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UPSC Mathematics 2024 Solutions Paper 1

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1a) Let H be a subspace of R^4 spanned by the vectors $v_1 = (1, -2, 5, -3)$. $v_2 = (2, 3, 1, -4)$, $v_3 = (3, 8, -3, -5)$ Then find a basis and dimension of H, and extend the basis of H to a basis of R^4.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis $\to (1, -2, 5, -3), (0, 7, -9, 2)$

dim $= 2$

Extension $\to$ clearly $\left.\begin{matrix} (0,0,1,0) \\ (0,0,0,1) \end{matrix}\right]$ to be added to form Basis

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1b) Let $T: R^3 \to R^3$ be a linear operator and $B = (v_1, v_2, v_3)$ be a basis of R^3 over R. Suppose that $Tv_1 = (1,1,0)$, $Tv_2 = (1,0,-1)$, $Tv_3 = (2,1,-1)$. Find a basis for the range space and null space of T.

Range space = Span of $\{T(v_1), T(v_2), T(v_3)\}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rage space Basis $= \{(1,1,0), (0,-1,-1)\}$

Rank $= 2$

$T(1,0,0) = (1,1,0)$

$T(0,1,0) = (1,0,-1)$

$T(0,0,1) = (2,1,-1)$

$T(x,y,z) = T[(x,0,0) + (0,y,0) + (0,0,z)]$

$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$= x(1,1,0) + y(1,0,-1) + z(2,1,-1)$

$= (x+y+2z, x+z, -y-z)$

Null space $T(x,y,z)=0$

$$x+y+2z=0$$
$$x+z=0$$
$$-y-z=0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad R_2 \to R_2 - R_1 \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{R_2}{-1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x+y+2z=0$$
$$y+z=0 \to \text{Let } z=\alpha$$
$$\Rightarrow y=-\alpha$$

$$x=-y-2z=\alpha-2\alpha=-\alpha$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Null space $= (-1,-1,1)$

Nullity $=1$

Rank + Nullity $= 2+1=3 \leftarrow R^3$ (Proved)

1c) Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{1}{1-e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for all values of x.

Question Bank: SC-B02-Qn-3

RHL: $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \dfrac{1}{1-e^{-1/h}}$

$h\to 0 \quad e^{-1/h} = \dfrac{1}{e^{1/h}} = \dfrac{1}{e^{\infty}} = \dfrac{1}{\infty} = 0$

$\lim_{h\to 0} \dfrac{1}{1-e^{-1/h}} = \dfrac{1}{1-0} = 1$

LHL: $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \dfrac{1}{1-e^{1/h}}$

$h\to 0 \quad e^{1/h} \to e^{1/0} \to \infty$

$\lim_{h\to 0} \dfrac{1}{1-e^{1/h}} \to \dfrac{1}{\infty} \to 0$

$\lim_{h\to 0} f(0+h) \neq \lim_{h\to 0} f(0-h)$ So Discontinuous at $x=0$

For $x\neq 0$ Continuous

1d) Expand $\ln(x)$ in powers of $(x-1)$ by Taylor's theorem and hence find the value of ln(1.1) correct up to four decimal places.

Taylor:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \cdots$$

$f(x) = \ln x \quad a = 1$

$f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$

$f^{iv}(x) = -\frac{6}{x^4} \quad f^{iv}(1) = -6$

$$f(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3 2}{3!} - \frac{(x-1)^4 6}{4!} + \cdots$$

$\frac{2}{3!} = \frac{1}{3} \quad \frac{6}{4!} = \frac{1}{4}$

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$x = 1.1 \quad x-1 = 0.1$

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

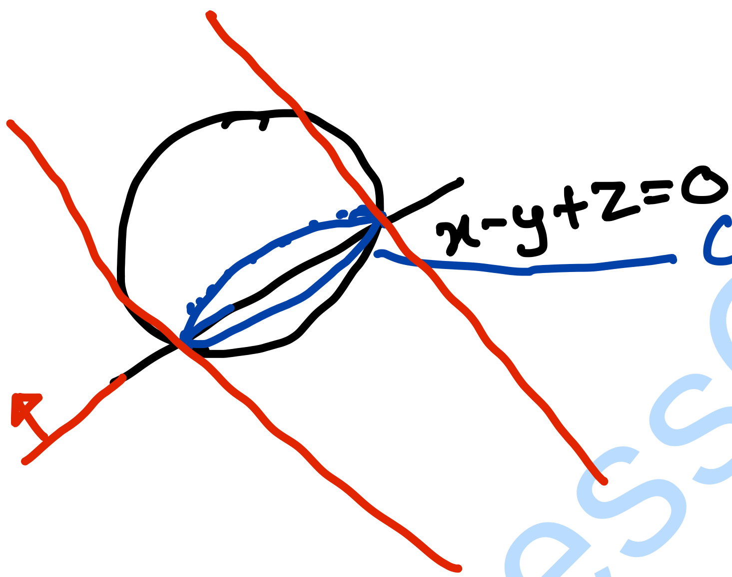
$$= 0.1 - 0.005 + 0.00033 - 0.000025$$

$$= 0.095305$$

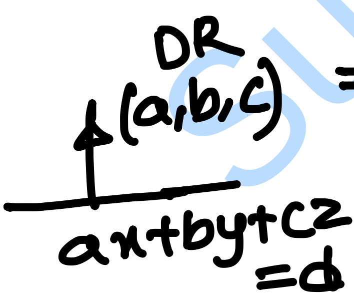
1e) Find the equation of the right circular cylinder which passes through the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$.

Question Bank SC-CO6 Qn-5

Guiding Circle $\rightarrow$ $\left.\begin{array}{l} x^2+y^2+z^2=9 \\ x-y+z=3 \end{array}\right\}$

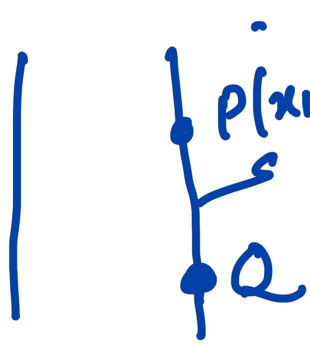


Axis of Cylinder $\Rightarrow$ perpendicular to plane



$\Rightarrow$ DR is $(1,-1,1)$

why $1.x-1.y+1.z=0$ (plane)



Eqn of Generator thru P is

$$\frac{x-x_1}{1} = \frac{y-y_1}{-1} = \frac{z-z_1}{1} = r$$

Another pt Q on generator is

$(r+x_1, -r+y_1, r+z_1)$

Generator pass thru circle $\left[\begin{array}{l} x^2+y^2+z^2=9 \\ x-y+z=3 \end{array}\right.$

↳ Let Q is on circle

$$\begin{cases} (r+x_1)^2+(-r+y_1)^2+(r+z_1)^2=9 & \text{— (1)} \\ (r+x_1)-(-r+y_1)+(r+z_1)=3 & \text{— (2)} \end{cases}$$

(1) $\Rightarrow 3r^2+2r(x_1+y_1+z_1)+x_1^2+y_1^2+z_1^2-9=0$ — (4)

(2) $\Rightarrow r=\frac{1}{3}(3-x_1+y_1-z_1)$ — (5)

Eliminate r by putting r from Eq 5 in Eq 4

$$\frac{1}{3}(3-x_1+y_1-z_1)^2+\frac{2}{3}(3-x_1+y_1-z_1)(x_1+y_1+z_1)$$
$$+x_1^2+y_1^2+z_1^2-9=0$$

$$\Rightarrow x_1^2+y_1^2+z_1^2+y_1z_1-z_1x_1+x_1y_1=9$$

Locus P is $x^2+y^2+z^2+yz-zx+xy=9$

2a) Consider a linear operator T on $\mathbf{R}^3$ over R defined by T(x,y,z) = $(2x, 4x - y, 2x + 3y - z)$ is T invertible ? If yes, justify your answer and find $\boldsymbol{T}^{-1}$.

