Test Copy of Mr Shivam Kumar ,AIR 19 ,CSE 2023

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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -5 PAPER 1

Name of the Candidate	SHIVAM KUMAR		
Email ID		Roll No	
Phone Number		Date	16th August.
Start Time :		Closing Time:	

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		6	5a		8	
1b		8	5b		8	
1c		8	5c		8	
1d		8	5d		8	
1e		8	5e		8	
2a			6a			
2b			6b			
2c			6c			
2d			6d			
3a		12	7a			
3b		12	7b			
3c		16	7c			
3d			7d			
4a		12	8a		12	
4b		10	8b		12	
4c		16	8c		16	
4d			8d			
Total						

$\frac{196}{250}$

Section A

1(a) If λ is an eigenvalue of the matrix $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$,

verify that $\frac{1}{\lambda}$ is also an eigenvalue of A.

Also verify that the eigenvalues are of unit modulus. (10)

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$A^T = A^{-1}$ —— (1)

if λ is eign value of A

$\Rightarrow \lambda$ is eigen value of A^T —— (2)

$\because \lambda$ is eign value of A

$\Rightarrow \frac{1}{\lambda}$ is eigen value of A^{-1}

$\Rightarrow \frac{1}{\lambda}$ is eign value of A^T (from 1)

$\boxed{\Rightarrow \frac{1}{\lambda} \text{ is eigen value of } A}$ → (from 2)

Part-2 incomplete

6

1(b) Let A be a square matrix of order 3 such that each of its diagonal elements is ' a ' and each of its off-diagonal elements is 1 . If $B = bA$ is orthogonal determine the values of a and b.

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$$

$$B = \begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix}$$

Since B is orthogonal

$\Rightarrow (ab)^2 + b^2 + b^2 = 1$

and $ab^2 + ab^2 + b^2 = 0$

$\Rightarrow a^2b^2 + 2b^2 = 1 \Rightarrow b^2(a^2+2) = 1$

and $2ab^2 + b^2 = 0$

$\Rightarrow b^2(2a+1) = 0$

$\Rightarrow \underline{b \neq 0} \Rightarrow \boxed{a = -\frac{1}{2}}$

$\therefore b^2\left(\frac{1}{4} + 2\right) = 1$

$\Rightarrow b^2\left(\frac{9}{4}\right) = 1$

$\Rightarrow b^2 = \frac{4}{9}$

$\Rightarrow b = \pm \frac{2}{3}$

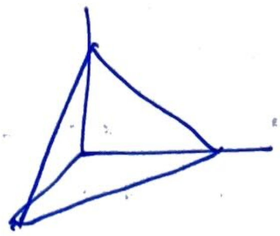
$\therefore \boxed{a = -\frac{1}{2},\ b = \frac{2}{3} \text{ or } -\frac{2}{3}}$

8

1(c) Evaluate $\iiint \frac{dxdydz}{(x+y+z+1)^3}$ over the region

$x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$ (10)

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$



$$= \int_R \int_{z_0}^{1-x-y} \frac{dz}{(x+y+z+1)^3}\, dy\, dx$$

$$= \iint \frac{1}{-2} \frac{1}{(x+y+z+1)^2} \Big|_0^{1-x-y} dy\, dx$$

$$= \frac{1}{-2} \iint \left(\frac{1}{2^2} - \frac{1}{(1+x+y)^2} \right) dy\, dx$$

$$= \frac{1}{+2} \iint \frac{1}{(1+x+y)^2} - \frac{1}{4}\ dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \frac{1}{(1+x+y)^2} dy\, dx - \frac{1}{8} \iint dy\, dx.$$

$$= \frac{1}{2} \int_0^1 \frac{-1}{1+x+y} \Big|_0^{1-x} dx - \frac{1}{8} \times \frac{1}{2} \times 1 \times 1$$

$$= \frac{1}{2} \int_0^1 -\left(\frac{1}{2} - \frac{1}{1+x} \right) dx - \frac{1}{16}$$

$$= \frac{1}{2} \int \left(\frac{1}{1+x} - \frac{1}{2} \right) dx - \frac{1}{16}$$

$$= \frac{1}{2} \left(\log(1+x) - \frac{1}{2} x \right) \Big|_0^1 - \frac{1}{16}$$

$$= \frac{1}{2} \left(\log 2 - \frac{1}{2} - 0 \right) - \frac{1}{16}$$

$$= \frac{\log 2}{2} - \frac{1}{4} - \frac{1}{16}$$

$$= \frac{\log 2}{2} - \frac{5}{16}$$

Use Dirichlet formula to get quick result

8

1(d) A function f defined on [0,1] is given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that it takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \frac{1}{2}$ (10)

$f(x) = \begin{cases} x & \text{if } x = \text{rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$

$\Rightarrow$ in $[0,1]$, $f(x)$ takes every rational values as for rational $f(x) = x$.

