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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -6 PAPER 2

Name of the Candidate			
Email ID		Roll No	
Phone Number		Date	16th August 2023
Start Time : 11:45 Am		Closing Time: 2:45 PM	

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		5	5a		8	
1b		8	5b			
1c		8	5c		8	
1d		8	5d		8	
1e		8	5e			
2a			6a			
2b			6b			
2c			6c			
2d			6d			
3a		13	7a		18	
3b		13	7b		13	
3c		18	7c		13	
3d			7d			
4a		18	8a			
4b		13	8b			
4c		10	8c			
4d			8d			
Total						

190/250

Section A

1(a) Every quotient group of a cyclic group is cyclic, but converse is not true (10)

Let ~~G~~ a cyclic group $G = \langle a \rangle$

$\therefore$ H be a ~~sub~~ normal subgroup of G

$H = \langle a^r \rangle$

$\frac{G}{H} = \{ H + b \mid \text{where } b \in G \}$

Since $H = \langle a^r \rangle$

$b = a^m$

$\therefore H + b = H + a^m = \cancel{(H+a) + (H+a) + \ldots}$

$= H + (a + a + a \cdots m \text{ times})$

$= (H+a) + (H+a) + \cdots + m \text{ times}$

$= (H+a)^m$

$\therefore \boxed{\frac{G}{H} = \langle H + a \rangle}$

$\frac{G}{H}$ is cyclic

5

1(b) If f(z) $= u +$ iv is analytic function and $u - v = e^x(\cos\ y - \sin\ y)$
Find f(z) (10)

$f(z) = u + iv$

$if(z) = iu - v$

$f(z) + if(z) = u - v + i(u+v)$

$(1+i)f(z) = U + iV$

$F(z) = U + iV$

here $U = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y)$

$U_y = e^x(-\sin y - \cos y)$

$U_x(z,0) = e^z(1)$

$U_y(z,0) = e^z(-1)$

$\therefore F'(z) = U_x(z,0) - iU_y(z,0)$

$(1+i)f'(z) = e^z + ie^z$

$f'(z) = e^z$

$\Rightarrow \boxed{f(z) = e^z + c}$ where c is constant

8

1(c) Show that $\int_0^{\pi/2} \log \sin x dx$ is convergent (10)

$I = \int_0^{\pi/2} \log \sin x \, dx.$

here at $\pi/2$ $\log \sin x = 0$

so convergent at $\pi/2$.

Convergent test at $x=0$:—

$I = \int_0^{\pi/2} \log \sin x \, dx$

let $f(x) = -\log \sin x$

$g(x) = \frac{1}{x^n}$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} -x^n \log(\sin x)$

$= \lim_{x\to 0} \frac{-\log(\sin x)}{x^{-n}}$

$= \lim_{x\to 0} \frac{-1}{\sin x} \cos x \times \frac{1}{-n x^{-n-1}}$

$= \lim_{x\to 0} \frac{+1}{\sin x} \frac{x^n \cdot x}{n}$

$= \lim_{x\to 0} \frac{x^n}{n} \frac{x}{\sin x}$

$= 0.$

~~when we~~ $\Rightarrow$ $f(x)$ is convergent when $g(x)$ is convergent

$\Rightarrow$ if we take $n = 1/2$

$\therefore \int_0^{\pi/2} g(x) \, dx = \int_0^{\pi/2} \frac{1}{x^{1/2}} dx$ is convergent (p-test)

$\Rightarrow \int_0^{\pi/2} \log(\sin x) dx$ is convergent

8

1(d) A firm can produce three types of cloth A, B and C. Three kinds of wool is required for it, say red, green and blue wools. One unit length of type A cloth needs 2 yards of red wool, 5 yards of blue wools, one unit length of type B cloth needs 3 yards of red wool, 4 yards of green wool, and 2 yards of blue wool, and one unit length of type C cloth needs 6 yards of green and 5 yards of blue wools. The firm has only a stock of 10 yards of red wool, 12 yards of green wool, and 17 yards of blue wool. It is assumed that the income obtained from one unit length of type A, B and C are Rs 4.00,5.00 and 6.00 respectively. Determine how the firm should use the available material, so as to maximize the income from the finished cloths. (10)