Similar On: Full Length Test-07, 2024, Qn1

Method 1:

To show invertible $\Rightarrow$ let $T(x,y,z)=0$
then show $x=0, y=0, z=0$

$T(x,y,z)=0 \Rightarrow 2x=0 \Rightarrow x=0$
$4x-y=0 \Rightarrow y=0$ as $x=0$
$2x+3y-z=0 \Rightarrow z=0$ as $x=y=0$

$T(x,y,z)=(p,q,r)$
$\Rightarrow 2x, 4x-y, 2x+3y-z = p, q, r$
$\Rightarrow 2x=p \Rightarrow x=\frac{p}{2}$
$4x-y=q \Rightarrow y=4x-q=2p-q$
$2x+3y-z=r \Rightarrow z=2x+3y-r$
$=p+6p-3q-r$
$=7p-3q-r$

$T^{-1}(p,q,r)=(x,y,z)$
$=\left(\frac{p}{2}, 2p-q, 7p-3q-r\right)$

$$T^{-1}(p,q,r) = \left(\frac{p}{2},\ 2p-q,\ 7p-3q-r\right)$$

Method 2

$$T(x,y,z) = (2x,\ 4x-y,\ 2x+3y-z)$$

$$T(1,0,0) = (2,4,2) = 2(1,0,0)+4(0,0,1)+2(0,0,1)$$

$$T(0,1,0) = (0,-1,3) = 0(1,0,0)-1(0,0,1)+3(0,0,1)$$

$$T(0,0,1) = (0,0,-1) = 0(1,0,0)+0(0,0,1)-1(0,0,1)$$

$$[T]_B = A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

$|A| = 2(1) = 2 \neq 0$

$\Rightarrow$ A is invertible

$\Rightarrow$ T is invertible

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 \\ 7 & -3 & -1 \end{bmatrix} \qquad A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x}{2},\ 2x-y,\ 7x-3y-z$$

$$T^{-1}(x,y,z) = \left(\frac{x}{2},\ 2x-y,\ 7x-3y-z\right)$$

2b) If $u = (x+y)/(1-xy)$ and $v = \tan^{-1}x + \tan^{-1} y$, then find $\partial(u,v)/\partial(x,y)$. Are u and v functionally related? If yes, find the relationship.

Questn Bank SCB13 Qn31

$$u = \frac{x+y}{1-xy} \qquad \frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2} \qquad \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$v = \tan^{-1}x + \tan^{-1}y$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)^2}\cdot\frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2}\cdot\frac{1}{\cancel{1+x^2}}$$

$$= 0$$

$J = 0 \Rightarrow$ They are not independent

$$v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}u$$

$\underline{u = \tan v}$ relatn b/w u & v

2c) Find the image of the line $x = 3 - 6t,\ y = 2t,\ \ z = 3 + 2t$ in the plane $3x + 4y - 5z + 26 = 0$.

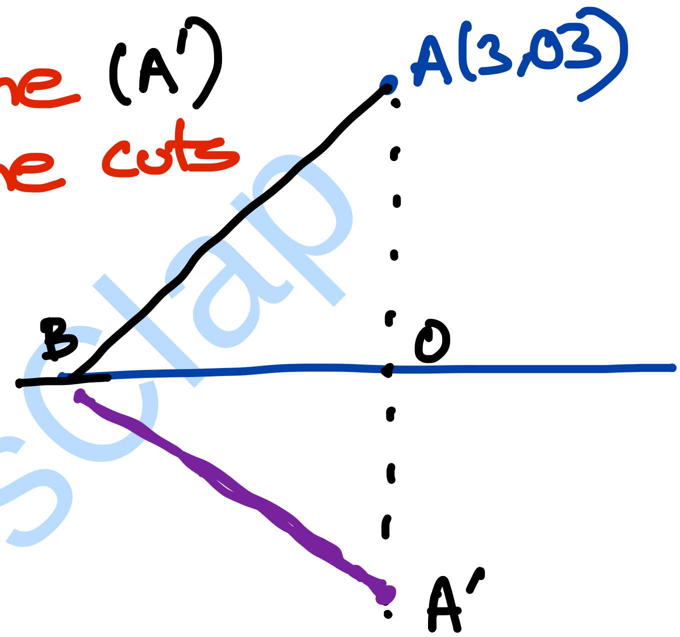
Line eqn is $\frac{x-3}{-6} = \frac{y-0}{2} = \frac{z-3}{2} = t$

↳ Clearly $A(3,0,3)$ lie on line

→ Find image of A on plane (A')

→ Find point where line cuts plane ie pt B

Our required line is A'B



To find A' :

Line perpendicular to plane has DRs $(3,4,-5)$ and also pass thru $(3,0,3)$

Line is $\frac{x-3}{3} = \frac{y-0}{4} = \frac{y-3}{-5} = k$

Line meets plane at O

Find O : Any pt on line is $(3k+3, 4k, -5k+3)$

pt lie on plane $3x+4y-5z+26=0$

$\underline{9}k+9+\underline{16}k+\underline{25}k-15+26=0$

$50k = -20 \qquad k = -\frac{2}{5}$

$-6+15$

O is $\left(-\frac{6}{5}+3, -\frac{8}{5}, 5\right)$

$\left(\frac{9}{5}, \frac{-8}{5}, 5\right)$

O is mid pt of A & A'

If A' is (α, β, γ) $\quad \left(O = \frac{A+A'}{2}\right)$

$\frac{9}{5} = \frac{3+\alpha}{2} \qquad \frac{-8}{5} = \frac{0+\beta}{2} \qquad 5 = \frac{3+\gamma}{2}$

$\alpha = \frac{18}{5} - 3 = \frac{3}{5} \qquad \beta = -\frac{16}{5} \qquad \gamma = 7$

$\left(\frac{3}{5}, \frac{-16}{5}, 7\right)$

Find pt B : $\frac{x-3}{-6} = \frac{y}{2} = \frac{z-3}{2} = t$

Any pt is $(-6t+3, 2t, 2t+3)$

B - lie on plane $3x+4y-5z+26=0$

$3(-6t+3) + 4(2t) - 5(2t+3) + 26 = 0$

$\Rightarrow t = 1$

pt B is $(-3, 2, 5)$

A' is $\left(\frac{3}{5}, \frac{-16}{5}, 7\right)$

B is $(-3, 2, 5)$

DRs of $A'B$ is $\frac{3}{5}+3, \frac{-16}{5}-2, 7-5)$

$$\left(\frac{18}{5}, \frac{-26}{5}, 2\right)$$

$$\sim (18, -26, 10)$$

$$\sim (9, -13, 5)$$

$A'B$ lie eqn is

$$\frac{x-3/5}{9} = \frac{y+16/5}{-13}, \frac{z-5}{7}$$

3a) Let $V = M_{2\times2}(R)$ denote a vector space over the field of real numbers. Find the matrix of the linear mapping $\phi: V \to V$ given by $\phi(v) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} v$ with respect to standard basis of $M_{2\times2}(\mathbb{R})$, and hence find the rank of φ. Is ϕ invertible? Justify your answer.

