 \neq 0$

step 5: print: stepsize(h) does not satisfy condition of $n = 2 \times$ integer. and stop, else

step 6: $x_0 = a$, $x_n = x_0 + nh$,

step 6: ~~$s = y_0 +$~~ $s = 0$

step 7: $i = 0$ to n

if $i = 0$ or $i = n$

$s = s + y(x_0 + ih)$

else,

if $i \pmod 2 = 0$

$s = s + 2 \times y(x_0 + ih)$

else $s = s + 4 \times y(x_0 + ih)$

step 8: next i

step 9: $s = s \times \frac{h}{3}$

step 10: print value of integration $\int_a^b f(x)dx = s$

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(b) $\int_0^1 x^2(1-x)\,dx$

$= \int_0^1 (x^2 - x^3)\,dx = \left.\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = 0.0833$

From Simpson ⅓rd rule with $h = 0.1$

n	0	1	2	3	4	5	6	7	8	9	10
x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0	9×10^{-3}	0.032	0.062	0.096	0.125	0.144	0.147	0.128	0.081	0

$\therefore \int_0^1 x^2(1-x)\,dx = \frac{0.1}{3}\Big[0+0+4(0.009 + 0.062 + 0.125 + 0.147 + 0.081)$
$+ 2(0.032 + 0.096 + 0.144 + 0.128)\Big]$

$= (1.693 + 0.2)\frac{1}{30} = 0.0611$

Result from Simpson $\frac{1}{3}$rd rule $= 0.0611$

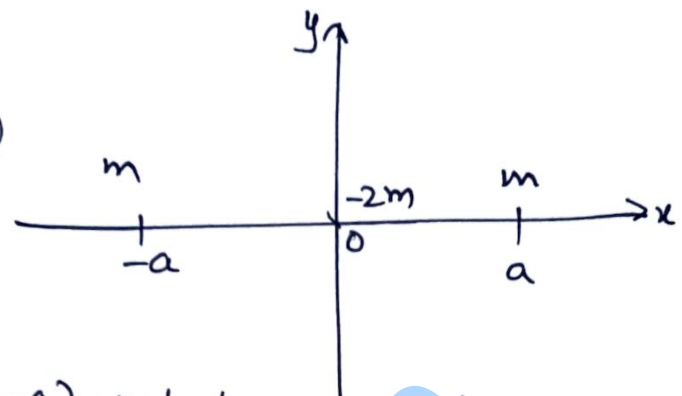
exact result $= 0.0833$

6(b) Two sources, each of strength m are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curves

$$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy) \text{ where } \lambda \text{ is a variable parameter.}$$

Show also that the fluid speed at any point is $(2ma^2)/(r_1, r_2, r_3)$ where r_1, r_2, r_3, are the distances of the points from the sources and the sink.
(10)

Complex potential due to all these system



$$\therefore W = -m\log(z-a) - m\log(z+a) + 2m\log z$$

$$\phi + i\psi = -m\log(x-a+iy) - m\log(x+a+iy) + 2m\log(x+iy)$$

$$\Rightarrow \phi + i\psi = -m\log\sqrt{(x-a)^2+y^2} - mi\tan^{-1}\left(\frac{y}{x-a}\right)$$
$$- m\log\sqrt{(x+a)^2+y^2} - mi\tan^{-1}\left(\frac{y}{x+a}\right)$$
$$+ 2m\log\sqrt{x^2+y^2} + 2mi\tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \psi = -m\tan^{-1}\left(\frac{y}{x-a}\right) - m\tan^{-1}\left(\frac{y}{x+a}\right) + 2m\tan^{-1}\left(\frac{y}{x}\right)$$

$$= -m\tan^{-1}\left(\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y^2}{x^2-a^2}}\right) + m\tan^{-1}\left(\frac{\frac{y}{x}+\frac{y}{x}}{1-\frac{y^2}{x^2}}\right)$$

$$= -m\tan^{-1}\left(\frac{yx+\cancel{ay}+yx-\cancel{ay}}{x^2-a^2-y^2}\right) + m\tan^{-1}\left(\frac{2yx}{x^2-y^2}\right)$$

$$= -m\tan^{-1}\left(\frac{2yx}{x^2-a^2-y^2}\right) + m\tan^{-1}\left(\frac{2yx}{x^2-y^2}\right)$$

$$\frac{\phi}{m} = \tan^{-1}\left(\frac{\frac{2yx}{x^2-y^2} - \frac{2yx}{x^2-y^2-a^2}}{\left(1+\frac{4y^2x^2}{(x^2-y^2)(x^2-y^2-a^2)}\right)}\right)$$

$$\frac{\phi}{m} = \tan^{-1}\left(\frac{2yx(\cancel{x^2}-\cancel{y^2}-a^2-\cancel{x^2}+\cancel{y^2})}{(x^2-y^2)^2-(x^2-y^2)a^2+4x^2y^2}\right)$$

$$\tan\frac{\phi}{m} = \frac{2yx(-a^2)}{(x^2+y^2)^2-(x^2-y^2)a^2}$$

$$\Rightarrow \frac{1}{\mu} = \frac{2yx(-a)^2}{(x^2+y^2)^2-(x^2-y^2)a^2}$$

$$\Rightarrow (x^2+y^2)^2 = (x^2-y^2)a^2 - \mu a^2(2xy))$$

let $\cdot\ \mu 2 = -\lambda$

$$\Rightarrow \boxed{(x^2+y^2)^2 = (x^2-y^2+\lambda xy)a^2}$$ is the streamlines

For speed

$$q = \left|\frac{\partial w}{\partial z}\right| = \left|-\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z}\right|$$