let firm produces x unit of A
y unit of B
z unit of C

maximize profit $Z = 4x + 5y + 6z$

subject to,

Red $2x + 3y + 0z \leq 10$

Blue $5x + 2y + 5z \leq 17$

green $0x + 4y + 6z \leq 12$

$x \geq 0,\ y \geq 0,\ z \geq 0$

Standard form $\max Z = 4x + 5y + 6z + 0s_1 + 0s_2 + 0s_3$

$2x + 3y + 0z + s_1 = 10$

$5x + 2y + 5z + s_2 = 17$

$0x + 4y + 6z + s_3 = 12$

IBFS, $S_1 = 10,\ S_2 = 17,\ S_3 = 12$

		4	5	6	0	0	0		
		x_1	x_2	z	s_1	s_2	s_3		
0	s_1	2	3	0	1	0	0	10	—
0	s_2	5	2	5	0	1	0	17	17/5
0	s_3	0	4	6	0	0	1	12	$\boxed{2.}$
	z_j	0	0	0	0	0	0	0	
	$c_j - z_j$	4	5	6	0	0	0		

0	s_1	2	3	0	1	0	0	10	.10/3
0	s_2	5	$-4/3$	0	0	1	$-5/6$	7	
6	z	0	$2/3$	1	0	0	$1/6$	2	6/2
		0	4	6	0	0	1	12	
		0	$\boxed{1}$	0	0	0	-1		

8

1(e) Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \text{ if } 0 < u < v,$$

and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$. (10)

consider

$$\int_u^v \frac{1}{1+x^2} dx$$

$$\int_u^v \frac{1}{1+v^2} dx < \int_u^v \frac{1}{1+x^2} dx < \int_u^v \frac{1}{1+u^2} dx \quad \text{as } \underline{0<u<v}$$

$$\Rightarrow \frac{v-u}{1+v^2} < \tan^{-1}(x)\Big|_u^v < \frac{v-u}{1+u^2}$$

$$\boxed{\Rightarrow \frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}}$$

let $v = \frac{4}{3}$, $u = 1$

$$\Rightarrow \frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{4/3-1}{1+1}$$

$$\Rightarrow \frac{1}{\cancel{3}} \times \frac{\cancel{9}\,3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{3} \times \frac{1}{2}.$$

$$\Rightarrow \boxed{\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}}$$

8

3(a) Evaluate the contour integral $\int_C f(z)dz$

using the parametric representations for C, where $f(z) = \frac{z^2-1}{z}$

and the curve C is

(a) the semicircle $z = 2e^{i\theta}(0 \leq \theta \leq \pi)$;

(b) the semicircle $z = 2e^{i\theta}(\pi \leq \theta \leq 2\pi)$;

(c) the circle $z = 2e^{i\theta}(0 \leq \theta \leq 2\pi)$. (15)

(a) $\int_C f(z)\,dz \qquad z = 2e^{i\theta} \Rightarrow dz = 2ie^{i\theta}\,d\theta$

$$= \int_0^{\pi} \frac{((2e^{i\theta})^2-1)}{2e^{i\theta}}\,d\theta \cdot 2ie^{i\theta}$$

$$= \int_0^{\pi} \frac{(4e^{2i\theta}-1)\cdot 2i}{2}\,d\theta = i\int_0^{\pi}(4e^{2i\theta}-1)\,d\theta$$

$$= i\left[\frac{4e^{i2\theta}}{2i} - \theta\right]_0^{\pi}$$

$$= i\left[4\times 0 - \pi\right]$$

$$= -i\pi$$

(b) $\int_C f(z)\,dz \qquad z = 2e^{i\theta} \quad \pi \leq \theta \leq 2\pi$

$$\Rightarrow \int_{\pi}^{2\pi} \frac{4e^{i2\theta}-1}{\cancel{2e^{i\theta}}} \times \cancel{2}i\cancel{e^{i\theta}}\,d\theta$$

$$= i\left[\int_{\pi}^{2\pi}(4e^{i2\theta}-1)\,d\theta\right] = i\left[\frac{4e^{i2\theta}}{i2} - \theta\right]_{\pi}^{2\pi}$$

$$= i\left[\frac{0}{2i} - \pi\right]$$

$$= -\pi i$$

(c) $\int_C f(z)\,dz$ where $0 \leq c \leq 2\pi$

$$= \int_0^{\pi} f(z)\,dz + \int_{\pi}^{2\pi} f(z)\,dz = -i\pi + (-i\pi)$$

$$= -2i\pi$$

13

3(b) If $f(x)$ be defined on $[0,2]$ as follows,

$$f(x) = x + x^2, \text{ when } x \text{ is rational.}$$
$$= x^2 + x^3, \text{ when } x \text{ is irrational}$$

then evaluate the upper and lower Riemann integrals of f over $[0,2]$ and show that f is not R integrable over $[0,2]$. (15)