Standard Basis

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad b_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad b_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$v = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \qquad \varphi(v) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} =$$

$$= \begin{bmatrix} x+2z & y+2t \\ 3x-z & 3y-t \end{bmatrix}$$

$$\varphi(b_1) = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1b_1 + 0b_2 + 3b_3 + 0b_4$$

$$\varphi(b_2) = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0b_1 + 1b_2 + 0b_3 + 3b_4$$

$$\varphi(b_3) = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} = 2b_1 + 0b_2 - b_3 + 0b_4$$

$$\varphi(b_4) = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} = 0b_1 + 2b_2 + 0b_3 - b_4$$

Matrix is
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$

Transpose values

Rank : Transposing

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

Rank $= 4$

Invertible :

Method 1 : we got rank $= 4$

rank + nullity $= 4$

$\Rightarrow$ nullity $= 0$ as rank $= 4$

$\Rightarrow$ Invertible

Method 2 : $\varphi(v) = \begin{bmatrix} x+2z & y+2t \\ 3x-z & 3y-t \end{bmatrix}$

$v = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ Now $\varphi(v) = 0 \Rightarrow$

$x+2z=0$ $y+2t=0$ $3x-z=0$ $3y-t=0$

$\Rightarrow x=y=z=t=0 \Rightarrow v=0$ matrix

so Invertible

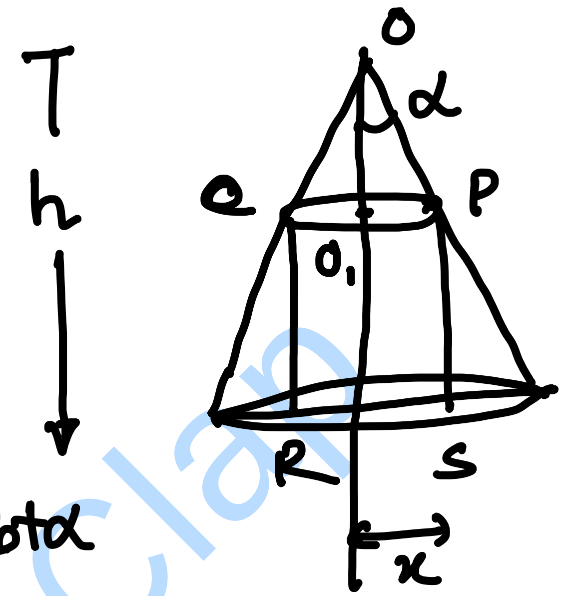
3b) Find the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α.

Question Bank : SCB04 Qn 40

PQRS Cylinder

Cone height $= h$

Cylinder radius $= x$
$= O_1P$



Cylinder height $=$
$h - OO_1 = h - x\cot\alpha$

$\tan\alpha = \dfrac{x}{OO_1}$

$OO_1 = x\cot\alpha$

Volume $V = \pi x^2 h =$
$= \pi x^2(h - x\cot\alpha)$
$= \pi[hx^2 - x^3\cot\alpha]$

$\dfrac{dV}{dx} = 0 \Rightarrow \pi[2hx - 3x^2\cot\alpha] = 0$
$\Rightarrow \pi x[2h - 3x\cot\alpha] = 0 \Rightarrow$
$\Rightarrow x = 0$ or $x = \dfrac{2h}{3}\tan\alpha$

$\dfrac{d^2V}{dx^2} = \pi[2h - 6x\cot\alpha]_{x = \frac{2h}{3}\tan\alpha} =$
$= \pi[2h - \dfrac{6.2h}{3}\tan\alpha.\cot\alpha] = -2\pi h =$ Negative
$\Rightarrow$ Max

$x = \dfrac{2h}{3}\tan\alpha$ $\quad V = \pi[hx^2 - x^3\cot\alpha]$
$= \dfrac{4}{27}\pi h^3\tan^2\alpha$

3c) Find the vertex of the cone $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$.

Make it Homogeneous

$F(x,y,z,t) = 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2$

$\frac{\partial F}{\partial x} = 8x + 2y + 12t = 0 \quad t=1 \Rightarrow 8x + 2y + 12 = 0$ Eq 1

$\frac{\partial F}{\partial y} = -2y + 2x - 3z - 11t = 0 \quad t=1 \Rightarrow 2x - 2y - 3z - 11 = 0$ Eq 2

$\frac{\partial F}{\partial z} = 0 = 4z - 3y + 6t \quad t=1 \Rightarrow 4z - 3y + 6 = 0$ Eq 3

$\frac{\partial F}{\partial t} = 0 = 12x - 11y + 6z + 8t \quad t=1 \Rightarrow 12x - 11y + 6z + 8 = 0$ Eq 4

Use only 3 eqns (Eq 1, 2, 3) to solve x, y, z gives $x = -1 \quad y = -2 \quad z = -3$

Put this value in Eqn 4 $\Rightarrow$ satisfy Eq 4

$\Rightarrow$ Cone is verified

Vertex is $(-1, -2, -3)$

4a) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ be a 3×3 matrix. Find the eigenvalues and the corresponding eigenvectors of A. Hence find the eigenvalues and the corresponding eigenvectors of A^{-15}, where $A^{-15} = (A^{-1})^{15}$.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} = -4 \quad \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4 \quad \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$$

$$-4-4-7 = -15$$

$$\text{Trace}(A) = 3+0+3 = 6$$

$$|A| = 8 \quad (\text{By Calculator})$$

$$\lambda^3 - (\text{Trace})\lambda^2 + (\text{Adj})\lambda - |A| = 0$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

$$\Rightarrow (\lambda - 8)(\lambda+1)^2 = 0 \Rightarrow \lambda = 8, -1, -1$$

$\underline{\lambda = 8}$:

$$A - \lambda I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \quad R_2 \to \frac{R_2}{2} \begin{bmatrix} -5 & 2 & 4 \\ 1 & -4 & 1 \\ 4 & 2 & -5 \end{bmatrix}$$

$$R_1 \to R_1 + 5R_2 \quad R_3 \to R_3 - 4R_2 \begin{bmatrix} 0 & -18 & 9 \\ 1 & -4 & 1 \\ 0 & 18 & -9 \end{bmatrix}$$

$R_3 \to R_3 + R_1 \begin{bmatrix} 0 & -18 & 9 \\ 1 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{9} \begin{bmatrix} 0 & -2 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$[\quad]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2y + z &= 0 \\ x - 4y + z &= 0 \end{aligned}\right]$$

Let $y = t \Rightarrow z = 2y = 2t$

$x = 4y - z = 4t - 2t = 2t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

E-vector : $t = 1 \Rightarrow (2, 1, 2)$

$A = -1$

$$A - \lambda I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 2R_2 \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

only eqn $2x + y + 2z = 0$

Let $x = \alpha \quad z = \beta \Rightarrow y = -2x - 2z$

$= -2\alpha - 2\beta$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\alpha - 2\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vectors are $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

Eigen value of A is $(8, -1, -1)$

Eigen value of A^{-15} is $\left(\frac{1}{8^{15}}, \frac{1}{(-1)^{15}}, \frac{1}{(-1)^{15}}\right)$

is $\left(\frac{1}{8^{15}}, -1, -1\right)$

Eigen vectors of A are

$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

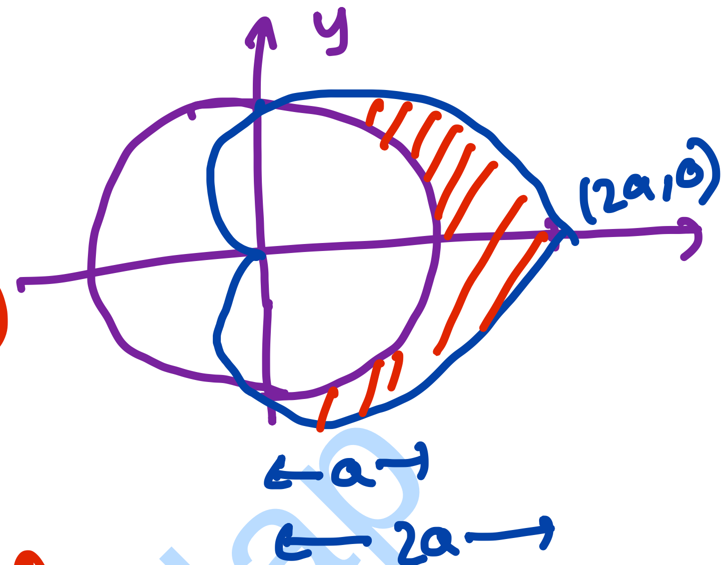
✓ Eigen vectors of A^{-15} are same as Eigen vector of A : NOTE

why $AX = \lambda X$ X : Eigen vector of A

$A^2X = \lambda AX = \lambda^2 X \Rightarrow A^3X = \lambda^3 X \Rightarrow A^9X = \lambda^9 X$

$\Rightarrow A^{-15}X = \lambda^{-15} X$ Imp

4b) Using double integration, find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.



