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UPSC MATHEMATICS STUDY MATERIAL

BOOK- 05 Vector Analysis

Important Links

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Curves in Space

1. INTRODUCTION

Differential geometry is that branch of geometry which is treated using the methods of calculus. In particular, we investigate curves and surfaces in space in differential geometry. Differential geometry plays an important role in engineering designs, geodesy, geograph and space travel. Formulae regarding vector algebra and vector calculus are frequently used in the study of differential geometry.

2. BRANCHES OF DIFFERENTIAL GEOMETRY

There are two branches of differential geometry.

(*i*) **Local Differential Geometry.** In this branch of differential geometry, we study the properties of curves and surfaces in space which depend only upon points close to a particular point of the geometric figure under consideration.

(*ii*) **Global Differential Geometry.** In this branch of differential geometry, we study the properties of curves and surfaces in space which involve the entire geometric figure under consideration.

In the present course, we shall study some of the fundamentals of local differential geometry.

3. FUNCTIONS OF CLASS C^m

A scalar valued (or vector valued) function defined on an interval I belongs to class C^m on the interval I if the m^{th} order derivative of the function exists and is continuous on I.

The class of continuous functions is denoted by C^0. The class of functions having derivatives of all orders is denoted by C^∞.

If a function belongs to the class C^m, then that function is called a **C^m function.**

We know that a vector function is continuous or has a derivative if and only if all components of the functions are continuous or have derivatives.

$\therefore$ A vector function $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ belongs to C^m on I if and only if its components $f_1(t)$, $f_2(t)$ and $f_3(t)$ belong to C^m on I.

Remark 1. We know that a differential function is always continuous.

$\therefore$ If a function belongs to C^m, then it belongs to C^k for all $k \leq m$.

Remark 2. In printing work, the vector quantity $\mathbf{f}$ is depicted by using bold letter. In writing work, the vector $\mathbf{f}$ is written as $\vec{f}$ or $\underline{f}$.

Example 1. *Show that the vector function* $\mathbf{f}(t) = (\cos t)\mathbf{i} + t^3\mathbf{j} + t^{5/3}\mathbf{k}$, $-\infty < t < \infty$ *belongs to* C^1 *on* $-\infty < t < \infty$ *and not* C^2 *on* $-\infty < t < \infty$.

Sol. We have $\mathbf{f}(t) = (\cos t)\mathbf{i} + t^3\mathbf{j} + t^{5/3}\mathbf{k}$

$$\therefore \quad \dot{\mathbf{f}}(t) = (-\sin t)\mathbf{i} + 3t^2\mathbf{j} + \frac{5}{3}t^{2/3}\mathbf{k} \qquad \left(\dot{\mathbf{f}} = \frac{d\mathbf{f}}{dt}\right)$$

$-\sin t$, $3t^2$, $\frac{5}{3}t^{2/3}$ are continuous functions of t, where $-\infty < t < \infty$.

$\therefore$ $\dot{\mathbf{f}}(t)$ is continuous on $-\infty < t < \infty$.

$\therefore$ $\mathbf{f}(t)$ belongs to C^1 on $-\infty < t < \infty$.

Also,
$$\ddot{\mathbf{f}}(t) = (-\cos t)\mathbf{i} + 6t\mathbf{j} + \frac{10}{9t^{1/3}}\mathbf{k}$$

The function $\frac{10}{9t^{1/3}}$ is not continuous at $t = 0$.

$\therefore$ The scalar function $t^{5/3}$ does not belong to C^2 on $-\infty < t < \infty$.

$\therefore$ $\mathbf{f}(t)$ does not belong to C^2 on $-\infty < t < \infty$.

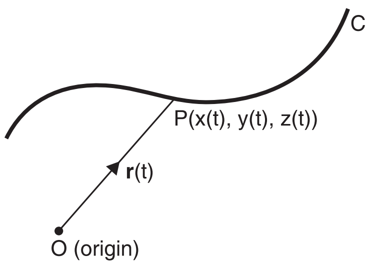
Remark. $\mathbf{f}(t)$ belongs to C^m for all $m \geq 0$ on any interval not containing '0'.

4. CURVE IN SPACE

A **curve in space** is defined as the locus of a point whose position vector relative to a fixed origin may be expressed as a function of a single parameter.

Thus, a curve C in space may be represented by a vector function

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where t is a parameter. Here $\mathbf{r}(t)$ is the position vector of the point P on the curve C and $x(t), y(t), z(t)$ are the cartesian coordinates of the point P. To each value t' of t there correspond a unique point of the curve C with position vector $\mathbf{r}(t')$ and cartesian coordinates $(x(t'), y(t'), z(t'))$.



As t increases, the direction of travelling along the curve C is called the **positive sense** on the curve C. Also as t decreases, the direction of travelling along the curve C is called the **negative sense** on the curve C.

If a curve in space lies wholly in a plane, then it is called a **plane curve.**

If a curve in space does not lie wholly in a plane then it is called a **skew curve** or a **tortous curve** or a **twisted curve.**

Example 2. *Show that the curve in space* $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + 0\mathbf{k}$ *is a plane curve.*

Sol. We have $\mathbf{r}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j} + 0\mathbf{k}$.

$\therefore$ Let (x, y, z) be the coordinates of the point with position vector $\mathbf{r}(t)$.

$$\therefore \quad x = a\cos t, \quad y = b\sin t, \quad z = 0$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0 \qquad (\because \cos^2 t + \sin^2 t = 1)$$

This represents an ellipse in the xy-plane.

$\therefore$ The given curve is a plane curve.

5. REGULAR CURVE

A curve $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$ is called a **regular curve** if

(*i*) $\dot{\mathbf{r}}(t)$ exists and is continuous on $a \le t \le b$ *i.e.*, $\mathbf{r}(t)$ is of class C^1 on $a \le t \le b$.

(*ii*) $\dot{\mathbf{r}}(t) \neq \mathbf{0}$ for all t in $a \le t \le b$.

For example, consider the curve

$$\mathbf{r} = \mathbf{r}(t) = 3t\mathbf{i} + t^4\mathbf{j} + 2\mathbf{k}, \ -\infty < t < \infty.$$

Here $\dot{\mathbf{r}}(t) = 3\mathbf{i} + 4t^3\mathbf{j} + 0\mathbf{k}$

$\dot{\mathbf{r}}(t)$ is continuous on $-\infty < t < \infty$ and also non-zero.

$\therefore$ The given curve is a regular curve.

Remark. If $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a regular curve then, $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}$ are never zero simultaneously.

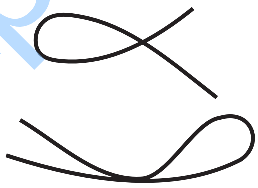
6. SIMPLE CURVE

A curve $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$ is called a **simple curve** if

(*i*) $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$ is a regular curve.

(*ii*) $t_1 \neq t_2 \Rightarrow \mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ *i.e.*, the curve is without points at which the curve intersects or touches itself.

Remark. A point where a curve intersects or touches itself is called a **multiple point.**



Curves with multiple points

EXERCISE 1.1

1. Show that the function $f(t) = t^2 + t^{5/2}$, belongs to:
   (*i*) C^2 on $(-\infty, \infty)$ (*ii*) C^3 on $(1, 4)$.
2. Show that the function $\mathbf{f}(t) = 3t^4\mathbf{i} + 6t^9\mathbf{j} + \mathbf{k}$ belongs to C^∞ on $(-\infty, \infty)$.
3. If the vector functions $\mathbf{f}$ and $\mathbf{g}$ belong to C^m on I, then show that the vector functions $\mathbf{f} + \mathbf{g}$, $\mathbf{f} \cdot \mathbf{g}$, $\mathbf{f} \times \mathbf{g}$ also belong to C^m on I.
4. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that the curve in space $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$ is a plane curve.
5. Show that the curve in space $\mathbf{r}(t) = 4 \sin t\mathbf{i} + 0\mathbf{j} + 3 \cos t\mathbf{k}$, $-\infty < t < \infty$ is a plane curve.
6. Show that the curve in space $\mathbf{r}(t) = 2t^2\mathbf{i} + (1 + t^3)\mathbf{j} + 7t\mathbf{k}$, $-\infty < t < \infty$ is a regular curve.

7. ARC OF A CURVE

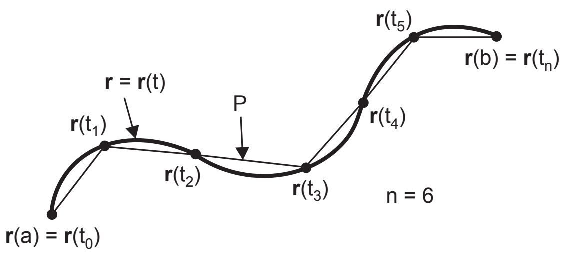
An **arc** of a curve is the portion of the curve between any two points of the curve. For simplicity, we shall say 'curve' for curves as well as for arcs.

8. LENGTH OF A CURVE

The length of a curve is defined in terms of the lengths of approximating polygonal arcs. Let $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$ be the given curve.

Let $a = t_0 < t_1 < \ldots < t_n = b$

be a subdivision of the interval $a \le t \le b$. This subdivision determines a sequence of points



$$\mathbf{r}_0 = \mathbf{r}(t_0),\ \mathbf{r}_1 = \mathbf{r}(t_1),\ \ldots\ldots,\ \mathbf{r}_n = \mathbf{r}(t_n).$$

These points are joined in sequence to form an approximating polygonal arc P as shown in the figure.

$\therefore$ Length of $$P = \sum_{i=1}^{n} \left| \mathbf{r}_i - \mathbf{r}_{i-1} \right| = \sum_{i=1}^{n} \left| \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) \right|$$

We make subdivisions of the interval arbitrarily small so that the greatest $| t_i - t_{i-1} |$ approaches 0 as $n \to \infty$.

If $\lim_{n\to\infty} \sum_{i=1}^{n} \left| \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) \right|$ exists finitely, then the given curve is said to a **rectifiable curve** and the value of this limit is called the **length** of the given curve.

Theorem. If $r = r(t)$, $a \le t \le b$ be a regular curve then prove that this curve is rectifiable and its length is given by the integral $\int_a^b | \dot{r}(t) |\, dt$.

Note. The proof of this theorem is beyond the scope of this book.

WORKING RULES FOR FINDING LENGTH OF THE CURVE $\mathbf{r} = \mathbf{r}(t)$ BETWEEN $a \le t \le b$

Step I. *Find* $\dot{\mathbf{r}}$ *and then* $|\dot{\mathbf{r}}|$.

Step II. *Evaluate* $\int_a^b |\dot{\mathbf{r}}|\, dt$. *This gives the required length of the curve.*

Example 1. *Find the length of the helix* $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}$, $0 \le t \le 2\pi$.

Sol. We have $$\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k},\ 0 \le t \le 2\pi.$$

$$\therefore \quad \dot{\mathbf{r}} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}$$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\therefore \quad \text{Length of the helix} = \int_0^{2\pi} |\dot{\mathbf{r}}|\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \sqrt{a^2 + b^2}\; t \Big|_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}.$$

Example 2. *Find the length of the curve* $\mathbf{r} = (4 \cosh 2t)\mathbf{i} + (4 \sinh 2t)\mathbf{j} + 8t\mathbf{k}$, $0 \le t \le \pi$.

Sol. We have $$\mathbf{r} = (4 \cosh 2t)\mathbf{i} + (4 \sinh 2t)\mathbf{j} + 8t\mathbf{k}, \quad 0 \le t \le \pi.$$

$$\therefore \quad \dot{\mathbf{r}} = (8 \sinh 2t)\mathbf{i} + (8 \cosh 2t)\mathbf{j} + 8\mathbf{k}$$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{64 \sinh^2 2t + 64 \cosh^2 2t + 64}$$

$$= 8\sqrt{2 \cosh^2 2t} = 8\sqrt{2}\, \cosh 2t$$

$\therefore$ Length of the curve $= \int_0^{\pi} |\dot{\mathbf{r}}|\, dt$

$$= \int_0^{\pi} 8\sqrt{2} \cosh 2t\, dt$$

$$= 4\sqrt{2} \sinh 2t \Big|_0^{\pi} = 4\sqrt{2}\, \sinh 2\pi.$$

Example 3. *Find the length of the semicubical parabola* $\mathbf{r} = t\mathbf{i} + t^{3/2}\,\mathbf{j}$ *from (0, 0, 0) to (4, 8, 0).*

Sol. We have $\mathbf{r} = t\mathbf{i} + t^{3/2}\,\mathbf{j}$.

The coordinates of the point with position vector $\mathbf{r}$ are $(t, t^{3/2}, 0)$.

$$t = 0, t^{3/2} = 0 \Rightarrow t = 0 \quad \text{and} \quad t = 4, t^{3/2} = 8 \Rightarrow t = 4.$$

$\therefore$ The given points correspond to the values 0 and 4 of t.

$$\dot{\mathbf{r}} = \mathbf{i} + \frac{3}{2} t^{1/2}\,\mathbf{j}$$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{1 + \frac{9}{4}t} = \frac{1}{2}\sqrt{4 + 9t}$$

$\therefore$ Length of the given curve

$$= \int_0^4 |\dot{\mathbf{r}}|\, dt = \int_0^4 \frac{1}{2}\sqrt{4 + 9t}\, dt$$

$$= \frac{1}{2} \cdot \frac{(4 + 9t)^{3/2}}{(3/2)9}\Bigg|_0^4 = \frac{1}{27}[(40)^{3/2} - (4)^{3/2}]$$

$$= \frac{1}{27}[80\sqrt{10} - 8] = \frac{8}{27}[\sqrt{1000} - 1] = 9.073.$$

Example 4. *Show that the length of the curve* $x = 2a(\sin^{-1} t + t\sqrt{1 - t^2}),\ y = 2at^2,\ z = 4at$ *between the points* $t = t_1$ *and* $t = t_2$ *is* $4\sqrt{2}\, a(t_2 - t_1)$.

Sol. We have $x = 2a(\sin^{-1} t + t\sqrt{1 - t^2}),\ y = 2at^2,\ z = 4at$.

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the given curve.

$$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2a(\sin^{-1} t + t\sqrt{1 - t^2})\mathbf{i} + 2at^2\mathbf{j} + 4at\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} = 2a\left(\frac{1}{\sqrt{1 - t^2}} + t \cdot \frac{1}{2}(1 - t^2)^{-1/2}(-2t) + \sqrt{1 - t^2}\,.1\right)\mathbf{i} + 4at\mathbf{j} + 4a\mathbf{k}$$

$$= 2a\left(\frac{1}{\sqrt{1 - t^2}} - \frac{t^2}{\sqrt{1 - t^2}} + \sqrt{1 - t^2}\right)\mathbf{i} + 4at\mathbf{j} + 4a\mathbf{k}$$

$$= 2a\left(\sqrt{1 - t^2} + \sqrt{1 - t^2}\right)\mathbf{i} + 4at\mathbf{j} + 4a\mathbf{k}$$

$$= 4a\sqrt{1 - t^2}\,\mathbf{i} + 4at\mathbf{j} + 4a\mathbf{k}$$

$$\therefore \qquad |\dot{\mathbf{r}}| = \sqrt{16a^2(1-t^2)+16a^2t^2+16a^2}$$

$$= 4a\sqrt{(1-t^2+t^2+1)} = 4\sqrt{2}\,a$$

$\therefore$ Length of the given curve

$$= \int_{t_1}^{t_2} |\dot{\mathbf{r}}|\,dt = \int_{t_1}^{t_2} 4\sqrt{2}\,a\,dt$$

$$= 4\sqrt{2}at\Big|_{t_1}^{t_2} = 4\sqrt{2}\,a(t_2-t_1).$$

Example 5. *Find the arc length as a function of* θ *along the epicycloid:*

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right),\ y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right),\ z = 0.$$

Sol. The given epicycloid is

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right),\ y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right),\ z = 0.$$

$$\therefore \qquad \dot{x} = -(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\theta\right),$$

$$\dot{y} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\theta\right) \quad \text{and} \quad \dot{z} = 0$$

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the epicycloid.

$$\therefore \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \therefore \quad \dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$

$$\therefore \qquad |\dot{\mathbf{r}}|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (a+b)^2\left(-\sin\theta + \sin\left(\frac{a+b}{b}\theta\right)\right)^2$$

$$+ (a+b)^2\left(\cos\theta - \cos\left(\frac{a+b}{b}\theta\right)\right)^2 + 0^2$$

$$= (a+b)^2\left[2 - 2\sin\theta\sin\frac{a+b}{b}\theta - 2\cos\theta\cos\frac{a+b}{b}\theta\right]$$

$$= (a+b)^2\left[2 - 2\cos\left(\frac{a+b}{b}\theta - \theta\right)\right] = (a+b)^2\left[2 - 2\cos\left(\frac{a}{b}\theta\right)\right]$$

$$= 4(a+b)^2\sin^2\left(\frac{a}{2b}\theta\right)$$

$$\therefore \qquad |\dot{\mathbf{r}}| = 2(a+b)\sin\left(\frac{a}{2b}\theta\right)$$

$$\therefore \qquad s = \int_0^\theta |\dot{\mathbf{r}}|\,d\theta = \int_0^\theta 2(a+b)\sin\left(\frac{a}{2b}\theta\right)d\theta = -\left.\frac{2(a+b)\cos\left(\frac{a}{2b}\theta\right)}{a/2b}\right|_0^\theta$$

$$= -\frac{4(a+b)b}{a}\left[\cos\left(\frac{a}{2b}\theta\right) - 1\right] = \frac{4(a+b)b}{a}\left[1 - \cos\left(\frac{a}{2b}\theta\right)\right].$$

Example 6. *Find the length of the curve given by the intersection of the surfaces* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x = a \cosh \frac{z}{a}$ *from the point (a, 0, 0) to the point (x, y, z).*

Sol. We have $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(1)$$

and $$x = a \cosh \frac{z}{a} \qquad ...(2)$$

Let $x = a \cosh t, \quad y = b \sinh t, \quad z = at$

$\therefore$ (1) and (2) are satisfied.

Let **r** be the position vector of the point (x, y, z) on the given curve.

$\therefore$ $$\mathbf{r} = (a \cosh t)\mathbf{i} + (b \sinh t)\mathbf{j} + at\mathbf{k}$$

$$a \cosh t = a,\ b \sinh t = 0,\ at = 0 \quad \Rightarrow \quad t = 0$$

$\therefore$ The initial point corresponds to $t = 0$.

$$\dot{\mathbf{r}} = (a \sinh t)\mathbf{i} + (b \cosh t)\mathbf{j} + a\mathbf{k}$$

$\therefore$ $$|\dot{\mathbf{r}}| = \sqrt{a^2 \sinh^2 t + b^2 \cosh^2 t + a^2}$$
$$= \sqrt{a^2 (\sinh^2 t + 1) + b^2 \cosh^2 t}$$
$$= \sqrt{a^2 \cosh^2 t + b^2 \cosh^2 t} = \sqrt{(a^2 + b^2)} \cosh t$$

$\therefore$ Length of the given curve

$$= \int_0^t |\dot{\mathbf{r}}|\, dt = \int_0^t \sqrt{a^2 + b^2} \cosh t\, dt$$
$$= \sqrt{a^2 + b^2} \sinh t \Big|_0^t = \sqrt{a^2 + b^2} \sinh t$$
$$= \frac{\sqrt{a^2 + b^2}}{b} y.$$

EXERCISE 1.2

1. Find the length of the helix $\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 4t\mathbf{k}$, $0 \le t \le 2\pi$.
2. (*i*) Find the length of one complete turn of the helix :
   $\mathbf{r} = (a \cos t, a \sin t, bt), -\infty < t < \infty, a > 0, b > 0.$
   (*ii*) Find the length of the helix $\mathbf{r} = a \cos u\mathbf{i} + a \sin u\mathbf{j} + cu\mathbf{k}$, $-\infty < u < \infty$ from the point $(a, 0, 0)$ to the point $(a, 0, 2\pi c)$.
3. Find the length of the curve $\mathbf{r} = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k}$, $0 \le t \le \pi$.
4. Find the length of the catenary $\mathbf{r} = t\mathbf{i} + \cosh t\mathbf{j}$ from $t = 0$ to $t = 1$.
5. Find the length of the curve $\mathbf{r} = (1 + 2t)\mathbf{i} + (2 + t)\mathbf{j} - \mathbf{k}$, $3 \le t \le 7$.
6. Find the length of the curve $\mathbf{r} = (2 + 9t)\mathbf{i} + (1 - 3t)\mathbf{j} + t\mathbf{k}$, $8 \le t \le 15$. Also, verify the result by using the distance formula to find the distance between two given points.

7. Find the length of the semicubical parabola $\mathbf{r} = t\mathbf{i} + t^{3/2}\mathbf{k}$ from (0, 0, 0) to (9, 0, 27).

8. Find the length of the curve $\mathbf{r} = (\sin^{-1} t + t\sqrt{1-t^2})\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$ from $t = 1$ to $t = 3$.

9. Find the length of the curve given by the intersection of the surfaces $x^2 - y^2 = 1$, $x = \cosh z$ from the point (1, 0, 0) to the point (x, y, z).

10. Find the length of the curve $\mathbf{r} = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}$, $0 \le t \le \pi$.

Answers

1. 10π	**2.** (*i*) $2\pi\sqrt{a^2+b^2}$	(*ii*) $2\pi\sqrt{a^2+c^2}$	**3.** $3\sqrt{2}\sinh 2\pi$
4. sinh 1	**5.** $4\sqrt{5}$	**6.** $7\sqrt{91}$	**7.** 28.73
8. $4\sqrt{2}$	**9.** $\sqrt{2}\,y$	**10.** $\sqrt{3}\,(e^{\pi} - 1)$.	

Hint

2. (*i*) The limits in the definite integral are to be t_0 and $t_0 + 2\pi$, where t_0 is any number.

9. ARC LENGTH AS PARAMETER IN REPRESENTATIONS OF CURVES

Let $\mathbf{r} = \mathbf{r}(t)$ be any regular curve. Let A($t = a$) be any arbitrary but fixed point on the curve. We define a function s of t as

$$s = s(t) = \int_a^t |\dot{\mathbf{r}}|\,dt \qquad \ldots(1)$$

$s(t)$ is called the **arc length function** of the curve $\mathbf{r} = \mathbf{r}(t)$. If $t_0 > a$, then $s(t_0)$ is the length of curve between the points with parametric values a and t_0. If $t_0 < a$, then $s(t_0)$ is negative of the length of the curve between the points with parametric values a and t_0. Thus $s(a) = 0$ and for points on one side of A the value of s will be positive ; for points on other side, negative. The choice of the fixed point A($t = a$) is arbitrary. Changing point A shall mean changing s by a constant quantity.

For simplicity, the arc length function s is written **arc length**. The use of arc length s as parameter in space curves would help us a lot in studying their curvature and torsion.

By the fundamental theorem of calculus, (1) implies

$$\frac{ds}{dt} = |\dot{\mathbf{r}}| = \left|\frac{d\mathbf{r}}{dt}\right| \qquad \ldots(2)$$

$$\therefore \quad \left|\frac{d\mathbf{r}}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\right|\left|\frac{dt}{ds}\right|$$

$$= \left|\frac{d\mathbf{r}}{dt}\right| \Big/ \left|\frac{ds}{dt}\right| = \left|\frac{d\mathbf{r}}{dt}\right| \Big/ \left|\frac{d\mathbf{r}}{dt}\right| = 1 \qquad \text{(Using (2))}$$

$$\therefore \quad \left|\frac{d\mathbf{r}}{ds}\right| = 1.$$

If the equation of a curve is given in terms of arc length, then we say that the equation of the curve is a **natural representation** of the curve.

If the parameter in the equation of a curve is other than, 'arc length s', then the equation of the curve is called an **arbitrary representation** of the curve.

In general, the geometric quantities along a curve are defined in terms of a natural representation of the curve. By using the chain rule and the relation $\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|$, these quantities can also be derived in terms of any arbitrary parameter.

WORKING RULES FOR WRITING r = r(t) IN TERMS OF s

Step I. *Find* $|\dot{\mathbf{r}}|$.

Step II. *Solve* $s = \int_0^t |\dot{\mathbf{r}}|\, dt$ *to find s in terms of t.*

Step III. *Using the relation found in* **Step II**, *find t in terms of s.*

Step IV. *Substitute the value of t in* **r** = **r**(*t*) *to get the required natural representation of the given curve.*

Example 1. *Find the equation of the helix* $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k}$, $-\infty < t < \infty$ *in terms of arc length s as parameter.*

Sol. We have $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k}$, $-\infty < t < \infty$ ...(1)

$\therefore \quad \dot{\mathbf{r}} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} + c\mathbf{k}$

$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} = \sqrt{a^2 + c^2}$

Let the point with $t = 0$ be the fixed point for the arc length parameter s.

$$\therefore \quad s = s(t) = \int_0^t |\dot{\mathbf{r}}|\, dt$$

$$= \int_0^t \sqrt{a^2 + c^2}\, dt = \sqrt{a^2 + c^2}\; t$$

$$\therefore \quad t = \frac{s}{\sqrt{a^2 + c^2}}$$

Substituting the value of t in (1), the equation of the given helix in terms of arc length s as parameter is

$$\mathbf{r}(s) = a \cos \frac{s}{\sqrt{a^2 + c^2}}\,\mathbf{i} + a \sin \frac{s}{\sqrt{a^2 + c^2}}\,\mathbf{j} + \frac{cs}{\sqrt{a^2 + c^2}}\,\mathbf{k}.$$

Example 2. *Express the curve* $\mathbf{r} = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}$, $-\infty < t < \infty$ *in terms of arc length s as parameter.*

Sol. We have $\mathbf{r} = e^t \cos t\,\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}$. ...(1)

$$\therefore \quad \dot{\mathbf{r}} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$$

$$= e^t(\cos t - \sin t)\mathbf{i} + e^t (\sin t + \cos t)\mathbf{j} + e^t\mathbf{k}$$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}}$$

$$= e^t\sqrt{\cos^2 t + \sin^2 t - 2 \cos t \sin t + \sin^2 t + \cos^2 t + 2 \sin t \cos t + 1}$$

$$= e^t\sqrt{3}$$

Let the point with $t = 0$ be the fixed point for the arc length parameter s.

$$\therefore \qquad s = s(t) = \int_0^t |\dot{\mathbf{r}}|\,dt$$

$$= \int_0^t e^t\sqrt{3}\,dt = \sqrt{3}e^t\Big|_0^t = \sqrt{3}\,(e^t - 1)$$

$$\Rightarrow \qquad e^t - 1 = \frac{s}{\sqrt{3}} \quad \Rightarrow \quad e^t = \frac{s}{\sqrt{3}} + 1 \quad \Rightarrow \quad t = \log\left(\frac{s}{\sqrt{3}} + 1\right)$$

Substituting the value of t in (1), the equation of the given curve in terms of arc length s as parameter is

$$\mathbf{r}(s) = \left(\frac{s}{\sqrt{3}} + 1\right)\cos\left(\log\left(\frac{s}{\sqrt{3}} + 1\right)\right)\mathbf{i} + \left(\frac{s}{\sqrt{3}} + 1\right)\sin\left(\log\left(\frac{s}{\sqrt{3}} + 1\right)\right)\mathbf{j} + \left(\frac{s}{\sqrt{3}} + 1\right)\mathbf{k}.$$

EXERCISE 1.3

1. Find the equation of the helix $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $-\infty < t < \infty$ in terms of arc length s as parameter.
2. Find the equation of the curve $\mathbf{r} = e^{2t}\cos t\mathbf{i} + e^{2t}\sin t\mathbf{j} + e^{2t}\mathbf{k}$, $-\infty < t < \infty$ in terms of arc length s as parameter.
3. For the helix $x = a\cos t$, $y = a\sin t$, $z = at\tan\alpha$, show that the length of the curve measured from the point $t = 0$ is $at\sec\alpha$. Also show that $\dfrac{ds}{dt} = a\sec\alpha$.
4. Show that $\mathbf{r} = \frac{1}{2}(s + \sqrt{s^2+1})\,\mathbf{i} + \dfrac{1}{2(s + \sqrt{s^2+1})}\mathbf{j} + \dfrac{1}{\sqrt{2}}\log(s + \sqrt{s^2+1})\,\mathbf{k}$ is a natural representation of a curve.

Answers

1. $\mathbf{r} = \cos\dfrac{s}{\sqrt{2}}\mathbf{i} + \sin\dfrac{s}{\sqrt{2}}\mathbf{j} + \dfrac{s}{\sqrt{2}}\mathbf{k}$

2. $\mathbf{r} = \left(\dfrac{2s}{3} + 1\right)\left[\cos\left(\dfrac{1}{2}\log\left(\dfrac{2s}{3} + 1\right)\right)\mathbf{i} + \sin\left(\dfrac{1}{2}\log\left(\dfrac{2s}{3} + 1\right)\right)\mathbf{j} + \mathbf{k}\right].$

Hint

4. $u = s + \sqrt{s^2+1}$ implies $\quad \mathbf{r} = \dfrac{1}{2}u\mathbf{i} + \dfrac{1}{2u}\mathbf{j} + \dfrac{1}{\sqrt{2}}(\log u)\mathbf{k}$

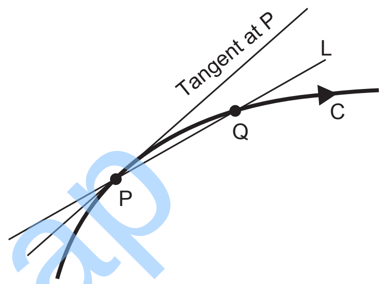
$$\therefore \quad \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du}\cdot\frac{du}{ds} = \left(\frac{1}{2}\mathbf{i} - \frac{1}{2u^2}\mathbf{j} + \frac{1}{\sqrt{2}u}\mathbf{k}\right)\left(1 + \frac{s}{\sqrt{s^2+1}}\right)$$

$$\therefore \quad \left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\frac{1}{4}+\frac{1}{4u^4}+\frac{1}{2u^2}}\cdot\left(\frac{\sqrt{s^2+1}+s}{\sqrt{s^2+1}}\right)$$

$$= \left(\frac{1}{2}+\frac{1}{2u^2}\right)\frac{u}{\sqrt{s^2+1}} = \frac{u^2+1}{2u\sqrt{s^2+1}} = 1.$$

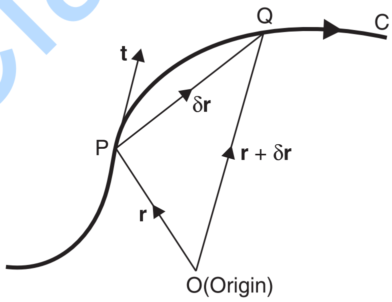
10. TANGENT TO A CURVE

Let C be a curve and P be any point on C. The **tangent** at P to the curve C is the limiting position of a straight line L through P and a point Q of C as Q approaches P along C.



11. UNIT TANGENT VECTOR

Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a regular curve C in terms of an arbitrary parameter t. Let P and Q be the points on the curve whose position vectors are $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ corresponding to the values t and $t + \delta t$ of the parameter respectively.



$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\mathbf{r} + \delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r}$$

$\therefore$ The quotient $\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector parallel to the line PQ. Since the given curve is regular, $\mathbf{r}(t)$ has continuous non-zero derivative.

$\therefore$ $\displaystyle\lim_{\delta t \to 0}\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}$ exists and is non-zero.

By the definition of a tangent to a curve at a point, the vector $\dfrac{d\mathbf{r}}{dt}$ *i.e.*, $\dot{\mathbf{r}}(t)$ is parallel to the tangent at the point P.

The vector $\dot{\mathbf{r}}(t)$ is called the **tangent vector** of C at the point P.

The corresponding unit vector $\dfrac{1}{|\dot{\mathbf{r}}|}\dot{\mathbf{r}}$ is called the **unit tangent vector** of C at the point P and it is denoted by $\mathbf{t}$. The vectors $\dot{\mathbf{r}}$ and $\mathbf{t}$ point in the direction of increasing t. Thus the directions of $\dot{\mathbf{r}}$ and $\mathbf{t}$ are same and depend upon the orientation of the curve C.

In particular, if the equation of the curve C is given in terms of the arc length s, then the tangent vector of C at P is $\dfrac{d\mathbf{r}}{ds}$.

The vector $\dfrac{d\mathbf{r}}{ds}$ is denoted by $\mathbf{r}'$. We know that $\dfrac{d\mathbf{r}}{ds}$ *i.e.*, $\mathbf{r}'$ is a unit vector.

$\therefore$ Unit tangent vector '$\mathbf{t}$' at P = $\mathbf{r}'$.

Example 1. *Find the unit tangent vector* **t** *and direction cosines of the tangent at a point on the circular helix* $x = a \cos t,\ y = a \sin t,\ z = bt,\ -\infty < t < \infty.$

Sol. The given helix is

$$x = a \cos t,\ y = a \sin t,\ z = bt,\ -\infty < t < \infty.$$

Let **r** be the position vector of the point (x, y, z) on the helix.

$$\therefore \quad \mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$$

Unit tangent vector, $\mathbf{t} = \dfrac{1}{|\dot{\mathbf{r}}|}\dot{\mathbf{r}}$

We have $\dot{\mathbf{r}} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k}$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\therefore \quad \mathbf{t} = \frac{1}{|\dot{\mathbf{r}}|}\dot{\mathbf{r}} = \frac{1}{\sqrt{a^2+b^2}}(-a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k})$$

$$= -\frac{a}{\sqrt{a^2+b^2}} \sin t\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}} \cos t\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}$$

Since tangent is parallel to **t** and **t** is a unit vector, the d.c.'s of the tangent are

$$-\frac{a}{\sqrt{a^2+b^2}} \sin t,\ \frac{a}{\sqrt{a^2+b^2}} \cos t,\ \frac{b}{\sqrt{a^2+b^2}}.$$

Example 2. *Show that the tangent vectors along the curve* $\mathbf{r} = at\mathbf{i} + bt^2\mathbf{j} + t^3\mathbf{k}$, *where* $2b^2 = 3a$ *make a constant angle with the vector* $\mathbf{i} + \mathbf{k}$.

Sol. Given curve is $\mathbf{r} = at\mathbf{i} + bt^2\mathbf{j} + t^3\mathbf{k}.$

$$\therefore \quad \dot{\mathbf{r}} = a\mathbf{i} + 2bt\mathbf{j} + 3t^2\mathbf{k}$$

The tangent vector at point 't' is $\dot{\mathbf{r}}$ *i.e.*, $a\mathbf{i} + 2bt\mathbf{j} + 3t^2\mathbf{k}$.

Given vector is $\mathbf{i} + \mathbf{k}$, *i.e.*, $1.\mathbf{i} + 0.\mathbf{j} + 1.\mathbf{k}$

Let θ be the angle between the tangent vector $\dot{\mathbf{r}}$ and the vector $\mathbf{i} + \mathbf{k}$.

$$\therefore \quad \cos\theta = \frac{a(1) + 2bt(0) + 3t^2(1)}{\sqrt{a^2 + 4b^2t^2 + 9t^4}\ \sqrt{1+0+1}}$$

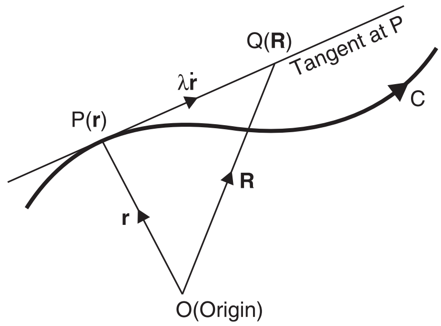
$$= \frac{a + 3t^2}{\sqrt{a^2 + 2(3a)t^2 + 9t^4}\ \sqrt{2}} = \frac{a+3t^2}{(a+3t^2)\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \pi/4, \text{ which is a constant angle.}$$

$\therefore$ The result holds.

12. EQUATION OF THE TANGENT AT A POINT ON A CURVE

Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a regular curve C in terms of an arbitrary parameter t. Let P($\mathbf{r}$) be any point on the curve. We know that $\dot{\mathbf{r}}$ is the tangent vector at the point P and the tangent at P is parallel to this vector. Let Q be a general point on the tangent at P. Let $\mathbf{R}$ be the position vector of the point Q.



∴ The equation of the tangent at the point P($\mathbf{r}$) is $\mathbf{R} = \mathbf{r} + \lambda\dot{\mathbf{r}}$, where λ is a scalar parameter.

Let the coordinates of P and Q be (x, y, z) and (X, Y, Z) respectively.

∴ $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad \mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$$

Also $$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

∴ The equation of the tangent at P is

$$X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + \lambda(\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k})$$

⇒ $$X = x + \lambda\dot{x},\ Y = y + \lambda\dot{y},\ Z = z + \lambda\dot{z}$$

∴ $$\frac{X - x}{\dot{x}} = \frac{Y - y}{\dot{y}} = \frac{Z - z}{\dot{z}} \quad (= \lambda).$$

These are the cartesian equations of the tangent at the point P(x, y, z). Here $\dot{x}, \dot{y}, \dot{z}$ are direction ratios of the tangent at the point P.

In particular, if the equation of the curve C is given in terms of the arc length s then, $\mathbf{r}'$ is the unit tangent vector of C at P.

∴ The tangent at P is parallel to the vector $\mathbf{r}'$.

∴ The equation of the tangent at P($\mathbf{r}$) is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{r}'$, where $\mathbf{R}$ is the position vector of the general point Q($\mathbf{R}$) on the tangent at P.

The cartesian form of the equations of tangent at P are

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} \quad (= \lambda)$$

Since $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ is a unit vector, x', y', z' are the direction cosines of the tangent at P.

Example 3. *Show that a curve is a straight line if all tangent lines are parallel.*

Sol. Let $\mathbf{r} = \mathbf{r}(s)$ be the given curve.

∴ Tangent vector = $\mathbf{r}'$

Since tangent line at a point is parallel to the tangent vector at that point and all tangent lines are parallel, the direction of $\mathbf{r}'$ is fixed. Also $\mathbf{r}'$ is a unit vector.

∴ $\mathbf{r}'$ is a non-zero constant vector, say $\mathbf{a}$.

∴ $$\frac{d\mathbf{r}}{ds} = \mathbf{a}$$

Integrating, we get

$$\mathbf{r} = \mathbf{a}s + \mathbf{b}, \text{ where } \mathbf{b} \text{ is a constant vector.}$$

∴ The curve is a straight line passing through the point with position vector **b** and parallel to the vector **a**.

Example 4. *Find the equation of the tangent to the ellipse* $\frac{1}{4}x^2 + y^2 = 1$ *at the point* $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$.

Sol. The given ellipse is

$$\frac{1}{4}x^2 + y^2 = 1.$$

The parametric equations of this ellipse are

$$x = 2\cos t,\ y = \sin t,\ z = 0$$

Let **r** be the position vector of the point (x, y, z).

$$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\mathbf{k}$$

$$2\cos t = \sqrt{2},\ \sin t = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad t = \frac{\pi}{4}$$

∴ The point $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$ in xy-plane corresponds to $t = \frac{\pi}{4}$.

$$\dot{\mathbf{r}} = -2\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 0\mathbf{k}$$

At $t = \frac{\pi}{4}$,

$$\dot{\mathbf{r}} = -2\sin\frac{\pi}{4}\mathbf{i} + \cos\frac{\pi}{4}\mathbf{j} + 0\mathbf{k}$$

$$= -\sqrt{2}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} + 0\mathbf{k}$$

∴ The tangent at $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$ passes through $\left(\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right)$ and has d.r.'s $-\sqrt{2}, \frac{1}{\sqrt{2}}, 0$.

∴ The equations of the tangent at $\left(\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right)$ are

$$\frac{x-\sqrt{2}}{-\sqrt{2}} = \frac{y-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{z-0}{0} \quad \text{or} \quad \frac{x-\sqrt{2}}{-\sqrt{2}} = \frac{\sqrt{2}y-1}{1} = \frac{z}{0}.$$

Example 5. *Find the equation of the tangent to the curve* $x = 1 + t,\ y = -t^2,\ z = 1 + t^2$, $-\infty < t < \infty$ *at the point for which* $t = 2$.

Sol. The given curve is

$$x = 1 + t,\ y = -t^2,\ z = 1 + t^2,\ -\infty < t < \infty.$$

$$t = 2 \quad \Rightarrow \quad x = 1 + 2 = 3, \quad y = -(2)^2 = -4, \quad z = 1 + (2)^2 = 5$$

∴ $t = 2$ corresponds to the point $(3, -4, 5)$ on the curve.

Also, $\dot{x} = 1, \quad \dot{y} = -2t, \quad \dot{z} = 2t$

$$t = 2 \Rightarrow \dot{x} = 1,\ \dot{y} = -2(2) = -4,\ \dot{z} = 2(2) = 4$$

∴ The tangent at (3, – 4, 5) passes through (3, – 4, 5) and has d.r.'s 1, – 4, 4.

∴ The equations of the tangent at (3, – 4, 5) are

$$\frac{x-3}{1} = \frac{y-(-4)}{-4} = \frac{z-5}{4} \quad \text{or} \quad x - 3 = \frac{y+4}{-4} = \frac{z-5}{4}.$$

Example 6. *Show that the equation of the tangent at any point on the curve whose equation referred to rectangular axes are $x = 3t$, $y = 3t^2$, $z = 2t^3$ makes a constant angle with the line $y = z - x = 0$.*

Sol. Given curve is $x = 3t$, $y = 3t^2$, $z = 2t^3$.

Let **r** be the position vector of the point (x, y, z) on the given curve.

$$\therefore \quad \mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} = 3\mathbf{i} + 6t\mathbf{j} + 6t^2\mathbf{k}$$

The tangent line at the point with parametric value t is parallel to the vector $\dot{\mathbf{r}}$.

∴ D.r.'s of the tangent are 3, $6t$, $6t^2$.

Given line is $y = z - x = 0$

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

∴ D.r.'s of the given line are 1, 0, 1.

Let θ be the angle between the tangent and the given line.

$$\therefore \quad \cos\theta = \frac{3(1) + 6t(0) + 6t^2(1)}{\sqrt{(3)^2 + (6t)^2 + (6t^2)^2}\ \sqrt{1^2 + 0^2 + 1^2}}$$

$$= \frac{3 + 6t^2}{\sqrt{(3+6t^2)^2}\ \sqrt{2}} = \frac{1}{\sqrt{2}}$$

∴ θ = π/4, which is a constant angle.

∴ The result holds.

13. DIRECTION RATIOS OF THE TANGENT AT A POINT ON THE CURVE OF INTERSECTION OF TWO SURFACES

Let the given curve be the intersection of the surfaces

$$F(x, y, z) = 0 \quad \text{...(1)}$$

and

$$G(x, y, z) = 0 \quad \text{...(2)}$$

Eliminating x, y, z from (1) and (2), let the equation of the given curve be

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where t is an arbitrary parameter.

∴ D.r.'s of the tangent to the curve at the point 't' are $\dot{x}, \dot{y}, \dot{z}$.

Differentiating (1) and (2) w.r.t. t, we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0$$

and

$$\frac{\partial G}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial G}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial G}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\Rightarrow \qquad (F_x)\dot{x} + (F_y)\dot{y} + (F_z)\dot{z} = 0$$

and
$$(G_x)\dot{x} + (G_y)\dot{y} + (G_z)\dot{z} = 0$$

Solving these equations for $\dot{x}, \dot{y}$ and $\dot{z}$, we get

$$\frac{\dot{x}}{F_yG_z - F_zG_y} = \frac{\dot{y}}{F_zG_x - F_xG_z} = \frac{\dot{z}}{F_xG_y - F_yG_x}$$

$\therefore$ $F_yG_z - F_zG_y,\ F_zG_x - F_xG_z,\ F_xG_y - F_yG_x$
are also d.r.'s of the tangent to the curve given by the equations F(x, y, z) = 0, G(x, y, z) = 0 at the point t.

Example 7. *Show that the equation of the tangent to the curve of intersection of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ *and the confocal* $\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda} = 1$ *is*

$$\frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)} = \frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)} = \frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)},$$

where (x, y, z) is an arbitrary point on the curve.

Sol. The given surfaces are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1 = 0 \qquad ...(1)$$

and
$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda} - 1 = 0 \qquad ...(2)$$

Let the equation of the curve of intersection of given surfaces be **r** = **r**(t), where t is an arbitrary parameter.

Differentiating the equations (1) and (2) w.r.t. t, we get

$$\frac{2x}{a^2}\dot{x} + \frac{2y}{b^2}\dot{y} + \frac{2z}{c^2}\dot{z} = 0$$

and
$$\frac{2x}{a^2-\lambda}\dot{x} + \frac{2y}{b^2-\lambda}\dot{y} + \frac{2z}{c^2-\lambda}\dot{z} = 0$$

$$\Rightarrow \qquad \frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} + \frac{z\dot{z}}{c^2} = 0$$

and
$$\frac{x\dot{x}}{a^2-\lambda} + \frac{y\dot{y}}{b^2-\lambda} + \frac{z\dot{z}}{c^2-\lambda} = 0$$

Solving these equations for $\dot{x}, \dot{y}$ and $\dot{z}$, we get

$$\frac{\dot{x}}{\frac{yz}{b^2(c^2-\lambda)} - \frac{yz}{c^2(b^2-\lambda)}} = \frac{\dot{y}}{\frac{zx}{c^2(a^2-\lambda)} - \frac{zx}{a^2(c^2-\lambda)}} = \frac{\dot{z}}{\frac{xy}{a^2(b^2-\lambda)} - \frac{xy}{b^2(a^2-\lambda)}}$$

$$\Rightarrow \quad \frac{\dot{x}}{\dfrac{\lambda yz(b^2-c^2)}{b^2c^2(c^2-\lambda)(b^2-\lambda)}} = \frac{\dot{y}}{\dfrac{\lambda zx(c^2-a^2)}{c^2a^2(a^2-\lambda)(c^2-\lambda)}} = \frac{\dot{z}}{\dfrac{\lambda xy(a^2-b^2)}{a^2b^2(b^2-\lambda)(a^2-\lambda)}}$$

$$\Rightarrow \quad \frac{\dot{x}}{\dfrac{a^2(b^2-c^2)(a^2-\lambda)}{x}} = \frac{\dot{y}}{\dfrac{b^2(c^2-a^2)(b^2-\lambda)}{y}} = \frac{\dot{z}}{\dfrac{c^2(a^2-b^2)(c^2-\lambda)}{z}}$$

$\therefore$ D.r.'s of the tangent at the point (x, y, z) on the curve are

$$\frac{a^2(b^2-c^2)(a^2-\lambda)}{x}, \frac{b^2(c^2-a^2)(b^2-\lambda)}{y}, \frac{c^2(a^2-b^2)(c^2-\lambda)}{z}.$$

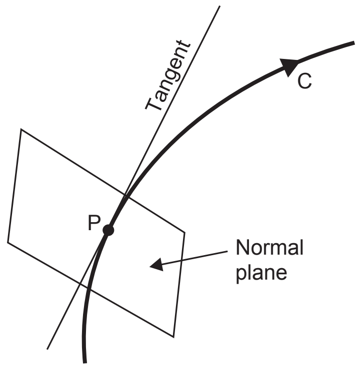
$\therefore$ The equations of the tangent at the point (x, y, z) on the curve are

$$\frac{X-x}{\dfrac{a^2(b^2-c^2)(a^2-\lambda)}{x}} = \frac{Y-y}{\dfrac{b^2(c^2-a^2)(b^2-\lambda)}{y}} = \frac{Z-z}{\dfrac{c^2(a^2-b^2)(c^2-\lambda)}{z}}$$

or
$$\frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)} = \frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)} = \frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}.$$

14. NORMAL PLANE TO A CURVE

Let C be a curve and P be any point on C. The **normal plane** at P to the curve C is the plane passing through P and perpendicular to the tangent at P.



15. EQUATION OF THE NORMAL PLANE AT A POINT ON A CURVE

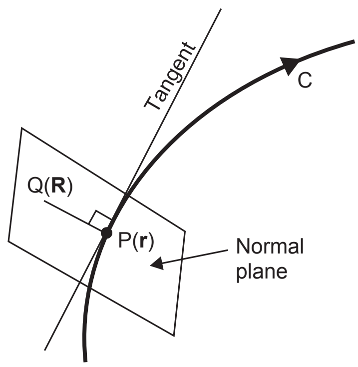
Let $\mathbf{r} = \mathbf{r}\,(t)$ be the equation of a regular curve C, where t is an arbitrary parameter. Let P($\mathbf{r}$) be any point on the curve. We know that the tangent at P is parallel to the tangent vector $\dot{\mathbf{r}}$.

Let Q be a general point on the normal plane at P. Let $\mathbf{R}$ be the position vector of the point Q.

$\therefore$ $\overrightarrow{\mathbf{PQ}}$ and $\dot{\mathbf{r}}$ are perpendicular.

$$\Rightarrow \quad \overrightarrow{\mathbf{PQ}} \cdot \dot{\mathbf{r}} = 0 \quad \Rightarrow \quad (\mathbf{R}-\mathbf{r}) \cdot \dot{\mathbf{r}} = 0 \qquad ...(1)$$

This represents the equation of the normal plane at the point P($\mathbf{r}$).



(1) $\Rightarrow$ $(\mathbf{R} - \mathbf{r}) \cdot \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = 0 \Rightarrow (\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$

$\therefore$ The equation of the normal plane at the point P(**r**) can also be written as

$$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0.$$

Example 8. *Find the equation of the normal plane to the curve* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ *at* $t = 1$.

Sol. The given curve is

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}.$$

$\therefore$ $\mathbf{r}(1) = 1.\mathbf{i} + (1)^2\,\mathbf{j} + (1)^3\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\therefore$ $t = 1$ corresponds to the point (1, 1, 1) on the curve.

Also $\dot{\mathbf{r}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

$\therefore$ $\dot{\mathbf{r}}(1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\therefore$ The equation of the normal plane at (1, 1, 1) is $(\mathbf{r} - \mathbf{r}(1)) \cdot \dot{\mathbf{r}}(1) = 0.$

$\Rightarrow$ $(\mathbf{r} - (\mathbf{i} + \mathbf{j} + \mathbf{k})) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 0$

$\Rightarrow$ $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (1(1) + 1(2) + 1(3)) = 0$

$\Rightarrow$ $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 6.$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. $\mathbf{t} = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \mathbf{r}'$ *is the unit tangent vector.*

Rule II. *The equation of the tangent to the curve* $\mathbf{r} = \mathbf{r}(t)$ *at the point* P(**r**) *is* $\mathbf{R} = \mathbf{r} + \lambda\dot{\mathbf{r}}$, *where* λ *is a scalar parameter. If* $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *and* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, *then this equation reduces to* $\dfrac{X - x}{\dot{x}} = \dfrac{Y - y}{\dot{y}} = \dfrac{Z - z}{\dot{z}}$ $(= \lambda)$.

Rule III. *The equation of the tangent to the curve* $\mathbf{r} = \mathbf{r}(s)$ *at the point* P(**r**) *is* $\mathbf{R} = \mathbf{r} + \lambda\mathbf{r}'$, *where* λ *is a scalar parameter. If* $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *and* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, *then this equation reduces to* $\dfrac{X - x}{x'} = \dfrac{Y - y}{y'} = \dfrac{Z - z}{z'}$ $(= \lambda)$.

Rule IV. *The equation of the normal plane to the curve* $\mathbf{r} = \mathbf{r}(t)$ *at the point* P(**r**) *is* $(\mathbf{R} - \mathbf{r}) \cdot \dot{\mathbf{r}} = 0$ *or equivalently* $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$.

EXERCISE 1.4

1. Find the unit tangent vector **t** and the direction cosines of the tangent to the helix $x = a \cos t$, $y = a \sin t$, $z = bt$, $-\infty < t < \infty$ at the point, where $t = \pi/4$.
2. Find the unit tangent vector **t** and the direction cosines of the tangent to the helix $x = a \cos t$, $y = a \sin t$, $z = at$, $-\infty < t < \infty$ at the point, where $t = \pi/3$.
3. Find the unit tangent vector **t** to the curve $\mathbf{r} = t\mathbf{i} + t^3\mathbf{j}$ at the point (1, 1, 0).
4. Find the unit tangent vector **t** to the curve $\mathbf{r} = \cos t\mathbf{i} + 2\sin t\mathbf{j}$ at the point $(1/2, \sqrt{3}, 0)$.
5. Find the unit tangent vector **t** to the curve $\mathbf{r} = \cosh t\mathbf{i} + \sinh t\mathbf{j}$ at the point (5/3, 4/3, 0).
6. Find the unit tangent vector **t** to the curve $\mathbf{r} = \log \cos t\mathbf{i} + \log \sin t\mathbf{j} + \sqrt{2}\,t\mathbf{k}$ at the point 't'.
7. Find the equation of the tangent to the curve $x = 1 + t$, $y = -t^2$, $z = 1 + t^2$, $-\infty < t < \infty$ at the point for which (*i*) $t = 1$ (*ii*) $t = 5$.
8. Find the equation of the tangent to the helix $\mathbf{r} = (a \cos t, a \sin t, bt)$, $-\infty < t < \infty$ at the point 't'.
9. Find the equation of the tangent to the curve $\mathbf{r} = t\mathbf{i} + t^3\mathbf{j}$ at the point (1, 1, 0).
10. Find the equation of the tangent to the curve $\mathbf{r} = \cos t\mathbf{i} + 2\sin t\mathbf{j}$ at the point $\left(\frac{1}{2}, \sqrt{3}, 0\right)$.
11. Find the equation of the tangent to the curve $\mathbf{r} = \cosh t\mathbf{i} + \sinh t\mathbf{j}$ at the point $\left(\frac{5}{3}, \frac{4}{3}, 0\right)$.
12. Find the point of intersection of the xy-plane and the tangent line to the curve $\mathbf{r} = (1 + t)\,\mathbf{i} - t^2\,\mathbf{j} + (1 + t^3)\,\mathbf{k}$ at $t = 1$.
13. Show that the tangent at any point on the curve $\mathbf{r} = at\mathbf{i} + bt^2\mathbf{j} + t^3\mathbf{k}$, $2b^2 = 3a$ makes a constant angle with the line $x - z = 0$, $y = 0$.
14. Find the equation of the normal plane to the curve $\mathbf{r} = (1 + t)\mathbf{i} - t^2\mathbf{j} + (1 + t^3)\mathbf{k}$ at $t = 1$.
15. Find the point of intersection of the xy-plane and the normal plane to the curve $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ at the point $t = \frac{\pi}{2}$.

Answers

1. $-\frac{a}{\sqrt{2(a^2+b^2)}}\mathbf{i} + \frac{a}{\sqrt{2(a^2+b^2)}}\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}\,;\; -\frac{a}{\sqrt{2(a^2+b^2)}}, \frac{a}{\sqrt{2(a^2+b^2)}}, \frac{b}{\sqrt{a^2+b^2}}$
2. $-\frac{\sqrt{6}}{4}\mathbf{i} + \frac{\sqrt{2}}{4}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}\,;\; -\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}$
3. $\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j} + 0\mathbf{k}$
4. $-\frac{\sqrt{3}}{\sqrt{7}}\mathbf{i} + \frac{2}{\sqrt{7}}\mathbf{j} + 0\mathbf{k}$
5. $\frac{4}{\sqrt{41}}\mathbf{i} + \frac{5}{\sqrt{41}}\mathbf{j} + 0\mathbf{k}$
6. $-\sin^2 t\mathbf{i} + \cos^2 t\mathbf{j} + \sqrt{2}\sin t\cos t\mathbf{k}$
7. (*i*) $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-2}{2}$ (*ii*) $\frac{x-6}{1} = \frac{y+25}{-10} = \frac{z-26}{10}$
8. $\frac{x - a\cos t}{-a\sin t} = \frac{y - a\sin t}{a\cos t} = \frac{z - bt}{b}$
9. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + 3\mathbf{j})$
10. $\mathbf{r} = \frac{1}{2}\mathbf{i} + \sqrt{3}\mathbf{j} + \lambda\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j}\right)$
11. $\mathbf{r} = \frac{5}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \lambda\left(\frac{4}{3}\mathbf{i} + \frac{5}{3}\mathbf{j}\right)$
12. $\left(\frac{4}{3}, \frac{1}{3}, 0\right)$
14. $\mathbf{r}\cdot(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - 10 = 0$
15. $(-\pi/2, k, 0)$, $-\infty < k < \infty$.

