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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -7 PAPER 1

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Email ID		Roll No	
Phone Number		Date	20th August 23
Start Time : 12:30 PM		Closing Time: 3:30 pm	

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		8	5a		8	
1b		8	5b		8	
1c		8	5c		6	
1d			5d		8	
1e		5	5e		8	
2a			6a		8	
2b			6b		12	
2c			6c		12	
2d			6d			
3a		16	7a		12	
3b		12	7b		8	
3c		12	7c			
3d			7d		8	
4a			8a			
4b			8b			
4c			8c			
4d			8d			
Total			167/250			

Section A

1(a) If V is a finite-dimensional vector space over F, then any two bases of V have the same number of elements. (10)

V is finite dimensional

~~let dim V = n~~

let $B_1 = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basic

let $B_2 = \{\beta_1, \beta_2 \ldots \beta_m\}$ be another basic

we need to show $n = m$

B_1 is L.I set

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \text{---} \ a_n\alpha_n = 0$.

$\alpha_1 = b_{11}\beta_1 + b_{12}\beta_2 + \text{---} + b_{1m}\beta_m$

$\vdots$

$\alpha_n = b_{n1}\beta_1 + b_{n2}\beta_2 + \ldots + b_{nm}\beta_m$

$\Rightarrow a_1[b_{11}\beta_1 + b_{12}\beta_2 + \ldots + b_{1m}\beta_m] + \ldots$

$+ \ldots [a_n][b_{n1}\beta_1 + b_{n2}\beta_2 + \ldots + b_{nm}\beta_m] = 0$

~~$\Rightarrow \beta_1[a_1b_{11} + a_1b_{12} + a_1b_{13} + \ldots + a_1 b_{1m}] +$~~

~~$\beta_2[a_2b_{21} + a_2b_{22} + \text{---} + a_2b_{2m}] + \ldots$~~ $+$

~~$\beta_m[a_n b_{n1} + a_n b_{n2} + \ldots + a_n b_{nm}] = 0$~~

$\therefore \beta_1[a_1b_{11} + a_2b_{21} + a_3b_{31} + \text{---} + a_nb_{n1}] +$

$\beta_2[a_1b_{12} + a_2b_{22} + a_3b_{32} + \text{---} + a_nb_{n2}] +$

$\vdots$

$\beta_m[a_1b_{1n} + a_2b_{2n} + \text{---} + a_nb_{nm}] = 0$

$\therefore a_1b_{11} + a_2b_{21} + a_3b_{31} + \text{---} + b_{n1}a_n = 0$

$a_1b_{12} + a_2b_{22} + \text{---} + a_nb_{n2} = 0$

$\vdots$

$a_1b_{1m} + a_2b_{2m} + \text{---} + a_mb_{nm} = 0$

n variable, m equation

if $n > m$

$\Rightarrow$ no. of equation are less then variable.

$\Rightarrow$ system will have a non zero solution

$\Rightarrow \exists\, a_i$ such that $a_i\alpha_i = 0 \Rightarrow \{\alpha_i\}$ are not basic $\Rightarrow$ contradiction

$\therefore m = n \Rightarrow$ Basic have same element.

8

1(b) Show that the system of equations

$$3x + 4y + 5z = \alpha$$
$$4x + 5y + 6z = \beta$$
$$5x + 6y + 7z = \gamma$$

is consistent only if α, β and λ are in arithmetic progression (A.P.) (10)

$$3x + 4y + 5z = \alpha$$
$$4x + 5y + 6z = \beta$$
$$5x + 6y + 7z = \gamma.$$

Write in $AX=B$ form

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{vmatrix} = 0.$$

$\therefore$ system will be consistent

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right]$$ has same rank.

$$\Rightarrow \sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta-\alpha \\ 1 & 1 & 1 & \gamma-\beta \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \end{array}$$

$$\Rightarrow \sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta-\alpha \\ 0 & 0 & 0 & \gamma-\beta-\beta+\alpha \end{array}\right]$$

For rank to be 2

$\gamma - \beta - (\beta - \alpha) = 0$

$\Rightarrow$ $\cancel{\gamma+\alpha=0}$ $(\gamma-\beta) = (\beta-\alpha)$ —— ①

$\Rightarrow$ $\boxed{\alpha, \beta, \gamma \text{ are in AP}}$

8

1(c) Verify Rolle's theorem for the function

$f(x) = \log\left\{\frac{x^2+ab}{x(a+b)}\right\}$ on $[a, b]$, where $0 < a < b$ (10)

