30th July 2023

SuccessClap

SuccessClap

Best Coaching for UPSC MATHEMATICS

UPSC Mathematics Test Series MAINS 2023

Topic: 01 Linear Algebra

Date :10-07-2022

Instructions:

Time: 90 Minutes
Maximum Marks: 150
All questions are compulsory
Each question carries Equal marks
Assume suitable data if considered necessary

102/150

→ Use Calculator to verify matrix operations det, Inverse, adj, Multiply, in Main Exam

→ Linear Eqns: Checkout Conditions for Unique Soln

1) Let $T: R^2 \rightarrow R^2$ such that $T(1,1) = (2,-3), T(1,-1) = (4,7)$.
Find the matrix of T relative to the basis $S = \{(1,0),(0,1)\}$

$T(1,1) = (2,-3)$
$T(1,-1) = (4,7)$

if $(x,y) = a(1,1) + b(1,-1)$

$x = a+b$
$y = a-b$

$\Rightarrow x+y = 2a$ and $x-y = 2b$.

$\Rightarrow a = \frac{x+y}{2}, \quad b = \frac{x-y}{2}$

$\therefore (x,y) = \frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)$

$\therefore (1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$

$\therefore T(1,0) = \frac{1}{2}T(1,1) + \frac{1}{2}T(1,-1)$
$= \frac{1}{2}(2,-3) + \frac{1}{2}(4,7) = \left(1+2, -\frac{3}{2}+\frac{7}{2}\right)$
$= (3,2)$

Again $T(0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$
$T(0,1) = \frac{1}{2}T(1,1) - \frac{1}{2}T(1,-1)$
$= \frac{1}{2}(2,-3) - \frac{1}{2}(4,7) = \left(1-2, -\frac{3}{2}-\frac{7}{2}\right) = (-1,-5)$

$\therefore T(1,0) = 3(1,0) + 2(0,1)$
$T(0,1) = -1(1,0) + (-5)(0,1)$

$\therefore$ Matrix of T relative to basis $\{(1,0), (0,1)\}$
is $\begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}$

12

2) Let $B = \{(1,0),(0,1)\}$ and $B' = \{(1,3),(2,5)\}$ be the bases of R^2. Find the transition matrices from B to B' and B' to B.

$B = \{(1,0),(0,1)\}$

Transition Matrix from B to B'

$(1,3) = a(1,0) + b(0,1)$

$1 = a + 0b \Rightarrow a = 1$

$b = 3$

$\therefore (1,3) = (1,0) + 3(0,1)$ — (I)

Similarly, $(2,5) = 2(1,0) + 5(0,1)$ — (II)

$\therefore$ Matrix from B to B'

$$\boxed{\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}}$$

Transition from B' to B

$(1,0) = a(1,3) + b(2,5)$

$\Rightarrow 1 = a + 2b$

$0 = 3a + 5b$

$\Rightarrow a = -5$

$b = 3$

$\Rightarrow (1,0) = -5(1,3) + 3(2,5)$ — (III)

again, $(0,1) = a(1,3) + b(2,5)$

$\Rightarrow 0 = a + 2b$

$1 = 3a + 5b$

$a = 2,\ b = -1$

$\therefore (0,1) = 2(1,3) - 1(2,5)$ — (IV)

From (III) & (IV)

Transition matrix from B' to B

$$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

12

3) Let T be the linear operator on $V_3(\mathrm{R})$ defined by
$T(x, y, z) = (2y + z, x - 4y, 3x)$
(i) Find the matrix of T relative to the basis $\mathrm{B} = \{(1,1,1), (1,1,0), (1,0,0)\}$
(ii) Verify $[T(\alpha)]_B = [T]_B[\alpha]_B$

~~Let (a, b, c) =~~ Given ~~T(2y+~~

$T(x,y,z) = (2y+z,\ x-4y,\ 3x)$

$\therefore T(1,1,1) = (3, -3, 3)$

$T(1,1,0) = (2, -3, 3)$

$T(1,0,0) = (0, 1, 3)$

Now, $(3,-3,3) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$$\Rightarrow \left[\begin{array}{l} a+b+c=3 \\ a+b=-3 \\ a=3 \end{array}\right]$$

$\Rightarrow a=3 \Rightarrow b=-6$

$c = 3-b-a = 3+6-3 = 6$

Similarly $(2,-3,3) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$\Rightarrow a=3$

$a+b=-3 \Rightarrow b=-6$

$a+b+c=2 \Rightarrow c = 2-a-b = 2-3+6 = 5$

Again, $(0,1,3) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$\Rightarrow a=3$