$dx\,dy = r\,dr\,d\theta$

$r : r = a$ to $r = a(1+\cos\theta)$

$\theta : \theta = -\frac{\pi}{2}$ to $\theta = \pi/2$

$$I = \iint dx\,dy = \iint r\,dr\,d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=a}^{r=a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\right) r^2 \Big|_a^{a(1+\cos\theta)} d\theta =$$

$$\left[\int_{-a}^{a} \to 2\int_0^a \text{ Even fn}\right]$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[a^2(1+\cos\theta)^2 - a^2\right] d\theta$$

$$= 2 \times \frac{a^2}{2} \int_0^{\pi/2} \left[\cos^2\theta + 2\cos\theta\right] d\theta$$

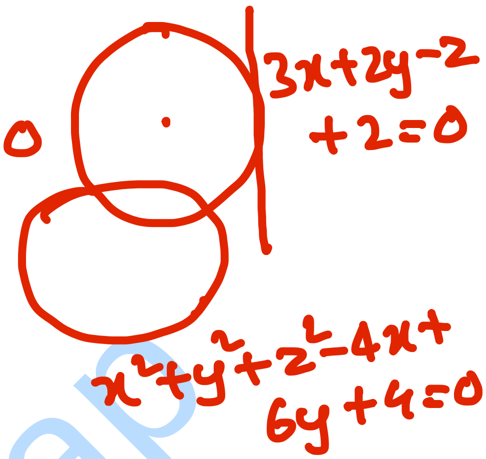
$$= a^2\left[\frac{1}{2} \times \frac{\pi}{2} + 2\,\frac{1}{\,}\right] = \frac{a^2(\pi+8)}{4}$$

$$\left[\int_0^{\pi/2} \cos^n\theta\,d\theta = \left(\frac{n-1}{n}\,\frac{n-3}{n-2} \cdots\cdots\right) \begin{cases} \pi/2 & \text{Even} \\ 1 & \text{odd} \end{cases}\right]$$

4c) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Let given sphere

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0$$



$\downarrow$

Tangent plane at $(1,-2,1)$

$$x-2y+z+g(x+1)+f(y-2)$$
$$+h(z+1)+c=0$$

$$(g+1)x+(f-2)y+(h+1)z+(g-2f+h+c)=0$$

But given tangent is $3x+2y-z+2=0$

$$\Rightarrow \frac{g+1}{3}=\frac{f-2}{2}=\frac{h+1}{-1}=\frac{g-2f+h+c}{2}=k$$

$$\Rightarrow g=3k-1 \quad f=2k+2 \quad h=-k-1,$$
$$g-2f+h+c=2k$$

Since sphere cuts orthogonally $x^2+y^2+z^2-4x+6y+4=0$

$$2g(-2)+2f(3)+2h(0)=c+4$$
$$-4g+6f=c+4$$

Put $g=3k-1$, $f=2k+2$ $h=-k-1$
in $g-2f+h+c=2k$

$$(3k-1)-2(2k+2)+(-k-1)+c=2k$$

$$\underline{-4k+c=6}$$

→ Put g, f, h in $-4g+6f=c+4$

$$\cancel{-12k}+4+\cancel{12k}+12=c+4$$

$$c=12$$

→ $c=12 \Rightarrow -4k+c=6 \Rightarrow 4k=c-6=6 \Rightarrow k=\frac{3}{2}$

$$k=\frac{3}{2} \Rightarrow g=3k-1=\frac{7}{2}$$

$$f=2k+2=5$$

$$h=-k-1=-5/2$$

Required sphere is

$$x^2+y^2+z^2+7x+10y-5z+12=0$$

5a) Find the orthogonal trajectories of the family of curves $r = c(\sec\,\theta + \tan\,\theta)$, where c is a parameter.

$$r = c(\sec\theta + \tan\theta)$$

$$\frac{dr}{d\theta} = c\left[\sec\theta\tan\theta + \sec^2\theta\right]$$

$$= c\sec\theta\left[\tan\theta + \sec\theta\right]$$

$$= r\sec\theta$$

$\downarrow$ orthogonal $\quad \frac{dr}{d\theta} \to -r^2\frac{d\theta}{dr}$

$$-r^2\frac{d\theta}{dr} = r\sec\theta \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$-\cos\theta\, d\theta = \frac{1}{r}dr$$

Integration

$$-\sin\theta = \ln r + c'$$

$$c - \sin\theta = \ln r$$

$$r = e^{c}\cdot e^{-\sin\theta}$$

$$= k e^{-\sin\theta}$$

5b) Solve the integral equation $y(t) = \cos\, t + \int_0^t y(x)\cos\,(t-x)dx$ using Laplace transform.

$$y(t)=\cos t + \int_0^t y(x)\cos(t-x)dx$$

Taking laplace

$$L(y(t)) = \frac{p}{p^2+1} + L\{y(t)\}\cdot L(\cos t)$$

$$= \frac{p}{p^2+1} + L(y(t))\left(\frac{p}{p^2+1}\right)$$

$$L(y(t))\left[1-\frac{p}{p^2+1}\right] = L(y(t))\left[\frac{p^2-p+1}{p^2+1}\right] = \frac{p}{p^2+1}$$

$$L(y(t)) = \frac{p}{p^2-p+1} = \frac{p}{\left(p-\frac{1}{2}\right)^2+3/4}$$

$$y(t)=L^{-1}\{\ \} = e^{t/2}\, L^{-1}\frac{p}{p^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= e^{t/2}\cos\frac{\sqrt{3}}{2}t$$

5c) At any time t (in seconds), the coterminous edges of a variable parallelepiped are represented by the vectors

$$\bar{\alpha} = t\hat{\imath} + (t+1)\hat{\jmath} + (2t+1)\hat{k}$$
$$\bar{\beta} = 2t\hat{\imath} + (3t-1)\hat{\jmath} + t\hat{k}$$
$$\bar{\gamma} = \hat{\imath} + 3t\hat{\jmath} + \hat{k}$$

What is the rate of change of the vectorial area of the parallelogram, whose coterminous edges are $\bar{\alpha}$ and $\bar{\gamma}$? Also find the rate of change of the volume of the parallelepiped at $t = 1$ second.

$$\alpha \times \gamma = \begin{vmatrix} i & j & k \\ t & t+1 & 2t+1 \\ 1 & 3t & 1 \end{vmatrix} = A$$

$$= i(-6t^2-2t+1) + j(t+1) + k(3t^2-t-1)$$

$$\frac{dA}{dt} = i(-12t-2) + j + k(6t-1)$$

$$\left.\frac{dA}{dt}\right)_{t=1} = -14i + j + 5k$$

$$\text{Volume} = A \cdot B \quad \text{or} \quad V = \begin{vmatrix} t & t+1 & 2t+1 \\ 2t & 3t-1 & t \\ 1 & 3t & 1 \end{vmatrix}$$

$$= (\alpha \times \gamma) \cdot B$$

$$= [\alpha \ \gamma \ \beta]$$

$$= (-6t^2-2t+1)(2t) + (t+1)(3t-1) + (3t^2-t-1)(t)$$

$= -12t^3 - 4t^2 + 2t + 3t^2 + 3t - t - 1$
$+ 3t^3 - t^2 - t$

$= -9t^3 - 2t^2 + 3t - 1$

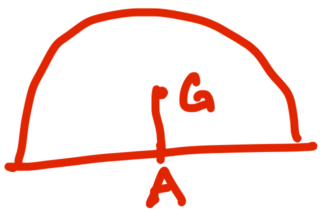
Rate of chge of volume $= \frac{dV}{dt}$

$= -27t^2 - 4t + 3$

$\left(\frac{dV}{dt}\right)_{t=1} = -27 - 4 + 3 = -28$

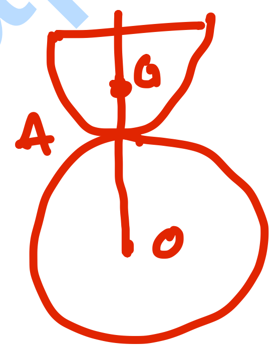
5d) A solid hemisphere rests in equilibrium on a solid sphere of equal radius. Determine the stability of the equilibrium in the two situations (i) when the curved surface and (ii) when the flat surface of the hemisphere rests on the sphere.