$f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = \log\left(\frac{a^2+ab}{a^2+ab}\right) = 0$

$f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right) = 0.$

as per Rolle's theorem

$c \in (a, b)$

Such that

$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$

$\Rightarrow f'(c) = 0$

$f(x) = \log(x^2+ab) - \log(x(a+b))$

$f'(x) = \frac{1 \times 2x}{x^2+ab} - \frac{(a+b)}{x(a+b)} = 0$

$\Rightarrow \frac{2x}{x^2+ab} = \frac{a+b}{x(a+b)}$

$\Rightarrow 2x^2a + 2x^2b = x^2a + a^2b + bx^2 + ab^2$

$\Rightarrow x^2a + x^2b = a^2b + ab^2$

$\Rightarrow x^2 = \frac{a^2b + ab^2}{a+b} = \frac{ab(a+b)}{a+b} = ab$

$\Rightarrow x = \sqrt{ab}$ as $0 < a < b$

~~we know~~

~~AM > GM~~

~~$\Rightarrow \frac{a+b}{2} > \sqrt{ab}$~~

$0 < a < b$

$\therefore \sqrt{ab} > a.$

$\sqrt{ab} < b$

$\therefore a < \sqrt{ab} < b$

here $c = \sqrt{ab}$

$\boxed{f'(c) = 0}$

and $c \in (a, b)$

Rolles theorem verified

8

1(d) Evaluate: $\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)$ (10)

$$\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)$$

$$= \cancel{\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{r}{n}\left(\frac{1}{\sqrt{1+\left(\frac{r}{n}\right)^2}}\right)}$$

$$\cancel{\lim_{n\to\infty} n\sum_{r=1}^{2n}\frac{r}{2n}\left(\frac{1}{\sqrt{1+(r/n)^2}}\right)}$$

$$= \cancel{\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)+\lim_{n\to\infty}\sum_{r=n+1}^{2n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)}$$

~~in Second Part change~~ $t = r-n$

$$= \cancel{\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{\sqrt{n^2+r^2}}+\lim_{n\to\infty}\sum_{r=1}}$$

$$= \lim_{n\to\infty} 2n\times\sum_{r=1}^{2n}\frac{1}{2n}\left(\frac{r/n}{\sqrt{1+(r/n)^2}}\right)$$

Since $\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\left(\frac{r/n}{\sqrt{1+(r/n)^2}}\right)$ is finite

$\therefore$ lim

$\frac{r}{n}\to x$

$\int_0^2 f\left(\frac{r}{n}\right)$

$\int_0^2 \frac{x}{\sqrt{1+x^2}}$

1(e) Find the equation of the plane passing through the points $(1,-1,1)$ and $(-2,1,-1)$ and perpendicular to the plane $2x+y+z+5=0$. (10)

Plane passing through
$(1,-1,1)$ & $(-2,1,-1)$
let (l, m, n) are d.r of the plane

$\Rightarrow l(1+2)+m(-1-1)+n(1+1)=0$
$3l-2m+2n=0$ —— (1)

Plane is ⊥r to $2x+y+z+5=0$
$\therefore 2l+m+n=0$ —— (2)
& $3l-2m+2n=0$ (from (1))

$\Rightarrow \frac{l}{2+2} = \frac{-m}{4-3} = \frac{n}{-4-3}$.
$\Rightarrow \frac{l}{4} = \frac{m}{-1} = \frac{n}{-7}$

$\therefore$ Plane

$\cancel{4(x-1)+(-1)(y+1)+7(z-1)=0}$
$\cancel{4x-y+7z-4-1-7=0}$
$\cancel{4x-y+7z-12=0}$

$4(x-1)-1(y+1)-7(z-1)=0$
$\Rightarrow 4x-y-7z-4-1+7=0$
$\Rightarrow \boxed{4x-y-7z+2=0}$
is required plane

5

3(a) Let $\mathcal{D} = \frac{d}{dx}$ be the differential operator on the space of differentiable functions in x on $\mathbb{R}$. Let

$$U = \text{Span}\{1, x, x^2, x^3\},$$
$$V = \text{Span}\{1, \sin x, \cos x, e^x\},$$
$$W = \text{Span}\{1, e^x, e^{2x}, e^{3x}\}.$$

Find the matrix representation relative to the basis. (20)

