Test Copy of Mr Shivam Kumar ,AIR 19 ,CSE 2023

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TEST SERIES FOR UPSC MATHEMATICS MAINS EXAM 2023

FULL LENGTH TEST -9 PAPER 1

Name of the Candidate	Shivam KUMAR		
Email ID		Roll No	
Phone Number		Date	30th August
Start Time :	12:00 PM	Closing Time:	3:00 PM

Index Table						Remarks
Section A			Section B			
Q.No	Max Marks	Marks Obtained	Q.No	Max Marks	Marks Obtained	
1a		8	5a		8	
1b		8	5b		8	
1c		8	5c		8	
1d		8	5d			
1e		8	5e		8	
2a		16	6a			
2b		5	6b			
2c		12	6c			
2d			6d			
3a			7a			
3b			7b			
3c			7c			
3d			7d			
4a		16	8a		12	
4b		12	8b		12	
4c		12	8c		16	
4d			8d			
Total						

185/250

Section A

1(a) Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}2\\3\end{bmatrix}$ and let $T: R^2 \to P_2$ be the linear transformation such that $T(\mathbf{v}_1) = 2 - 3x + x^2$ and $T(\mathbf{v}_2) = 1 - x^2$. Find $T\begin{bmatrix}a\\b\end{bmatrix}$ and then find $T\begin{bmatrix}-1\\2\end{bmatrix}$. (10)

let

$$\begin{bmatrix}a\\b\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}2\\3\end{bmatrix}$$

$$a = x + 2y.$$
$$b = x + 3y$$

$$\Rightarrow b - a = y$$

$$x = a - 2y = a - 2(b-a)$$
$$= a - 2b + 2a = 3a - 2b$$

$$x = 3a - 2b$$
$$y = b - a.$$

8

$$\therefore \begin{bmatrix}a\\b\end{bmatrix} = (3a-2b)\begin{bmatrix}1\\1\end{bmatrix} + (b-a)\begin{bmatrix}2\\3\end{bmatrix}$$

$$\therefore T\begin{bmatrix}a\\b\end{bmatrix} = (3a-2b)T\begin{bmatrix}1\\1\end{bmatrix} + (b-a)T\begin{pmatrix}2\\3\end{pmatrix}$$

$$= (3a-2b)(2-3x+x^2) + (b-a)(1-x^2)$$

$$= x^2(3a-2b-b+a) + x(6b-9a) + 6a-4b+b-a$$

$$\boxed{T\begin{pmatrix}a\\b\end{pmatrix} = x^2(4a-3b) + x(6b-9a) + \cancel{7a}-5b}$$ 5a-3b

$$\therefore T\begin{pmatrix}-1\\2\end{pmatrix} = x^2(-4-6) + x(6\times2-9\times-1) + 7(-1) - \cancel{5\times2}$$

$$= x^2(-10) + 21x - 17$$

$$\boxed{T\begin{pmatrix}-1\\2\end{pmatrix} = -17 + 21x - 10x^2}$$

-11

1(b) Show that the matrices $\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ are similar but that $\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ are not (10)

$A = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$|A| = 4+1 = 5$

$|B| = 6-1 = 5$

Since, A & B are non singular

eigen value of A $\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda)+1 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 4 + 1 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 5 = 0$.

eigen value of B $\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) - 1 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6 - 1 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 5 = 0$

$\therefore$ eigen values of A & B are same.

$\therefore$ $\boxed{\text{A is similar to B}}$

$C = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$, $|C| = -6+6 = 0$

$D = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$, $|D| = 0-2 = -2$

C is singular & D is non singular

$\Rightarrow$ $\boxed{\text{C cannot be similar to D}}$

8

1(c) Prove that $\frac{\pi^3}{24} \le \int_0^\pi \frac{x^2}{5+3\cos x}dx \le \frac{\pi^3}{6}$ (10)

in $[0, \pi]$

$-1 \le \cos x \le 1$

$\Rightarrow 5-3 \le 5+3\cos x \le 5+3$

$\Rightarrow 2 \le 5+3\cos x \le 8$

$\Rightarrow \frac{1}{8} \le \frac{1}{5+3\cos x} \le \frac{1}{2}$

$\Rightarrow \frac{x^2}{8} \le \frac{x^2}{5+3\cos x} \le \frac{x^2}{2} \quad (\because x^2 > 0)$

$\Rightarrow \int_0^\pi \frac{x^2}{8}dx \le \int_0^\pi \frac{x^2}{5+3\cos x}dx \le \int_0^\pi \frac{x^2}{2}dx$

$\Rightarrow \left[\frac{x^3}{24}\right]_0^\pi \le \int_0^\pi \frac{x^2}{5+3\cos x}dx \le \left[\frac{x^3}{6}\right]_0^\pi$

$\Rightarrow \boxed{\frac{\pi^3}{24} \le \int_0^\pi \frac{x^2}{5+3\cos x}dx \le \frac{\pi^2}{6}}$

Write By MVT

8

1(d) Find the value of a, b and c such that

$$\lim_{x\to 0} \frac{axe^{x} - b\log(1+x) + cxe^{-x}}{x^2 \sin x} = 2$$ (10)

$$\lim_{x\to 0} \frac{axe^{x} - b\log(1+x) + cxe^{-x}}{x^2 \sin x}$$ is in $\frac{0}{0}$ form.

