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MATHEMATICS

28th July 2023

Test Series 2023

7

120/150

Topic: 05 Ordinary Differential Equations

Instructions:

Time: 90 Minutes
Maximum Marks: 150
All questions are compulsory
Each question carries Equal marks
Assume suitable data if considered necessary

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1) Find the orthogonal trajectories of the family of curves $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$, where λ is a parameter.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad \text{—(i)}$$

differentiate w.r.t x

$$\Rightarrow \frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0$$

$$\Rightarrow \frac{\cancel{2}x}{a^2+\lambda}=-\frac{\cancel{2}yy'}{b^2+\lambda}$$

$$\Rightarrow x(b^2+\lambda)=-yy'(a^2+\lambda)$$

$$\Rightarrow xb^2+yy'a^2=-yy'\lambda-x\lambda$$

$$\Rightarrow \lambda=\frac{xb^2+yy'a^2}{-(x+yy')} \quad \text{—(ii)}$$

put in equation (i)

$$\frac{x^2}{a^2-\frac{xb^2+yy'a^2}{x+yy'}}+\frac{y^2}{b^2-\frac{xb^2+yy'a^2}{x+yy'}}=1$$

$$\Rightarrow \frac{x^2(x+yy')}{xa^2-xb^2+\cancel{a^2yy'}-\cancel{a^2yy'}}+\frac{y^2(x+yy')}{\cancel{xb^2}+b^2yy'-\cancel{xb^2}-yy'a^2}=1$$

$$\Rightarrow \frac{x^{\cancel{2}}(x+yy')}{\cancel{x}(a^2-b^2)}+\frac{y^{\cancel{2}}(x+yy')}{(b^2-a^2)\cancel{y}\cdot y'}=1$$

$$\Rightarrow (x^2+xyy')+-\left(\frac{xy+y^2y'}{y'}\right)=a^2-b^2$$

$$\Rightarrow x^2+xyy'-\frac{xy}{y'}-y^2=a^2-b^2$$

$$\Rightarrow \boxed{x^2-y^2+xyy'-\frac{xy}{y'}=a^2-b^2} \quad \text{—(iii)}$$

To get orthogonal trajectory put y' as $\frac{-1}{y'}$ in (11)

$$x^2 - y^2 + xy\left(\frac{-1}{y'}\right) - xy(-y') = (a^2 - b^2)$$

$$\Rightarrow x^2 - y^2 + xyy' - \frac{xy}{y'} = a^2 - b^2$$

which is same as equation (11)

Thus $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal

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2) Solve the following differential equations:

$$p^3 - 4xyp + 8y^2 = 0$$

$4xyp = p^3 + 8y^2$

$\Rightarrow 4x = \frac{p^3+8y^2}{yp} = \frac{p^2}{y} + \frac{8y}{p}$

differential w.r.t y

$\Rightarrow 4\frac{dx}{dy} = \frac{2p}{y}\frac{dp}{dy} + \frac{p^2}{-y^2} + \frac{8}{p} + \frac{8y}{-p^2}\frac{dp}{dy}$

$\Rightarrow \frac{4}{p} = \frac{dp}{dy}\left(\frac{2p}{y} - \frac{8y}{p^2}\right) + \frac{8}{p} - \frac{p^2}{y^2}$

$\Rightarrow \frac{p^2}{y^2} - \frac{4}{p} = \frac{dp}{dy}\frac{2y}{p}\left(\frac{p^2}{y^2} - \frac{4}{p}\right)$

$\Rightarrow \cancel{dy}\frac{dp}{dy} \cdot \frac{2y}{p} = 1$

$\Rightarrow \frac{dp}{p} = \frac{dy}{2y}$

$\Rightarrow \log p = \frac{1}{2}\log y + \log c$

$\Rightarrow \boxed{p = c\sqrt{y}}$ Put in original PDE

$\Rightarrow c y^{3/2} - 4xy\, c y^{1/2} + 8y^2 = 0$

$\Rightarrow c y^{3/2}[4x] + 8y^2 = 0$

$\Rightarrow \boxed{c4x + 8y^{+1/2} = 0}$

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3) Solve by Laplace

$(D^4 - 1)y = 1$, if $y = Dy = D^2y = D^3y = 0$ when $t = 0$.