~~f(x) For~~
~~For upper riem~~

$f(x) = x + x^2$ when x is ~~irration~~
$= x^3 + x^2$ when x is irration

in $[0,1]$ $x + x^2 > x^2 + x^3$

in $[1,2]$ $x^2 + x^3 > x + x^2$

$\therefore$ upper riemann itegral $= \int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$$= \int_0^1 (x+x^2)\,dx + \int_1^2 (x^2+x^3)\,dx$$

$$= \frac{x^2}{2} + \frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3} + \frac{x^4}{4}\Big|_1^2$$

$$= \frac{1}{2} + \cancel{\frac{1}{3}} + \frac{8}{3} + \frac{16}{4} - \cancel{\frac{1}{3}} - \frac{1}{4}$$

$$= \frac{83}{12}.$$

Lower Riemann itegral $= \int_0^2 f(x)dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)dx$

$$= \int_0^1 (x^2+x^3)\,dx + \int_1^2 (x+x^2)\,dx$$

$$= \frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1 + \frac{x^2}{2} + \frac{x^3}{3}\Big|_1^2$$

$$= \cancel{\frac{1}{3}} + \frac{1}{4} + \frac{4}{2} + \frac{8}{3} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$= \frac{63}{12}$$

13

$\therefore$ Lower Riemann integral $\neq$ upper Riemann integral

$\therefore$ f is not R integrable over $[0,2]$

3(c) Solve the following LP problem by using the two-phase simplex method.
Minimize $Z = x_1 - 2x_2 - 3x_3$
subject to the constraints
(i) $-2x_1 + x_2 + 3x_3 = 2$,
(ii) $2x_1 + 3x_2 + 4x_3 = 1$
and
$x_1, x_2, x_3 \geq 0$. (20)

minimize $z = x_1 - 2x_2 - 3x_3$

maximize $z^* = -x_1 + 2x_2 + 3x_3 - MA_1 - MA_2$

Subject to
$$-2x_1 + x_2 + 3x_3 + A_1 = 2$$
$$2x_1 + 3x_2 + 4x_3 + A_2 = 1$$

Phase I

maximize $z^* = \cancel{\to} + 0x_1 + 0x_2 + 0x_3 - 1A_1 - 1A_2$

	c_j	0	0	0	−1	−1		
		x_1	x_2	x_3	A_1	A_2	b	θ
−1	A_1	−2	1	3	1	0	2	2/3
−1	A_2	2	3	$\boxed{4}$	0	1	1	$\boxed{1/4}$ →
	z_j	0	−4	−7	−1	−1	−3	
	$c_j - z_j$	0	4	$\boxed{7}$	0	0		
−1	A_1	$-\frac{7}{2}$	−5/4	0	1		5/4	
0	x_3	1/2	3/4	1	0		1/4	
	z_j	7/2	$\frac{5}{4}$	0	−1		−5/4	
	$c_j - z_j$	−7/2	$\frac{-5}{4}$	0	0			

here $z^* < 0$ & all $c_j - z_j \leq 0$

and A_1 is still in basis

∴ This LPP has no OBFS

18

4(a) If F is a field, then $F[x]$ is a Euclidean domain (20)

F is a field ⇒ F is an integral domain

⇒ F is an ID ⇒ F[x] is an ID

to Prove For ED if $f(x), \& g(x) \in F$

⇔ $\exists\ d: F[x]-\{0\} \to \mathbb{Z}$

such that $d(f(x)) = \deg(f(x))$

and we need to prove (i) $d(f(x)) \leq d(f \cdot g)$

(ii) $d(f(x)) \geq 0$

(iii) for any $f, g \in F[x]$

$\exists\ h(x)$ such that

$f(x) = g(x) \cdot h(x) + r(x)$

$d(r(x)) < d(g(x))$

or $d(r(x)) = 0$.

let 1) Since $f(x) \in F[x]-\{0\}$

degree of $f(x) \geq 0$.