~~apply L'Hôpital rule~~

$$\lim_{x\to 0} \frac{ax\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right] - b\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right] + cx\left[1-x+\frac{x^2}{2!}-\frac{x^3}{6}+\cdots\right]}{x^2\left[x-\frac{x^3}{3!}+\cdots\right]}$$

$$= \lim_{x\to 0} \frac{x[a-b+c] + x^2\left[a+\frac{b}{2}-c\right] + x^3\left[\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right] + x^4[\cdots]}{x^3\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right]}$$

For limit to exist

$a-b+c=0$ — (1), $a+\frac{b}{2}-c=0$ — (2)

and since limit $=2$

$\Rightarrow \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$ — (3)

solving (1), (2) & (3)

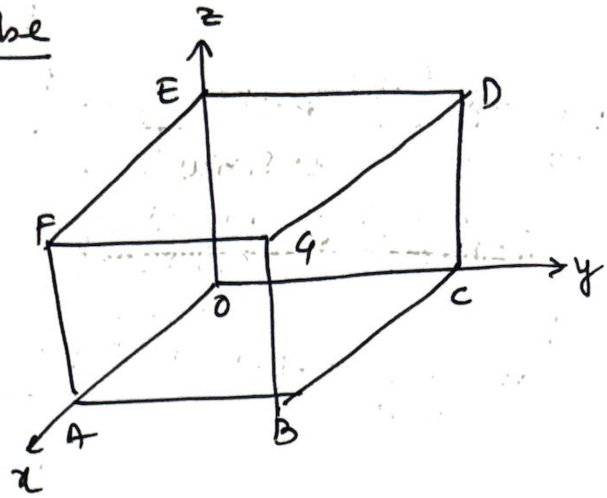
$$\boxed{\begin{aligned} a &= 3 \\ b &= 12 \\ c &= 9 \end{aligned}}$$

8

1(e) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$. (10)

Let OABCDEFG is a cube

O(0,0,0)
A(a,0,0)
B(a,a,0)
C(0,a,0)
D(0,a,a)
E(0,0,a)
F(a,0,a)
G(a,a,a)



Diagonals

OG, AD, EB, CF

D.r of OG = (a, a, a)
D.r of AD = (a, −a, −a)
Dr. of BE = (a, a, −a)
Dr. of CF = (−a, a, −a)

Let line has D.c of (l, m, n)

$\therefore \cos\alpha = \dfrac{la+ma+na}{\sqrt{a^2+a^2+a^2}} = \dfrac{la+ma+na}{\sqrt{3}a}$

$\cos\beta = \dfrac{la-ma-na}{\sqrt{3}a}$

$\cos\gamma = \dfrac{la+ma-na}{\sqrt{3}a}$

$\cos\delta = \dfrac{-la+ma-na}{\sqrt{3}a}$

$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \dfrac{(l+m+n)^2+(l-m-n)^2+(l+m-n)^2+(-l+m-n)^2}{3}$

$= \dfrac{1}{3}(4l^2+4m^2+4n^2+2lm+2mn+2ln+2mn-2lm-2ln+2lm-2ln-2mn-2mn-2lm+2lm)$

$= \dfrac{4}{3}(l^2+m^2+n^2) = \dfrac{4}{3}$

$\therefore \boxed{\cos^2\alpha+\cos^2\beta+\cos^2\delta+\cos^2\gamma = \frac{4}{3}}$

8

2(a) Let T be a multiplication by the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}$$

(i) Find a basis for the range of T.
(ii) Find a basis for the kernel of T.
(iii) Find the rank and nullity of A.
(iv) Find the rank and nullity of T.
(v) Verify the dimension theorem.

(20)

(i) Given $T[x] = AX$

$$\text{Range of } T = \begin{bmatrix} A \end{bmatrix}[x] = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x+2y+5z \\ 3x+5y+13z \\ -2x-y-4z \end{bmatrix}$$

$$\text{Range } T = L\{(x+2y+5z), (3x+5y+13z), (-2x-y-4z)\}$$

$$= \{x(1,3,-2) + y(2,5,-1) + z(5,13,-4)\}$$

$$\text{Range } T = L\{(1,3,-2), (2,5,-1), (5,13,-4)\}$$

consider Row matrix of $A(T)$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -1 \\ 5 & 13 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1 - 2R_2} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $R(T) = \{(1,3,-2), (0,-1,3)\}$

(ii) For kernel of T

$$T(X) = 0 \Rightarrow AX = 0$$

$$\Rightarrow \begin{bmatrix} x+2y+5z \\ 3x+5y+13z \\ -2x-y-4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ matrix form

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 3 & 5 & 13 & 0 \\ -2 & -1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 3 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad x + 2y + 5z = 0$