$(D^4-1)y = 1$

apply laplace transformation

$L(D^4y) - L(y) = L(1)$

$\Rightarrow p^4 Y - p^3 y(0) - p^2 y'(0) - p y''(0) - y'''(0) - Y = \frac{1}{p}$ where $Y = L(y)$

$\Rightarrow p^4 Y - Y = \frac{1}{p}$

$\Rightarrow Y(p^4 - 1) = \frac{1}{p}$

$\Rightarrow Y = \frac{1}{p(p^4-1)} = \frac{1}{p(p^2-1)(p^2+1)}$

~~using partials fraction~~

$\cancel{\frac{1}{p(p^2-1)(p^2+1)} = \frac{A}{p} + \frac{Bp+C}{p^2-1} +}$

$L^{-1}\left(\frac{1}{p(p^2+1)}\right) = \int_0^t L^{-1}\left(\frac{1}{p^2+1}\right) dx = \int_0^t \sin x \, dx = -\cos x \Big|_0^t = 1 - \cos t$

$L^{-1}\left(\frac{1}{p^2-1}\right) = \frac{L^{-1}}{2}\left(\frac{1}{p-1} - \frac{1}{p+1}\right) = \frac{e^t - e^{-t}}{2}.$

$\therefore L^{-1}\left(\frac{1}{p^2-1} \cdot \frac{1}{p(p^2+1)}\right) = L^{-1}\left(\frac{1}{p^2-1}\right) * L^{-1}\left(\frac{1}{p(p^2+1)}\right)$

$= \int_0^t (1-\cos x)\left(\frac{e^{t-x} - e^{-t+x}}{2}\right) dx.$

$= \int_0^t \frac{e^{t-x} - e^{-t+x}}{2} - \frac{e^{t}}{2}\cos x \, e^{-x} + \frac{e^{-t}}{2}\cos x \, e^{x} dx$

$= \frac{-e^t e^{-x} - e^{-t} \cdot e^{x}}{2}\Big|_0^t - \frac{e^t}{2} \frac{e^{-x}(-\cos x) + \sin x)}{2}$

$+ \frac{e^{-t}}{2} \frac{e^x(\cos x + \sin x)}{2}$

$$y(t) = \left[\frac{-e^{t-x} - e^{-t+x}}{2} - \frac{e^t}{2} \frac{e^{-x}(\sin x - \cos x)}{2} + \frac{e^{-t}}{2} \frac{e^{x}(\cos x + \sin x)}{2} \right]_0^t$$

$$= -\frac{1 + e^t - 1 + e^{-t}}{2} - \frac{\cancel{e^t}}{2} \frac{\cancel{e^{-t}}(\sin t - \cos t)}{2} + \frac{e^t}{2}\left(\frac{-1}{2}\right)$$

$$+ \frac{\cancel{e^{-t}}\cancel{e^t}(\cos t + \sin t)}{4} - \frac{e^{-t}}{4}(\cancel{\cos x})$$

$$= \frac{-2}{2} + \frac{e^t + e^{-t}}{2} + \frac{1}{4}(2\cos t) + \frac{e^t + e^{-t}}{(-4)}$$

$$= -1 + \cancel{\sinh} \cosh t + \frac{\cos t}{2} + \left(\frac{-1}{2}\right)\cosh t\,)$$

$$\boxed{y(t) = \frac{1}{2}\cosh t + \frac{\cos t}{2} - 1}$$

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4) Evaluate $L^{-1}\{s/(s^2+4)^3\}$

$$L^{-1}\left(\frac{s}{(s^2+4)^3}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2+4)^2}\cdot\frac{1}{s^2+4}\right)$$