Now since $f(x) = 1-x$ for all irrational x in zero to 1

if $g(x) = 1 - f(x)$
$= 1-(1-x) = x$.

$g(x)$ takes all irrational value in $[0,1]$

$\Rightarrow$ $f(x)$ takes all irrational value in $[0,1]$ as $f(x) = 1-g(x)$ in $[0,1]$

Now, at any point $c \neq \frac{1}{2}$.

if c_n is sequence of irrational number converging to a rational number c

$\Rightarrow \lim\limits_{n\to\infty} f(c_n) = \lim\limits_{n\to\infty} 1-c_n = 1-\lim\limits_{n\to\infty} c_n = 1-c$

where as $f(c) = c$

$\therefore \lim\limits_{n\to\infty} f(c_n) \neq f(c)$

thus from sequential criteria

$f(x)$ is not continuous at $c \neq \frac{1}{2}$

when $c = 1/2$

$$f(c) = \lim_{n \to \infty} f(c_n) = 1/2 .$$

and $\left| f(x) - f(1/2) \right| = \left| x - 1/2 \right| < \epsilon_1$ if $|x - 1/2| < \epsilon_1$

take ~~$|x-1/2|$~~ $\epsilon_1 = \delta$.

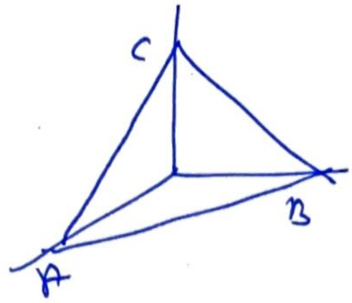
$\therefore \left| f(x) - f(1/2) \right| < \epsilon_1$ if $|x - 1/2| < \delta$.

$\boxed{\therefore f(x) \text{ is continuous at } x = 1/2}$

8

1(e) A sphere of radius k passes through the origin and meets the axes in A, B, C. Show that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$. (10)

Let $A = (a, 0, 0)$
$B = (0, b, 0)$
$C = (0, 0, c)$



$\therefore$ sphere OABC

$\Rightarrow x^2+y^2+z^2-ax-by-cz=0$

radius $= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}}$.

$4k^2 = a^2+b^2+c^2$ —— (1)

Let centroid $\Rightarrow (\alpha, \beta, \gamma)$

$\therefore \alpha = \frac{a}{3}, \beta = \frac{b}{3}, \gamma = \frac{c}{3}$

$\Rightarrow a = 3\alpha, b = 3\beta, c = 3\gamma$ put in (1)

$4k^2 = 9(\alpha^2+\beta^2+\gamma^2)$

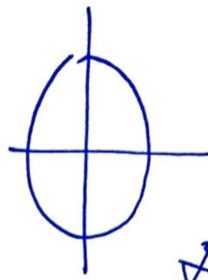
$\therefore$ locus of centroid

$\Rightarrow \boxed{9(x^2+y^2+z^2) = 4k^2}$ Proved

8

3(a) Show that the condition that the plane $ux + vy + wz = 0$ may cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators is $(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$. (15)

to show that
~~for~~ Plane ~~to~~ cut $ax^2+by^2+cz^2=0$
in Perpendicular generator



let l_1, m_1, n_1 be d.r of generator
& l_2, m_2, n_2

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ —— (1)

now, $ax^2+by^2+cz^2=0$
if, (l, m, n) dir lies on it

$\Rightarrow al^2+bm^2+cn^2=0$

& $ul+vm+wn=0$

$\Rightarrow l = \dfrac{-vm-wn}{u}$

$\Rightarrow a\left(\dfrac{-vm-wn}{u}\right)^2 + bm^2 + cn^2 = 0$

$\Rightarrow a\left(v^2m^2+w^2n^2+2vwnm\right) + bu^2m^2 + cu^2n^2 = 0$

$\Rightarrow m^2(av^2+bu^2) + 2mn(vwa) + n^2(cu^2+aw^2) = 0$

$\therefore$ ~~m_1~~ if $\dfrac{m_1}{n_1}, \dfrac{m_2}{n_2}$ are two root

$\Rightarrow \dfrac{m_1 m_2}{n_1 n_2} = \left(\dfrac{cu^2+aw^2}{av^2+bu^2}\right)^{+1} \Rightarrow \cancel{\dfrac{m_1 m_2}{av^2+bu^2} = \dfrac{n_1 n_2}{cu^2+aw^2}}$