$-y - 2z = 0$

Let $z = k \Rightarrow y = -2z$
$= -2k$

$x = -2y - 5z = -2(-2k) - 5(k)$
$= 4k - 5k = -k$

$\therefore (x, y, z) = (-k, -2k, k) = k(-1, -2, 1)$

$\therefore \boxed{\text{Nullspace} = L\{(-1, 2, 1)\}}$

Basis of nullspace $= (-1, -2, 1)$

(iii) ~~Dimension of~~ Rank of A = Rank of T = 2

Rank of T = 2

Nullity of A = 1

Nullity of T = 1

$\therefore$ (v) Dimention theorem

Dim (u) = Rank + nullity

here Dim (u) = 3 as $u = R^3$

Rank + nullity = 2 + 1 = 3

$\therefore \boxed{\text{Dim } U = \text{Rank} + \text{nullity}}$

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2(b) Determine the extreme values of the function
$f(x,y) = 3x^2 - 6x + 2y^2 - 4y$ in the region $\{(x,y) \in \mathbb{R}^2 : 3x^2 + 2y^2 \le 20\}$. (15)

UPSC asked similar earlier

$f(x,y) = 3x^2 - 6x + 2y^2 - 4y$

$\frac{\partial F}{\partial x} = 6x - 6$ & $\frac{\partial^2 F}{\partial x^2} = 6$

$\frac{\partial F}{\partial y} = 4y - 4$ & $\frac{\partial^2 F}{\partial y^2} = 4$

$\frac{\partial^2 F}{\partial x \partial y} = 0$.

$\frac{\partial F}{\partial x} = 0 \Rightarrow x = 1$

$\frac{\partial F}{\partial y} = 0 \Rightarrow y = 1$

$(x,y) = (1,1)$ lies in $(3x^2 + 2y^2 < 20)$

here $\frac{\partial^2 F}{\partial x^2} > 0$

& $\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = 6 \times 4 - 0 > 0$

$\boxed{\therefore (1,1) \text{ is point of minima.}}$

minimum value $= 3\times 1 - 6\times 1 + 2\times 1 - 4$
$= 3 - 6 + 2 - 4 = -5$

$\boxed{\text{minimum value} = -5}$

5

On $3x^2 + 2y^2 = 20$

consider Lagrange's multiplier

$F(x,y) = 3x^2 - 6x + 2y^2 - 4y + \lambda(3x^2 + 2y^2 - 20)$

$\frac{\partial F}{\partial x} = 6x - 6 + 6\lambda(x) = 0 \Rightarrow x - 1 + \lambda x = 0$

$\frac{\partial F}{\partial y} = 4y - 4 + 4\lambda y = 0 \Rightarrow y - 1 + \lambda y = 0$

$\Rightarrow x = \frac{1}{1+\lambda}$

$y = \frac{1}{1+\lambda}$

$\because 3x^2 + 2y^2 = 20$

$\Rightarrow 3\left(\frac{1}{1+\lambda}\right)^2 + 2\left(\frac{1}{1+\lambda}\right)^2 = 20$

$\Rightarrow 5 = 20(1+\lambda)^2 \Rightarrow 2(1+\lambda) = \pm 1 \Rightarrow 2(1+\lambda) = 1,\ 2(1+\lambda) = -1$

$\Rightarrow 1+\lambda = \frac{1}{2}$ or $1 + \lambda = -\frac{1}{2}$

$\boxed{\frac{1}{1+\lambda} = 2}$, $\boxed{\frac{1}{(1+\lambda)} = -2}$

$\therefore (x,y) = (2,2)$ or $(-2,-2)$

at $(2,2) \rightarrow F(x,y) = 12 - 12 + 8 - 8 = \boxed{0}$ at $(-2,-2) \Rightarrow f(x,y) = (12 + 12 + 8 + 8) = \boxed{40}$

2(c) A square $ABCD$ of diagonal $2a$ is folded along the dlagonal AC so thut the pianes DAC, BAC are at right angles. Find the shortest distance between DC and AB.

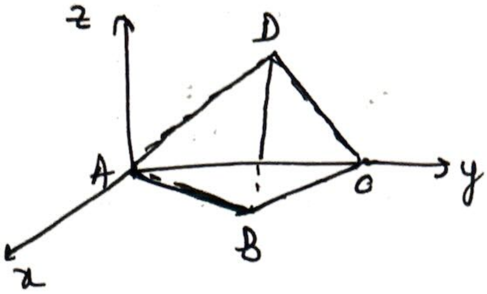
(15)

$A \equiv (0,0,0)$

$C \equiv (0,2a,0)$

$D \equiv (0,a,a)$

$B \equiv (a,a,0)$



equation of DC $\Rightarrow \frac{x-0}{0} = \frac{y-a}{a} = \frac{z-a}{-a}$

$\Rightarrow \frac{x}{0} = \frac{y-a}{a} = \frac{z-a}{-a}$ —— (I)

equation of AB $\Rightarrow \frac{x-0}{a} = \frac{y-0}{a} = \frac{z-0}{0}$

$\Rightarrow \frac{x}{a} = \frac{y}{a} = \frac{z}{0}$ —— (II)

we need to find SD between line (I) & (II)

let d-r of S.D (l, m, n)