$\therefore\ d(f(x)) \geq 0$ —— (1).

ii) $d(f(x) \cdot g(x)) = \deg(f(x) \cdot g(x))$

if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$g(x) = b_0 + b_1 x + b_2 x^2 + \text{———} + b_m x^m$

$f(x) \cdot g(x) = a_0 b_0 + (a_0 b_1 + b_0 a_1) x + \text{———} + (a_n b_m x^{n+m})$

Since F[x] is an ID

⇒ if $a_n \neq 0$ & $b_m \neq 0$

⇒ $a_n * b_m \neq 0$ (as no zero divisor in ID)

⇒ $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg(g(x))$

since $\deg(g(x)) \geq 0$

⇒ $\deg(f(x) \cdot g(x)) \geq \deg(f(x))$ —— (2)

⇒ $d(f(x) \cdot g(x)) \geq d(f(x))$ —— (2)

(iii) let $f(x), g(x), h(x) \in F[x]$

such that

$f(x) = h(x)\, g(x) + r(x)$

here $f(x) \in F[x]$

$g(x) \in F[x]$

~~Since~~ if $r(x) = 0$

$\Rightarrow \deg(r(x)) = 0$

$d(r(x)) = 0$ — (3)

if $r(x) \neq 0$

$r(x) = f(x) - h(x) g(x)$

in any field F, and $F[x]$

division algorithm is applicable

$\Rightarrow f(x) = h(x) g(x) + r(x)$

where $\deg(r(x)) = 0$

or $\deg(r(x)) < \deg(g(x))$

$\Rightarrow d(r(x)) = 0$

or $d(r(x)) < d(g(x))$ —— (4)

from (1), (2) & (4)

$\boxed{F[x] \text{ is a Euclidian domain}}$

18

4(b) Prove that $\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2} d\theta = \pi \frac{1-p+p^2}{1-p}, 0 < p < 1.$ (15)

$$I = \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2} = \cancel{\text{Real}} \int_0^{2\pi} \frac{1+\cos 6\theta}{2(1-2p\cos 2\theta + p^2)} d\theta$$

$$= 2\int_0^{\pi} \frac{1+\cos 6\theta}{2(1-2p\cos 2\theta+p^2)} d\theta$$

put $2\theta = \phi$.
$\Rightarrow 2d\theta = d\phi$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 3\phi}{(1-2p\cos\phi+p^2)} d\phi.$$

$$I = \frac{1}{2} \text{Re} \int_0^{2\pi} \frac{1+e^{3i\phi}}{1-2p\cos\phi+p^2} d\phi.$$

~~Now con~~ Put $z = e^{i\phi}$ $\quad |z| = 1$

$dz = ie^{i\phi} d\phi$
$\Rightarrow d\phi = \frac{dz}{iz}$

$$\cancel{I = \frac{1}{2}} \therefore \int_0^{\pi} \frac{1+e^{3i\phi}}{1-2p\cos\phi+p^2} d\phi = \int_c \frac{1+z^3}{\left(1-\cancel{2}p\left(\frac{z^2+1}{\cancel{2}z}\right)+p^2\right)(iz)} dz$$

$$= \frac{1}{i}\int_c \frac{(1+z^3)dz}{\cancel{z}\left(z(1+p^2)-pz^2-p\right)}$$

$$= \frac{1}{-i}\int_c \frac{(1+z^3)\,dz}{pz^2-(1+p^2)z+p}$$

$$= \frac{1}{-i}\int_c \frac{1+z^3}{pz^2-z-p^2z+p} dz = -\frac{1}{i}\int \frac{(1+z^3)dz}{\cancel{p(z-1)(z+1)}\; z(pz-1)-p(pz-1)}$$

$$= -\frac{1}{i}\int_c \frac{dz\,(1+z^3)}{p(z-p)\left(z-\frac{1}{p}\right)}$$

$0<p<1$
$\Rightarrow \boxed{\frac{1}{p} > 1}$

$\therefore z = p$ is only singularity in c

~~test~~ $= -\frac{1}{i} \times 2\pi i$ (residue at $z=p$)

$$= \cancel{i}\cdot -2\pi \times \frac{1+p^3}{p(p-1/p)} = \frac{-2\pi(1+p^3)}{(p^2-1)}$$

$$\therefore I = \frac{1}{2} \mathrm{Re}\left(\frac{(-)(p^3+1)}{(p^2-1)}\right) \times 2\pi$$

$$= \frac{1}{2} \frac{(\cancel{p+1})(p^2+1-p)}{(1-p)(\cancel{1+p})} \times 2\pi$$

$$I = \frac{\pi(p^2-p+1)}{1-p}$$

13

$$\boxed{\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\, d\theta = \frac{\pi(p^2-p+1)}{1-p}}$$