Similarly $\Rightarrow \boxed{\dfrac{m_1 m_2}{av^2+bu^2} = \dfrac{n_1 n_2}{cu^2}}$

$\therefore \dfrac{l_1 l_2}{bw^2+cv^2} = \dfrac{m_1 m_2}{aw^2+cu^2} = \dfrac{n_1 n_2}{av^2+bu^2}$

$\therefore$ From (1)

$bw^2+cv^2+aw^2+cu^2+av^2+bu^2=0$

$\Rightarrow \boxed{u^2(b+c) + v^2(a+c) + w^2(a+b) = 0}$ is required condition

12

3(b) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$ (15)

$u = \log(x^3+y^3+z^3-3xyz)$

$\Rightarrow \cancel{\log} e^u = x^3+y^3+z^3-3xyz$

$\Rightarrow$ ~~let be~~ let $e^u = f = x^3+y^3+z^3-3xyz$

$\frac{\partial f}{\partial x} = \frac{\partial e^u}{\partial x} = e^u \frac{\partial u}{\partial x}$

$\Rightarrow 3x^2 - 3yz = e^u \frac{\partial u}{\partial x}$

similarly $\frac{\partial u}{\partial y} \cdot e^u = 3y^2 - 3xz$ and,

$\frac{\partial u}{\partial z} e^u = 3z^2 - 3xy$

$\frac{\partial^2 f}{\partial x^2} = e^u \frac{\partial^2 u}{\partial x^2} + e^u\left(\frac{\partial u}{\partial x}\right)^2$

$\Rightarrow 6x = e^u\left[\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$ — (1)

$\frac{\partial^2 f}{\partial y \partial x} = -3z = e^u\left[\frac{\partial^2 u}{\partial y \partial x}\right] + e^u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)$ — (a)

$\therefore e^u\left(\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right) = 6y$ — (2)

$e^u\left(\frac{\partial^2 u}{\partial z^2} + \left(\frac{\partial u}{\partial z}\right)^2\right) = 6z$ — (3)

$e^u\left(\frac{\partial^2 u}{\partial y \partial z} + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial z}\right)\right) = -3x \rightarrow$ (b)

$e^u\left(\frac{\partial^2 u}{\partial x \partial z} + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial z}\right)\right) = -3y \rightarrow$ (c)

$e^u\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + 2\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial x \partial z}\right)\right]$

$+ e^u\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + 2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial x}\right)\right.$

$\left. + 2\left(\frac{\partial u}{\partial y}\right) * \left(\frac{\partial u}{\partial z}\right)\right] = \cancel{6x^2+y^2+z^2}$

$6x + 6y + 6z$

$- 6x - 6y - 6z$

$= 0$

$$\Rightarrow \cancel{e^u}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u + \cancel{e^u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)^2 = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)^2$$

$$= -\frac{1}{e^{2u}}\left(3x^2-3yz+3y^2-3xz+3z^2-3zy\right)^2$$

$$= -\frac{1}{e^{2u}}\, 9\left(x^2+y^2+z^2-xz-yz-xy\right)^2$$

$$= \frac{-9\left(x^2+y^2+z^2-xy-yz-xz\right)^2}{\left(x^3+y^3+z^3-3xyz\right)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$$\therefore \boxed{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}}$$

12

3(c) Solve the following system of non-linear equations for the unknown angles α, β and γ, where $0 \le \alpha \le 2\pi, 0 \le \beta \le 2\pi$ and $0 \le \gamma < \pi$.

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$
$$4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$$
$$6\sin\alpha - 3\cos\beta + \tan\gamma = 9$$

(20)

Let $\sin\alpha = x$, $\cos\beta = y$, $\tan\gamma = z$

$-1 \le x \le 1$, $-1 \le y \le 1$, $-\infty < z < \infty$

$$2x - y + 3z = 3$$
$$4x + 2y - 2z = 2$$
$$6x - 3y + z = 9$$

consider in Matrix form $AX = B$ then $[A|B]$ is

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$\Rightarrow -8z = 0 \Rightarrow \boxed{z = 0}$

and $4y + (-8z) = -4$

$\Rightarrow \boxed{y = -1}$

and $2x + (-y) + 3z = 3$

$\Rightarrow 2x + 1 = 3$

$\boxed{x = 1}$

$\therefore$ $\sin\alpha = 1$
$\cos\beta = -1$
$\tan\gamma = 0$

$\therefore$ $\boxed{\begin{array}{l} \alpha = \frac{\pi}{2} \\ \beta = \pi \\ \gamma = 0 \end{array}}$