For Basic U :-

$D(1) = 0 = 0\times 1 + 0x + 0x^2 + 0x^3$

$D(x) = 1 = 1\cdot 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0x^2 + 0x^3$

$D(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0x^3$

$\therefore$ ~~Matri~~ Matrix representation for U

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For Basic V :$\rightarrow$

$D(1) = 0 = 0\cdot 1 + 0\cdot \sin x + 0\cdot \cos x + 0\cdot e^x$

$D(\sin x) = \cos x = 0\cdot 1 + 0\cdot \sin x + 1\cdot \cos x + 0\cdot e^x$

$D(\cos x) = -\sin x = 0\cdot 1 + (-1)\sin x + 0\cdot \cos x + 0\cdot e^x$

$D(e^x) = e^x = 0\cdot 1 + 0\cdot \sin x + 0\cdot \cos x + 1\cdot e^x$

Matric for V

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For ~~matri~~ Basic W

$d(1) = 0 = 0\cdot 1 + 0\cdot e^{x} + 0\cdot e^{2x} + 0\cdot e^{3x}$

$d(e^{x}) = e^{x} = 0\cdot 1 + 1\cdot e^{x} + 0\cdot e^{2x} + 0\cdot e^{3x}$

$d(e^{2x}) = 2e^{2x} = 0\cdot 1 + 0\cdot e^{x} + 2\cdot e^{2x} + 0\cdot e^{3x}$

$d(e^{3x}) = 3e^{3x} = 0\cdot 1 + 0\cdot e^{x} + 0\cdot e^{2x} + 3e^{3x}$

Matrix representation for W

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

16

3(b) Find the points of extremum of the function $f(x)$ for which

$f'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4$ (15)

$f'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4$

$f'(x) = 0$ at $x = 1, 2, 3, 4$

Stationary points are $x=1, x=2, x=3, x=4$

$f'(x)$: $+$ | 1 | $-$ | 2 | $-$ | 3 | $+$ | 4 | $+$

$\therefore f'(x) > 0 \quad x < 1$

$f'(x) < 0 \quad 1 < x < 3$

$f'(x) > 0 \quad 3 < x < 4$

$f'(x) > 0 \quad x > 4$

$\therefore$ local minimal at $x = 3$

local maxima at $x = 1$

and inflection point at $x = 2, x = 4$

$\therefore$ point of extremum are $x = 1, x = 3$

12

3(c) Find the equations of the tangent planes to the ellipsoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

which pass through the line $x - y - z = 0 = x - y + 2z - 9$ (15)

Any Plane through $x-y-z=0$
$x-y+2z-9=0$

$\Rightarrow x-y+2z-9+\lambda(x-y-z)=0$

$x(1+\lambda)+y(-1-\lambda)+z(2-\lambda)-9=0$ —— (1)

ellipsoid $\frac{2}{27}x^2+\frac{6}{27}y^2+\frac{3}{27}z^2=1$

condition for tangency for $ax^2+by^2+cz^2=1$ & $lx+my+nz=p$

is $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

$\therefore (1+\lambda)^2\times\frac{27}{2}+(-1-\lambda)^2\left(\frac{27}{6}\right)+\frac{27}{3}(2-\lambda)^2=9\times9$

$\Rightarrow \frac{(1+\lambda)^2}{2}+\frac{(1+\lambda)^2}{6}+\frac{(2-\lambda)^2}{3}=3$

$\Rightarrow \frac{2}{3}(1+\lambda)^2+(2-\lambda)^2=3 \Rightarrow 2(1+\lambda)^2+(2-\lambda)^2=9$

$\Rightarrow 2+\cancel{4\lambda}+2\lambda^2+4-\cancel{4\lambda}+\lambda^2=9$

$\Rightarrow 3\lambda^2=3$

$\Rightarrow \lambda^2=1$

$\Rightarrow \boxed{\lambda=\pm1}$

$\therefore$ (1) becomes

$\lambda=1$

$\Rightarrow \boxed{2x-2y+z=9}$

$\lambda=-1$

$\Rightarrow 0+0+3z=9$

$\boxed{z=3}$

$\therefore$ required planes are

$\boxed{\begin{array}{c}2x-2y+z=9\\ \text{and } z=3\end{array}}$

12

SECTION B

5(a) Find an integrating factor and solve the differential equation

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ (10)

$M = y^4 + 2y$

$N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2.$

$\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y^3 - 4 - 4y^3 - 2 = -3y^3 - 6 = -3[y^3+2]$