$$= \left|\frac{m(\cancel{z^2}-a\cancel{z}+\cancel{z^2}+a)z-2(\cancel{z^2}-a^2))}{(z-a)(z+a)z}\right|$$

$$= \left|\frac{m\cdot 2\cdot a^2}{r_1 r_2 r_3}\right|$$ where r_1, r_2, r_3 are distance from ~~fix~~ point to source & sink

$$\therefore \boxed{q = \frac{2ma^2}{r_1 r_2 r_3}}$$

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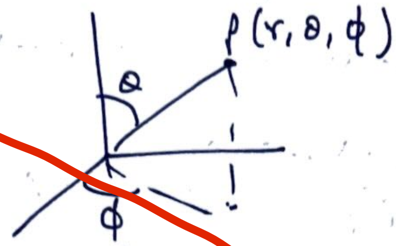
6(c) A particle is moving in a central force field given by the potential, $V = -k\frac{e^{-ar}}{r}$, where, k and a are positive constants. Construct the Hamiltonian of the particle and obtain the equation of motion. (20)

$$V = \frac{-k e^{-ar}}{r}$$

Position at P in spherical coordinates



$$\left(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta\right)$$

∴ Velocity

$$v = \dot{r}\sin\theta\cos\phi + r\sin\theta(-\sin\phi)\dot{\phi} + r\cos\theta\cos\phi\,\dot{\theta},$$
$$\dot{r}\sin\theta\sin\phi + r\sin\theta\cos\phi\,\dot{\phi} + r\cos\theta\sin\phi\,\dot{\theta},$$
$$\dot{r}\cos\theta + r(-\sin\theta)\dot{\theta}$$

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2\left[\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi\right] + r^2\dot{\theta}^2[1]$$
$$+ 2r\dot{r}(\cos^2\phi\sin\theta\cos\theta)\dot{\theta} + 2r\dot{r}(\cos\theta\sin\theta\sin^2\phi)\dot{\theta}$$
$$+ 2r\dot{r}(\cos\theta)(-\sin\theta)\dot{\theta}$$
$$+ r^2(0)\dot{\theta}\dot{\phi} + r$$

$$\boxed{v^2 = \dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2}$$

Write directly this formula
$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 s^2\theta\,\dot{\phi}^2$

$$\therefore KE = T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2\right]$$

$$PE = V = \frac{-k e^{-ar}}{r}.$$

$$\therefore L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2\right] + \frac{k e^{-ar}}{r}$$

Let $P_r = \frac{\partial L}{\partial \dot{r}}$

$$\Rightarrow P_r = m\dot{r}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta.$$

Central forces are Always Planar

ϕ is constant

$\dot{\phi} = 0$

$v = \dot{r} + r\dot{\theta}$

$\therefore H = \cancel{p_r \dot{r}} \, p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$

$= m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\dot{\phi}^2\sin^2\theta - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - \frac{ke^{-ar}}{r}$

$$\boxed{H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}\sin^2\theta) - \frac{ke^{-ar}}{r}}$$

ϕ + Noted

~~we know $\frac{\partial H}{\partial p}$~~

$\dot{r} = \frac{p_r}{m}$

$\dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$

$\dot{\theta} = \frac{p_\theta}{mr^2}$

$$\therefore \boxed{H = \frac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\phi^2}{mr^2\sin^2\theta} + \frac{p_\theta^2}{mr^2}\right) - \frac{ke^{-ar}}{r}}$$

$\therefore \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}.$

$\cancel{\dot{r} = \frac{p_x}{m}},$

$\frac{\partial H}{\partial r} = \frac{\partial}{\partial r}\left(\frac{-ke^{-ar}}{r}\right) = \frac{-ke^{-ar}}{-ar} + \frac{-ke^{-ar}}{-r^2}$

$$\boxed{-\dot{p}_r = \frac{ke^{-ar}}{ar} + \frac{ke^{-ar}}{r^2}}$$

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$\frac{\partial H}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{p_\phi^2}{mr^2\sin^2\theta}\right) = \frac{p_\phi^2}{mr^2}\left(\frac{-2\cos\theta}{\sin^3\theta}\right) = \dot{p}_\theta$

$\frac{\partial H}{\partial \phi} = 0 = \dot{p}_\phi$

$$\therefore \boxed{-m\ddot{r} = \frac{ke^{-ar}}{r}\left[\frac{1}{a} + \frac{1}{r}\right]} \Rightarrow \boxed{m\ddot{r} + \frac{ke^{-ar}}{r}\left[\frac{1}{a} + \frac{1}{r}\right]}$$

$\frac{\partial}{\partial t}(mr^2\dot{\phi}\sin^2\theta) = 0 \Rightarrow \boxed{m\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right) = 0}$

$\frac{d}{dt}(mr^2\dot{\theta}) = \left(\frac{m^2r^4\dot{\phi}^2}{mr^2}\left(\frac{-2\cos\theta}{\sin^3\theta}\right)\sin^4\theta\right)$

$$\Rightarrow \boxed{\frac{d}{dt}(mr^2\dot{\theta}^2) + 2mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0}$$