4(c) Apply MODI method to obtain optimal solution of transportation problem

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

(15)

using vogal method for initial feasible solution

19) 5	30) x	50) x	10) 2	~~7~~ ~~5~~ 2	~~(9)~~ (40)
70) x	30) x	40) 7	60) 2	9	~~(10)~~ (20)
40) x	8) 8	70) x	20) 10	~~18~~ 10	~~(12)~~ ~~20~~ (50)
~~5~~	~~8~~	7	~~14~~ ~~12~~ 4		
~~(21)~~	~~(22)~~	~~(10)~~	~~(10)~~		
		(30)	(40)		

Modi method :— u

				u
19) 5	30) (–)	50) (–)	10) 2	−50
70) (–)	30) (-18) θ	40) 7	60) 2 −θ	0
40) (–)	8) 8 −θ	70) (–)	20) 10 +θ	−40
V 69	48	40	60	

19) 5	30) (–)	50) (–)	10) 2	−32
70) (–)	30) 2	40) 7	60) (–)	0
40) (–)	8) 6	70) (–)	20) 12	−22
51	30	40	42	

optimal reached

~~$S_1D_1 = 5$~~ units	cost/unit	cost
$S_1 \rightarrow D_1 \Rightarrow 5$	19	95
$S_1 \rightarrow D_4 \Rightarrow 2$	10	20
$S_2 \rightarrow D_2 \rightarrow 2$	30	60
$S_2 \rightarrow D_3 \rightarrow 7$	40	280
$S_3 \rightarrow D_2 \rightarrow 6$	8	48
$S_3 \rightarrow D_4 \rightarrow 12$	12	144
		647

743

10

Section B

5(a) Find the differential equation of all spheres whose radii are the same (10)

consider standard equation

$$x^2+y^2+z^2+2ax+2by+2cz+d=0 \quad \text{--- (1)}$$

radius $= \sqrt{a^2+b^2+c^2-d}$

$\Rightarrow a^2+b^2+c^2-d = \text{constant} = k$

P. D. (1) with respect to x.

$2x+2a+2cp=0 \Rightarrow x+a+cp=0$

and $2y+2b+2cq=0 \Rightarrow y+b+cq=0.$

let standard equation

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ where r is constant

P.D with respect to x

$2(x-a)+2(z-c)p=0 \Rightarrow x-a+(z-c)p=0$

$2(y-b)+2(z-c)q=0 \Rightarrow y-a+(z-c)q=0$

$0+(z-c)\frac{\partial^2 z}{\partial y \partial z}+p.q=0$

$(z-c) = \frac{-p.q}{\frac{\partial^2 z}{\partial x \partial y}}$

$(x-a) = \frac{+p^2q}{\frac{\partial^2 z}{\partial x \partial y}}$

$y-b = \frac{q^2p}{\frac{\partial^2 z}{\partial x \partial y}}$

$\therefore (x-a)^2+(y-b)^2+(z-c)^2=r^2$

$\Rightarrow \boxed{pq^2+p^4q^2+q^4p^2=r^2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2}$

where r is constant

8

5(b) Simplify the expression $AB + \overline{AC} + A\bar{B}C(AB + C)$ (10)

$F = AB + \overline{AC} + A\bar{B}C(AB+C)$

$= \cancel{AB + \overline{AC} +}$

$= AB + \bar{A} + \bar{C} + A\bar{B}C(AB) + A\bar{B}CC$

$= \bar{A} + B + \bar{C} + 0 + A\bar{B}C$

$= \bar{A} + B + AC + \bar{C}$

$= \bar{A} + B + C + \bar{C}$

$= \bar{A} + B$

1

5(c) Find the necessary and sufficient condition that vortex lines may be at right angles to the streamlines. (10)

Let velocity is (u, v, w)