16. MOVING TRIHEDRON OF A CURVE

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a regular curve C with arc length s as parameter. We assume that $\mathbf{r}''(s)$ exists and $|\mathbf{r}''(s)| \neq 0$. We know that $\mathbf{r}'(s)$ equals the unit tangent vector $\mathbf{t}(s)$ at the point $\mathbf{r}(s)$ on the curve C.

$\therefore \quad \mathbf{t} = \mathbf{r}'$

We define $\quad \mathbf{n} = \dfrac{\mathbf{t}'}{|\mathbf{t}'|}$.

$\mathbf{n}$ is meaningful, because $|\mathbf{t}'| = |\mathbf{r}''| \neq 0$.

Also, $\mathbf{t}$ is a unit vector.

$\Rightarrow \quad |\mathbf{t}| = 1 \quad \Rightarrow \quad |\mathbf{t}|^2 = 1 \quad \Rightarrow \quad \mathbf{t} \cdot \mathbf{t} = 1$

$\Rightarrow \quad \mathbf{t} \cdot \mathbf{t}' + \mathbf{t}' \cdot \mathbf{t} = 0 \quad \Rightarrow \quad 2\mathbf{t} \cdot \mathbf{t}' = 0 \quad \Rightarrow \quad \mathbf{t} \cdot \mathbf{t}' = 0$

$\Rightarrow$ $\mathbf{t}'$ is perpendicular to $\mathbf{t}$ $\Rightarrow$ $\mathbf{n}$ is perpendicular to $\mathbf{t}$.

Also, $\quad |\mathbf{n}| = \left|\dfrac{\mathbf{t}'}{|\mathbf{t}'|}\right| = \dfrac{1}{|\mathbf{t}'|}|\mathbf{t}'| = 1$

$\therefore$ $\mathbf{n}$ is a unit vector and is perpendicular to the unit tangent vector $\mathbf{t}$.

$\therefore$ $\mathbf{n}$ lies in the normal plane at the point under consideration. The vector $\mathbf{n}$ is called the **unit principal normal vector** to the curve C at the point $\mathbf{r}(s)$. In terms of $\mathbf{r}$, we have

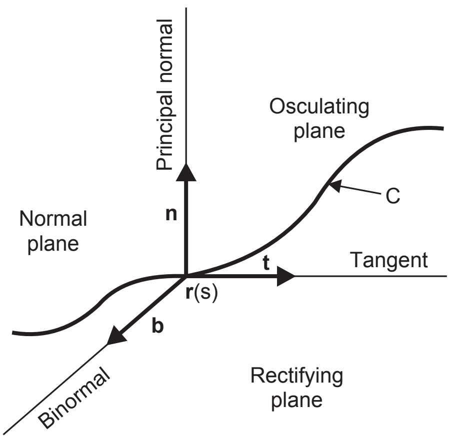
$$\mathbf{n} = \frac{\mathbf{r}''}{|\mathbf{r}''|}. \qquad (\because \ \mathbf{t} = \mathbf{r}')$$

We define $\quad \mathbf{b} = \mathbf{t} \times \mathbf{n}$.

$\therefore \quad |\mathbf{b}| = |\mathbf{t} \times \mathbf{n}| = |\mathbf{t}|\,|\mathbf{n}| \sin\dfrac{\pi}{2} = 1 \times 1 \times 1 = 1$

Also by the definition of vector cross product, $\mathbf{b}$ is perpendicular to vectors $\mathbf{t}$ and $\mathbf{n}$ both and the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ form a right handed triad.

The vector $\mathbf{b}$ is called the **unit binormal vector** to the curve C at the point $\mathbf{r}(s)$.



$\therefore$ We have

unit tangent vector, $\quad \mathbf{t} = \mathbf{r}'$

unit principal normal vector $\mathbf{n} = \dfrac{\mathbf{t}'}{|\mathbf{t}'|} = \dfrac{\mathbf{r}''}{|\mathbf{r}''|}$

$(\because \ |\mathbf{t}'| = |\mathbf{r}''| \neq 0)$

unit binormal vector, $\mathbf{b} = \mathbf{t} \times \mathbf{n}$

$$= \mathbf{r}' \times \frac{\mathbf{r}''}{|\mathbf{r}''|} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}''|}.$$

Thus $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ forms a right-handed orthonormal triplet as shown in the figure. The triplet $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is called the **moving trihedron** of the given curve $\mathbf{r} = \mathbf{r}(s)$.

Remark. Since the unit vectors **t, n, b** form a right handed triad, we have

(*i*) $\mathbf{t} \cdot \mathbf{n} = 0 \quad \mathbf{n} \cdot \mathbf{b} = 0 \quad \mathbf{b} \cdot \mathbf{t} = 0$

(*ii*) $\mathbf{t} \times \mathbf{n} = \mathbf{b} \quad \mathbf{n} \times \mathbf{b} = \mathbf{t} \quad \mathbf{b} \times \mathbf{t} = \mathbf{n}$.

The straight lines in the directions of **t**, **n** and **b** are respectively called the **tangent,** the **principal normal** and the **binormal** of the curve C at the point **r**(*s*). The equations of the tangent at the point **r**(*s*) is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$, where **R** is a general point on the tangent and λ is a scalar. The equation of the principal normal at the point **r**(*s*) is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}$, where **R** is a general point on the principal normal and λ is a scalar. The equation of the binormal at the point **r**(*s*) is $\mathbf{R} = \mathbf{r} + \lambda\mathbf{b}$, where **R** is a general point on the binormal and λ is a scalar.

We know that the unit vectors **n** and **b** are both perpendicular to the unit vector **t**.

$\therefore$ The normal plane of C at **r** is parallel to the vectors **n** and **b** both at the point **r** and its equation is $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$, where **R** is a general point on the normal plane.

If **R** is the position vector of a general point on the normal plane at the point **r**, then the vectors $\mathbf{R} - \mathbf{r}$, **n** and **b** lie in the normal plane and are thus coplanar vectors.

$$\therefore \qquad [\mathbf{R} - \mathbf{r} \quad \mathbf{n} \quad \mathbf{b}] = 0.$$

This also gives the equation of the normal plane at the point **r.**

The plane through the point **r** and parallel to the vectors **t** and **b** at the point **r** is called the **rectifying plane** of C at the point **r.** The rectifying plane is perpendicular to the vector **n.**

$\therefore$ The equation of the rectifying plane at the point **r** is $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$, where **R** is a general point on the rectifying plane.

If **R** is the position vector of a general point on the rectifying plane at the point **r**, then the vectors $\mathbf{R} - \mathbf{r}$, **t** and **b** lie in the rectifying plane and are thus coplanar vectors.

$$\therefore \qquad [\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{b}] = 0$$

This also gives the equation of the rectifying plane at the point **r.**

The plane through the point **r** and parallel to the vectors **t** and **n** at the point **r** is called the **osculating plane** of C at the point **r**. The osculating plane is perpendicular to the vector **b.**

$\therefore$ The equation of the osculating plane at the point **r** is $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$, where **R** is a general point on the osculating plane.

If **R** is the position vector of a general point on the osculating plane at the point **r**, then the vectors $\mathbf{R} - \mathbf{r}$, **t** and **n** lie in the osculating plane and are thus coplanar vectors.

$$\therefore \qquad [\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{n}] = 0$$

This also gives the equation of the osculating plane at the point **r.**

Thus at each point **r** on the curve C we have the following three characteristic lines and three characteristic planes :

Tangent	$\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$		
Principal normal	$\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}$		
Binormal	$\mathbf{R} = \mathbf{r} + \lambda\mathbf{b}$		
Normal plane	$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$	or	$[\mathbf{R} - \mathbf{r} \quad \mathbf{n} \quad \mathbf{b}] = 0$
Rectifying plane	$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$	or	$[\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{b}] = 0$
Osculating plane	$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$	or	$[\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{n}] = 0.$

17. CARTESIAN EQUATIONS OF CHARACTERISTIC LINES AND PLANES

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a regular curve with arc length *s* as parameter. We assume that $\mathbf{r}''(s)$ exists and $|\mathbf{r}''(s)| \neq 0$. Let $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$.

$$\therefore \qquad \mathbf{t} = \mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$$

and

$$\mathbf{t}' = x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}$$

$$\therefore \quad \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|} = \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}}(x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k})$$

$$= \frac{x''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{i} + \frac{y''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{j} + \frac{z''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{k}$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ \dfrac{x''}{\sqrt{x''^2 + y''^2 + z''^2}} & \dfrac{y''}{\sqrt{x''^2 + y''^2 + z''^2}} & \dfrac{z''}{\sqrt{x''^2 + y''^2 + z''^2}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

$$= \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}} ((y'z'' - y''z')\mathbf{i} + (z'x'' - z''x')\mathbf{j} + (x'y'' - x''y')\mathbf{k})$$

$\therefore$ The equation of the tangent $\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$ at the point $\mathbf{r}$ reduces to

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}.$$

The equation of the principal normal $\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}$ at the point $\mathbf{r}$ reduces to

$$\frac{X - x}{x''} = \frac{Y - y}{y''} = \frac{Z - z}{z''},$$

because x'', y'', z'' are d.r.'s of the principal normal at the point (x, y, z).

The equation of the binormal $\mathbf{R} = \mathbf{r} + \lambda\mathbf{b}$ at the point $\mathbf{r}$ reduces to

$$\frac{X - x}{y'z'' - y''z'} = \frac{Y - y}{z'x'' - z''x'} = \frac{Z - z}{x'y'' - x''y'},$$

because $y'z'' - y''z'$, $z'x'' - z''x'$, $x'y'' - x''y'$ are d.r.'s of the binormal at the point (x, y, z).

The equation of the normal plane $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$ at the point $\mathbf{r}$ reduces to

$$(X - x)x' + (Y - y)y' + (Z - z)z' = 0.$$

The equation of the rectifying plane $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$ at the point $\mathbf{r}$ reduces to

$$(X - x)x'' + (Y - y)y'' + (Z - z)z'' = 0.$$

The equation of the osculating plane $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$ at the point $\mathbf{r}$ reduces to

$$(X - x)(y'z'' - y''z') + (Y - y)(z'x'' - z''x') + (Z - z)(x'y'' - x''y') = 0.$$

In the above equations, the point $\mathbf{R}$ with coordinates (X, Y, Z) is a general point on the corresponding line (or plane).

18. VALUES OF UNIT VECTORS t, n AND b ALONG A CURVE GIVEN IN TERMS OF AN ARBITRARY PARAMETER

Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a regular curve C, where t is an arbitrary parameter. We assume that $\ddot{\mathbf{r}}(t)$ exists and $|\ddot{\mathbf{r}}(t)| \neq 0$.

In terms of arc length s, we have

$$\mathbf{t} = \mathbf{r}', \quad \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|} \quad \text{and} \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}.$$

(i) $\mathbf{t} = \mathbf{r}' = \dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{\dot{\mathbf{r}}}{\dfrac{ds}{dt}} = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$ $\quad\left(s = \int_a^t |\dot{\mathbf{r}}|\, dt \;\Rightarrow\; \dfrac{ds}{dt} = |\dot{\mathbf{r}}|\right)$

(ii) $\mathbf{t}' = \dfrac{d\mathbf{t}}{ds} = \dfrac{d\mathbf{t}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{\dot{\mathbf{t}}}{\dfrac{ds}{dt}} = \dfrac{\dot{\mathbf{t}}}{|\dot{\mathbf{r}}|}$

and $\quad |\mathbf{t}'| = \left|\dfrac{\dot{\mathbf{t}}}{|\dot{\mathbf{r}}|}\right| = \dfrac{|\dot{\mathbf{t}}|}{|\dot{\mathbf{r}}|}$

$$\therefore \qquad \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|} = \frac{\dot{\mathbf{t}}/|\dot{\mathbf{r}}|}{|\dot{\mathbf{t}}|/|\dot{\mathbf{r}}|} = \frac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|}$$

(iii) $\mathbf{b} = \mathbf{t} \times \mathbf{n} = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \times \dfrac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|} = \dfrac{\dot{\mathbf{r}} \times \dot{\mathbf{t}}}{|\dot{\mathbf{r}}||\dot{\mathbf{t}}|}$

$\therefore$ In terms of arbitrary parameter t, we have

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}, \quad \mathbf{n} = \frac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|} \quad \text{and} \quad \mathbf{b} = \frac{\dot{\mathbf{r}} \times \dot{\mathbf{t}}}{|\dot{\mathbf{r}}||\dot{\mathbf{t}}|}.$$

WORKING RULES FOR SOLVING PROBLEMS

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a regular curve with arc length s as parameter. Let $\mathbf{r}''(s)$ exists and $|\mathbf{r}''(s)| \neq 0$.

Rule I. (i) $\mathbf{t} = \mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$

(ii) $\mathbf{n} = \dfrac{\mathbf{t}'}{|\mathbf{t}'|} = \dfrac{1}{\sqrt{x''^2 + y''^2 + z''^2}}(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k})$

(iii) $\mathbf{b} = \mathbf{t} \times \mathbf{n} = \dfrac{1}{\sqrt{x''^2 + y''^2 + z''^2}}((y'z'' - y''z')\mathbf{i} + (z'x'' - z''x')\mathbf{j} + (x'y'' - x''y')\mathbf{k})$

Rule II. *Equation of tangent:*

(i) $\mathbf{R} = \mathbf{r} + \lambda\mathbf{t}$ $\qquad$ (ii) $\dfrac{X - x}{x'} = \dfrac{Y - y}{y'} = \dfrac{Z - z}{z'}$

Rule III. *Equation of principal normal:*

(i) $\mathbf{R} = \mathbf{r} + \lambda\mathbf{n}$ $\qquad$ (ii) $\dfrac{X - x}{x''} = \dfrac{Y - y}{y''} = \dfrac{Z - z}{z''}$

Rule IV. *Equation of binormal:*

(i) $\mathbf{R} = \mathbf{r} + \lambda\mathbf{b}$ $\qquad$ (ii) $\dfrac{X - x}{y'z'' - y''z'} = \dfrac{Y - y}{z'x'' - z''x'} = \dfrac{Z - z}{x'y'' - x''y'}$

Rule V. *Equation of normal plane:*

(*i*) $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$ (*ii*) $[\mathbf{R} - \mathbf{r} \quad \mathbf{n} \quad \mathbf{b}] = 0$

(*iii*) $(X - x)x' + (Y - y)y' + (Z - z)z' = 0$

Rule VI. *Equation of rectifying plane:*

(*i*) $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{n} = 0$ (*ii*) $[\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{b}] = 0$

(*iii*) $(X - x)x'' + (Y - y)y'' + (Z - z)z'' = 0$

Rule VII. *Equation of osculating plane:*

(*i*) $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{b} = 0$ (*ii*) $[\mathbf{R} - \mathbf{r} \quad \mathbf{t} \quad \mathbf{n}] = 0$

(*iii*) $(X - x)(y'z'' - y''z') + (Y - y)(z'x'' - z''x') + (Z - z)(x'y'' - x''y') = 0.$

Theorem 1. *Let $r = r(s)$ be the equation of a regular curve with arc length s as parameter. If r'' exists and $|r''| \neq 0$ at a point r, prove that the equation of the osculating plane at the point r is*

$$[R - r \quad r' \quad r''] = 0.$$

Proof. We know that the plane through the point **r** and parallel to the vectors **t** and **n** at the point **r** is the osculating plane at the point **r**.

We have $$\mathbf{r}' = \frac{d\mathbf{r}}{ds} = \mathbf{t}$$

∴ $\mathbf{r}'$ is parallel to the osculating plane.

Also $$\mathbf{r}'' = \frac{d}{ds}(\mathbf{t}) = \mathbf{t}' = |\mathbf{t}'|\,\mathbf{n} \qquad \left(\because \ \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|}\right)$$

∴ $\mathbf{r}''$ is parallel to the osculating plane.

Let **R** be the position vector of a general point on the osculating plane at the point **r**.

∴ The vector $\mathbf{R} - \mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ lie in the osculating plane and are thus coplanar vectors.

∴ $$[\mathbf{R} - \mathbf{r} \quad \mathbf{r}' \quad \mathbf{r}''] = 0.$$

This is the equation of the required osculating plane.

Corollary. If $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ and $\mathbf{r}'' = x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}$.

∴ The equation of the osculating plane is $$\begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0.$$

Theorem 2. *Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a regular curve, where t is an arbitrary parameter. If $\ddot{\mathbf{r}}$ exists and $|\ddot{\mathbf{r}}| \neq 0$ at a point $\mathbf{r}$, prove that the equation of the osculating plane at the point $\mathbf{r}$ is*

$$[\mathbf{R} - \mathbf{r} \quad \dot{\mathbf{r}} \quad \ddot{\mathbf{r}}] = 0.$$

Proof. We know that the plane through the point **r** and parallel to the vectors **t** and **n** at the point **r** is the osculating plane at the point **r**.

We have $$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = \mathbf{t}\dot{s} = \dot{s}\mathbf{t}$$

∴ $\dot{\mathbf{r}}$ is parallel to the osculating plane.

Also, $$\ddot{\mathbf{r}} = \frac{d}{dt}(\dot{s}\,\mathbf{t}) = \dot{s}\frac{d\mathbf{t}}{dt} + \frac{d\dot{s}}{dt}\mathbf{t}$$

$$= \dot{s}\,\dot{\mathbf{t}} + \ddot{s}\,\mathbf{t} = \ddot{s}\,\mathbf{t} + \dot{s}\,|\dot{\mathbf{t}}|\,\mathbf{n} \qquad \left(\because\ \mathbf{n} = \frac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|}\right)$$

$\therefore$ $\ddot{\mathbf{r}}$ lies in the plane of $\mathbf{t}$ and $\mathbf{n}$.

$\therefore$ $\ddot{\mathbf{r}}$ is parallel to the osculating plane.

Let $\mathbf{R}$ be the position vector of a general point on the osculating plane at the point $\mathbf{r}$.

$\therefore$ The vectors $\mathbf{R} - \mathbf{r}$, $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ lie in the osculating plane and are thus coplanar vectors.

$\therefore$ $$[\mathbf{R} - \mathbf{r} \quad \dot{\mathbf{r}} \quad \ddot{\mathbf{r}}] = 0.$$

This is the equation of the required osculating plane.

Corollary. If $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$

and $$\ddot{\mathbf{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}.$$

$\therefore$ The equation of the osculating plane is $\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$

Example 1. *For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, (i) show that any plane meets it in three points and (ii) find the equation of the osculating plane at the point t_1.*

Sol. The given curve is

$$x = 3t,\ y = 3t^2,\ z = 2t^3.$$

(*i*) Let $ax + by + cz + d = 0$ be any plane in space.

Putting $x = 3t$, $y = 3t^2$, $z = 2t^3$, we get $3at + 3bt^2 + 2ct^3 + d = 0$

$\Rightarrow$ $$2ct^3 + 3bt^2 + 3at + d = 0$$

This is a cubic equation in t and gives three values of t.

$\therefore$ The plane $ax + by + cz + d = 0$ meets the given curve in three points.

(*ii*) We have $$x = 3t,\ y = 3t^2,\ z = 2t^3$$

$\therefore$ $$\dot{x} = 3,\ \dot{y} = 6t,\ \dot{z} = 6t^2 \quad \text{and} \quad \ddot{x} = 0,\ \ddot{y} = 6,\ \ddot{z} = 12t$$

Let (x, y, z) be a general point on the osculating plane at the point t_1.

$\therefore$ The equation of the osculating plane is

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 2t_1^3 \\ 3 & 6t_1 & 6t_1^2 \\ 0 & 6 & 12t_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 3t_1 & y - 3t_1^2 & z - 2t_1^3 \\ 1 & 2t_1 & 2t_1^2 \\ 0 & 1 & 2t_1 \end{vmatrix} = 0$$

$$(x - 3t_1)(2t_1^2) - (y - 3t_1^2)(2t_1) + (z - 2t_1^3)(1) = 0$$

$\Rightarrow$ $$2t_1^2 x - 2t_1 y + z = 2t_1^3.$$

Example 2. *Find the equation of the osculating plane to the curve $x = 2 \log t$, $y = 4t$, $z = 2t^2 + 1$ at the point t.*

Sol. The given curve is

$$x = 2 \log t,\ y = 4t,\ z = 2t^2 + 1.$$

$$\therefore \qquad \dot{x} = \frac{2}{t},\ \dot{y} = 4,\ \dot{z} = 4t \quad \text{and} \quad \ddot{x} = -\frac{2}{t^2},\ \ddot{y} = 0,\ \ddot{z} = 4$$

Let (X, Y, Z) be a general point on the osculating plane at the point t.

$\therefore$ The equation of the osculating plane is $\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$

$$\Rightarrow \quad \begin{vmatrix} X - 2\log t & Y - 4t & Z - (2t^2 + 1) \\ \frac{2}{t} & 4 & 4t \\ -\frac{2}{t^2} & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} X - 2\log t & Y - 4t & Z - (2t^2 + 1) \\ 1 & 2t & 2t^2 \\ 1 & 0 & -2t^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (X - 2 \log t)(-4t^3) - (Y - 4t)(-4t^2) + (Z - 2t^2 - 1)(-2t) = 0$$

$$\Rightarrow \quad 2t^2(X - 2 \log t) - 2t(Y - 4t) + Z - 2t^2 - 1 = 0$$

$$\Rightarrow \quad 2t^2X - 2tY + Z = 4t^2 \log t - 6t^2 + 1.$$

Example 3. *Let* $\mathbf{r} = \mathbf{r}(t)$ *be the equation of a regular curve. By using the equation* $[\mathbf{R} - \mathbf{r}\ \mathbf{r}'\ \mathbf{r}''] = 0$, *show that the equation of the osculating plane at the point* $\mathbf{r}$ *is* $[\mathbf{R} - \mathbf{r}\ \dot{\mathbf{r}}\ \ddot{\mathbf{r}}] = 0$.

Sol. Given equation of the osculating plane at point $\mathbf{r}$ is

$$[\mathbf{R} - \mathbf{r} \quad \mathbf{r}' \quad \mathbf{r}''] = 0 \qquad \ldots(1)$$

$$\mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds} = \frac{\dot{\mathbf{r}}}{\dot{s}}$$

$$\mathbf{r}'' = \frac{d\mathbf{r}'}{ds} = \frac{d}{ds}\left(\frac{\dot{\mathbf{r}}}{\dot{s}}\right) = \frac{d}{dt}\left(\frac{\dot{\mathbf{r}}}{\dot{s}}\right) \cdot \frac{dt}{ds} = \frac{\dot{s}\,\ddot{\mathbf{r}} - \ddot{s}\,\dot{\mathbf{r}}}{\dot{s}^2} \cdot \frac{1}{\dot{s}} = \frac{1}{\dot{s}^2}\ddot{\mathbf{r}} - \frac{\ddot{s}}{\dot{s}^3}\dot{\mathbf{r}}$$

$$\therefore \quad (1) \quad \Rightarrow \qquad \left[\mathbf{R} - \mathbf{r} \quad \frac{1}{\dot{s}}\dot{\mathbf{r}} \quad \left(\frac{1}{\dot{s}^2}\ddot{\mathbf{r}} - \frac{\ddot{s}}{\dot{s}^3}\dot{\mathbf{r}}\right)\right] = 0$$

$$\Rightarrow \quad \left[\mathbf{R} - \mathbf{r}\ \frac{1}{\dot{s}}\dot{\mathbf{r}}\ \frac{1}{\dot{s}^2}\ddot{\mathbf{r}}\right] - \left[\mathbf{R} - \mathbf{r}\ \frac{1}{\dot{s}}\dot{\mathbf{r}} - \frac{\ddot{s}}{\dot{s}^3}\dot{\mathbf{r}}\right] = 0$$

$$\Rightarrow \quad \frac{1}{\dot{s}^3}[\mathbf{R} - \mathbf{r}\ \dot{\mathbf{r}}\ \ddot{\mathbf{r}}] - \left(-\frac{\ddot{s}}{\dot{s}^4}\right)[\mathbf{R} - \mathbf{r}\ \dot{\mathbf{r}}\ \dot{\mathbf{r}}] = 0$$

($\because$ Determinant with two equal rows is zero)

$$\Rightarrow \qquad \frac{1}{\dot{s}^3}[\mathbf{R}-\mathbf{r} \quad \dot{\mathbf{r}} \quad \ddot{\mathbf{r}}] + \frac{\ddot{s}}{\dot{s}^4}\,.\,0 = 0$$

$$\Rightarrow \qquad \frac{1}{\dot{s}^3}[\mathbf{R}-\mathbf{r} \quad \dot{\mathbf{r}} \quad \ddot{\mathbf{r}}] = 0$$

$$\Rightarrow \qquad [\mathbf{R}-\mathbf{r} \quad \dot{\mathbf{r}} \quad \ddot{\mathbf{r}}] = 0.$$

∴ The result holds.

Example 4. *For the curve $x = 4a\cos^3 t$, $y = 4a\sin^3 t$, $z = 3c\cos 2t$, find*

(*i*) *the equation of the principal normal at the point t.*

(*ii*) *the equation of the osculating plane at the point t.*

Sol. (*i*) Let **r** be the position vector of the point (x, y, z) on the given curve.

$$\therefore \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 4a\cos^3 t\,\mathbf{i} + 4a\sin^3 t\,\mathbf{j} + 3c\cos 2t\,\mathbf{k}$$

$$\therefore \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = (-12a\cos^2 t\sin t\,\mathbf{i} + 12a\sin^2 t\cos t\,\mathbf{j} - 6c\sin 2t\,\mathbf{k})\frac{dt}{ds}$$

$$= 12\sin t\cos t\,(-a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} - c\mathbf{k})\frac{dt}{ds} \qquad \text{(Using } \sin 2t = 2\sin t\cos t)$$

$$\therefore \qquad |\,\mathbf{r}'\,| = 12\sin t\cos t\sqrt{a^2\cos^2 t + a^2\sin^2 t + c^2}\,\frac{dt}{ds}$$

$$\Rightarrow \qquad 1 = 12\sin t\cos t\sqrt{a^2+c^2}\,\frac{dt}{ds} \qquad (\because\ |\,\mathbf{t}\,| = |\,\mathbf{r}'\,| = 1)$$

$$\therefore \qquad \frac{ds}{dt} = 12\sin t\cos t\sqrt{a^2+c^2}$$

$$\therefore \qquad \mathbf{r}' = 12\sin t\cos t\,(-a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} - c\mathbf{k})\,.\,\frac{1}{12\sin t\cos t\sqrt{a^2+c^2}}$$

$$= \frac{1}{\sqrt{a^2+c^2}}(-a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} - c\mathbf{k})$$

$$\therefore \qquad \mathbf{r}'' = \frac{1}{\sqrt{a^2+c^2}}(a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j})\frac{dt}{ds}$$

$$= \frac{a}{\sqrt{a^2+c^2}}(\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,\frac{1}{12\sin t\cos t\sqrt{a^2+c^2}}$$

$$= \frac{a}{12(a^2+c^2)}(\sec t\,\mathbf{i} + \operatorname{cosec} t\,\mathbf{j})$$

$$\therefore \qquad \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|} = \frac{\mathbf{r}''}{|\mathbf{r}''|} = \frac{a}{12(a^2+c^2)\,|\mathbf{r}''|}(\sec t\,\mathbf{i} + \operatorname{cosec} t\,\mathbf{j})$$

∴ D.r.'s of principal normal are $\sec t$, $\operatorname{cosec} t$, 0.

∴ The equations of the principal normal at the point t are

$$\frac{x-4a\cos^3 t}{\sec t} = \frac{y-4a\sin^3 t}{\operatorname{cosec} t} = \frac{z-3c\cos 2t}{0}.$$

(*ii*) The equation of the osculating plane at the point t is

$$\begin{vmatrix} x-4a\cos^3 t & y-4a\sin^3 t & z-3c\cos 2t \\ -\dfrac{a}{\sqrt{a^2+c^2}}\cos t & \dfrac{a}{\sqrt{a^2+c^2}}\sin t & \dfrac{-c}{\sqrt{a^2+c^2}} \\ \dfrac{a}{12(a^2+c^2)}\sec t & \dfrac{a}{12(a^2+c^2)}\operatorname{cosec} t & 0 \end{vmatrix} = 0$$

$$\Rightarrow \qquad \begin{vmatrix} x-4a\cos^3 t & y-4a\sin^3 t & z-3c\cos 2t \\ -a\cos t & a\sin t & -c \\ \sec t & \operatorname{cosec} t & 0 \end{vmatrix} = 0$$

$$(x-4a\cos^3 t)(c\operatorname{cosec} t) - (y\ 4a\sin^3 t)(c\sec t)$$
$$+ (z-3c\cos 2t)(-a\sin t\cos t - a\sin t\cos t) = 0$$

$$\Rightarrow \qquad c\operatorname{cosec} t\cdot x - c\sec t\cdot y - 2\sin t\cos t\, z$$
$$= 4a\sin t\cos^3 t - 4a\sin^3 t\cos t - 6ac\sin t\cos t\cos 2t$$

$$\Rightarrow \qquad c\operatorname{cosec} t\cdot x - c\sec t\cdot y - 2\sin t\cos t\, z$$
$$= 2a\sin 2t\cos^2 t - 2a\sin 2t\sin^2 t - 3ac\sin 2t\cos 2t.$$

Example 5. *Find the vectors* **t**, **n** *and* **b** *along the curve* $\mathbf{r} = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j} + (3t + t^3)\mathbf{k}$.

Sol. We have $\mathbf{r} = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j} + (3t + t^3)\mathbf{k}$.

$$\therefore \qquad \dot{\mathbf{r}} = (3 - 3t^2)\mathbf{i} + 6t\mathbf{j} + (3 + 3t^2)\mathbf{k}$$

or

$$\dot{\mathbf{r}} = 3[(1 - t^2)\mathbf{i} + 2t\mathbf{j} + (1 + t^2)\mathbf{k}]$$

$$|\,\dot{\mathbf{r}}\,| = 3\sqrt{(1-t^2)^2 + 4t^2 + (1+t^2)^2}$$

$$= 3\sqrt{2 + 2t^4 + 4t^2} = 3\sqrt{2}\,(1 + t^2)$$

$$\therefore \qquad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\,\dot{\mathbf{r}}\,|} = \frac{1}{3\sqrt{2}(1+t^2)}\,.\,3[(1 - t^2)\mathbf{i} + 2t\mathbf{j} + (1 + t^2)\mathbf{k}]$$

$$= \frac{1-t^2}{\sqrt{2}(1+t^2)}\mathbf{i} + \frac{\sqrt{2}\,t}{1+t^2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

$$\dot{\mathbf{t}} = \frac{(1+t^2)(-2t) - (1-t^2)2t}{\sqrt{2}(1+t^2)^2}\mathbf{i} + \frac{\sqrt{2}((1+t^2)\cdot 1 - t\,.\,2t)}{(1+t^2)^2}\mathbf{j} + 0\mathbf{k}$$

$$= \frac{\sqrt{2}}{(1+t^2)^2}\,(-2t\,\mathbf{i} + (1 - t^2)\,\mathbf{j})$$

$$\therefore \qquad |\,\dot{\mathbf{t}}\,| = \frac{\sqrt{2}}{(1+t^2)^2}\,[4t^2 + (1 - t^2)^2]^{1/2} = \frac{\sqrt{2}}{1+t^2}$$

$$\therefore \qquad \mathbf{n} = \frac{\dot{\mathbf{t}}}{|\,\dot{\mathbf{t}}\,|} = \frac{\sqrt{2}}{(1+t^2)^2}\,(-2t\mathbf{i} + (1 - t^2)\mathbf{j})\,.\,\frac{1+t^2}{\sqrt{2}} = -\frac{2t}{1+t^2}\mathbf{i} + \frac{1-t^2}{1+t^2}\mathbf{j}$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{1-t^2}{\sqrt{2}(1+t^2)} & \dfrac{\sqrt{2}\,t}{1+t^2} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{2t}{1+t^2} & \dfrac{1-t^2}{1+t^2} & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)(1+t^2)} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 1-t^2 & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2} [-(1-t^4)\mathbf{i} - 2t(1+t^2)\mathbf{j} + (1+t^2)^2\mathbf{k}]$$

$$= \frac{1}{\sqrt{2}(1+t^2)} [(t^2-1)\mathbf{i} - 2t\mathbf{j} + (1+t^2)\mathbf{k}].$$

Example 6. *Show that the points on the helix* $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$, $a > 0$, $b \neq 0$ *at which the osculating planes pass through a fixed point are all coplanar.*

Sol. The given helix is

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}.$$

$$\therefore \quad \dot{\mathbf{r}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}$$

$$\ddot{\mathbf{r}} = -a\cos t\mathbf{i} - a\sin t\mathbf{j} + 0\mathbf{k}$$

The equation of the osculating plane is $[\mathbf{R} - \mathbf{r} \;\; \dot{\mathbf{r}} \;\; \ddot{\mathbf{r}}] = 0$.

Let $\qquad \mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$

$\therefore$ The equation of the osculating plane is

$$\begin{vmatrix} X - a\cos t & Y - a\sin t & Z - bt \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad ab\sin t(X - a\cos t) - ab\cos t(Y - a\sin t) + a^2(Z - bt) = 0$

$\Rightarrow \quad b\sin t\, X - b\cos t\, Y + aZ = abt$

Let the osculating plane at the point $\mathbf{r}$ passes through the fixed point (α, β, γ).

$\therefore \quad (b\sin t)\alpha - (b\cos t)\beta + a\gamma = abt$

$\Rightarrow \quad -b\beta(a\cos t) + b\alpha(a\sin t) - a^2(bt) = -a^2\gamma$

$\Rightarrow \quad b\beta(a\cos t) - b\alpha(a\sin t) + a^2(bt) = a^2\gamma$

$\therefore$ The locus of the point $\mathbf{r}$ (= $(a\cos t, a\sin t, bt)$) is

$$b\beta x - b\alpha y + a^2 z = a^2\gamma, \text{ which is a plane.}$$

$\therefore$ The result holds.

Example 7. *Find the equations of characteristic lines and planes to the helix* $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ *at the point where* $t = \pi/2$.

Sol. We have $\qquad \mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$

$$\therefore \quad \mathbf{r}\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2}\mathbf{i} + \sin\frac{\pi}{2}\mathbf{j} + \frac{\pi}{2}\mathbf{k} = \mathbf{j} + \frac{\pi}{2}\mathbf{k}$$

$\therefore$ The point under consideration is $(0, 1, \pi/2)$.

$$\dot{\mathbf{r}} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$\therefore \qquad |\,\dot{\mathbf{r}}\,| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\therefore \qquad \mathbf{t} = \dfrac{\dot{\mathbf{r}}}{|\,\dot{\mathbf{r}}\,|} = \dfrac{1}{\sqrt{2}}\,(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})$

$$\dot{\mathbf{t}} = \frac{1}{\sqrt{2}}\,(-\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{k})$$

and $\qquad |\,\dot{\mathbf{t}}\,| = \dfrac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \dfrac{1}{\sqrt{2}}$

$\therefore \qquad \mathbf{n} = \dfrac{\dot{\mathbf{t}}}{|\,\dot{\mathbf{t}}\,|} = \dfrac{1}{\sqrt{2}}\,(-\cos t\mathbf{i} - \sin t\mathbf{j})\,.\,\sqrt{2} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dfrac{\sin t}{\sqrt{2}} & \dfrac{\cos t}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{\sin t}{\sqrt{2}}\,\mathbf{i} - \frac{\cos t}{\sqrt{2}}\,\mathbf{j} + \frac{1}{\sqrt{2}}\,\mathbf{k}$$

$\therefore \qquad \mathbf{t}(\pi/2) = \dfrac{1}{\sqrt{2}}(-1\cdot\mathbf{i} + 0\cdot\mathbf{j} + \mathbf{k}) = -\dfrac{1}{\sqrt{2}}\,\mathbf{i} + \dfrac{1}{\sqrt{2}}\,\mathbf{k}$

$$\mathbf{n}(\pi/2) = -0\cdot\mathbf{i} - 1\cdot\mathbf{j} = -\mathbf{j}$$

$$\mathbf{b}(\pi/2) = \frac{1}{\sqrt{2}}\,\mathbf{i} - 0\cdot\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Tangent. The equation of the tangent is

$$\mathbf{r} = \mathbf{r}(\pi/2) + \lambda\mathbf{t}(\pi/2) \quad i.e., \quad \mathbf{r} = \mathbf{j} + \frac{\pi}{2}\mathbf{k} + \lambda\left(-\frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{k}\right).$$

Equations in cartesian form are $\dfrac{x-0}{-1} = \dfrac{y-1}{0} = \dfrac{z-\pi/2}{1}$.

Principal normal. The equation of the principal normal is

$$\mathbf{r} = \mathbf{r}(\pi/2) + \lambda\mathbf{n}(\pi/2) \quad i.e., \quad \mathbf{r} = \mathbf{j} + \frac{\pi}{2}\mathbf{k} + \lambda(-\mathbf{j}).$$

Equations in the cartesian form are $\dfrac{x-0}{0} = \dfrac{y-1}{-1} = \dfrac{z-\pi/2}{0}$.

Binormal. The equation of the binormal is

$$\mathbf{r} = \mathbf{r}(\pi/2) + \lambda\mathbf{b}(\pi/2) \quad i.e., \quad \mathbf{r} = \mathbf{j} + \frac{\pi}{2}\mathbf{k} + \lambda\left(\frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{k}\right).$$

Equations in the cartesian form are $\dfrac{x-0}{1} = \dfrac{y-1}{0} = \dfrac{z-\pi/2}{1}$.

Normal plane. The equation of the normal plane is $(\mathbf{r} - \mathbf{r}(\pi/2))\cdot\mathbf{t}(\pi/2) = 0$

i.e., $\qquad \left(\mathbf{r} - \left(\mathbf{j} + \dfrac{\pi}{2}\mathbf{k}\right)\right).\left(-\dfrac{1}{\sqrt{2}}\,\mathbf{i} + \dfrac{1}{\sqrt{2}}\,\mathbf{k}\right) = 0 \quad \text{or} \quad \left(\mathbf{r} - \left(\mathbf{j} + \dfrac{\pi}{2}\mathbf{k}\right)\right)\cdot(-\mathbf{i} + \mathbf{k}) = 0$

or $\qquad \mathbf{r}\cdot(-\mathbf{i} + \mathbf{k}) - \left(0\,(-1) + 1(0) + \dfrac{\pi}{2}\,(1)\right) = 0 \quad \text{or} \quad \mathbf{r}\cdot(-\mathbf{i} + \mathbf{k}) = \dfrac{\pi}{2}.$

Equation in the cartesian form is $-x + z = \frac{\pi}{2}$ *i.e.*, $x - z + \frac{\pi}{2} = 0$.

Rectifying plane. The equation of the rectifying plane is $(\mathbf{r} - \mathbf{r}(\pi/2)) \,.\, \mathbf{n}(\pi/2) = 0$

i.e., $$\left(\mathbf{r} - \left(\mathbf{j} + \frac{\pi}{2}\mathbf{k}\right)\right) \cdot (-\mathbf{j}) = 0 \quad \text{or} \quad -\mathbf{r} \cdot \mathbf{j} + 1 = 0 \quad \text{or} \quad \mathbf{r} \cdot \mathbf{j} - 1 = 0.$$

Equation in the cartesian form is $y - 1 = 0$.

Osculating plane. The equation of the osculating plane is

$$(\mathbf{r} - \mathbf{r}(\pi/2)) \cdot \mathbf{b}(\pi/2) = 0 \quad i.e., \left(\mathbf{r} - \left(\mathbf{j} + \frac{\pi}{2}\mathbf{k}\right)\right) . \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = 0$$

or $$\left(\mathbf{r} - \left(\mathbf{j} + \frac{\pi}{2}\mathbf{k}\right)\right) \cdot (\mathbf{i} + \mathbf{k}) = 0 \quad \text{or} \quad \mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) = \frac{\pi}{2}.$$

Equation in the cartesian form is $x + z = \frac{\pi}{2}$.

Example 8. *For the curve* $\mathbf{r} = (e^{-t} \sin t, e^{-t} \cos t, e^{-t})$, *find the following at the point t:*

(*i*) *the unit tangent vector* **t**

(*ii*) *the equation of the tangent*

(*iii*) *the unit principal normal vector* **n**

(*iv*) *the equation of the normal plane*

(*v*) *the unit binormal vector* **b**

(*vi*) *the equation of the binormal.*

Sol. (*i*) The given curve is

$$\mathbf{r} = e^{-t} \sin t\mathbf{i} + e^{-t} \cos t\mathbf{j} + e^{-t}\mathbf{k}.$$

$$\therefore \quad \dot{\mathbf{r}} = (e^{-t} \cos t - e^{-t} \sin t)\mathbf{i} + (-e^{-t} \sin t - e^{-t} \cos t)\mathbf{j} - e^{-t}\mathbf{k}$$

$$\therefore \quad = e^{-t} [(\cos t - \sin t)\mathbf{i} - (\sin t + \cos t)\mathbf{j} - \mathbf{k}]$$

$$\therefore \quad |\dot{\mathbf{r}}| = e^{-t} \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \sqrt{3}\, e^{-t}$$

$$\therefore \quad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{3}e^{-t}} \cdot e^{-t} [(\cos t - \sin t)\mathbf{i} - (\sin t + \cos t)\mathbf{j} - \mathbf{k}]$$

$$\therefore \quad \mathbf{t} = \frac{1}{\sqrt{3}} [(\cos t - \sin t)\mathbf{i} - (\sin t + \cos t)\mathbf{j} - \mathbf{k}]$$

(*ii*) Using **t**, the d.r.'s of the tangent are $\cos t - \sin t, -(\sin t + \cos t), -1$.

$\therefore$ The equations of the tangent at the point t are

$$\frac{x - e^{-t} \sin t}{\cos t - \sin t} = \frac{y - e^{-t} \cos t}{-(\sin t + \cos t)} = \frac{z - e^{-t}}{-1}.$$

(*iii*) $$\dot{\mathbf{t}} = \frac{1}{\sqrt{3}} [(-\sin t - \cos t)\mathbf{i} - (\cos t - \sin t)\mathbf{j} + 0\mathbf{k}]$$

$$= \frac{1}{\sqrt{3}} [-(\sin t + \cos t)\mathbf{i} + (\sin t - \cos t)\mathbf{j}]$$

$$\therefore \qquad |\dot{\mathbf{t}}| = \frac{1}{\sqrt{3}}\sqrt{(\sin t + \cos t)^2 + (\sin t - \cos t)^2} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \qquad \mathbf{n} = \frac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|} = \frac{1}{\sqrt{3}}\,[-(\sin t + \cos t)\mathbf{i} + (\sin t - \cos t)\mathbf{j}]\,.\,\frac{\sqrt{3}}{\sqrt{2}}$$

$$\therefore \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\,[-(\sin t + \cos t)\mathbf{i} + (\sin t - \cos t)\mathbf{j}].$$

(*iv*) The equation of the normal plane is $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$.

$$\Rightarrow \quad [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (e^{-t}\sin t\,\mathbf{i} + e^{-t}\cos t\,\mathbf{j} + e^{-t}\,\mathbf{k})] \cdot \left[\frac{1}{\sqrt{3}}((\cos t - \sin t)\mathbf{i} - (\sin t + \cos t)\mathbf{j} - \mathbf{k})\right] = 0$$

$$\Rightarrow \quad x(\cos t - \sin t) - y(\sin t + \cos t) - z + e^{-t}(-\sin t\cos t + \sin^2 t + \sin t\cos t + \cos^2 t + 1) = 0$$

$$\Rightarrow \quad (\cos t - \sin t)x - (\sin t + \cos t)y - z = -2e^{-t}$$

$$\Rightarrow \quad (\sin t - \cos t)x + (\sin t + \cos t)y + z = 2e^{-t}.$$

(*v*) $\mathbf{b} = \mathbf{t} \times \mathbf{n}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}}(\cos t - \sin t) & -\frac{1}{\sqrt{3}}(\sin t + \cos t) & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}}(\sin t + \cos t) & \frac{1}{\sqrt{2}}(\sin t - \cos t) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t - \cos t & \sin t + \cos t & 1 \\ \sin t + \cos t & \cos t - \sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\,[(\sin t - \cos t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} - 2\mathbf{k}].$$

(*vi*) Using $\mathbf{b}$, the d.r.'s of the binormal are $\sin t - \cos t,\ \sin t + \cos t,\ -2$.

$\therefore$ The equation of the binormal at the point t are

$$\frac{x - e^{-t}\sin t}{\sin t - \cos t} = \frac{y - e^{-t}\cos t}{\sin t + \cos t} = \frac{z - e^{-t}}{-2}.$$

Example 9. *Find the equation of the osculating plane at a general point on the curve* $\mathbf{r} = (t, t^2, t^3)$. *Show that the osculating planes at three points on this curve meet at a point lying in the plane determined by these three points.*

Sol. The given curve is $\mathbf{r} = (t, t^2, t^3)$. Parametric equations of the curve are

$$x = t,\ y = t^2,\ z = t^3.$$

$$\therefore \qquad \dot{x} = 1,\ \dot{y} = 2t,\ \dot{z} = 3t^2,\ \ddot{x} = 0,\ \ddot{y} = 2,\ \ddot{z} = 6t$$

$\therefore$ Equation of the osculating plane at point 't' is

$$\begin{vmatrix} X - t & Y - t^2 & Z - t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 0$$

$\Rightarrow \quad 6t^2(X - t) - 6t(Y - t^2) + 2(Z - t^3) = 0$

$\Rightarrow \quad 3t^2X - 3tY + Z - t^3 = 0.$ ...(1)

Let t_1, t_2, t_3 be the values of the parameter t at any three points on the curve.

$\therefore$ The equations of the osculating planes at these points are

$$3t_1^2X - 3t_1Y + Z - t_1^3 = 0$$
$$3t_2^2X - 3t_2Y + Z - t_2^3 = 0$$

and
$$3t_3^2X - 3t_3Y + Z - t_3^3 = 0$$

Let these planes intersect at the point (α, β, γ).

$\therefore$
$$3t_1^2\alpha - 3t_1\beta + \gamma - t_1^3 = 0$$
$$3t_2^2\alpha - 3t_2\beta + \gamma - t_2^3 = 0$$

and
$$3t_3^2\alpha - 3t_3\beta + \gamma - t_3^3 = 0$$

$\therefore$ t_1, t_2, t_3 are the roots of the cubic equation

$$3t^2\alpha - 3t\beta + \gamma - t^3 = 0$$

or $\quad t^3 - 3\alpha t^2 + 3\beta t - \gamma = 0$ ...(2)

Let the equation of the plane passing through the points t_1, t_2 and t_3 be

$$ax + by + cz + d = 0 \qquad ...(3)$$

$\Rightarrow \quad at + bt^2 + ct^3 + d = 0$

$\Rightarrow \quad ct^3 + bt^2 + at + d = 0,$

where $t = t_1, t_2, t_3$ ...(4)

$\therefore$ Equations (2) and (4) are same.

$\therefore \quad \dfrac{1}{c} = \dfrac{-3\alpha}{b} = \dfrac{3\beta}{a} = \dfrac{-\gamma}{d}$

$\therefore \quad a = 3\beta c,\ b = -3\alpha c,\ d = -\gamma c$

$\therefore$ (3) $\Rightarrow \quad 3\beta cx - 3\alpha cy + cz - \gamma c = 0$

$\Rightarrow \quad 3\beta x - 3\alpha y + z - \gamma = 0$

(α, β, γ) lies on this plane if

$$3\beta\alpha - 3\alpha\beta + \gamma - \gamma = 0 \quad \text{if } 0 = 0, \text{ which is true.}$$

$\therefore$ The result holds.

Example 10. *Show that there are three points on the curve $x = at^3 + b$, $y = 3ct^2 + 3dt$, $z = 3et + f$ such that their osculating planes pass through the origin and that the three points lie on the plane $3cex + afy = 0$.*

Sol. The given curve is

$$x = at^3 + b,\ y = 3ct^2 + 3dt,\ z = 3et + f.$$

$\therefore \quad \dot{x} = 3at^2,\ \dot{y} = 6ct + 3d,\ \dot{z} = 3e,\ \ddot{x} = 6at,\ \ddot{y} = 6c,\ \ddot{z} = 0$

$\therefore$ Equation of the osculating plane at point 't' is

$$\begin{vmatrix} X - (at^3 + b) & Y - (3ct^2 + 3dt) & Z - (3et + f) \\ 3at^2 & 6ct + 3d & 3e \\ 6at & 6c & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad -18ce(X - at^3 - b) + 18aet(Y - 3ct^2 - 3dt)$
$\qquad + (18act^2 - 36act^2 - 18adt)(Z - 3et - f) = 0$

$\Rightarrow \quad ce(X - at^3 - b) - aet(Y - 3ct^2 - 3dt) + (act^2 + adt)(Z - 3et - f) = 0$

Let this plane pass through the origin.

$\therefore$ $$ce(0 - at^3 - b) - aet(0 - 3ct^2 - 3dt) + (act^2 + adt)(0 - 3et - f) = 0$$

$\Rightarrow$ $$-acet^3 - bce + 3acet^3 + 3adet^2 - 3acet^3 - acft^2 - 3adet^2 - adft = 0$$

$\Rightarrow$ $$acet^3 + acft^2 + adft + bce = 0 \quad ...(1)$$

This is a cubic in t. Let the roots of this equation be t_1, t_2 and t_3.

$\therefore$ There are three points on the given curve whose osculating planes passes through the origin. Let $x_1 = at_1^3 + b$, $y_1 = 3ct_1^2 + 3dt_1$, $z_1 = 3et_1 + f$

$\therefore$ $$t_1^3 = \frac{x_1 - b}{a},\ t_1 = \frac{z_1 - f}{3e}$$

$$t_1^2 = \frac{y_1 - 3dt_1}{3c} = \frac{1}{3c}\left(y_1 - 3d\left(\frac{z_1 - f}{3e}\right)\right) = \frac{ey_1 - dz_1 + df}{3ce}$$

(1) $\Rightarrow$ $$acet_1^3 + acft_1^2 + adft_1 + bce = 0$$

$\Rightarrow$ $$ace\left(\frac{x_1 - b}{a}\right) + acf\left(\frac{ey_1 - dz_1 + df}{3ce}\right) + adf\left(\frac{z_1 - f}{3e}\right) + bce = 0$$

$\Rightarrow$ $$ce(x_1 - b) + \frac{af}{3e}(ey_1 - dz_1 + df) + \frac{adf}{3e}(z_1 - f) + bce = 0$$

$\Rightarrow$ $$3ce^2(x_1 - b) + af(ey_1 - dz_1 + df) + adf(z_1 - f) + 3bce^2 = 0$$

$\Rightarrow$ $$3ce^2x_1 - 3bce^2 + aefy_1 - adf z_1 + adf^2 + adf z_1 - adf^2 + 3bce^2 = 0$$

$\Rightarrow$ $$3ce^2x_1 + aefy_1 = 0$$

$\Rightarrow$ $$3cex_1 + afy_1 = 0$$

$\therefore$ (x_1, y_1, z_1) lies in the plane $3cex + afy = 0$.

Similarly, the points corresponding to t_2 and t_3 also lie on the plane $3cex + afy = 0$.

EXERCISE 1.5

1. Find the intersection of the xy-plane and the tangent lines to the helix $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $(t > 0)$.
2. Find the equation of the osculating plane at any point on the curve $\mathbf{r} = (t, t^2, t^3)$.
3. Find the equation of the osculating plane to the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point for which $t = 1$.
4. Find the equation of the osculating plane to the curve $x = 3t, y = 3t^2, z = 2t^3$ at the points (3, 3, 2), (– 3, 3, – 2) and (6, 12, 16).
5. Find the equation of the osculating plane at the point 't' on the helix $x = a\cos t, y = a \sin t, z = ct$.
6. Show that the osculating plane at the point $t = 1$ of the curve $\mathbf{r} = (3at, 3bt^2, ct^3)$ is $\frac{x}{a} - \frac{y}{b} + \frac{z}{c} = 1$.
7. Find the equation of the osculating plane at the point t of the curve
   $$x = a\cosh t,\ y = a\sinh t,\ z = bt.$$
8. Find the equation of the osculating plane at the point t of the curve
   $$\mathbf{r} = 4a\cos^3 t\mathbf{i} + 4a\sin^3 t\mathbf{j} + 2a\cos 2t\mathbf{k}.$$
9. Find the osculating plane at the point t of the curve $x = a\cos 2t, y = a\sin 2t, z = 2a\sin t$.
10. Find the basic unit vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ of the curve $\mathbf{r} = (t, t^2, t^3)$ at the point $t = 1$. Find also the equations of the characteristic lines and planes at this point.

Answers

1. $(\cos t + t \sin t, \sin t - t \cos t, 0)$ **2.** $3t^2x - 3ty + z = t^3$

3. $3x - 3y + z = 1$

4. $2x - 2y + z = 2,\ 2x + 2y + z = -2,\ 8x - 4y + z = 16$

5. $c(x \sin t - y \cos t - at) + az = 0$ **7.** $bx \sinh t - by \cosh t + az = abt$

8. $2x \cos t - 2y \sin t - 3z = 2a \cos 2t$

9. $(\sin 3t + 3 \sin t)x - (\cos 3t + 3 \cos t)y + 4z = 6a \sin t$

10. $\mathbf{t} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),\ \mathbf{n} = \frac{1}{\sqrt{266}}(-11\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}),\ \mathbf{b} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3},\ \frac{x-1}{11} = \frac{y-1}{8} = \frac{z-1}{-9},\ \frac{x-1}{3} = \frac{y-1}{-3} = \frac{z-1}{1},$$

$x + 2y + 3z = 6,\ 11x + 8y - 9z = 10,\ 3x - 3y + z = 1.$

2

Curvature and Torsion

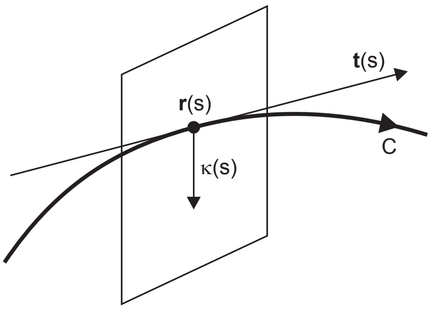
1. INTRODUCTION

For curves in space, the concepts of curvature and torsion are of fundamental importance. We know that line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side etc. In geometry, we look for geometric quantities which distinguish one figure from another. The importance of curvature and torsion can easily be estimated from the fact that it can be proved that a curve is uniquely determined (except for its position in space) if its curvature and torsion are given as continuous functions of arc length 's'.

2. CURVATURE OF A CURVE

Let $\mathbf{r} = \mathbf{r}(s)$ be a regular curve C of class $C^m (m \geq 2)$, where s is the parameter 'arc length'.

The vector $\mathbf{r}''(s)$ is called the **curvature vector** on the curve C at the point $\mathbf{r}(s)$ and it is denoted by $\boldsymbol{\kappa}(s)$ (or by $\boldsymbol{\kappa}$). The magnitude of the curvature vector is called the **curvature** of the curve C at the point $\mathbf{r}(s)$ and it is denoted by $\kappa(s)$ (or by κ).



$\therefore \quad \kappa(s) = |\mathbf{r}''(s)|$

Also $\quad \mathbf{t}(s) = \mathbf{r}'(s)$, so we have

$\boldsymbol{\kappa}(s) = \mathbf{t}'(s).$

We know that $\mathbf{t}(s)$ is a unit vector.

$\Rightarrow \quad \mathbf{t}(s) \cdot \mathbf{t}(s) = 1$

$\Rightarrow \quad \mathbf{t}(s) \cdot \mathbf{t}'(s) + \mathbf{t}'(s) \cdot \mathbf{t}(s) = 0$

$\Rightarrow \quad 2\mathbf{t}'(s) \cdot \mathbf{t}(s) = 0$

$\Rightarrow \quad \boldsymbol{\kappa}(s) \cdot \mathbf{t}(s) = 0$

$\therefore$ The curvature vector $\boldsymbol{\kappa}(s)$ is orthogonal to $\mathbf{t}(s)$ and hence parallel to the normal plane at $\mathbf{r}(s)$. When $\boldsymbol{\kappa}(s)$ is non-zero, it is in the direction in which the curve is turning.