$a+b=1 \Rightarrow b=-2$

$a+b+c=0 \Rightarrow$ ~~c=1~~ $c=-1$

$\therefore$ Matrix $[T]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

(ii) Let $\alpha = (x,y,z)$

$T(\alpha) = (2y+z,\ x-4y,\ 3x)$

For $[T]_B[\alpha]_B$:

$(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$\Rightarrow z = a$

$a+b=y \Rightarrow b = y-z$

$a+b+c = x \Rightarrow c = x-z-(y-z)$

$= x-y.$

4)

$$\therefore \alpha_B = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$\therefore [T_B][\alpha]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} = \begin{bmatrix} 3x \\ -4y-2x \\ -x+6y+z \end{bmatrix} \quad \text{--- (a)}$$

$$T(\alpha) = (2y+z,\ x-4y,\ 3x) = a(1,1,1) + b(1,1,0) + c(1,0,0)$$

$$a = 3x$$

$$a+b = x-4y \Rightarrow b = x-4y-3x = -4y-2x$$

$$c = -a-b+2y+z = -3x+4y+2x+2y+z = -x+6y+z$$

$$[T(\alpha)]_B = \begin{bmatrix} 3x \\ -4y-2x \\ -x+6y+z \end{bmatrix} \quad \text{--- (b)}$$

From (a) & (b)

$$\therefore \boxed{[T(\alpha)]_B = [T_B][\alpha]_B}$$

12

4) Suppose $\mathbf{S}$ is an n-rowed real skew-symmetric matrix and I is the unit matrix of order n. Then show that
(i) I-S is non-singular ;
(ii) $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}$ is orthogonal ;
(iii) $\mathbf{A} = (\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})$;
(iv) If $\mathbf{X}$ is a charasteristic vector of $\mathbf{S}$ corresponding to the characteristic root λ, then $\mathbf{X}$ is also a characteristic vector of $\mathbf{A}$ and $(1+\lambda)/(1-\lambda)$ is the corresponding characteristic root.

S is real skew symmetric

(i) Since S is real skew symmetric
$\Rightarrow$ S is skew hermitian
so, eign value of S will be zero or pure imaginary
$\therefore |S-I| \neq 0$
$\Rightarrow |I-S| \neq 0 \Rightarrow (I-S)$ is non singular

(ii) $(I-S)(I+S) = I - S + IS - S.S$
$= I - S.S = (I+S)(I-S)$ —— (a)

Now $A = (I+S)(I-S)^{-1}$
for orthogonal
$AA^T = I$
$\Rightarrow A^T = A^{-1}$
$\Rightarrow A^T = [(I+S)(I-S)^{-1}]^T$
$= ((I-S)^T)^{-1}(I+S)^T$
$= (I-S^T)^{-1}(I+S^T)$
$= (I+S)^{-1}(I-S)$ (skew symmetric)
$\therefore AA^T = (I+S)(I-S)^{-1}(I+S)^{-1}(I-S)$
$= (I+S)[(I+S)(I-S)]^{-1}(I-S)$
$= (I+S)((I-S)(I+S))^{-1}(I-S)$
$= (I+S)(I+S)^{-1}(I-S)^{-1}(I-S)$
$= I$
$\therefore A$ is orthogonal

(iii) $A = (I+S)(I-S)^{-1}$

$A^T = (I+S)^{-1}(I-S)$

$(A^T)^{-1} = [(I+S)^{-1}(I-S)]^{-1}$

$\boxed{A = (I-S)^{-1}(I+S)}$

(iv) $\boxed{SX = \lambda X}$

if λ is C.R of S for vector X

$\therefore$ $1-\lambda$ is C.R of $I-S$ for vector X

$1+\lambda$ is CR of $I+S$ for vector X

$\therefore$ $(1-\lambda)^{-1}$ is CR of $(I-S)^{-1}$

$\therefore$ $(1-\lambda)^{-1}(1+\lambda)$ is CR of $(I-S)^{-1}(I+S)$ for vector X

$\therefore$ $\boxed{CR = \frac{1+\lambda}{(1-\lambda)} \text{ for vector } X}$

12

5) Show that the set of all convergent sequences is a vector space over the field of real numbers.

Let $V(R)$ is set of all convergent sequence over R.

$V = \{\{x_n\}$, where $\{x_n\}$ is convergent $\}$

To show V is vector space

(i) if vector additon & scalar multiplication is defined

(ii) $(V, +)$ is abelian group

(iii) $\forall\ a, b \in R \quad \{\alpha_n\}, \{\beta_n\} \in V$.