16

4(a) Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. (15)

$F = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$\frac{\partial F}{\partial x} = 4x^3 - 4x + 4y = 0$

$\frac{\partial F}{\partial y} = 4y^3 + 4x - 4y = 0$

$\Rightarrow$ $x^3 - x + y = 0$ and

$y^3 - y + x = 0$

adding both

$x^3 + y^3 = 0$

$\Rightarrow x^3 = -y^3$

$\Rightarrow \boxed{x = -y}$ or $\boxed{x = y = 0}$

when $x = -y$

$\Rightarrow x^3 - x - x = 0$

$\Rightarrow x^3 - 2x = 0$

$\Rightarrow x = 0$ or $x^2 - 2 = 0$

$x = 0$ or $x = +\sqrt{2}, -\sqrt{2}$

$\therefore$ stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

$\frac{\partial^2 f}{\partial x \partial y} = 4 = \frac{\partial^2 f}{\partial y \partial x}$.

at $(0,0)$ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -4$

$\frac{\partial^2 f}{\partial x \partial y} = 4$

$\therefore f_{xx} \cdot f_{yy} - (f_{xy})^2 = 16 - 16 = 0$

$\therefore (0,0)$ is neither maxima nor minima at $(\sqrt{2}, -\sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2} = 24-4=20$$

$$\frac{\partial^2 f}{\partial y^2} = 24-4=20$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4$$

$\Rightarrow$ $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 20 \times 20 - 16 > 0$

and $f_{xx} > 0$

$\therefore (\sqrt{2}, -\sqrt{2})$ is minimum point

Similarly $(-\sqrt{2}, \sqrt{2})$ is minimum point

12

4(b) Trace the curve $y^2(x^2+y^2)+a^2(x^2-y^2)=0$ (15)

$$y^2x^2+y^4+a^2x^2-a^2y^2=0$$

$$\Rightarrow y^4+y^2x^2+a^2x^2-a^2y^2=0 \quad \text{——} (1)$$

For asymptots :- $\phi_4(m)=m^4+m^2$

$\phi_4(m)=0 \Rightarrow m^4+m^2=0$

$m=0,0,\ m^2=-1$

$\phi_3(m)=0 \qquad \phi_2(m)=a^2-a^2m^2$

$\therefore \boxed{c=-\frac{\phi_3'(m)}{\phi_3(m)}=\frac{0}{0}}$, since $c=\frac{\phi_3(m)}{\phi_4(m)}=\frac{0}{0}$;

$\therefore \frac{c^2}{2!}\phi_4''(m)+c\,\phi_{3}'(m)+\phi_2(m)=0$

$\Rightarrow \frac{c^2}{2}(12m^2+2)+c\,0+-a^2\times 2\cdot m=0$

$\Rightarrow \frac{c^2}{2}(2)+0=0.$

$\boxed{c=0}$

$\Rightarrow y=0$ is asymptots.