$\therefore$ equation of streamlines

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{--- (1)}$$

streamlines

vortex lines $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} = 0$. --- (2)

~~here~~

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\therefore \Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$\therefore$ vortex lines (2) & streamline (1) are $\perp^r$ then :-

$$u\Omega_x + v\Omega_y + w\Omega_z = 0$$

$$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$\Rightarrow u\,dx + v\,dy + w\,dz = 0$ is exact differential equation

$$u\,dx + v\,dy + w\,dz = \mu\, d\phi = \mu\left[\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right]$$

$$\Rightarrow u = \frac{\partial \phi}{\partial x}\mu, \quad v = \mu\frac{\partial \phi}{\partial y}, \quad w = \mu\frac{\partial \phi}{\partial z}$$

$$\boxed{\therefore (u, v, w) = \mu\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)}$$

8

5(d) Explain the main steps of the Gauss-Jordan method and apply this method to find the inverse of the matrix $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$. (10)

Gauss Jordan Method

① in Gauss Jordan method we write a matrix in format [A|B] for system AX = B

② apply row operation to covert in form [D|B'] where D is diagonal matrix.

③ apply substitution to get value

Now For inverse

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 6 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

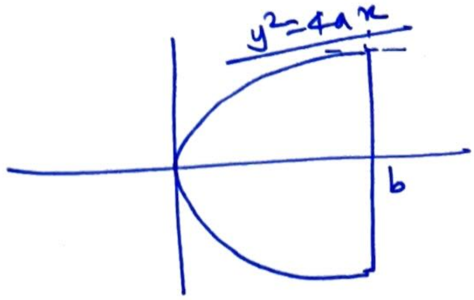
$$\xrightarrow{R_1 \to R_1 - 3R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1+3+3 & 0-3 & 0-3 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

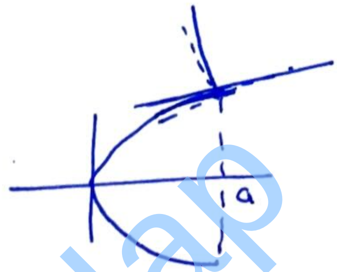
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & -3/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 7/2 & -3/2 & -3/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}}$$

8

5(e) A uniform lamina is bounded by a parabolic are of latus rectum $4a$, and a double ordinates at a distance b from the vertex. If $b = (a/3)(7 + 4\sqrt{7})$. Show that two of the principal axes at the end of a latus rectum are the tangent and normal there (10)





7(a) Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Further it is given that $y(x,0) = 1 - x, 0 < x < 1$. (20)

$y(0,t) = 10^\circ c$

$\left.\frac{\partial y}{\partial x}\right|_{x=1} = 0 \qquad y(x,0) = 1-x \quad 0<x<1$

Let $u(x,t) = y(x,t) - 10 \Rightarrow y(x,t) = u(x,t)+10$

$\therefore u(0,t) = 0$

$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$

$u(x,0) = 1-x-10 = \underline{-9-x}$

heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Let $u = X(x) \cdot T(t)$

$\therefore X'' T = \frac{1}{c^2} X T'$

$\Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} = k.$

From variable seperable method

$\Rightarrow X'' - kX = 0$ and $T' - c^2 T k = 0$

$\boxed{\text{case I}}$ if $\boxed{k=0}$

$X = ax+b, \; T = d$

here $X(0) = 0$

$\Rightarrow b = 0$

$\left.\frac{\partial X}{\partial x}\right|_{x=1} = 0 \Rightarrow \underline{a=0}$

$\therefore u(x,t) = 0$

not possible

$\boxed{\text{Case 2}}$ if $k = \lambda^2$

$\Rightarrow X'' - \lambda^2 X = 0, \quad T' - (c^2\lambda^2 T) = 0$

$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ and $T = c_3 e^{c^2\lambda^2 t}$

T is increasing f^n with time

not feasible

Case 3 $k = -\lambda^2$

$X'' + \lambda^2 X = 0$ and $T' + c^2\lambda^2 T = 0$

$X = c_1 \cos\lambda x + c_2 \sin\lambda x$, $T = c_3 e^{-c^2\lambda^2 t}$

given $X(0) = 0$

$\Rightarrow c_1 = 0$

$X(x) = c_2 \sin\lambda x$. $\Rightarrow u(x,t) = c_2 c_3 \sin\lambda x \, e^{-\lambda^2 c^2 t}$