1) $a\cdot(\alpha+\beta) = a\alpha + b\beta$

2) $(a+b)\alpha = a\alpha + b\alpha$

3) $(ab)\alpha = a(b\alpha)$

4) $1.\alpha = \alpha$.

Now (i) vector addition $\{x_n\}, \{y_n\} \in V$

$\{x_n\} + \{y_n\}$ is sum of two convergent ~~sta~~ sequence

$\Rightarrow \{x_n\} + \{y_n\}$ is convergent sequencs

$\Rightarrow \{x_n\} + \{y_n\} \in V$.

Scalar ~~Pat~~ Multiplication

$a \in R$, $\{x_n\} \in V$.

$\Rightarrow a\{x_n\}$ will also be convergen

$\Rightarrow a\{x_n\} \in V$.

(ii) for V to be abelian

closure $\rightarrow \{x_n\}, \{y_n\} \in V$

shown that $\{x_n\}, \{y_n\} \in V \Rightarrow$ closed.

associative $\rightarrow \{x_n\}\ \{y_n\}\ \{z_n\} \in V$.

since, $(\{x_n\} + \{y_n\}) + \{z_n\} = \{x_n\} + (\{y_n\} + \{z_n\})$

$\Rightarrow$ $\boxed{\text{associative}}$

(iii) Identity since null sequence $\{0\}$ is convergent sequence

$\Rightarrow \{0\} \in V$.

Now, $\{x_n\} + \{0\} = \{x_n\} = \{0\} + \{x_n\} \quad \forall \{x_n\} \in V$

$\Rightarrow$ $\{0\}$ is identity

(iv) Inverse.

$\{x_n\}$ is convergent

$\Rightarrow -\{x_n\}$ is convergent

$\Rightarrow \{-x_n\}$ is convergent

$\{x_n\} + \{-x_n\} = \{x_n\} - \{x_n\} = \{0\}$

$\therefore \{-x_n\}$ is inverse of $\{x_n\} \Rightarrow$ Inverse exist

(v) sum of addition of sequences is commutative

$\therefore (V, +)$ is abelian group.

Now, $a, b \in R$, $\{x_n\}, \{y_n\} \in V$

1) $a(\{x_n\} + \{y_n\}) = a\{x_n\} + a\{y_n\}$

2) $(a+b)\{x_n\} = a\{x_n\} + b\{x_n\}$

c) $(ab)\{x_n\} = (ab)\{x_n\} = a(b\{x_n\})$

(d) $1 \cdot \{x_n\} = \{x_n\}$

$\therefore$ $\boxed{V \text{ is a vector space}}$

12

6) Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$$

by using elementary transformations.

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = I \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_1$

$\therefore$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 2R_1$
$R_4 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$
$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} A$$

$R_4 \to R_4 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & -3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ -1 & 1 & -3/2 & 1 \end{bmatrix} A$$

$R_4 \to -\frac{2}{3} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 2/3 & -2/3 & 1 & -2/3 \end{bmatrix} A$$

$R_1 \to R_1 - R_4$
$R_2 \to R_2 + R_4$
$R_3 \to R_3 - 3/2 R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -1 & 5/3 \\ 5/3 & -8/3 & 2 & -2/3 \\ -1 & 2 & -2 & 1 \\ -1 & 1 & -3/2 & 1 \end{bmatrix} A$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 1/3 & 2/3 & -1 & 5/3 \\ 5/3 & -8/3 & 2 & -2/3 \\ -1 & \cancel{2} & -2 & 1 \\ -1 & 1 & -3/2 & 1 \end{bmatrix}}$$

Use Calculator to verify matrix operation

7) Verify Cayley -Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}.

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left((4-\lambda)(6-\lambda) \underset{-25}{}\right) - 2(12-2\lambda-15) + 3(10-12+3\lambda) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 10\lambda + \underset{-25}{24}) - 2(-2\lambda-3) + 3(-2+3\lambda) = 0$

$\Rightarrow \lambda^2 - 10\lambda + \cancel{24} - \lambda^3 + 10\lambda^2 + \cancel{24}\lambda + 4\lambda + \cancel{6} - \cancel{6} + 9\lambda = 0$

$\Rightarrow -\lambda^3 + 11\lambda^2 + 4\lambda + (-1) = 0$ —— (I)

$\therefore$ for CH eqn

$A^3 = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$

$A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$

NOW $-A^3 + 11A^2 + 4A + (-1)I = -\begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$

$+ 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore -A^3 + 11A^2 + 4A - I = 0$ —— (II)

From (I) & (II) ~~eq~~ cayley - Hamiton equation is satisfied.