Now, $L^{-1}\left(\frac{s}{(s^2+4)^2}\right) = L^{-1}\left(\frac{1}{-2}\frac{d}{ds}\left(\frac{1}{s^2+4}\right)\right)$

$$= \frac{1}{2}L^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s^2+4}\right)\right\}$$

$$= \frac{1}{2}t\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{2}t\frac{\sin 2t}{2} = \frac{t\sin 2t}{4}$$

$$L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t.$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+4)^2}\cdot\frac{1}{s^2+4}\right\} = \frac{1}{8}\sin 2t * t\sin 2t$$

$$= \frac{1}{8}\int_0^t x\sin 2x\sin(2t-2x)\,dx.$$

$$= \frac{1}{16}\int_0^t x\, 2\sin 2x\sin(2t-2x)\,dx$$

$$= \frac{1}{16}\int_0^t x\left[\cos(2x-2t+2x) - \cos(\cancel{2x}+2t-\cancel{2x})\right]dx$$

$$= \frac{1}{16}\int_0^t \left(x\cos(4x-2t) - x\cos 2t\right)dx$$

$$= \frac{1}{16}\left[x\frac{\sin(4x-2t)}{4}\Big|_0^t - \int_0^t\frac{\sin(4x-2t)}{4}dt - \frac{x^2}{2}\cos 2t\Big|_0^t\right]$$

$$= \frac{1}{16}\left[\frac{t\sin 2t}{4} + \frac{\cos(4x-2t)}{16}\Big|_0^t - \frac{t^2}{2}\cos 2t\right]$$

$$= \frac{1}{16}\left[\frac{t\sin 2t}{4} + \cancel{\frac{1}{16}\cos 2t} - \cancel{\frac{1}{16}\cos 2t} - \frac{t^2\cos 2t}{2}\right]$$

$$= \frac{1}{64}t\sin 2t + \frac{1}{16}\cos 2t\left[\cancel{\frac{1}{16}} - \frac{t^2}{2}\cancel{\cos 2t}\right]$$

$$\boxed{L^{-1}\left(\frac{s}{(s^2+4)^3}\right) = \frac{1}{64}t\sin 2t + \frac{1}{32}(-t^2)\cos 2t}$$

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5) Solve by Normal Method the following differential equations:

$$\frac{d^2y}{dx^2}+\frac{1}{x^{1/3}}\frac{dy}{dx}+\left(\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}\right)=0.$$

~~let $\frac{dy}{dx}=z$~~

$\Rightarrow \cancel{\frac{dz}{dx}+\frac{1}{x^{1/3}}dz}\ \phi = \frac{6}{x^2}+\frac{1}{6x^{4/3}}-\frac{1}{4x^{2/3}}$

$IF = \cancel{e^{\int x^{-1/3}dx}} = e^{\frac{x^{-1/3+1}}{-1/3+1}} = e^{x^{2/3}\cdot\frac{3}{2}}$

$P = x^{-1/3}$

$u = e^{\int -\frac{1}{2}Pdx} = e^{-\int \frac{1}{2}x^{-1/3}dx} = e^{-\frac{1}{2}\frac{x^{2/3}}{2/3}}$

$= e^{-\frac{1}{2}\times\frac{3}{2}x^{2/3}}$

$= e^{-\frac{3}{4}x^{2/3}}$

$I = Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}$

$= \left(\cancel{\frac{1}{4x^{2/3}}}-\cancel{\frac{1}{6x^{4/3}}}-\frac{6}{x^2}\right)-\cancel{\frac{1}{4}x^{-2/3}}-\cancel{\frac{1}{2}\left(-\frac{1}{3}\right)x^{-4/3}}$

$I = -\frac{6}{x^2}$

$S = \frac{R}{u} = 0$

$\therefore \frac{d^2v}{dx^2}+Iv=0$

~~$\Rightarrow \frac{d^2v}{dx^2}\ \phi-\frac{6}{x^2}dv=0$~~

~~$\Rightarrow d\left(\frac{dv}{dx}\right)-\frac{6}{x^2}dx=0$~~

~~$\Rightarrow \frac{dv}{dx}+\frac{6}{x}=c_1$~~

~~$\Rightarrow v+6\log x = c_1x+c_2$~~

~~$\Rightarrow v=$~~

$\Rightarrow \frac{d^2v}{dx^2}-\frac{6}{x^2}v=0$

$\Rightarrow x^2\frac{d^2v}{dx^2}-6v=0$

Cauchy euler eqn

$\Rightarrow (D'(D'-1)-6)v=0$ where $D'=\frac{d}{dz}$, $x=e^z$

$(D'^2-D'-6)v=0$

$\Rightarrow (D'^2-3D'+2D'-6)v=0$

$\Rightarrow (D'-3)(D'+2)v=0$

$v=c_1e^{3z}+c_2e^{-2z}$

$$\boxed{v=c_1x^3+\frac{c_2}{x^2}}$$

$$\therefore \boxed{y=uv=\left(c_1x^3+\frac{c_2}{x^2}\right)e^{-\frac{3}{4}x^{2/3}}}$$