Questn Bank SCE06 Qn-1



Hemissphere
$CG: AG = \frac{3a}{8}$

(a) Curved surface on sphere



$AG = a - \frac{3a}{8} = \frac{5a}{8}$

Radius of lower body $R = a$
radiu of upper body $r = a$

$\frac{1}{R} + \frac{1}{r} = \frac{1}{a} + \frac{1}{a} = \frac{2}{a} = \frac{10}{5a}$

$\frac{1}{h} = \frac{8}{5a}$ $\quad \frac{1}{R} + \frac{1}{r} > \frac{1}{h}$ Unstable

(b) Flat surface $r = \infty$
radi of lower body $R = a$

$\frac{1}{R} + \frac{1}{r} = \frac{1}{a} + \frac{1}{\infty} = \frac{1}{a} = \frac{3a}{3a}$

$h = \frac{3a}{8}$: CG of upperbody $\frac{1}{h} = \frac{8}{3a}$

$\frac{1}{h} > \frac{1}{R} + \frac{1}{a} \rightarrow$ Stable

5e) (i) Let C be a plane curve $\bar{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$, where f and g have second-order derivatives. Show that the curvature at a point is given by

$$K = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}$$

What is the value of torsion τ at any point of this curve?

$$r = (f(t), g(t), 0)$$

$$\dot{r} = (f'(t), g'(t), 0)$$

$$\ddot{r} = [f''(t), g''(t), 0]$$

$$\dot{r} \times \ddot{r} = \begin{bmatrix} i & j & k \\ f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \end{bmatrix}$$

$$= k[f'(t)g''(t) - f''(t)g'(t)]$$

$$|\dot{r}|^2 = f'(t)^2 + g'(t)^2$$

$$K = \frac{\dot{r} \times \ddot{r}}{|\dot{r}|^3} = \frac{f'(t)g''(t) - f''(t)g'(t)}{[f'(t)^2 + g'(t)^2]^{3/2}}$$

$$\dddot{r} = [f'''(t), g'''(t), 0]$$

$$\tau = \text{Torsion} = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = [\dot{r} \times \ddot{r}] \cdot [\dddot{r}]$$

But $\dot{r} \times \ddot{r} = [\ ]\hat{k}$

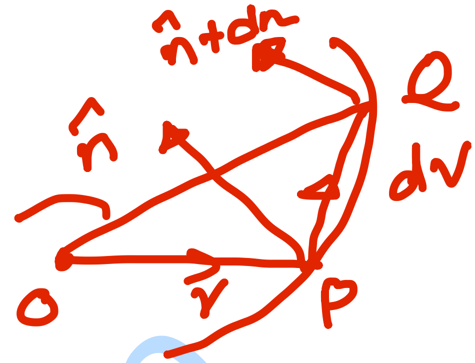
$$= [\ ]\ \hat{k} \cdot [f'''(t)\hat{i} + g'''(t)\hat{j} + 0\hat{k}]$$

$$= [\ ] \times 0 = 0$$

Torsion is zero

(ii) Show that the principal normals at two consecutive points of a curve do not intersect unless torsion τ is zero.

P, Q be two consecutive pts on curve,



Position vector is r, $r+dr$

Principal normal is n & $n+dn$

Three vectors $n, n+dn, dr$ lie on plane

$$[n, n+dn, dr] = 0$$

$$[n, n, dr] + [n, dn, dr] = 0$$

$$\Rightarrow [n, dn, dr] = 0$$

$$\Rightarrow \left[n, \frac{dn}{ds}, \frac{dr}{ds}\right] = 0$$

$$[n, n', r'] = 0 \qquad n' = \tau b - kt$$
$$r' = t$$

$$[n, \tau b - kt, t] = 0$$

$$[n, \tau b, t] - [n, kt, t] = 0$$

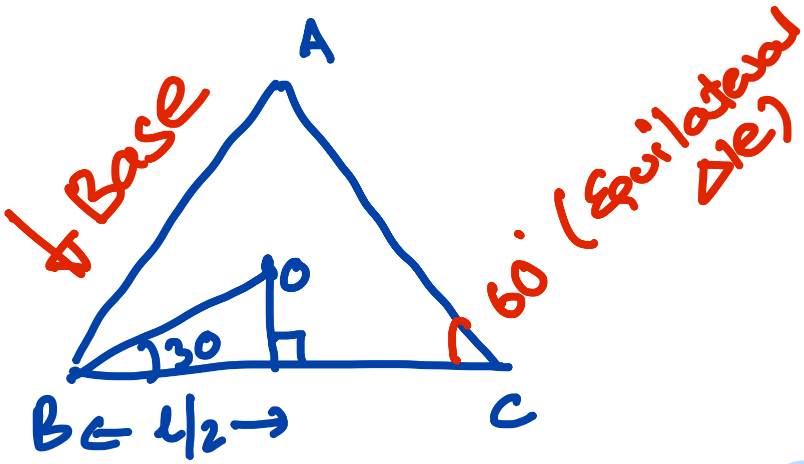
$$\tau[n\ b\ t] - k[n, t, t] = 0$$

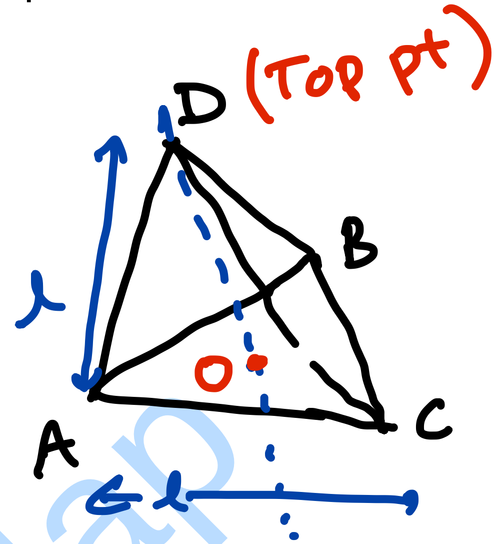
$$[n\ b\ t] = 1 \qquad [n\ t\ t] = 0$$

$$\tau = 0$$

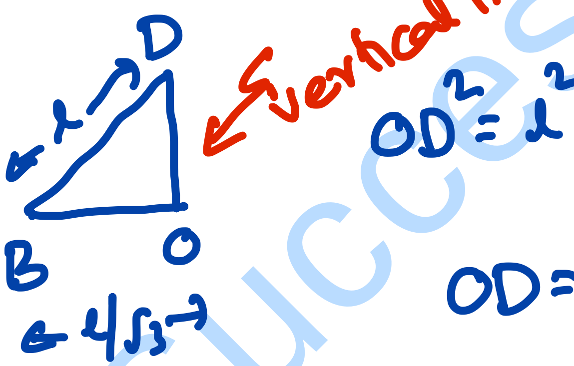
Principal normal at two consecutive pt do not intersect unless $\tau = 0$

6a) A regular tetrahedron, formed of six light rods, each of length l, rests on a smooth horizontal plane. A ring of weight W and radius r is supported by the slant sides. Using the principle of virtual work, find the stress in any of the horizontal sides.