8

$-A^3 + 11A^2 + A4 - I = 0$

Multiply with A^{-1}

$\Rightarrow -A^2 + 11A + 4I - A^{-1} = 0$

$\therefore A^{-1} = -A^2 + 11A + 4I$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ -3 & 0 & -1 \\ 2 & -1 & -3 \end{bmatrix}}$$

8) Find the values of 'a' and 'b' for which the equations $x+y+z=3; x+2y+2z=6; x+ay+3z=b$ have

(i) No solution (ii) a unique solution (iii) infinite number of solutions.

$x+y+z=3$
$x+2y+2z=6$
$x+ay+3z=b$

Writing in $[A|B]$ ~~[b]~~ form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{array}\right]$$

reducing to row-echlon form

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{array}\right]$$

i) will have no solution if $Rank(A) \neq Rank(A|B)$

$\Rightarrow$ if $Rank(A) = 2$ & $Rank(A|B) = 3$

~~or Rank(A~~

$\Rightarrow$ $a-3=0$ and $b-9 \neq 0$

$\Rightarrow$ $\boxed{a=3,\ b\neq 9}$

ii) unique solution if

$Rank(A) = Rank(A|B) = 3$

$\Rightarrow$ $a-3 \neq 0$, $b-9 \neq 0$

$\boxed{\Rightarrow a \neq 3,\ b \neq 9}$

iii) Infinitely many solution

$Rank(A) = Rank(A|B) = 2$

$\Rightarrow$ $a-3=0$, $b-9=0$

$\boxed{a=3,\ b=9}$

10

Note b can be any thing

9) Show that the martix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonaizable.

Also find the diagonal form and a diagonalizing matrix P.

To find eigan value λ

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (-9-\lambda)\left((3-\lambda)(7-\lambda)-32\right)$

$-4(-56+8\lambda+64)+4(-64+48-16\lambda) = 0$

$\Rightarrow (-9-\lambda)(21+\lambda^2-10\lambda-32) - 4(8+8\lambda) + 4(-16-16\lambda) = 0$

$\Rightarrow (-9-\lambda)(\lambda^2-10\lambda-11) - 32(\lambda+1) - 64(\lambda+1) = 0$

$\Rightarrow (-9-\lambda)(\lambda+1)(\lambda-11) - 96(\lambda+1) = 0$

$\Rightarrow (\lambda+1)\left(-(9+\lambda)(\lambda-11) - 96\right) = 0$

$\Rightarrow (\lambda+1)(-\lambda^2+2\lambda+99-96) = 0$

$\Rightarrow (\lambda+1)(\lambda^2-2\lambda-3) = 0 \Rightarrow (\lambda+1)(\lambda+1)(\lambda-3) = 0$

$\boxed{\lambda = -1, -1, 3}$

when $\lambda = -1$

$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -8x + 4y + 4z = 0$

$\Rightarrow -2x + y + z = 0$.

if $x = k_1$

$y = k_2$

$\Rightarrow z = 2k_1 - k_2$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 2k_1 - k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ eigen vectors for $\lambda = -1$ are $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

when $\lambda = 3$

$$\left\{\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\right\}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -3x + y + z &= 0 \\ 2x + z &= 0 \\ -4x + 2y + z &= 0. \end{aligned}$$

let $x = k$

$\therefore \quad z = -2k.$

let

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$2y - z = 0$

$-x + y = 0$

$y = k$

$\Rightarrow z = 2k$

$x = k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ Algebraic Multiplicity = Geometric multiplicity

$\Rightarrow$ diagonalizable.

$$\boxed{D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}}$$

12

10) Let W_1 be the subspace of $V_4(R)$ generated by the set of vectors

$$S = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$$

and W_2 the subspace of $V_4(\mathrm{R})$ generated by the set of vectors

$$T = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$$

Find:

(i) $\dim.(W_1 + W_2)$

(ii) $\dim.(W_1 \cap W_2)$

(i) Since W_1 is subspace & W_2 is subspace

W_1+W_2 is also a subspace

W_1+W_2 will be spaned by $S \cup T$

$$S_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & +1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ R_2 \to -R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

(1) W_1+W_2 will be spanned by

$\{(1,1,0,-1),(0,1,3,1),(0,1,2,-1)$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$\therefore$ dimension of $W_1+W_2 = 3$

$$\boxed{\dim(W_1+W_2) = 3}$$

12

(ii) $W_1 \cap W_2$ will be generated by common vectors from row echolon from of S & T

i.e, $(1,1,0,-1)$

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

Also, $\dim(W_1+W_2) = \dim(W_1)+\dim(W_2) - \dim(W_1 \cap W_2)$

$\therefore \boxed{\dim(W_1 \cap W_2) = 2+2-3 = 1}$