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6) Solve the following differential equations:

$$(x\sin x + \cos x)y'' - x\cos x \cdot y' + y\cos x = 0.$$

$$y'' - \frac{x\cos x}{x\sin x+\cos x} y' + \frac{y\cos x}{x\sin x+\cos x} = 0$$

$$P = -\frac{x\cos x}{x\sin x+\cos x} \qquad Q = \frac{\cos x}{x\sin x+\cos x}$$

$$\boxed{P + Qx = 0}$$

$\therefore u = x$ is one solution where $y = uv$

$$\therefore \frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0.$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{-x\cos x}{x\sin x+\cos x} + \frac{2}{x}\right)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{2}{x} + \frac{-x\cos x}{x\sin x+\cos x}\right) z = 0$$

$$\Rightarrow \frac{dz}{z} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x+\cos x}\right)dx = 0$$

Integrate both side

$$\Rightarrow \log z + \log x^2 - \log(x\sin x+\cos x) = \log c$$

$$\Rightarrow z = \frac{c(x\sin x+\cos x)}{x^2}.$$

$$\Rightarrow \frac{dv}{dx} = c\frac{d}{dx}\left(\frac{-\cos x}{x}\right)$$

$$\therefore \boxed{v = -c\frac{\cos x}{x} + d}$$

$$\therefore y = u \cdot v = x\left(\frac{-\cos x}{x} \cdot c + d\right)$$

$$\boxed{y = \cos x\, c_1 + x c_2}$$

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7) Use the variation of paraneters method to show that the solution of equation $d\overset{2}{y}dx^2 + k^2y = \phi(x)$ satisfying the initial conditions $y(0) = 0, y'(0) = 0$ is $y(x) = \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)dt$.

$$\frac{d^2y}{dx^2} + k^2y = \phi(x)$$

Homogenous part of D·E

$$\frac{d^2y}{dx^2} + k^2y = 0$$

$\Rightarrow m^2 + k^2 = 0 \Rightarrow m = \pm ik$.

$\Rightarrow y_{CF} = c_1\cos kx + c_2 \sin kx$.

Now let $y_p = A\cos kx + B\sin kx$

$$W = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix} = k$$

$$\therefore A = \int \frac{-\sin kx \cdot \phi(x)}{k} dx$$

$$B = \int \frac{\cos kx\,(\phi(x))}{k} dx.$$

$$\therefore y_p = \cos kx \int \frac{-\sin kx\,\phi(x)}{k} dx + \sin kx \int \frac{\cos kx\,\phi(x)}{k} dx.$$

$$= \cos kx \int_0^x \frac{-\sin(kt)\,\phi(t)dt}{k} + \sin kx \int_0^x \cos(kt)\,\phi(t)\,dt$$

$$= \int_0^x \frac{\cos(kt)\cdot\sin kx \cdot \phi(t) - \cos kx \cdot \sin kt\;\phi(t)}{k} dt$$

$$y_p = \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)\,dt$$

$y_{CF} = c_1\cos kx + c_2\sin kx$

$\therefore y = y_{cf} + y_p = c_1\cos kx + c_2\sin kx + \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)dt$

Now, $y(0) = 0$

$\Rightarrow 0 = c_1 \Rightarrow c_1 = 0$

$y = c_2\sin kx + \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)dt$

$y'(x) = c_2 k\cos kx + \frac{1}{k}\,\phi(t)\sin k(x-x)\;\phi = 0$

$c_2 = 0$

$$\therefore\ y = \frac{1}{K}\int_0^x \phi(t)\sin(K(\cancel{t}-\cancel{t}))\,dt$$

$$\boxed{\Rightarrow y = \frac{1}{K}\int_0^x \phi(t)\sin K(\cancel{x}-t)\,dt}$$

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8) Solve $(D^2 + a^2)y = \tan ax$