$\Rightarrow l \times 0 + m \times a + n(-a) = 0$ and

$l \times a + m \times a + n(0) = 0$

from cramer's rule

$\frac{l}{0+a^2} = \frac{-m}{+a^2} = \frac{n}{-a^2}$

$\Rightarrow \frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}$

d.c of S.D line $= \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

$\therefore$ D.r of AC $= (0, 2a, 0)$

$\therefore$ shortest distance $= \left| 0 \times \frac{1}{\sqrt{3}} + \frac{-1}{\sqrt{3}} \times 2a + \frac{-1}{\sqrt{3}} \times 0 \right|$

$\boxed{\text{Shortest Distance} = \frac{2a}{\sqrt{3}}}$

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4(a) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1 \quad 1 \quad 0 \quad 0)^T$.

Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation

$$A^4 + p\,A^3 + q\,A^2 + r\,A = 0.$$

(20)

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Trace of $A = -1+5-10+8$

$= -11+13 = 2$

Sum of eigen value $= \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2$

let $\lambda_1 = 4$

one eigen vector $(1,1,0,0)^T$

$\Rightarrow AX = \lambda_2 X$

$$\Rightarrow \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+3 \\ -3+5 \\ 10-10 \\ 4-4 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_2 = -1+3 = 2$$

$\Rightarrow \boxed{\lambda_2 = 2}$

determinant of A

$$|A| = (-1)\begin{vmatrix} 5 & 1 & -1 \\ -10 & -10 & 14 \\ -4 & -4 & 8 \end{vmatrix} - 3\begin{vmatrix} -3 & 1 & -1 \\ 10 & -10 & 14 \\ 4 & -4 & 8 \end{vmatrix} - 1\begin{vmatrix} -3 & 5 & -1 \\ 10 & -10 & 14 \\ 4 & -4 & 8 \end{vmatrix}$$

$$-1\begin{vmatrix} -3 & 5 & 1 \\ 10 & -10 & -10 \\ 4 & -4 & -4 \end{vmatrix}$$

$= (-1)(-96) - 3(48) - 1(-48) - 1(0)$

$= 0$

$\Rightarrow \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = 0$

$\Rightarrow \lambda_3 = 0$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2$

$\Rightarrow 4 + 2 + 0 + \lambda_4 = 2$

$\Rightarrow \boxed{\lambda_4 = -4}$

$(\lambda-4)(\lambda+4) = \lambda^2 - 16$

$\therefore$ characteristic equation

$(\lambda-4)(\lambda-2)(\lambda+4)(\lambda-0)=0$

$\Rightarrow (\lambda^2-4)(\lambda^2-2\lambda)=0$

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$\Rightarrow \lambda^4 - 4\lambda^2 - 2\lambda^3 + 8\lambda = 0$

$\Rightarrow \lambda^4 - 2\lambda^3 - 4\lambda^2 + 8\lambda = 0$

16

From C.H ~~eq~~ theorem.

$A^4 - 2A^3 - 4A^2 + 8A = 0$

$\Rightarrow$ ~~p=2~~ $\boxed{p=-2,\ q=-4,\ r=8}$

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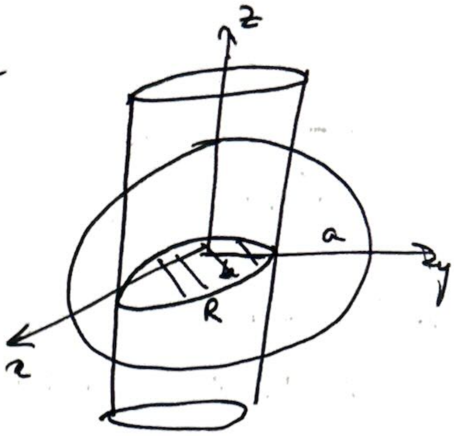
4(b) Find the volume cut from a sphere of radius a by a right circular cylinder with b as radius of the base and whose axis passes through the centre of the sphere. (15)

let sphere $x^2+y^2+z^2=a^2$

cylinder $x^2+y^2=b^2$

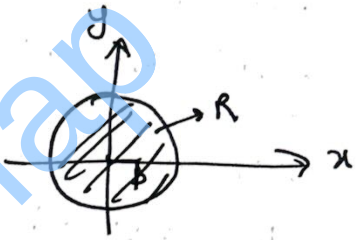


Volume cut

$$= 2\iint_R z\,dx\,dy \qquad x^2+y^2+z^2=a^2.$$

$$= 2\iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy.$$

Put $x=r\cos\theta$
$y=r\sin\theta$



$$= 2\int_0^b\int_0^{2\pi}\sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$= 2\int_0^b 2\pi\sqrt{a^2-r^2}\,r\,dr.$$

$$= 4\pi\int_0^b r\sqrt{a^2-r^2}\,dr.$$

$$= -2\pi\int_0^b (-2r)\sqrt{a^2-r^2}\,dr.$$

$$= -2\pi\left.\frac{(a^2-r^2)^{3/2}}{3/2}\right|_0^b \quad = -2\pi\left[(a^2-b^2)^{3/2}-a^3\right]\frac{2}{3}$$