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$

$\therefore$ Integrating factor $= e^{\int -\frac{3}{y} dy} = e^{-\log y^3}$

$= \frac{1}{y^3}$

$\therefore \frac{y^4+2y}{y^3} dx + \frac{(xy^3 + 2y^4 - 4x)}{y^3} dy = 0$

$\Rightarrow \left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$

$\Rightarrow y\,dx + x\,dy + 2y\,dy + \frac{2}{y^2}dx - \frac{4x}{y^3}dy = 0$

$\Rightarrow d(xy) + d(y^2) + d\left(\frac{2x}{y^2}\right) = 0$

Integrating

$\Rightarrow \boxed{xy + y^2 + \frac{2x}{y^2} = c}$

8

5(b) Form the differential equation corresponding to $Ax^2 + By^2 = 1$ by eliminating A and B . (10)

$$Ax^2 + By^2 = 1$$

$$\boxed{\cancel{x^2 + \frac{B}{A}y^2 = \frac{1}{A}}}$$

differentiating w.r.t x

$$2Ax + 2Byp = 0 \quad \text{--- (I)}$$

again w.r.t x differentiation

$$2A + 2B\left[p.p + y\frac{d^2y}{dx^2}\right] = 0 \quad \text{--- (II)}$$

From (I) & (II)

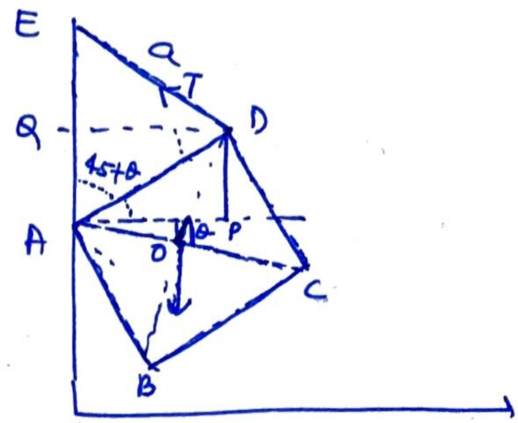
$$\begin{vmatrix} x & yp \\ 1 & p\left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right) \end{vmatrix} = 0$$

$$\Rightarrow x\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] - y\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0}$$

8

5(c) A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable. (10)



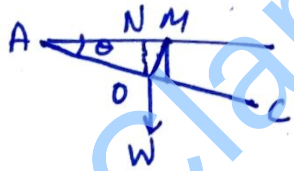
At equilibrium,

Let say AC makes angle θ with horizontal

$AO = \frac{a}{\sqrt{2}}$

$ON = \frac{a}{\sqrt{2}}\cos\theta$.

$AE = 2 \times a\cos(45+\theta)$



∴ Principle of virtual work

$W\delta(ON+AE)=0$

$\Rightarrow W\delta\left(\frac{a}{\sqrt{2}}\cos\theta + 2a\cos(45+\theta)\right)=0$

$\Rightarrow \delta\left(\frac{\cos\theta}{\sqrt{2}} + 2\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\right)=0$

$\Rightarrow -\frac{\sin\theta}{\sqrt{2}} + \sqrt{2}\{-\sin\theta \pm \cos\theta\}=0$

$\Rightarrow \sin\theta\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right) = \sqrt{2}\cos\theta$.

$\sin\theta \cancel{\times}\frac{(1+2)}{\sqrt{2}} = \sqrt{2}\cos\theta$

$\Rightarrow \sin\theta(3) = 2\cos\theta$

$\boxed{\tan\theta = 2/3}$

1/3

Deapth of com below E $= \frac{a}{\sqrt{2}}\cos\theta + \sqrt{2}(\cos\theta - \sin\theta)$

$\frac{dh}{d\theta} = -\frac{a}{\sqrt{2}}\sin\theta + \sqrt{2}(-\sin\theta - \cos\theta)$

$\frac{d^2h}{d\theta^2} = -\frac{\cos\theta}{\sqrt{2}} + \sqrt{2}(-\cos\theta + \sin\theta)$

$= -\cos\theta\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right) + \sqrt{2}\sin\theta$