The reciprocal of the curvature at a point is called the **radius of curvature** at that point and it is denoted by ρ.

$$\therefore \quad \rho = \frac{1}{\kappa} \qquad \text{(Assuming } \kappa \neq 0\text{)}$$

A point on the curve C is called a **point of inflexion** if the curvature κ at that point is zero.

Remark. We have $\quad \dfrac{\boldsymbol{\kappa}}{\kappa} = \dfrac{\mathbf{r}''}{|\mathbf{r}''|} = \dfrac{\mathbf{t}'}{|\mathbf{t}'|} = \mathbf{n}.$

$$\therefore \quad \mathbf{n} = \frac{\boldsymbol{\kappa}}{\kappa}.$$

Example 1. *Show that:* $\kappa = |\mathbf{r}' \times \mathbf{r}''|$.

Sol. $|\mathbf{r}' \times \mathbf{r}''| = |\mathbf{t} \times \mathbf{t}'| = |\mathbf{t}| \cdot |\mathbf{t}'| \sin \frac{\pi}{2} = 1 \cdot \kappa \cdot 1 = \kappa$

$\therefore \quad \kappa = |\mathbf{r}' \times \mathbf{r}''|$.

Example 2. *For the helix* $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$, $a > 0$, $b \neq 0$, *find the curvature at the point t.*

Sol. We have $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$.

$\therefore \quad \dot{\mathbf{r}} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k}$

$\Rightarrow \quad |\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$

$\therefore \quad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{a^2+b^2}}(-a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k})$

$\therefore \quad \boldsymbol{\kappa} = \mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt} \cdot \frac{dt}{ds} = \frac{d\mathbf{t}}{dt} \Big/ \frac{ds}{dt}$

$= \frac{1}{\sqrt{a^2+b^2}}(-a \cos t\mathbf{i} - a \sin t\mathbf{j}) \Big/ \frac{ds}{dt}$

$= -\frac{a}{\sqrt{a^2+b^2}}(\cos t\mathbf{i} + \sin t\mathbf{j}) \Big/ \sqrt{a^2+b^2} \qquad \left(\because \frac{ds}{dt} = |\dot{\mathbf{r}}|\right)$

$= -\frac{a}{a^2+b^2}(\cos t\mathbf{i} + \sin t\mathbf{j})$

$\therefore$ Curvature, $\kappa = |\boldsymbol{\kappa}| = \frac{a}{a^2+b^2}\sqrt{(-\cos t)^2 + (-\sin t)^2} = \frac{a}{a^2+b^2}$.

Example 3. *For the curve* $\mathbf{r} = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$, *find the curvature vector and curvature at the point* $t = 1$.

Sol. We have $\mathbf{r} = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$.

$\therefore \quad \dot{\mathbf{r}} = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ and $|\dot{\mathbf{r}}| = \sqrt{1+t^2+t^4}$

$\therefore \quad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{1+t^2+t^4}}(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})$

$\dot{\mathbf{t}} = \frac{1}{\sqrt{1+t^2+t^4}}(\mathbf{j} + 2t\mathbf{k}) + \left(-\frac{2t+4t^3}{2(1+t^2+t^4)^{3/2}}\right)(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})$

$$= \frac{1}{(1+t^2+t^4)^{3/2}}\,[(1+t^2+t^4)(\mathbf{j}+2t\mathbf{k})-(t+2t^3)(\mathbf{i}+t\mathbf{j}+t^2\mathbf{k})]$$

$$= \frac{1}{(1+t^2+t^4)^{3/2}}\,[-(t+2t^3)\mathbf{i}+(1-t^4)\mathbf{j}+(2t+t^3)\mathbf{k}]$$

$$\therefore \qquad \kappa = \mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\,\frac{dt}{ds} = \dot{\mathbf{t}} \Big/ \frac{ds}{dt} = \dot{\mathbf{t}}\,/\,|\,\dot{\mathbf{r}}\,|$$

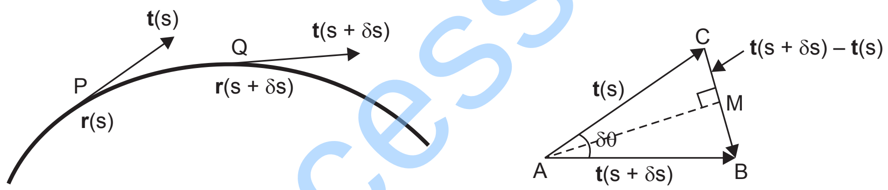
$$= \frac{1}{(1+t^2+t^4)^2}\,[-(t+2t^3)\mathbf{i}+(1-t^4)\mathbf{j}+(2t+t^3)\mathbf{k}]$$

∴ At $t = 1$,

$$\kappa = \frac{1}{(3)^2}[-3\mathbf{i}+0\mathbf{j}+3\mathbf{k}] = -\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{k} \text{ and } \kappa = |\,\kappa\,| = \sqrt{\frac{1}{9}+\frac{1}{9}} = \frac{\sqrt{2}}{3}.$$

Theorem 1. *Prove that the curvature of a regular curve at a point is equal to the rate of change of direction of the tangent with respect to arc length.*

Proof. Let $\mathbf{r} = \mathbf{r}(s)$ be a regular curve of class $C^m(m \geq 2)$, where s is the parameter 'arc length'. Let $\mathbf{r}(s)$ be any point P on the given curve. Let $\mathbf{r}(s+\delta s)$, $\delta s > 0$ be a neighbouring point Q of $\mathbf{r}(s)$. Let $\mathbf{t}(s)$ and $\mathbf{t}(s+\delta s)$ be the unit tangent vectors at the points $\mathbf{r}(s)$ and $\mathbf{r}(s+\delta s)$ respectively.



Let $\delta\theta$ denote the angle between the tangent vectors $\mathbf{t}(s)$ and $\mathbf{t}(s+\delta s)$.

By definition,

$$\kappa = |\,\mathbf{t}'\,| = \left|\lim_{\delta s \to 0}\frac{\mathbf{t}(s+\delta s)-\mathbf{t}(s)}{\delta s}\right| = \lim_{\delta s\to 0}\left|\frac{\mathbf{t}(s+\delta s)-\mathbf{t}(s)}{\delta s}\right|$$

$$\therefore \qquad \kappa = \lim_{\delta s \to 0}\frac{|\,\mathbf{t}(s+\delta s)-\mathbf{t}(s)\,|}{\delta s} \qquad \text{...(1)}$$

Since $\mathbf{t}'$ is a unit vector, we have AC = AB = 1.

In ΔABC, $\quad CB = |\,\mathbf{t}(s+\delta s)-\mathbf{t}(s)\,|$

Also $\quad CB = 2\,CM = 2\sin\angle CAM = 2\sin\frac{\delta\theta}{2}$

$$= 2\left[\frac{\delta\theta}{2}-\frac{(\delta\theta/2)^3}{3!}+\ldots\ldots\right]$$

(By using Taylor's expansion for the sine function.)

$$= \delta\theta - \frac{(\delta\theta)^3}{24} + \ldots\ldots$$

$$= \delta\theta\left(1 - \frac{(\delta\theta)^2}{24} + \ldots\ldots\right)$$

$\therefore$ (1) $\Rightarrow$
$$\kappa = \lim_{\delta s \to 0} \frac{\delta\theta\left(1 - \frac{(\delta\theta)^2}{24} + \ldots\ldots\right)}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} \cdot \lim_{\delta\theta \to 0}\left(1 - \frac{(\delta\theta)^2}{24} + \ldots\ldots\right) \qquad (\because \quad \delta\theta \to 0 \text{ as } \delta s \to 0)$$

$$= \frac{d\theta}{ds} \cdot 1 = \frac{d\theta}{ds} = \text{rate of change of } \theta \text{ w.r.t. } s$$

$\therefore$ The curvature at a point is equal to the rate of change of the tangent with respect to the arc length.

Theorem 2. *Prove that a regular curve of class $C^m (m \geq 2)$ is a straight line if and only if its curvature is identically zero.*

Proof. Let $\mathbf{r} = \mathbf{r}(s)$ be a regular curve of class $C^m (m \geq 2)$, where s is the parameter 'arc length'.

Let the curve be a straight line.

Let the curve passes through the point whose position vector is $\mathbf{a}$ and is parallel to vector $\mathbf{b}$.

$\therefore$ $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is a parameter.

$\Rightarrow$ $\frac{d\mathbf{r}}{dt} = \mathbf{b}$ and $\left|\frac{d\mathbf{r}}{dt}\right| = |\mathbf{b}|$

$\therefore$ $\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\mathbf{b}}{|\mathbf{b}|}$

$\therefore$ $\boldsymbol{\kappa} = \mathbf{t}' = \frac{d\mathbf{t}}{dt} \cdot \frac{dt}{ds} = \mathbf{0} \cdot \frac{dt}{ds} = \mathbf{0}$

$\therefore$ $\kappa = |\mathbf{0}| = 0$ *i.e.*, the curvature is identically zero.

Conversely, let the curvature of the curve be identically zero *i.e.*, $\kappa = 0$.

$\Rightarrow$ $\boldsymbol{\kappa} = \mathbf{0}$ $\Rightarrow$ $\mathbf{t}' = \mathbf{0}$ $\Rightarrow$ $\mathbf{t} = \mathbf{c}$, a constant vector.

$\Rightarrow$ $\mathbf{r}' = \mathbf{c}$ $\Rightarrow$ $\mathbf{r} = \mathbf{c}s + \mathbf{d}$, where $\mathbf{d}$ is a constant vector.

$\therefore$ The curve is a straight line passing through the point whose position vector is $\mathbf{d}$ and is parallel to the vector $\mathbf{c}$.

$\therefore$ The result holds.

Theorem 3. *Let* $\mathbf{r} = \mathbf{r}(t)$ *be a regular curve of class* C^m $(m \geq 2)$, *where t is an arbitrary parameter. Prove that*

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}.$$

Proof. We have $\quad \dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt} = \mathbf{r}'\dot{s} = \dot{s}\mathbf{r}'$

and $\quad \ddot{\mathbf{r}} = \dfrac{d\dot{\mathbf{r}}}{dt} = \dfrac{d}{dt}(\dot{s}\mathbf{r}') = \dot{s}\dfrac{d\mathbf{r}'}{dt} + \ddot{s}\mathbf{r}' = \dot{s}(\mathbf{r}''\dot{s}) + \ddot{s}\mathbf{r}' = \ddot{s}\mathbf{r}' + \dot{s}^2\mathbf{r}''$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \dot{s}\mathbf{r}' \times (\ddot{s}\mathbf{r}' + \dot{s}^2\mathbf{r}'') = (\dot{s}\,\ddot{s})(\mathbf{r}' \times \mathbf{r}') + (\dot{s}^3)(\mathbf{r}' \times \mathbf{r}'')$$

$$= (\dot{s}\,\ddot{s})\,\mathbf{0} + (\dot{s}^3)(\mathbf{r}' \times \mathbf{r}'') = \dot{s}^3(\mathbf{r}' \times \mathbf{r}'')$$

$$= |\dot{\mathbf{r}}|^3(\mathbf{r}' \times \mathbf{r}'') \qquad (\because \dot{s} = |\dot{\mathbf{r}}|)$$

$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = |\dot{\mathbf{r}}|^3 |\mathbf{r}'||\mathbf{r}''| \sin\theta$, where θ is the angle between $\mathbf{r}'$ and $\mathbf{r}''$.

Now $\mathbf{r}' = \mathbf{t}$, $\mathbf{r}'' = \mathbf{t}'$ and $\mathbf{t}$ and $\mathbf{t}'$ are orthogonal.

$\therefore \quad \theta = \pi/2$

Also $\quad |\mathbf{r}'| = |\mathbf{t}| = 1$ and $|\mathbf{r}''| = \kappa$

$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = |\dot{\mathbf{r}}|^3 .1.\kappa.1$

$$\therefore \quad \kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}.$$

Corollary. If $\quad \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = \sqrt{\Sigma(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2}$$

and $$|\dot{\mathbf{r}}| = \sqrt{\Sigma\dot{x}^2}.$$

$$\therefore \quad \kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{\Sigma(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2}}{(\Sigma\dot{x}^2)^{3/2}}.$$

Remarks 1. If $\mathbf{r} = \mathbf{r}(s)$, then

$$\kappa = |\mathbf{r}''| = |\mathbf{r}' \times \mathbf{r}''|.$$

2. If $\mathbf{r} = \mathbf{r}(t)$, then $\quad \kappa = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}.$

Example 4. *For the circle* $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $a > 0$, *find the radius of curvature at point t.*

Sol. We have $\quad \mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $a > 0$.

$\therefore \quad \dot{\mathbf{r}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$

$\Rightarrow \quad |\dot{\mathbf{r}}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$$\therefore \quad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{1}{a}(-a\sin t\mathbf{i} + a\cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$\therefore \qquad \boldsymbol{\kappa} = \mathbf{t}' = \dfrac{d\mathbf{t}}{ds} = \dfrac{d\mathbf{t}}{dt} \cdot \dfrac{dt}{ds}$

$= (-\cos t\mathbf{i} - \sin t\mathbf{j}) \Big/ \dfrac{ds}{dt}$

$= -(\cos t\mathbf{i} + \sin t\mathbf{j}) / |\dot{\mathbf{r}}|$

$= -\dfrac{1}{a}(\cos t\mathbf{i} + \sin t\mathbf{j})$

$\therefore$ Curvature, $\qquad \kappa = |\boldsymbol{\kappa}| = \dfrac{1}{a}\sqrt{(-\cos t)^2 + (-\sin t)^2} = \dfrac{1}{a}$

$\therefore$ Radius of curvature $= \dfrac{1}{\kappa} = \dfrac{1}{1/a} = a$

$\therefore$ Radius of curvature is equal to the radius of the given circle.

Alternative method

We have $\qquad \mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j},\ a > 0.$

$\therefore \qquad \dot{\mathbf{r}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$

and $\qquad \ddot{\mathbf{r}} = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$\therefore \qquad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (a^2\sin^2 t + a^2\cos^2 t)\mathbf{k} = a^2\mathbf{k}$

$\therefore \qquad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = a^2$

Also $\qquad |\dot{\mathbf{r}}|^2 = a^2\sin^2 t + a^2\cos^2 t = a^2 \quad \therefore \quad |\dot{\mathbf{r}}| = a$

$\therefore$ Radius of curvature $= \dfrac{1}{\kappa} = \dfrac{|\dot{\mathbf{r}}|^3}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|} = \dfrac{a^3}{a^2} = a.$

Example 5. *Show that along the plane curve* $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$,

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Sol. We have $\qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}. \qquad$ ...(1)

Also, $\qquad \kappa = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \qquad$ ...(2)

(1) $\Rightarrow \qquad \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and $\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

$\therefore \qquad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix} = (\dot{x}\ddot{y} - \ddot{x}\dot{y})\mathbf{k}$

$\therefore \qquad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = |\dot{x}\ddot{y} - \ddot{x}\dot{y}|$ and $|\dot{\mathbf{r}}| = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$

$\therefore$ (2) $\Rightarrow \qquad \kappa = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[(\dot{x}^2 + \dot{y}^2)^{1/2}]^3} = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$

Example 6. *For the curve*

$$\mathbf{r} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + t\mathbf{k},$$

find the curvature at point t.

Sol. We have $\mathbf{r} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + t\mathbf{k}.$

$\therefore$ $\dot{\mathbf{r}} = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$

and $\ddot{\mathbf{r}} = \sin t\mathbf{i} + \cos t\mathbf{j} + 0\mathbf{k}$

Now
$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \qquad \text{...(1)}$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - \cos t & \sin t & 1 \\ \sin t & \cos t & 0 \end{vmatrix} = -\cos t\mathbf{i} + \sin t\mathbf{j} + (\cos t - 1)\mathbf{k}$$

$\therefore$
$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = \sqrt{\cos^2 t + \sin^2 + (\cos t - 1)^2}$$
$$= \left(1 + \left(-2\sin^2 \frac{t}{2}\right)^2\right)^{1/2} = \left(1 + 4\sin^4 \frac{t}{2}\right)^{1/2}$$
$$|\dot{\mathbf{r}}| = \sqrt{((1 - \cos t)^2 + \sin^2 t + 1)}$$
$$= (1 + \cos^2 t - 2\cos t + \sin^2 t + 1)^{1/2}$$
$$= (1 + 2(1 - \cos t))^{1/2} = \left(1 + 4\sin^2 \frac{t}{2}\right)^{1/2}$$

$\therefore$ (1) $\Rightarrow$
$$\kappa = \frac{\left(1 + 4\sin^4 \frac{t}{2}\right)^{1/2}}{\left(\left(1 + 4\sin^2 \frac{t}{2}\right)^{1/2}\right)^3} = \frac{\left(1 + 4\sin^4 \frac{t}{2}\right)^{1/2}}{\left(1 + 4\sin^2 \frac{t}{2}\right)^{3/2}}.$$

Example 7. *For the curve $x = 4a\cos^3 t$, $y = 4a\sin^3 t$, $z = 3c\cos 2t$, show that*

$$\kappa = \frac{a}{6(a^2 + c^2)\sin 2t}.$$

Sol. Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$\therefore$ $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 4a\cos^3 t\mathbf{i} + 4a\sin^3 t\mathbf{j} + 3c\cos 2t\mathbf{k}$

$\therefore$
$$\mathbf{t} = \mathbf{r}' = \frac{d\mathbf{r}}{dt}\frac{dt}{ds}$$
$$= (-12a\cos^2 t\sin t\mathbf{i} + 12a\sin^2 t\cos t\mathbf{j} - 6c\sin 2t\mathbf{k})\frac{dt}{ds}$$
$$= (-6a\cos t\sin 2t\mathbf{i} + 6a\sin t\sin 2t\mathbf{j} - 6c\sin 2t\mathbf{k})\frac{dt}{ds}$$

$\therefore$
$$\mathbf{t} = 6\sin 2t(-a\cos t\mathbf{i} + a\sin t\mathbf{j} - c\mathbf{k})\frac{dt}{ds} \qquad \text{...(1)}$$

$$\Rightarrow \qquad |\,\mathbf{t}\,| = 6 \sin 2t \,.\, \sqrt{a^2 \cos^2 t + a^2 \sin^2 t + c^2} \left| \frac{dt}{ds} \right|$$

$$\Rightarrow \qquad 1 = 6 \sin 2t \sqrt{a^2 + c^2} \left| \frac{dt}{ds} \right| \qquad (\because \ |\,\mathbf{t}\,| = 1)$$

$$\Rightarrow \qquad \left| \frac{ds}{dt} \right| = 6\sqrt{a^2 + c^2} \sin 2t$$

$$\Rightarrow \qquad \frac{ds}{dt} = 6\sqrt{a^2 + c^2} \sin 2t \qquad \text{(Assuming } \sin 2t > 0)$$

$$\therefore \ (1) \Rightarrow \qquad \mathbf{t} = 6 \sin 2t\,(-a \cos t\mathbf{i} + a \sin t\mathbf{j} - c\mathbf{k}) \,.\, \frac{1}{6\sqrt{a^2 + c^2} \sin 2t}$$

$$= \frac{1}{\sqrt{a^2 + c^2}} (-a \cos t\mathbf{i} + a \sin t\mathbf{j} - c\mathbf{k})$$

Now
$$\mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt} \frac{dt}{ds}$$

$$= \frac{1}{\sqrt{a^2 + c^2}} (a \sin t\mathbf{i} + a \cos t\mathbf{j} - 0\mathbf{k}) \frac{1}{6\sqrt{a^2 + c^2} \sin 2t}$$

$$= \frac{a}{6(a^2 + c^2) \sin 2t} (\sin t\mathbf{i} + \cos t\mathbf{j})$$

$$\therefore \qquad \kappa = |\,\mathbf{t}'\,| = \frac{a}{6(a^2 + c^2) \sin 2t} \sqrt{\sin^2 t + \cos^2 t} = \frac{a}{6(a^2 + c^2) \sin 2t}.$$

Example 8. *Find the radius of curvature at any point of the curve*

$$x^2 + y^2 = a^2, \ x^2 - y^2 = az.$$

Sol. The given curve is

$$x^2 + y^2 = a^2 \qquad \ldots(1) \qquad\qquad x^2 - y^2 = az \qquad \ldots(2)$$

Let $\quad x = a \cos t, \ y = a \sin t$

$\therefore \quad$ (1) is satisfied.

(2) $\Rightarrow \quad a^2 \cos^2 t - a^2 \sin^2 t = az$

$$\Rightarrow \qquad z = a \cos 2t$$

$\therefore \quad$ The given curve is

$$x = a \cos t, \ y = a \sin t, \ z = a \cos 2t.$$

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$$\therefore \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + a \cos 2t\,\mathbf{k}$$

$$\therefore \qquad \dot{\mathbf{r}} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} - 2a \sin 2t\mathbf{k}$$

and
$$\ddot{\mathbf{r}} = -a \cos t\mathbf{i} - a \sin t\mathbf{j} - 4a \cos 2t\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & -2a\sin 2t \\ -a\cos t & -a\sin t & -4a\cos 2t \end{vmatrix}$$

$$= a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & -\cos t & 2\sin 2t \\ \cos t & \sin t & 4\cos 2t \end{vmatrix}$$

$$= a^2 [(-4\cos t\cos 2t - 2\sin t\sin 2t)\mathbf{i} - (4\sin t\cos 2t - 2\cos t\sin 2t)\mathbf{j} + (\sin^2 t + \cos^2 t)\mathbf{k}]$$

$$= a^2 [(-2\cos t\cos 2t - 2\cos(t - 2t))\mathbf{i} + (-2\sin t\cos 2t + 2\sin(2t - t))\mathbf{j} + \mathbf{k}]$$

$$= a^2[-4\cos^3 t\mathbf{i} + 4\sin^3 t\mathbf{j} + \mathbf{k}]$$

$$\therefore \quad |\dot{\mathbf{r}}|^2 = a^2\sin^2 t + a^2\cos^2 t + 4a^2\sin^2 2t$$

$$= a^2(1 + 4\sin^2 2t) = a^2(5 - 4\cos^2 2t)$$

$$= a^2\left(5 - \frac{4z^2}{a^2}\right) = 5a^2 - 4z^2$$

$$\therefore \quad |\dot{\mathbf{r}}| = \sqrt{5a^2 - 4z^2}$$

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = a^4(16\cos^6 t + 16\sin^6 t + 1)$$

$$= a^4[16\{(\cos^2 t + \sin^2 t)^3 - 3\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)\} + 1]$$

$$= a^4[16\{1^3 - 3\cos^2 t\sin^2 t \,.\, 1\} + 1] = a^4[17 - 48\cos^2 t\sin^2 t]$$

$$= a^4[17 - 12\sin^2 2t] = a^4[5 + 12\cos^2 2t]$$

$$= a^4\left[5 + \frac{12z^2}{a^2}\right] = a^2(5a^2 + 12z^2)$$

$$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = a\sqrt{5a^2 + 12z^2}$$

$$\therefore \quad \rho = \frac{1}{\kappa} = \frac{|\dot{\mathbf{r}}|^3}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|} = \frac{(5a^2 - 4z^2)^{3/2}}{a\sqrt{5a^2 + 12z^2}}.$$

Example 9. *Find the equation of the osculating plane and curvature at point t of the curve $x = a\cos 2t$, $y = a\sin 2t$, $z = 2a\sin t$.*

Sol. Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\cos 2t\mathbf{i} + a\sin 2t\mathbf{j} + 2a\sin t\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} = -2a\sin 2t\mathbf{i} + 2a\cos 2t\mathbf{j} + 2a\cos t\mathbf{k}$$

and

$$\ddot{\mathbf{r}} = -4a\cos 2t\mathbf{i} - 4a\sin 2t\mathbf{j} - 2a\sin t\,\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2a\sin 2t & 2a\cos 2t & 2a\cos t \\ -4a\cos 2t & -4a\sin 2t & -2a\sin t \end{vmatrix}$$

$$= -4a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin 2t & \cos 2t & \cos t \\ 2\cos 2t & 2\sin 2t & \sin t \end{vmatrix}$$

$$= -4a^2[(\cos 2t \sin t - 2 \sin 2t \cos t)\mathbf{i} - (-\sin 2t \sin t - 2 \cos 2t \cos t)\mathbf{j} + (-2 \sin^2 2t - 2 \cos^2 2t)\mathbf{k}]$$
$$= -4a^2[(\sin (t - 2t) - \sin 2t \cos t)\mathbf{i} + (\cos (2t - t) + \cos 2t \cos t)\mathbf{j} - 2\mathbf{k}]$$
$$= 4a^2[(\sin t + \sin 2t \cos t)\mathbf{i} - (\cos t + \cos 2t \cos t)\mathbf{j} + 2\mathbf{k}]$$

Equation of the osculating plane is $[\mathbf{R} - \mathbf{r}\ \dot{\mathbf{r}}\ \ddot{\mathbf{r}}] = 0$.

$$\Rightarrow \quad (\mathbf{R} - \mathbf{r}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) = 0 \quad \ldots(1)$$

Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\therefore$ (1) $\Rightarrow$ $[(x - a \cos 2t)\mathbf{i} + (y - a \sin 2t)\mathbf{j} + (z - 2a \sin t)\mathbf{k}] \cdot 4a^2[(\sin t + \sin 2t \cos t)\mathbf{i} - (\cos t + \cos 2t \cos t)\mathbf{j} + 2\mathbf{k}] = 0$

$\Rightarrow$ $(x - a \cos 2t)(\sin t + \sin 2t \cos t) - (y - a \sin 2t)(\cos t + \cos 2t \cos t) + (z - 2a \sin t)2 = 0$

$\Rightarrow$ $(\sin t + \sin 2t \cos t)x - (\cos t \cdot 2 \cos^2 t)y + 2z$

$= a \cos 2t \sin t + a \cos 2t \sin 2t \cos t - a \sin 2t \cos t - a \sin 2t \cos 2t \cos t + 4a \sin t$

$= a \sin (t - 2t) + 4a \sin t$

$= 3a \sin t$

$\therefore$ $(\sin t + \sin 2t \cos t)x - 2 \cos^3 t\, y + 2z = 3a \sin t$

This is the equation of the osculating plane.

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = 16a^4[(\sin t + \sin 2t \cos t)^2 + (\cos t + \cos 2t \cos t)^2 + 4]$$
$$= 16a^4[\sin^2 t + \sin^2 2t \cos^2 t + 2 \sin t \sin 2t \cos t + \cos^2 t + \cos^2 2t \cos^2 t + 2 \cos t \cos 2t \cos t + 4]$$
$$= 16a^4[1 + (\sin^2 2t + \cos^2 2t) \cos^2 t + 2 \cos t (\sin t \sin 2t + \cos t \cos 2t) + 4]$$
$$= 16a^4[1 + \cos^2 t + 2 \cos t \cos (t - 2t) + 4]$$
$$= 16a^4[5 + 3 \cos^2 t]$$

$$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = 4a^2\sqrt{5 + 3\cos^2 t}$$

Also, $|\dot{\mathbf{r}}|^2 = 4a^2 \sin^2 2t + 4a^2 \cos^2 2t + 4a^2 \cos^2 t$
$= 4a^2 + 4a^2 \cos^2 t = 4a^2(1 + \cos^2 t)$

$$\therefore \quad |\dot{\mathbf{r}}|^3 = 8a^3(1 + \cos^2 t)^{3/2}$$

$$\therefore \quad \kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{4a^2\sqrt{5 + 3\cos^2 t}}{8a^3(1 + \cos^2 t)^{3/2}} = \frac{\sqrt{5 + 3\cos^2 t}}{2a\,(1 + \cos^2 t)^{3/2}}.$$

Example 10. *Show that a curve* $\mathbf{r} = \mathbf{r}(s)$ *of class* $C^m (m \geq 2)$ *is a straight line if all tangent lines are concurrent.*

Sol. The equation of the tangent line at the point $\mathbf{r}(s)$ is

$$\mathbf{R}(s) = \mathbf{r}(s) + \lambda(s)\mathbf{t}(s),$$

where $\mathbf{R}(s)$ is a general point on the tangent line and $\lambda(s)$ is a parameter.

Let all tangent lines intersect at the point $\mathbf{r}_0(s)$.

$$\therefore \quad \mathbf{r}_0(s) = \mathbf{r}(s) + \lambda_0(s)\mathbf{t}(s) \quad \text{for some value } \lambda_0(s) \text{ of } \lambda(s)$$

Differentiating w.r.t. s, we get

$$0 = \mathbf{r}'(s) + \lambda_0(s)\mathbf{t}'(s) + \lambda_0'(s)\mathbf{t}(s)$$
$$\Rightarrow \quad 0 = \mathbf{t}(s) + \lambda_0(s)\mathbf{t}'(s) + \lambda_0'(s)\mathbf{t}(s)$$
$$\Rightarrow \quad 0 = (1 + \lambda_0'(s))\mathbf{t}(s) + \lambda_0(s)\mathbf{t}'(s)$$

Multiplying by $\mathbf{t}'(s)$, we get

$\Rightarrow \quad 0 = (1 + \lambda_0'(s))\,(\mathbf{t}(s) \cdot \mathbf{t}'(s)) + \lambda_0(s)(\mathbf{t}'(s) \,.\, \mathbf{t}'(s))$

$\Rightarrow \quad 0 = (1 + \lambda_0'(s)) \cdot 0 + \lambda_0(s)\,|\mathbf{t}'(s)|^2$

$\Rightarrow \quad \lambda_0(s)\,|\mathbf{t}'(s)|^2 = 0 \quad \Rightarrow \quad |\mathbf{t}'(s)| = 0 \quad \Rightarrow \quad \mathbf{t}'(s) = 0$ (Assuming $\lambda_0(s) \neq 0$)

$\Rightarrow \quad \mathbf{t}(s) = \mathbf{c}$, a constant vector

$\Rightarrow \quad \mathbf{r}'(s) = \mathbf{c} \quad \Rightarrow \quad \mathbf{r}(s) = s\mathbf{c} + \mathbf{d}$, where $\mathbf{d}$ is a constant vector

$\therefore$ The curve $\mathbf{r} = \mathbf{r}(s)$ is a straight line passing through the point whose position vector is $\mathbf{d}$ and is parallel to the vector $\mathbf{c}$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (*i*) $\boldsymbol{\kappa} = \mathbf{r}'' = \mathbf{t}'$ (*ii*) $\kappa = |\mathbf{r}''| = |\mathbf{t}'|$

Rule II. *Radius of curvature*, $\rho = \dfrac{1}{\kappa}$

Rule III. (*i*) $\kappa = |\mathbf{r}' \times \mathbf{r}''|$ (*ii*) $\kappa = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$.

Rule IV. *Curve is a straight line if and only if its curvature is identically zero.*

EXERCISE 2.1

1. For the curve $\mathbf{r} = a \cos t\mathbf{i} + b \sin t\mathbf{j}$, $a, b > 0$, find the curvature at point t.
2. For the curve $\mathbf{r} = \cosh t\mathbf{i} + \sinh t\mathbf{j}$, find the curvature at point t'.
3. For the curve $\mathbf{r} = t\mathbf{i} + t^{3/2}\mathbf{j}$, $t > 0$, find the curvature at point t.
4. Show that a curve $\mathbf{r} = \mathbf{r}(t)$ of class C^m $(m \geq 2)$, where t is an arbitrary parameter, is a straight line if $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ are linearly dependent for all t.
5. For the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, find the curvature at the point (0, 0, 0).
6. Show that the curvature of a circle of radius a is equal to $1/a$.
7. Let $\mathbf{r} = \mathbf{r}(t)$ be a regular curve of class $C^m (m \geq 2)$, where t is an arbitrary parameter. Show that:
$$\kappa = \frac{\sqrt{(\dot{\mathbf{r}} \,.\, \dot{\mathbf{r}})(\ddot{\mathbf{r}} \,.\, \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \,.\, \ddot{\mathbf{r}})^2}}{(\dot{\mathbf{r}} \,.\, \dot{\mathbf{r}})^{3/2}}.$$
8. Show that for a curve $y = y(x)$ in the xy-plane:
$$\kappa(x) = \frac{|y''|}{(1 + y'^2)^{3/2}}.$$
9. For the following curves in the xy-plane, find curvature: (*i*) $y = x^2$ (*ii*) $xy = \lambda$.
10. For the curve $x = a(3t - t^3)$, $y = 3at^2$, $z = a(3t + t^3)$, show that:
$$\kappa = \frac{1}{3a(1 + t^2)^2}.$$
11. For the curve $x = t$, $y = t^2$, $z = t^3$, show that:
$$\kappa^2 = \frac{4(9t^4 + 9t^2 + 1)}{(9t^4 + 4t^2 + 1)^3}.$$

12. For the helix $x = a \cos t$, $y = a \sin t$, $z = at \cot \alpha$, show that: $\kappa = \dfrac{1}{a} \sin^2 \alpha$.

13. Find the curvature of the curve given by $\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j} + bt\mathbf{k}$.

14. For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, show that: $\rho = \dfrac{3}{2}(1 + 2t^2)^2$.

Answers

1. $\dfrac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$ **2.** $\dfrac{1}{\cosh^2 2t}$ **3.** $\dfrac{6}{\sqrt{t}\,(4 + 9t)^{3/2}}$

5. 2 **9.** (*i*) $\dfrac{2}{(1 + 4x^2)^{3/2}}$ (*ii*) $\dfrac{2\lambda x^3}{(x^4 + \lambda^2)^{3/2}}$

13. $\dfrac{a\left(b^2 + 4a^2 \sin^4 \dfrac{t}{2}\right)^{1/2}}{\left(b^2 + 4a^2 \sin^2 \dfrac{t}{2}\right)^{3/2}}$.

Hints

4. Let $\dot{\mathbf{r}}(t) = \lambda \ddot{\mathbf{r}}(t)$.

7. $(\dot{\mathbf{r}} . \dot{\mathbf{r}})(\ddot{\mathbf{r}} . \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} . \ddot{\mathbf{r}})^2 = |\dot{\mathbf{r}}|^2 |\ddot{\mathbf{r}}|^2 - (|\dot{\mathbf{r}}||\ddot{\mathbf{r}}|\cos\theta)^2 = |\dot{\mathbf{r}}|^2 |\ddot{\mathbf{r}}|^2 (1 - \cos^2\theta) = |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2$.

3. TORSION OF A CURVE

Let $\mathbf{r} = \mathbf{r}(s)$ be a regular curve C of class $C^m (m \geq 3)$ and $\mathbf{r}''(s) \neq 0$, where s is the parameter 'arc length'.

$$\mathbf{r}''(s) \neq \mathbf{0} \quad \Rightarrow \mathbf{n}(s) = \frac{\mathbf{t}'(s)}{|\mathbf{t}'(s)|} = \frac{\mathbf{t}'(s)}{|\mathbf{r}''(s)|} \quad \Rightarrow \quad \mathbf{n}(s) \text{ is defined.}$$

Also, $$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s) = \mathbf{r}'(s) \times \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|} = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{|\mathbf{r}''(s)|} \quad \Rightarrow \quad \mathbf{b}(s) \text{ is defined.}$$

∴ $\mathbf{n}(s)$ and $\mathbf{b}(s)$ exist at the point $\mathbf{r}(s)$.

Since $\mathbf{r}'''(s)$ exists, the binormal $\mathbf{b}(s)$ is differentiable w.r.t. s.

∴ $\mathbf{b}'(s)$ exists.

The scalar quantity $-\mathbf{n}(s) . \mathbf{b}'(s)$ is called the **torsion** of the curve C at the point $\mathbf{r}(s)$ and it is denoted by $\tau(s)$ (or by τ).

The reciprocal of the torsion is called the **radius of torsion** at that point and it is denoted by σ.

$$\therefore \qquad \sigma = \frac{1}{\tau}. \qquad \text{(Assuming } \tau \neq 0)$$

Remark. It can be proved that a curve is uniquely determined (except for its position in space) if we are given its curvature $\kappa(\neq 0)$ and torsion τ as continuous functions of arc length s. This result shows the importance of curvature and torsion in the study of differential geometry of space curves.

Example 1. *For the helix* $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$, $a > 0$, $b \neq 0$, *find the torsion at the point t.*

Sol. We have $\mathbf{r} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$

$\therefore$ $\dot{\mathbf{r}} = -a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k}$

$$\Rightarrow \quad |\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\therefore \quad \mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{a^2 + b^2}} (-a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k})$$

$$\mathbf{t}' = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \frac{d\mathbf{t}}{dt}\bigg/\frac{ds}{dt}$$

$$= \frac{1}{\sqrt{a^2 + b^2}}(-a \cos t\mathbf{i} - a \sin t\mathbf{j}) \;\; /|\dot{\mathbf{r}}|$$

$$= -\frac{a}{\sqrt{a^2 + b^2}} (\cos t\mathbf{i} + \sin t\mathbf{j}) \bigg/ \sqrt{a^2 + b^2}$$

$$= -\frac{a}{a^2 + b^2}(\cos t\mathbf{i} + \sin t\mathbf{j})$$

$$\therefore \quad |\mathbf{t}'| = \frac{a}{a^2 + b^2}(\cos^2 t + \sin^2 t)^{1/2} = \frac{a}{a^2 + b^2}$$

$$\therefore \quad \mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|} = -\frac{a}{a^2 + b^2}(\cos t\mathbf{i} + \sin t\mathbf{j}) \cdot \frac{a^2 + b^2}{a} = -(\cos t\mathbf{i} + \sin t\mathbf{j})$$

$$\therefore \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{\sqrt{a^2+b^2}}\sin t & \frac{a}{\sqrt{a^2+b^2}}\cos t & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= -\frac{1}{\sqrt{a^2 + b^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ \cos t & \sin t & 0 \end{vmatrix}$$

$$= -\frac{1}{\sqrt{a^2 + b^2}} [-b \sin t\mathbf{i} + b \cos t\mathbf{j} - a\mathbf{k}]$$

$$\therefore \quad \mathbf{b}' = \frac{d\mathbf{b}}{ds} = \frac{d\mathbf{b}}{dt}\frac{dt}{ds} = \frac{d\mathbf{b}}{dt}\bigg/\frac{ds}{dt}$$

$$= -\frac{1}{\sqrt{a^2 + b^2}}(-b \cos t\mathbf{i} - b \sin t\mathbf{j} - 0\mathbf{k}) \;\; /|\dot{\mathbf{r}}|$$

$$= \frac{1}{\sqrt{a^2+b^2}}(b \cos t\mathbf{i} + b \sin t\mathbf{j}) \Big/ \sqrt{a^2+b^2}$$

$$= \frac{b}{a^2+b^2}(\cos t\mathbf{i} + \sin t\mathbf{j})$$

$\therefore$ Torsion, $\tau = -\mathbf{n} \cdot \mathbf{b}'$

$$= (\cos t\mathbf{i} + \sin t\mathbf{j}) \,.\, \frac{b}{a^2+b^2}(\cos t\mathbf{i} + \sin t\mathbf{j})$$

$$= \frac{b}{a^2+b^2} \,.\, (\cos^2 t + \sin^2 t) = \frac{b}{a^2+b^2}.$$

Note. Torsion at each point of a helix is always constant.

Theorem 1. *(Serret-Frenet formulae). Let r = r(s) be a regular curve of class $C^m (m \geq 3)$, where s is the parameter 'arc length' and $r''(s) \neq 0$. Prove that*

(*i*) $t' = \kappa n$ (*ii*) $n' = -\kappa t + \tau b$ (*iii*) $b' = -\tau n$.

Proof. (*i*) $\kappa\mathbf{n} = |\mathbf{r}''|\mathbf{n} = |\mathbf{t}'| \left(\frac{\mathbf{t}'}{|\mathbf{t}'|}\right) = \mathbf{t}'$

$\therefore$ $\mathbf{t}' = \kappa\mathbf{n}$.

(*iii*) $(\mathbf{b} \cdot \mathbf{t})' = \mathbf{b} \cdot \mathbf{t}' + \mathbf{b}' \cdot \mathbf{t} = \mathbf{b} \cdot (\kappa\mathbf{n}) + \mathbf{b}' \cdot \mathbf{t}$

$= \kappa(\mathbf{b} \cdot \mathbf{n}) + \mathbf{b}' \cdot \mathbf{t} = 0 + \mathbf{b}' \cdot \mathbf{t} = \mathbf{b}' \cdot \mathbf{t}$

Also, $(\mathbf{b} \,.\, \mathbf{t})' = 0' = 0$

$\Rightarrow$ $\mathbf{b}' \,.\, \mathbf{t} = 0$ $\Rightarrow$ $\mathbf{b}'$ is perpendicular to $\mathbf{t}$.

Also, $|\,\mathbf{b}\,| = 1$ $\Rightarrow$ $\mathbf{b} \cdot \mathbf{b} = 1$ $\Rightarrow$ $\mathbf{b} \cdot \mathbf{b}' + \mathbf{b}' \cdot \mathbf{b} = 0$ $\Rightarrow$ $\mathbf{b}' \cdot \mathbf{b} = 0$

$\Rightarrow$ $\mathbf{b}'$ is perpendicular to $\mathbf{b}$.

$\therefore$ $\mathbf{b}'$ is perpendicular to the plane determined by $\mathbf{t}$ and $\mathbf{b}$.

$\therefore$ $\mathbf{b}'$ is parallel to $\mathbf{n}$.

Let $\mathbf{b}' = \lambda\mathbf{n}$.

$\Rightarrow$ $\mathbf{n} \,.\, \mathbf{b}' = \mathbf{n} \cdot (\lambda\mathbf{n}) = \lambda(\mathbf{n} \cdot \mathbf{n}) = \lambda \cdot 1 = \lambda$

$\Rightarrow$ $-\tau = \lambda$ $(\because \quad \tau = -\mathbf{n} \,.\, \mathbf{b}')$

$\therefore$ $\mathbf{b}' = -\tau\mathbf{n}$.

(*ii*) We have $\mathbf{n} = \mathbf{b} \times \mathbf{t}$.

$\therefore$ $\mathbf{n}' = (\mathbf{b} \times \mathbf{t})' = \mathbf{b} \times \mathbf{t}' + \mathbf{b}' \times \mathbf{t} = \mathbf{b} \times (\kappa\mathbf{n}) + (-\tau\mathbf{n}) \times \mathbf{t}$

(Using (*i*) and (*iii*))

$= \kappa(\mathbf{b} \times \mathbf{n}) - \tau(\mathbf{n} \times \mathbf{t}) = \kappa(-\mathbf{t}) - \tau(-\mathbf{b}) = -\kappa\mathbf{t} + \tau\mathbf{b}$

$\therefore$ $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$.

Remark 1. Serret-Frenet equations shows that we can express the vectors $\mathbf{t}'$, $\mathbf{n}'$, $\mathbf{b}'$ as linear combinations of the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$.

Remark 2. We have proved equation (*iii*) before proving equation (*ii*) because the result of (*iii*) is used in proving (*ii*).

Remark 3. The Serret-Frenet equations can also be written as

$$\mathbf{t}' = 0\mathbf{t} + \kappa\mathbf{n} + 0\mathbf{b}$$
$$\mathbf{n}' = -\kappa\mathbf{t} + 0\mathbf{n} + \tau\mathbf{b}$$
$$\mathbf{b}' = 0\mathbf{t} - \tau\mathbf{n} + 0\mathbf{b}.$$

In the above equations, the coefficients of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ form the matrix $\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$.

Remark 4. In the above discussion, **t**, **n**, **b**, **t**′, **n**′, **b**′, κ, τ are all functions of the parameter *s*. For the sake of simplicity, we have written **t**(*s*) as **t** etc.

Example 2. *Show that along the curve* $\mathbf{r} = \mathbf{r}(s)$, $\kappa\tau = |\,\mathbf{t}' \cdot \mathbf{b}'\,|$.

Sol. We have $\mathbf{t}' \cdot \mathbf{b}' = (\kappa\mathbf{n}) \cdot (-\tau\mathbf{n}) = -\kappa\tau(\mathbf{n} \,.\, \mathbf{n}) = -\kappa\tau(1) = -\kappa\tau$

∴ $|\,\mathbf{t}' \cdot \mathbf{b}'\,| = |\,-\kappa\tau\,| = \kappa\tau.$

Example 3. *Show that along the curve* $\mathbf{r} = \mathbf{r}(s)$, $\tau = [\mathbf{t}\ \mathbf{n}\ \mathbf{n}']$, *provided* $\kappa \neq 0$.

Sol.
$$\mathbf{n} \times \mathbf{n}' = \mathbf{n} \times (-\kappa\mathbf{t} + \tau\mathbf{b}) = -\kappa(\mathbf{n} \times \mathbf{t}) + \tau(\mathbf{n} \times \mathbf{b})$$
$$= -\kappa(-\mathbf{b}) + \tau(\mathbf{t}) = \kappa\mathbf{b} + \tau\mathbf{t}$$

∴ $$[\mathbf{t}\ \mathbf{n}\ \mathbf{n}'] = \mathbf{t} \cdot (\mathbf{n} \times \mathbf{n}') = \mathbf{t} \cdot (\kappa\mathbf{b} + \tau\mathbf{t})$$
$$= \kappa(\mathbf{t} \cdot \mathbf{b}) + \tau(\mathbf{t} \,.\, \mathbf{t}) = \kappa(0) + \tau(1) = \tau$$

∴ $$\tau = [\mathbf{t}\ \mathbf{n}\ \mathbf{n}'].$$

Example 4. *Show that along the curve* $\mathbf{r} = \mathbf{r}(s)$, $\tau = \dfrac{[\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}''']}{\kappa^2}$, *provided* $\kappa \neq 0$.

Sol. We have $\mathbf{r}' = \mathbf{t}$ and $\mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}$

∴ $$\mathbf{r}''' = \frac{d\mathbf{r}''}{ds} = \frac{d\mathbf{t}'}{ds} = \frac{d}{ds}(\kappa\mathbf{n}) = \kappa\mathbf{n}' + \kappa'\mathbf{n}$$
$$= \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \kappa'\mathbf{n} \qquad \text{(Using } \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b})$$
$$= -\kappa^2\mathbf{t} + \kappa\tau\mathbf{b} + \kappa'\mathbf{n}$$

∴ $$\mathbf{r}'' \times \mathbf{r}''' = \kappa\mathbf{n} \times (-\kappa^2\mathbf{t} + \kappa\tau\mathbf{b} + \kappa'\mathbf{n})$$
$$= -\kappa^3(\mathbf{n} \times \mathbf{t}) + \kappa^2\tau(\mathbf{n} \times \mathbf{b}) + \kappa\kappa'(\mathbf{n} \times \mathbf{n})$$
$$= -\kappa^3(-\mathbf{b}) + \kappa^2\tau\mathbf{t} + \kappa\kappa'\mathbf{0} = \kappa^3\mathbf{b} + \kappa^2\tau\mathbf{t}$$

∴ $$[\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}'''] = \mathbf{r}' \cdot (\mathbf{r}'' \times \mathbf{r}''') = \mathbf{t} \cdot (\kappa^3\mathbf{b} + \kappa^2\tau\mathbf{t})$$
$$= \kappa^3(\mathbf{t} \cdot \mathbf{b}) + \kappa^2\tau(\mathbf{t} \cdot \mathbf{t}) = 0 + \kappa^2\tau \cdot 1 = \kappa^2\tau$$

∴ $$\tau = \frac{[\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}''']}{\kappa^2}.$$

Theorem 2. *Let* $\mathbf{r} = \mathbf{r}(t)$ *be a regular curve of class* $C^m (m \geq 3)$, *where t is an arbitrary parameter. Prove that*

$$\tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}, \text{ provided } |\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,| \neq 0.$$

Proof. We have
$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'\dot{s} = \dot{s}\mathbf{t}$$
$$\ddot{\mathbf{r}} = \frac{d\dot{\mathbf{r}}}{dt} = \frac{d}{dt}(\dot{s}\mathbf{t}) = \dot{s}\frac{d\mathbf{t}}{dt} + \ddot{s}\mathbf{t} = \dot{s}(\mathbf{t}'\dot{s}) + \ddot{s}\mathbf{t}$$
$$= \dot{s}^2(\kappa\mathbf{n}) + \ddot{s}\mathbf{t} = \ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}$$

∴ $$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (\dot{s}\,\mathbf{t}) \times (\ddot{s}\,\mathbf{t} + \kappa\dot{s}^2\mathbf{n}) = \dot{s}\ddot{s}(\mathbf{t} \times \mathbf{t}) + \kappa\dot{s}^3(\mathbf{t} \times \mathbf{n})$$
$$= \dot{s}\ddot{s}(\mathbf{0}) + \kappa\dot{s}^3\mathbf{b} = \kappa\dot{s}^3\mathbf{b}$$

Differentiating w.r.t. *t*, we get

$$\dot{\mathbf{r}} \times \dddot{\mathbf{r}} + \ddot{\mathbf{r}} \times \ddot{\mathbf{r}} = \kappa'\dot{s}^3\mathbf{b} + \kappa 3\dot{s}^2\ddot{s}\mathbf{b} + \kappa\dot{s}^3(\mathbf{b}'\dot{s})$$

$\Rightarrow \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} + \mathbf{0} = (\kappa' \dot{s}^3 + 3\kappa \dot{s}^2 \ddot{s})\mathbf{b} + \kappa \dot{s}^4(-\tau \mathbf{n})$

$\Rightarrow \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (\kappa' \dot{s}^3 + 3\kappa \dot{s}^2 \ddot{s})\mathbf{b} - \kappa \tau \dot{s}^4 \mathbf{n}$

$\therefore \quad \ddot{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \dddot{\mathbf{r}}) = (\ddot{s}\mathbf{t} + \kappa \dot{s}^2 \mathbf{n}) \cdot \left[(\kappa' \dot{s}^3 + 3\kappa \dot{s}^2 \ddot{s})\mathbf{b} - \kappa \tau \dot{s}^4 \mathbf{n}\right]$

$= -\kappa^2 \dot{s}^6 \tau (\mathbf{n} \cdot \mathbf{n})$ (Using $\mathbf{t} \cdot \mathbf{b} = 0$, $\mathbf{t} \cdot \mathbf{n} = 0$, $\mathbf{n} \cdot \mathbf{b} = 0$)

$\Rightarrow \quad [\ddot{\mathbf{r}} \;\; \dot{\mathbf{r}} \;\; \dddot{\mathbf{r}}] = -\kappa^2 \dot{s}^6 \tau$

$\Rightarrow \quad -[\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}] = -\kappa^2 \dot{s}^6 \tau$

$\Rightarrow \quad [\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}] = \kappa^2 \dot{s}^6 \tau$ ...(1)

Also $\quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = \kappa \dot{s}^3 |\mathbf{b}| = \kappa \dot{s}^3$

$\therefore$ (1) $\Rightarrow \quad [\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}] = |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 \tau$

$$\therefore \quad \tau = \frac{[\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}.$$

Corollary. If $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$[\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}] = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} \quad \text{and} \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = \Sigma(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2.$$

$$\therefore \quad \tau = \frac{[\dot{\mathbf{r}} \; \ddot{\mathbf{r}} \; \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} \div \Sigma(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2.$$

Remarks 1. If $\mathbf{r} = \mathbf{r}(s)$, then

$$\tau = \frac{[\mathbf{r}' \;\; \mathbf{r}'' \;\; \mathbf{r}''']}{|\mathbf{r}' \times \mathbf{r}''|^2}. \qquad (\text{Using } \kappa = |\mathbf{r}' \times \mathbf{r}''|)$$

2. If $\mathbf{r} = \mathbf{r}(t)$, then

$$\tau = \frac{[\dot{\mathbf{r}} \;\; \ddot{\mathbf{r}} \;\; \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}.$$

Theorem 3. *Prove that a regular curve of class $C^m (m \geq 3)$ is a plane curve if and only if its torsion is identically zero.*

Proof. Let $\mathbf{r} = \mathbf{r}(s)$ be a regular curve of class $C^m (m \geq 3)$, where s is the parameter 'arc length'.

Let the curve be a plane curve.

Let the plane of the curve be normal to the vector $\mathbf{a}$.

$\therefore \quad \mathbf{r} \cdot \mathbf{a} = \lambda$, where λ is some constant and $\mathbf{r} = \mathbf{r}(s)$.

$$\Rightarrow \quad \frac{d}{ds}(\mathbf{r} \cdot \mathbf{a}) = 0$$

$\Rightarrow \quad \mathbf{r}' \cdot \mathbf{a} = 0 \Rightarrow \mathbf{t} \cdot \mathbf{a} = 0$ ...(1)

(1) $\Rightarrow \quad \dfrac{d}{ds}(\mathbf{t}\,.\,\mathbf{a}) = 0 \quad \Rightarrow \quad \mathbf{t}'\,.\,\mathbf{a} = 0$

$\Rightarrow \quad \dfrac{\mathbf{t}'}{|\mathbf{t}'|}\,.\,\mathbf{a} = 0 \quad \Rightarrow \quad \mathbf{n}\,.\,\mathbf{a} = 0$ ...(2)

(1) and (2) imply that **a** is perpendicular to the plane of **t** and **n.**

$\therefore$ **a** is parallel to the unit binormal vector **b.**

Let $\mathbf{b} = \mu\,\mathbf{a}$.

$\therefore \quad \mathbf{b}' = \dfrac{d\mathbf{b}}{ds} = \dfrac{d}{ds}(\mu\,\mathbf{a}) = \mathbf{0}$

$\therefore \quad \tau = -\,\mathbf{n}\,.\,\mathbf{b}' = -\,\mathbf{n}\,.\,\mathbf{0} = 0$ *i.e.*, the torsion is identically zero.

Conversely, let the torsion of the curve be identically zero.

By **Serret-Frenet equation**, $\mathbf{b}' = -\,\tau\mathbf{n}$.

$\therefore \quad \mathbf{b}' = 0\mathbf{n} = \mathbf{0} \quad \therefore \quad \mathbf{b} = \mathbf{b}_0$, a constant vector.

Let **r** be any point on the curve.

$\therefore \quad \dfrac{d}{ds}(\mathbf{r}\,.\,\mathbf{b}_0) = \mathbf{r}'.\,\mathbf{b}_0 = \mathbf{t}\,.\,\mathbf{b}_0 = 0$ $(\because \ \mathbf{t}\,.\,\mathbf{b} = 0)$

$\therefore \quad \mathbf{r}\,.\,\mathbf{b}_0 = \text{constant}.$

This equation represents a plane.

$\therefore$ The curve $\mathbf{r} = \mathbf{r}(s)$ lies on a plane.

$\therefore$ The result holds.

Remark. For a curve to lie in a plane it is sufficient to show that its unit binormal vector **b** is a constant vector.

Example 5. *Show that the curve* $\mathbf{r} = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + b\mathbf{k}$, $a > 0$, $b \neq 0$ *is a plane curve.*

Sol. We have $\mathbf{r} = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + b\mathbf{k}$

$\therefore \quad \dot{\mathbf{r}} = -\,a \sin t\,\mathbf{i} + a \cos t\,\mathbf{j} + 0\mathbf{k}$

$$|\dot{\mathbf{r}}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$$

$\therefore \quad \mathbf{t} = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \dfrac{1}{a}(-\,a \sin t\,\mathbf{i} + a \cos t\,\mathbf{j})$

$= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

$\therefore \quad \dot{\mathbf{t}} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ and $|\dot{\mathbf{t}}| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\therefore \quad \mathbf{n} = \dfrac{\dot{\mathbf{t}}}{|\dot{\mathbf{t}}|} = -\dfrac{1}{1}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) = -\,(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$

$\therefore \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}$

$\therefore$ **b** is a constant vector.

$$\Rightarrow \quad \mathbf{b}' = \mathbf{0} \quad \Rightarrow \quad \tau = -\mathbf{n}\,.\,\mathbf{b}' = -\mathbf{n}\,.\,\mathbf{0} = 0.$$

$\therefore$ The given curve is a plane curve.