$$= +\frac{4}{3}\pi\left[a^3-(a^2-b^2)^{3/2}\right]$$

$$\boxed{\text{Volume cut} = \frac{4}{3}\pi\left[a^3-(a^2-b^2)^{3/2}\right]}$$

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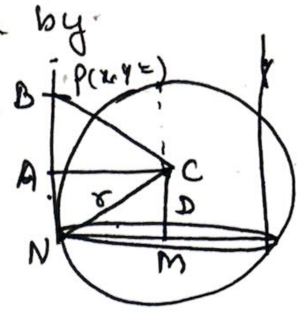
4(c) Find the equation of a right circular cylinder described on the circle through the points $A(a,0,0), B(0,a,0)$ and $C(0,0,a)$ as the guiding curve. (15)

Circle ABC equation is given by plane ABC and sphere OABC where O is origin



$\therefore \; x^2+y^2+z^2-ax-by-cz=0$

$\Rightarrow x^2+y^2+z^2-ax-ay-az=0$

& $\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$

$\therefore$ Circle ABC $\Rightarrow x^2+y^2+z^2-ax-ay-az=0$ and
$x+y+z=a$

$CN = r = \sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2} = \sqrt{\frac{3a^2}{4}} = \frac{\sqrt{3}}{2}a.$

$\boxed{r^2 = \frac{3}{4}a^2}$ $\qquad C=\left(a/2, a/2, a/2\right)$

$D = \frac{\left|\frac{a}{2}+\frac{a}{2}+\frac{a}{2}-a\right|}{\sqrt{1+1+1}} = \frac{a}{2\sqrt{3}}$

$D^2 = \frac{a^2}{4\times 3} = \frac{a^2}{12}$

$\therefore NM^2 = \frac{3}{4}a^2 - \frac{a^2}{12} = \frac{9-1}{12}a^2 = \frac{8}{12}a^2 = \frac{2}{3}a^2.$

DC of line through CM = (DC of Plane ABC
$=\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\therefore \; AB = \left|\left(x-\frac{a}{2}\right)\frac{1}{\sqrt{3}} + \left(y-\frac{a}{2}\right)\frac{1}{\sqrt{3}} + \left(z-\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)\right|$

$AB^2 = \frac{1}{3}\left[\left(x-\frac{a}{2}\right)+\left(y-\frac{a}{2}\right)+\left(z-\frac{a}{2}\right)\right]^2$

$\therefore \; BC^2 = AB^2 + AC^2 = AB^2 + NM^2$

$\Rightarrow \left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2}\right)^2+\left(z-\frac{a}{2}\right)^2 = \frac{1}{3}\left[\left(x-a/2\right)+\left(y-\frac{a}{2}\right)+\left(z-\frac{a}{2}\right)\right]^2 + \frac{2}{3}a^2$

$\Rightarrow x^2+y^2+z^2+\frac{3a^2}{4}+(-ax-ay-az) = \frac{1}{3}\left(x+y+z-\frac{3a}{2}\right)^2+\frac{2}{3}a^2$

$\Rightarrow x^2+y^2+z^2+\cancel{\frac{3a^2}{4}}-(ax+ay+az) = \frac{1}{3}(x+y+z)^2 - \frac{3a}{3}(x+y+z) + \frac{9a^2}{4}\times\frac{1}{3}+\frac{2}{3}a^2$

$$\Rightarrow x^2+y^2+z^2 - \frac{1}{3}(x+y+z)^2 = \frac{2}{3}a^2$$

$$\Rightarrow 3(x^2+y^2+z^2) - \frac{1}{3}(x^2+y^2+z^2+2xy+2yz+2xz) = 2a^2$$

$$\Rightarrow 2(x^2+y^2+z^2) + -2(xy+yz+zx) = 2a^2$$

$$\Rightarrow \boxed{x^2+y^2+z^2-xy-yz-zx = a^2}$$

is required equation of cylinder

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SECTION B

5(a) Prove that the orthogonal trajectories of the family of conics $y^2 - x^2 + 4xy - 2cx = 0$ consists of a family of cubics with the common asymptote $x + y = 0$. (10)

$y^2 - x^2 + 4xy - 2cx = 0$

$\boxed{\text{~~differentiate~~}} \Rightarrow \frac{y^2}{x} - x + 4y - 2c = 0$

differentiate w.r.t x

$\Rightarrow \frac{2yy'}{x} - \frac{y^2}{x^2} - 1 + 4y' = 0$ —(1)