when $y=0 \Rightarrow a^2x^2=0 \Rightarrow x=0$

when $x=0 \Rightarrow y^4+a^2(-y^2)=0$

$\Rightarrow y=0$ or $y^2-a^2=0$

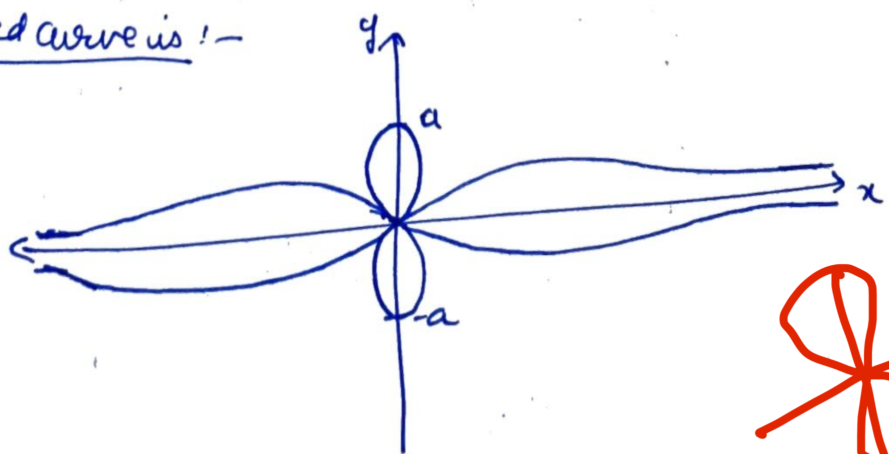
$\boxed{y=\pm a}$

(1) is symmetric about x and y axis

$\Rightarrow$ when $y=x \Rightarrow y^2(2y^2)+a^2\times 0=0$

$\Rightarrow y=0$

$\therefore$ required curve is :-



10

4(c) Find A^{100}, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (20)

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda)-18$$

Charectaristic equation

$$\lambda^2-5\lambda+4-6=0$$

$$\lambda^2-5\lambda-2=0 \quad \text{—} ①$$

$\therefore$ ~~fro~~ from CH equation $A^2-5A-2I=0$ — ②

consider.

$$\lambda^{100} = \phi(\lambda)(\lambda^2-5\lambda-2) + a\lambda+b \quad \text{—} ③$$

From ① $\lambda = \frac{5+\sqrt{33}}{2}, \frac{5-\sqrt{33}}{2}$

Let $\frac{5+\sqrt{33}}{2} = \alpha, \frac{5-\sqrt{33}}{2} = \beta$

$\Rightarrow \alpha^{100} = a\alpha+b$

$\beta^{100} = a\beta+b$

$\Rightarrow \frac{\alpha^{100}-\beta^{100}}{\alpha-\beta} = a$

$\Rightarrow b = \alpha^{100} - \left(\frac{\alpha^{100}-\beta^{100}}{\alpha-\beta}\right)\alpha$

$$= \frac{\cancel{\alpha^{101}} - \beta\alpha^{100} - \cancel{\alpha^{101}} + \alpha\beta^{100}}{\alpha-\beta}$$

$$= \frac{\alpha\beta(\beta^{99}-\alpha^{99})}{\alpha-\beta} = \frac{\alpha\beta^{100}-\beta\alpha^{100}}{\alpha-\beta}$$

$\therefore A^{100} = aA+bI$

$$= \frac{(\alpha^{100}-\beta^{100})A + (\alpha\beta^{100}-\beta\alpha^{100})I}{\alpha-\beta}$$

$$= \frac{1}{\alpha-\beta}\left[[\alpha^{100}-\beta^{100}]A + 2I(\alpha^{99}-\beta^{99})\right]$$

$$A^{100} = \frac{1}{\alpha-\beta}\left[[\alpha^{100}-\beta^{100}]A + 2(\alpha^{99}-\beta^{99})I\right]$$

where $\alpha = \frac{5+\sqrt{33}}{2}$

$\beta = \frac{5-\sqrt{33}}{2}$

16

Section B

5(a) Solve $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0.$ (10)

$M = 2xy^4e^y + 2xy^3 + y$

$N = x^2y^4e^y - x^2y^2 - 3x$

$\frac{\partial M}{\partial x} = 2y^4e^y + 2y^3$

$\frac{\partial M}{\partial y} = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 6xy^2 + 2xy^2 + 4$

$= 8xy^3e^y + 8xy^2 + 4$

$= \frac{4}{y}[2y^4e^y + 2xy^3 + y]$

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-4}{y} = f(y)$

$\therefore$ $IF = e^{-\int 4/y\, dy} = e^{-4\log y}$

$= e^{\log 1/y^4} = \frac{1}{y^4}$

$\therefore$ D.E becomes

$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$

$\Rightarrow 2xe^y dx + x^2e^y dy + \frac{2x}{y}dx - \frac{x^2}{y^2}dy + \frac{1}{y^3}dx - \frac{3x}{y^4}dy = 0$

$\Rightarrow d(x^2e^y) + d\left(\frac{x^2}{y^2}\right) + d\left(\frac{x}{y^3}\right) = 0$

$\Rightarrow \boxed{x^2e^y + \frac{x^2}{y^2} + \frac{x}{y^3} = c}$

8

5(b) Solve the following system of differential equations by the method of Laplace transforms: $x' + y = 2\cos t, x + y' = 0, x(0) = 0, y(0) = 1$. (10)