Alternative method

$$[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = 0$$

$$\therefore \quad \tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = \frac{0}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = 0.$$

$\therefore$ The given curve is a plane curve.

Example 6. *Show that the curve* $\mathbf{r} = \left(t, \dfrac{1+t}{t}, \dfrac{1-t^2}{t}\right)$ *lies on a plane.*

Sol. The given curve is

$$\mathbf{r} = t\mathbf{i} + \frac{1+t}{t}\mathbf{j} + \frac{1-t^2}{t}\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}} = \mathbf{i} + \left(-\frac{1}{t^2}\right)\mathbf{j} + \left(-\frac{1}{t^2} - 1\right)\mathbf{k}$$

$$\ddot{\mathbf{r}} = \frac{2}{t^3}\mathbf{j} + \frac{2}{t^3}\mathbf{k}$$

and
$$\dddot{\mathbf{r}} = -\frac{6}{t^4}\mathbf{j} - \frac{6}{t^4}\mathbf{k}$$

$$\therefore \quad [\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} 1 & -\dfrac{1}{t^2} & -\dfrac{1}{t^2} - 1 \\ 0 & \dfrac{2}{t^3} & \dfrac{2}{t^3} \\ 0 & -\dfrac{6}{t^4} & -\dfrac{6}{t^4} \end{vmatrix} = -\frac{12}{t^7} + \frac{12}{t^7} = 0$$

$$\therefore \quad \tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = \frac{0}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = 0.$$

$\therefore$ The given curve lies on a plane.

Example 7. *Find the torsion of the curve* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point 't'.

Sol. We have $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

$$\therefore \quad \dot{\mathbf{r}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k},$$

$$\ddot{\mathbf{r}} = 2\mathbf{j} + 6t\mathbf{k} \quad \text{and} \quad \dddot{\mathbf{r}} = 6\mathbf{k}$$

Now,
$$\tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} \qquad \ldots(1)$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}$$

$$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = 36t^4 + 36t^2 + 4 = 4(1 + 9t^2 + 9t^4)$$

$$[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = 12$$

$$\therefore \ (1) \Rightarrow \quad \tau = \frac{12}{4(1+9t^2+9t^4)} = \frac{3}{1+9t^2+9t^4}.$$

Example 8. *For the curve* $\mathbf{r} = (at - a \sin t)\mathbf{i} + (a - a \cos t)\mathbf{j} + bt\mathbf{k}$, *find the torsion at the point 't'.*

Sol. We have $\quad \mathbf{r} = (at - a \sin t)\mathbf{i} + (a - a \cos t)\mathbf{j} + bt\mathbf{k}$

$$\therefore \quad \dot{\mathbf{r}} = (a - a \cos t)\mathbf{i} + a \sin t\mathbf{j} + b\mathbf{k}$$

$$\ddot{\mathbf{r}} = a \sin t\ \mathbf{i} + a \cos t\mathbf{j} + 0\mathbf{k} \quad \text{and} \quad \dddot{\mathbf{r}} = a \cos t\mathbf{i} - a \sin t\mathbf{j} + 0\mathbf{k}$$

Now, $$\tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} \qquad ...(1)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a - a\cos t & a \sin t & b \\ a \sin t & a \cos t & 0 \end{vmatrix}$$

$$= -ab \cos t\ \mathbf{i} + ab \sin t\ \mathbf{j} + (a^2 \cos t - a^2)\mathbf{k}$$

$$\therefore \quad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2 = a^2b^2 \cos^2 t + a^2b^2 \sin^2 t + a^4 (\cos t - 1)^2$$

$$= a^2b^2 + a^4\left(-2\sin^2\frac{t}{2}\right)^2 = a^2\left[b^2 + 4a^2 \sin^4\frac{t}{2}\right]$$

Also, $$[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} a - a\cos t & a \sin t & b \\ a \sin t & a \cos t & 0 \\ a \cos t & -a \sin t & 0 \end{vmatrix} = b[-a^2\sin^2 t - a^2\cos^2 t] = -a^2b$$

$$\therefore \ (1) \Rightarrow \quad \tau = \frac{-a^2b}{a^2\left(b^2 + 4a^2\sin^4\frac{t}{2}\right)} = -\frac{b}{b^2 + 4a^2\sin^4\frac{t}{2}}.$$

Example 9. *For the curve* $x = a \tan t, y = a \cot t, z = \sqrt{2}\ a \log \tan t$, *prove that*

$$\rho = \sigma = \frac{2\sqrt{2}\ a}{\sin^2 2t}.$$

Sol. The given curve is

$$x = a \tan t, y = a \cot t, z = \sqrt{2}\ a \log \tan t.$$

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a \tan t\mathbf{i} + a \cot t\mathbf{j} + \sqrt{2}a \log \tan t\mathbf{k}$$

$$\therefore \quad \mathbf{t} = \mathbf{r}' = \dot{\mathbf{r}}\frac{dt}{ds}$$

$$= \left(a \sec^2 t\mathbf{i} - a \operatorname{cosec}^2 t\mathbf{j} + \sqrt{2}\ a \frac{\sec^2 t}{\tan t}\mathbf{k}\right)\frac{dt}{ds}$$

$$\therefore \quad \mathbf{t} = a\left(\sec^2 t\mathbf{i} - \operatorname{cosec}^2 t\mathbf{j} + \frac{\sqrt{2}}{\sin t \cos t}\mathbf{k}\right)\frac{dt}{ds} \quad \text{...(1)}$$

$$\Rightarrow \quad |\,\mathbf{t}\,| = a\left(\sec^4 t + \operatorname{cosec}^4 t + \frac{2}{\sin^2 t \cos^2 t}\right)^{1/2}\frac{dt}{ds}$$

$$\Rightarrow \quad \frac{ds}{dt}.1 = a\left(\frac{\sin^4 t + \cos^4 t + 2\sin^2 t\cos^2 t}{\sin^4 t \cos^4 t}\right)^{1/2} = \frac{a}{\sin^2 t \cos^2 t}$$

$$\therefore \quad \frac{dt}{ds} = \frac{\sin^2 t \cos^2 t}{a}$$

$$\therefore \ (1) \Rightarrow \quad \mathbf{t} = a\left(\sec^2 t\mathbf{i} - \operatorname{cosec}^2 t\mathbf{j} + \frac{\sqrt{2}}{\sin t \cos t}\mathbf{k}\right).\frac{\sin^2 t \cos^2 t}{a}$$

or

$$\mathbf{t} = (\sin^2 t\mathbf{i} - \cos^2 t\mathbf{j} + \sqrt{2}\sin t\cos t\mathbf{k})$$

$$\therefore \quad \mathbf{t}' = \dot{\mathbf{t}}\frac{dt}{ds} = (2\sin t\cos t\mathbf{i} + 2\cos t\sin t\mathbf{j} + \sqrt{2}\cos 2t\mathbf{k}).\frac{\sin^2 t\cos^2 t}{a}$$

$$\Rightarrow \quad \kappa\mathbf{n} = \frac{\sin^2 t\cos^2 t}{a}(\sin 2t\mathbf{i} + \sin 2t\mathbf{j} + \sqrt{2}\cos 2t\mathbf{k}) \quad \text{...(2)}$$

(Using $\mathbf{t}' = \kappa\mathbf{n}$)

$$\Rightarrow \quad \kappa|\,\mathbf{n}\,| = \frac{\sin^2 t\cos^2 t}{a}(\sin^2 2t + \sin^2 2t + 2\cos^2 2t)^{1/2} = \frac{\sqrt{2}\sin^2 t\cos^2 t}{a}$$

$$\therefore \quad \kappa = \frac{\sqrt{2}\sin^2 t\cos^2 t}{a} \quad \text{and} \quad \rho = \frac{1}{\kappa} = \frac{a}{\sqrt{2}\sin^2 t\cos^2 t}$$

$$\therefore \ (2) \Rightarrow \quad \mathbf{n} = \frac{1}{\sqrt{2}}(\sin 2t\mathbf{i} + \sin 2t\mathbf{j} + \sqrt{2}\cos 2t\mathbf{k})$$

$$\Rightarrow \quad \mathbf{n}' = \dot{\mathbf{n}}\frac{dt}{ds} = \frac{1}{\sqrt{2}}(2\cos 2t\mathbf{i} + 2\cos 2t\mathbf{j} - 2\sqrt{2}\sin 2t\mathbf{k})\frac{\sin^2 t\cos^2 t}{a}$$

$$\Rightarrow \quad \tau\mathbf{b} - \kappa\mathbf{t} = \frac{\sqrt{2}\sin^2 t\cos^2 t}{a}(\cos 2t\mathbf{i} + \cos 2t\mathbf{j} - \sqrt{2}\sin 2t\mathbf{k})$$

(Using $\mathbf{n}' = \tau\mathbf{b} - \kappa\mathbf{t}$)

$$\Rightarrow \quad (\tau\mathbf{b} - \kappa\mathbf{t})\cdot(\tau\mathbf{b} - \kappa\mathbf{t})$$

$$= \frac{2\sin^4 t\cos^4 t}{a^2}(\cos^2 2t + \cos^2 2t + 2\sin^2 2t)$$

$$\Rightarrow \quad \tau^2 + \kappa^2 = \frac{4\sin^4 t\cos^4 t}{a^2}$$

$$\Rightarrow \quad \tau^2 = \frac{4\sin^4 t\cos^4 t}{a^2} - \frac{2\sin^4 t\cos^4 t}{a^2} = \frac{2\sin^4 t\cos^4 t}{a^2}$$

$\therefore \qquad \tau = \dfrac{\sqrt{2}\sin^2 t \cos^2 t}{a}$ and $\sigma = \dfrac{1}{\tau} = \dfrac{a}{\sqrt{2}\sin^2 t\cos^2 t}$

$\therefore \qquad \rho = \sigma = \dfrac{a}{\sqrt{2}\sin^2 t\cos^2 t} = \dfrac{2\sqrt{2}\,a}{\sin^2 2t}$.

Example 10. *Prove that for the curve of intersection of the surfaces $x^2 + y^2 = z^2$ and $z = a\tan^{-1}\dfrac{y}{x}$:*

$$\rho = \frac{a(2+\theta^2)^{3/2}}{(8+5\theta^2+\theta^4)^{1/2}} \quad \textit{and} \quad \sigma = \frac{a(8+5\theta^2+\theta^4)}{6+\theta^2}, \quad \textit{where} \quad y = x\tan\theta.$$

Sol. Given surfaces are

$$x^2 + y^2 = z^2 \qquad \text{...(1)} \qquad \text{and} \qquad z = a\tan^{-1}\frac{y}{x} \qquad \text{...(2)}$$

Let $\quad y = x\tan\theta \quad \therefore \quad (2) \Rightarrow z = a\tan^{-1}(\tan\theta) = a\theta$

(1) $\Rightarrow \quad x^2 + x^2\tan^2\theta = a^2\theta^2 \quad \Rightarrow x^2\sec^2\theta = a^2\theta^2 \quad \Rightarrow \quad x = a\theta\cos\theta$

$\therefore \quad y = (a\theta\cos\theta)\tan\theta = a\theta\sin\theta$

$\therefore$ The parametric equations of the given curve are

$$x = a\theta\cos\theta,\ y = a\theta\sin\theta,\ z = a\theta.$$

Let **r** be the position vector of the point (x, y, z) on the curve.

$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\theta\cos\theta\mathbf{i} + a\theta\sin\theta\mathbf{j} + a\theta\mathbf{k}$

$\therefore \quad \dot{\mathbf{r}} = a\,[(\cos\theta - \theta\sin\theta)\mathbf{i} + (\sin\theta + \theta\cos\theta)\mathbf{j} + \mathbf{k}]$

$\ddot{\mathbf{r}} = a\,[(-2\sin\theta - \theta\cos\theta)\mathbf{i} + (2\cos\theta - \theta\sin\theta)\mathbf{j}]$

$\dddot{\mathbf{r}} = a\,[(-3\cos\theta + \theta\sin\theta)\mathbf{i} + (-3\sin\theta - \theta\cos\theta)\mathbf{j}]$

$$\therefore \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = a^2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta - \theta\sin\theta & \sin\theta + \theta\cos\theta & 1 \\ -2\sin\theta - \theta\cos\theta & 2\cos\theta - \theta\sin\theta & 0 \end{vmatrix}$$

$$\begin{aligned}
&= a^2\,[(-2\cos\theta + \theta\sin\theta)\mathbf{i} + (-2\sin\theta - \theta\cos\theta)\mathbf{j} \\
&\quad + (2\cos^2\theta - \theta\sin\theta\cos\theta - 2\theta\sin\theta\cos\theta + \theta^2\sin^2\theta \\
&\quad + 2\sin^2\theta + 2\theta\sin\theta\cos\theta + \theta\sin\theta\cos\theta + \theta^2\cos^2\theta)\mathbf{k}] \\
&= a^2\,[(-2\cos\theta + \theta\sin\theta)\mathbf{i} - (2\sin\theta + \theta\cos\theta)\mathbf{j} + (2 + \theta^2)\mathbf{k}]
\end{aligned}$$

$$\begin{aligned}
[\dot{\mathbf{r}}\ \ \ddot{\mathbf{r}}\ \ \dddot{\mathbf{r}}] &= \dot{\mathbf{r}}\,.\,(\ddot{\mathbf{r}} \times \dddot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})\,.\,\dddot{\mathbf{r}} \\
&= a^2\,[(-2\cos\theta + \theta\sin\theta)\mathbf{i} - (2\sin\theta + \theta\cos\theta)\mathbf{j} \\
&\quad + (2 + \theta^2)\mathbf{k}]\,.\,a\,[(-3\cos\theta + \theta\sin\theta)\mathbf{i} - (3\sin\theta + \theta\cos\theta)\mathbf{j}] \\
&= a^3\,[(6\cos^2\theta + \theta^2\sin^2\theta - 5\theta\sin\theta\cos\theta) \\
&\quad + (6\sin^2\theta + \theta^2\cos^2\theta + 5\theta\sin\theta\cos\theta)] \\
&= a^3(6 + \theta^2)
\end{aligned}$$

$$|\dot{\mathbf{r}}|^2 = a^2[(\cos^2\theta + \theta^2\sin^2\theta - 2\theta\cos\theta\sin\theta) + \sin^2\theta + \theta^2\cos^2\theta + 2\theta\sin\theta\cos\theta + 1]$$
$$= a^2(2+\theta^2)$$
$$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2 = a^4[(4\cos^2\theta + \theta^2\sin^2\theta - 4\theta\cos\theta\sin\theta) + (4\sin^2\theta + \theta^2\cos^2\theta + 4\theta\sin\theta\cos\theta) + (2+\theta^2)^2]$$
$$= a^4[4+\theta^2+4+\theta^4+4\theta^2] = a^4[8+5\theta^2+\theta^4]$$

$\therefore$ $$\rho = \frac{1}{\kappa} = \frac{|\dot{\mathbf{r}}|^3}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|} = \frac{a^3(2+\theta^2)^{3/2}}{a^2(8+5\theta^2+\theta^4)^{1/2}} = \frac{a(2+\theta^2)^{3/2}}{(8+5\theta^2+\theta^4)^{1/2}}$$

and $$\sigma = \frac{1}{\tau} = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}{[\dot{\mathbf{r}}\;\ddot{\mathbf{r}}\;\dddot{\mathbf{r}}]} = \frac{a^4(8+5\theta^2+\theta^4)}{a^3(6+\theta^2)} = \frac{a(8+5\theta^2+\theta^4)}{6+\theta^2}.$$

Example 11. *For a point on the curve of intersection of the surfaces* $x^2 - y^2 = c^2$ *and* $y = x\tanh\frac{z}{c}$, *show that* $\rho = \sigma = \frac{2x^2}{c}$.

Sol. Given curve is

$$x^2 - y^2 = c^2 \quad ...(1) \qquad y = x\tanh\frac{z}{c} \quad ...(2)$$

Let $x = c\cosh t,\quad y = c\sinh t.$

$\therefore$ (1) is satisfied.

(2) $\Rightarrow$ $c\sinh t = c\cosh t\tanh\frac{z}{c}$

$\Rightarrow$ $\tanh\frac{z}{c} = \tanh t \Rightarrow z = ct$

$\therefore$ The parametric equations of the given curve are

$$x = c\cosh t,\ y = c\sinh t,\ z = ct.$$

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$\therefore$ $$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c\cosh t\,\mathbf{i} + c\sinh t\,\mathbf{j} + ct\mathbf{k}$$

$\therefore$ $$\dot{\mathbf{r}} = c\sinh t\,\mathbf{i} + c\cosh t\,\mathbf{j} + c\mathbf{k}$$
$$\ddot{\mathbf{r}} = c\cosh t\,\mathbf{i} + c\sinh t\,\mathbf{j}$$
and $$\dddot{\mathbf{r}} = c\sinh t\,\mathbf{i} + c\cosh t\,\mathbf{j}$$

$\therefore$ $$|\dot{\mathbf{r}}| = c(\sinh^2 t + \cosh^2 t + 1)^{1/2} = c(2\cosh^2 t)^{1/2} = \sqrt{2}\,c\cosh t$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c\sinh t & c\cosh t & c \\ c\cosh t & c\sinh t & 0 \end{vmatrix} = c^2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix}$$
$$= c^2[-\sinh t\mathbf{i} + \cosh t\mathbf{j} - \mathbf{k}]$$

$\therefore$ $$|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}| = c^2(\sinh^2 t + \cosh^2 t + 1)^{1/2} = c^2(2\cosh^2 t)^{1/2} = \sqrt{2}\,c^2\cosh t$$

$\therefore$ $$\kappa = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{2}\,c^2\cosh t}{(\sqrt{2}\,c\cosh t)^3} = \frac{1}{2c\cosh^2 t}$$

$$\therefore \qquad \rho = \frac{1}{\kappa} = 2c\cosh^2 t = 2c\left(\frac{x}{c}\right)^2 = \frac{2x^2}{c}.$$

$$[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} c\sinh t & c\cosh t & c \\ c\cosh t & c\sinh t & 0 \\ c\sinh t & c\cosh t & 0 \end{vmatrix} = c\,(c^2\cosh^2 t - c^2\sinh^2 t) = c^3$$

$$\therefore \qquad \tau = \frac{[\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = \frac{c^3}{(\sqrt{2}\,c^2\cosh t)^2} = \frac{1}{2c\cosh^2 t}$$

$$\therefore \qquad \sigma = \frac{1}{\tau} = 2c\cosh^2 t = 2c\left(\frac{x}{c}\right)^2 = \frac{2x^2}{c}.$$

$$\therefore \qquad \rho = \sigma = \frac{2x^2}{c}.$$

Example 12. *Determine the function f(u) so that the curve given by* $\mathbf{r} = (a\cos u, a\sin u, f(u))$ *should be a plane curve.*

Sol. The given curve is

$$\mathbf{r} = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + f(u)\mathbf{k}.$$

Given curve is a plane curve iff $\tau = 0$

$$\text{iff} \qquad \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2} = 0 \quad \text{iff} \quad [\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = 0.$$

$\therefore$ Given curve is a plane curve iff $[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = 0$.

We shall choose $f(u)$ so that we may have $[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = 0$.

$$\dot{\mathbf{r}} = -a\sin u\,\mathbf{i} + a\cos u\,\mathbf{j} + \dot{f}(u)\mathbf{k}$$

$$\ddot{\mathbf{r}} = -a\cos u\,\mathbf{i} - a\sin u\,\mathbf{j} + \ddot{f}(u)\mathbf{k}$$

$$\dddot{\mathbf{r}} = a\sin u\,\mathbf{i} - a\cos u\,\mathbf{j} + \dddot{f}(u)\mathbf{k}$$

$$[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = \begin{vmatrix} -a\sin u & a\cos u & \dot{f}(u) \\ -a\cos u & -a\sin u & \ddot{f}(u) \\ a\sin u & -a\cos u & \dddot{f}(u) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \dot{f}(u)+\dddot{f}(u) \\ -a\cos u & -a\sin u & \ddot{f}(u) \\ a\sin u & -a\cos u & \dddot{f}(u) \end{vmatrix} \qquad (\text{Operating } R_1 \to R_1 + R_3)$$

$$= (\dot{f}(u)+\dddot{f}(u))\,(a^2\cos^2 u + a^2\sin^2 u) = a^2(\dot{f}(u)+\dddot{f}(u))$$

$$\therefore \qquad a^2(\dot{f}(u)+\dddot{f}(u)) = 0 \qquad (\because\ [\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}] = 0)$$

$$\Rightarrow \qquad \dot{f}(u)+\dddot{f}(u) = 0$$

Integrating, we get $f(u) + \ddot{f}(u) = c_1$

$$\Rightarrow \qquad \ddot{f}(u) = c_1 - f(u) \quad \Rightarrow \quad 2\dot{f}(u)\,\ddot{f}(u) = 2(c_1 - f(u))\,\dot{f}(u)$$

Integrating, we get

$$(\dot{f}(u))^2 = -(c_1 - f(u))^2 + c_2$$

$$\Rightarrow \qquad \dot{f}(u) = \sqrt{c_2 - (c_1 - f(u))^2}$$

$$\Rightarrow \qquad \frac{d(f(u))}{\sqrt{c_2 - (c_1 - f(u))^2}} = du$$

Integrating, we get

$$-\sin^{-1}\frac{c_1 - f(u)}{\sqrt{c_2}} = u + c_3$$

$$\Rightarrow \qquad \sin(-(u + c_3)) = \frac{c_1 - f(u)}{\sqrt{c_2}}$$

$$\Rightarrow \qquad -\sqrt{c_2}\sin(u + c_3) = c_1 - f(u)$$

$$\Rightarrow \qquad f(u) = \sqrt{c_2}\sin(u + c_3) + c_1$$

$$= \sqrt{c_2}(\sin u \cos c_3 + \cos u \sin c_3) + c_1$$

$$= \sqrt{c_2}\cos c_3 \sin u + \sqrt{c_2}\sin c_3 \cos u + c_1$$

$$\therefore \qquad f(u) = A \sin u + B \cos u + C,$$

where $A = \sqrt{c_2}\cos c_3$, $B = \sqrt{c_2}\sin c_3$, $C = c_1$ are arbitrary constants.

Example 13. *If the tangent and binormal at a point of a curve make angles θ and φ with a fixed direction, show that:* $\dfrac{\sin\theta}{\sin\phi}\dfrac{d\theta}{d\phi} = -\dfrac{\kappa}{\tau}$.

Sol. Let the equation of the curve be $\mathbf{r} = \mathbf{r}(s)$, where the parameter '$s$' is arc length.

Let the tangent and binormal at the point P of the curve make angles θ and φ with vector **a** which is along the given fixed direction.

∴ Angle between **t** and **a** is θ and angle between **b** and **a** is φ.

$$\therefore \qquad \mathbf{t}\cdot\mathbf{a} = a\cos\theta \qquad \ldots(1) \qquad (\because\ |\mathbf{t}| = 1)$$

and

$$\mathbf{b}\cdot\mathbf{a} = a\cos\phi \qquad \ldots(2) \qquad (\because\ |\mathbf{b}| = 1)$$

Differentiating (1) w.r.t. s, we get

$$\mathbf{t}'\cdot\mathbf{a} = -a\sin\theta\frac{d\theta}{ds}$$

$$\Rightarrow \qquad \kappa\mathbf{n}\cdot\mathbf{a} = -a\sin\theta\frac{d\theta}{ds} \qquad \ldots(3)$$

Differentiating (2) w.r.t. s, we get

$$\mathbf{b}'\cdot\mathbf{a} = -a\sin\phi\frac{d\phi}{ds}$$

$$\Rightarrow \qquad -\tau\mathbf{n}\cdot\mathbf{a} = -a\sin\phi\frac{d\phi}{ds} \qquad \ldots(4)$$

Dividing (3) by (4), we get

$$\frac{\kappa(\mathbf{n}.\mathbf{a})}{-\tau(\mathbf{n}.\mathbf{a})} = \frac{-a\sin\theta\dfrac{d\theta}{ds}}{-a\sin\phi\dfrac{d\phi}{ds}}$$

$$\Rightarrow \qquad -\frac{\kappa}{\tau} = \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}.$$

Example 14. *For the curve* $\mathbf{r} = \mathbf{r}(s)$, *if*

$$\frac{d\mathbf{t}}{ds} = \mathbf{w}\times\mathbf{t},\quad \frac{d\mathbf{n}}{ds} = \mathbf{w}\times\mathbf{n} \text{ and } \frac{d\mathbf{b}}{ds} = \mathbf{w}\times\mathbf{b}, \text{ find the vector } \mathbf{w}.$$

Sol. Given equations are

$$\frac{d\mathbf{t}}{ds} = \mathbf{w}\times\mathbf{t} \quad ...(1) \qquad \frac{d\mathbf{n}}{ds} = \mathbf{w}\times\mathbf{n} \quad ...(2) \qquad \frac{d\mathbf{b}}{ds} = \mathbf{w}\times\mathbf{b} \quad ...(3)$$

By *Frenet formula*, $\dfrac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$.

$$\Rightarrow \qquad \frac{d\mathbf{t}}{ds} = \mathbf{0} + \kappa\mathbf{n} = \tau(\mathbf{t}\times\mathbf{t}) + \kappa(\mathbf{b}\times\mathbf{t}) = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{t}$$

$$\therefore \qquad \frac{d\mathbf{t}}{ds} = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{t} \qquad ...(4)$$

By *Frenet formula*, $\dfrac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} + \tau\mathbf{b}$.

$$\Rightarrow \qquad \frac{d\mathbf{n}}{ds} = -\kappa(\mathbf{n}\times\mathbf{b}) + \tau(\mathbf{t}\times\mathbf{n}) = \kappa(\mathbf{b}\times\mathbf{n}) + \tau(\mathbf{t}\times\mathbf{n})$$

$$= (\kappa\mathbf{b} + \tau\mathbf{t})\times\mathbf{n} = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{n}$$

$$\therefore \qquad \frac{d\mathbf{n}}{ds} = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{n} \qquad ...(5)$$

By *Frenet formula*, $\dfrac{d\mathbf{b}}{ds} = -\tau\mathbf{n}$.

$$\Rightarrow \qquad \frac{d\mathbf{b}}{ds} = -\tau(\mathbf{b}\times\mathbf{t}) + \mathbf{0} = \tau(\mathbf{t}\times\mathbf{b}) + \kappa(\mathbf{b}\times\overline{\mathbf{b}}) = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{b}$$

$$\therefore \qquad \frac{d\mathbf{b}}{ds} = (\tau\mathbf{t} + \kappa\mathbf{b})\times\mathbf{b} \qquad ...(6)$$

If $\mathbf{w} = \tau\mathbf{t} + \kappa\mathbf{b}$, then given equations (1), (2) and (3) are satisfied.

Note. The vector $\mathbf{w} = \tau\mathbf{t} + \kappa\mathbf{b}$ is called the **Darboux vector** for the curve $\mathbf{r} = \mathbf{r}(s)$.

Example 15. *Using Serret-Frenet formula, find the direction cosines of the unit principal normal vector and the unit binormal vector at the point 's' for the curve* $\mathbf{r} = \mathbf{r}(s)$.

Sol. Given curve is $\quad \mathbf{r} = \mathbf{r}(s)$.

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the curve.

$$\therefore \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

We have $\mathbf{t}' = \kappa\mathbf{n}$.

$$\Rightarrow \quad \mathbf{n} = \frac{\mathbf{t}'}{\kappa} = \frac{(\mathbf{r}')'}{\kappa} = \frac{\mathbf{r}''}{\kappa} = \frac{1}{\kappa}(x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}) = \frac{x''}{\kappa}\mathbf{i} + \frac{y''}{\kappa}\mathbf{j} + \frac{z''}{\kappa}\mathbf{k}$$

$$|\,\mathbf{n}\,| = 1 \quad \Rightarrow \quad \left(\frac{x''^2}{\kappa^2} + \frac{y''^2}{\kappa^2} + \frac{z''^2}{\kappa^2}\right)^{1/2} = 1 \quad \Rightarrow \quad \kappa^2 = x''^2 + y''^2 + z''^2$$

$$\therefore \quad \kappa = \sqrt{x''^2 + y''^2 + z''^2}$$

$$\therefore \quad \mathbf{n} = \frac{x''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{i} + \frac{y''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{j} + \frac{z''}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{k}$$

Since **n** is a unit vector, the d.c.'s of **n** are

$$\frac{x''}{\sqrt{x''^2 + y''^2 + z''^2}}, \quad \frac{y''}{\sqrt{x''^2 + y''^2 + z''^2}}, \quad \frac{z''}{\sqrt{x''^2 + y''^2 + z''^2}}.$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \mathbf{r}' \times \frac{\mathbf{t}'}{\kappa} = \frac{1}{\kappa}(\mathbf{r}' \times \mathbf{r}'') = \frac{1}{\kappa}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

$$\left(\kappa = \mathbf{t}' \quad \Rightarrow \quad \kappa = |\,\mathbf{t}'\,| \quad \Rightarrow \quad \mathbf{n} = \frac{\mathbf{t}'}{|\,\mathbf{t}'\,|} = \frac{\mathbf{t}'}{\kappa}\right)$$

$$= \frac{1}{\kappa}\,[(y'z'' - y''z')\mathbf{i} + (z'x'' - z''x')\mathbf{j} + (x'y'' - x''y')\mathbf{k}]$$

$$= \frac{y'z'' - y''z'}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{i} + \frac{z'x'' - z''x'}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{j} + \frac{x'y'' - x''y'}{\sqrt{x''^2 + y''^2 + z''^2}}\mathbf{k}$$

Since **b** is a unit vector, the d.c.'s of **b** are

$$\frac{y'z'' - y''z'}{\sqrt{x''^2 + y''^2 + z''^2}}, \quad \frac{z'x'' - z''x'}{\sqrt{x''^2 + y''^2 + z''^2}}, \quad \frac{x'y'' - x''y'}{\sqrt{x''^2 + y''^2 + z''^2}}.$$

Example 16. *Let* $\mathbf{r} = \mathbf{r}(t)$ *be a curve. Prove that:*

(*i*) $\dot{\mathbf{r}} = \dot{s}\mathbf{t}$ (*ii*) $\ddot{\mathbf{r}} = \ddot{s}\mathbf{t} + \kappa\dot{s}^2\,\mathbf{n}$

(*iii*) $\dddot{\mathbf{r}} = (\dddot{s} - \kappa^2\dot{s}^3)\,\mathbf{t} + \dot{s}\,(3\kappa\ddot{s} - \dot{\kappa}\dot{s})\,\mathbf{n} + \kappa\tau\dot{s}^3\mathbf{b}.$

Hence deduce that:

(*a*) $\mathbf{n} = \dfrac{\dot{s}\,\ddot{\mathbf{r}} - \ddot{s}\,\dot{\mathbf{r}}}{\kappa\dot{s}^3}$ (*b*) $\mathbf{b} = \dfrac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\kappa\dot{s}^3}$

(*c*) $\kappa^2 = \dfrac{|\,\ddot{\mathbf{r}}\,|^2 - \ddot{s}^2}{\dot{s}^4}$ (*d*) $\tau = \dfrac{[\dot{\mathbf{r}}\;\ddot{\mathbf{r}}\;\dddot{\mathbf{r}}]}{\kappa^2\dot{s}^6}$.

Sol. (*i*) $\dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt} = \mathbf{r}'\dot{s} = \dot{s}\mathbf{t}.$

(*ii*) $\ddot{\mathbf{r}} = \dfrac{d}{dt}(\dot{\mathbf{r}}) = \dfrac{d}{dt}(\dot{s}\mathbf{t}) = \dot{s}\dfrac{d\mathbf{t}}{dt} + \ddot{s}\mathbf{t}$

$$= \dot{s}\,(\mathbf{t}'\,\dot{s}) + \ddot{s}\mathbf{t} = \dot{s}^2\,(\kappa\mathbf{n}) + \ddot{s}\mathbf{t} = \ddot{s}\,\mathbf{t} + \kappa\dot{s}^2\,\mathbf{n}.$$

(*iii*)
$$\dddot{\mathbf{r}} = \frac{d}{dt}(\ddot{\mathbf{r}}) = \frac{d}{dt}(\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n})$$
$$= \left(\ddot{s}\frac{d\mathbf{t}}{dt} + \dddot{s}\mathbf{t}\right) + \left(\dot{\kappa}\dot{s}^2\mathbf{n} + \kappa 2\dot{s}\,\ddot{s}\,\mathbf{n} + \kappa\dot{s}^2\frac{d\mathbf{n}}{dt}\right)$$
$$= \ddot{s}(\mathbf{t}'\dot{s}) + \dddot{s}\mathbf{t} + \dot{\kappa}\dot{s}^2\mathbf{n} + 2\kappa\dot{s}\ddot{s}\mathbf{n} + \kappa\dot{s}^2\mathbf{n}'\dot{s}$$
$$= \dot{s}\ddot{s}(\kappa\mathbf{n}) + \dddot{s}\mathbf{t} + \dot{\kappa}\dot{s}^2\mathbf{n} + 2\kappa\dot{s}\ddot{s}\mathbf{n} + \kappa\dot{s}^3(-\kappa\mathbf{t} + \tau\mathbf{b})$$
$$= (\dddot{s} - \kappa^2\dot{s}^3)\mathbf{t} + \dot{s}(3\kappa\ddot{s} + \dot{\kappa}\dot{s})\mathbf{n} + \kappa\tau\,\dot{s}^3\mathbf{b}.$$

(*a*)
$$\dot{s}\ddot{\mathbf{r}} - \ddot{s}\dot{\mathbf{r}} = \dot{s}(\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}) - \ddot{s}(\dot{s}\mathbf{t}) = \kappa\dot{s}^3\mathbf{n}$$
$$\therefore \quad \mathbf{n} = \frac{\dot{s}\ddot{\mathbf{r}} - \ddot{s}\dot{\mathbf{r}}}{\kappa\dot{s}^3}.$$

(*b*)
$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (\dot{s}\mathbf{t}) \times (\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}) = \dot{s}\ddot{s}(\mathbf{t} \times \mathbf{t}) + \kappa\dot{s}^3(\mathbf{t} \times \mathbf{n})$$
$$= \dot{s}\ddot{s}(\mathbf{0}) + \kappa\dot{s}^3\mathbf{b} = \kappa\dot{s}^3\,\mathbf{b}$$
$$\therefore \quad \mathbf{b} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\kappa\dot{s}^3}.$$

(*c*)
$$|\ddot{\mathbf{r}}|^2 = \ddot{\mathbf{r}}\,.\,\ddot{\mathbf{r}} = (\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n})\,.\,(\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n})$$
$$= \ddot{s}^2(\mathbf{t}\,.\,\mathbf{t}) + \kappa\dot{s}^2\ddot{s}(\mathbf{t}\,.\,\mathbf{n}) + \kappa\dot{s}^2\ddot{s}(\mathbf{n}\,.\,\mathbf{t}) + \kappa^2\dot{s}^4(\mathbf{n}\,.\,\mathbf{n})$$
$$= \ddot{s}^2(1) + 0 + 0 + \kappa^2\dot{s}^4(1) = \ddot{s}^2 + \kappa^2\dot{s}^4$$
$$\therefore \quad |\ddot{\mathbf{r}}|^2 - \ddot{s}^2 = \kappa^2\dot{s}^4$$
$$\therefore \quad \kappa^2 = \frac{|\ddot{\mathbf{r}}|^2 - \ddot{s}^2}{\dot{s}^4}.$$

(*d*)
$$[\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}] = \dot{\mathbf{r}}\,.\,(\ddot{\mathbf{r}} \times \dddot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})\,.\,\dddot{\mathbf{r}}$$
$$= (\kappa\dot{s}^3\mathbf{b})\,.\,[(\dddot{s} - \kappa^2\dot{s}^3)\mathbf{t} + \dot{s}(3\kappa\ddot{s} + \dot{\kappa}\dot{s})\mathbf{n} + \kappa\tau\dot{s}^3\mathbf{b}] \quad \text{(Using (iii) and (b))}$$
$$= \kappa^2\dot{s}^6\tau(\mathbf{b}\,.\,\mathbf{b}) = \kappa^2\dot{s}^6\tau \quad \text{(Using } \mathbf{b}\,.\,\mathbf{t} = 0, \mathbf{b}\,.\,\mathbf{n} = 0)$$
$$\therefore \quad \tau = \frac{[\dot{\mathbf{r}}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]}{\kappa^2\dot{s}^6}.$$

Example 17. *Let* $\mathbf{r} = \mathbf{r}(s)$ *be a curve. Prove that:*

(*i*) $\mathbf{r}'\,.\,\mathbf{r}'' = 0$ (*ii*) $\mathbf{r}''' = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}$

(*iii*) $\mathbf{r}'\,.\,\mathbf{r}''' = -\kappa^2$ (*iv*) $\mathbf{r}''\,.\,\mathbf{r}''' = \kappa\kappa'$

(*v*) $\mathbf{r}'''' = -3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\kappa'\tau + \tau'\kappa)\mathbf{b}$

(*vi*) $\mathbf{r}'\,.\,\mathbf{r}'''' = -3\kappa\kappa'$ (*vii*) $\mathbf{r}''\,.\,\mathbf{r}'''' = \kappa(\kappa'' - \kappa^3 - \kappa\tau^2)$

(*viii*) $\mathbf{r}'''\,.\,\mathbf{r}'''' = \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa^2\tau\tau' + \kappa\kappa'\tau^2$

(*ix*) $[\mathbf{t}'\,\mathbf{t}''\,\mathbf{t}'''] = \kappa^3(\kappa\tau' - \kappa'\tau) = \kappa^5\dfrac{d}{ds}\left(\dfrac{\tau}{\kappa}\right)$ (*x*) $[\mathbf{b}'\,\mathbf{b}''\,\mathbf{b}'''] = \tau^3(\kappa'\tau - \kappa\tau') = \tau^5\dfrac{d}{ds}\left(\dfrac{\kappa}{\tau}\right)$.

Sol. (*i*) $\mathbf{r}' \cdot \mathbf{r}'' = \mathbf{t} \cdot \mathbf{t}' = \mathbf{t} \cdot (\kappa\mathbf{n}) = \kappa(\mathbf{t} \cdot \mathbf{n}) = \kappa \cdot 0 = 0.$ $(\because \ \mathbf{r}' = \mathbf{t} \text{ and } \mathbf{t}' = \kappa\mathbf{n})$

(*ii*) $\mathbf{r}''' = (\mathbf{r}')'' = \mathbf{t}'' = (\mathbf{t}')' = (\kappa\mathbf{n})' = \kappa\mathbf{n}' + \kappa'\mathbf{n} = \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \kappa'\mathbf{n}$

$= -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}.$ $(\because \ \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b})$

(*iii*) $\mathbf{r}' \cdot \mathbf{r}''' = \mathbf{t} \cdot (-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b})$ (Using (*iii*))

$= -\kappa^2(\mathbf{t} \cdot \mathbf{t}) + \kappa'(\mathbf{t} \cdot \mathbf{n}) + \kappa\tau(\mathbf{t} \cdot \mathbf{b}) = -\kappa^2 \cdot 1 + 0 + 0 = -\kappa^2.$

(*iv*) $\mathbf{r}'' \cdot \mathbf{r}''' = (\mathbf{r}')' \cdot \mathbf{r}''' = \mathbf{t}' \cdot \mathbf{r}'''$

$= (\kappa\mathbf{n}) \cdot (-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b})$ (Using (*iii*))

$= -\kappa^3(\mathbf{n} \cdot \mathbf{t}) + \kappa\kappa'(\mathbf{n} \cdot \mathbf{n}) + \kappa^2\tau(\mathbf{n} \cdot \mathbf{b}) = 0 + \kappa\kappa' \cdot 1 + 0 = \kappa\kappa'.$

(*v*) $r'''' = (r''')' = (-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b})'$ (Using (*iii*))

$= -(\kappa^2\mathbf{t}' + 2\kappa\kappa'\mathbf{t}) + (\kappa'\mathbf{n}' + \kappa''\mathbf{n}) + (\kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} + \kappa\tau\mathbf{b}')$

$= -\kappa^2(\kappa\mathbf{n}) - 2\kappa\kappa'\mathbf{t} + \kappa'(-\kappa\mathbf{t} + \tau\mathbf{b}) + \kappa''\mathbf{n} + \kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} + \kappa\tau(-\tau\mathbf{n})$

$= -\kappa^3\mathbf{n} - 2\kappa\kappa'\mathbf{t} - \kappa\kappa'\mathbf{t} + \kappa'\tau\mathbf{b} + \kappa''\mathbf{n} + \kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} - \kappa\tau^2\mathbf{n}.$

$= -3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\kappa'\tau + \tau'\kappa)\,\mathbf{b}.$

(*vi*) $\mathbf{r}' \cdot \mathbf{r}'''' = \mathbf{t} \,.\, [-3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\kappa'\tau + \tau'\kappa)\mathbf{b}]$ (Using (*v*))

$= -3\kappa\kappa'(\mathbf{t} \cdot \mathbf{t}) + (\kappa'' - \kappa^3 - \kappa\tau^2)(\mathbf{t} \,.\, \mathbf{n}) + (2\kappa'\tau + \tau'\kappa)(\mathbf{t} \,.\, \mathbf{b})$

$= -3\kappa\kappa' \cdot 1 + 0 + 0 = -3\kappa\kappa'.$

(*vii*) $\mathbf{r}'' \cdot \mathbf{r}'''' = \mathbf{t}' \cdot r''''$

$= (\kappa\mathbf{n}) \cdot (-3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\kappa'\tau + \tau'\kappa)\mathbf{b}]$

$= -3\kappa^2\kappa'(\mathbf{n} \cdot \mathbf{t}) + \kappa(\kappa'' - \kappa^3 - \kappa\tau^2)(\mathbf{n} \cdot \mathbf{n}) + \kappa(2\kappa'\tau + \tau'\kappa)(\mathbf{n} \cdot \mathbf{b})$

$= 0 + \kappa(\kappa'' - \kappa^3 - \kappa\tau^2) \cdot 1 + 0 = \kappa(\kappa'' - \kappa^3 - \kappa\tau^2).$

(*viii*) $\mathbf{r}''' \cdot \mathbf{r}'''' = (-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}).\,(-3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\,\mathbf{n} + (2\kappa'\tau + \tau'\kappa)\mathbf{b})$

(Using (*ii*) and (*v*))

$= 3\kappa^3\kappa'(\mathbf{t} \cdot \mathbf{t}) + \kappa'(\kappa'' - \kappa^3 - \kappa\tau^2)(\mathbf{n} \cdot \mathbf{n}) + \kappa\tau(2\kappa'\tau + \tau'\kappa)(\mathbf{b} \cdot \mathbf{b})$

$= 3\kappa^3\kappa' + \kappa'\kappa'' - \kappa'\kappa^3 - \kappa\kappa'\tau^2 + 2\kappa\kappa'\tau^2 + \kappa^2\tau\tau'$

$= \kappa'\kappa'' + 2\kappa^3\kappa' + \kappa^2\tau\tau' + \kappa\kappa'\tau^2.$

(*ix*) $[\mathbf{t}'\ \mathbf{t}''\ \mathbf{t}'''] = [\mathbf{r}''\ \mathbf{r}'''\ \mathbf{r}'''']$

$$= \begin{vmatrix} 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \\ -3\kappa\kappa' & \kappa'' - \kappa^3 - \kappa\tau^2 & 2\kappa'\tau + \tau'\kappa \end{vmatrix}$$

$(\because \ \mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}$ and the vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ form a right handed triad)

$= -\kappa[-\kappa^2(2\kappa'\tau + \tau'\kappa) + 3\kappa\kappa'(\kappa\tau)] = 2\kappa^3\kappa'\tau + \kappa^4\tau' - 3\kappa^3\kappa'\tau$

$= \kappa^4\tau' - \kappa^3\kappa'\tau = \kappa^3(\kappa\tau' - \kappa'\tau)$

$$= \kappa^5 \cdot \left(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2}\right) = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

(*x*) We have $\mathbf{b}' = -\tau\mathbf{n} = 0\mathbf{t} - \tau\mathbf{n} + 0\mathbf{b}$

$\mathbf{b}'' = (\mathbf{b}')' = (-\tau\mathbf{n})' = -\tau\mathbf{n}' - \tau'\mathbf{n}$

$= -\tau(-\kappa\mathbf{t} + \tau\mathbf{b}) - \tau'\mathbf{n} = \tau\kappa\mathbf{t} - \tau'\mathbf{n} - \tau^2\mathbf{b}$

$\mathbf{b}''' = (\mathbf{b}'')' = (\tau\kappa\mathbf{t} - \tau'\mathbf{n} - \tau^2\mathbf{b})'$

$= (\tau'\kappa\mathbf{t} + \tau\kappa'\mathbf{t} + \tau\kappa\mathbf{t}') - (\tau''\mathbf{n} + \tau'\mathbf{n}') - (2\tau\tau'\mathbf{b} + \tau^2\mathbf{b}')$

$= \tau'\kappa\mathbf{t} + \tau\kappa'\mathbf{t} + \tau k(\kappa\mathbf{n}) - \tau''\mathbf{n} - \tau'(-\kappa\mathbf{t} + \tau\mathbf{b}) - 2\tau\tau'\mathbf{b} - \tau^2(-\tau\mathbf{n})$

$= \tau'\kappa\mathbf{t} + \tau\kappa'\mathbf{t} + \tau\kappa^2\mathbf{n} - \tau''\mathbf{n} + \tau'\kappa\mathbf{t} - \tau'\tau\,\mathbf{b} - 2\tau\tau'\mathbf{b} + \tau^3\mathbf{n}$

$= (2\tau'\kappa + \tau\kappa')\,\mathbf{t} + (\tau\kappa^2 - \tau'' + \tau^3)\,\mathbf{n} - 3\tau\tau'\mathbf{b}$

$$\therefore \qquad [\mathbf{b}'\ \mathbf{b}''\ \mathbf{b}'''] = \begin{vmatrix} 0 & -\tau & 0 \\ \tau\kappa & -\tau' & -\tau^2 \\ 2\tau'\kappa + \tau\kappa' & \tau\kappa^2 - \tau'' + \tau^3 & -3\tau\tau' \end{vmatrix}$$

($\because$ **t**, **n**, **b** form a right handed triad)

$$= \tau\,[\tau\kappa\,(-3\tau\tau') + (2\tau'\kappa + \tau\kappa')\tau^2]$$
$$= \tau\,[-3\tau^2\tau'\kappa + 2\tau^2\tau'\kappa + \tau^3\kappa']$$
$$= \tau[-\tau^2\tau'\kappa + \tau^3\kappa'] = \tau^3(\kappa'\tau - \kappa\tau')$$
$$= \tau^5\left(\frac{\tau\kappa' - \kappa\tau'}{\tau^2}\right) = \tau^5\frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. $\tau = -\mathbf{n}\cdot\mathbf{b}'$

Rule II. *Radius of torsion,* $\sigma = \dfrac{1}{\tau}$

Rule III. *Serret-Frenet Formula:*

(*i*) $\mathbf{t}' = \kappa\mathbf{n}$ (*ii*) $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$ (*iii*) $\mathbf{b}' = -\tau\mathbf{n}$

Rule IV. (*i*) $\tau = \dfrac{[\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}''']}{|\mathbf{r}'\times\mathbf{r}''|^2}$ (*ii*) $\tau = \dfrac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}$

Rule V. *Curve is a plane curve if and only if its torsion is identically zero.*

EXERCISE 2.2

1. Find the torsion of the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point where $t = 2$.
2. Show that the torsion of a plane curve (with $\kappa > 0$) is identically zero.
3. Find the torsion of the curve $\mathbf{r} = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j} + (3t + t^3)\mathbf{k}$ at point t.
4. Find the torsion of the curve $\mathbf{r} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + t\mathbf{k}$ at point t.
5. For the curve $x = a(3t - t^3)$, $y = 3at^2$, $z = a(3t + t^3)$, show that curvature κ and torsion τ each is equal to $\dfrac{1}{3a(1+t^2)^2}$.
6. Find the torsion of the helix $x = a\cos t$, $y = a\sin t$, $z = at\tan\alpha$ at point t.
7. For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, show that:
$$\kappa = \tau = \frac{2}{3(1+2t^2)^2}.$$
8. For a point on the curve of intersection of the surfaces $x^2 + y^2 = a^2$, $x^2 - y^2 = az$, find the torsion.
9. Find the torsion at any point t of the curve $x = a\cos 2t$, $y = a\sin 2t$, $z = 2a\sin t$.
10. Let $\mathbf{r} = \mathbf{r}(t)$ be a regular curve of class C^m $(m \geq 3)$, where t is an arbitrary parameter. Prove that
$$\tau = \frac{[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}]}{(\dot{\mathbf{r}}\,.\,\dot{\mathbf{r}})\,(\ddot{\mathbf{r}}\,.\,\ddot{\mathbf{r}}) - (\dot{\mathbf{r}}\,.\,\ddot{\mathbf{r}})^2},\ \text{provided } \kappa \neq 0.$$
11. Show that the curve $\mathbf{r} = \mathbf{r}(t)$ is a plane curve if and only if $[\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}}] = 0$.

12. Is $x = a \cos t, y = a \sin t, z = ct, a > 0, c \neq 0$ a plane curve ? Calculate the curvature and torsion of the above curve at point 't'.

13. Show that the position vector of the current point on the curve $\mathbf{r} = \mathbf{r}(s)$ satisfies the differential equation:

$$\frac{d}{ds}\left[\sigma \frac{d}{ds}\left(\rho \frac{d^2\mathbf{r}}{ds^2}\right)\right] + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{d\mathbf{r}}{ds}\right] + \frac{\rho}{\sigma}\frac{d^2\mathbf{r}}{ds^2} = \mathbf{0}.$$

Answers

1. $\frac{3}{181}$ **3.** $\frac{2}{3(1+t^2)^2}$ **4.** $-\frac{1}{1+4\sin^4 \frac{t}{2}}$

6. $a \sec^2 \alpha$ **8.** $\frac{6\sqrt{a^2 - z^2}}{5a^2 + 12z^2}$ **9.** $\frac{3}{a\,(5 \sec t + 3 \cos t)}$

12. $\frac{a}{a^2 + c^2}, \frac{c}{a^2 + c^2}$.

Hint

13. Using $\rho = \frac{1}{\kappa}, \sigma = \frac{1}{\tau}, \frac{d\mathbf{r}}{ds} = \mathbf{r}' = \mathbf{t}$ and $\frac{d^2\mathbf{r}}{ds^2} = \mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}$, we get

$$\frac{d}{ds}\left[\sigma \frac{d}{ds}\left(\rho \frac{d^2\mathbf{r}}{ds^2}\right)\right] + \frac{d}{ds}\left[\frac{\sigma}{\rho}\frac{d\mathbf{r}}{ds}\right] + \frac{\rho}{\sigma}\frac{d^2\mathbf{r}}{ds^2}$$

$$= \left[\frac{1}{\tau}\left(\frac{1}{\kappa}.\kappa\mathbf{n}\right)'\right]' + \left[\frac{\kappa}{\tau}\mathbf{t}\right]' + \frac{\tau}{\kappa}(\kappa\mathbf{n}) = \left[\frac{1}{\tau}(-\kappa\mathbf{t} + \tau\mathbf{b})\right]' + \left[\frac{\kappa}{\tau}\mathbf{t}\right]' + \tau\mathbf{n}$$

$$= -\left[\frac{\kappa}{\tau}\mathbf{t}\right]' + \mathbf{b}' + \left[\frac{\kappa}{\tau}\mathbf{t}\right]' + \tau\mathbf{n} = -\tau\mathbf{n} + \tau\mathbf{n} = \mathbf{0}.$$

4. CONTACT OF A CURVE WITH A SURFACE

In this section, we shall study the degree of contact of a curve with a surface.

A curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of sufficiently high class is said to have n-**point contact** (or **contact of** $(n-1)^{\text{th}}$ **order**) with a surface $F(x, y, z) = 0$ at the point corresponding to t_0 if the function $f(t) = F(x(t), y(t), z(t))$ satisfies :

$$f(t_0) = \dot{f}(t_0) = \ddot{f}(t_0) = \ldots\ldots = f^{(n-1)}(t_0) = 0 \quad \text{and} \quad f^{(n)}(t_0) \neq 0.$$

Example 1. *Show that the curve* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ *has 6-point contact with the paraboloid* $x^2 + z^2 - y = 0$ *at the origin.*

Sol. Given curve is $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

$$t = 0, t^2 = 0, t^3 = 0 \quad \Rightarrow \quad t = 0$$

$\therefore$ The origin corresponds to $t = 0$.

Given surface is $x^2 + z^2 - y = 0$.

Let $f(t) = t^2 + (t^3)^2 - t^2 = t^6$

$\therefore$ $\dot{f}(t) = 6t^5,\ \ddot{f}(t) = 30t^4,\ \dddot{f}(t) = 120t^3, f^{(4)}(t) = 360t^2, f^{(5)}(t) = 720t, f^{(6)}(t) = 720$

$\therefore$ $\dot{f}(0) = 0, \ddot{f}(0) = 0, \dddot{f}(0) = 0, f^{(4)}(0) = 0, f^{(5)}(0) = 0$ and $f^{(6)}(0) \neq 0$.

$\therefore$ The given curve has 6-point contact with the given paraboloid at the origin.

Example 2. *If the circle $lx + my + nz = 0$, $x^2 + y^2 + z^2 = 2cz$ has 3-point contact with the paraboloid $ax^2 + by^2 = 2z$ at the origin then show that: $c = \dfrac{l^2 + m^2}{bl^2 + am^2}$.*

Sol. Given circle is $lx + my + nz = 0$ ...(1) $x^2 + y^2 + z^2 = 2cz$ ...(2)

Let the parametric equations of this circle be $x = \phi_1(t), y = \phi_2(t), z = \phi_3(t)$.

Substituting the values of x, y, z in (1) and (2) and differentiating w.r.t. t, we get

$$l\dot{x} + m\dot{y} + n\dot{z} = 0 \quad \text{...(3)}$$

and $$2x\dot{x} + 2y\dot{y} + 2z\dot{z} = 2c\dot{z}$$

or $$x\dot{x} + y\dot{y} + z\dot{z} = c\dot{z} \quad \text{...(4)}$$

The circle passes through the origin.

$\therefore$ (4) $\Rightarrow$ $0\dot{x} + 0\dot{y} + 0\dot{z} = c\dot{z}$ $\Rightarrow$ $c\dot{z} = 0$ $\Rightarrow$ $\dot{z} = 0$

$\therefore$ (3) $\Rightarrow$ $l\dot{x} + m\dot{y} + n(0) = 0$ $\Rightarrow$ $l\dot{x} + m\dot{y} = 0$

$$\Rightarrow \quad \frac{\dot{x}}{m} = \frac{\dot{y}}{-l} = \lambda, \text{ say} \quad \text{...(5)}$$

The paraboloid is $ax^2 + by^2 = 2z$. After substituting the values of x, y, z in terms of t, let

$$f(t) = ax^2 + by^2 - 2z \quad \text{...(6)}$$

$\therefore$ $$\dot{f}(t) = 2ax\dot{x} + 2by\dot{y} - 2\dot{z} \quad \text{...(7)}$$

$$\ddot{f}(t) = 2a[\dot{x}^2 + x\ddot{x}] + 2b\,[\dot{y}^2 + y\ddot{y}] - 2\ddot{z}$$

or $$\ddot{f}(t) = 2a\dot{x}^2 + 2ax\ddot{x} + 2b\dot{y}^2 + 2by\ddot{y} - 2\ddot{z} \quad \text{...(8)}$$

Since the circle has 3-point contact with the paraboloid at the origin, we have

$$f(t) = \dot{f}(t) = \ddot{f}(t) = 0,\ \dddot{f}(t) \neq 0 \text{ at the origin.}$$

$\therefore$ (6) $\Rightarrow$ $a(0)^2 + b(0)^2 - 2(0) = 0$ ...(9)

(7) $\Rightarrow$ $2a(0)\dot{x} + 2b(0)\dot{y} - 2\dot{z} = 0$ ...(10)

(8) $\Rightarrow$ $2a\dot{x}^2 + 2a(0)\ddot{x} + 2b\dot{y}^2 + 2b(0)\ddot{y} - 2\ddot{z} = 0$ ...(11)

(9) $\Rightarrow$ $0 = 0$, which is always true.