$$\cos 30 = \frac{l/2}{OB} = \frac{\sqrt{3}}{2} \Rightarrow \frac{l}{OB} = \sqrt{3} \qquad OB = \frac{l}{\sqrt{3}}$$



$$OD^2 = l^2 - \left(\frac{l}{\sqrt{3}}\right)^2 = l^2 - \frac{l^2}{3} = \frac{2l^2}{3}$$

$$OD = \sqrt{\frac{2}{3}}\, l = \text{Height}$$

$$\frac{PD}{r} = \frac{OD}{OB}$$

$$PD = r\frac{OD}{OB} = r\frac{\sqrt{\frac{2}{3}}\, l}{l/\sqrt{3}} = r\sqrt{2}$$

$$OP = OD - PD$$

$$= \sqrt{\frac{2}{3}}\, l - r\sqrt{2} = \text{Distance of Ring from Base}$$

Ring : $W \times \delta P O$

Horizontal rod stress $= 3\,\delta l \cdot T$

Virtual method

$$-W\,\delta PO + 3T\,\delta l = 0$$

$$-W\left(\sqrt{\frac{2}{3}}\right)\delta l + 3\,\delta l\,T = 0$$

$$W\sqrt{\frac{2}{3}} \times \frac{1}{3} = T \qquad T = W\sqrt{\frac{2}{27}}$$

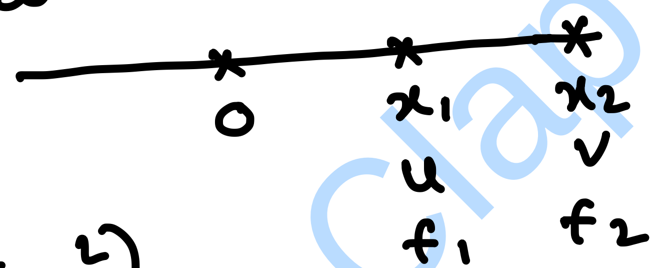
* Soln is to be verified again

6b) A particle executes simple harmonic motion such that in two of its positions, velocities are u and v, and the two corresponding accelerations are f_1 and f_2. For what value(s) of k, the distance between the two positions is $k(v^2 - u^2)$? Show also that the amplitude of the motion is $\frac{1}{f_2^2 - f_1^2}[(u^2 - v^2)(u^2 f_2^2 - v^2 f_1^2)]^{1/2}$

Queston Bank SCE02-Qn-10

SHM: $v^2 = \omega^2(A^2 - x^2)$ $\quad$ $A \to$ amp

$a = -\omega^2 x$ $\quad$ $a \to$ acceleration



$$\left[\begin{aligned} u^2 &= \omega^2(A^2 - x_1^2) \\ v^2 &= \omega^2(A^2 - x_2^2) \end{aligned}\right.$$

$$\left[\begin{aligned} f_1 &= \omega^2 x_1 \\ f_2 &= \omega^2 x_2 \end{aligned}\right.$$

Add

$f_1 + f_2 = \omega^2(x_1 + x_2)$

$$v^2 - u^2 = \omega^2[x_1^2 - x_2^2]$$

$$= \omega^2[x_1 - x_2][x_1 + x_2]$$

$$= \omega^2(x_1 + x_2)(x_1 - x_2)$$

$$= (f_1 + f_2)(x_1 - x_2)$$

$$x_1 - x_2 = \frac{v^2 - u^2}{f_1 + f_2}$$

Qn: what value of K, the distance b/w two position is $K(v^2-u^2)$

$K = ?$ $\quad x_1 - x_2 = K(v^2-u^2) = \dfrac{(v^2-u^2)}{f_1+f_2}$

$$\boxed{K = \frac{1}{f_1+f_2}}$$

$$\left.\begin{aligned} f_1 &= \omega^2 x_1 \Rightarrow \\ f_2 &= \omega^2 x_2 \Rightarrow \end{aligned}\right] \Rightarrow f_1 - f_2 = \omega^2(x_1 - x_2)$$

$$= \omega^2\left(\frac{v^2-u^2}{f_1+f_2}\right)$$

$$(f_1+f_2)(f_1-f_2) = \omega^2(v^2-u^2)$$

$$f_1^2 - f_2^2 = \omega^2(v^2-u^2)$$

$$\omega^2 = \frac{f_1^2-f_2^2}{v^2-u^2}$$

why $x_1 - x_2 = \dfrac{v^2-u^2}{f_1+f_2}$ proved above

we have $u^2 = \omega^2(A^2 - x_1^2)$ $\quad x_1 = \dfrac{f_1}{\omega^2}$

$$= \omega^2\left(A^2 - \frac{f_1^2}{\omega^4}\right)$$

$$\frac{u^2}{\omega^2} = A^2 - \frac{f_1^2}{\omega^4}$$

$$A^2 = \frac{u^2}{\omega^2} + \frac{f_1^2}{\omega^4} \qquad \frac{1}{\omega^2} = \frac{v^2-u^2}{f_1^2-f_2^2}$$

$$A^2 = u^2\left(\frac{v^2-u^2}{f_1^2-f_2^2}\right) + f_1^2\left(\frac{v^2-u^2}{f_1^2-f_2^2}\right)^2$$

$$= \frac{v^2-u^2}{f_1^2-f_2^2}\left[u^2 + \frac{(v^2-u^2)f_1^2}{f_1^2-f_2^2}\right]$$

$$= \frac{v^2-u^2}{f_1^2-f_2^2}\left[\frac{u^2f_1^2-u^2f_2^2+v^2f_1^2-u^2f_1^2}{f_1^2-f_2^2}\right]$$

$$= \frac{v^2-u^2}{f_1^2-f_2^2}\left[\frac{v^2f_1^2-u^2f_1^2}{f_1^2-f_2^2}\right]$$

$$= \frac{1}{(f_1^2-f_2^2)^2}(v^2-u^2)(v^2f_1^2-u^2f_2^2)$$

As per answer $(f_1^2-f_2^2)^2 = (f_2^2-f_1^2)^2$

$(-1a)^2 = a^2$

$$(v^2-u^2)(v^2f_1^2-u^2f_2^2) = (u^2-v^2)(u^2f_2^2-v^2f_1^2)$$

$-1 \times -1 = 1$

$$A = \frac{1}{f_2^2-f_1^2}\left[(u^2-v^2)(u^2f_2^2-v^2f_1^2)\right]^{1/2}$$

6c) (a) Find the second solution of the differential equation $xy'' + (x-1)y' - y = 0$ using $u(x) = -e^{-x}$ as one of the solutions.

$y = uv \qquad u = -e^{-x}$

$\frac{du}{dx} = (-1)(-1)e^{-x} = e^{-x}$

$\frac{2}{u}\frac{du}{dx} = \frac{2}{-e^{-x}} \times e^{-x} = -2$

$y'' + \left(\frac{x-1}{x}\right)y' - \left(\frac{1}{x}\right)y = 0$

$y'' + Py' + Qy = R \qquad P = \frac{x-1}{x} \quad Q = -\frac{1}{x}$

$R = 0$

$P + \frac{2}{u}\frac{du}{dx} = \frac{x-1}{x} - 2 = 1 - \frac{1}{x} - 2 = -1 - \frac{1}{x} = -\left(1 + \frac{1}{x}\right)$

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0$

$\frac{dv}{dx} = q$

$\frac{dq}{dx} - \left(1 + \frac{1}{x}\right)q = 0 \quad \Rightarrow \frac{dq}{q} = \left(1 + \frac{1}{x}\right)dx$

$\ln q = x + \ln x + \ln c$

$q = e^{x} \cdot xc = cxe^{x}$

$\frac{dv}{dx} = cxe^{x}$

$$v = c\int xe^{x}\,dx$$

$$= c\left[x\int e^{x} - \int\left(1 \cdot \int e^{x}\right) dx\right]$$

$$= c\left[xe^{x} - e^{x}\right] = c(x-1)e^{x}$$

$$y = uv \qquad u = -e^{-x}$$

$$= c(x-1)e^{x}\left(-e^{-x}\right) = c(1-x)$$

second soln is $y = c(1-x)$

(b) Find the general solution of the differential equation $x^2y'' - 2xy' + 2y = x^3 \sin x$ by the method of variation of parameters.