$\cancel{\Rightarrow 2xyy' - y^2 - x^2 + 4x^2y' = 0}$

$\Rightarrow$ For orthogonal trajectory of (1) put y' as $-1/y'$

$\Rightarrow -\frac{2y}{xy'} - \frac{y^2}{x^2} - 1 + \frac{4}{-y'} = 0$

$\Rightarrow -2xy - y^2 x y' - x^2y' - 4x^2 = 0$

$\Rightarrow y'(y^2+x^2) + 2xy + 4x^2 = 0$

$\Rightarrow dx(2xy + 4x^2) + dy(y^2+x^2) = 0$

$\Rightarrow 2xy\,dx + x^2dy + 4x^2dx + y^2dy = 0$

$\Rightarrow d(x^2y) + 4x^2dx + y^2dy = 0$

$\Rightarrow x^2y + \frac{4x^3}{3} + \frac{y^3}{3} = c_1$

$\Rightarrow 3x^2y + 4x^3 + y^3 = c_2$ —(2)

For asymptot $\boxed{y = mx + c}$

$\phi_3(m) = m^3 + 4 + 3m = 0 \Rightarrow m^3 + m^2 - m^2 - m + 4m + 4 = 0$

$(m+1)(m^2 - m + 4) = 0$

m = −1 is real root

$\phi_2(m) = 0$

$\therefore c = -\frac{\phi_2(m)}{\phi_3'(m)} = 0$

$\therefore y = -x$ is one asymptot

$\boxed{y + x = 0}$ is asymptot to cubic curve (2)

8

5(b) Solve $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$. (10)

$2ydx + 3xdy + 2xy(3ydx + 4xdy) = 0$

$\Rightarrow dx(2y + 6xy^2) + dy(3x + 8x^2y) = 0$

$M = 2y + 6xy^2$, $N = 3x + 8x^2y$

$\frac{\partial M}{\partial y} = 2 + 12xy$ & $\frac{\partial N}{\partial x} = 3 + 16xy$

$\Rightarrow dx \cdot y(2 + 6xy) + dy\, x(3 + 8xy) = 0$

is of form of $M \cdot dx + N \cdot dy$

$f(xy) \cdot y\,dx + g(xy) \cdot x\,dy = 0$

$\therefore IF = \frac{1}{Mx - Ny} = \frac{1}{xy(2+6xy) - xy(3+8xy)}$

$= \frac{1}{2xy + 6x^2y^2 - 3xy - 8x^2y^2}$

$= \frac{1}{-xy - 2x^2y^2}$

$= \frac{-1}{xy(1+2xy)}$

$\therefore \frac{dx\, y(2+6xy)}{-xy(1+2xy)} + \frac{x\, dy(3+8xy)}{-xy(1+2xy)} = 0$

$\Rightarrow \frac{dx}{x}\left(\frac{2+6xy}{1+2xy}\right) + \frac{dy}{y}\left(\frac{3+8xy}{1+2xy}\right) = 0$

$\Rightarrow \frac{dx}{x}\left[2 + \frac{2xy}{1+2xy}\right] + \frac{dy}{y}\left[3 + \frac{2xy}{1+2xy}\right] = 0$

$\Rightarrow \frac{2dx}{x} + \frac{3dy}{y} + \frac{2xy}{1+2xy}\left[\frac{dx}{x} + \frac{dy}{y}\right] = 0$

$\Rightarrow \frac{2dx}{x} + \frac{3dy}{y} + \frac{2}{1+2xy}\, d(xy) = 0$

$\Rightarrow$ Integrate

$\Rightarrow 2\log x + 3\log y + \log(1+2xy) = \log c$

$\Rightarrow \boxed{x^2 y^3 (1+2xy) = c}$

Also Try $x^\alpha y^\beta$ & find IF

8

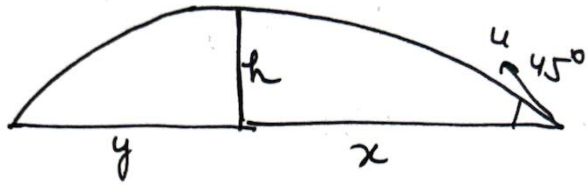
5(c) From a point on the ground at a distance x from the foot of a vertical wall, a ball is thrown at an angle of 45° which just clears the top of the wall and afterwards strikes the ground at a distance y on the other side. Find the height of the wall.

(10)

h = maximum height of projectile

$$\Rightarrow h = \frac{u^2 \sin^2 45}{2g}$$

$$= \frac{u^2}{2g} \cdot \times \frac{1}{2}$$



$$\boxed{h = \frac{u^2}{4g}}$$

$$\text{Range} = y + x = \frac{u^2 \sin(2 \times 45)}{g}$$

$$y + x = \frac{u^2 \times 1}{g}$$

$$\Rightarrow h = \left(\frac{u^2}{g}\right) \times \frac{1}{4} = \frac{(y+x)}{4}$$

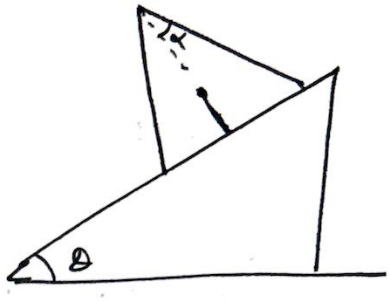
$$\therefore \boxed{h = \frac{y+x}{4}}$$

$$\boxed{\text{height of wall} = \frac{y+x}{4}}$$

8

5(d) A heavy right cone of semi-vertical angle α rests in limiting equilibrium with its plane base upon an inclined plane of inclination θ to the horizon. Show that the cone will topple over or not, according as $\tan\theta >$ or $< 4\tan\alpha$. Examine the case when $\tan\theta = 4\tan\alpha$.