(10) $\Rightarrow$ $\dot{z} = 0$, which is also true.

(11) $\Rightarrow$ $2a\dot{x}^2 + 2b\dot{y}^2 - 2\ddot{z} = 0$ $\Rightarrow$ $a\dot{x}^2 + b\dot{y}^2 = \ddot{z}$ ...(12)

Differentiating (4) w.r.t. t, we get

$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} + \dot{z}^2 + z\ddot{z} = c\ddot{z}.$$

Since the circle passes through the origin, we have

$$\dot{x}^2 + 0\ddot{x} + \dot{y}^2 + 0\ddot{y} + \dot{z}^2 + 0\ddot{z} = c\ddot{z}.$$

$\Rightarrow$ $$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = c\ddot{z}$$

$\Rightarrow$ $$\dot{x}^2 + \dot{y}^2 = c\ddot{z} \quad \text{...(13)}(\because \ \dot{z} = 0)$$

Dividing (13) by (12), we get

$$c = \frac{\dot{x}^2 + \dot{y}^2}{a\dot{x}^2 + b\dot{y}^2} = \frac{m^2\lambda^2 + l^2\lambda^2}{am^2\lambda^2 + bl^2\lambda^2} = \frac{m^2 + l^2}{am^2 + bl^2}. \quad \text{(Using (5))}$$

$\therefore$ $$c = \frac{l^2 + m^2}{bl^2 + am^2}.$$

Example 3. *Find the equation of the plane that has 3-point contact with the curve $x = t^4 - 1, y = t^3 - 1, z = t^2 - 1$ at the origin.*

Sol. Given curve is

$$x = t^4 - 1, y = t^3 - 1, z = t^2 - 1.$$

$$t^4 - 1 = 0, t^3 - 1 = 0, t^2 - 1 = 0 \quad \Rightarrow \quad t = 1$$

$\therefore$ The origin corresponds to $t = 1$.

Let the equation of the required plane through the origin be $ax + by + cz = 0$.

Let $$f(t) = a(t^4 - 1) + b(t^3 - 1) + c(t^2 - 1)$$

$\therefore$ $$\dot{f}(t) = 4at^3 + 3bt^2 + 2ct$$

$$\ddot{f}(t) = 12at^2 + 6bt + 2c$$

$$\dddot{f}(t) = 24at + 6b$$

Since the plane has 3-point contact at $t = 1$, we have

$$f(1) = 0, \ \dot{f}(1) = 0, \ddot{f}(1) = 0 \ \text{ and } \ \dddot{f}(1) \neq 0.$$

$$f(1) = 0 \quad \Rightarrow \quad 0 = 0$$

$$\dot{f}(1) = 0 \quad \Rightarrow \quad 4a + 3b + 2c = 0 \quad \text{...(1)}$$

$$\ddot{f}(1) = 0 \quad \Rightarrow \quad 12a + 6b + 2c = 0 \quad \text{...(2)}$$

(1) and (2) $\Rightarrow$ $$\frac{a}{6 - 12} = \frac{b}{24 - 8} = \frac{c}{24 - 36}$$

$\Rightarrow$ $$\frac{a}{-6} = \frac{b}{16} = \frac{c}{-12} \quad \Rightarrow \quad \frac{a}{3} = \frac{b}{-8} = \frac{c}{6}$$

Let $$a = 3, b = -8, c = 6.$$

Also $$\dddot{f}(1) = 24a + 6b = 24(3) + 6(-8) = 24 \neq 0$$

$\therefore$ The equation of the required plane is $3x - 8y + 6z = 0$.

Example 4. *Find the lines that have 4-point contact with the surface* $x^4 + 3xyz + x^2 - y^2 - z^2 + 2yz - 3xy - 2y + 2z - 1 = 0$ *at the point (0, 0, 1).*

Sol. Let $$\frac{x-0}{a} = \frac{y-0}{b} = \frac{z-1}{c} = t \quad ...(1)$$

be a line passing through (0, 0, 1).

$\therefore \quad x = at, y = bt, z = ct + 1$

$\therefore$ The point (0, 0, 1) corresponds to the value $t = 0$.

Given surface is

$$x^4 + 3xyz + x^2 - y^2 - z^2 + 2yz - 3xy - 2y + 2z - 1 = 0.$$

Let $f(t) = (at)^4 + 3(at)(bt)(ct + 1) + (at)^2 - (bt)^2 - (ct + 1)^2 + 2(bt)(ct + 1) - 3(at)(bt) - 2(bt) + 2(ct + 1) - 1$

$\therefore \quad f(t) = a^4t^4 + 3abt^2(ct + 1) + a^2t^2 - b^2t^2 - (ct + 1)^2 + 2bt(ct + 1) - 3abt^2 - 2bt + 2ct + 1$

$\therefore \quad \dot{f}(t) = 4a^3t^3 + 9abct^2 + 6abt + 2a^2t - 2b^2t - 2(ct + 1)c + 4bct + 2b - 6abt - 2b + 2c$

$\ddot{f}(t) = 12a^3t^2 + 18abct + 6ab + 2a^2 - 2b^2 - 2c^2 + 4bc - 6ab$

$\dddot{f}(t) = 24a^3t + 18abc$

$\ddddot{f}(t) = 24a^3.$

Let the line (1) has 4-point contact with the given surface at (0, 0, 1).

$\therefore \quad f(0) = 0, \dot{f}(0) = 0, \ddot{f}(0) = 0, \dddot{f}(0) = 0, \ddddot{f}(0) \neq 0.$

$f(0) = 0 \quad \Rightarrow \quad -1 + 1 = 0,$ which is true.

$\dot{f}(0) = 0 \quad \Rightarrow \quad -2c + 2b - 2b + 2c = 0$, which is true.

$\ddot{f}(0) = 0 \quad \Rightarrow \quad 6ab + 2a^2 - 2b^2 - 2c^2 + 4bc - 6ab = 0$

$\Rightarrow \quad a^2 - b^2 - c^2 + 2bc = 0 \quad ...(2)$

$\dddot{f}(0) = 0 \quad \Rightarrow \quad 18abc = 0 \quad \Rightarrow \quad abc = 0 \quad \Rightarrow \quad a = 0$ or $b = 0$ or $c = 0$

Case I. $\mathbf{a = 0}$

(2) $\Rightarrow \quad b^2 + c^2 - 2bc = 0 \quad \Rightarrow \quad b = c$

$\therefore$ The line is $\dfrac{x}{0} = \dfrac{y}{c} = \dfrac{z-1}{c}$ or $\dfrac{x}{0} = \dfrac{y}{1} = \dfrac{z-1}{1}$.

Case II. $\mathbf{b = 0}$

(2) $\Rightarrow \quad a^2 - c^2 = 0 \quad \Rightarrow \quad a = \pm c$

$\therefore$ The lines are $\dfrac{x}{\pm c} = \dfrac{y}{0} = \dfrac{z-1}{c}$ or $\dfrac{x}{\pm 1} = \dfrac{y}{0} = \dfrac{z-1}{1}$.

Case III. $\mathbf{c = 0}$

(2) $\Rightarrow \quad a^2 - b^2 = 0 \quad \Rightarrow \quad a = \pm b$

$\therefore$ The lines are $\dfrac{x}{\pm b} = \dfrac{y}{b} = \dfrac{z-1}{0}$ or $\dfrac{x}{\pm 1} = \dfrac{y}{1} = \dfrac{z-1}{0}$.

$\therefore$ There are five possible lines.

Theorem 1. *Let r = r(s) be any curve and P(s_0) be any point on the curve. Prove that the curve r = r(s) has at least 2-point contact with a plane through P at the point P iff the plane contains the tangent line at P.*

Proof. Let the equation of a plane through P(s_0) be $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N} = 0$, where $\mathbf{r_0} = \mathbf{r}(s_0)$ *i.e.,* the position vector of P and **N** is a unit vector perpendicular to the plane.

Let $\qquad f(s) = (\mathbf{r}(s) - \mathbf{r}_0) \cdot \mathbf{N}$

$\therefore \qquad f'(s) = \mathbf{r}'(s) \cdot \mathbf{N} = \mathbf{t}(s) \cdot \mathbf{N}$

Now $\qquad f(s_0) = (\mathbf{r}_0 - \mathbf{r}_0) \cdot \mathbf{N} = 0$ and $f'(s_0) = \mathbf{t}(s_0) \cdot \mathbf{N}$

$\therefore \qquad f'(s_0) = 0$ iff $\mathbf{t}(s_0) \cdot \mathbf{N} = 0$

iff **N** is orthogonal to $\mathbf{t}(s_0)$ iff the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N} = 0$ contains the tangent line at P.

$\therefore$ The curve $\mathbf{r} = \mathbf{r}(s)$ has at least 2-point contact with a plane through P at the point P on the curve iff the plane contains the tangent line at P.

Theorem 2. *Let r = r(s) be any curve and P(s_0) be any non-inflexional point on the curve. Prove that the curve r = r(s) has at least 3-point contact with a plane through P at the point P iff the plane is the osculating plane at P.*

Proof. Let the equation of a plane through P be $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N} = 0$, where $\mathbf{r}_0 = \mathbf{r}(s_0)$ *i.e.,* the position vector of P and **N** is a unit vector perpendicular to the plane.

Let $\qquad f(s) = (\mathbf{r}(s) - r_0) \cdot \mathbf{N}$

$\therefore \qquad f'(s) = \mathbf{r}'(s) \cdot \mathbf{N} = \mathbf{t}(s) \cdot \mathbf{N}$

and $\qquad f''(s) = \mathbf{t}'(s) \cdot \mathbf{N} = \kappa(s)\, \mathbf{n}(s) \cdot \mathbf{N}$

Now $\qquad f(s_0) = (\mathbf{r}_0 - \mathbf{r}_0) \cdot \mathbf{N} = 0, \mathbf{f}'(s_0) = \mathbf{t}(s_0) \cdot \mathbf{N}$ and $f''(s_0) = \kappa(s_0)\, \mathbf{n}(s_0) \cdot \mathbf{N}.$

$\therefore \qquad f'(s_0) = 0,\ f''(s_0) = 0$ iff $\mathbf{t}(s_0) \cdot \mathbf{N} = 0, \kappa(s_0)\, \mathbf{n}(s_0) \cdot \mathbf{N} = 0$

iff $\qquad \mathbf{t}(s_0) \cdot \mathrm{N} = 0 \quad \mathbf{n}(s_0) \cdot \mathbf{N} = 0 \qquad (\because \kappa(s_0) \neq 0)$

iff **N** is orthogonal to $\mathbf{t}(s_0)$ and $\mathbf{n}(s_0)$ iff the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{N} = 0$ is the osculating plane at P.

$\therefore$ The curve $\mathbf{r} = \mathbf{r}(s)$ has at least 3-point contact with a plane through P at the point P on the curve iff the plane is the osculating plane at P.

Remark. If P is an inflexional point, then $f''(s_0) = \kappa(s_0)\, \mathbf{n}(s_0) \cdot \mathbf{N} = 0$ even if the plane contains only the tangent line at P and is not the osculating plane at P.

Example 5. *Show that the osculating plane has at least 4-point contact with a curve at P iff either the curvature or the torsion vanishes at P.*

Sol. Let the equation of the curve be $\mathbf{r} = \mathbf{r}(s)$. Let $\mathbf{r}_0 = \mathbf{r}(s_0)$ be the position vector of the point P on the curve.

The equation of the osculating plane at P is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{b}_0 = 0$.

Let $\qquad f(s) = (\mathbf{r}(s) - \mathbf{r}_0) \cdot \mathbf{b}_0$

$\therefore \qquad f'(s) = \mathbf{r}'(s) \cdot \mathbf{b}_0 = \mathbf{t} \cdot \mathbf{b}_0,$

$\qquad f''(s) = \mathbf{t}' \cdot \mathbf{b}_0 = (\kappa\mathbf{n}) \cdot \mathbf{b}_0 = \kappa(\mathbf{n} \cdot \mathbf{b}_0)$

and
$$f'''(s) = \kappa'(\mathbf{n} \,.\, \mathbf{b}_0) + \kappa(\mathbf{n}' \,.\, \mathbf{b}_0)$$
$$= \kappa'(\mathbf{n} \,.\, \mathbf{b}_0) + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) \,.\, \mathbf{b}_0$$
$$= \kappa'(\mathbf{n} \,.\, \mathbf{b}_0) - \kappa^2(\mathbf{t} \,.\, \mathbf{b}_0) + \kappa\tau(\mathbf{b} \,.\, \mathbf{b}_0)$$

∴
$$f(s_0) = (\mathbf{r}_0 - \mathbf{r}_0) \,.\, \mathbf{b}_0 = 0,$$
$$f'(s_0) = \mathbf{t}_0 \,.\, \mathbf{b}_0 = 0,$$
$$f''(s_0) = \kappa_0(\mathbf{n}_0 \,.\, \mathbf{b}_0) = 0$$

and
$$f'''(s_0) = \kappa_0'(\mathbf{n}_0 \,.\, \mathbf{b}_0) - \kappa_0^2(\mathbf{t}_0 \,.\, \mathbf{b}_0) + \kappa_0\tau_0(\mathbf{b}_0 \,.\, \mathbf{b}_0) = \kappa_0\tau_0$$

∴
$$f'''(s_0) = 0 \text{ iff } \kappa_0 = 0 \quad \text{or} \quad \tau_0 = 0.$$

∴ The osculating plane at P has at least 4-point contact with the curve at P iff either the curvature or the torsion vanishes at P.

EXERCISE 2.3

1. Show that the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ has 2-point contact with the paraboloid $x^2 + y^2 = z$ at the origin.
2. Find the equation of the plane that has 3-point contact with the curve $x = 2t + 1$, $y = 3t^2 + 2$, $z = 4t^3 + 3$ at the point (3, 5, 7).
3. Let $\mathbf{r} = \mathbf{r}(s)$ be any curve and $P(s_0)$ be any point of inflexion on the curve. Prove that the curve $\mathbf{r} = \mathbf{r}(s)$ has at least 3-point contact with a plane through P at the point P iff the plane contains the tangent line at P.
4. Show that the osculating plane at P has 3-point contact with a curve at P iff neither the curvature nor the torsion vanishes at P.
5. Show that the osculating plane at P has at least 3-point contact with a curve at P.

Answer

2. $6x - 4y + z = 5$.

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5. CONTACT OF A CURVE WITH A CURVE

A curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of sufficiently high class is said to have ***n*-point contact** (or **contact of** $(n-1)$**th order**) with a curve $F(x, y, z) = 0$, $G(x, y, z) = 0$ at the point corresponding to t_0 if the functions $f(t) = F(x(t), y(t), z(t))$ and $g(t) = G(x(t), y(t), z(t))$ satisfy:

$$f(t_0) = \dot{f}(t_0) = \ddot{f}(t_0) = \ldots = f^{(n-1)}(t_0) = 0$$
$$g(t_0) = \dot{g}(t_0) = \ddot{g}(t_0) = \ldots = g^{(n-1)}(t_0) = 0$$

and either
$$f^{(n)}(t_0) \neq 0 \quad \text{or} \quad g^{(n)}(t_0) \neq 0.$$

Thus, the curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ has n-point contact with the curve $F(x, y, z) = 0$, $G(x, y, z) = 0$ if and only if the curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ has n-point contact with one of the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ and at least n-point contact with the other surface.

6. OSCULATING CIRCLE TO A CURVE

A circle having at least 3-point contact with a given curve C at a point P on the curve is called the **osculating circle** to the curve C at the point P.

The centre of the osculating circle at point P is called the **centre of curvature** of the curve C at the point P.

By the definition of contact between curves, the osculating circle to the curve C at point P can be considered as the intersection of a sphere with at least 3-point contact with the curve C at point P and a plane with at least 3-point contact with C at P. If $\kappa \neq 0$ at P, then the osculating plane at P is the unique plane having at least 3-point contact with the curve C at P. In particular, if $\tau \neq 0$ in addition to $\kappa \neq 0$ at P, then the osculating plane is the unique plane having exactly 3-point contact with C at P.

Therefore the osculating circle to a curve at a point always lies on the osculating plane to the curve at that point, provided $\kappa \neq 0$ at the point under consideration.

Thus, the osculating circle to a curve at a point can be considered as the intersection of a sphere with at least 3-point contact with the curve at that point and the osculating plane to the curve at the point under consideration, provided $\kappa \neq 0$.

7. EQUATION OF OSCULATING CIRCLE

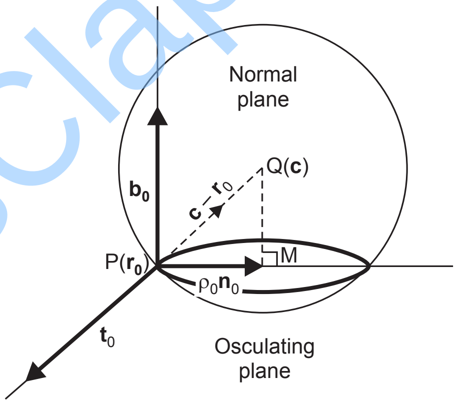
Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a curve C, where s is the parameter 'arc length'. Let P be any point on the curve C for the value s_0 of s. Let $\mathbf{r_0} = \mathbf{r}(s_0)$. Let curvature $\kappa_0 (= \kappa(s_0))$ be non-zero at P.

The equation of the osculating plane at P is $(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{b_0} = 0$, where $\mathbf{b_0} = \mathbf{b}(s_0)$.

Let the osculating circle at P be the intersection of the osculating plane at P and the sphere

$$|\mathbf{r} - \mathbf{c}|^2 = a^2$$

with centre at Q(**c**) and passing through P and having at least 3-point contact with the curve $\mathbf{r} = \mathbf{r}(s)$ at P.



$\therefore \quad |\mathbf{r_0} - \mathbf{c}|^2 = a^2$

Let $\quad f(s) = |\mathbf{r}(s) - \mathbf{c}|^2 - a^2.$

$\therefore \quad f(s) = (\mathbf{r}(s) - \mathbf{c}) \cdot (\mathbf{r}(s) - \mathbf{c}) - a^2$

$\therefore \quad f'(s) = (\mathbf{r}(s) - \mathbf{c}) \cdot (\mathbf{r}'(s)) + \mathbf{r}'(s) \cdot (\mathbf{r}(s) - \mathbf{c}) - 0 = 2(\mathbf{r}(s) - \mathbf{c}) \cdot \mathbf{t}(s)$

$f''(s) = 2(\mathbf{r}(s) - \mathbf{c}) \cdot \mathbf{t}'(s) + 2(\mathbf{t}(s)) \cdot \mathbf{t}(s)$

$\quad = 2(\mathbf{r}(s) - \mathbf{c}) \cdot \kappa(s)\, \mathbf{n}(s) + 2(1) = 2\kappa(s)\, (\mathbf{r}(s) - \mathbf{c}) \cdot \mathbf{n}(s) + 2$

Since the sphere has at least 3-point of contact at P, we have $f(s_0) = f'(s_0) = f''(s_0) = 0$.

$f(s_0) = 0 \quad \Rightarrow \quad (\mathbf{r}_0 - \mathbf{c}) \cdot (\mathbf{r}_0 - \mathbf{c}) - a^2 = 0 \;\Rightarrow\; |\mathbf{r}_0 - \mathbf{c}|^2 - a^2 = 0.$

$\Rightarrow \quad a^2 - a^2 = 0$, which is true.

$f'(s_0) = 0 \quad \Rightarrow \quad 2(\mathbf{r}_0 - \mathbf{c}) \cdot \mathbf{t}_0 = 0 \;\Rightarrow\; (\mathbf{c} - \mathbf{r_0}) \cdot \mathbf{t}_0 = 0$

$\Rightarrow \quad \mathbf{c} - \mathbf{r}_0$ lies in the normal plane at P

$\Rightarrow$ centre (Q) of the sphere is in the normal plane at P.

$f''(s_0) = 0 \quad \Rightarrow \quad 2\kappa_0(\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{n_0} + 2 = 0$

$$\Rightarrow \quad (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{n_0} = -\frac{1}{\kappa_0} = -\rho_0 \qquad \text{(Using } \kappa_0 \neq 0\text{)}$$

$\Rightarrow \quad (\mathbf{c} - \mathbf{r_0}) \cdot \mathbf{n_0} = \rho_0$

$\therefore$ Projection of $\mathbf{c} - \mathbf{r_0}$ on $\mathbf{n_0} = \rho_0$ ...(1)

Let QM be perpendicular to the principal normal at P.

$\therefore$ M is the centre of the osculating circle at P.

Also, (1) $\Rightarrow$ $\mathbf{PM} = \rho_0 \mathbf{n}_0 \Rightarrow$ PM $= \rho_0 \Rightarrow$ M is the centre of curvature at P.

Also, the centre of curvature (M) lies on the principal normal at P and at a distance ρ_0 from P.

$\therefore$ The radius of the osculating circle at P is ρ_0 which is also equal to the radius of curvature of the given curve at P.

Also, the position vector of the centre of the osculating circle at P($\mathbf{r_0}$)

$$= \text{P.V. of M} = \mathbf{r_0} + \rho_0 \mathbf{n_0}$$

and it lies on the principal normal of the given curve at P.

Remark. If $\kappa_0 = 0$, then $f''(s_0) = 0 \Rightarrow 0 + 2 = 0$, which is impossible.

$\therefore$ If $\kappa_0 = 0$, then there does not exist any sphere having at least 3-point of contact with the given curve at P.

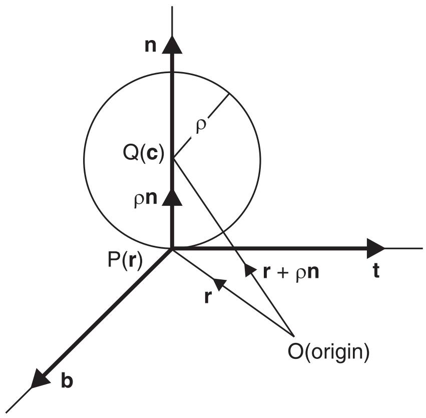
8. LOCUS OF CENTRE OF CURVATURE

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a curve C. For each point P with non-zero curvature, on the curve C, there exists an osculating circle. Let C_1 denote the locus of the centre of curvature *i.e.,* the centre of osculating circle as the point P moves along the curve C. We shall prove two properties regarding the curve C_1, the locus of centre of curvature.

Property I. *The tangent to the locus of centre of curvature lies in the normal plane of the original curve.*

Proof. Let P($\mathbf{r}$) be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$ at P. Let $\mathbf{c}$ be the position vector of the centre of curvature Q at the point P.

$\therefore \quad \mathbf{c} = \mathbf{r} + \rho\mathbf{n}$ ...(1)



We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{c}}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n}) \frac{ds}{ds_1}.$$

$$\Rightarrow \quad \mathbf{t_1} = (\mathbf{r}' + \rho\mathbf{n}' + \rho'\mathbf{n}) \frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t_1} = (\mathbf{r}' + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + \rho'\mathbf{n}) \frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t_1} = (\mathbf{t} - \mathbf{t} + \rho\tau\mathbf{b} + \rho'\mathbf{n}) \frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t_1} = (\rho\tau\mathbf{b} + \rho'\mathbf{n}) \frac{ds}{ds_1}$$

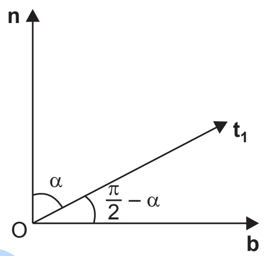
$\therefore$ $\mathbf{t_1}$ lies in the plane of $\mathbf{b}$ and $\mathbf{n}$.

$\therefore$ The tangent at a point to the curve C_1 lies in the corresponding normal plane of curve C.

$\therefore$ The tangent to the locus of the centre of curvature lies in the normal plane of the original curve.

Remark. Let α be the angle between the tangent to the locus of the centre of curvature at the centre of curvature at point P and the principal normal at P.

$$\therefore \qquad \mathbf{t}_1 \cdot \mathbf{n} = 1 \,.\, 1. \cos\alpha = \cos\alpha$$



$$\Rightarrow \qquad (\rho\tau\mathbf{b} + \rho'\mathbf{n})\frac{ds}{ds_1} \cdot \mathbf{n} = \cos\alpha$$

$$\Rightarrow \qquad 0 + \rho'\frac{ds}{ds_1} = \cos\alpha, \textit{ i.e.}, \cos\alpha = \rho'\frac{ds}{ds_1}.$$

Also, angle between $\mathbf{t}_1$ and $\mathbf{b} = \dfrac{\pi}{2} - \alpha$

$$\therefore \qquad \mathbf{t}_1 \cdot \mathbf{b} = 1 \,.\, 1. \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$

$$\Rightarrow \qquad (\rho\tau\mathbf{b} + \rho'\mathbf{n})\frac{ds}{ds_1} \cdot \mathbf{b} = \sin\alpha$$

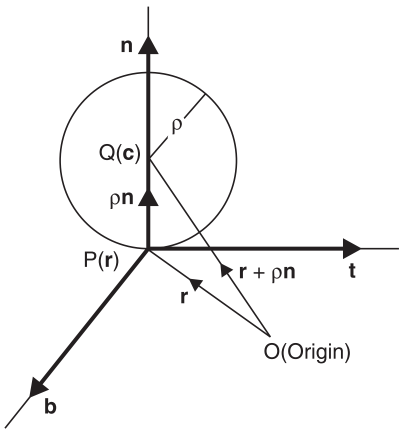
$$\Rightarrow \qquad \rho\tau\frac{ds}{ds_1} + 0 = \sin\alpha, \textit{ i.e.}, \sin\alpha = \rho\tau\frac{ds}{ds_1}.$$

Dividing, we get $\quad \tan\alpha = \dfrac{\rho\,\tau}{\rho'} = \dfrac{\rho}{\rho'\,\sigma}$

$$\therefore \qquad \alpha = \tan^{-1}\left(\frac{\rho}{\rho'\,\sigma}\right).$$

Property II. *If the original curve C has a constant curvature* κ, *then the curvature of the locus* C_1 *of centre of curvature is also constant and the torsion of* C_1 *varies inversely as that of C.*

Proof. Let P($\mathbf{r}$) be any point on the curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\mathbf{c}$ be the position vector of the centre of curvature at the point P.



$$\therefore \qquad \mathbf{c} = \mathbf{r} + \rho\mathbf{n} \qquad \text{...(1)}$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{c}}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n})\frac{ds}{ds_1}.$$

$$\Rightarrow \qquad \mathbf{t}_1 = (\mathbf{r}' + \rho\mathbf{n}' + \rho'\mathbf{n})\frac{ds}{ds_1} \qquad \text{...(2)}$$

Since κ is constant, we have $\quad \rho' = \left(\dfrac{1}{\kappa}\right)' = 0.$

$$\therefore \quad (2) \quad \Rightarrow \qquad \mathbf{t}_1 = (\mathbf{r}' + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}))\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\mathbf{t} - \mathbf{t} + \rho\tau\mathbf{b})\,\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = \rho\tau\mathbf{b}\,\frac{ds}{ds_1} \qquad ...(3)$$

$$\Rightarrow \quad \mathbf{t}_1 \cdot \mathbf{t}_1 = \left(\rho\tau\mathbf{b}\,\frac{ds}{ds_1}\right)\cdot\left(\rho\tau\mathbf{b}\,\frac{ds}{ds_1}\right)$$

$$\Rightarrow \quad 1 = \rho^2\tau^2\left(\frac{ds}{ds_1}\right)^2 \quad \Rightarrow \quad \frac{ds}{ds_1} = \frac{1}{\rho\tau}$$

$$\therefore \quad (3) \Rightarrow \quad \mathbf{t}_1 = (\rho\tau\mathbf{b})\,\frac{1}{\rho\tau} = \mathbf{b} \qquad ...(4)$$

Differentiating w.r.t. s_1, we get

$$\frac{d\mathbf{t}_1}{ds_1} = \frac{d\mathbf{b}}{ds}\,\frac{ds}{ds_1}.$$

$$\Rightarrow \quad \mathbf{t}_1' = \mathbf{b}'\,\frac{1}{\rho\tau} \quad \Rightarrow \quad \rho\tau\mathbf{t}_1' = -\tau\mathbf{n} \quad \Rightarrow \quad \mathbf{t}_1' = -\frac{1}{\rho}\,\mathbf{n}$$

$$\Rightarrow \quad \mathbf{t}_1' = -\kappa\mathbf{n}$$

$$\Rightarrow \quad \kappa_1\mathbf{n}_1 = -\kappa\mathbf{n}$$

$\therefore$ Vectors $\mathbf{n}_1$ and $\mathbf{n}$ are parallel. Choosing the direction of $\mathbf{n}_1$ opposite to that of $\mathbf{n}$, we have $\mathbf{n}_1 = -\mathbf{n}$.

$$\therefore \quad \kappa_1 = \kappa$$

$\therefore$ The curvature of the curve C_1 is also constant. ($\because$ κ is constant)

Also $\quad \mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 = \mathbf{b} \times (-\mathbf{n}) = -\mathbf{b} \times \mathbf{n} = \mathbf{n} \times \mathbf{b} = \mathbf{t}$ (Using (4))

Differentiating w.r.t. s_1, we get

$$\frac{d\mathbf{b}_1}{ds_1} = \frac{d\mathbf{t}}{ds}\,\frac{ds}{ds_1}$$

$$\Rightarrow \quad -\tau_1\,\mathbf{n}_1 = \mathbf{t}'\,\frac{ds}{ds_1} \quad \Rightarrow \quad -\tau_1\,\mathbf{n}_1 = (\kappa\mathbf{n})\,\frac{1}{\rho\tau}$$

$$\Rightarrow \quad \tau_1\,\mathbf{n} = \frac{\kappa}{\rho\tau}\,\mathbf{n} \quad \Rightarrow \quad \tau_1 = \left(\frac{\kappa}{\rho}\right)\frac{1}{\tau} = \kappa^2\cdot\frac{1}{\tau} \qquad (\because\ \mathbf{n}_1 = -\mathbf{n})$$

$\therefore$ Torsion of curve C_1 varies inversely as that of C.

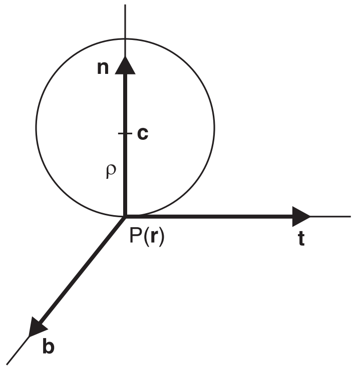
Example 1. *Show that the principal normal to a curve is perpendicular to the locus of the centre of curvature at points, where curvature* κ *is constant.*

Sol. Let P($\mathbf{r}$) be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\mathbf{c}$ be the position vector of the centre of curvature at the point P.

$$\therefore \quad \mathbf{c} = \mathbf{r} + \rho\mathbf{n} \qquad ...(1)$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of curvature.

Differentiating (1) w.r.t. s_1, we get



$$\frac{d\mathbf{c}}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\mathbf{r}' + \rho\mathbf{n}' + \rho'\mathbf{n})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\mathbf{t} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + 0\mathbf{n})\frac{ds}{ds_1}$$

$$\left(\rho' = 0 \text{ because } \rho = \frac{1}{\kappa} \text{ is constant}\right)$$

$$\Rightarrow \quad \mathbf{t}_1 = \left(\rho\tau\frac{ds}{ds_1}\right)\mathbf{b} \qquad (\because \rho\kappa = 1)$$

$$\Rightarrow \quad \mathbf{n}.\,\mathbf{t}_1 = \left(\rho\tau\frac{ds}{ds_1}\right)\mathbf{n}\,.\,\mathbf{b} = 0$$

$\Rightarrow$ **n** is perpendicular to the tangent vector to the curve C_1 at the point with position vector **c**.

$\therefore$ Principal normal to the curve C is perpendicular to the curve C_1 *i.e.,* the locus of centre of curvature.

Example 2. *If s_1 is the arc length of the locus of centre of curvature, show that*

$$\frac{ds_1}{ds} = \frac{\sqrt{\kappa^2\tau^2 + \kappa'^2}}{\kappa^2} = \sqrt{\left(\frac{\rho}{\sigma}\right)^2 + \rho'^2}\,.$$

Sol. Let the given curve be $\mathbf{r} = \mathbf{r}(s)$. Let suffix '1' be used for quantities corresponding to the locus of centre of curvature.

Let $\mathbf{r}_1$ be the position vector of the centre of curvature corresponding to the point **r** on the curve $\mathbf{r} = \mathbf{r}(s)$.

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n}$$

$$\Rightarrow \quad \mathbf{r}_1 = \mathbf{r} + \frac{1}{\kappa}\mathbf{n}$$

Differentiating w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}\left(\mathbf{r} + \frac{1}{\kappa}\mathbf{n}\right)\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = \left(\mathbf{r}' + \frac{1}{\kappa}\mathbf{n}' + \left(-\frac{\kappa'}{\kappa^2}\right)\mathbf{n}\right)\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = \left(\mathbf{t} + \frac{1}{\kappa}(-\kappa\mathbf{t} + \tau\mathbf{b}) - \frac{\kappa'}{\kappa^2}\mathbf{n}\right)\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = \left(-\frac{\kappa'}{\kappa^2}\mathbf{n} + \frac{\tau}{\kappa}\mathbf{b}\right)\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1\,.\,\mathbf{t}_1 = \left(-\frac{\kappa'}{\kappa^2}\mathbf{n} + \frac{\tau}{\kappa}\mathbf{b}\right).\left(-\frac{\kappa'}{\kappa^2}\mathbf{n} + \frac{\tau}{\kappa}\mathbf{b}\right)\left(\frac{ds}{ds_1}\right)^2$$

$$\Rightarrow \qquad 1 = \left(\frac{\kappa'^2}{\kappa^4} + \frac{\tau^2}{\kappa^2}\right)\left(\frac{ds}{ds_1}\right)^2 = \left(\frac{\kappa'^2 + \kappa^2\tau^2}{\kappa^4}\right)\left(\frac{ds}{ds_1}\right)^2$$

$$\Rightarrow \qquad \left(\frac{ds_1}{ds}\right)^2 = \frac{\kappa^2\tau^2 + \kappa'^2}{\kappa^4} \qquad \text{...(1)}$$

$$\Rightarrow \qquad \frac{ds_1}{ds} = \frac{\sqrt{\kappa^2\tau^2 + \kappa'^2}}{\kappa^2}$$

$$(1) \Rightarrow \qquad \left(\frac{ds_1}{ds}\right)^2 = \frac{\tau^2}{\kappa^2} + \frac{\kappa'^2}{\kappa^4} = \frac{\rho^2}{\sigma^2} + \left(\left(\frac{1}{\kappa}\right)'\right)^2 = \frac{\rho^2}{\sigma^2} + \rho'^2$$

$$\therefore \qquad \frac{ds_1}{ds} = \sqrt{\left(\frac{\rho}{\sigma}\right)^2 + \rho'^2}\,.$$

$\therefore$ The result holds.

9. OSCULATING SPHERE TO A CURVE

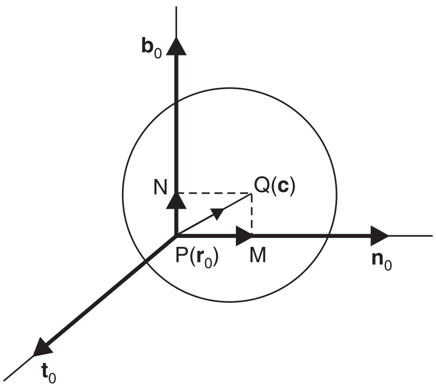
A sphere having at least 4-point contact with a given curve C at a point P on the curve is called the **osculating sphere** to the curve C at the point P.

The centre of the osculating sphere at point P is called the **centre of spherical curvature** of the curve C at the point P.

Remark. The radius of osculating sphere is also referred as the **radius of spherical curvature.**

10. EQUATION OF OSCULATING SPHERE

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a curve C, where s is the parameter 'arc length'. Let P be any point on the curve C for the value s_0 of s. Let $\mathbf{r_0} = \mathbf{r}(s_0)$. Let curvature κ_0 $(= \kappa(s))$ and torsion τ_0 $(= \tau(s_0))$ be non-zero at P.



Let the equation of the osculating sphere be

$$|\,\mathbf{r} - \mathbf{c}\,|^2 = a^2$$

with centre at Q($\mathbf{c}$) and passing through P and having at least 4-point contact with the curve $\mathbf{r} = \mathbf{r}(s)$ at P.

$\therefore \quad |\,\mathbf{r}_0 - \mathbf{c}\,|^2 = a^2$

Let $\qquad f(s) = |\,\mathbf{r}(s) - \mathbf{c}\,|^2 - a^2.$

$\therefore \qquad f(s) = (\mathbf{r}(s) - \mathbf{c}) \,.\, (\mathbf{r}(s) - \mathbf{c}) - a^2$

$$f'(s) = (\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{r}'(s)) + \mathbf{r}'(s) \,.\, (\mathbf{r}(s) - \mathbf{c}) - 0 = 2\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{t}(s)$$

$$f''(s) = 2(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{t}'(s) + 2\,(\mathbf{t}(s)) \,.\, \mathbf{t}(s)$$
$$= 2(\mathbf{r}(s) - \mathbf{c}) \,.\, \kappa(s)\,\mathbf{n}(s) + 2(1) = 2\kappa(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{n}(s) + 2$$

$$f'''(s) = 2\kappa'(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{n}(s) + 2\kappa(s)\,(\mathbf{t}(s) - 0) \,.\, \mathbf{n}(s) + 2\kappa(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{n}'(s) + 0$$
$$= 2\kappa'(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{n}(s) + 2\kappa(s)\,(0) + 2\kappa(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, (-\,\kappa(s)\,\mathbf{t}(s) + \tau(s)\,\mathbf{b}(s))$$
$$= 2\kappa'(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{n}(s) - 2\kappa^2(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{t}(s) + 2\kappa(s)\,\tau(s)\,(\mathbf{r}(s) - \mathbf{c}) \,.\, \mathbf{b}(s)$$

Since the sphere has at least 4-point contact at P, we have

$$f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = 0.$$

$f(s_0) = 0 \quad \Rightarrow \quad (\mathbf{r}_0 - \mathbf{c}) \cdot (\mathbf{r_0} - \mathbf{c}) - a^2 = 0 \quad \Rightarrow \quad |\,\mathbf{r_0} - \mathbf{c}\,|^2 - a^2 = 0$

$\Rightarrow \quad a^2 - a^2 = 0$, which is true.

$f'(s_0) = 0 \quad \Rightarrow \quad 2(\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{t_0} = 0 \quad \Rightarrow \quad (\mathbf{c} - \mathbf{r_0}) \cdot \mathbf{t_0} = 0$ ...(1)

$\Rightarrow$ $\mathbf{c} - \mathbf{r_0}$ lies in the normal plane at P

$\Rightarrow$ centre (Q) of the sphere is in the normal plane at P.

$f''(s_0) = 0 \quad \Rightarrow \quad 2\kappa_0(\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{n_0} + 2 = 0$

$$\Rightarrow \quad (\mathbf{r}_0 - \mathbf{c}) \cdot \mathbf{n_0} = -\frac{1}{\kappa_0} = -\rho_0 \qquad \text{...(2)} \quad (\text{Using } \kappa_0 \neq 0)$$

$\Rightarrow \quad (\mathbf{c} - \mathbf{r}_0) \cdot \mathbf{n}_0 = \rho_0$

$\Rightarrow$ Projection of $\mathbf{c} - \mathbf{r_0}$ on $\mathbf{n_0} = \rho_0$

$f'''(s_0) = 0 \quad \Rightarrow \quad 2\kappa_0' (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{n_0} - 2\kappa_0^2 (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{t}_0 + 2\kappa_0\tau_0 (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{b_0} = 0$

$$\Rightarrow \quad 2\kappa_0'\left(-\frac{1}{\kappa_0}\right) - 2\kappa_0^2(0) + 2\kappa_0\tau_0(\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{b_0} = 0$$

(Using (1) and (2))

$$\Rightarrow \quad -\frac{2\kappa_0'}{\kappa_0} + 2\kappa_0\tau_0 (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{b_0} = 0$$

$$\Rightarrow \quad (\mathbf{r_0} - \mathbf{c}) \cdot \mathbf{b_0} = \frac{\kappa_0'}{\kappa_0^2\tau_0} \qquad (\text{Using } \tau_0 \neq 0)$$

$$\Rightarrow \quad (\mathbf{c} - \mathbf{r}_0) \cdot \mathbf{b_0} = -\frac{\kappa_0'}{\kappa_0^2\tau_0} = \left(\frac{d}{ds}\left(\frac{1}{\kappa_0}\right)\right)\sigma_0 = \rho_0'\sigma_0$$

$\Rightarrow$ projection of $\mathbf{c} - \mathbf{r_0}$ on $\mathbf{b_0} = \rho_0'\sigma_0$

$\therefore$ The components of the vector $\mathbf{c} - \mathbf{r_0}$ along the vectors $\mathbf{t}_0$, $\mathbf{n}_0$ and $\mathbf{b}_0$ are 0, ρ_0 and $\rho_0'\,\sigma_0$ respectively.

$$\therefore \quad \mathbf{c} - \mathbf{r_0} = 0\mathbf{t}_0 + \rho_0\mathbf{n} + \rho_0'\sigma_0\mathbf{b}$$

$$\therefore \quad |\,\mathbf{c} - \mathbf{r_0}\,| = \sqrt{\rho_0^2 + \rho_0'^2\,\sigma_0^2}$$

and
$$\mathbf{c} = \mathbf{r_0} + \rho_0\,\mathbf{n} + \rho_0'\sigma_0\,\mathbf{b}_0.$$

The centre Q($\mathbf{c}$) of the osculating sphere is the centre of spherical curvature of the curve C at P.

Radius of the osculating sphere at P = PQ = $|\,\mathbf{c} - \mathbf{r}_0\,| = \sqrt{\rho_0^2 + \rho_0'^2\,\sigma_0^2}$.

Also, the position vector of the centre of spherical curvature of the curve C at P($\mathbf{r_0}$) = P.V. of Q = $\mathbf{c} = \mathbf{r}_0 + \rho_0\mathbf{n}_0 + \rho_0'\sigma_0\mathbf{b_0}$ and it lies on the normal plane of the given curve at P.

Remark 1. In terms of κ_0 and τ_0, we have

(*i*) radius of osculating sphere at $P(\mathbf{r_0}) = \sqrt{\left(\frac{1}{\kappa_0}\right)^2 + \left(\frac{\kappa_0'}{\kappa_0^2\,\tau_0}\right)^2}$ and

(*ii*) p.v. of centre of spherical curvature $= \mathbf{r}_0 + \frac{1}{\kappa_0}\mathbf{n_0} - \frac{\kappa_0'}{\kappa_0^2\,\tau_0}\mathbf{b_0}$.

Remark 2. If curvature of $\mathbf{r} = \mathbf{r}(s)$ at P is constant, then radius of osculating sphere at

$P(\mathbf{r}_0) = \sqrt{\rho_0^2 + 0.\sigma_0^2} = \rho_0$ and p.v. of centre of spherical curvature at

$P(\mathbf{r}_0) = \mathbf{r}_0 + \rho_0\mathbf{n}_0 + (0)\sigma_0\mathbf{b} = \mathbf{r}_0 + \rho_0\mathbf{n}_0$.

∴ Centre of the osculating sphere coincides with the centre of osculating circle at points where curvature vanishes.

11. LOCUS OF CENTRE OF SPHERICAL CURVATURE

Let $\mathbf{r} = \mathbf{r}(s)$ be the equation of a curve C. For each point P with non-zero curvature and torsion on the curve C there exists an osculating sphere. Let C_1 denote the locus of the centre of spherical curvature *i.e.*, the centre of osculating sphere as the point P moves along the curve C. We shall prove some properties regarding the curve C_1, the locus of centre of spherical curvature.

Property I. *The tangent to the locus of centre of spherical curvature is parallel to the corresponding binormal to the original curve.*

Proof. Let $P(\mathbf{r})$ be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$, $\tau \neq 0$ at P. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P.

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b} \qquad \text{...(1)}$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of spherical curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\mathbf{t} + \rho\mathbf{n}' + \rho'\mathbf{n} + \rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} + \rho'\sigma\mathbf{b}')\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\mathbf{t} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + \rho'\mathbf{n} + \rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} - \rho'\sigma\tau\mathbf{n})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = ((1 - \rho\kappa)\mathbf{t} + \rho'(1 - \sigma\tau)\mathbf{n} + (\rho\tau + \rho''\sigma + \rho'\sigma')\mathbf{b})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\mathbf{b} \qquad (\because \kappa\rho = 1, \tau\sigma = 1)$$

∴ $\mathbf{t}_1$ is parallel to $\mathbf{b}$.

∴ The tangent to C_1 is parallel to the corresponding binormal to C.

Property II. *The principal normal to the locus of centre of spherical curvature is parallel to the corresponding principal normal to the original curve.*

Proof. Let $P(\mathbf{r})$ be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$, $\tau \neq 0$ at P. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P.

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b} \qquad \text{...(1)}$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of spherical curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b})\frac{ds}{ds_1}$$

$$\Rightarrow \qquad \mathbf{t}_1 = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,\mathbf{b} \qquad ...(2)$$

(For detail see **property I**)

$$\Rightarrow \qquad |\mathbf{t}_1| = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,|\mathbf{b}| \quad \Rightarrow \quad 1 = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,.\,1$$

$$\therefore \qquad \frac{ds}{ds_1} = \frac{1}{\rho\tau + \rho''\sigma + \rho'\sigma'}$$

$$\therefore \quad (2) \quad \Rightarrow \qquad \mathbf{t}_1 = \mathbf{b} \qquad ...(3)$$

Differentiating (3) w.r.t. s_1, we get

$$\frac{d\mathbf{t}_1}{ds_1} = \frac{d\mathbf{b}}{ds}\frac{ds}{ds_1}$$

$$\Rightarrow \qquad \kappa_1\mathbf{n}_1 = -\tau\mathbf{n}\frac{ds}{ds_1}$$

$\therefore$ $\mathbf{n}_1$ is parallel to $\mathbf{n}$.

$\therefore$ The principal normal to C_1 is parallel to the corresponding principal normal to C.

Property III. *The binormal to the locus of centre of spherical curvature is parallel to the corresponding tangent to the original curve.*

Proof. Let P($\mathbf{r}$) be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$, $\tau \neq 0$ at P. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P.

$$\therefore \qquad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b} \qquad ...(1)$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of spherical curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b})\frac{ds}{ds_1}$$

$$\Rightarrow \qquad \mathbf{t}_1 = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,\mathbf{b} \qquad ...(2)$$

(For detail see **property I**)

$$\Rightarrow \qquad |\mathbf{t}_1| = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}|\mathbf{b}|$$

$$\Rightarrow \qquad \frac{ds}{ds_1} = \frac{1}{\rho\tau + \rho''\sigma + \rho'\sigma'}$$

$$\therefore \quad (2) \quad \Rightarrow \qquad \mathbf{t}_1 = \mathbf{b} \qquad ...(3)$$

Differentiating (3) w.r.t. s_1, we get

$$\frac{d\mathbf{t}_1}{ds_1} = \frac{d\mathbf{b}}{ds}\frac{ds}{ds_1}$$

$$\Rightarrow \qquad \kappa_1\mathbf{n}_1 = -\tau\mathbf{n}\frac{ds}{ds_1} \qquad ...(4)$$

$$\Rightarrow \quad \kappa_1 |\mathbf{n}_1| = |-1| \, \tau \frac{ds}{ds_1} |\mathbf{n}|$$

$$\Rightarrow \quad \kappa_1 \,.\, 1 = 1 \,.\, \tau \frac{ds}{ds_1} \,.\, 1 \quad \Rightarrow \quad \kappa_1 = \tau \frac{ds}{ds_1}$$

$\therefore$ 4) $\Rightarrow$ $\mathbf{n}_1 = -\mathbf{n}$

$\Rightarrow$ $\mathbf{t}_1 \times \mathbf{n}_1 = \mathbf{b} \times (-\mathbf{n})$ (Using (3))

$\Rightarrow$ $\mathbf{b}_1 = \mathbf{t}$

$\therefore$ $\mathbf{b}_1$ is parallel to $\mathbf{t}$.

$\therefore$ The binormal to C_1 is parallel to the corresponding tangent to C.

Property IV. *The product of curvatures at the corresponding points on the locus of centre of spherical curvature and the original curve is equal to the product of their torsions.*

Proof. Let P($\mathbf{r}$) be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$, $\tau \neq 0$ at P. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P.

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\,\mathbf{b} \qquad ...(1)$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of locus of centre of spherical curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b}) \frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\rho\tau + \rho''\sigma + \rho'\sigma') \frac{ds}{ds_1} \mathbf{b} \qquad ...(2)$$

(For detail see **property I**)

$$\Rightarrow \quad |\mathbf{t}_1| = (\rho\tau + \rho''\sigma + \rho'\sigma') \frac{ds}{ds_1} |\mathbf{b}|$$

$$\Rightarrow \quad \frac{ds}{ds_1} = \frac{1}{\rho\tau + \rho''\sigma + \rho'\sigma'}$$

$$\therefore \ (2) \Rightarrow \quad \mathbf{t}_1 = \mathbf{b} \qquad ...(3)$$

Differentiating (3) w.r.t. s_1, we get

$$\frac{d\mathbf{t}_1}{ds_1} = \frac{d\mathbf{b}}{ds} \frac{ds}{ds_1}$$

$$\Rightarrow \quad \kappa_1\mathbf{n}_1 = -\tau\mathbf{n} \frac{ds}{ds_1} \qquad ...(4)$$

$$\Rightarrow \quad \kappa_1 |\mathbf{n}_1| = |-1| \, \tau \frac{ds}{ds_1} |\mathbf{n}|$$

$$\Rightarrow \quad \kappa_1 \,.\, 1 = 1 \,.\, \tau \frac{ds}{ds_1} \,.\, 1 \quad \Rightarrow \quad \kappa_1 = \tau \frac{ds}{ds_1}$$

$\therefore$ (4) $\Rightarrow$ $\mathbf{n}_1 = -\mathbf{n}$

$\Rightarrow$ $\mathbf{t}_1 \times \mathbf{n}_1 = \mathbf{b} \times (-\mathbf{n})$ (Using (3)

$\Rightarrow$ $\mathbf{b}_1 = \mathbf{t}$

Differentiating w.r.t. s_1, we get

$$\frac{d\mathbf{b}_1}{ds_1} = \frac{d\mathbf{t}}{ds}\frac{ds}{ds_1}$$

$$\Rightarrow \quad -\tau_1\mathbf{n}_1 = (\kappa\mathbf{n})\frac{\kappa_1}{\tau} \qquad \left(\text{Using } \kappa_1 = \tau\frac{ds}{ds_1}\right)$$

$$\Rightarrow \quad -\tau_1(-\mathbf{n}) = \frac{\kappa\kappa_1}{\tau}\mathbf{n} \qquad (\text{Using } \mathbf{n}_1 = -\mathbf{n})$$

$$\Rightarrow \quad \tau\tau_1\mathbf{n} = \kappa\kappa_1\mathbf{n} \quad \Rightarrow \quad \kappa\kappa_1 = \tau\tau_1.$$

∴ The product of curvatures at the corresponding points is equal to the product of the torsions.

Property V. *If curvature κ of a curve C is constant, then the curvature κ_1 of the curve C_1 of the locus of centre of spherical curvature is also constant.*

Proof. Let P(**r**) be any point on a curve C given by $\mathbf{r} = \mathbf{r}(s)$. Let $\kappa \neq 0$, $\tau \neq 0$ at P. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P.

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b} \qquad ...(1)$$

We shall use suffix '1' with quantities corresponding to the curve C_1 of the locus of centre of spherical curvature.

Differentiating (1) w.r.t. s_1, we get

$$\frac{d\mathbf{r}_1}{ds_1} = \frac{d}{ds}(\mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b})\frac{ds}{ds_1}$$

$$\Rightarrow \quad \mathbf{t}_1 = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,\mathbf{b} \qquad ...(2)$$

(For detail see **property I**)

$$\Rightarrow \quad |\,\mathbf{t}_1\,| = (\rho\tau + \rho''\sigma + \rho'\sigma')\frac{ds}{ds_1}\,|\,\mathbf{b}\,|$$

$$\Rightarrow \quad \frac{ds}{ds_1} = \frac{1}{\rho\tau + \rho''\sigma + \rho'\sigma'}$$

$$\therefore \quad (2) \Rightarrow \quad \mathbf{t}_1 = \mathbf{b} \qquad ...(3)$$

Differentiating (3) w.r.t. s_1, we get

$$\frac{d\mathbf{t}_1}{ds_1} = \frac{d\mathbf{b}}{ds}\frac{ds}{ds_1}$$

$$\Rightarrow \quad \kappa_1\mathbf{n}_1 = -\tau\mathbf{n}\frac{ds}{ds_1}$$

$$\Rightarrow \quad \kappa_1\mathbf{n}_1 = -\tau\left(\frac{\kappa}{\tau}\right)\mathbf{n} \quad \left(\rho' = \left(\frac{1}{\kappa}\right)' = 0, \rho'' = 0 \Rightarrow \frac{ds}{ds_1} = \frac{1}{\rho\tau + 0\sigma + 0.\sigma'} = \frac{\kappa}{\tau}\right)$$

$$\Rightarrow \quad \kappa_1\mathbf{n}_1 = -\kappa\mathbf{n}$$

$$\Rightarrow \quad \kappa_1\,|\,\mathbf{n}_1\,| = |\,-1\,|\,\kappa\,|\,\mathbf{n}\,| \quad \Rightarrow \quad \kappa_1 = \kappa$$

$\Rightarrow$ κ_1 is constant because κ is constant.

∴ The curvature of the curve C_1 of the locus of centre of spherical curvature is also constant.

Example 3. *If R is the radius of osculating sphere to a curve* $\mathbf{r} = \mathbf{r}(s)$ *at point 's', then show that*

$$R = \left|\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau}\right|.$$

Sol. We have $\mathbf{t}' = \kappa\mathbf{n}$

and $\mathbf{t}'' = \kappa\mathbf{n}' + \kappa'\mathbf{n} = \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \kappa'\mathbf{n} = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}$

$$\therefore \quad \mathbf{t}\times\mathbf{t}'' = \mathbf{t}\times(-\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b})$$
$$= -\kappa^2(\mathbf{t}\times\mathbf{t}) + \kappa'(\mathbf{t}\times\mathbf{n}) + \kappa\tau(\mathbf{t}\times\mathbf{b})$$
$$= -\kappa^2(\mathbf{0}) + \kappa'\mathbf{b} + \kappa\tau(-\mathbf{n}) = -\kappa\tau\mathbf{n} + \kappa'\mathbf{b}$$

$$\therefore \quad \frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau} = -\frac{\kappa\tau}{\kappa^2\tau}\mathbf{n} + \frac{\kappa'}{\kappa^2\tau}\mathbf{b} = -\rho\mathbf{n} - \left(\frac{1}{\kappa}\right)'\sigma\,\mathbf{b} = -\rho\mathbf{n} - \rho'\sigma\mathbf{b}$$

$$\Rightarrow \quad -\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau} = \rho\mathbf{n} + \rho'\sigma\mathbf{b}$$

$$\Rightarrow \quad (-1)^2\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau}\cdot\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau} = (\rho\mathbf{n} + \rho'\sigma\mathbf{b})\cdot(\rho\mathbf{n} + \rho'\sigma\mathbf{b})$$

$$\Rightarrow \quad \left|\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau}\right|^2 = \rho^2\,\mathbf{n}\cdot\mathbf{n} + \rho'^2\sigma^2\,\mathbf{b}\cdot\mathbf{b} = \rho^2 + \rho'^2\sigma^2$$
$$= R^2 \qquad (\because \quad R^2 = \rho^2 + \rho'^2\sigma^2)$$

$$\therefore \quad R = \left|\frac{\mathbf{t}\times\mathbf{t}''}{\kappa^2\tau}\right|.$$

Example 4. *Show that the radius of osculating sphere of the circular helix*

$$x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\cot\alpha$$

is equal to the radius of curvature at each point on the helix.