$\frac{dx}{dt} + y = 2\cos t$ and $x + \frac{dy}{dt} = 0$

Take laplace both side

$L(x') + L(y) = 2\frac{p}{p^2+1}$ and $L(x) + L(y') = 0$

$\Rightarrow pX - x(0) + Y = \frac{2p}{p^2+1}$ and $X + pY - y(0) = 0$

where $L(x) = X$
$L(y) = Y$

$\Rightarrow pX + Y = \frac{2p}{p^2+1}$ and $X + pY = 1$

$\Rightarrow p^2X + pY = \frac{2p^2}{p^2+1}$ and $X + pY = 1$

$\Rightarrow p^2X + 1 - X = \frac{2p^2}{p^2+1}$

$\Rightarrow X = \left(\frac{2p^2}{p^2+1} - 1\right)\frac{1}{p^2-1} = \frac{\cancel{p^2-1}}{\cancel{(p^2-1)}(p^2+1)} = \frac{1}{p^2+1}$

$\therefore x = L^{-1}(X) = L^{-1}\left(\frac{1}{p^2+1}\right) = \sin t$

$pY = 1 - X$
$= 1 - \frac{1}{p^2+1}$

$Y = \frac{1}{p} - \frac{1}{p(p^2+1)}$

Since $x = \sin t$
$\frac{dx}{dt} = \cos t$

$\therefore y = 2\cos t - \frac{dx}{dt} = 2\cos t - \cos t$
$= \cos t$

$\therefore \boxed{\begin{aligned} x &= \sin t \\ y &= \cos t \end{aligned}}$

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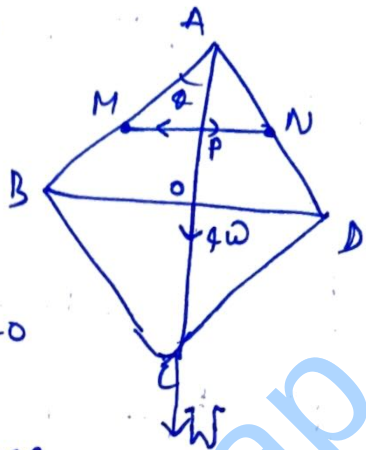
5(c) Four rods of equal weights w form a rhombus $ABCD$, with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that the thrust in this stiffening rod is $(2W+4w)\tan\alpha$ (10)

when in equilibrium
$\theta = \alpha$.



Principle of virtual work

$T\delta(MN) + 4w\,\delta(AO)$
$+ W\delta(AC) = 0$

let $AB = a$

$MP = \frac{a}{2}\sin\theta \Rightarrow MN = a\sin\theta$

$AO = a\cos\theta$.

$AC = 2a\cos\theta$.

$\therefore T\delta(a\sin\theta) + 4w\,\delta(a\cos\theta) + W\delta(2a\cos\theta) = 0$

$T a\cos\theta\,\delta\theta + [4w + 2W]a(-\sin\theta)\,\delta\theta = 0$

$\Rightarrow T = \underline{(4w + 2W)}\tan\theta$

when in equilibrium
$\theta = \alpha$

$$\boxed{T = (4w + 2W)\tan\alpha}$$

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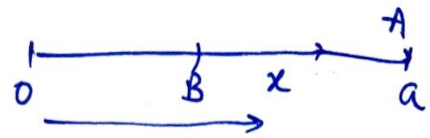
5(d) Show that the time of descent to the centre of force, the force varying inversely as the square of the distance from the centre, through the first half of its initial distance is to that through the last half as $\pi + 2 : \pi - 2$. (10)

given $F = -\frac{K}{x^2}$

Let Particle start from A



$\boxed{OA = a}$

$\therefore \boxed{OB = \frac{a}{2}}$

$\therefore \quad \frac{d^2x}{dt^2} = -\frac{K}{x^2}$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \int \frac{2K}{x^2}\, dx + B$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{+2K}{x} + B.$

when $x = a$ $\frac{dx}{dt} = 0$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 0 = \frac{+2K}{a} + B \Rightarrow B = -\frac{2K}{a}$

$\therefore \left(\frac{dx}{dt}\right)^2 = +\frac{2K}{x} - \frac{2K}{a} \Rightarrow \frac{dx}{dt} = -\sqrt{\frac{2K}{x} - \frac{2K}{a}}$

~~$\Rightarrow \frac{dx}{dt} = -\sqrt{\frac{2K}{a} - \frac{2K}{x}} = \sqrt{2K}\cdot\sqrt{\frac{x-a}{ax}}$.~~