$\therefore$ $\frac{\partial u}{\partial x} = c_2 c_3 \frac{\cos\lambda x}{\lambda} e^{-\lambda^2 c^2 t}$

at $x = l \Rightarrow \frac{\partial u}{\partial x} = 0$

$\Rightarrow \cos\lambda l = 0 \Rightarrow \cos\lambda l = \cos(2n+1)\frac{\pi}{2}$

$$\boxed{\lambda = \frac{(2n+1)\pi}{2l}}$$

$\therefore u_n(x,t) = E_n \sin\left(\frac{2n+1}{2l}\right)\pi x \, e^{-\lambda^2 c^2 t}$

$u(x,t) = \sum E_n \sin\left(\frac{2n+1}{2l}\right)\pi x \, e^{-\lambda^2 c^2 t}$

$u(x,0) = -9 - x \Rightarrow$

$-9 - x = \sum E_n \sin\left(\frac{2n+1}{2}\right)\pi x$ $(\because l = 1)$

$\therefore E_n = \frac{2}{1}\int_0^1 (-9-x)\sin\left(\frac{2n+1}{2}\pi x\right) dx$.

$= 2\left[\left[(-9-x)\cos\left(\frac{2n+1}{2}\right)\pi x \times \frac{2(-1)}{\pi(2n+1)}\right]_0^1 - \int_0^1 (-1)\frac{(-1)\,2}{\pi(2n+1)}\cos\left(\frac{2n+1}{2}\right)\pi x \, dx\right]$

$= 2\left[\frac{-10(-2)}{(2n+1)\pi}\times 0 - \frac{(-9)(-2)}{(2n+1)\pi} - \frac{2^2}{\pi^2(2n+1)^2}\sin\left(\frac{2n+1}{2}\right)\pi x \Big|_0^1\right]$

$= 2\left[\frac{-18}{(2n+1)\pi} - \frac{4}{\pi^2(2n+1)^2}\sin\left(\frac{2n+1}{2}\right)\pi\right]$

$= \frac{-36}{(2n+1)\pi} - \frac{8}{\pi^2(2n+1)^2}(-1)^n$ where $n = 0, 1, 2, 3$

$= \frac{8(-1)^n}{\pi^2(2n-1)^2} - \frac{36}{(2n-1)\pi}$ $n = 1, 2, 3$.