Sol. Given helix is $x = a\cos\theta,\ y = a\sin\theta,\ z = a\,\theta\cot\alpha$.

Let $\mathbf{r}$ be the position vector of the point $P(x, y, z)$ on the helix.

$$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j} + a\theta\cot\alpha\mathbf{k}$$

We know that $\kappa = \dfrac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$ (A standard formula)

Now $\dot{\mathbf{r}} = -a\sin\theta\mathbf{i} + a\cos\theta\mathbf{j} + a\cot\alpha\mathbf{k}$

$$\ddot{\mathbf{r}} = -a\cos\theta\mathbf{i} - a\sin\theta\mathbf{j} + 0\mathbf{k}$$

$$\therefore \quad \dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta & a\cos\theta & a\cot\alpha \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= -a^2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & \cot\alpha \\ \cos\theta & \sin\theta & 0 \end{vmatrix}$$

$$= -a^2[-\cot\alpha\sin\theta\mathbf{i} + \cot\alpha\cos\theta\mathbf{j} - \mathbf{k}]$$

$$\therefore \qquad |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = a^2\sqrt{\cot^2\alpha\sin^2\theta + \cot^2\alpha\cos^2\theta + 1}$$

$$= a^2\sqrt{\cot^2\alpha + 1} = a^2 \operatorname{cosec}\alpha$$

Also $\qquad |\dot{\mathbf{r}}| = (a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\alpha)^{1/2}$

$$= (a^2 + a^2\cot^2\alpha)^{1/2} = a \operatorname{cosec}\alpha$$

$$\therefore \qquad \kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{a^2 \operatorname{cosec}\alpha}{a^3 \operatorname{cosec}^3\alpha} = \frac{\sin^2\alpha}{a}$$

$\therefore$ κ is a constant quantity.

$$\therefore \qquad \rho' = \left(\frac{1}{\kappa}\right)' = 0$$

Let R be the radius of osculating sphere at the point P on the curve.

$$\therefore \qquad R = \sqrt{\rho^2 + \rho'^2\sigma^2} = \sqrt{\rho^2 + (0)^2\sigma^2} = \rho$$

Also the radius of curvature at P is ρ.

$\therefore$ The result holds.

Example 5. *Find the equation of the osculating sphere to the curve $x = 2t + 1$, $y = 3t^2 + 2$, $z = 4t^3 + 3$ at the point (1, 2, 3).*

Sol. The given curve is

$$x = 2t + 1, y = 3t^2 + 2, z = 4t^3 + 3.$$

The point P(1, 2, 3) corresponds to the value 0 of t.

Let $\mathbf{r}$ be the position vector of the point (x, y, z) on the given curve.

$$\therefore \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (2t + 1)\mathbf{i} + (3t^2 + 2)\mathbf{j} + (4t^3 + 3)\mathbf{k}$$

Let (α, β, γ) and R be the centre and the radius of the osculating sphere at P(1, 2, 3) respectively.

$\therefore$ The equation of the osculating sphere is $|\mathbf{r} - \mathbf{c}|^2 = R^2$, where $\mathbf{c} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$.

This sphere passes through (1, 2, 3) and has at least 4-point contact with the given curve at P.

$$\therefore \qquad |(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})|^2 = R^2$$

$$\Rightarrow \qquad (1 - \alpha)^2 + (2 - \beta)^2 + (3 - \gamma)^2 = R^2 \qquad \text{...(1)}$$

Let $\qquad f(t) = |\,(2t + 1)\mathbf{i} + (3t^2 + 2)\mathbf{j} + (4t^3 + 3)\mathbf{k} - (\alpha i + \beta j + \gamma k)\,|^2 - R^2$

$$= (2t + 1 - \alpha)^2 + (3t^2 + 2 - \beta)^2 + (4t^3 + 3 - \gamma)^2 - R^2$$

$$\therefore \qquad f'(t) = 2(2t + 1 - \alpha)2 + 2(3t^2 + 2 - \beta)6t + 2(4t^3 + 3 - \gamma)12t^2$$

$$f''(t) = 8 + 108t^2 + 12(2 - \beta) + 480t^4 + 48(3 - \gamma)t$$

$$f'''(t) = 216t + 1920t^3 + 48(3 - \gamma)$$

$$f^{iv}(t) = 216 + 5760t^2$$

Since the sphere has at least 4-point contact at $t = 0$, we have

$$f(0) = f'(0) = f''(0) = f'''(0) = 0 \text{ and } f^{iv}(0) \neq 0$$

$f(0) = 0 \Rightarrow (1 - \alpha)^2 + (2 - \beta)^2 + (3 - \gamma)^2 - R^2 = 0$, which is true.

$$f'(0) = 0 \Rightarrow 4(1 - \alpha) = 0 \Rightarrow \alpha = 1 \quad \text{(By using (1))}$$
$$f''(0) = 0 \Rightarrow 8 + 12\,(2 - \beta) = 0 \Rightarrow \beta = 8/3$$
$$f'''(0) = 0 \Rightarrow 48(3 - \gamma) = 0 \Rightarrow \gamma = 3$$

$$\therefore \quad \mathbf{c} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \mathbf{i} + \frac{8}{3}\mathbf{j} + 3\mathbf{k}$$

$$(1) \Rightarrow \quad R^2 = (1-1)^2 + \left(2 - \frac{8}{3}\right)^2 + (3-3)^2 = \frac{4}{9} \Rightarrow R = \frac{2}{3}$$

$\therefore$ Centre and radius of the osculating sphere at the point P are (1, 8/3, 3) and 2/3 respectively.

$\therefore$ The equation of the osculating sphere is

$$\left|(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - \left(\mathbf{i} + \frac{8}{3}\mathbf{j} + 3\mathbf{k}\right)\right| = \left(\frac{2}{3}\right)^2.$$

$$\Rightarrow \quad (x-1)^2 + \left(y - \frac{8}{3}\right)^2 + (z-3)^2 = \frac{4}{9}$$

$$\Rightarrow \quad 3x^2 + 3y^2 + 3z^2 - 6x - 16y - 18z + 50 = 0.$$

Example 6. *If the radius of the osculating sphere of a curve is constant, prove that the curve lies on a sphere or has constant curvature.*

Sol. Let the given curve be $\mathbf{r} = \mathbf{r}(s)$.

Let R be the radius of the osculating sphere at each point on the curve $\mathbf{r} = \mathbf{r}(s)$. Let P($\mathbf{r}$) be any point on this curve.

$$\therefore \quad R = \sqrt{\rho^2 + \rho'^2\sigma^2}$$

$$\Rightarrow \quad R^2 = \rho^2 + \rho'^2\sigma^2 \qquad \text{...(1)}$$

Differentiating (1) w.r.t. s, we get

$$0 = 2\rho\rho' + \rho'^2(2\sigma\sigma') + (2\rho'\rho'')\sigma^2$$

$$\Rightarrow \quad 0 = 2\rho'(\rho + \rho'\sigma\sigma' + \rho''\sigma^2)$$

$\Rightarrow$ Either $\rho' = 0$ ...(2) or $\rho + \rho'\sigma\sigma' + \rho''\sigma^2 = 0$ ...(3)

(2) $\Rightarrow$ ρ is constant $\Rightarrow$ κ is constant.

Let (3) hold. Let $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at the point P($\mathbf{r}$).

$$\therefore \quad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b}$$

Differentiating w.r.t. s, we get

$$\frac{d\mathbf{r}_1}{ds} = \mathbf{r}' + (\rho\mathbf{n}' + \rho'\mathbf{n}) + (\rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} + \rho'\sigma\mathbf{b}')$$
$$= \mathbf{t} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + \rho'\mathbf{n} + \rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} + \rho'\sigma(-\tau\mathbf{n})$$
$$= (1 - \rho\kappa)\mathbf{t} + (\rho' - \rho'\sigma\tau)\mathbf{n} + (\rho\tau + \rho''\sigma + \rho'\sigma')\mathbf{b}$$
$$= 0\mathbf{t} + 0\mathbf{n} + (\rho\tau + \rho''\sigma + \rho'\sigma')\mathbf{b} \qquad (\because \ \rho\kappa = 1, \sigma\tau = 1)$$
$$= \left(\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma'\right)\mathbf{b}$$

$$= \frac{1}{\sigma}(\rho + \rho''\sigma^2 + \rho'\sigma'\sigma)\mathbf{b}$$

$$= \frac{1}{\sigma}(0)\mathbf{b} = \mathbf{0} \qquad \text{(Using (3))}$$

$$\Rightarrow \qquad \frac{d\mathbf{r}_1}{ds} = \mathbf{0} \quad \Rightarrow \quad \mathbf{r}_1 \text{ is constant.}$$

∴ The centre of the osculating sphere is independent of the point P(**r**) on the curve **r** = **r**(s).

∴ The centre of osculating sphere is same at every point on the curve **r** = **r**(s).

Also, the radius of the osculating sphere is same for each point on the curve **r** = **r**(s).

∴ The osculating sphere is a fixed sphere for each point on the curve **r** = **r**(s).

∴ The curve **r** = **r**(s) lies itself on this sphere.

Example 7. *Show that the necessary and sufficient condition for a curve* **r** = **r**(*s*) *to lie on a sphere is s* $\frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma) = 0$ *at every point on the curve.*

Sol. Necessity. Let the curve **r** = **r**(s) lie on a sphere.

∴ This sphere is the osculating sphere to the curve **r** = **r**(s) at every point of the curve.

∴ The radius R of the osculating sphere at each point is constant.

We have $\quad R^2 = \rho^2 + \rho'^2\sigma^2$

Differentiating w.r.t. s, we get

$$0 = 2\rho\rho' + \rho'^2(2\sigma\sigma') + (2\rho'\rho'')\sigma^2$$

$$\Rightarrow \qquad 0 = 2\rho'\sigma\left(\frac{\rho}{\sigma} + \rho'\sigma' + \rho''\sigma\right)$$

$$\Rightarrow \qquad \frac{\rho}{\sigma} + \rho'\sigma' + \rho''\sigma = 0 \qquad (\text{Assuming } \rho' \neq 0, \sigma \neq 0)$$

$$\Rightarrow \qquad \frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma) = 0.$$

Sufficiency. Let $\quad \frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma) = 0.$

$$\Rightarrow \qquad \frac{\rho}{\sigma} + \rho'\sigma' + \rho''\sigma = 0$$

$$\Rightarrow \qquad 2\rho'\sigma\left(\frac{\rho}{\sigma} + \rho'\sigma' + \rho''\sigma\right) = 0$$

$$\Rightarrow \qquad 2\rho\rho' + \rho'^2(2\sigma\sigma') + (2\rho'\rho'')\sigma^2 = 0$$

$$\Rightarrow \qquad \frac{d}{ds}(\rho^2 + \rho'^2\sigma^2) = 0$$

$$\Rightarrow \qquad \rho^2 + \rho'^2\sigma^2 = \lambda, \text{ a constant.}$$

$$\Rightarrow \qquad R^2 = \lambda \quad \textit{i.e.,} \ \ R \text{ is constant.}$$

∴ Radius of osculating sphere is independent of the point on the curve **r** = **r**(s).

Let $\mathbf{r}_1$ be the position vector of the centre of osculating sphere at the point s.

$\therefore \qquad \mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b}$

Differentiating w.r.t. s, we get

$$\frac{d\mathbf{r}_1}{ds} = \mathbf{r}' + (\rho\mathbf{n}' + \rho'\mathbf{n}) + (\rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} + \rho'\sigma\mathbf{b}')$$
$$= \mathbf{t} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + \rho'\mathbf{n} + \rho''\sigma\mathbf{b} + \rho'\sigma'\mathbf{b} + \rho'\sigma(-\tau\mathbf{n})$$
$$= (1 - \rho\kappa)\,\mathbf{t} + (\rho' - \rho'\sigma\tau)\,\mathbf{n} + (\rho\tau + \rho''\sigma + \rho'\sigma')\,\mathbf{b}$$
$$= 0\mathbf{t} + 0\mathbf{n} + \left(\frac{\rho}{\sigma} + \frac{d}{ds}(\rho'\sigma)\right)\mathbf{b} = 0\,\mathbf{b} = \mathbf{0}$$

$$\Rightarrow \qquad \frac{d\mathbf{r}_1}{ds} = 0 \quad \Rightarrow \quad \mathbf{r}_1 \text{ is constant.}$$

The centre of the osculating sphere is independent of the point on the curve $\mathbf{r} = \mathbf{r}(s)$.

$\therefore$ The centre of the osculating sphere is same at every point on the curve $\mathbf{r} = \mathbf{r}(s)$.

Also, the radius of the osculating sphere is same for each point on the curve $\mathbf{r} = \mathbf{r}(s)$.

$\therefore$ The osculating sphere is a fixed sphere for each point on the curve $\mathbf{r} = \mathbf{r}(s)$.

$\therefore$ The curve $\mathbf{r} = \mathbf{r}(s)$ lies itself on this sphere.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *A curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of sufficiently high class is said to have n-***point contact** *with a surface $F(x, y, z) = 0$ at the point t_0 if the function $f(t) = F(x(t), y(t), z(t))$ satisfies:*

$$f(t_0) = \dot{f}(t_0) = \ddot{f}(t_0) = \ldots = f^{(n-1)}(t_0) = 0 \text{ and } f^{(n)}(t_0) \neq 0.$$

Rule II. *A curve $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ of sufficiently high class is said to have n-***point contact** *with a curve $F(x, y, z) = 0$, $G(x, y, z) = 0$ at the point t_0 if the functions $f(t) = F(x(t), y(t), z(t))$ and $g(t) = G(x(t), y(t), z(t))$ satisfy:*

$$f(t_0) = \dot{f}(t_0) = \ddot{f}(t_0) = \ldots = f^{(n-1)}(t_0) = 0$$
$$g(t_0) = \dot{g}(t_0) = \ddot{g}(t_0) = \ldots = g^{(n-1)}(t_0) = 0$$

and either $f^{(n)}(t_0) \neq 0$ or $g^{(n)}(t_0) \neq 0$.

Rule III. *A circle having at least 3-point contact with a given curve C at a point P on the curve is called the* **osculating circle** *to the curve C at the point P. The centre of the osculating circle at the point P is called the* **centre of curvature** *of the curve C at the point P.*

If $\mathbf{r}_1$ be the position vector of the centre of curvature at point $\mathbf{r}$ then,

$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n}$ and radius of osculating circle $= \rho$.

Rule IV. *A sphere having at least 4-point contact with a given curve C at a point P on the curve is called the* **osculating sphere** *to the curve C at the point P. The centre of the osculating sphere at the point P is called the* **centre of spherical curvature** *of the curve C at the point P.*

If $\mathbf{r}_1$ be the position vector of the centre of spherical curvature at point $\mathbf{r}$, then

$\mathbf{r}_1 = \mathbf{r} + \rho\mathbf{n} + \rho'\sigma\mathbf{b}$ *and radius of osculating sphere* $= \sqrt{\rho^2 + \rho'^2\sigma^2}$.

EXERCISE 2.4

1. Show that the tangent to the locus of the centre of curvature lies in the normal plane of the original curve and makes an angle $\tan^{-1} \dfrac{\rho}{\sigma\rho'}$ with the principal normal of the original curve.
2. If C is a curve of constant curvature κ, show that the locus C_1 of its centre of curvature is also a curve of constant curvature κ_1 such that $\kappa_1 = \kappa$ and its torsion τ_1 is given by the relation $\tau_1 = \dfrac{\kappa^2}{\tau}$.
3. For a curve of constant curvature, show that the centre of spherical curvature coincides with the centre of circular curvature.
4. If R is the radius of the osculating sphere to a curve $\mathbf{r} = \mathbf{r}(s)$ at point s, then show that:
$$R^2 = \rho^4\sigma^2 \,|\mathbf{r}'''|^3 - \sigma^2.$$
5. For the curve $\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$, show that:
$$x'''^2 + y'''^2 + z'''^2 = \frac{1}{\rho^2\sigma^2} + \frac{1+\rho'^2}{\rho^4}.$$
6. For the curve $\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$, show that:
$$\rho^4(x'''^2 + y'''^2 + z'''^2) = 1 + \frac{R^2}{\sigma^2},$$
where R is the radius of spherical curvature at the point (x, y, z).
7. Show that the radius of spherical curvature of a circular helix is equal to the radius of its circular curvature.

Hint

4. $\mathbf{r}''' = (\mathbf{r}'')' = (\mathbf{t}')' = (\kappa\mathbf{n})' = \kappa\mathbf{n}' + \kappa'\mathbf{n} = \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \kappa'\mathbf{n} = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\,\mathbf{b}$

$\therefore \quad |\mathbf{r}'''|^2 = (-\kappa^2)^2 + (\kappa')^2 + (\kappa\tau)^2$
$$= \frac{1}{\rho^4} + \left(-\frac{\rho'}{\rho^2}\right)^2 + \frac{1}{\rho^2\sigma^2} = \frac{\sigma^2 + \rho'^2\sigma^2 + \rho^2}{\rho^4\sigma^2} = \frac{\sigma^2 + R^2}{\rho^4\sigma^2}.$$

3

Differential Operators

3.1 PARTIAL DERIVATIVES

In chapter 2 we have considered a vector function of a single scalar variable t *i.e.*, $\mathbf{f}(t)$. Now we shall consider a vector function of several scalar variables. A vector function of two scalar variables say u, v is $\mathbf{f}(u, v)$ and $\frac{\partial \mathbf{f}}{\partial u}$ is the partial derivative of $\mathbf{f}(u, v)$ with respect to u,

i.e.,
$$\frac{\partial \mathbf{f}}{\partial u} = \lim_{\delta u \to 0} \frac{\mathbf{f}(u + \delta u, v) - \mathbf{f}(u, v)}{\delta u}$$

Similarly,
$$\frac{\partial \mathbf{f}}{\partial v} = \lim_{\delta v \to 0} \frac{\mathbf{f}(u, v + \delta v) - \mathbf{f}(u, v)}{\delta v}$$

In case
$$\mathbf{f}(u, v) = f_1(u, v)\,\mathbf{i} + f_2(u, v)\,\mathbf{j} + f_3(u, v)\,\mathbf{k}$$

then
$$\frac{\partial \mathbf{f}}{\partial u} = \frac{\partial f_1}{\partial u}\mathbf{i} + \frac{\partial f_2}{\partial u}\mathbf{j} + \frac{\partial f_3}{\partial u}\mathbf{k}$$

and
$$\frac{\partial \mathbf{f}}{\partial v} = \frac{\partial f_1}{\partial v}\mathbf{i} + \frac{\partial f_2}{\partial v}\mathbf{j} + \frac{\partial f_3}{\partial v}\mathbf{k}$$

Partial derivatives of second and higher order.

Again $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ both are vector functions of two scalar variables u and v and these possesss partial derivatives with respect to u and v.

$$\frac{\partial^2 \mathbf{f}}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \mathbf{f}}{\partial u}\right), \quad \frac{\partial^2 \mathbf{f}}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{f}}{\partial v}\right)$$

$$\frac{\partial^2 \mathbf{f}}{\partial u\, \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial \mathbf{f}}{\partial v}\right), \quad \frac{\partial^2 \mathbf{f}}{\partial v\, \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial \mathbf{f}}{\partial u}\right)$$

Also,
$$\frac{\partial^2 \mathbf{f}}{\partial u\, \partial v} = \frac{\partial^2 \mathbf{f}}{\partial v\, \partial u}.$$

Again, if $\mathbf{r} = \mathbf{f}(u, v)$, where $u = \phi(p, q)$, $v = \psi(p, q)$, *i.e.*, u and v are scalar functions of two scalar variables p and q then

$$\frac{\partial \mathbf{r}}{\partial p} = \frac{\partial \mathbf{f}}{\partial u} \cdot \frac{\partial u}{\partial p} + \frac{\partial \mathbf{f}}{\partial v} \cdot \frac{\partial v}{\partial q}$$

The total change in $\mathbf{f}$ due to simultaneous change in variables u and v is given by

$$d\,\mathbf{f} = \frac{\partial \mathbf{f}}{\partial u}\,du + \frac{\partial \mathbf{f}}{\partial v}\,d\,v.$$

In partial differentiation of vectors the same laws are followed as in ordinary calculus for scalar functions.

If $\mathbf{r}$ and $\mathbf{s}$ be two vectors functions of x, y, z then we have

(i)
$$\frac{\partial}{\partial x}(\mathbf{r} + \mathbf{s}) = \frac{\partial \mathbf{r}}{\partial x} + \frac{\partial \mathbf{s}}{\partial x}$$

(ii) $$\frac{\partial}{\partial x}(\mathbf{r}\,.\,\mathbf{s}) = \mathbf{r}\,.\,\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\,.\,\mathbf{s}$$

(iii) $$\frac{\partial}{\partial x}(\mathbf{r}\times\mathbf{s}) = \mathbf{r}\times\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\times\mathbf{s}$$

(iv) $$\frac{\partial^2}{\partial y\,\partial x}(\mathbf{r}\,.\,\mathbf{s}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{r}\,.\,\mathbf{s})\right\}$$
$$= \frac{\partial}{\partial y}\left\{\mathbf{r}\,.\,\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\,.\,\mathbf{s}\right\}$$
$$= \frac{\partial}{\partial y}\left(\mathbf{r}\,.\,\frac{\partial \mathbf{s}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\,.\,\mathbf{s}\right)$$
$$= \mathbf{r}\,.\,\frac{\partial^2 \mathbf{s}}{\partial y\,\partial x} + \frac{\partial \mathbf{r}}{\partial y}\,.\,\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial^2 \mathbf{r}}{\partial y\,\partial x}\,.\,\mathbf{s} + \frac{\partial \mathbf{r}}{\partial x}\,.\,\frac{\partial \mathbf{s}}{\partial y}$$

(v) $$\frac{\partial^2}{\partial y\,\partial x}(\mathbf{r}\times\mathbf{s}) = \mathbf{r}\times\frac{\partial^2 \mathbf{s}}{\partial y\,\partial x} + \frac{\partial \mathbf{r}}{\partial y}\times\frac{\partial \mathbf{r}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\times\frac{\partial \mathbf{s}}{\partial y} + \frac{\partial^2 \mathbf{r}}{\partial y\,\partial x}\times\mathbf{s}$$

EXAMPLE

If $\mathbf{a} = xyz\,\mathbf{i} + x^2z\,\mathbf{j} - y^3\mathbf{k}$ *and* $\mathbf{b} = x^3\,\mathbf{i} - xyz\,\mathbf{j} + x^3z\,\mathbf{k}$, *calculate* $\frac{\partial^2\mathbf{a}}{\partial y^2}\times\frac{\partial^2\mathbf{b}}{\partial x^2}$ *at the point* (1, 1, 0).

Sol. We have

$$\frac{\partial \mathbf{a}}{\partial y} = xz\,\mathbf{i} - 3y^2\mathbf{k}$$

$\therefore$ $$\frac{\partial^2 \mathbf{a}}{\partial y^2} = -6y\mathbf{k}$$

$$\frac{\partial \mathbf{b}}{\partial x} = 3x^2\mathbf{i} - yz\,\mathbf{j} + 2xz\,\mathbf{k}$$

$\therefore$ $$\frac{\partial^2 \mathbf{b}}{dx^2} = 6x\mathbf{i} + 2x\mathbf{k}$$

$\therefore$ $$\frac{\partial^2 \mathbf{a}}{\partial y^2}\times\frac{\partial^2 \mathbf{b}}{\partial x^2} = (-6y\,\mathbf{k})\times(6x\,\mathbf{i} + 2z\,\mathbf{k})$$
$$= -36xy\,\mathbf{j}$$
$$= -36\,\mathbf{j} \text{ at the point } (1, 1, 0)$$

EXERCISES

1. If $\mathbf{a} = x^2yz\,\mathbf{i} - 2xz^3\mathbf{j} + xz^2\mathbf{k}$ and $\mathbf{b} = 2z\,\mathbf{i} + y\,\mathbf{j} - x^2\mathbf{k}$, find $\frac{\partial^2}{\partial x\,\partial y}(\mathbf{a}\times\mathbf{b})$ at (1, 0, – 2).

[**Ans.** $-4(\mathbf{i} + 2\mathbf{j})$]

2. If $\mathbf{P} = e^{xy}\mathbf{i} + (x - 2y)\,\mathbf{j} + (x\sin y)\,\mathbf{k}$, find the values of $\frac{\partial \mathbf{P}}{\partial x}, \frac{\partial \mathbf{P}}{\partial y}, \frac{\partial^2 \mathbf{P}}{\partial x^2}, \frac{\partial^2 \mathbf{P}}{\partial y^2}$ and $\frac{\partial^2 \mathbf{P}}{\partial x\,\partial y}$.

[**Ans.** $\frac{\partial \mathbf{P}}{\partial x} = (ye^{xy})\,\mathbf{i} + \mathbf{y} + (\sin y)\,\mathbf{k}$, $\frac{\partial \mathbf{P}}{\partial y} = (xe^{xy})\,\mathbf{i} - 2\mathbf{j} + (x\cos y)\,\mathbf{k}$

$\frac{\partial^2 \mathbf{P}}{\partial x^2} = (y^2e^{xy})\,\mathbf{i}$, $\frac{\partial^2 \mathbf{P}}{\partial y^2} = x^2e^{xy}\,\mathbf{i} - (x\sin y)\,\mathbf{k}$]

3. If $\phi(x, y, z) = xy^2z$ and $\mathbf{A} = xz\,\mathbf{i} - xy\,\mathbf{j} + yz^2\mathbf{k}$ find $\dfrac{\partial^3}{\partial x^2\,\partial z}(\phi\,\mathbf{A})$ at $(2, -1, -1)$.

[**Ans.** $4\,\mathbf{i} + 2\,\mathbf{j}$]

3.2 THE OPERATOR DEL (∇)

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

and it operates distributively.

Hence $$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

may be thought of as ∇ operating on f, *i.e.*,

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f$$

∇ is a vector operator and is called differential operator. As ∇ is made up of three symbolic components along the three axes **i, j, k** and the symbolic magnitude of these are $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ respectively. Hence it may be looked upon as symbolic vector itself.

3.3 SCALAR POINT FUNCTION

If corresponding to each point P of a region R of space there corresponds a scalar denoted by $\phi(P)$ then ϕ is said to be a scalar point function for the region R. If the co-ordinates of P be (x, y, z) then

$$\phi(P) = \phi(x, y, z).$$

As an example, the density $\phi(P)$ at any point P of a certain body occupying given region R is a scalar piont function. Similarly the temperature $\phi(P)$ at any point P of a fluid occupying a certain region is a scalar point function. As another example we may say that the distance of any point P in space from a fixed point P_0 is scalar function.

$$\phi(P) = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}$$

3.3.1 Vector Point Function

If corresponding to each point P of a region P of space there corresponds a vector defined by $\mathbf{f}(P)$ then $\mathbf{f}$ is called a vector point function for the region R.

If the coordinates of P be (x, y, z) then

$$\mathbf{f}(P) = \mathbf{f}(x, y, z) = f_1(x, y, z)\,\mathbf{i} + f_2(x, y, z)\,\mathbf{j} + f_3(x, y, z)\,\mathbf{k}$$

For example, if the velocity of a particle at any time t occupying the postiion P in a certain region is $\mathbf{f}(P)$ then $\mathbf{f}(P)$ is a vector point function for that region.

3.4 GRADIENT OR SLOPE OF A SCALAR POINT-FUNCTION

If $f(x, y, z)$ be a scalar point function and continuously differentiable then the vector

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

is called the gradient of f and is written as grad f.

It should be noted that ∇f is a vector whose three components are $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$. Thus if f is a scalar point function, then ∇f is a vector point function.

3.4 OPERATOR $\mathbf{a} \cdot \nabla, \mathbf{a}$ BEING ANY VECTOR

The operator $\mathbf{a} \cdot \nabla$ is defined by the quantity

$$\mathbf{a} \cdot \nabla = \mathbf{a} \cdot \mathbf{i} \frac{\partial}{\partial x} + \mathbf{a} \cdot \mathbf{j} \frac{\partial}{\partial y} + \mathbf{a} \cdot \mathbf{k} \frac{\partial}{\partial z}$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

so that we have

$$(\mathbf{a} \cdot \nabla)\, f = \mathbf{a} \cdot \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{a} \cdot \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{a} \cdot \mathbf{k} \frac{\partial f}{\partial z}$$

For a unit vector **a**,

$$\mathbf{a} \cdot \mathbf{i}, \mathbf{a} \cdot \mathbf{j}, \mathbf{a} \cdot \mathbf{k}$$

are the direction cosines of **a** and hence

$$(\mathbf{a} \cdot \nabla)\, f, (\mathbf{a} \cdot \nabla)\, \mathbf{f}$$

stand for the directional derivatives of the respective functions along the directions of the unit vector **a**.

3.4 TOTAL DIFFERENCE df, WHERE f IS A SCALAR POINT FUNCTION

We have

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad \text{...(i)}$$

Also,

$$d\mathbf{r} \cdot (\nabla f) = (\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz) \cdot \left(\mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$df = d\mathbf{r} \cdot \nabla f = d\mathbf{r} \cdot \text{grad } f.$$

where total differential is df, and **f** is a vector point function.

We have

$$d\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x} dx + \frac{\partial \mathbf{f}}{\partial y} dy + \frac{\partial \mathbf{f}}{\partial z} dz$$

$$= d\mathbf{r} \cdot \mathbf{i} \frac{\partial f}{\partial x} + d\mathbf{r} \cdot \mathbf{j} \frac{\partial f}{\partial y} + d\mathbf{r} \cdot \mathbf{k} \frac{\partial f}{\partial z}$$

$$= (d\mathbf{r} \cdot \nabla)\, \mathbf{f}$$

3.4.3 Theorem

If f and g are two functions then

$$\text{grad } (f \pm g) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right)(f \pm g)$$

$$= \mathbf{i} \frac{\partial}{\partial x}(f \pm g) + \mathbf{j} \frac{\partial}{\partial y}(f \pm g) + \mathbf{k} \frac{\partial}{\partial z}(f \pm g)$$

$$= \left(\mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z}\right) \pm \left(\mathbf{i} \frac{\partial g}{\partial x} + \mathbf{j} \frac{\partial g}{\partial y} + \mathbf{k} \frac{\partial g}{\partial z}\right)$$

$$= \text{grad } f \pm \text{grad } g$$

3.4.4 Gradient of a Scalar Product

We have

$$\text{grad}\,(fg) = \nabla\,(fg) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) fg$$

$$= \mathbf{i}\frac{\partial}{\partial x}(fg) + \mathbf{j}\frac{\partial}{\partial y}(fg) + \mathbf{k}\frac{\partial}{\partial z}(fg)$$

$$= \mathbf{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \mathbf{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \mathbf{k}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$$

$$= f\left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right) + g\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial x} + \mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$= f\,(\text{grad}\,g) + g\,(\text{grad}\,f)$$

or $$\nabla(fg) = f\nabla\,(g) + g\,\nabla(f)$$

3.4.5 Gradient of a Quotient

We have

$$\nabla\left(\frac{f}{g}\right) = \mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

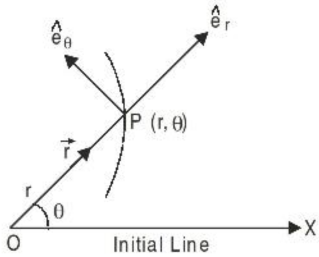
$$= \mathbf{i}\left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right) + \mathbf{j}\left(\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right) + \mathbf{k}\left(\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)$$

$$= \frac{1}{g^2}\left[g\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) - f\left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right)\right]$$

$$= \frac{1}{g^2}[g\,\nabla f - f\,\nabla g]$$

$\therefore$ $$\textbf{grad}\left(\frac{f}{g}\right) = \frac{g\,.\,\textbf{grad}\,f - f\,.\,\textbf{grad}\,g}{g^2}$$

3.5 GRADIENT IN POLAR CO-ORDINATES



Let **r** be the position vector of a point $P\,(r, \theta)$. Let $\hat{\mathbf{e}}_\mathbf{r}$ be the unit vector along **r** (in the sense of r increasing). Let $\hat{\mathbf{e}}_\theta$ be the unit vector along perpendicular to **r** (in the sense of θ increasing). Then the distance ds in the direction of **r** is dr and directional derviative along $\hat{\mathbf{e}}_\mathbf{r}$ is $\hat{\mathbf{e}}_\mathbf{r}.\nabla\phi$ where $\phi\,(x, y, z) = 0$, is the level surface.

Hence, $$\frac{\partial\phi}{\partial r} = \hat{\mathbf{e}}_\mathbf{r}\,.\,\nabla\phi \qquad ...(1)$$

Again, the distance ds in the direction perpendicular to **r** is $r\,d\theta$. Also, directional derivative along $\hat{\mathbf{e}}_0\,.\,\nabla\,\phi$.

Hence, $$\frac{\partial\phi}{r\,\partial\theta} = \hat{\mathbf{e}}_0\,.\,\nabla\phi. \qquad ...(2)$$

Clearly, the components of $\nabla\phi$ along $\hat{\mathbf{e}}_\mathbf{r}$ and along $\hat{\mathbf{e}}_\theta$ are respectively $\hat{\mathbf{e}}_\mathbf{r}.\nabla\phi$ and $\hat{\mathbf{e}}_\theta.\nabla\phi$.

Hence, $$\nabla\phi = (\hat{\mathbf{e}}_\mathbf{r}.\nabla\phi)\,\hat{\mathbf{e}}_\mathbf{r} + (\hat{\mathbf{e}}_0\,.\,\nabla\theta)\,\hat{\mathbf{e}}_0$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta \qquad ...(3)$$

[using (1) and (2)]

Equation (3) expresses gradient in polar coordinates.

EXAMPLES

1. *Find grad* ϕ, *if* $\phi = r^n = (x^2 + y^2 + z^2)^{n/2}$.

Sol. We have

$$\frac{\partial\phi}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} . 2x$$
$$= n . (r^2)^{(n-2)/2} . x$$
$$= n\, r^{n-2} . x$$

Similarly,
$$\frac{\partial\phi}{\partial y} = nr^{n-2}y,\ \frac{\partial\phi}{\partial z} = nr^{n-2}z$$

$\therefore$
$$\text{grad}\ \phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$
$$= nr^{n-2}x\,\mathbf{i} + nr^{n-2}y\,\mathbf{j} + nr^{n-2}z\,\mathbf{k}$$
$$= nr^{n-2}(\mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z)$$
$$= nr^{n-2}\mathbf{r}.$$

2. *Prove that grad* $f(r) \times \mathbf{r} = 0$.

Sol. We have

$$\text{grad}\ \{f(r)\} = \mathbf{i}\frac{\partial}{\partial x}f(r) + \mathbf{j}\frac{\partial}{\partial y}f(r) + \mathbf{k}\frac{\partial}{\partial z}f(r)$$
$$= \mathbf{i} . f'(r)\frac{\partial r}{\partial x} + \mathbf{j}\,f'(r)\frac{\partial r}{\partial y} + \mathbf{k}\,f'(r)\frac{\partial r}{\partial z}$$
$$= f'(r)\left[\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right] \qquad ...(i)$$

We have $r^2 = x^2 + y^2 + z^2$

$\therefore$ $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$.

Hence from (i)

$$\text{grad}\ \{f(r)\} = f'(r)\left[\mathbf{i}\frac{x}{r} + \mathbf{j}\frac{y}{r} + \mathbf{k}\frac{z}{r}\right]$$
$$= \frac{f'(r)}{r}\mathbf{r}$$

Hence
$$\text{grad}\ f(r) \times \mathbf{r} = \frac{f'(r)}{r}\mathbf{r} \times \mathbf{r}$$
$$= 0 \qquad [\because\ \mathbf{r} \times \mathbf{r} = 0]$$

3. *If* $\phi(x, y) = \log\sqrt{(x^2 + y^2)}$, *show that*

$$\text{grad}\ \phi = \frac{\mathbf{r} - (\mathbf{k} . \mathbf{r})\,\mathbf{k}}{\{\mathbf{r} - (\mathbf{k} . \mathbf{r})\,\mathbf{k}\} . \{\mathbf{r} - (\mathbf{k} . \mathbf{r})\,\mathbf{k}\}}$$

Sol. We have

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \therefore \qquad \mathbf{r} . \mathbf{k} = z \qquad ...(i)$$

Now, $\phi = \frac{1}{2}\log(x^2 + y^2)$

$\therefore$ $\frac{\partial\phi}{\partial x} = \frac{1}{2(x^2+y^2)} \cdot 2x = \frac{x}{x^2+y^2}$

Similarly, $\frac{\partial\phi}{\partial y} = \frac{y}{x^2+y^2}, \frac{\partial\phi}{\partial z} = 0$

$\therefore$
$$\text{grad } \phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$
$$= \frac{x}{x^2+y^2}\mathbf{i} + \frac{y}{(x^2+y^2)}\mathbf{j} + 0.\mathbf{k}$$
$$= \frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2} = \frac{\mathbf{r}-z\mathbf{k}}{(x\mathbf{i}+y\mathbf{j}).(x\mathbf{i}+y\mathbf{j})}$$
$$= \frac{\mathbf{r}-z\mathbf{k}}{(\mathbf{r}-z\mathbf{k}).(\mathbf{r}-z\mathbf{k})}, \text{ By (i).}$$

Now, by replacing z by **r.k,** we get
$$\text{grad } \phi = \frac{\mathbf{r}-(\mathbf{k}.\mathbf{r})\mathbf{k}}{\{\mathbf{r}-(\mathbf{k}.\mathbf{r})\mathbf{k}\}.\{\mathbf{r}-(\mathbf{k}.\mathbf{r})\mathbf{k}\}}$$

4. *If* $\phi = \log|\mathbf{r}|$, *then show that grad* $\phi = \frac{\mathbf{r}}{r^2}$.

Sol. Let $|\mathbf{r}| = r$, then $r^2 = x^2 + y^2 + z^2$

$\Rightarrow$ $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad $\{\log|\mathbf{r}|\}$ = grad $(\log r)$
$$= \mathbf{i}\frac{\partial}{\partial x}(\log r) + \mathbf{j}\frac{\partial}{\partial y}(\log r) + \mathbf{k}\frac{\partial}{\partial z}(\log r)$$
$$= \mathbf{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \mathbf{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \mathbf{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$$
$$= \mathbf{i}\left(\frac{1}{r}.\frac{x}{r}\right) + \mathbf{j}\left(\frac{1}{r}.\frac{y}{r}\right) + \mathbf{k}\left(\frac{1}{r}.\frac{z}{r}\right)$$
$$= \frac{(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)}{r^2} = \frac{\mathbf{r}}{r^2}.$$

5. *If* **a** *and* **b** *be constant vectors then show that* **grad [r a b] = a × b.**

Sol. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then
$$\phi = [\mathbf{r}\,\mathbf{a}\,\mathbf{b}] = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= x(a_2b_3 - a_3b_2) + y(a_3b_1 - a_1b_3) + z(a_1b_2 - a_2b_1)$$

Now,

$$\frac{\partial \phi}{\partial z} = (a_2 b_3 - a_3 b_2), \; \frac{\partial \phi}{\partial y} = a_3 b_1 - a_1 b_3$$

and
$$\frac{\partial \phi}{\partial z} = (a_1 b_2 - a_2 b_1)$$

$\therefore$
$$\text{grad } \phi = \text{grad } [\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$
$$= \mathbf{i}\,(a_2 b_3 - a_3 b_2) + \mathbf{j}\,(a_3 b_1 - a_1 b_3) + \mathbf{k}\,(a_1 b_2 - a_2 b_1)$$
$$= \mathbf{a} \times \mathbf{b}.$$

EXERCISES

1. Prove the grad $\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$.
2. Show that grad $(\mathbf{r} \cdot \mathbf{r}) = 2\,\mathbf{r}$.
3. If $\phi = f(r)$, then show that grad $\phi = \frac{f'(r)}{r}\,\mathbf{r}$.
4. If $f = 3x^2 y - y^3 z^2$ find grad f at the point $(1, -2, -1)$. [**Ans.** $-12\,\mathbf{i} - 9\,\mathbf{j} - 16\,\mathbf{k}$]
5. If $\phi = (3r^2 - 4\sqrt{r} + 6r^{-1/3})$ find $\nabla\phi$. [**Ans.** $2\,(3 - r^{-3/2} - r^{-7/3})\,\mathbf{r}$]

3.6 SCALAR AND VECTORS FIELDS

If to every point in a region, finite or infinite there corresponds a definite value of some physical property, the region is called a **field.**

If this property is a scalar, the field is called a *scalar field* for example density at all points, or potential at all points, or temperature at any given instant are scalar fields.

If this property is a vector, the field is known as *vector field.* For example the velocity at all points of a fluid or intensity of electric field at all points are the vector fields.

Equipotential or Level Surfaces

Let $\phi\,(x, y, z)$ be a scalar point function over a certain region. All those points which satisfy an equation of the type.

$\phi\,(x, y, z)$ = constant = c will constitute a family of surfaces which are called *level surfaces.* For all points on a member of the above family of surfaces the function $\phi(x, y, z)$ will be the same.

3.7 DIRECTIONAL DERIVATIVE OF A FUNCTION

If s represents a distance from any point $P\,(x, y, z)$ on the level surface $f(x, y, z) = 0$, in the direction of a unit vector $\hat{\mathbf{a}}$ then $\frac{df}{ds}$ is defined as the directional derivative of f in the direction of $\hat{\mathbf{a}}$.

The directional derivatives of $f(x, y, z)$ along the positive directions of x, y and z axes are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ respectively.

Also the directional derivatives of a vector function $\mathbf{f}$ along the coordinate axes are $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \frac{\partial \mathbf{f}}{\partial y}, \frac{\partial \mathbf{f}}{\partial z}$.

3.8 SOME THEOREMS

Theorem I. *grad $f(=\nabla f)$ is vector normal to the surface $f(x, y, z) = c$, where c is a constant.*

Let $A\,(x, y, z)$ be a point on the surface $f(x, y, z) = c$, and $B\,(x + \delta x, y + \delta y, z + \delta z)$ be another point in the neighbourhood of point A.

Let $$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$\therefore$ $$\mathbf{r} + \delta\mathbf{r} = (x+\delta x)\,\mathbf{i} + (y+\delta y)\,\mathbf{j} + (z+\delta z)\,\mathbf{k}$$

on subtracting, we get

$$\overrightarrow{AB} = \delta\mathbf{r} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k}$$

When $B \to A$, AB tends to the tangent at A to the given surface. Therefore in the limit, (i) becomes

$$d\,\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k},$$

and this lies in the tangent plane to the surface at A. But we know that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right).(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \nabla f\,.\,d\,\mathbf{r}.$$

Since $f(x, y, z) = \text{constant}$, $df = 0$, hence

$$\nabla f\,.\,d\,\mathbf{r} = 0.$$

Hence ∇f is perpendicular to $d\mathbf{r}$, *i.e.*, perpendicular to the tangent plane at A. *i.e.*, normal to the surface $f(x, y, z) = c$.

Theorem II. *The directional derivatives of a scalar field f at a point $A(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by*

$$\frac{df}{ds} = \hat{\mathbf{a}}\,.\,\text{grad}\ f = (\hat{\mathbf{a}}\,.\,\nabla)\,f$$

i.e., directional derivative $\frac{df}{ds}$ is the resolved part of ∇f (or grad f) in the direction of $\hat{\mathbf{a}}$.

As $\hat{\mathbf{a}}$ is unit vector at $A\,(x, y, z)$.

Hence $$\hat{\mathbf{a}} = \mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{dz} + \mathbf{k}\frac{dz}{ds}.$$

Where s is a length in the direction of $\hat{\mathbf{a}}$

Also, $$\hat{\mathbf{a}}\,.\,\text{grad}\ f = \left(\mathbf{i}\frac{dx}{dx} + \mathbf{j}\frac{dy}{ds} + \mathbf{k}\frac{dz}{ds}\right).\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dz} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= \frac{df}{ds}.$$

But $\mathbf{a}\,.\,\text{grad}\ f = \text{grad}\ f\ \cos\theta$, where θ is the angel grad f makes with $\mathbf{a}$

Hence the second result.

Similarly the directional derivative of a vector field $\mathbf{f}$ at a point (x, y, z) in the direction of unit vector $\hat{\mathbf{a}}$ is $(\hat{\mathbf{a}}\,\nabla)\,\mathbf{f}$.

Theorem III. *If $\hat{n}$ be a unit vector normal to the level surface $f(x, y, z) = c$ at a point A in the direction f increasing and n be a distance along the normal, then*

$$\text{grad}\ f = \frac{df}{dn}\,\hat{\mathbf{n}}$$

Since grad f is normal to $f(x, y, z) = c$, hence grad f is of the form

$$\text{grad}\ f = A\,\hat{\mathbf{n}}$$

where A is some constant and $\hat{\mathbf{n}}$ is the unit vector along the tangent.

Now from theorem III,

$$\hat{n} \cdot \text{grad } f = \frac{df}{dn}$$

By using (i), (ii) becomes

$$\hat{n} \cdot A\hat{n} = \frac{df}{dn} \qquad \text{or} \qquad A = \frac{df}{dn}.$$

Hence from (i)

$$\text{grad } f = \frac{df}{dn}\,\mathbf{n}$$

Hence the magnitude of ∇f is equal to $\frac{df}{dn}$.

Thus, *the gradient of scalar field f is a vector nromal to the surface f = constant and having a magnitude equal to the rate of change of f along the normal.*

Theorem IV. *grad f is a vector in the direction in which the maximum value of* $\frac{df}{ds}$ *occurs.*

The directional derivative in the direction of $\hat{\mathbf{a}}$ is given by

$$\frac{df}{ds} = \hat{\mathbf{a}} \cdot \text{grad } f = \hat{\mathbf{a}} \cdot \frac{df}{ds}\hat{n}, \text{ by theorem III}$$

$$= \frac{df}{ds}\,\hat{\mathbf{a}} \cdot \hat{\mathbf{n}} = \frac{df}{ds}\cos\theta,$$

where θ is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$.

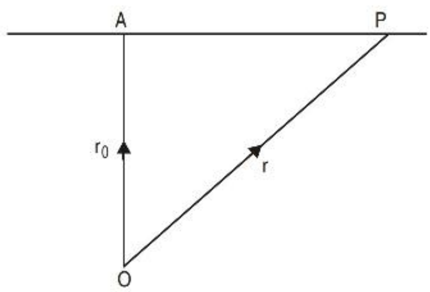
This value will be maximum when $\cos\theta = 1$, *i.e.*, the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is zero, *i.e.*, $\hat{\mathbf{a}}$ is along the normal.

Thus the directional derivative is maximum along the normal to the surface. Its maximum value is $\frac{df}{dn}$ *i.e.*, $|\text{grad } f|$.

3.9 TANGENT PLANE AND NORMAL LINE

To find the vector equations of the tangent plane and normal line to the surface $f(x, y, z) = k$ *where k is a constant.*

(i) Tangent Plane : Let the point of contact of the tangent plane with the given surface be A. Let $\mathbf{r}_0$ be the position vector of A. Let P be any point on the tangent plane. Let $\mathbf{r}$ be the position vector of P. Then the vector $\mathbf{r} - \mathbf{r}_0$ will lie in the tangent plane. Again, we know that grad f is in a direction normal to the tangent plane. Hence, $\mathbf{r} - \mathbf{r}_0$ will be perpendicular to grad f [If a line is perpendicular to a plane, then, it is perpendicular to every line lying in the plane.]



$$\therefore \qquad (\mathbf{r} - \mathbf{r}_0) \cdot \text{grad } f = 0. \qquad ...(1)$$

This equation is satisfied by any piont $\mathbf{r}$ lying in the tangent plane and is not satisfied by any other point. Hence, (1) is the required equation of the tangent plane to the given surface at the point $\mathbf{r}_0$.

(ii) Normal Line : Let P be any point on the normal line at $\mathbf{r}_0$. Let $\mathbf{r}$ be the position vector of P. Then $\mathbf{r} - \mathbf{r}_0$ lies along the normal line.

Since, grad f is normal to the tangent plane, hence, $\mathbf{r} - \mathbf{r}_0$, is parallel to grad f.

$$\therefore \qquad (\mathbf{r} - \mathbf{r}_0) \times \text{grad } f = 0.$$

A discussed in (1), we can show that equation (2) represents the equation of the normal line to the given surface at $\mathbf{r}_0$.

3.9.1 Tangent Line and Normal Plane

To find the vector equations of the tangent line and normal plane at a given point of the curve represented by the intersection of the two surfaces $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$.

(i) Tangent Line : Let $\mathbf{r}_0$ be the position vector of the given point of the curve and $\mathbf{r}$ be the position vector of any point on the tangent line. Then $\mathbf{r} - \mathbf{r}_0$ is a vector along the tangent line. Hence, $\mathbf{r} - \mathbf{r}_0$ is perpendicular to both grad f_1 and grad f_2. Hence, $\mathbf{r} - \mathbf{r}_0$ will be parallel to grad $f_1 \times$ grad f_2.

$$\therefore \quad (\mathbf{r} - \mathbf{r}_0) \times (\text{grad } f_1 \times \text{grad } f_2) = 0. \qquad ...(1)$$

This equation is satisfied by any piont $\mathbf{r}$ lying on the tangent line and is not satisfied by any other point. Hence, equation (1) represents the equation of the tangent line at $\mathbf{r}_0$.

(iii) Normal Plane : Let $\mathbf{r}$ be the position vector of any point on the normal plane through $\mathbf{r}_0$. Then the vector $\mathbf{r} - \mathbf{r}_0$ will lie in the normal plane. Hence, the vector $\mathbf{r} - \mathbf{r}_0$ will be parallel to the plane through grad f_1 and grad f_2, *i.e.*, perpendicular to grad $f_1 \times$ grad f_2.

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\text{grad } f_1 \times \text{grad } f_2) = 0. \qquad ...(2)$$

Obviously, (2) represents the equation of the normal plane at $\mathbf{r}_0$.

EXAMPLES

1. *Find the directional derivative of the function* $\phi = x^2 - y^2 + 2z^2$ *at the point* $P(1, 2, 3)$ *in the direction of the line PQ where Q is the point* $(5, 0, 4)$.

Sol. Here,

$$\text{grad } \phi = \mathbf{i} \cdot 2x - \mathbf{j} \cdot 2y + \mathbf{k} \cdot 4z$$
$$= 2\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} \quad \text{at} \quad P(1, 2, 3).$$

and
$$\mathbf{a} = \overrightarrow{PQ} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\therefore \quad \hat{\mathbf{a}} = \frac{4\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{(21)}}$$

$\therefore$ directional derivative along the given direction $= \mathbf{a} \cdot \text{grad } \phi$

$$= \frac{1}{\sqrt{(21)}} (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 4\mathbf{j} + 12\mathbf{k})$$
$$= \frac{1}{\sqrt{(21)}} (8 + 8 + 12) = \frac{28}{\sqrt{(21)}}$$
$$= 4\sqrt{\left(\frac{7}{3}\right)}.$$

2. *Find the directional derivative of* $\phi = x^2yz + 4xz^2$ *at* $(1, -2, -1)$ *in the direction* $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. *In what direction the directional derivative will be maximum and what is its magnitude ? Also find a unit normal to the surface* $x^2yz + 4xz^2 = 6$ *at the point* $(1, -2, -1)$.

Sol.
$$\phi = x^2yz + 4xz^2$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2xyz + 4z^2$$

$$\frac{\partial \phi}{\partial y} = x^2 z,$$

$$\frac{\partial\phi}{\partial z} = x^2 y + 8xz$$

$\therefore$
$$\text{grad } \phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$
$$= (2xyz + 4z^2)\,\mathbf{i} + (x^2 z)\,\mathbf{j} + (x^2 y + 8xy)\,\mathbf{k}$$
$$= 8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k} \text{ at the point } (1, -2, -1)$$

Let $\hat{\mathbf{a}}$ be the unit vector in the given direction.

Then
$$\hat{\mathbf{a}} = \frac{2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}}{\sqrt{4+1+4}} = \frac{2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}}{3}.$$

$\therefore$ Directional dervative $= \dfrac{\partial\phi}{\partial s} = \hat{\mathbf{a}} \,.\, \text{grad } \phi$
$$= \left(\frac{2\mathbf{i} - \mathbf{j} - \mathbf{k}}{3}\right).(8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k})$$
$$= \frac{16+1+20}{3} = \frac{37}{3}.$$

Again, we know that the directional derivative is maximum in the direction of normal which is the direction of grad ϕ. Hence, the directional derivative is maximum along grad $\phi = 8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k}$.

Further, maximum value of the directional derivative
$$= |\text{grad } \phi| = |8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k}|$$
$$= \sqrt{64+1+100} = \sqrt{165}.$$

Again, a unit vector normal to the surface $= \dfrac{\text{grad } \phi}{|\text{grad } \phi|} = \dfrac{8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k}}{\sqrt{165}}$

3. *Find the equation of the tangent plane and normal line to the surface* $x^2 + y^2 + z^2 = 25$ *at the point* $(4, 0, 3)$.

Sol. Let $f = x^2 + y^2 + z^2 - 25$

Then
$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
$$= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$
$$= 8\,\mathbf{i} + 2\,\mathbf{k}, \text{ at the point } (4, 0, 3)$$

Also
$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},\ \mathbf{r}_0 = 4\,\mathbf{i} + 0\,\mathbf{j} + 3\,\mathbf{k}$$

$\therefore$
$$\mathbf{r} - \mathbf{r}_0 = (x-4)\,\mathbf{i} + 4\,\mathbf{j} + (z-3)\,\mathbf{k}$$

Equation of tangent plane is
$$(\mathbf{r} - \mathbf{r}_0)\,.\,\text{grad } f = 0$$

$$\Rightarrow \quad [(x-4)\,\mathbf{i} + y\mathbf{j} + (z-3)\,\mathbf{k}]\,.\,[8\mathbf{i} + 6\mathbf{k}] = 0$$

$$\Rightarrow \quad 8\,(x-4) + 6\,(z-3) = 0 \ \Rightarrow 4x + 3z = 25$$

The equation of normal line is $(\mathbf{r} - \mathbf{r}_0) \times \text{grad } f = 0$

$$\Rightarrow \quad [(x-4)\,\mathbf{i} + y\mathbf{j} + (z-3)\,\mathbf{k}] \times [8\,\mathbf{i} + 6\,\mathbf{k}] = 0 \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-4 & y & z-3 \\ 8 & 0 & 6 \end{vmatrix} = 0$$

$$\Rightarrow \quad 3y\,\mathbf{i} + [4\,(z-3) - 3\,(x-4)]\,\mathbf{j} + (-4y)\,\mathbf{k} = 0 = 0$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from both sides, we get
$$3y = 0,\ 4\,(z-3) - 3\,(x-4) = 0,$$
$$-4y = 0$$

$$\Rightarrow \qquad y = 0, \frac{x-4}{4} = \frac{z-3}{3}$$

$\therefore$ Required equation of normal is $\dfrac{x-4}{4} = \dfrac{y}{0} = \dfrac{z-3}{3}$

4. *Find the directional derivative of* $f = x^2 + y^2 + z^2$ *at* $(1, 2, 3)$ *in the direction of line* $x/3 = y/4 = z/5$.