Auxiliary equation of homogeneous DE

$m^2+a^2=0$

$\Rightarrow m = \pm ia$

$\therefore y_{CF} = c_1\cos ax + c_2 \sin ax$

$$y_p = \frac{1}{D^2+a^2}\tan ax$$

$$= \frac{1}{(D+ia)(D-ia)}\tan ax$$

$$= \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right)\tan ax$$

$$y_p = \frac{1}{2ia}\left(\frac{1}{D-ia}\tan ax - \frac{1}{D+ia}\tan ax\right)$$

Now, $\frac{1}{D-ia}\tan ax = e^{iax}\int e^{-iax}\tan ax\, dx$

$$= e^{iax}\int[\cos ax - i\sin ax]\tan ax\, dx$$

$$= e^{iax}\int\left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\left[\frac{-\cos ax}{a} - i\int\frac{1-\cos^2 ax}{\cos ax}dx\right]$$

$$= e^{iax}\left[\frac{-\cos ax}{a} - i\int(\sec ax - \cos ax)dx\right]$$

$$= e^{iax}\left[\frac{-\cos ax}{a} - \frac{i\log(\sec ax + \tan ax)}{a} + \frac{i\sin ax}{a}\right]$$

$$= \frac{e^{iax}}{a}\left[-\cos ax + i\sin ax - \frac{i\log(\sec ax+\tan ax)}{a}\right]$$

$$= \frac{e^{iax}}{-a}\left[e^{-iax} + i\log(\sec ax + \tan ax)\right]$$

again ~~$\frac{1}{D+ia}\tan ax = e^{iax}\int e^{iax}\tan ax$~~

put $i = -i$ in above

$$\frac{1}{D+ia} = \frac{e^{-iax}}{-a}\left[e^{iax} - i\log(\sec ax + \tan ax)\right]$$

$$\therefore \left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] = \frac{e^{iax}}{-a}\;\frac{i\log(\sec ax+\tan ax)}{-a}\left(e^{iax} - e^{-iax}\right)$$

$$= \frac{i \log(\sec ax + \tan ax)}{-a} 2i \sin ax$$

$$= \frac{\log(\sec ax + \tan ax)}{} \frac{2 \sin ax}{a}$$

$$\therefore y_p = \frac{1}{2ia} \times \frac{2}{a} \sin ax \log(\sec ax + \tan ax)$$

$$= \frac{1}{+ia^2} \sin ax \log(\sec ax + \tan ax)$$

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$$\therefore \boxed{y = c_1 \cos ax + c_2 \sin ax + \frac{1}{ia^2} \sin ax \log|\sec ax + \tan ax|}$$

9) Solve $(D^3+1)y = e^{2x}\sin x + e^{x/2}\sin(\sqrt{3}x/2)$.

$(D^3+1)y = e^{2x}\sin x + e^{x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)$

~~m^3+1~~ Auxilary eqn of homogenous DE part

$m^3+1=0$

$m=-1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$

$\therefore \boxed{y_{cf} = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)} \quad -(1)$

$y_p = \frac{1}{D^3+1} e^{2x}\sin x + \frac{1}{D^3+1} e^{x/2}\sin\frac{\sqrt{3}x}{2}$

Now, $\frac{1}{D^3+1} e^{2x}\sin x$

$= e^{2x}\cdot\frac{1}{(D+2)^3+1}\sin x = e^{2x}\frac{1}{D^3+8+3D^2\cdot 2+3D\cdot 4+1}\sin x$

$= e^{2x}\frac{1}{-D+8-6+12D+1}\sin x$

$= e^{2x}\frac{1}{11D+3}\sin x$

$= e^{2x}\frac{11D-3}{121D^2-9}\sin x$

$= e^{2x}\frac{(11\cos x - 3\sin x)}{-130}$

$= \frac{e^{2x}}{130}(3\sin x - 11\cos x)$

Again, $\frac{1}{D^3+1}e^{x/2}\sin\frac{\sqrt{3}x}{2}$

$= e^{x/2}\frac{1}{(D+\frac{1}{2})^3+1}\sin\frac{\sqrt{3}x}{2} = e^{x/2}\frac{1}{D^3+\frac{1}{8}+\frac{3}{4}D+\frac{3}{2}D^2+1}\sin\frac{\sqrt{3}x}{2}$