$\therefore y(x,t) = u(x,t) + 10$

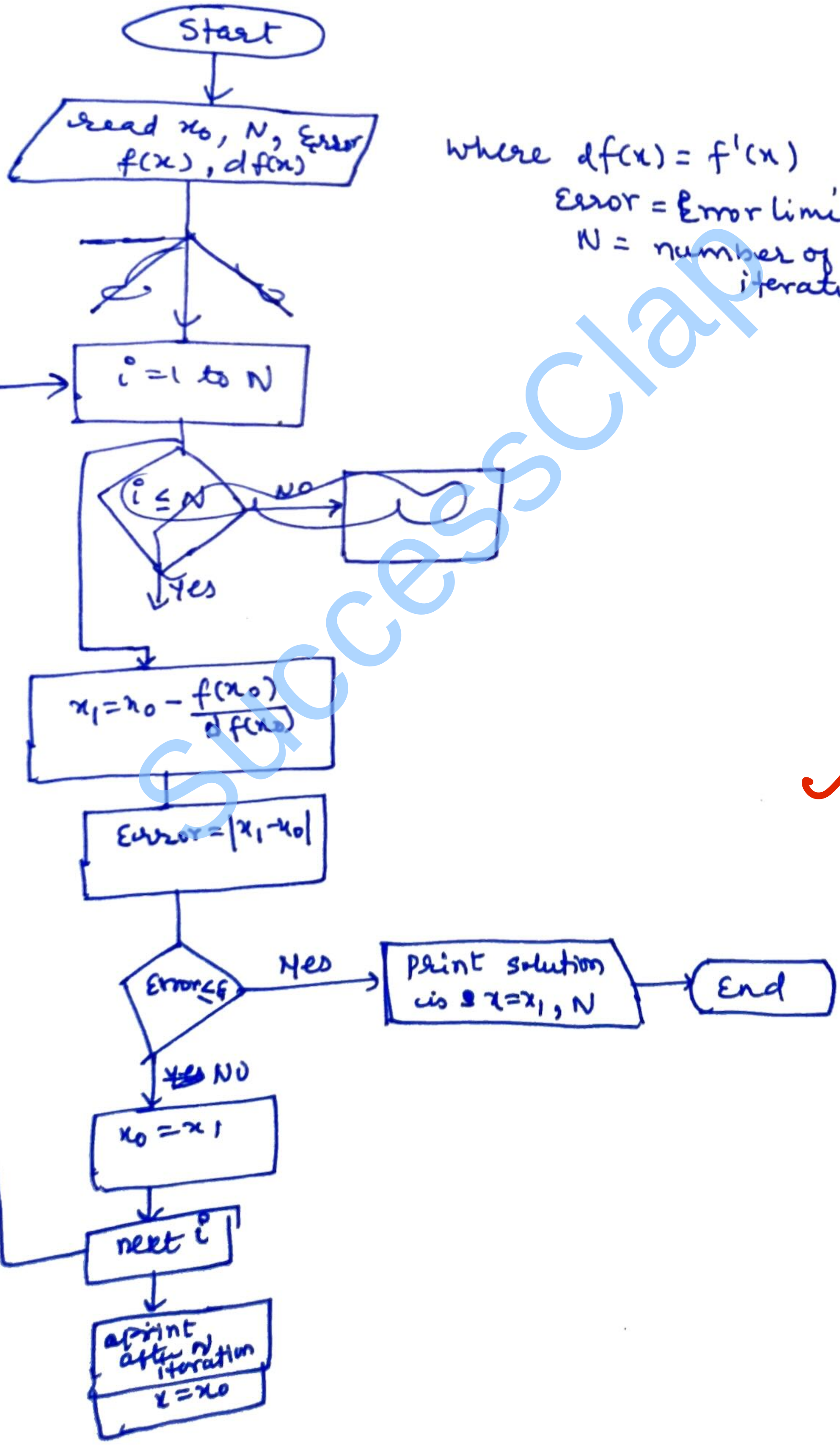
$$\boxed{y(x,t) = 10 + \sum_{n=1}^{\infty}\left(\frac{8(-1)^n}{\pi^2(2n-1)^2} - \frac{36}{(2n-1)\pi}\right)\sin\left(\frac{(2n-1)\pi}{2}\right)x \, e^{-\frac{(2n-1)^2}{2^2}c^2 t}}$$

18

7(b) Write an algorithm in the form of a flow chart for Newton-Raphson method. Describe the cases of failure of this method. (15)

Newton Raphson method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



where $df(x) = f'(x)$
Error = Error limit
N = number of iteration

13

$\boxed{\text{Failure condition}}$

Newton Raphson method fails when $f'(x) = 0$ at some x_i.

7(c) A two-dimensional flow field is given by $\psi = xy$.

(a) Show that the flow is irrotational.

(b) Find the velocity potential.

(c) Verify that ψ and ϕ satisfy the Laplace equation.

(d) Find the streamlines and potential lines. (15)

$\psi = xy$

(a) Flow is irrotational if $\nabla \times \bar{V} = 0$.

we know $\quad u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$.

$v = -\frac{\partial \phi}{\partial y} = +\frac{\partial \psi}{\partial x}$.

$u = -x$

$v = y$

Now $\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = \hat{k}(0-0) = 0$

$\therefore$ Flow is irrotational.

(b) $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = x$.

$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -y$.

$\Rightarrow \phi = \frac{x^2}{2} + f(y)$

$\frac{\partial \phi}{\partial y} = 0 + f'(y) \Rightarrow f'(y) = -y$

$f(y) = \frac{-y^2}{2}$.

$\therefore \boxed{\phi = \frac{x^2 - y^2}{2}}$

velocity potential is $\boxed{\phi = \frac{x^2 - y^2}{2}}$

(c) $\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$.

$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) = 1 - 1 = 0$

$\therefore \phi$ & ψ satisfy laplace equation

(d) stream lines where

$\psi = \text{constant}$

$\Rightarrow \boxed{xy = c_1}$

potential lines where

$\phi = \text{constant}$

$\Rightarrow \frac{x^2 - y^2}{2} = c_2$

$\Rightarrow \boxed{x^2 - y^2 = 2c_2}$

13