~~$\frac{dx\sqrt{x}}{\sqrt{x-a}} = dt\sqrt{\frac{2K}{a}}$~~

$\int \frac{dx\sqrt{x}}{\sqrt{a-x}} = -\int dt \sqrt{\frac{2K}{a}}$

Put $x = a\sin^2\theta$

$dx = 2a\sin\theta\cos\theta\, d\theta.$

when $a = x$
$\theta = \pi/2$

when $x = 0$
$\theta = 0$

when $x = \frac{a}{2}$
$\Rightarrow \theta = \pi/4$

$-T_1 \cdot \sqrt{\frac{2K}{a}} = \int_{\pi/2}^{\pi/4} \frac{2a\sin\theta\cos\theta \cdot \sqrt{a}\sin\theta}{\sqrt{a}\cos\theta}\, d\theta$

$= a\int_{\pi/2}^{\pi/4} 1 - \cos 2\theta\, d\theta = a\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_{\pi/2}^{\pi/4} = a\left(-\pi/4 - \left(\frac{1}{2}\right)\right)$

$= -a\left(\frac{\pi + 2}{2}\right)$

$-T_2\sqrt{\frac{2K}{a}} = a\int_{\pi/4}^{0} (1 - \cos 2\theta)\, d\theta = a\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_{\pi/4}^{0} = 0 - a\left(\frac{\pi}{4} - \frac{1}{2}\right)$

$= -a\left(\frac{\pi - 2}{2}\right)$

$\therefore \boxed{\frac{T_1}{T_2} = \frac{\pi + 2}{\pi - 2}}$

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5(e) Determine constants a, b, c so that the directional derivative of $\phi(x,y,z) = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude 88 in a direction parallel to z-axis. (10)

$\nabla\phi = (ay^2 + 3x^2z^2c)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)$

at (1,2,-1)

$\nabla\phi = (4a + 3c)\hat{i} + (4a + (-b))\hat{j} + (2b - 2c)\hat{k}$

directional derivative in direction ‖r to z axis

~~$(\nabla\phi)_{\parallel z\,axis} = (2b-2c)\hat{k}$~~

Directional derivative is max in $\nabla\phi$ direction itself

$\Rightarrow$ $4a + 3c = 0$

$4a - b = 0$

$2b - 2c = 88$

$\Rightarrow b - c = \cancel{88}\ 44$

$\therefore 4a + 0b + 3c = 0$

$4a - b + 0c = 0$

$0 + b - c = 44$

$$\boxed{a = \frac{33}{4},\ b = 33,\ c = -11}$$

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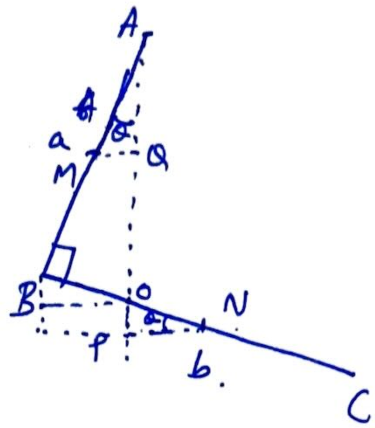
8 (a)



System will be in equilibrium if COM lies just below A (vertically below.

$MQ = \frac{a}{2}\sin\theta.$

$BO = a\tan\theta$

$ON = -a\tan\theta + \frac{b}{2}$

$\therefore NP = \left(\frac{b}{2} - a\tan\theta\right)\cos\theta = \frac{b}{2}\cos\theta - a\sin\theta$

$$\Rightarrow \bar{x} = \frac{\omega a(MQ) - \omega b(NP)}{\omega a - \omega b} = 0$$

$\Rightarrow a\,\frac{a}{2}\sin\theta - b\left(\frac{b}{2}\cos\theta - a\sin\theta\right) = 0$

$\Rightarrow a^2\sin\theta = b^2\cos\theta - 2ab\sin\theta$

$\Rightarrow (a^2+2ab)\sin\theta = b^2\cos\theta$

$\therefore \boxed{\tan\theta = \frac{b^2}{a^2+2ab}}$

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8(b) For the curve $x = 4a\cos^3 t, y = 4a\sin^3 t, z = 3c\cos 2t$, show that