$= e^{x/2}\frac{1}{-D\cancel{\frac{3}{4}}+\cancel{\frac{3}{4}D}+\frac{1}{8}-\frac{3}{2}\times\frac{3}{4}+1}\sin\frac{\sqrt{3}x}{2}$

$= e^{x/2}\frac{1}{}$

$$= e^{x/2} \frac{1}{(D+\frac{1}{2})^3+1} \sin\frac{\sqrt{3}x}{2}$$

$$= e^{x/2} \frac{1}{D^3+\frac{1}{8}+\frac{3}{4}D+\frac{3}{2}D^2+1} \sin\frac{\sqrt{3}x}{2}$$

$$\cancel{= e^{x/2} \frac{1}{D^3-\frac{\sqrt{3}}{2}D^2+\frac{\sqrt{3}}{2}D^2-\frac{3}{4}D+}}$$

$$= e^{x/2} \frac{1}{D(D^2+\frac{3}{4})+\frac{3}{2}(D^2+\frac{3}{4})} \sin\frac{\sqrt{3}x}{2}$$

$$= e^{x/2} \frac{1}{(D^2+3/4)(D+\frac{3}{2})} \sin\frac{\sqrt{3}x}{2}$$

$$= e^{x/2} \frac{1}{D^2+3/4} \frac{D-3/2}{D^2-9/4} \sin\frac{\sqrt{3}x}{2}$$

$$= e^{x/2} \frac{1}{(D^2+3/4)(-3)} \left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x - \frac{3}{2}\cdot\sin\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{e^{x/2}}{-3}\left[\frac{\sqrt{3}}{2}\cdot\frac{x}{\cancel{2}} \sin\frac{\sqrt{3}}{2}x \cdot \frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2}\cdot\frac{x}{2}\cos\frac{\sqrt{3}}{2}x\cdot\frac{2}{\sqrt{3}}\right]$$

$$= \frac{e^{x/2}}{-3}\left[\frac{x}{2}\sin\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}x}{2}\cos\frac{\sqrt{3}}{2}x\right]$$

$$\therefore y_p = \frac{e^{x/2}}{-3} x \left[\frac{1}{2}\sin\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x\right] + \frac{e^{2x}}{130}(3\sin x - 11\cos x)$$

$$= \frac{e^{x/2}}{-3} x\left(\sin\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{3}\right)\right) + \frac{e^{2x}}{130}(3\sin x - 11\cos x)$$

$$\boxed{y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_3\sin\frac{\sqrt{3}}{2}x + c_2\cos\frac{\sqrt{3}}{2}x\right) - \frac{e^{x/2}}{3}x\sin\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{3}\right) + \frac{e^{2x}}{130}(3\sin x - 11\cos x)}$$

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10) Solve $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

Put $x+y=u \Rightarrow 1+\frac{dy}{dx}=du$

$x^2+y^2=2v$.

$\Rightarrow 2x+2\frac{dy}{dx}y=2dv$

$\Rightarrow 1+p=\frac{du}{dx}$

& $x+py=\frac{dv}{dx}$

$\Rightarrow \frac{1+p}{x+py}=\frac{du}{dv}$

$\Rightarrow$ ~~x^2+y^2 v (~~

$\frac{(x^2+y^2)(1+p)^2}{(x+yp)^2} - \frac{2(x+y)(1+p)}{(x+yp)}+1=0$

$\Rightarrow 2v\left(\frac{du}{dv}\right)^2 - 2(u)\frac{du}{dv}+1=0$

$\Rightarrow 2v\left(\frac{du}{dv}\right)^2 - 2u\frac{du}{dv}+1=0$

$\Rightarrow p^2v\cdot 2 = 2up+1=0$

$\Rightarrow \boxed{u = pv+\frac{1}{2p}}$

Clarait form

$\therefore u = cv+\frac{1}{2c}$

$\Rightarrow \boxed{(x+y)=2c(x^2+y^2)+\frac{1}{2c}}$

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