$$x^2y'' - 2xy' + 2y = x^3 \sin x$$

$$x = e^z \quad D_1 \equiv \frac{d}{dz} \quad (D_1(D_1-1) - 2D_1 + 2)y = 0$$

$$(D_1^2 - 3D_1 + 2)\,y = 0 \quad = (D_1-1)(D_1-2) = 0$$

$$D_1 = 2, 1$$

$$y_c = c_1 e^z + c_2 e^{2z}$$

$$= c_1 x + c_2 x^2$$

$$u = x \quad v = x^2 \quad u' = 1 \quad v' = 2x$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

$$PI = Au + Bv$$

Main eqn is $y'' - \frac{2y'}{x} + \frac{2}{x^2}y = x \sin x$
$= R$

$$R = x \sin x$$

$$A = -\int \frac{vR}{W}\,dx \qquad B = \int \frac{uR}{W}\,dx$$

$$A = -\int \frac{(x^2)(x\sin x)}{x^2}\,dx = -\int x\sin x\,dx$$

$$= (-1)\left[\, x(-\cos x) - \int (-\cos x)\,dx \right]$$

$$= (-1)\left[-x\cos x + \sin x\right]$$

$$= x\cos x - \sin x$$

$$B = \int \frac{uR}{W}\,dx = \int \frac{x \cdot x\sin x}{x^2}\,dx = \int \sin x\,dx$$

$$= -\cos x$$

$$PI = Au + Bv \qquad u = x \quad v = x^2$$

$$= x^2\cos x - x\sin x - x^2\cos x$$

$$= -x\sin x$$

$$Z = c_1 x + c_2 x^2 - x\sin x$$

7a) State uniqueness theorem for the existence of unique solution of the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ in the rectangular region $R: |x - x_0| \leq a, |y - y_0| \leq b$. Test the existence and uniqueness of the solution of the initial value problem $\frac{dy}{dx} = 2\sqrt{y}, \quad y(1) = 0$, in a suitable rectangle R. If more than one solution exists, then find all the solutions.

Existence Thm: If f is continuous function in open rectangle $R = \{(x,y)\ a<x<b,\ c<y<d\}$ that contains point (x_0, y_0), then initial value problem $y' = f(x,y)$ $y(x_0) = y_0$ has atleast a solution in some open sub-interval of (a,b) which contains the point x_0

Uniqueness Thm: If f and f_y are continuous function in open rectangle $R = \{(x,y)\ a<x<b\ \ c<y<d\}$ that contains a point (x_0, y_0), then the initial value problem $y' = f(x,y)$ $y(x_0) = y_0$ has a unique solution on some open sub-interval of (a,b) which contains the point x_0.

→ It provide info about existence of soln to Initial Value problem, but does not state how to find solutions or find the open interval

→ It does not provide info about how many solutions may have.

→ $\frac{dy}{dx} = 2\sqrt{y} = f(x,y) \quad y(1)=0$

$f'(y) = 2\left(\frac{-1}{2\sqrt{y}}\right) = \frac{-1}{\sqrt{y}}$

$f'(y)$ is not continous at $y=0$

Thm : f & $f'(y)$ to be continous then uniqueness

Here $f'(y)$ is not continous at $y=0$

So No-Unique soln

→ <u>other solns</u> : $\frac{dy}{dx} = 2\sqrt{y}$

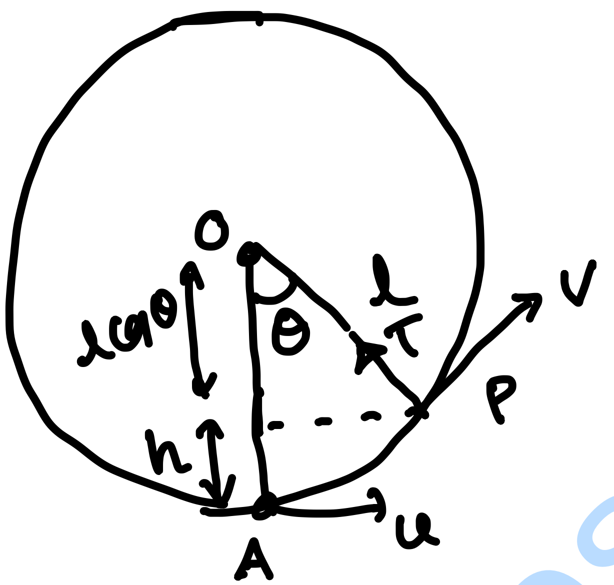
$\frac{dy}{\sqrt{y}} = 2dx = 2\sqrt{y} = 2x + c$

But $y(1)=0 \Rightarrow x=1\ y=0 \Rightarrow c=-2$

Soln $2\sqrt{y} = 2x-2 \Rightarrow \underline{\sqrt{y} = x-1}$

7b) A heavy particle hanging vertically from a fixed point by a light inextensible string of length l starts to move with initial velocity u in a circle so as to make a complete revolution in a vertical plane. Show that the sum of tensions at the lends of any diameter is constant.

UPSC - 1998



A: velocity u

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$$

$$h = l - l\cos\theta$$

$$= l(1-\cos\theta)$$

$$u^2 = v^2 + 2gl(1-\cos\theta)$$

$$v^2 = u^2 - 2gl(1-\cos\theta)$$

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$T = mg\cos\theta + \frac{mv^2}{l}$$

$$= mg\cos\theta + \frac{m\left[u^2 - 2gl(1-\cos\theta)\right]}{l}$$

$$= mg\cos\theta + \frac{mu^2}{l} - 2mg(1-\cos\theta)$$

$$= \frac{mu^2}{l} + mg\cos\theta - 2mg + 2mg\cos\theta$$

$$T = \frac{mu^2}{l} - 2mg + 2mg\cos\theta$$

$$[T]_{at\ \theta} = \frac{mu^2}{l} - 2mg + 2mg\cos\theta$$

At other end of diameter

$$\theta \to \theta + \pi$$

$$\cos(\theta + \pi) = -\cos\theta$$

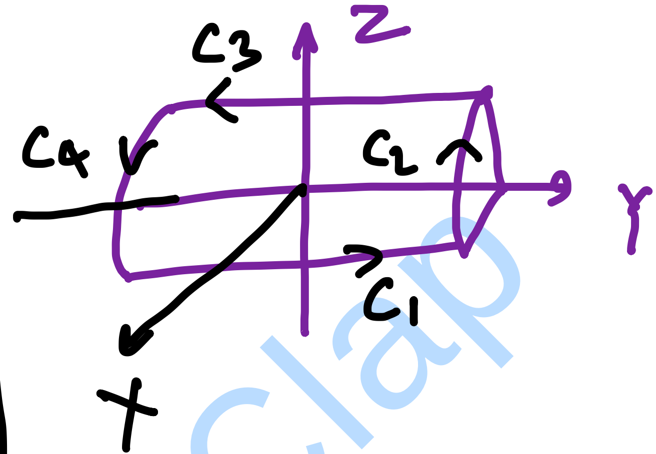
$$[T]_{at\ \theta+\pi} = \frac{mu^2}{l} - 2mg - 2mg\cos\theta$$

$$[T]_{at\ \theta} + [T]_{at\ \theta+\pi} = \frac{2mu^2}{l} - 4mg$$

$$= \text{constant}$$

7c) State Stokes' theorem and verify it for the vector field $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$ over the surface S, which is the upwardly oriented part of the cylinder $z = 1 - x^2$, for $0 \le x \le 1, -2 \le y \le 2$.