Sol. We have directional derivative $= \hat{\mathbf{a}} \cdot \text{grad } f$

Now, vector in direction of line $x/3 = y/4 = z/5$,

$$\mathbf{a} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$$

$$\therefore \qquad \hat{\mathbf{a}} = \frac{3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}}{\sqrt{9 + 16 + 25}} = \frac{3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}}{5\sqrt{2}}$$

and
$$\text{grad } f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$
$$= \mathbf{i}\,(2x) + \mathbf{j}\,(2y) + \mathbf{k}\,(2z)$$
$$= 2\,\mathbf{i} + 4\,\mathbf{j} + 6\mathbf{k}\ (1, 2, 3)$$

$\therefore$ directional deerivative $= \hat{\mathbf{a}} \cdot \text{grad } f$

$$= \frac{1}{5\sqrt{2}}(3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}) = \frac{52}{5\sqrt{2}}$$
$$= \frac{52\sqrt{2}}{10}.$$

EXERCISES

1. (a) Find the directional derivative of $\phi = x^3 + y^3 + z^3$ at the point $(1, 1, -2)$ in the direction of the vector $\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. [**Ans.** $21/\sqrt{6}$]

(b) Find the directional derivative of $\phi = 3x^2 - 2y - 3z$ at the point (1, 1, 1) in the direction specified by $2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$. [**Ans.** 19/3]

2. In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2 y z^2$ is a maximum and what is the magnitude ? [**Ans.** $-4\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}; 4\sqrt{(11)}$]

3. (a) Find the maxtimum value of the directional derivative of $\phi = 2x^2 + 3y^2 + 5z^2$ at the point $(1, 1, -4)$.

(b) Find the maximum value of the directional derivative of $\phi = xy + yz + zx$ at the point (1, 0, 2).

4. (a) Find the unit vector normal to $\phi = x^2 + y^2 + z$ at the point $(1, -1, 2)$.

$$\left[\textbf{Ans. } \frac{1}{3}(2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k})\right]$$

(b) Find a unit ector normal to the surface $\phi = x^2 + y^2 - z^2$ at the point (1, 1, 1).

$$\left[\textbf{Ans. } \frac{2\,(\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{3}}\right]$$

5. Find the equation of tangent plane and normal line to the surface $xyz = 4$ at the point $(2, -1, 5)$.

$$\left[\textbf{Ans. } 2x + y + z = 6, \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}\right]$$

3.10 DIVERGENCE OF A VECTOR

If $\mathbf{f}(x, y, z)$ is any given continuously differentiable vector point function, then the scalar function defined by

$$\nabla . \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(\mathbf{i}\, f_1 + \mathbf{j}\, f_2 + \mathbf{k}\, f_3)$$

is called the *divergence of* **f**, and is written as div. **f**. We read it as del dot **f** or divergence of **f**. It is clear that *div* **f** *is scalar.*

Solenoidal vector. A vector **f** is called a solenoidal vector if div **f** vanishes. *i.e.*, where div. $\mathbf{f} = 0$.

3.10.1 Divergence of a Sum

Let **f** and **F** be two vector functions, then

$$\text{div}\,(\mathbf{f} + \mathbf{F}) = \mathbf{i}\,.\frac{\partial}{\partial x}(\mathbf{f} + \mathbf{F}) + \mathbf{j}\,.\frac{\partial}{\partial y}(\mathbf{f} + \mathbf{F}) + \mathbf{k}\,.\frac{\partial}{\partial z}(\mathbf{f} + \mathbf{F})$$

$$= \mathbf{i}\,.\left(\frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{F}}{\partial x}\right) + \mathbf{j}\,.\left(\frac{\partial \mathbf{f}}{\partial y} + \frac{\partial \mathbf{F}}{\partial y}\right) + \mathbf{k}\,.\left(\frac{\partial \mathbf{f}}{\partial z} + \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial f_2}{\partial y} + \frac{\partial F_2}{\partial y}\right) + \left(\frac{\partial f_3}{\partial z} + \frac{\partial F_3}{\partial z}\right).$$

where $\mathbf{f} = \mathbf{i}\, f_1 + \mathbf{j}\, f_2 + \mathbf{k}\, f_3$

and $\mathbf{F} = \mathbf{i}\, F_1 + \mathbf{j}\, F_2 + \mathbf{k}\, F_3$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

Note. *If* $\mathbf{a} = \mathbf{i}\, a_1 + \mathbf{j}\, a_2 + \mathbf{k}\, a_3$ *is a constant vector, then* $\frac{\partial a_1}{\partial x}, \frac{\partial a_2}{\partial y}, \frac{\partial a_3}{\partial z}$ *are all zero. Hence for a constant vector* **a**,

$$\text{div}\, \mathbf{a} = 0.$$

3.11 CURL OF A VECTOR

If **f** is any given continuously differentiable vector point function, then the vector function defined by

$$\nabla \times \mathbf{f} = \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z}$$

is called the curl of **f** and is written as curl **f**. We read it as *del cross* **f** *or curl of* **f**.

3.11.1 Expression of Curl f in terms of the components of f

Let $\mathbf{f} = \mathbf{i}\, f_1 + \mathbf{j}\, f_2 + \mathbf{k}\, f_3$

Then curl $\mathbf{f} = \nabla \times \mathbf{f}$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (\mathbf{i}\, f_1 + \mathbf{j}\, f_2 + \mathbf{k}\, f_3)$$

$$= \mathbf{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Obviously, the components of curl **f** along the co-ordinate axes are

$$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right), \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right), \text{ and } \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

3.11.2 Curl of Sums

$$\text{Curl}\,(\mathbf{f}+\mathbf{F}) = \nabla \times \{\mathbf{i}\,(f_1+F_1)+\mathbf{j}\,(f_1+F_2)+\mathbf{k}\,(f_3+F_3)\}$$

Here $\mathbf{f} = \mathbf{i}\, f_1 + \mathbf{j}\, f_2 + \mathbf{k}\, f_3$

and $\mathbf{F} = \mathbf{i}\, F_1 + \mathbf{j}\, F_2 + \mathbf{k}\, F_3$

so curl $(\mathbf{f}+\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1+F_1 & f_2+F_2 & f_3+F_3 \end{vmatrix}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

By the property of determinants

$$= \text{curl}\,\mathbf{f} + \text{curl}\,\mathbf{F}.$$

Note. If **a** is a constant vector then all differentials of its components are zero. Hence for a constant vector

$$\text{Curl}\,\mathbf{a} = 0.$$

Irrotational Vector

A vector **f** is said to be irrotational if curl $\mathbf{f} = 0$. (or if curl **f** vanishes).

3.12 THE LAPLACIAN OPERATOR ∇^2

The operator ∇^2 is defined by the equation

$$(\nabla\,.\,\nabla)\, f = \nabla\,.\,(\nabla f)$$

The operator ∇^2 is called the **Laplacian**

Now, $\nabla^2 f = \nabla\,(\nabla f)$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad ...(i)$$

since $\mathbf{i}\,.\,\mathbf{i} = \mathbf{j}\,.\,\mathbf{j} = \mathbf{k}\,.\,\mathbf{k} = 1$ and $\mathbf{i}\,.\,\mathbf{j} = \mathbf{j}\,.\,\mathbf{k} = 0$

Again, $\nabla\,.\,\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad ...(ii)$$

Hence from (i) and (ii) $\nabla\,.\,\nabla f = (\nabla f)$

$\nabla^2 f$ is read as *del square of f*. Again if $\nabla^2 f = 0$, then $\nabla^2 f_1 = \nabla^2 f_2 = \nabla^2 f_3 = 0$.

EXAMPLES

1. *If* $\mathbf{f} = (x+y+1)\,\mathbf{i} + \mathbf{j} + (-x-y)\,\mathbf{k}$, *find curl* **f** and **f**. *curl* **f**.

Sol. We have

$$\text{curl}\,\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \mathbf{i}\,(-1-0) - \mathbf{j}\,(-1-0) + \mathbf{k}\,(0-1)$$

$$= -\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\therefore \quad \mathbf{f}\,.\,\text{curl}\,\mathbf{f} = [(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}]\,.\,(-\mathbf{i}+\mathbf{j}-\mathbf{k})$$
$$= -(x+y+1)+1-(-x-y)$$
$$= -x-y-1+1+x+y=0.$$

2. *If* $\mathbf{F} = xy^2\mathbf{i} + 2x^2yz\,\mathbf{j} - 3yz^2\mathbf{k}$, *find div* $\mathbf{f}$ *and curl* $\mathbf{f}$ *at* $(1, -1, 1)$.

Sol.
$$\text{div}\,.\,\mathbf{f} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$
$$= y^2 + 2x^2z - 6yz.$$
$$(\text{div}\,\mathbf{f})_{(1,-1,1)} = 1+2+6 = 9.$$
$$\mathbf{curl\,f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$
$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$
$$= \mathbf{i}\,[-3z^2 - 2x^2y] + \mathbf{j}\,[0+0] + \mathbf{k}\,[4xyz - 2xy]$$
$$= -\mathbf{i}\,(3z^2 + 2x^2y) + \mathbf{k}\,(4xyz - 2xy)$$
$$= -\mathbf{i}\,(3-2) + \mathbf{k}\,(-4+2)$$
$$(\text{curl}\,\mathbf{f})_{(1,-1,1)} = -\mathbf{i} - 2\,\mathbf{k}$$

3. *If* $\mathbf{u} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{(x^2+y^2+z^2)}} = \hat{\mathbf{r}}$, *show that* $\nabla\,.\,\mathbf{u} = \dfrac{2}{\sqrt{(x^2+y^2+z^2)}} = \dfrac{2}{r}$ *and* $\nabla \times \mathbf{u} = 0$.

Sol. We have $r^2 = x^2+y^2+z^2$, then $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$.

Now
$$\mathbf{u} = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}$$
$$\nabla.\,\mathbf{u} = \text{div}\,\mathbf{u} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$
$$= \frac{r - x\dfrac{\partial r}{\partial x}}{r^2} + \frac{r - y\dfrac{\partial r}{\partial y}}{r^2} + \frac{r - z\dfrac{\partial r}{\partial z}}{r^2}$$
$$= \frac{1}{r^2}\left[3r - \left(x\,.\,\frac{x}{r} + y\,.\,\frac{y}{r} + z\,.\,\frac{z}{r}\right)\right]$$
$$= \frac{1}{r^2}\left[3r - \left(\frac{x^2+y^2+z^2}{r}\right)\right] = \frac{1}{r^2}\left[3r - \frac{r^2}{r}\right]$$
$$= \frac{1}{r^2}(3r - r) = \frac{1}{r^2}\,.\,2r = \frac{2}{r}.$$
$$\nabla\,.\,\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/r & y/r & z/r \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial}{\partial y}(z/r) - \frac{\partial}{\partial z}(y/r)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial z}(x/r) - \frac{\partial}{\partial x}(z/r)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial x}(y/r) - \frac{\partial}{\partial y}(x/r)\right\}$$

$$= \mathbf{i}\left\{-\frac{1}{r^2}\frac{\partial r}{\partial y}\cdot z + \frac{1}{r^2}\frac{\partial r}{\partial z}\cdot y\right\} + \mathbf{j}\left\{-\frac{1}{r^2}\frac{\partial r}{\partial z}\cdot x + \frac{1}{r^2}\frac{\partial r}{\partial x}\cdot z\right\} + \mathbf{k}\left\{-\frac{1}{r^2}\frac{\partial r}{\partial x}\cdot y + \frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\cdot x\right\}$$

$$= \mathbf{i}\left\{-\frac{yz}{r^3} + \frac{yz}{r^3}\right\} + \mathbf{j}\left\{-\frac{xz}{r^3} + \frac{xz}{r^3}\right\} + \mathbf{k}\left\{-\frac{xy}{r^3} + \frac{xy}{r^3}\right\}$$

$$= \mathbf{i}\,(0) + \mathbf{j}\,(0) + \mathbf{k}\,(0) = \mathbf{0}$$

4. *Find the constants a, b, c, so that*

$$\mathbf{f} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k} \text{ is irrotational.}$$

Sol. A vector **f** is said to be irrotational if curl $\mathbf{f} = 0$.

Now,
$$\text{curl}\,\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + ax & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= (c+1)\,\mathbf{i} + (a-4)\,\mathbf{j} + (b-2)\,\mathbf{k}$$

Now curl $\mathbf{f} = 0$ if $c + 1 = 0$, $a - 4 = 0$ and $b - 2 = 0$.

$\therefore$ $a = 4, b = 2, c = -1.$

5. *If the vector* $\mathbf{f} = 3x\,\mathbf{i} + (x+y)\,\mathbf{j} - ax\,\mathbf{k}$ *is solenoidal, find* **a**.

Sol. A vector **f** is said to be solenoidal if div $\mathbf{f} = 0$

$$\therefore \quad \text{div}\,\mathbf{f} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(-az) = 3 + 1 - a = 0$$

$\therefore$ $a = 4.$

EXERCISES

1. Find the curl of the vector function $\mathbf{f} = y(x+z)\,\mathbf{i} + z(x+y)\,\mathbf{j} + x(y+z)\,\mathbf{k}$ and hence find the value of curl (curl **f**). [**Ans.** curl $\mathbf{f} = -y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$, curl (curl **f**) $= \mathbf{i} + \mathbf{j} + \mathbf{k}$]

2. (a) If, $\mathbf{f} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, then show that curl (cur **f**) $= 0$.

   (b) If $\mathbf{f} = x^2y\,\mathbf{i} + y^2z\,\mathbf{j} - z^2x\,\mathbf{k}$, find $\nabla \times \mathbf{f}$ at (1, 2, 3). [**Ans.** $-4\,\mathbf{i} + 9\,\mathbf{j} - \mathbf{k}$]

   (c) If $\mathbf{f} = xy^2\,\mathbf{i} - 2y^2z^3\,\mathbf{j} + xyz^2\,\mathbf{k}$, find div. **f** at (1, – 1, 1). [**Ans.** 3]

3. (a) If $\mathbf{f} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (y^2 - 2xy)\,\mathbf{k}$, find div. **f** and curl **f**.

   [**Ans.** div. $\mathbf{f} = 2x + 2y$, curl $\mathbf{f} = (2y - 2x)\,\mathbf{i} + 2y\,\mathbf{j} + 4y\,\mathbf{k}$]

   (b) Find the divergence and curl of the vector function $\mathbf{f} = (2z - 3y)\,\mathbf{i} + (3x - z)\,\mathbf{j} + (y - 2x)\,\mathbf{k}$. [**Ans.** div $\mathbf{f} = 0$, curl $\mathbf{f} = 2\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}$]

   (c) Compute divergence and curl of the vector $\mathbf{f} = x^2y\,\mathbf{i} + xz\,\mathbf{j} + 2yz\,\mathbf{k}$ at (–1, 1, 1).

   [**Ans.** div. $\mathbf{f} = 0$, curl $\mathbf{f} = 3\,\mathbf{i}$]

4. (a) If $\mathbf{f} = 3xy\,\mathbf{i} + 20yz^2\mathbf{j} - 15xz\,\mathbf{k}$ and $\phi = y^2 - xz$, than find div. ($\phi\,\mathbf{f}$).

   [**Ans.** $3y^3 - 20xz^3 - 15xy^2 + 30xz^2 + 60y^2z^2 - 6xyz$]

(b) If $\mathbf{f} = x^2\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\phi = xy^2$, find div $(\phi\,\mathbf{f})$. [**Ans.** $xy^2\,(3x + 4y)$]

5. Determine the constant a so that the vector

$\mathbf{V} = (x + 3y)\,\mathbf{i} + (y - 2z)\,\mathbf{j} + (x + az)\,\mathbf{k}$ is solenoidal. [**Ans.** -2]

6. Prove that $\phi = 2x^2 - 5y^2 + 3z^2$ satisfies Laplace's equation $\nabla^2\phi = 0$.

7. Show that the vectors

(a) $\mathbf{f} = (4xy - z^3)\,\mathbf{i} + 2x^2\mathbf{j} - 3xz^2\mathbf{k}$

and (b) $\mathbf{f} = (y^2 \cos x + z^3)\,\mathbf{i} + (2y \sin x - 4)\,\mathbf{j} + (3xz^2 + 2)\,\mathbf{k}$

are irrotational.

8. Determine a, b, c so that the vector $\mathbf{f}$ given by

$\mathbf{f} = (2x + 3y + az)\,\mathbf{i} + (bx + 2y + 3z)\,\mathbf{j} + (2x + cy + 3z)\,\mathbf{k}$ is irrotational.

[**Ans.** $a = 2, b = 3, c = 3$]

EXAMPLES

1. *Prove that div* $\hat{\mathbf{r}} = \dfrac{2}{r}$

Sol. We have

$$\hat{\mathbf{r}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{r} = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}$$

$$\therefore \quad \text{div}\,\hat{\mathbf{r}} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r\,.1 - x\,.\dfrac{\partial r}{\partial x}}{r^2} + \frac{r\,.1 - y\,.\dfrac{\partial r}{\partial y}}{r^2} + \frac{r\,.1 - z\,.\dfrac{\partial r}{\partial z}}{r^2}$$

$$= \frac{1}{r^2}\left[3r - \left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)\right] \qquad \text{...(i)}$$

we have $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

hence from (i)

$$\text{div}\,\hat{\mathbf{r}} = \frac{1}{r^2}\left[3r - \left(x\,.\frac{x}{r} + y\,.\frac{y}{r} + z\,.\frac{z}{r}\right)\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{x^2 + y^2 + z^2}{r}\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{r^2}{r}\right]$$

$$= \frac{1}{r^2}\,.\,2r = \frac{2}{r}$$

2. *Prove that* div $r^n\mathbf{r} = (n + 3)\,r^n$.

Sol. We have

$$r^n\mathbf{r} = r^n x\,\mathbf{i} + r^n y\,\mathbf{j} + r^n z\,\mathbf{k}$$

$$\therefore \quad \text{div}\,r^n\mathbf{r} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n\,.1 + nr^{n-1}x\,.\frac{\partial r}{\partial x} + r^n\,.1 + nr^{n-1}\,y\,.\frac{\partial r}{\partial y} + r^n\,.1 + nr^{n-1}z\,.\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3r^n + nr^n = (n+3)\, r^n.$$

3. *Prove that* curl $(r^n \mathbf{r}) = \mathbf{0}$, *i.e.*, $r^n\mathbf{r}$ *is irrotational.*

Sol. We have, $\mathbf{r} = \mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z,\ r^2 + y^2 + z^2.$

Then $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{curl}\,(r^n\mathbf{r}) = \nabla \times (i\, xr^n + j\, yr^n + k\, zr^n)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(zr^n) - \frac{\partial}{\partial z}(yr^n)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(xr^n) - \frac{\partial}{\partial x}(zr^n)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(yr^n) - \frac{\partial}{\partial y}(xr^n)\right]$$

$$= \mathbf{i}\left[nr^{n-1}\frac{\partial r}{\partial y}.\, z - nr^{n-1}\frac{\partial r}{\partial z}.\, y\right] + \mathbf{j}\left[nr^{n-1}\frac{\partial r}{\partial z}\, x - nr^{n-1}\frac{\partial r}{\partial x}.\, z\right] + \mathbf{k}\left[nr^{n-1}\frac{\partial r}{\partial x}\, y - nr^{n-1}\frac{\partial x}{\partial y}.\, x\right]$$

$$= nr^{n-1}\left[\mathbf{i}\left(\frac{yz}{r} - \frac{zy}{r}\right) + \mathbf{j}\left(\frac{zx}{r} - \frac{xz}{r}\right) + \mathbf{k}\left(\frac{xy}{r} - \frac{yx}{r}\right)\right]$$

$$= nr^{n-1}[\mathbf{i}\,.\,0 + \mathbf{j}\,.\,0 + \mathbf{k}\,.\,0] = \mathbf{0}.$$

4. *Prove that* div $(\mathbf{r} \times \mathbf{a}) = 0$ *or* div $(\mathbf{a} \times \mathbf{r}) = 0$.

Sol. We know, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$$

$$\mathbf{r} \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \mathbf{i}\,(a_3 y - a_2 z) - \mathbf{j}\,(a_3 x - a_1 z) + \mathbf{k}\,(a_2 x - a_1 y)$$

Now, $\text{div}\,(\mathbf{r} \times \mathbf{a}) = \nabla\,.\,(\mathbf{r} \times \mathbf{a})$

$$= \left\{\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right\}\{\mathbf{i}\,(a_3 y - a_2 z) - \mathbf{j}\,(a_3 x - a_1 z) + \mathbf{k}\,(a_2 x - a_1 c)\}$$

$$= \frac{\partial}{\partial x}(a_3 y - a_2 z) - \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_2 x - a_1 y) = 0.$$

5. *If* **a** *and* **b** *are constant vectors, then show that* curl $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \mathbf{b} \times \mathbf{a}$.

Sol. We have

$$\text{curl}\,[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \text{curl}\,[(\mathbf{b}\,.\,\mathbf{r})\,\mathbf{a} - (\mathbf{b}\,.\,\mathbf{a})\,\mathbf{r}]$$

$$= \text{curl}\,[(b_1x + b_2y + b_3z)(a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) - (a_1b_1 + a_2b_2 + a_3b_3)\,.\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1\,(b_1x + b_2y + b_3z) & a_2\,(b_1x + b_2y + b_3z) & a_3\,(b_1x + b_2y + b_3z) \end{vmatrix}$$

$$- \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (a_1b_1 + a_2b_2 + a_3b_3)\,x & (a_1b_1 + a_2b_2 + a_3b_3)\,y & (a_1b_1 + a_2b_2 + a_3b_3)\,z \end{vmatrix}$$

$$= \mathbf{i}\,(a_3b_2 - a_2b_3) - \mathbf{j}\,(a_3b_1 - a_1b_3) + \mathbf{k}\,(a_2b_1 - a_1b_2) = \mathbf{b} \times \mathbf{a}$$

EXERCISES

1. If $\mathbf{f} = \hat{\mathbf{r}}$, then prove that curl $\mathbf{f} = 0$.
2. Show that div $\dfrac{\mathbf{r}}{r^3} = 0$.
3. If $\mathbf{v} = \omega \times \mathbf{r}$ prove that $\omega = \dfrac{1}{2}$ curl $\mathbf{v}$ where ω is a constant vector.
4. Prove that div $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = -2\mathbf{b}\,.\,\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.
5. Prove that div $\dfrac{\mathbf{a} \times \mathbf{r}}{r^3} = 0$ and curl $\dfrac{\mathbf{a} \times \mathbf{r}}{r^3} = -\dfrac{a}{r^3} + \dfrac{3\,\mathbf{r}}{r^5}\,(\mathbf{a}\,.\,\mathbf{r})$
6. Prove that $\mathbf{a}\,.\,\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{a}\,.\,\mathbf{r}}{r^3}$.

3.13 SECOND ORDER DIFFERENTIAL FUNCTIONS

The three quantities which have been specially studied in this chapter are grad ϕ, div $\mathbf{f}$ and curl $\mathbf{f}$. For these quantities certain improtant points are to be noted :

(i) grad is associated with a *scalar point function* ϕ and grad ϕ is a *vector.*

(ii) Divergence is associated with vector point function $\mathbf{f}$ and divergence $\mathbf{f}$ is a *scalar.*

(iii) Curl is associated with vector function $\mathbf{f}$ and curl $\mathbf{f}$ is a *vector.*

Since grad ϕ and curl $\mathbf{f}$ are vector points functions and as such we can find their divergence as well as curl. Also div $\mathbf{f}$ is a scalar point function we can find its grad we may thus form the following functions :

curl grad $\phi \equiv \nabla \times (\nabla\,\phi)$

div curl $\mathbf{f} \equiv \nabla\,.\,(\nabla \times \mathbf{f})$

div grad $\phi \equiv \nabla\,.\,(\nabla\,\phi)$

curl curl $\mathbf{f} = \nabla \times (\nabla \times \mathbf{f})$

grad div $\mathbf{f} = \nabla\,(\nabla\,.\,\mathbf{f})$

These are called *second order differential functions.*

Properties of Second Order Differential Operators

Property 1. Curl (grad ϕ) $\equiv \nabla \times (\nabla\,\phi) = 0$.

$$\nabla \times (\nabla\,\phi) = \nabla \times \left(\mathbf{i}\,\frac{\partial\phi}{\partial x} + \mathbf{j}\,\frac{\partial\phi}{\partial y} + \mathbf{k}\,\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right) + \mathbf{j}\left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right) + \mathbf{k}\left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)$$

$$= 0, \text{ as } \frac{\partial^2\phi}{\partial y\,\partial z} = \frac{\partial^2\phi}{\partial z\,\partial y} \text{ etc.}$$

Hence $\nabla \times (\nabla\,\phi) = 0$.

Property 2. div (curl f) $\equiv \nabla\,.\,(\nabla \times \mathbf{f}) = 0$

Let $\mathbf{f} = \mathbf{i}\,f_1 + \mathbf{j}\,f_2 + \mathbf{k}\,f_3$

$$\therefore \qquad \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Therefore

$$\nabla\,.\,(\nabla \times \mathbf{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\,\partial y} - \frac{\partial^2 f_2}{\partial x\,\partial z} + \frac{\partial^2 f_1}{\partial y\,\partial z} - \frac{\partial^2 f_3}{\partial y\,\partial x} + \frac{\partial^2 f_2}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial z\,\partial y}.$$

Hence $\quad \nabla\,.\,(\nabla \times \mathbf{f}) = 0$

Property 3. div (grad ϕ) $\equiv \nabla\,.\,(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

We have

$$\nabla\,.\,(\nabla\,\phi) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial y}\right).\left(\mathbf{i}\,\frac{\partial\phi}{\partial x} + \mathbf{j}\,\frac{\partial\phi}{\partial y} + \mathbf{k}\,\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$$

$$\therefore \qquad \nabla\,.\,(\nabla\,\phi) = \nabla^2\phi$$

Property 4. Curl (curl f) = grad (div) f $-\nabla^2$f

i.e., $\qquad \nabla \times (\nabla \times \mathbf{f}) = (\nabla\,.\,\mathbf{f}) - \nabla^2\mathbf{f}$

Let $\quad \mathbf{f} = \mathbf{i}\,f_1 + \mathbf{j}\,f_2 + \mathbf{k}\,f_3$

$$\therefore \qquad \nabla \times \mathbf{f} = \mathbf{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\therefore \qquad \nabla \times (\nabla \times \mathbf{f}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right\}$$

$$+ \mathbf{j}\left\{\frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right\}$$

$$+\mathbf{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\right\}$$

$$=\mathbf{i}\left\{\frac{\partial^2 f_2}{\partial y\,\partial x}-\frac{\partial^2 f_1}{\partial y^2}-\frac{\partial^2 f_1}{\partial z^2}+\frac{\partial^2 f_3}{\partial z\,\partial x}\right\}+\mathbf{j}\left\{\frac{\partial^2 f_3}{\partial z\,\partial y}-\frac{\partial^2 f_2}{\partial z^2}-\frac{\partial^2 f_2}{\partial x^2}+\frac{\partial^2 f_1}{\partial x\,\partial y}\right\}$$

$$+\mathbf{k}\left\{\frac{\partial^2 f_1}{\partial x\,\partial z}-\frac{\partial^2 f_3}{\partial x^2}-\frac{\partial^2 f_3}{\partial y^2}+\frac{\partial^2 f_2}{\partial y\,\partial z}\right\}$$

$$=\mathbf{i}\,\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)+\mathbf{j}\,\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)$$

$$+\mathbf{k}\,\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)-(i\,\nabla^2 f_1+j\,\nabla^2 f_2+k\,\nabla^2 f_3)$$

$$=\nabla(\nabla\,.\,\mathbf{f})-\nabla^2(\mathbf{i}\,f_1+\mathbf{j}\,f_2+\mathbf{k}\,f_3)$$

$$=\nabla(\nabla.\,\mathbf{f})-\nabla^2\mathbf{f}$$

so $\nabla\times(\nabla\times\mathbf{f})=\nabla(\nabla\,.\,\mathbf{f})-\nabla^2\mathbf{f}.$

EXAMPLES

1. *Prove that div grad* $r^n=\nabla^2 r^n=n\,(n+1)\,r^{n-2}$.

Sol. Let $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$

$\therefore$ $$r=\sqrt{(x^2+y^2+z^2)}$$

$\therefore$ $$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{(x^2+y^2+z^2)}}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r}\ \text{and}\ \frac{\partial r}{\partial z}=\frac{z}{r} \quad ...(i)$$

Let $\phi=r^n$, then

$$\frac{\partial\phi}{\partial x}=nr^{n-1}\,\frac{\partial r}{\partial x}=nr^{n-1}\,.\,\frac{x}{r}=nxr^{n-2} \quad ...(ii)$$

$$\frac{\partial^2\phi}{\partial x^2}=n\left[1\,.\,r^{n-2}+x\,(n-2)\,r^{n-3}\,.\,\frac{x}{r}\right]$$

$$=nr^{n-2}\left[1+\frac{(n-2)}{r^2}\,x^2\right]$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2}=nr^{n-2}\left[1+\frac{(n-2)}{r^2}\,y^2\right]$$

and $$\frac{\partial^2\phi}{\partial z^2}=nr^{n-2}=nr^{n-2}\left[1+\frac{(n-2)}{r^2}\,z^2\right]$$

$\therefore$ $$\text{div grad } r^n=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$=nr^{n-2}\left[1+\frac{(n-2)}{r^2}+x^2\right]+nr^{n-2}\left[1+\frac{(n-2)}{r^2}\,y^2\right]$$

$$+nr^{n-2}\left[1+\frac{(n-2)}{r^2}\,z^2\right]$$

$$=nr^{n-2}\left[3+\frac{(n-2)}{r^2}\,(x^2+y^2+z^2)\right]$$

$$=nr^{n-2}\left[3+\frac{(n-2)}{r^2}\,.\,r^2\right]$$

$$= nr^{n-2}(3+n-2)$$
$$= n(n+1)r^{n-2}$$

2. *Prove that curl grad* $r^n = 0$.

Sol. We have : $\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$ and $r^2 = x^2 + y^2 + z^2$.

Also, $$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\text{grad } r^n = \mathbf{i}\frac{\partial}{\partial x}(r^n) + \mathbf{j}\frac{\partial}{\partial y}(r^n) + \mathbf{k}\frac{\partial}{\partial z}(r^n)$$
$$= \mathbf{i}\,.\,nr^{n-1}\frac{\partial r}{\partial x} + \mathbf{j}\,.\,nr^{n-1}\frac{\partial r}{\partial y} + \mathbf{k}\,.\,nr^{n-1}\frac{\partial r}{\partial z}$$
$$= nr^{n-1}\frac{x}{r}\mathbf{i} + nr^{n-1}\frac{y}{r}\mathbf{j} + nr^{n-1}\frac{z}{r}\mathbf{k}$$
$$= nr^{n-2}(\mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z) = nr^{n-2}\mathbf{r}.$$

$$\therefore \quad \text{curl grad } r^n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ nr^{n-2}x & nr^{n-2}y & nr^{n-2}z \end{vmatrix}$$
$$= n\left[\mathbf{i}\left\{\frac{\partial}{\partial y}(r^{n-2}z) - \frac{\partial}{\partial z}(r^{n-2}y)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial z}(r^{n-2}x) - \frac{\partial}{\partial x}(r^{n-2}z)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial x}(r^{n-2}y) - \frac{\partial}{\partial y}(r^{n-2}x)\right\}\right]$$
$$= n\left[\mathbf{i}\left\{(n-2)r^{n-3}\frac{\partial r}{\partial y}z - (n-2)r^{n-3}\frac{\partial r}{\partial z}y\right\} + \mathbf{j}\left\{(n-2)r^{n-3}\frac{\partial r}{\partial z}x - (n-2)r^{n-3}\frac{\partial r}{\partial x}z\right\} + \mathbf{k}\left\{(n-2)r^{n-2}\frac{\partial r}{\partial x}y - (n-2)r^{n-3}\frac{\partial r}{\partial y}x\right\}\right]$$
$$= n(n-2)r^{n-3}\left[\mathbf{i}\left(\frac{y}{r}z - \frac{z}{r}y\right) + \mathbf{j}\left(\frac{z}{r}x - \frac{x}{r}z\right) + \mathbf{k}\left(\frac{x}{r}y - \frac{y}{r}x\right)\right]$$
$$= n(n-2)r^{n-3}[\mathbf{i}\,.\,0 + \mathbf{j}\,.\,0 + \mathbf{k}\,.\,0]$$
$$= 0.$$

3. *Prove that* $\nabla^2\left(\dfrac{x}{r^2}\right) = -\dfrac{2x}{r^4}$.

Sol. We have

$$\text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^2}\right) \quad \text{...(i)}$$

We have

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)\right]$$
$$= \frac{\partial}{\partial x}\left[1.\frac{1}{r^2} - \frac{2x}{r^3}.\frac{\partial r}{\partial x}\right]$$
$$= \frac{\partial}{\partial x}\left[\frac{1}{r^2} - \frac{2x}{r^3}.\frac{x}{r}\right] = \frac{\partial}{\partial x}\left[\frac{1}{r^2} - \frac{2x^2}{r^4}\right]$$

$$= \left[-\frac{2}{r^3}\,\frac{x}{r} - \frac{4x}{r^4} + \frac{8x^2}{r^5}\cdot\frac{x}{r} \right]$$

$$= \left[-\frac{2}{r^4}\,x - \frac{4x}{r^4} + \frac{8x^3}{r^6} \right]$$

$$= -\frac{6x}{r^4} + \frac{8x^3}{r^6} = -2x\left[\frac{3}{r^4} - \frac{4x^2}{r^6} \right] \qquad \text{...(i)}$$

Now
$$\frac{\partial}{\partial y^2}\left(\frac{x}{r^2}\right) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\right] = \frac{\partial}{\partial y}\left[\frac{-2x}{r^3}\cdot\frac{\partial r}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{-2x}{r^3}\cdot\frac{y}{r}\right] = \frac{\partial}{\partial y}\left(\frac{-2xy}{r^4}\right)$$

$$= -2x\left[\frac{1}{r^4} - \frac{4y}{r^5}\cdot\frac{y}{r}\right]$$

$$= -2x\left(\frac{1}{r^4} - \frac{4y^2}{r^6}\right) \qquad \text{...(ii)}$$

Similarly,

$$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^2}\right) = -2x\left(\frac{1}{r^4} - \frac{4z^2}{r^6}\right) \qquad \text{...(iii)}$$

$\therefore$
$$\nabla^2\left(\frac{x}{r^2}\right) = -2x\left[\frac{3}{r^4} + \frac{1}{r^4} + \frac{1}{r^4} - \frac{4}{r^6}(x^2+y^2+z^2)\right]$$

$$= -2x\left[\frac{5}{r^4} - \frac{4}{r^4}\right] = \frac{-2x}{r^4}.$$

EXERCISES

1. Verify that curl grad $\phi = 0$ and div $\phi\,\mathbf{A} = \text{grad}\,\phi\,.\,\mathbf{A} + \phi\,\text{div}\,\mathbf{A}$ given that $\phi = x^3 + y^3 + z^3 + 3xyz$ and $= x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.
2. If $\mathbf{f} = x^2 y\,\mathbf{i} + xz\,\mathbf{j} + 2yz\,\mathbf{k}$ prove that div (curl $\mathbf{f}$) = 0 .
3. Prove that div grad $\left(\frac{1}{r}\right) = 0$ or $\nabla^2\left(\frac{1}{r}\right) = 0$.
4. Show that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$
5. Prove that curl $\{f(r)\,\mathbf{r}\} = 0$.
6. Prove that curl curl $\mathbf{F} = 0$ where $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$.

3.14 VECTOR IDENTITIES

If u and v be two scalar functions and $\mathbf{a}$ and $\mathbf{b}$ be two vector functions, then we can have the products $u\,v$ and $\mathbf{a.b}$ both scalar. So we shall find

grad (uv) and grad $(\mathbf{a.b})$. Similarly products $u\,\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}$ are vetors, so we can find btoh their divergence as well as curl, *i.e.,*

$$\text{div}\,.(u\,\mathbf{a}),\ \text{div}\,(\mathbf{a}\times\mathbf{b}) \ \text{ and } \ \text{curl}\,(u\,\mathbf{a}),\ \text{curl}\,(\mathbf{a}\times\mathbf{b})$$

These results are known as *vector identities* and we shall find these one by one.

(I) grad (u v) = u grad v + u grad v

We have, $$\text{grad}(u\,v) = \Sigma\,\mathbf{i}\,\frac{\partial}{\partial x}(uv)$$

$$= \Sigma\,\mathbf{i}\left(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial u}{\partial x}\right)$$

$$= u\left(\mathbf{i}\,\frac{\partial v}{\partial x} + \mathbf{j}\,\frac{\partial v}{\partial y} + \mathbf{k}\,\frac{\partial v}{\partial z}\right) + v\left(\mathbf{i}\,\frac{\partial u}{\partial x} + \mathbf{j}\,\frac{\partial u}{\partial y} + \mathbf{k}\,\frac{\partial u}{\partial z}\right)$$

$$= u\;\text{grad}\;v + v\;\text{grad}\;u$$

i.e., $\nabla(u\,v) = u\,\nabla\,v + v\,\nabla\,u$

(II) grad (a . b) = a × curl b + b × curl a + (a . ∇) b + (b . ∇) a

We have $$\text{grad}\,(\mathbf{a}\,.\,\mathbf{b}) = \Sigma\,\mathbf{i}\,\frac{\partial}{\partial x}(\mathbf{a}\,.\,\mathbf{b})$$

$$= \Sigma\,\mathbf{i}\left(\mathbf{a}\,.\,\frac{\partial \mathbf{b}}{\partial x} + \mathbf{b}\,.\,\frac{\partial \mathbf{a}}{\partial x}\right)$$

$$= \Sigma\,\mathbf{i}\left(\mathbf{a}\,.\,\frac{\partial \mathbf{b}}{\partial x}\right) + \Sigma\,\mathbf{i}\left(\mathbf{b}\,.\,\frac{\partial \mathbf{a}}{\partial x}\right) \qquad \text{...(i)}$$

Now, $$\mathbf{a} \times \left(\mathbf{i} \times \frac{\partial \mathbf{b}}{\partial x}\right) = \left(\mathbf{a}\,.\,\frac{\partial \mathbf{b}}{\partial x}\right)\mathbf{i} - (\mathbf{a}\,.\,\mathbf{i})\,\frac{\partial \mathbf{b}}{\partial x}$$

or $$\left(\mathbf{a}\,.\,\frac{\partial \mathbf{b}}{\partial x}\right)\mathbf{i} = \mathbf{a} \times \left(\mathbf{i} \times \frac{\partial \mathbf{b}}{\partial x}\right) + (\mathbf{a}\,.\,\mathbf{i})\,\frac{\partial \mathbf{b}}{\partial x}$$

∴ $$\Sigma\left(\mathbf{a}\,.\,\frac{\partial \mathbf{b}}{\partial x}\right)\mathbf{i} = \mathbf{a} \times \Sigma\left(\mathbf{i} \times \frac{\partial \mathbf{b}}{\partial x}\right) + \Sigma\,(\mathbf{a}\,.\,\mathbf{i})\,\frac{\partial \mathbf{b}}{\partial x}$$

$$= \mathbf{a} \times \text{curl}\;\mathbf{b} + (\mathbf{a}\,.\,\nabla)\,\mathbf{b} \qquad \text{...(ii)}$$

Similarly,

$$\Sigma\left(\mathbf{b}\,.\,\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i} = \mathbf{b} \times \text{curl}\;\mathbf{a} + (\mathbf{b}\,.\,\nabla)\,\mathbf{b} \qquad \text{...(iii)}$$

Hence from (i), (ii), and (iii), we obtain

$$\nabla\,(\mathbf{a}\,.\,\mathbf{b}) = \mathbf{a} \times \text{curl}\;\mathbf{b} + \mathbf{b} \times \text{curl}\;\mathbf{a} + (\mathbf{a}\,.\,\nabla)\,\mathbf{b} + (\mathbf{b}\,.\,\nabla)\,\mathbf{a}$$

(III) div (*u* a) = *u* div a + a . grad *u*

We have

$$\text{div}\,(u\,\mathbf{a}) = \mathbf{i}\,.\,\frac{\partial}{\partial x}(u\,\mathbf{a}) + \mathbf{j}\,.\,\frac{\partial}{\partial y}(u\,\mathbf{a}) + \mathbf{k}\,.\,\frac{\partial}{\partial z}(u\,\mathbf{a})$$

$$= \mathbf{i}\,.\left(\mathbf{a}\,\frac{\partial u}{\partial x} + u\,\frac{\partial \mathbf{a}}{\partial x}\right) + \mathbf{j}\,.\left(\mathbf{a}\,\frac{\partial u}{\partial y} + u\,\frac{\partial \mathbf{a}}{\partial y}\right) + \mathbf{k}\left(\mathbf{a}\,\frac{\partial u}{\partial x} + u\,\frac{\partial \mathbf{a}}{\partial z}\right)$$

$$= \mathbf{i}\,.\left(\mathbf{a}\,\frac{\partial u}{\partial x}\right) + \mathbf{j}\,.\left(\mathbf{a}\,\frac{\partial u}{\partial y}\right) + \mathbf{k}\left(\mathbf{a}\,\frac{\partial u}{\partial z}\right) + u\left(\mathbf{i}\,\frac{\partial \mathbf{a}}{\partial x} + \mathbf{j}\,\frac{\partial \mathbf{a}}{\partial y} + \mathbf{k}\,\frac{\partial \mathbf{a}}{\partial z}\right)$$

$$= \mathbf{a}\,.\left(\mathbf{i}\,\frac{\partial u}{\partial x} + \mathbf{j}\,\frac{\partial u}{\partial y} + \mathbf{k}\,\frac{\partial u}{\partial z}\right) + u\left(\mathbf{i}\,\frac{\partial \mathbf{a}}{\partial x} + \mathbf{j}\,\frac{\partial \mathbf{a}}{\partial y} + \frac{\partial \mathbf{a}}{\partial z}\right)$$

$$= \mathbf{a}\,.\;\text{grad}\;u + u\;\text{div}\;\mathbf{a}.$$

(IV) div (**a** × **b**) = **b** . curl **a** − **a** . curl **b**.

We have $$\text{div}\;(\mathbf{a} \times \mathbf{b}) = \mathbf{i}\,.\,\frac{\partial}{\partial x}(\mathbf{a} \times \mathbf{b}) + \mathbf{j}\,.\,\frac{\partial}{\partial y}(\mathbf{a} + \mathbf{b}) + \mathbf{k}\,.\,\frac{\partial}{\partial z}(\mathbf{a} \times \mathbf{b})$$

$$= \Sigma\,\mathbf{i}\left(\frac{\partial \mathbf{a}}{\partial x} \times \mathbf{b} + \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial x}\right)$$

$$= \Sigma \mathbf{i} \frac{\partial \mathbf{a}}{\partial x} + \mathbf{b} + \Sigma \mathbf{i} . \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial x}$$

$$= \left(\Sigma \mathbf{i} \times \frac{\partial \mathbf{a}}{\partial x} \right) . \mathbf{b} - \left(\Sigma \mathbf{i} \times \frac{\partial \mathbf{b}}{\partial x} \right) . \mathbf{a}$$

$$= \mathbf{b} . \text{curl } \mathbf{a} - \mathbf{a} . \text{curl } \mathbf{b}$$

(V) curl (u a) = (grad u) × a + u curl a

We have

$$\text{curl } (u\,\mathbf{a}) = \mathbf{i} \times \frac{\partial}{\partial x}(u\,\mathbf{a}) + \mathbf{j} \times \frac{\partial}{\partial y}(u\,\mathbf{a}) + \mathbf{k} \times \frac{\partial}{\partial z}(u\,\mathbf{a})$$

$$= \Sigma \mathbf{i} \times \left(\frac{\partial u}{\partial x} \mathbf{a} + u \frac{\partial \mathbf{a}}{\partial x} \right)$$

$$= \Sigma \mathbf{i} \times \left(\frac{\partial u}{\partial x} \mathbf{a} \right) + \Sigma \mathbf{i} \times u \frac{\partial \mathbf{a}}{\partial x}$$

$$= \Sigma \left(\mathbf{i} \frac{\partial u}{\partial x} \right) \times \mathbf{a} + \left(\Sigma \mathbf{i} \times \frac{\partial \mathbf{a}}{\partial x} \right) u$$

$$= (\text{grad } u) \times \mathbf{a} + u \text{ curl } \mathbf{a}$$

(VI) curl (a × b) = a div b − b div a + (b . ∇) a − (a . ∇) b

We have

$$\text{curl } (\mathbf{a} \times \mathbf{b}) = \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{a} \times \mathbf{b}) + \mathbf{j} \times \frac{\partial}{\partial y}(\mathbf{a} \times \mathbf{b}) + \mathbf{k} \times \frac{\partial}{\partial z}(\mathbf{a} \times \mathbf{b})$$

$$= \Sigma \mathbf{i} \times \left(\mathbf{a} \times \frac{\partial b}{\partial x} + \frac{\partial \mathbf{a}}{\partial x} \times \mathbf{b} \right)$$

$$= \Sigma \mathbf{i} \times \left(\mathbf{a} \times \frac{\partial \mathbf{b}}{\partial x} \right) + \Sigma \mathbf{i} \times \left(\frac{\partial \mathbf{a}}{\partial x} \times \mathbf{b} \right)$$

$$= \Sigma \left(\mathbf{i} . \frac{\partial \mathbf{b}}{\partial x} \right) \mathbf{a} - \Sigma (\mathbf{i} . \mathbf{a}) \frac{\partial \mathbf{b}}{\partial x} + \Sigma (\mathbf{i} . \mathbf{b}) \frac{\partial \mathbf{a}}{\partial x} - \Sigma \left(\mathbf{i} . \frac{\partial \mathbf{a}}{\partial x} \right) \mathbf{b}$$

as $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} . \mathbf{c})\, \mathbf{b} - (\mathbf{a} . \mathbf{b})\, \mathbf{c}$

$$= \left(\Sigma \mathbf{i} . \frac{\partial \mathbf{b}}{\partial x} \right) \mathbf{a} - \left(\Sigma \mathbf{i} . \frac{\partial \mathbf{a}}{\partial x} \right) \mathbf{b} + \Sigma (\mathbf{i} . \mathbf{b}) \frac{\partial \mathbf{a}}{\partial x} - \Sigma (\mathbf{i} . \mathbf{a}) \frac{\partial \mathbf{b}}{\partial x}$$

$$= \mathbf{a} \text{ div } \mathbf{b} - \mathbf{b} \text{ div } \mathbf{a} + (\mathbf{b} . \nabla)\, \mathbf{a} - (\mathbf{a} . \nabla)\, \mathbf{b}$$

EXAMPLES

1. *If* **a** *be a constant vector find grad* **(a.f)**, *div* **(a × f)** *and curl* **(a × f)**.

Sol. We have

$$\text{grad } (\mathbf{a} . \mathbf{f}) = \mathbf{a} \times \text{curl } \mathbf{f} + \mathbf{f} \times \text{curl } \mathbf{a} + (\mathbf{a} . \nabla)\, \mathbf{f} + (\mathbf{f} . \nabla)\, \mathbf{a}, \text{ by Identity II}$$

$$= \mathbf{a} \times \text{curl } \mathbf{a} + (\mathbf{a} . \nabla)\, \mathbf{f}$$

as **a** is constant vector, hence

$$\text{curl } \mathbf{a} = 0 \text{ and } (\mathbf{f} . \nabla)\, \mathbf{a} = 0$$

$$\text{div } (\mathbf{a} \times \mathbf{f}) = (\text{curl } \mathbf{a}) . \mathbf{f} - (\text{curl } \mathbf{f}) . \mathbf{a} \text{ by identity IV}$$

$$= -(\text{curl } \mathbf{f}) . \mathbf{a} \qquad \text{as curl } \mathbf{a} = 0$$

Hence $\text{curl } (\mathbf{a} \times \mathbf{f}) = \mathbf{a} \text{ div } \mathbf{a} + (\mathbf{f} . \nabla)\, \mathbf{a} - (\mathbf{a} . \nabla)\, \mathbf{f}$

by Identity VI as div $\mathbf{a} = \mathbf{0}$ and $(\mathbf{f} . \nabla)\, \mathbf{a} = \mathbf{0}$

2. *If* $\mathbf{f} = \psi \text{ grad } \phi$, *show that* $\mathbf{f} . \text{curl } \mathbf{f} = 0$.

Sol. We have

$$\text{curl } \mathbf{f} = \text{curl } (\psi \text{ grad } \phi)$$

$$= (\text{grad } \psi) \times (\text{grad } \phi) + \psi \text{ curl } (\text{grad } \phi)$$

$$[\text{as curl } u\,\mathbf{a} = (\text{grad } u) \times \mathbf{a} + u \text{ curl } \mathbf{a}]$$

$$= (\text{grad } \psi) \times (\text{grad } \phi) \quad \text{as curl (grad } \phi) = 0$$

$$\therefore \quad \mathbf{f} \cdot \text{curl } \mathbf{f} = \psi\, (\text{grad } \phi) \cdot \{(\text{grad } \psi) \times (\text{grad } \phi)\} = 0$$

as scalar triple product in which two vectors are equal is zero.

3. *If*

$$\mathbf{F} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k}$$

then prove that

(i) $\mathbf{F} = \mathbf{r} \times \nabla f$, (ii) $\mathbf{F} \cdot \mathbf{r} = \mathbf{0}$ and (iii) $\mathbf{F}$. grad = 0.

Sol. We have

$$\mathbf{r} = \mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z$$

and
$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

(i) $\therefore$
$$\mathbf{r} \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) + \mathbf{j}\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right) + \mathbf{k}\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)$$

$$= \mathbf{F}$$

(ii)
$$\mathbf{F} \cdot \mathbf{r} = \left[\left(t\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\mathbf{k}\right)\right] \cdot (\mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z)$$

$$= \left(xy\frac{\partial f}{\partial z} - xz\frac{\partial f}{\partial y} + yz\frac{\partial f}{\partial x} - xy\frac{\partial f}{\partial z} + zx\frac{\partial f}{\partial y} - yz\frac{\partial f}{\partial x}\right)$$

$$= 0$$

(ii) $\mathbf{F}$. grad $f = \left[\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\mathbf{j}\right.$

$$\left. + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k}\right] \cdot \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$= y\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} + z\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} - x\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial z} + x\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial z} - y\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial z}$$

$$= 0.$$

EXERCISES

1. If $\mathbf{f}$ and $\mathbf{g}$ are irrotational, show that $\mathbf{f} \times \mathbf{g}$ is solenoidal.
2. Prove that $\nabla \cdot \left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$
3. Prove that $\nabla^2\left[\left(\frac{\mathbf{r}}{r^2}\right)\right] = 2r^{-4}$
4. Prove that $\nabla\left(\mathbf{a} \cdot \frac{\mathbf{r}}{r^n}\right) = \frac{\mathbf{a}}{r^n} - n\frac{(\mathbf{a} \cdot \mathbf{r})\,\mathbf{r}}{n+2}$
5. Prove that div $\frac{\mathbf{a} \times \mathbf{r}}{r^3} = 0$ and curl $\frac{\mathbf{a} \times \mathbf{r}}{r^3} = -\frac{\mathbf{a}}{r^3} + \frac{3\,\mathbf{r}}{r^5}(\mathbf{a} \cdot \mathbf{r})$.

6. Prove that $\nabla \cdot \left(\mathbf{a} \times \dfrac{\mathbf{r}}{r^n} \right) = 0$

7. Show that curl $(\mathbf{a} \cdot \mathbf{r})\, \mathbf{a} = 0$

8. Prove that curl $(\psi \nabla \phi) = \nabla\psi \times \nabla\phi = -$ curl $(\phi \nabla \psi)$.

9. Show that $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is conservative and find ϕ such that $F = \nabla\phi$.

10. (i) Prove that $\mathbf{v} \cdot \nabla \mathbf{v} = \dfrac{1}{2} \nabla \mathbf{v}^2 - \mathbf{v} \times \text{curl}\, \mathbf{v}$.

(ii) If **a** is a constant unit vector, prove that

$$\hat{\mathbf{a}} \cdot [\nabla (\mathbf{v} \cdot \mathbf{a}) - \text{curl}\,(\mathbf{v} \times \hat{\mathbf{a}})] = \text{div}\,.\, \mathbf{v}.$$

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4

Integration of Vectors

4.1 INTEGRATION OF VECTORS

Integration is the inverse process of differentiation. If two functions **B** (t) and **f** (t) are connected together such that

$\frac{d}{dt}\{\mathbf{F}(t)\} = f(t)$, then $\mathbf{F}(t)$ is called integral of $\mathbf{f}(t)$ and in symbol

$$\mathbf{F}(t) = \int \mathbf{f}(t)\, dt$$

f (t), the function to be integrated is called the integrand and t is the variable of integration.

If **c** is an arbitrary constant vector, then we have

$$\frac{d}{dt}[\mathbf{F}(t) \pm \mathbf{c}] = \mathbf{f}(t)$$

i.e., $$\int \mathbf{f}(t)\, dt = \mathbf{F}(t) \pm \mathbf{c}$$

The arbitrary constant **c** is called the constant of integration.

The following standard results have been derived :

$$\frac{d}{dt}(\mathbf{r}\,.\,\mathbf{s}) = \frac{d\,\mathbf{r}}{dt}.\mathbf{s} + \mathbf{r}.\frac{d\,\mathbf{s}}{dt}$$

$$\therefore \quad \int \left(\frac{d\,\mathbf{r}}{dt}.\mathbf{s} + \mathbf{r}.\frac{d\,\mathbf{s}}{dt}\right) dt = \mathbf{r}\,.\,\mathbf{s} + \mathbf{c} \qquad \text{...(i)}$$

Particularly, if **s** = **r**,

$$\int \left(2\,\mathbf{r}.\frac{d\,\mathbf{r}}{dt}\right) dt = \mathbf{r}^2 + \mathbf{c} \qquad \text{...(ii)}$$

Since the derivative of $\left(\frac{d\,\mathbf{r}}{dt}\right)^2$ is $2\left(\frac{d\,\mathbf{r}}{dt}.\frac{d^2\mathbf{r}}{dt^2}\right)$

hence $$\int 2\left(\frac{d\,\mathbf{r}}{dt}.\frac{d^2\mathbf{r}}{dt^2}\right) dt = \left(\frac{d\,\mathbf{r}}{dt}\right)^2 + \mathbf{c} \qquad \text{...(iii)}$$

Again, the derivative of the unit vector $\hat{\mathbf{r}}$ may be written as

$$\frac{d}{dt}(\hat{\mathbf{r}}) = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{1}{r}\frac{d\,\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}$$

$$\therefore \quad \int \left(\frac{1}{r}\frac{d\,\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right) = \frac{\mathbf{r}}{r} + \mathbf{c} = \hat{\mathbf{r}} + \mathbf{c} \qquad \text{...(iv)}$$

Note : *It should be borne in mind that the constant of integration is of the same nature as the integrand, i.e., if integrand is a vector* **c** *is a vector and if integrand is a scalar* **c** *is a scalar.*

EXAMPLES

1. *Find the value of* **r** *satisfying the equation* $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}t + \mathbf{b}$, *where* **a** *and* **b** *are constant vectors.*

Sol. Integrating the $$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}t + \mathbf{b},$$

we get $$\frac{d\mathbf{r}}{dt} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{b}t + \mathbf{c},$$
Where **c** is a constant.

Again integration, we get $$\mathbf{r} = \frac{1}{6}\mathbf{a}t^3 + \frac{1}{2}\mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$
Where **d** is a constant.