(10)



5(e) Determine the constants a and b such that the curl of vector $\bar{A} = (2xy + 3yz)\hat{\imath} + (x^2 + axz - 4z^2)\hat{\jmath} - (3xy + byz)\hat{k}$ is zero. (10)

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+3yz & x^2+axz-4z^2 & -3xy-byz \end{vmatrix}$$

$$= \hat{i}\left[-3x-bz-(ax-8z)\right] - \hat{j}\left[-3y-3y\right] + \hat{k}\left(\cancel{2x}+az-\cancel{2x}-3z\right)$$

$$= \hat{i}\left[\cancel{-3+0}\,(-3-a)x + (-b+8)z\right] - \hat{j}\left[-6y\right] + \hat{k}\,(a-3)z$$

wrong question

8

8(a) Solve $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given $y(0) = 1$ using L. T (15)

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$$

take laplace both side and $L(y) = Y$

$$\therefore L\left\{\frac{dy}{dt}\right\} + L(2y) + L\left\{\int_0^t y\,dt\right\} = L\{\sin t\}$$

$$\Rightarrow pY - y(0) + 2Y + \frac{Y}{p} = \frac{1}{1+p^2} \quad \left[L\left\{\int_0^t f(t)dt\right\} = \frac{F(p)}{p}\right]$$

$\Rightarrow$

$$\Rightarrow pY - 1 + 2Y + \frac{Y}{p} = \frac{1}{p^2+1}$$

$$\Rightarrow Y\left[p + 2 + \frac{1}{p}\right] = \frac{1}{p^2+1} + 1$$

$$\Rightarrow Y\left[\frac{(p+1)^2}{p}\right] = \frac{1}{p^2+1} + 1$$

$$\Rightarrow Y = \frac{1 \times p}{(p^2+1)(p+1)^2} + \frac{p}{(p+1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{p^2+1} - \frac{1}{(p+1)^2}\right] + \frac{1}{p+1} - \frac{1}{(p+1)^2}$$

$$= \frac{1}{2(p^2+1)} - \frac{1}{2(p+1)^2} - \frac{1}{(p+1)^2} + \frac{1}{p+1}$$

$$= \frac{1}{2(p^2+1)} - \frac{3}{2(p+1)^2} + \frac{1}{p+1}$$

$$y = L^{-1}(Y) = \frac{1}{2}\sin t - \frac{3}{2}e^{-t}t + e^{-t}$$

$$\boxed{y(t) = \frac{1}{2}\sin t + e^{-t} - \frac{3t}{2}e^{-t}}$$

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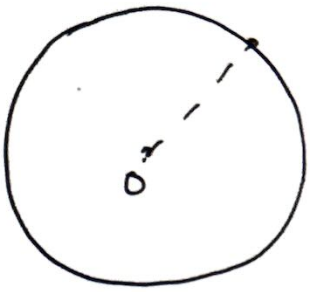
8(b) If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in time which is $\sqrt{2}/8$ times the period of the planet's revolution (15)



$$F = -\frac{\mu}{r^2}$$

$$\frac{d^2r}{dt^2} = -\frac{\mu}{r^2}$$

$$\Rightarrow 2\frac{dr}{dt}\left(\frac{d^2r}{dt^2}\right) = -\frac{2\mu}{r^2}\frac{dr}{dt}$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = -\frac{2\mu}{-r} + B.$$

at $r = a$, $\frac{dr}{dt} = 0$

$$\Rightarrow 0 = -\frac{2\mu}{-a} + B \Rightarrow B = -\frac{2\mu}{a}.$$

$$\therefore \left(\frac{dr}{dt}\right)^2 = -\frac{2\mu}{a} + \frac{2\mu}{r} = -2\mu\left(\frac{r-a}{ar}\right) = \frac{2\mu(a-r)}{ar}$$

$$\therefore \left(\frac{dr}{dt}\right) = -\sqrt{2\mu}\,\frac{\sqrt{a-r}}{\sqrt{ar}}$$

$$\Rightarrow \int_a^0 \frac{dr\sqrt{r}}{\sqrt{a-r}} = \int_0^T -\sqrt{2\mu/a}\,dt$$