$= \frac{1}{\sqrt{2}}\left[-3\cos\theta + 2\sin\theta\right]$

When $\tan\theta = \frac{2}{3}$

$\sin\theta = \frac{2}{\sqrt{13}}$

$\cos\theta = \frac{3}{\sqrt{13}}$

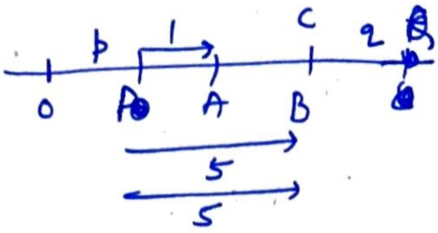
$\Rightarrow \frac{d^2h}{d\theta^2} < 0$

∴ $\boxed{\text{Equilibrium is stable}}$

6

5(d) At the ends of three successive seconds the distances of a point moving with S.H.M. from its mean position measured in the same direction are 1,5 and 5. Show that the period of a complete oscillation is $\frac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$. (10)

Θ at end of 3 successive second particle be at A, B, C



$OA = 1$
$OB = 5$
$OC = 5 \Rightarrow$ B, C are at same location

in SHM with amplitude a

$$\boxed{v^2 = \mu(a^2 - x^2)}$$

$$\frac{dx}{dt} = +\sqrt{\mu}\sqrt{a^2 - x^2}$$

$$\Rightarrow \frac{dx}{\sqrt{a^2-x^2}} = +\sqrt{\mu}\, t \Rightarrow \sin^{-1}\left(\frac{x}{a}\right) = +\sqrt{\mu}\, t + \phi$$

$$\boxed{\frac{x}{a} = \sin(\phi + \omega t)}$$

$$\sin^{-1}(x/a) = \omega t$$

From A → B

$$\Rightarrow \sin^{-1}\left(\frac{x}{a}\right)\Big|_1^5 = \omega(1)$$

$$\Rightarrow \sin^{-1}\left(\frac{5}{a}\right) - \sin^{-1}\left(\frac{1}{a}\right) = \omega \quad \text{— (1)}$$

From A to Q $\Rightarrow \sin^{-1}\left(\frac{x}{a}\right)\Big|_1^a = \omega \times \left(\frac{3}{2}\right)$

$$\Rightarrow \sin^{-1}\left(\frac{a}{1}\right) - \sin^{-1}\frac{1}{a} = \omega \times 3/2$$

$$\Rightarrow \sin^{-1}\frac{1}{a} = \pi/2 - \omega\, 3/2 \Rightarrow$$

$$\frac{1}{a} = \cos\left(3/2\, \omega\right) \quad \text{— (2)}$$

From (1) $\sin^{-1}\left(\frac{5}{a}\right) = \omega + \frac{\pi}{2} - \omega\, 3/2 = \frac{\pi}{2} - \frac{\omega}{2}$

$$\frac{5}{a} = \cos(\omega/2) \quad \text{— (3)}$$

$(3)/(2) \Rightarrow 5 = \dfrac{\cos \omega/2}{\cos 3\omega/2} \Rightarrow 5\left(4\cos^3\frac{\omega}{2} - 3\cos\frac{\omega}{2}\right) = \cos\frac{\omega}{2}$

$$\Rightarrow 20\cos^2\frac{\omega}{2} = 16$$

$$\cos^2\frac{\omega}{2} = \frac{4}{5}$$

$$1 + \cos\omega = 8/5 \Rightarrow \cos\omega = \frac{3}{5}$$

$$\omega = \cos^{-1}(3/5)$$

$$\therefore \boxed{T = \frac{2\pi}{\omega} = \frac{2\pi}{\cos^{-1}(3/5)}}$$

8

5(e) Find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change if the temperature at any point is given by $\phi(x, y, z) = xy + yz + zx.$ (10)

$\nabla\phi(x,y,z) = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

Temperature will change most rapidly in direction where $\nabla\phi$ will be maximum