$\kappa = \frac{a}{6(a^2+c^2)\sin 2t}$. (15)

$$\frac{d\vec{r}}{dt} = 4a\cdot 3\cos^2 t(-\sin t),\ 12a\sin^2 t\cos t,\ -6c\sin 2t$$

$$= -12a\cos^2 t\sin t,\ 12a\sin^2 t\cos t,\ -6c\sin 2t$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(12a)^2\sin^2 t\cos^4 t + (12a)^2\sin^4 t\cos^2 t + 36c^2\sin^2 2t}$$

$$= \sqrt{144a^2\sin^2 t\cos^2 t + 144c^2\sin^2 t\cos^2 t}$$

$$= \cancel{\sin t\cos t}\sqrt{36a^2\sin^2 2t + 36c^2\sin^2 2t}$$

$$= \cancel{6a}\ 6\sin 2t\sqrt{a^2+c^2}.$$

$$\vec{T} = \frac{dr}{ds} = \frac{d\vec{r}}{dt}\Big/\left|\frac{d\vec{r}}{dt}\right| = \frac{-12a\sin t\cos^2 t\,\hat{i} + 12a\sin^2 t\cos t - 6c\sin 2t}{6\sin 2t\sqrt{a^2+c^2}}$$

$$= \frac{-a\cos t\,\hat{i} + a\sin t\,\hat{j} - c\,\hat{k}}{\sqrt{a^2+c^2}}$$

$$\therefore \frac{dT}{ds} = \frac{dT}{dt}\Big/\left|\frac{dr}{dt}\right| = \frac{1}{\sqrt{a^2+c^2}}\left[a\sin t\,\hat{i} + a\cos t\,\hat{j}\right] \times \frac{1}{6\sin 2t\sqrt{a^2+c^2}}$$

$$= \frac{1}{(a^2+c^2)6\sin 2t}(a\sin t\,\hat{i} + a\cos t\,\hat{j})$$

$$\therefore \kappa = \left|\frac{dT}{ds}\right| = \frac{1}{(a^2+c^2)6\sin 2t} \times \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$\boxed{\kappa = \frac{a}{6(a^2+c^2)\sin 2t}}$$

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8(c) Solve $(3-x)y'' - (9-4x)y' + (6-3x)y = 0.$ (20)

$$y'' - \left(\frac{9-4x}{3-x}\right)y' + \frac{6-3x}{3-x}y = 0.$$

$$P = -\frac{(9-4x)}{3-x} \qquad Q = \frac{6-3x}{3-x}.$$

$$1+P+Q = \frac{\cancel{3}-\cancel{x}-\cancel{9}+\cancel{4x}+\cancel{6}-\cancel{3x}}{3-x} = 0$$

$\therefore$ ~~y=e~~ $u = e^x$ is one solution

$\therefore$ Let $y = u \cdot v$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0.$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{4x-9}{3-x} + 2\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{4x-9+6-2x}{3-x}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{2x-3}{3-x}\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$

$$\Rightarrow \frac{dz}{dx} + z\left(\frac{2x-6+3}{3-x}\right)d = 0$$

$$\Rightarrow \frac{dz}{z} + \left(-2 + \frac{3}{3-x}\right)dx = 0$$

$$\Rightarrow \log z + -2x - 3\log(3-x) = \log a.$$

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$$\Rightarrow z = a e^{2x}(3-x)^3.$$

$$\frac{dv}{dx} = a e^{2x}(3-x)^3$$

$$\Rightarrow dv = a e^{2x}(3-x)^3 \cdot dx$$

$$\therefore v = a (3-x)^3\frac{e^{2x}}{2} + \int 3(3-x)^2\frac{e^{2x}}{2}dx + b$$

$$= a(3-x)^3 \cdot \frac{e^{2x}}{2} + \frac{3}{2}\left[(3-x)^2\frac{e^{2x}}{2} + \int \cancel{2}(3-x)\frac{e^{2x}}{\cancel{2}}dx\right] + b$$

$$= \frac{a}{2}(3-x)^3e^{2x} + \frac{3}{4}(3-x)^2e^{2x} + \frac{3}{2}\left[(3-x)\frac{e^{2x}}{2} + \int\frac{e^{2x}}{2}dx\right] + b$$

$$= \frac{a}{2}(3-x)^3e^{2x} + \frac{3}{4}(3-x)^2e^{2x} + \frac{3}{4}(3-x)e^{2x} + e^{2x}\frac{3}{8} + b$$

$$\therefore\ y = e^{x}\left[e^{2x}a\left(\frac{(3-x)^3}{3} + \frac{3}{4}(3-x)^2 + \frac{3}{4}(3-x) + \frac{3}{8}\right)\right] + be^{x}$$

$$\boxed{y = ae^{3x}\left[\frac{(3-x)^3}{3} + \frac{3}{4}(3-x)^2 + \frac{3}{4}(3-x) + \frac{3}{8}\right] + be^{x}}$$

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