Put $r = a\sin^2\theta$.
$dr = a\,2\sin\theta\cos\theta\,d\theta$

$$\Rightarrow \int_{\pi/2}^0 \frac{d\theta \cdot 2a\sin\theta \cancel{\cos\theta}\,\cancel{\sqrt{a}}\sin\theta}{\cancel{\sqrt{a}}\,\cancel{\cos\theta}} = -\int_0^T \sqrt{2\mu/a}\,dt$$

$$\Rightarrow \int_0^{\pi/2} 2a\sin^2\theta\,d\theta. = T\sqrt{\frac{2\mu}{a}}$$

$$\Rightarrow \int_0^{\pi/2} a(1-\cos 2\theta)\,d\theta = \sqrt{\frac{2\mu}{a}}\,T$$

$$\Rightarrow \frac{a}{2}\times\frac{\pi}{2} = \sqrt{\frac{2\mu}{a}}\,T$$

$$\Rightarrow T = \frac{a^{3/2}\pi}{2\sqrt{2}\sqrt{\mu}}$$

$$T_{revolution} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}.$$

$$\frac{T}{T_{rev}} = \frac{a^{3/2}\cancel{\pi}}{2\sqrt{2}\sqrt{\mu}} \times \frac{\sqrt{\mu}}{2\cancel{\pi}\cancel{a^{3/2}}}$$

$$= \frac{1}{4\sqrt{2}}$$

$$T/T_{rev} = \frac{\sqrt{2}}{8}$$

$$\Rightarrow \boxed{T = \frac{\sqrt{2}}{8} T_{rev}}$$

proved

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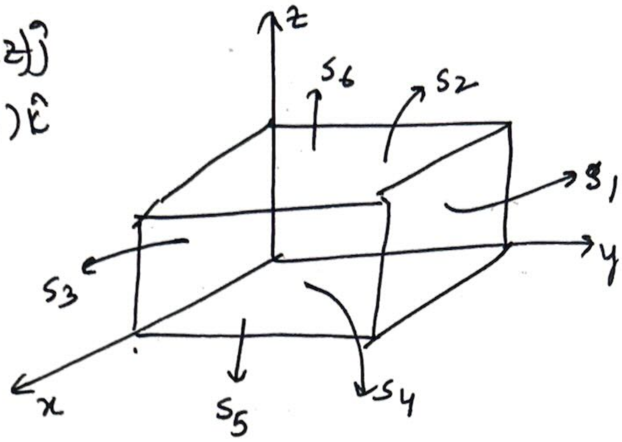
8(c) Verify the Gauss divergence Theorem for $\vec{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ (20)

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$$



$$\nabla\cdot\vec{F} = 2x\hat{i} + 2y + 2z$$

$$= 2(x+y+z)$$

$$\iiint \nabla\cdot\vec{F}\, dV$$

$$= \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dx\,dy\,dz$$

$$= \int_0^c\int_0^b \left(\frac{x^2}{2} + 2yx + 2xz\right)\Big|_0^a dy\,dz$$

$$= \int_0^c\int_0^b a^2 + 2ay + 2az \; dy\,dz$$

$$= \int_0^c \left[a^2b + ab^2 + 2abz\right] dz$$

$$= \underline{a^2bc + ab^2c + abc^2} \quad \text{— (a)}$$

__Now__ Surface integral

on, $\underline{S_1} \rightarrow \quad \hat{n} = \hat{j}, \; y = b$

$$\therefore \iint \vec{F}\cdot\hat{n}\, ds = \iint dx\cdot dz\,(y^2 - xz)$$

$$= \int_0^a\int_0^c (b^2 - xz)\,dx\,dz$$

$$= \int_0^a \left(b^2c - \frac{xc^2}{2}\right) dx$$

$$= ab^2c - \frac{a^2c^2}{4} \quad \text{— (1)}$$

On Surface S_2 $\quad x=0, \; \hat{n} = -\hat{i}$

$$\iint \vec{F}\cdot\hat{n}\, ds = \iint -(x^2 - yz)\,dy\,dz$$

$$= \int_0^b\int_0^c yz\,dy\,dz = \frac{b^2c^2}{4} \quad \text{— (2)}$$

on Surface S_3

$S_3 \to \hat{n} = -\hat{j}, \; y=0$

$$\Rightarrow \iint \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b zx\,dx\,dz = \frac{b^2a^2}{4} \quad \text{--- (3)}$$

onsurface S_4

$\hat{n} = \hat{i}, \quad x=a$

$$\Rightarrow \iint \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^c (a^2 - yz)\,dx\,dy$$

$$= a^2bc - \frac{b^2c^2}{4} \quad \text{--- (4)}$$

onsurface S_5

$\hat{n} = -\hat{k}, \; z=0$

$$\Rightarrow \iint \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b + xy\,dx\,dy = \frac{a^2b^2}{4} \quad \text{--- (5)}$$

onsurface S_6

$\hat{n} = \hat{k}, \; z=c$

$$\Rightarrow \iint \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b (c^2 - xy)\,dx\,dy = c^2ab - \frac{a^2b^2}{4} \quad \text{--- (6)}$$

adding (1), (2), (3), (4), (5) and (6)

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = abc^2 + acb^2 + a^2bc - \cancel{\frac{a^2b^2}{4}} - \cancel{\frac{a^2c^2}{4}} - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{c^2a^2}{4}} + \cancel{\frac{b^2c^2}{4}}$$

$$= abc^2 + ab^2c + a^2cb \quad \text{--- (b)}$$

$\therefore$ from (a) & (b)

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint (\nabla\cdot\vec{F})\,dV}$$

this varifies stroke's theorem