$\nabla\phi(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{i}$

Maximum temperature change in $(\hat{i}+\hat{j}+\hat{k})$ direction

and magnitude of maximum rate of change $= \sqrt{4+4+4} = \sqrt{12}$

$= 4\sqrt{3}$

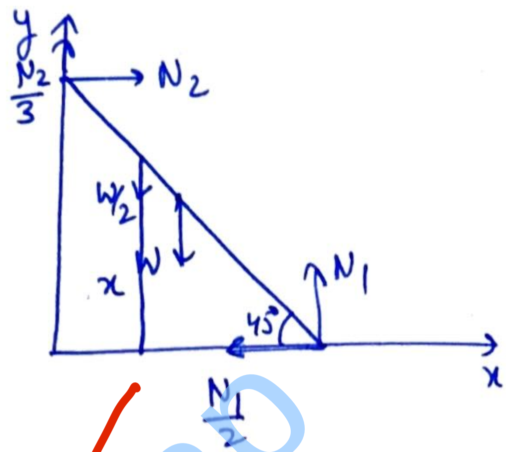
8

6(a) A uniform ladder of length 70 metres rests against a rough vertical wall with which it makes an angle 45°, the co-efficient of friction between the ladder and the wall is $\frac{1}{3}$ and between the ladder and the ground is $\frac{1}{2}$. If a man whose weight is one half that of the ladder ascends the ladder, how high will he be when the ladder slips? (20)

let length of ladder

$2L = 70$ m

let person climb to height x



in x direction

$$N_2 = \frac{N_1}{2}$$

in Y direction

$$\frac{N_2}{3} + N_1 = \frac{3W}{2}.$$

$$\Rightarrow \frac{N_2}{3} + 2N_2 = \frac{3W}{2}$$

$$\Rightarrow 7N_2 = 3W \text{ — } (1).$$

Moment of force about Bottom point

$$\Rightarrow N_2 \times \frac{2l}{\sqrt{2}} + \frac{N_2}{3} \times \frac{2l}{\sqrt{2}} = W \times \frac{l}{\sqrt{2}} + \frac{W}{2} \times x$$

$$\Rightarrow \frac{3\cancel{W}}{7}\left[\frac{2l}{\sqrt{2}} + \frac{2l}{3\sqrt{2}}\right] = \cancel{W}\left[\frac{l}{\sqrt{2}} + \frac{x}{2}\right]$$

$$\Rightarrow \frac{3 \times 2l}{7\sqrt{2}}\left[\frac{4}{3}\right] = \frac{l}{\sqrt{2}} + \frac{x}{2}.$$

$l = 35$

$$\Rightarrow \frac{\cancel{3} \times \cancel{70}^{10}}{\cancel{7} \times \sqrt{2}} \times \frac{4}{\cancel{3}} = \frac{35}{\sqrt{2}} + \frac{x}{2}$$

$$\frac{40}{\sqrt{2}} - \frac{35}{\sqrt{2}} = \frac{x}{2}$$

$$\Rightarrow \boxed{x = \frac{10}{\sqrt{2}}}$$

∴ Man will be able to climb verticle height $\frac{10}{\sqrt{2}}$ or 10 m along the ladder length

8

6(b) Under what conditions for the constants A, B, C, D is the following equation exact?

$$(dx + By)dx + (Cx + Dy)dy = 0$$

Solve the exact equation. (15)

$(Ax+By)dx + (Cx+Dy)\,dy = 0$

$M = Ax + By$

$N = Cx + Dy$.

$\frac{\partial M}{\partial y} = B$

$\frac{\partial N}{\partial x} = C$

exact when $\boxed{B = C}$

Since A, B, C, D are constant

$\Rightarrow$ $Ax + By$ & $Cx + Dy$ are hmogenous.

$(Ax + By)dx + (Cx + Dy)dy = 0$ (using $B = C$)

$\Rightarrow$ $Ax\,dx + By\,dx + Bx\,dy + Dy\,dy = 0$

$\Rightarrow$ $\boxed{A\frac{x^2}{2} + Bxy + D\frac{y^2}{2} = \text{constant}}$

12

6(c) If ϕ and ψ are two scalar point functions, show that

$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi.$ (15)

$\nabla^2(\phi\psi)$

$= \nabla(\nabla\phi\psi) = \nabla(\psi\nabla\phi + \phi\nabla\psi)$

we know

$\nabla(\phi\cdot\vec{A}) = \phi(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla\phi)$

$\nabla^2(\phi\psi) = \nabla(\psi(\nabla\phi)) + \nabla(\phi(\nabla\psi))$

$= \psi\nabla\cdot(\nabla\phi) + \nabla\phi\cdot\nabla\psi + \nabla\phi\cdot\nabla\psi + \phi\nabla\cdot(\nabla\psi)$

$= \psi\nabla^2\phi + 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$

$\boxed{\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi}$

12

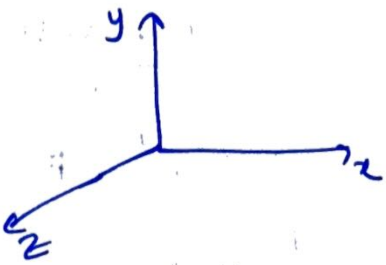
7(a) Verify Stokes' theorem for $\bar{F} = 4y\hat{\imath} - 4x\hat{\jmath} + 3\hat{k}$, where S is a disk of 1-unit radius lying on the plane $z = 1$ and C is its boundary. (15)

Strokes theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds$$

$C:\quad x^2+y^2=1,\ z=1$

$\hat{n} = \hat{k}$.