Repeated UPSC Qn - UPSC-CSE-2020



$$F = xy\,i + yz\,j + zx\,k$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\,i - z\,j - x\,k$$

$$g(x,y,z) = z + x^2 - 1 = 0$$

$$\hat{n} = \frac{\nabla g}{|\nabla g|} = \frac{2x\,i + k}{\sqrt{4x^2+1}}$$

$$I = \int_S \left[(\nabla \times F).n\right] ds = \iint_S \frac{-2xy - x}{\sqrt{4x^2+1}}\, ds$$

$$ds = \frac{dx\,dy}{|\hat{n}.k|} = \sqrt{4x^2+1}\, dx\,dy$$

$$I = \iint_S (-2xy - x)\frac{1}{\sqrt{4x^2+1}} \times \sqrt{4x^2+1}\, dx\,dy$$

$$= \int_{x=0}^{x=1} \int_{y=-2}^{y=2} (-2xy - x)\, dy\, dx$$

$$= \int_{x=0}^{x=1} -xy^2 - xy \Big|_{-2}^{2} dx = \int_0^1 -4x\, dx$$
$$= \underline{-2}$$

Line Integral $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

$F \cdot dr = xy\,dx + yz\,dy + zx\,dz$

$C_1 : x=1 \quad z=0 \quad dx=0 \quad dz=0$

$F \cdot dr = 0 \Rightarrow \int_{C_1} = 0$

$C_2 : y=2 \quad z=1-x^2 \quad dy=0 \quad dz=-2x\,dx$

$F \cdot dr = 2x\,dx + 0 + x(1-x^2)(-2x\,dx)$

$= 2x - 2x^2 + 2x^4\, dx$

$\int_{C_2} = \int_1^0 (2x - 2x^2 + 2x^4)\, dx = \frac{-11}{15}$

$C_3 : x=0 \quad z=1 \quad dx=0 \quad dz=0$

$\int_{C_3} y\,dy = \int_2^{-2} y\,dy = 0$

$C_4 : y=-2 \quad z=1-x^2 \quad dy=0 \quad dz=-2x\,dx$

$\int_{C_4} (-2x - 2x^2 + 2x^4)\, dx = -19/15$

$\oint F \cdot dr = 0 - 11/15 + 0 - 19/15 = \underline{-2}$

Proved

8a) Using Laplace transform, solve the initial value problem

$$y'' + 2y' + 5y = \delta(t-2),\ y(0) = 0,\ y'(0) = 0$$

where $\delta(t-2)$ denotes the Dirac delta function.

$$y'' + 2y' + 5y = \delta(t-2)$$

$$L(y'') + 2L(y') + 5L(y) = L\{\delta(t-2)\}$$

$$p^2 L(y) - p\,y(0) - y'(0) + 2[pL(y) - y(0)] + 5L(y) = L\{\delta(t-2)\}$$

$$p^2 L(y) + 2p(L(y)) + 5L(y) = e^{-2p}$$

$$(p^2+2p+5)L(y) = e^{-2p}$$

$$L(y) = \frac{e^{-2p}}{p^2+2p+5} = \frac{e^{-2p}}{(p+1)^2+4}$$

Take inverse

$$L^{-1}\left(\frac{1}{(p+1)^2+4}\right) = e^{-t}\sin 2t$$

Time shifting property

$$L^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

$$y(t) = u(t-2)e^{-(t-2)}\sin(2(t-2))$$

8b) Using Gauss divergence theorem, evaluate the integral

$$\iint_S \left(y^2\hat{\imath} + xz^3\hat{\jmath} + (z-1)^2\hat{k}\right) \cdot \hat{n}\,dS$$

over the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1$ and $z = 5$.

$$\int_S F.n\,ds = \int_V (\nabla.F)\,dv$$

$$F = y^2 i + xz^3 j + (z-1)^2 k$$

$$\nabla.F = 0 + 0 + 2(z-1)$$

$$I = \int_V (\nabla.F)\,dv = \int_V 2(z-1)\,dv$$

Cylindrical Coord System

$$= \int\iint 2(z-1)\,dz\,dx\,dy$$

$$= \iiint 2(z-1)\,\rho\,d\rho\,d\theta\,dz$$

$$= \left[\int_{\rho=0}^{\rho=4} \rho\,d\rho\right]\left[\int_{\theta=0}^{2\pi} d\theta\right]\int_{z=1}^{z=5} 2(z-1)\,dz$$

$$= \frac{\rho^2}{2}\Big|_0^4 \times 2\pi \times (z-1)^2\Big|_1^5$$

$$= 8 \times 2\pi \times \left[4^2 - 0\right] = 16\pi \times 16$$

$$= 256\pi$$

8c) A particle moves with a central acceleration $\mu\left(\frac{3}{r^3}+\frac{d^2}{r^5}\right)$ being projected from a distance d at an angle $45°$ with a velocity equal to that in a circle at the same distance. Prove that the time it takes to reach the centre of force is $\frac{d^2}{\sqrt{2\mu}}\left(2-\frac{\pi}{2}\right)$

$u=\frac{1}{r}$ central acceleration $P=\mu\left(\frac{3}{r^3}+\frac{d^2}{r^5}\right)$

V : velocity in a circle at distance d

$$\frac{V^2}{d}=[P]_{r=d}=\mu\left(\frac{3}{d^3}+\frac{d^2}{d^5}\right)$$

$a=d$

$$V^2=\frac{4\mu}{d^2} \qquad V=\frac{2\sqrt{\mu}}{d}$$

Diff eqn

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left[3u^3+d^2u^5\right]$$

$$=\mu(3u+d^2u^3]$$

Multiply both side by $2\frac{du}{d\theta}$ & integrating

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(3u^2+\frac{d^2}{2}u^4\right]+A$$

Initial $r=d$ $u=1/d$ $v=\frac{2\sqrt{\mu}}{d}$ $\theta=45°$

$$p=r\sin\phi=d\sin\frac{\pi}{4}=\frac{d}{\sqrt{2}}$$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2}{d^2}$$

$$\frac{4\mu}{d^2} = h^2 \cdot \frac{2}{d^2} = \mu\left(\frac{3}{d^2} + \frac{d^2}{2d^4}\right) + A$$

$$h^2 = 2\mu \qquad A = \frac{\mu}{2d^2}$$

Putting h^2, A in eqn

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + \frac{d^2}{2}u^4\right) + \frac{\mu}{2d^2}$$

$$2\left(\frac{du}{d\theta}\right)^2 = u^2 + \frac{d^2}{2}u^4 + \frac{1}{2d^2}$$

$$u = \frac{1}{r} \qquad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{d^2}{2r^4} + \frac{1}{2d^2}$$

$$4d^2\left(\frac{dr}{d\theta}\right)^2 = 2d^2r^2 + d^4 + r^4 = \left(r^2 + d^2\right)^2$$

$$\frac{dr}{d\theta} = -\frac{r^2 + d^2}{2d}$$

negative sign because r decrease when θ increase

$$h = r^2\frac{d\theta}{dt} = r^2\frac{d\theta}{dr} \cdot \frac{dr}{dt}$$

$$\sqrt{2\mu} = -r^2\frac{2d}{r^2 + d^2} \cdot \frac{dr}{dt}$$

$$dt = -\frac{2d}{\sqrt{2H}} \frac{r^2 dr}{r^2+d^2}$$

$$t_1 = \frac{-2d}{\sqrt{2H}} \int_{r=d}^{0} \frac{r^2 dr}{r^2+d^2}$$

$$= \frac{-2d}{\sqrt{2H}} \int_{d}^{0} \left[1 - \frac{d^2}{r^2+d^2} \right] dr$$

$$= \frac{-2d}{\sqrt{2H}} \left[r - d\tan^{-1}\frac{r}{d} \right]_d^0$$

$$= \frac{-2d}{\sqrt{2H}} \left[\left(0 - d\tan^{-1}0\right) - \left(d - d\tan^{-1}\frac{d}{d}\right) \right]$$

$$= \frac{2d}{\sqrt{2H}} \left[d - d \cdot \frac{\pi}{4} \right]$$

$$= \frac{d^2}{\sqrt{2H}} \left[2 - \frac{\pi}{2} \right]$$