$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & -4x & 3 \end{vmatrix}$$

$$= \hat{i}(0+0) - \hat{j}(0+0) + \hat{k}(-4-4) = -8\hat{k}$$

$$\iint_S (\nabla\times\vec{F})\cdot n\, ds = \iint_{x^2+y^2=1} -8\hat{k}\cdot\hat{k}\cdot dx\,dy$$

$$= -8\times\pi\times 1 = -8\pi \quad \text{—— (1)}$$

$$\oint \vec{F}\cdot d\vec{r} = \int_C 4y\,dx - 4x\,dy + 3dz$$

since $z=1, \Rightarrow dz=0$

$$= \int 4y\,dx - 4x\,dy$$

put $x=\cos\theta,\ y=\sin\theta$

$$= \int_0^{2\pi} 4\sin\theta(-\sin\theta\,d\theta) - 4\cos\theta\cos\theta\,d\theta$$

$$= -4\int_0^{2\pi} d\theta = -4\times 2\pi = -8\pi \quad \text{—— (2)}$$

from (1) & (2)

$$\boxed{\iint_S (\nabla\times\vec{F})\cdot\hat{n}\cdot ds = \oint_C \vec{F}\cdot d\vec{r}}$$

12

7(b) Solve the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ (10)

$$(x + 2y^3)\frac{dy}{dx} = y$$

$$\Rightarrow (x+2y^3)dy - y\,dx = 0$$

$$\Rightarrow x\,dy - y\,dx + 2y^3 dy = 0$$

$$\Rightarrow \frac{x\,dy - y\,dx}{y^2} + 2y\,dy = 0.$$

$$\Rightarrow -d\left(\frac{x}{y}\right) \mp 2y\,dy = 0$$

$$\boxed{\Rightarrow -\frac{x}{y} + y^2 = c_1}$$

$$\Rightarrow \boxed{y^2 - \frac{x}{y} = c_1}$$

8

7(c) A spherical raindrop, falling freely, receives in each instant an increase of volume equal to λ times its surface at that instant; find the velocity at the end of time t, and the distance fallen through in that time. (15)

$$\frac{dv}{dt} = \lambda \cdot 4\pi r^2$$ where r is radius

$$\Rightarrow \frac{4}{3}\pi \frac{dr^3}{dt} = \lambda 4\pi r^2$$

$$\Rightarrow \frac{3r^2}{3}\frac{dr}{dt} = \lambda r^2$$

$$\boxed{\frac{dr}{dt} = \lambda}$$

at any time

$$mg = \frac{d^2x}{dt^2} \times m.$$

7(d) Solve using Laplace transforms, $\frac{d^2y}{dt^2} + y = t$ Given $y(0) = 1$ and $y''(0) = -2$. (10)

$\frac{d^2y}{dt^2} + y = t$ $\boxed{y(0) = 1,\ y'(0) = -2}$

Laplace both side $L(y) = Y$

$\Rightarrow p^2Y - py(0) - y'(0) + Y = \frac{1}{p^2}$ $\left\{L(t) = \frac{1}{p^2}\right\}$

$\Rightarrow p^2Y - p + 2Y = \frac{1}{p^2}$

$\Rightarrow \cancel{(p^2+2)Y = \frac{1}{p^2} + p}$

$(p^2+1)Y = \frac{1}{p^2} + p - 2.$

$Y = \frac{1}{p^2(1+p^2)} + \frac{p}{(1+p^2)} - \frac{2}{p^2+1}$

$= \frac{-1}{1+p^2} + \frac{1}{p^2} + \frac{p}{1+p^2} - \frac{2}{p^2+1}$

$Y = \frac{1}{p^2} - \frac{3}{1+p^2} + \frac{p}{1+p^2}$

$\boxed{y(t) = t - 3\sin(t) + \cos(t)}$

~~y'(0) = 1 - 3\cos t~~
~~= 1 - 3~~

9