2. *If* $\mathbf{r}(t) = 5t^2\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$,

$$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = -14\,\mathbf{i} + 75\,\mathbf{j} - 15\,\mathbf{k}$$

Sol. We have $$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{c}$$

$$\therefore \quad \int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \left[\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right]_1^2$$

Now, $$\frac{d\mathbf{r}}{dt} = 10t\,\mathbf{i} + \mathbf{j} - 3t^2\,\mathbf{k}$$

$$\therefore \quad \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= -2t^3\,\mathbf{i} + 5t^4\,\mathbf{j} - 5t^2\,\mathbf{k}$$

$$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \left[-2t^3\,\mathbf{i} + 5t^4\,\mathbf{j} - 5t^2\,\mathbf{k}\right]_1^2$$

$$= -14\,\mathbf{i} + 75\,\mathbf{j} - 15\,\mathbf{k}$$

3. *Given that*

$$\mathbf{r}(t) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \text{ when } t = 2$$
$$= 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \text{ when } t = 3$$

Show that $$\int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) dt = 10.$$

Sol. We have $$\int \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \frac{1}{2}\mathbf{r}^2 + \mathbf{c}$$

$$\therefore \quad \int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \left[\frac{1}{2}\mathbf{r}^2\right]_{t=2}^{t=3}$$

When $t = 3$, $\mathbf{r} = 4\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$

$$\mathbf{r}^2 = (4\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot (4\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k})$$
$$= 16 + 4 + 9 = 29$$

When $t = 2$, $\mathbf{r} = 2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$

$$\mathbf{r}^2 = (2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}) \cdot (2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k})$$
$$= 4 + 1 + 4 = 9$$

$$\therefore \quad \int_2^3 \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) dt = \frac{1}{2}[29 - 9] = 10.$$

4. *If* $\mathbf{r} \times d\mathbf{r} = \mathbf{0}$, *show that* $\hat{\mathbf{r}} = $ *constant.*

Sol. Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

then $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

$$\therefore \quad \mathbf{r} \times d\mathbf{r} = \mathbf{0}$$

$$\Rightarrow \quad (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$$

$$\Rightarrow \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ dx & dy & dz \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \quad (y\,dz - z\,dy)\,\mathbf{i} + (z\,dx - x\,dz)\,\mathbf{j} + (x\,dy - y\,dx)\,\mathbf{k} = \mathbf{0}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}.$$

Equating the coefficients of **i, j, k** on both sides, we get

$$y\,dz - z\,dy = 0 \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$z\,dx - x\,dz = 0 \Rightarrow \frac{dz}{z} = \frac{dx}{x}$$

$$x\,dy - y\,dx = 0 \Rightarrow \frac{dx}{x} = \frac{dy}{y}.$$

The three results on combination admit

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

Taking first two,
Integrating both sides, we get

$$\log x = \log y + \log c_1,$$

where $\log c_1$ is an arbitrary scalar constant of integration.

$$\Rightarrow \quad \log x = \log (c_1 y)$$

$$\Rightarrow \quad x = c_1 y.$$

Taking last two,
Integrating yields $\log z = \log y + \log c_2,$
where $\log c_2$ is an arbitrary scalar constant of integration.

$$\Rightarrow \quad z = c_2 y.$$

Hence,
$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}.$$

$$= \frac{c_1 y\,\mathbf{i} + y\,\mathbf{j} + c_2 y\,\mathbf{k}}{\sqrt{c_1^2 y^2 + y^2 + c_2^2 y^2}}$$

$$= \frac{c_1\,\mathbf{i} + \mathbf{j} + c_2\,\mathbf{k}}{\sqrt{c_1^2 + 1 + c_2^2}}$$

which is clearly a constant vector being independent of x, y and z.

5. *Show that necessary and sufficient condition that direction of given vector* **r** *is constant is that*

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = 0.$$

Sol. Let $|\mathbf{r}| = r_1$

$$\hat{\mathbf{r}} = \mathbf{R}$$

Hence, $\mathbf{r} = r_1\,\mathbf{R}$ ...(1)

Necessary condition : Given that direction of **r** is constant, we have to prove that

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = 0$$

$$\mathbf{r} \times \frac{d\,\mathbf{r}}{dt} = r_1\,\mathbf{R} + \frac{d}{dt}\{r_1\,\mathbf{R}\}$$

$$= r_1\,\mathbf{R} \times \left\{\frac{dr_1}{dt}\,\mathbf{R} + r_1\,\frac{d\,\mathbf{R}}{dt}\right\}$$

$$= r_1\,\frac{dr_1}{dt}\,(\mathbf{R} \times \mathbf{R}) + r_1^2\left(\mathbf{R} \times \frac{d\,\mathbf{R}}{dt}\right)$$

$$= r_1^2\left(\mathbf{R} \times \frac{d\,\mathbf{R}}{dt}\right) \quad [\because\ \mathbf{R} \times \mathbf{R} = 0]$$

$$= 0.\ \text{as } \mathbf{R} \text{ is constant, so } \frac{d\,\mathbf{R}}{dt} = 0.$$

Sufficient Condition : Given that $\mathbf{r} \times \dfrac{d\,\mathbf{r}}{dt} = 0.$

We have to prove direction of $\mathbf{r}$ is constant

$$\mathbf{r} \times \frac{d\,\mathbf{r}}{dt} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} & \dfrac{dz}{dt} \end{vmatrix}$$

$$\Rightarrow \quad = \mathbf{i}\left\{y\,\frac{dz}{dt} - z\,\frac{dy}{dt}\right\} - \mathbf{j}\left\{x\,\frac{dz}{dt} - z\,\frac{dx}{dt}\right\} + \mathbf{k}\left\{x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right\}$$

$$\Rightarrow \quad = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$$

Comparing the coefficient of **i, j, k**

$$y\,\frac{dz}{dt} - z\,\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dz}{z} = \frac{dy}{y}$$

$$\Rightarrow \quad \log z = \log y + \log c_1$$

$$\Rightarrow \quad z = c_1 y$$

$$x\,\frac{dz}{dt} - z\,\frac{dx}{dt} = 0$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dz}{z}$$

$$\Rightarrow \quad \log z = \log x + \log c_2$$

$$\Rightarrow \quad z = c_2 x$$

$$x\,\frac{dy}{dt} - y\,\frac{dx}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \quad \log y = \log x + \log c_3$$

$$\Rightarrow \quad y = c_3 x$$

Hence,

$$\mathbf{r} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\,\mathbf{i} + c_3 x\,\mathbf{j} + c_2 x\,\mathbf{k}}{\sqrt{x^2 + c_3^2 x^2 + c_2^2 x^2}}$$

$$= \frac{\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}}{\sqrt{1 + c_3^2 + c_2^2}}$$

which is independent of x, y, z so it is constnat vector.

EXERCISES

1. If $\mathbf{f}(t) = t\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + (2t^2 + 3t^3)\,\mathbf{k}$,

find $\int_0^1 \mathbf{f}(t)\,dt$. $\left[\textbf{Ans. } \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{4}\mathbf{k}\right]$

2. If $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + (t-1)\,\mathbf{k}$ and $\mathbf{s} = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, evaluate

(i) $\int_0^2 (\mathbf{r}\,.\,\mathbf{s})\,dt$, (ii) $\int_0^2 (\mathbf{r} \times \mathbf{s})\,dt$. $\left[\textbf{Ans. } \text{(i) } 12,\ \text{(ii) } -24\,\mathbf{i} - \frac{40}{3}\mathbf{j} + \frac{64}{5}\mathbf{k}\right]$

3. Evaluate $\int_0^1 (e^t\,\mathbf{i} + e^{-2t}\mathbf{j} + t\,\mathbf{k})\,dt$. $\left[\textbf{Ans. } (e-1)\,\mathbf{i} - \frac{1}{2}(e^{-2} - 1)\,\mathbf{j} + \frac{1}{2}\mathbf{k}\right]$

4. The acceleration of a particle at any time t is $e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + \mathbf{k}$ find $\mathbf{v}$ given that $\mathbf{v} = \mathbf{i} + \mathbf{j}$, at $t = 0$. $\left[\textbf{Ans. } e^t\,\mathbf{i} + \frac{1}{2}(2^{2t} + 1)\,\mathbf{j} + t\,\mathbf{k}\right]$

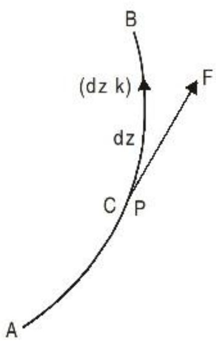
5. If $\mathbf{a} = t\,\mathbf{i} - 3\,\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$, $\mathbf{c} = 3\,\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$
show that

(i) $\int_1^2 \mathbf{a}\,.\,(\mathbf{b} \times \mathbf{c})\,dt = 0$

(ii) $\int_1^2 \mathbf{a} \times (\mathbf{b} \times \mathbf{c})\,dt = -\frac{87}{2}\mathbf{i} - \frac{44}{3}\mathbf{j} + \frac{15}{2}\mathbf{k}$

4.2 LINE INTEGRALS

Let $\mathbf{r} = \mathbf{f}(t)$ represents, a continuously differentiable curve denoted by C and $\mathbf{f}(\mathbf{r})$ be a continuous vector point function. Then $\frac{d\mathbf{r}}{ds}$ is a unit vector function along the tangent at and point P on the curve. The component of the vector function $\mathbf{F}$ along this tangent is $\mathbf{F}.\ \frac{d\mathbf{r}}{ds}$ which is a function of s for points on the curve. Then



$$\int_C \mathbf{F}\,.\frac{d\mathbf{r}}{ds}\,ds = \int_C \mathbf{F}\,.\,d\mathbf{r},$$

is called the line integral or tangent line integral of $\mathbf{F}(\mathbf{r})$ along C.

Let $\mathbf{F} = \mathbf{i}\,F_1 + \mathbf{j}\,F_2 + \mathbf{k}\,F_3$

and $\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$

$\therefore$ $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$

$$\therefore \int_C \mathbf{F}\,.\,d\mathbf{r} = \int (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})\,.\,(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

$$\therefore \int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{t_1}^{t_2} \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt,$$

Where t_1 and t_2 are the values of the parameter t for extremities A and B of the arc of the curve C.

Again, if $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\therefore \quad \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\, ds$$

$$= \int_{s_1}^{s_2} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) ds$$

Where s_1 and s_2 are the values of s for the extremities of A and B of the arc C.

Physical Interpretation of $\int_C \mathbf{F} \cdot d\mathbf{r}$

If $\mathbf{F}$ represents a force acting on a particle moving along the curve C then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the *work done by the force*. If $\mathbf{F}$ represents the velocity of fluid, it is called the *circulation of* $\mathbf{F}$ about C.

Other types of line integrals

(i) $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \times \frac{d\mathbf{r}}{ds} \cdot ds = \int_{s_1}^{s_2} \mathbf{F} \times \mathbf{t}\, ds,$$

Where $\mathbf{t}$ is a unit tangent vector.

Now, $$\mathbf{F} \cdot d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix}$$

$$= \mathbf{i}\,(F_2\, dz - F_3\, dy) + \mathbf{j}\,(F_3\, dx - F_1\, dz) + \mathbf{k}\,(F_1\, dy - F_2\, dx)$$

$$\therefore \quad \int_C \mathbf{F} \times d\mathbf{r} = \mathbf{i} \int_C (F_2\, dz - F_3\, dy) + \mathbf{j} \int_C (F_3\, dx - F_1\, dz) + \mathbf{k} \int_C (F_1\, dy - F_2\, dx)$$

(ii) $$\int_C \phi\, d\mathbf{r} = \mathbf{i} \int_C \phi\, dx + \mathbf{j} \int_C \phi\, dy + \mathbf{k} \int_C \phi\, dz,$$

Where ϕ is a scalar point function.

EXAMPLES

1. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ *and where C is* $\mathbf{r} = \mathbf{i}\,t + \mathbf{j}\,t^2 + \mathbf{k}\,t^3$, t *varying from* -1 *to* $+1$.

Sol. The equation of the curve in parameteric from is

$$x = t, \quad y = t^2, \; z = t^3$$

$$\therefore \quad \mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$$

$$= t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Also $$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$= \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\therefore \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \cdot \frac{d\mathbf{r}}{dt} = \int_{-1}^{1} (t^3 + 5t^6)\, dt$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7} \right]_{-1}^{1} = \frac{10}{7}.$$

2. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ *and* **C** *is the portion of the curve* $\mathbf{r} = (a \cos t)\,\mathbf{i} + (b \sin t)\,\mathbf{j} + (ct)\,\mathbf{k}$ *from* $t = 0$ *to* $\pi/2$.

Sol. We have $$\mathbf{r} = (a \cos t)\,\mathbf{i} + (b \sin t)\,\mathbf{j} + (ct)\,\mathbf{k}.$$

Hence, the parametric equations of the given curve are

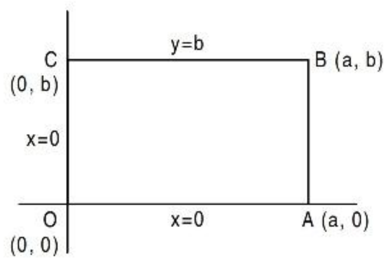
$$x = a\cos t$$
$$y = b\sin t$$
$$z = ct$$

Also, $\dfrac{dr}{dt} = (-a\sin t)\,\mathbf{i} + (b\cos t)\,\mathbf{j} + c\,\mathbf{k}$

Now, $\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_C \mathbf{F}.\dfrac{d\mathbf{r}}{dt}\,dt$

$$= \int_C (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k})\,.\,(-a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k})\,dt$$
$$= \int_C (bc\,t\sin t\,\mathbf{i} + a\,ct\cos t\,\mathbf{j} + ab\sin t\cos t\,\mathbf{k})\,.\,(-a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + c\,\mathbf{k})\,dt$$
$$= \int_C (-abc\,t\sin^2 t + abc\,t\cos^2 t + abc\sin t\cos t)\,dt$$
$$= abc\int_C [t(\cos^2 t - \sin^2 t) + \sin t\cos t]\,dt$$
$$= abc\int_C \left(t\cos 2t + \frac{\sin 2t}{2}\right)dt$$
$$= abc\int_0^{\pi/2} \left(t\cos 2t + \frac{\sin 2t}{2}\right)dt$$
$$= abc\left[t\,\frac{\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\pi/2}$$
$$= \frac{abc}{2}(t\sin 2t)_0^{\pi/2}$$
$$= 0.$$

3. *Evaluate* $\int_C \mathbf{F}\,.\,d\mathbf{r}$, *where* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$ *and the curve C is the rectangle in the xy plane bounded by* $y = 0,\ x = a,\ y = b,\ x = 0$.



Sol.
In the xy-plane,
$$z = 0$$
$\therefore\quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$

or $\quad d\,\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$

$\therefore\quad \int_C \mathbf{F}\,.\,d\mathbf{r} = \int_C [(x^2 + y^2)\,dx - 2xy\,dy]$ ...(i)

Now,
$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{OA} \mathbf{F}\,.\,d\mathbf{r} + \int_{AB} \mathbf{F}\,.\,d\mathbf{r} + \int_{BC} \mathbf{F}\,.\,d\mathbf{r} + \int_{CO} \mathbf{F}\,.\,d\mathbf{r}$$

Along OA, $y = 0$

$\therefore\quad dy = 0$ and x varies from 0 to a.

Along AB, $x = a$

$\therefore\quad dx = 0$, and y varies from 0 to b.

Along BC, $y = b$

$\therefore\quad dy = 0$ and x varies from a to 0

Along CO, $x = 0$

$\therefore\quad dx = 0$ and y varies from b to 0.

Hence from (i) and (ii), we get
$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2 + b^2)\,dx + \int_b^0 0\,.\,dy$$

$$= \frac{a^3}{3} - 2a \cdot \frac{b^2}{2} + \left[\frac{x^3}{3} + b^2 x\right]_a^0 + 0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2 a = -2ab^2.$$

4. *If* $\mathbf{F} = (x^2 + y^3)\,\mathbf{i} + (x^3 - y^2)\,\mathbf{j}$, *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *the following paths :*

(a) $y^2 = x$, *joining* (0, 0) *to* (1, 1)

(b) $x^2 = y$, *joining* (0, 0) *to* (1, 1)

(c) Along the straight line joining (0, 0) to (1, 0) and then to (1, 1).

(d) Along the straight line joining (0, 0) to (2, – 2) then to (0, – 1) and then to (1, 1).

Sol. Here we have

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \text{ so that } d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$$

Hence $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2 + y^3)\,dx + (x^3 - y^2)\,dy$ ...(i)

(a) We have $y^2 = x$, $\therefore$ $dx = 2y\,dy$ and y varies from 0 to 1

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (y^4 + y^3)(2y\,dy) + (y^6 - y^2)\,dy$$

$$= \int_0^1 (y^6 + 2y^5 + 2y^4 - y^2)\,dy$$

$$= \frac{1}{7} + \frac{1}{3} + \frac{2}{5} - \frac{1}{3} = \frac{19}{35}$$

(b) We have $x^2 = y$ $\therefore$ $2x\,dy = dx$ and x varies from 0 to 1

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x^2 + x^6)\,dx + (x^3 - x^4)\,2x\,dx$$

$$= \int_0^1 (x^6 - 2x^5 + 2x^4 + x^2)\,dx$$

$$= \frac{1}{7} - \frac{1}{3} + \frac{2}{5} + \frac{1}{3} = \frac{19}{35}$$

(c) Along the line joint (0, 0) to (1, 0) $y = 0$, $\therefore dy = 0$ and x varies from 0 to 1

$$\therefore \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x^2 + y^3)\,dx$$

$$= \int_0^1 x^2\,dx = \frac{1}{3}, \text{ because } y = 0$$

Along the line (1, 0) to (1, 1), $x = 1$ $\therefore dx = 0$ and y varies form 0 to 1.

$$\therefore \quad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x^3 - y^2)\,dy$$

$$= \int_0^1 (1 - y^2)\,dy, \quad \text{as } x = 1$$

$$= \left[y - \frac{y^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \frac{1}{3} + \frac{2}{3} = 1.$$

(d) The equation of the line joining (0, 0) and (2, – 2) is

$$y = -x$$

$\therefore$ $dy = -dx$ and x *varies from 0 to 2.*

Now put $y = -x$ and $dy = -dx$ in (i) and integrate with in limits 0 to 2.

$$\therefore \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (x^2 - x^3)\, dx - (x^3 - x^2)\, dx$$

$$= 2\int_0^2 (x^2 - x^3)\, dx = -\frac{8}{3}.$$

Along C_2 the line joining (2, – 2) to (0, – 1) has the equation

$$y + 1 = \frac{-2+1}{2+0}(x - 0)$$

or $$y = -\frac{(x+2)}{2}$$

$$\therefore \quad dy = -\frac{1}{2}dx \text{ and } x \text{ varies from 2 to 0.}$$

Putting the above data in (i) and integrating with respect to x within the above limits, we get

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\frac{9}{2},$$

Similarly, $$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{6}$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = -\frac{8}{3} - \frac{9}{2} - \frac{1}{6} = -\frac{22}{3}.$$

EXERCISES

1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = x^2\mathbf{i} - xy\,\mathbf{j}$ from the point (0, 0) to (1, 1) along the parabola $y^2 = x$. [**Ans.** 1/12]
2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$ and curve C is the arc of $y = x^2 - 4$ from (2, 0) to (4, 12) in the xy-plane. [**Ans.** 732]
3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = x^2\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the arc of the curve $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ from $t = 0$ to $t = 1$. [**Ans.** 1]
4. Evalute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the arc of the curve $r = (a\cos\theta)\,\mathbf{i} + (a\sin\theta)\,\mathbf{j} + (a\,\theta)\,\mathbf{k}$ to $\theta = \frac{\pi}{2}$. $\left[\textbf{Ans. } a^3\left(\frac{5\pi}{8} - \frac{4}{3}\right)\right]$
5. If $\mathbf{F} = (2x + y)\,\mathbf{i} + (3y - x)\,\mathbf{j}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve in the xy plane consisting of straight line from O (0, 0) to A (2, 0) and then to B (2, 2). [**Ans.** 13]
6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$ and C is the curve in xy-plane consisting of $x = 2$ to $x = 4$ and $y = 12$. [**Ans.** 768]
7. If $\mathbf{F} = (2y + x)\,\mathbf{i} + xy\,\mathbf{j} + (yz - x)\,\mathbf{k}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the following C,

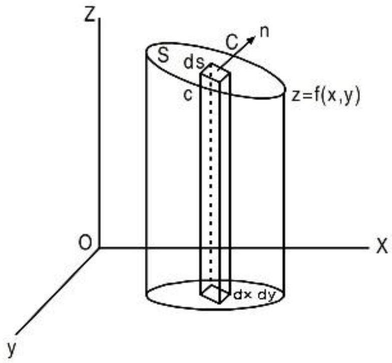
   (a) $x = 2t^2,\ y = t,\ z = t^3$ from $t = 0$ to $t = 1$.
   (b) The straight line form (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (2, 1, 1).
   (c) The straight line joining (0, 0, 0) to (2, 1, 1). $\left[\textbf{Ans. } \text{(a) } \frac{106}{35}, \text{(b) } 10, \text{(c) } 8\right]$

4.3 NORMAL SURFACE INTEGRAL

Let $\mathbf{F}$ ($\mathbf{r}$) be a continuous vector point function and $\mathbf{r} = \mathbf{f}\ (u, v)$ be a smooth surface such that $\mathbf{f}\ (u, v)$ possesses continuous first order partial derivatives.

Consider any portion of the surface which may be closed or not.

Divide this surface into a number of sub-surface $\delta S_1, \delta S_2, \delta S_3$ and so on. Let δS_p be one of the sub-surfaces.



Take any point P in this sub-surface and let n_p denote the positive unit normal vector to this sub-surface at the point P.

δS_p is the magnitude of the sub-surface and the corresponding vector area be denoted by δa_p.

$\therefore$ $$\delta a_p = n_p \, \delta S_p$$

Consider the sum

$$\Sigma F_p \, \delta a_p = \Sigma F_p . n_p \, \delta S_p \quad \text{...(i)}$$

The summation extending to various sub-surfaces into which S has been divided. Also $\mathbf{F}_p . \mathbf{n}_p$ denotes the normal component of $\mathbf{F}_p$ at $\mathbf{P}$.

The limit of the above sum when the number of sub-surface tends to infinity and the area of each sub-surface tends to zero is defined as the *normal surface integral* of $F(\mathbf{r})$ over S and is denoted as

$$\int_S \mathbf{F} . d\mathbf{a} = \int_S \mathbf{F} . \mathbf{n} \, dS.$$

the sign of the above integral will change if we choose the normal on the other side.

Cartesian Form

If F_1, F_2, F_3 be the components of $\mathbf{F}$ along the coordinate axes, then

$$\int_S \mathbf{F} . \mathbf{n} \, dS = \iint (F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy).$$

The above formula can also be put into the form

$$\iint \left[F_1 \frac{\partial (y, z)}{\partial (u, v)} + F_2 \frac{\partial (z, x)}{\partial (u, v)} + F_3 \frac{\partial (x, y)}{\partial (u, v)} \right] du \, dv$$

When the integration is to be performed over the region in the $u - v$ plane. Corresponding to the surface S given by

$$\mathbf{r} = \mathbf{f}(u, v)$$

and $\dfrac{\partial (y, z)}{\partial (u, v)}$ is the Jacobian $= \begin{vmatrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{vmatrix}$ etc.

Other forms of surface integral are

$$\int_S \mathbf{F} \times d\mathbf{a} \quad \text{and} \quad \int_S \phi \, d\mathbf{a}.$$

Where $\mathbf{F}$ is a continuous vector point function and ϕ is a continuous scalar point function.

Various Other Forms of Surface Integral

$$\int_S \mathbf{F} . d\mathbf{a} = \int_S \mathbf{F} . \mathbf{n} \, dS$$

Now $\mathbf{n}_p \, \delta S_p$ is the vector area of δS_p and hence its projection on xy-plane whose unit normal is $\mathbf{k}$ is

$$n_p \, \delta S_p . \mathbf{k} = (n_p . \mathbf{k}) \, \delta S_p$$

But projection δS_p on xy plane is $\delta x \, \delta y$

$\therefore$ $$(n_p . \mathbf{k}) \, \delta S_p = \delta x \, \delta y,$$

$\therefore$ $$\delta S_p = \frac{\delta x \, \delta y}{n_p . \mathbf{k}}$$

$\therefore$ Surface Integral $\int_S \mathbf{F} . \mathbf{n} \, dS = \Sigma \mathbf{F}_p . \mathbf{n}_p \, \delta S_p$

$$= \Sigma \mathbf{F}_p \cdot \mathbf{n}_p \frac{\partial x \partial y}{\mathbf{n}_p \cdot \mathbf{k}}$$

$$= \int \int_{S_3} \mathbf{F} \cdot \mathbf{n} \frac{dx\,dy}{\mathbf{n} \cdot \mathbf{k}}$$

where S_3 is the projection of S on xy-plane.

Similarly, $$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int \int_{S_1} \mathbf{F} \cdot \mathbf{n} \frac{dy\,dz}{\mathbf{n} \cdot \mathbf{i}}$$

where S_1 is the projection of S on yz-plane or whose normal is **i.**

or $$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int \int_{S_2} \mathbf{F} \cdot \mathbf{n} \frac{dz\,dx}{\mathbf{n} \cdot \mathbf{j}}$$

where S_2 is the projection of S on zx-plane whose normal is **j.**

4.4 VOLUME INTEGRAL

Let $\mathbf{F}(\mathbf{r})$ be a continuous vector point function and a volume V be enclosed by a surface given by $\mathbf{r} = \mathbf{r}(u, v)$. Divide the given volume into various $\partial v_1, \partial v_2, \ldots$ elements. Let δv_p be one such element and P be any point on it.

Consider the sum $\Sigma F_p \partial v_p$ ...(i)

Where the summation is to be extended to all the elements into which V has been divided. The limit of the above sum when the number of volume elements tends to infinity and each element tends to zero is defined as the *volume integral* and is written as

$$\int_V \mathbf{F}\, dv.$$

In cartesian form

$$\int_V \mathbf{F}\, dv = \mathbf{i} \int \int \int_V F_1\, dx\,dy\,dz + \mathbf{j} \int \int \int_V F_2\, dx\,dy\,dz + \mathbf{k} \int \int \int_V F_3\, dx\,dy\,dz$$

EXAMPLES

1. *Evaluate* $\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{a}$, *where S denotes the sphere of radius a with centre at the origin.*

Sol. Let the equation to the sphere be

$$x^2 + y^2 + z^2 = a^2.$$

A normal to the above surface is given by

$$\text{grad}\,(x^2 + y^2 + z^2) = \mathbf{i} \frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \mathbf{j} \frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \mathbf{k} \frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

$$\therefore \quad \text{Unit normal} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}}.$$

$$= \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} = \mathbf{n}$$

Again, $$\mathbf{F} = \frac{\mathbf{r}}{r^3} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3}$$

$$\therefore \quad \int_S \mathbf{F} \cdot d\mathbf{a} = \int_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \int_S \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\, dS$$

$$= \int \frac{x^2 + y^2 + z^2}{a^4}\, dS = \int_S \frac{a^2}{a^4}\, dS$$

$$= \frac{1}{a^2} \int_S dS = \frac{1}{a^2} \cdot 4\pi a^2 = 4\pi.$$

2. *If* $\mathbf{f} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\int_S (\nabla \times \mathbf{f}) \cdot \mathbf{n}\, dS$, *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *above the xy-plane.*

Sol. Let

$$\mathbf{F} = \nabla \times \mathbf{f} = \text{curl}\, \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$$

Also, we know that the normal to the surface $x^2 + y^2 + z^2 = a^2$ will be

$$\text{grad}\,(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

$$\therefore \quad \mathbf{n} = \text{unit normal} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}}$$

$$= \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$$

$$\therefore \quad \mathbf{F} \cdot \mathbf{n} = (x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}) \cdot \left(\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}\right)$$

$$= \frac{x^2 + y^2 - 2z^2}{a}$$

Also, we know that

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int_{S_3} \mathbf{F} \cdot \mathbf{n} \frac{dx\,dy}{\mathbf{n} \cdot \mathbf{k}}$$

Where S_3 is the projection of S on xy-plane.

$$\mathbf{n} \cdot \mathbf{k} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} \cdot \mathbf{k} = \frac{z}{a}$$

$$= \frac{\sqrt{(a^2 - x^2 - y^2)}}{a}$$

Also, $$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2 - 2z^2}{a}$$

$$= \frac{x^2 + y^2 - 2(a^2 - x^2 - y^2)}{a}$$

$$= \frac{3(x^2 + y^2) - 2a^2}{a}$$

$$\therefore \quad \text{Surface Integral} = \int\int_{S_3} \frac{3(x^2 + y^2) - 2a^2}{a} \cdot \frac{dx\,dy}{\sqrt{(a^2 - x^2 - y^2)}}\, a \qquad \text{...(i)}$$

Now, S_3 is the projection of $x^2 + y^2 + z^2 = a^2$ in the xy-plane and is given by $x^2 + y^2 = a^2$.

In order to integrate (i), put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad \int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^a \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}}\, r\, d\theta\, dr$$

$$= 2\pi \int_0^a \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}}\, r\, dr$$

Put $a^2 - r^2 = t^2$, $\therefore\ -2r\,dr = 2t\,dt$

$$\therefore \quad \int_S \mathbf{F}\,.\,\mathbf{n}\,dS = 2\pi \int_0^a \frac{3(a^2 - t^2) - 2a^2}{t}(-t)\,dt$$

$$= 2\pi \int_0^a (a^2 - 3t^2)\,dt = 2\pi\,[a^3 - a^3]_0^a$$

$$= 2\pi\,(a^3 - a^3) = 0.$$

3. *If* $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$, *then evaluate* $\int_V \nabla\,.\,\mathbf{F}\,dV$ *and* $\int\int\int_V \nabla \times \mathbf{F}\,dV$, *where V is the closed region bounded by the plane* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$.

Sol. We have

$$\nabla\,.\,\mathbf{f} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x - 0 = 2x$$

and $\quad dV = dx\,dy\,dz$.

Limits of z are from 0 to $4 - (2x + 2y)$, limits of y are from 0 to $2 - x$ and limits of x are from 0 to 2.

$$\therefore \quad \int_V = \nabla\,.\,\mathbf{f}\,dV$$

$$= \int_0^{4-(2x+2y)} \int_0^{(2-x)} \int_0^2 2x\,dx\,dy\,dz$$

$$= \int_0^{(2-x)} \int_0^2 2x\,(4 - 2x - 2y)\,dx\,dy$$

$$= \int_0^2 \left[8xy - 4x^{2y} - 2xys\right]_0^{2-x} dx$$

$$= \int_0^2 [8x\,(2 - x) - 4x^2\,(2 - x) - 2x\,(2 - x)^2]\,dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x)\,dx = \left[2\,.\,\frac{2^4}{4} - 8\,.\,\frac{2^3}{3} + 8\,.\,\frac{2^2}{2}\right] = \frac{8}{3}.$$

Again,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3x & -2xy & -4x \end{vmatrix}$$

$$= \mathbf{j} - 2\,\mathbf{k}\,y$$

$$\therefore \quad \int_V \nabla \times \mathbf{f}\,dV = \int\int\int (\mathbf{j} - 2\mathbf{k}\,y)\,dx\,dy\,dz$$

$$= \int_0^{(4-2x-2y)} \int_0^{2-x} \int_0^2 (\mathbf{j} - 2\mathbf{k}\,y)\,dx\,dy\,dz$$

$$= \int_0^{2-x} \int_0^2 (\mathbf{j} - 2\mathbf{k}\,y)\,(4 - 2x - 2y)\,dx\,dy$$

$$= \int_0^2 \left[\mathbf{j}\,(4y - 2xy - y^2) - 2\mathbf{k}\left(2y^2 - xy^2 - \frac{2y^3}{3}\right)\right]_0^{2-x} dx$$

$$= \int_0^2 \left[\mathbf{j}\,(2 - x)\,(4 - 2x - 2 + x) - 2\mathbf{k}\,(2 - x^2)\left\{2 - x - \frac{2}{3}(2 - x)\right\}\right] dx$$

$$= \int_0^2 \left[\mathbf{j}\,(2 - x)^2 - \frac{2\mathbf{k}}{3}(2 - x)^3\right] dx$$

$$= \left[\mathbf{j}\,\frac{(x - 2)^3}{3} + \frac{2\mathbf{k}}{3}\,\frac{(x - 2)^4}{4}\right]_0^2$$

$$= \mathbf{j}\,.\,\frac{8}{3} + \frac{2\mathbf{k}}{3}\left(-\frac{16}{4}\right) = \frac{8}{3}(\mathbf{j} - \mathbf{k}).$$

EXERCISES

1. Evaluate $\int_S \mathbf{f} \cdot \mathbf{n}\, dS$ where $\mathbf{f} = y\mathbf{i} + 2x\,\mathbf{j} + z\mathbf{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$. [**Ans.** 108]

2. Evaluate $\int_S \mathbf{f} \cdot \mathbf{n}\, dS$ over the surface of the cylinder $x^2 + y^2 = 9$ included in the first octant between $z = 0$ and $z = 4$ where $\mathbf{f} = z\,\mathbf{i} + x\,\mathbf{j} - yz\,\mathbf{k}$. [**Ans.** 42]

3. Evaluate $\int_S \mathbf{f} \cdot \mathbf{n}\, dS$ where $\mathbf{f} = 4x\mathbf{i} - 2y^2\,\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. [**Ans.** 84p]

4. Evaluate $\int_S \mathbf{f} \cdot \mathbf{n}\, dS$ where $\mathbf{f} = 2xy\,\mathbf{i} - 2zy\,\mathbf{j} + x^2\,\mathbf{k}$ over the surface of cube bounded by the coordinate planes and the planes $x = a$, $y = a$, $z = a$. $\left[\textbf{Ans. } \frac{1}{2}a^4\right]$

5. Evaluate $\int_V \mathbf{f}\, dV$ for $\mathbf{f} = x\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$ where V is the region bounded by the surface $x = 0$, $y = 0$, $y = 6$, $z = 4$ and $z = x^2$. $\left[\textbf{Ans. } 24\,\mathbf{i} + 96\,\mathbf{j} + \frac{384}{5}\mathbf{k}.\right]$

5

Gauss's, Green's and Stoke's Theorem

5.1 GAUSS'S DIVERGENCE THEOREM

Reduction of Surface Integral to Volume Integral

Statement : *The normal surface integral of a vector function* **F** *over the boundary of a closed region is equal to the volume integral of div* **F** *taken throughout the region.*

In symbols it may be stated as follows :

If **F** *be a continuously differentiable vector point function in a region V and S is a closed surface enclosing the region V, then*

$$\int_S \mathbf{F}.\mathbf{n}\, dS = \int_V div\, \mathbf{F}\, dV,$$

where **n** *is the unit outward drawn normal vector of the surface.*

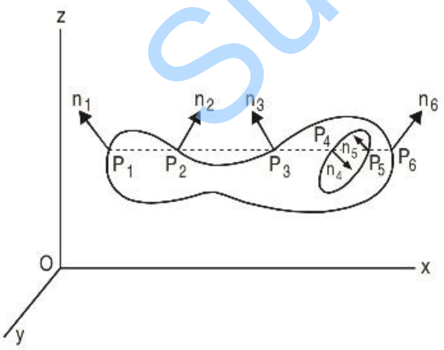
In cartesian co-ordinates the Divergence theorem may be written as

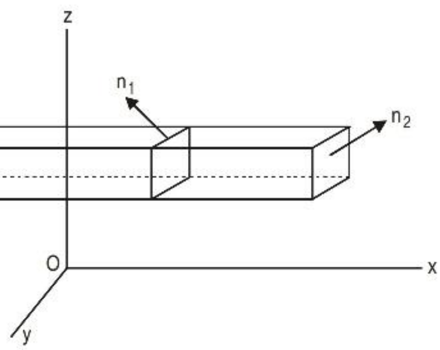
$$\int_S \mathbf{F}.\mathbf{n}\, dS = \int_V div\, \mathbf{F}\, dV$$

or $$\int\int_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

$$= \int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

Proof : Let $\mathbf{F} = U\,\mathbf{i} + V\,\mathbf{j} + W\,\mathbf{k}$, where U, V, W and their derivatives in any direction are assumed to be uniform, finite and continuous.





Let us consider the volume integral

$$I = \int\int\int \frac{\partial U}{\partial x}\, dx\, dy\, dz,$$

where $dx\, dy\, dz$ has been written for the volume element dV. For fixed values of y and z, take a rectangular prism parallel to x-axis, bounded by the planes y, $y + dy$, z, $z + dz$, the area of the normal section being $dy\, dz$.

The prism so formed cuts the boundary an even number of times at the points $P_1, P_2, ..., P_{2n}$.

If a point moves along the prism in the direction of x increasing, it enters the region at $P_1, P_3, ..., P_{2n-1}$ and leaves $P_2, P_4, ..., P_{2n}$.

Then taking the integral and integrating with respect to x, we obtain

$$I = \int\int (-U_1 + U_2 - U_3 + ... - U_{2n-1} + U_{2n})\, dy\, dz$$

where U_r is the value of U at that point P_r.

Let dS_r is the value of U at that point P_r.

Let dS_r be the area of the element of the boundary intercepted by the prism at the point P_r. Then

$$\begin{aligned} dy\, dz &= \text{area of projection of } dS_r \text{ on the } yz\text{-plane} \\ &= -\mathbf{i}\,.\,\mathbf{n}_r\, dS_r \quad \text{for } r \text{ odd} \\ &= \mathbf{i}\,.\,\mathbf{n}_r\, dS_r \quad \text{for } r \text{ even,} \end{aligned}$$

as the angle $\mathbf{n}_r$ makes with $\mathbf{i}$ is acute or obtuse according as r is even or odd (when the line parallel to x-axis enters the surface, the outward normal makes an obtuse angle with it and acute angle when the line leaves the surface).

$$\therefore \quad I = \int \mathbf{i}\,.(U_1\,\mathbf{n}_1\, dS_1 + U_1\,\mathbf{n}_2\, dS_2 + ... + U_{2n}\,\mathbf{n}_{2n}\, dS_{2n})$$

On summing for all the rectangular prism, we obtain

$$\int \frac{\partial U}{\partial x}\, dv = \int U\,\mathbf{i}\,.\,\mathbf{n}\, dS = \int U\,\mathbf{i}\,.\,d\mathbf{S} \quad ...(1)$$

Similarly

$$\int \frac{\partial V}{\partial y}\, dv = \int V\,\mathbf{j}\,.\,\mathbf{n}\, dS \quad ...(2)$$

and

$$\int \frac{\partial W}{\partial z}\, dv = \int W\,\mathbf{k}\,.\,\mathbf{n}\, dS \quad ...(3)$$

Adding (1), (2) and (3), we get

$$\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) dv = \int (U\,\mathbf{i} + V\,\mathbf{j} + W\,\mathbf{k})\,.\,\mathbf{n}\, dS$$

i.e.,

$$\int \text{div}\,\mathbf{F}\, dv = \int \mathbf{F}\,.\,\mathbf{n}\, dS$$

Cartesian Representation of Gauss's Theorem

Let $\mathbf{F}(P) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$ and

$$d\mathbf{S} = dS(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$$

where α, β, γ are the direction angles of $d\mathbf{S}$. Therefore, $dS\cos\alpha$, $dS\cos\beta$ and $dS\cos\gamma$ are the orthogonal projections of the elementary area dS on the yz-plane, zx-plane and xy-plane respectively.

Since the mode of sub division of the surface is arbitrary, we choose a sub-division formed by the planes parallel to the yz-plane, the zx-plane and the xy-plane. Then the projections on the coordinates planes will be rectangles with sides dy and dz on the yz-plane, dz and dx on the zx-plane, dx and dy on the xy-plane. Hence the projected surface elements are $dy\,dz$ on the yz-plane, $dz\,dx$ on the zy-plane and $dx\,dy$ on the xy-plane.

$$\therefore \quad \int_S \mathbf{F}\,.\,\mathbf{n}\, dS = \int\int_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

Also by Gauss's divergence theorem, we have

$$\int_S \mathbf{F}\,.\,\mathbf{n}\, dS = \int_V \text{div}\,\mathbf{F}\, dV$$

In cartesian coordinate $dV = dx\,dy\,dz$. Also

$$\text{div}\,\mathbf{F} = \nabla\,.\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\therefore \quad \int_V \text{div}\,\mathbf{F}\, dV = \int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

Hence the Cartesian form of Gauss's theorem is

$$\int\int_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) = \int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

EXAMPLES

1. *If* $\mathbf{F} = 2xy\,\mathbf{i} - yz\,\mathbf{j} + x^2\,\mathbf{k}$, *evaluate* $\int_S \mathbf{F}\,.\,\mathbf{n}\, dS$, *where S denotes the entire surface of the cube bounded by the coordinate planes and the planes* $x = a, y = a, z = a$ *by the application of Gauss's theorem.*

Sol. We have

$$\mathbf{F} = 2xy\,\mathbf{i} - yz\,\mathbf{j} + x^2\,\mathbf{k}$$

$$\therefore \quad \text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial x}(x^2)$$

$$= 2y - z$$

$$\therefore \quad \int_S \mathbf{F}\,.\,\mathbf{n}\, dS = \int_V div\,\mathbf{F}\, dV, \text{ by Gauss's divergence theorem}$$

$$= \int_0^a \int_0^a \int_0^a (2y - z)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left[2yz - \frac{z^2}{2}\right]_0^a dx\, dy$$

$$= \int_0^a \int_0^a \left(2ay - \frac{a^2}{y}\right) dx\, dy$$

$$= \int_0^a \left[ay^2 - \frac{1}{2}a^2 y\right]_0^a dx$$

$$= \int_0^a \left(a^3 - \frac{1}{2}a^3\right) dx = \frac{1}{2}a^3\,[x]_0^a$$

$$= \frac{1}{2}a^4.$$

2. *Verify Gauss divergence theorem for*

$$\int\int_S \{(x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx + z\, dx\, dy\}$$

over the surface of cube bounded by coordinate planes and the planes $x = y = z = a$.

Sol. Let $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$.

From Gauss divergence theorem, we know

$$\int_S \mathbf{F}\,.\,\mathbf{n}\, dS = \int_S [F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy] = \int_V div\,\mathbf{F}\, dV. \quad ...(i)$$

Here, $F_1 = x^3 - yz,\ F_2 = -2x^2 y,\ F_3 = z$

Hence, $\mathbf{F} = (x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + z\,\mathbf{k}$

$$\text{div}\,\mathbf{F} = \left\{\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right\}.\{(x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + z\,\mathbf{k}\}$$

$$= \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2 y) + \frac{\partial}{\partial z}(z)$$

$$= 3x^2 - 2x^2 + 1 = x^2 + 1$$

Hence, $\int_S \mathbf{F}\,.\,\mathbf{n}\, dS = \int_V (x^2 + 1)\, dV$

$$= \int_0^a \int_0^a \int_0^a (x^2 + 1)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^a (x^2 + 1) \{z\}_0^a \, dx \, dy$$

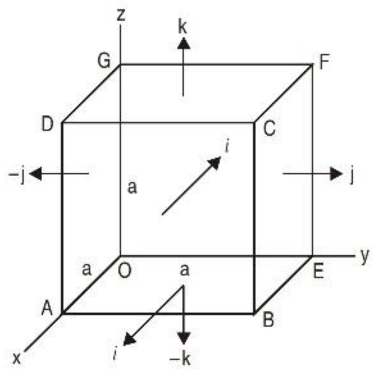
$$= a \int_0^a \int_0^a (x^2 + 1) \, dx \, dy$$

$$= a \int_0^a (x^2 + 1) \{y\}_0^a \, dx$$

$$= a^2 \int_0^a (x^2 + 1) \, dx$$

$$= a^2 \left\{ \frac{x^3}{3} + x \right\}_0^a$$

$$= a^2 \left\{ \frac{a^3}{3} + a \right\} = \frac{a^5}{3} + a^3 \qquad \text{...(ii)}$$



Verification by Direct Integral : Outward drawn unit vector normal to face *OEFG* is $-\mathbf{i}$ and dS is $dy\,dz$.

If I_1 is integral along this face,

$$I_1 = \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int \int_S \mathbf{F} \cdot (-\mathbf{i}) \, dy \, dz$$

$$= \int \int_S (x^3 - yz) \, dy \, dz \quad [\text{as } x = 0 \text{ for this face}]$$

$$= \int_0^a \int_0^a yz \, dy \, dz = \int_0^a y \left\{ \frac{z^2}{2} \right\}_0^a dy$$

$$= \frac{a^2}{2} \int_0^a y \, dy = \frac{a^2}{2} \left[\frac{y^2}{2} \right]_0^a = \frac{a^4}{4}$$

For face *ABCD*, its equation is $x = a$ and $\mathbf{n} \, dS = \mathbf{i} \, dy \, dz$,

If I_2 is integral along this face

$$I_2 = \int \int_S \mathbf{F} \cdot \mathbf{i} \, dy \, dz$$

$$= \int \int_S (x^3 - yz) \, dy \, dz$$

$$= \int_0^a \int_0^a (a^3 - yz) \, dy \, dz$$

$$= \int_0^a \left\{ a^3 z - y \frac{a^2}{2} \right\}_0^a dy$$

$$= \left[a^4 y - \frac{a^2}{2} \frac{y^2}{2} \right]_0^2 = a^5 - \frac{a^4}{4}$$

If I_3 is integral along face *OGDA* whose equation is

$$y = 0$$

$$\mathbf{n} \, dS = -\mathbf{j} \, dx \, dz$$

Hence,
$$I_3 = \int \int_S F \cdot (-\mathbf{j}) \, dx \, dz$$

$$= -\int \int_S -2x^2 y \, dx \, dz$$

$$= 0, \text{ as } y = 0.$$

If I_4 is integral along face *BEFC* whose equation is

$$y = a$$

$$\mathbf{n} \, dS = \mathbf{j} \, dx \, dz$$

Then $$I_4 = -\int\int_S 2x^2 y\, dx\, dz$$
$$= -2a \int_0^a \int_0^a x^2\, dx\, dz$$
$$= -2a \int_0^a x^2 \{z\}_0^a\, dx$$
$$= -2a \int_0^a x^2\, dx$$
$$= -2a^2 \left[\frac{x^3}{3}\right]_0^a = -\frac{2}{3}a^5.$$

If I_5 is integral along face $OABF$ whose equation is
$$z = 0$$
$$\mathbf{n}\, dS = -\mathbf{k}\, dx\, dy$$
$$I_5 = \int\int_S \mathbf{F} \cdot (-\mathbf{k}\, dx\, dy)$$
$$= -\int\int_S z\, dx\, dy = 0 \text{ as } z = 0,$$

If I_6 is integral along face $OFGD$ whose equation is
$$z = a\ \mathbf{n}\, dS = \mathbf{k}\, dx\, dy$$
$$I_6 = \int\int_S z\, dx\, dy = \int_0^a \int_0^0 a\, dx\, dy$$
$$= a\int_0^a [y]_0^a\, dx = a^2 \int_0^a dx = a^3$$

Total surface $$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$
$$= \frac{a^4}{4} + a^5 - \frac{a^4}{4} + 0 - \frac{2}{3}a^5 + 0 + a^3$$
$$= \frac{a^5}{3} + a^3 \qquad \text{...(iii)}$$

which is equal to volume integral. Hence Gauss theorem is verified.

3. *Show that*
$$\int_S (ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}) \cdot \mathbf{n}\, dS = \frac{4}{3}\pi(a+b+c)$$
where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Sol. We have by Gauss's divergence theorem
$$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_V \text{div}\,\mathbf{F}\, dV$$

Now, $$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(a\,x) + \frac{\partial}{\partial y}(b\,y) + \frac{\partial}{\partial z}(c\,z) = a + b + c$$

$\therefore$ $$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_V (a+b+c)\, dV$$
$$= (a+b+c)V.$$

Now $V =$ Volume of sphere of unit radius
$$= \frac{4}{3} \cdot \pi \cdot 1^3 = \frac{4}{3}\pi$$

$\therefore$ $$\int_S (ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}) \cdot dS = (a+b+c) \cdot \frac{4}{3}\pi = \frac{4}{3}(a+b+c)\pi.$$

EXERCISES

1. Show that $\frac{1}{3}\int_S \mathbf{r}\,.\,\mathbf{n}\, dS = V$.

2. Evaluate $\int_S \mathbf{F}\,.\,\mathbf{n}\, ds$ when $\mathbf{F} = 4xy\,\mathbf{i} + yz\,\mathbf{j} - xz\,\mathbf{k}$ and S is the surface of the cube bounded by the plane $x = 0, x = 2, y = 2, y = 0$ and $z = 0, z = 2$. [**Ans.** 32]

3. Evaluate $\int_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,.\,\mathbf{n}\, dS$ where S denotes the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$ by the application of Gauss's theorem. [**Ans.** $3a^3$]

4. Verify Divergence theorem for $\mathbf{f} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$.

5. Verify the divergence theorem for the function $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ over the cylindrical region S bounded by $x^2 + y^2 = a^2, z = 0$ and $z = h$.

6. Evaluate $\int_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\,.\,\mathbf{n}\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane. [**Ans.** 3/12]

5.2 GREEN'S THEOREM IN THE PLANE

Relation between Plane Surface Integral and Line Integral

If S is a closed region in the xy-plane bounded by a simple closed curve C and if $\phi(x, y)$ *and* $\psi(x, y)$ *are continuous functions having continuous partial derivatives in R, then*

$$\oint_C (\psi\, dx + \phi\, dy) = \int\int_S \left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) dx\, dy,$$

where C is traversed in the positive (anti-clockwise) direction.

In vector notation, Green's theorem is

$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_S \text{curl}\,\mathbf{F}\,.\,\mathbf{n}\, dS$$

where $\mathbf{n} = \mathbf{k}$ for xy-plane and $dS = dx\, dy$ and $\text{curl}\,\mathbf{F}\,.\,\mathbf{n} = \left\{\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right\}$ and $\mathbf{F} = i\psi + j\phi$

Proof : By Stoke's theorem, we have

$$\int\int_S \text{curl}\,\mathbf{F}\,.\,\mathbf{n}\, dS = \int_C \mathbf{F}\,.\,d\mathbf{r} \quad ...(1)$$

Let $\mathbf{F} = \mathbf{i}\,\psi + \mathbf{j}\phi$, then we have

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \psi & \phi & 0 \end{vmatrix}$$

$$= -\mathbf{i}\frac{\partial\phi}{\partial z} + \mathbf{j}\frac{\partial\psi}{\partial z} + \mathbf{k}\left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right)$$

Also, since $\mathbf{n} = \mathbf{k}$, we have

$$\text{curl}\,\mathbf{F}\,.\,\mathbf{n} = \text{curl}\,\mathbf{F}\,.\,\mathbf{k} = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}.$$

$$\therefore \quad \int\int_S \text{curl}\,\mathbf{F}\,.\,\mathbf{n}\, dS = \int\int_S \left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) dx\, dy \quad ...(2)$$

Also, $$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_C (\psi\mathbf{i} + \phi\mathbf{j})\,(\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz)$$

$$= \int_C (\psi\, dx + \phi\, dy)$$

Hence from (1), (2) and (3), we get

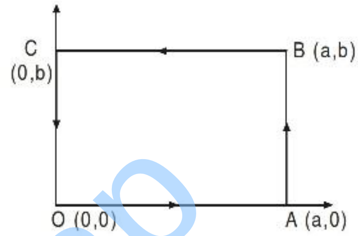
$$\int\int_S \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy = \int_C (\psi\, dx + \phi\, dy)$$

EXAMPLES

1. *Verify Green's theorem in the plane for*

$$\oint_C [(x^2 + y^2)\, dx - 2xy\, dy],$$

where C is the rectangle bounded by $y = 0, x = 0, y = b, x = a$.



Sol. By Green's theorem we have

$$\oint_C [(x^2 + y^2)\, dx - 2xy\, dy]$$

$$= \int\int_S \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2 + y^2)\right] dx\, dy$$

$$= \int\int_S (-2y - 2y)\, dx\, dy - \int\int -4y\, dx\, dy - \int\int -4y\, dx\, dy \quad \text{...(i)}$$

Now to verify Green's theorem, first we shall evaluate Line integral of L.H.S. along the rectangle *OABC*.

For this, if $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$

then $\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_C \{(x^2 + y^2)\, dx - 2xy\, dy\}$

$$= \int_{OA} \mathbf{F}\,.\,d\mathbf{r} + \int_{AB} \mathbf{F}\,.\,d\mathbf{r} + \int_{BC} \mathbf{F}\,.\,d\mathbf{r} + \int_{CO} \mathbf{F}\,.\,d\mathbf{r} \quad \text{...(ii)}$$

Along OA, $y = 0$, $\therefore\ dy = 0$ and y varies from 0 to a

Along AB, $x = a$, $\therefore\ dx = 0$ and y varies from 0 to b

Along BC, $y = b$, $\therefore\ dy = 0$ and x varies from a to 0

Along CO, $x = 0$, $\therefore\ dx = 0$ and y varies from b to 0

$\therefore$ From (ii), we get

$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_0^a x^2\, dx - \int_0^b 2ay\, dy + \int_a^0 (x^2 + b^2)\, dx \int_b^0 0\,.\,dy$$

$$= \frac{a^3}{3} - 2a\,.\,\frac{b^2}{2} + \left[\frac{x^3}{3} + b^2 x\right]_a^0 = -2ab^2. \quad \text{...(iii)}$$

$$\text{R.H.S.} = \int_{x=0}^a \int_{y=0}^b (-4y)\, dx\, dy$$

$$= -2b \int_0^a dx = -2ab^2 \quad \text{...(iv)}$$

Hence the Green's theorem is verified.

2. *Show that the area bounded by a simple closed curve C is given by* $\frac{1}{2}\int_C (x\, dy - y\, dx)$ *and hence find the area of an ellipse.*

Sol. We know that

$$\int_C (\psi\, dx + \phi\, dy) = \int\int_S \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy$$

where S is the plane area A enclosed by a curve C. Choosing

$$\psi = -y \text{ and } \phi = x$$

$$\therefore \quad \frac{\partial \psi}{\partial y} = -1 \text{ and } \frac{\partial \phi}{\partial x} = 1$$

$$\therefore \quad \int_C (-y\,dx + x\,dy) = \int\int_S 2\,dx\,dy$$

$$= 2\int\int_S dx\,dy = 2A$$

$$\therefore \quad A = \frac{1}{2}\int (x\,dy - y\,dx) \qquad \text{...(i)}$$

Let the parametric equation of the ellipse be

$$x = a\cos t, \quad y = b\sin t$$

and in going round C, t varies form 0 to 2π.

$$\therefore \quad A = \frac{1}{2}\int_0^{2\pi} a\cos t\,(b\cos t\,dt) - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}ab\,.\,2\pi = \pi ab$$

EXERCISES

1. Evaluate by Green's theorem in the plane

$$\oint_C [(x^2 - \cosh y)\,dx + (y + \sin x)\,dy]$$

where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$. [**Ans.** $\pi(\cosh 1 - 1)$]

2. Verify Green's theorem in the plane for $\int_C (xy + y^2)\,dx + x^2 dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

3. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

4. Evaluate by Green's theorem $\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$, where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$, $(0, \pi/2)$ and hence verigfy Green's Theorem.

[**Ans.** $2(e^{-\pi} - 1)$]

5.3 STOKES THEOREM

Relation between Line integral and Surface Integral.

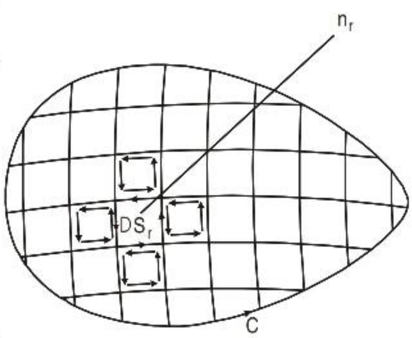
Statement : *The line integral of the tangential component of a vector function* **F** *taken around a simple closed curve C is equal to the normal surface integral of curl* **F** *taken over any surface S having C as its boundary.*

In symbolic form we can state the above theorem as follows :

If **F** is any continuously differentiable vector function and S is a surface bounded by a curve C then,

$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_S \text{curl } \mathbf{F}\,.\,\mathbf{n}\,dS$$

where **n** is the unit normal vector at any point of S, which is drawn in the sense in which a right handed screw would move when rotated in the sense of description of C.



Proof : Consider a space curve C bounding an open surface S. Divide S into m sub-regions so small that they may be assumed to be planar with areas $\Delta S_1, \Delta S_2, ..., \Delta S_m$. Choose any point (ξ_r, η_r, ζ_r) inside or on the boundary C_r of ΔS_r.

Assume that C is described in the positive sense. Then an orientation for each C_r is determined as follows :

(i) If C_r and C have an edge in common, this edge is described in the same direction along both boundaries, and

(ii) If C_r and C_s have an edge in common, this edge is described in opposite directions.

Let the unit normal vector at (ξ_r, η_r, ζ_r) be $\mathbf{n}_r$ with positive direction such that this and the direction of C_r are related by the right hand screw rule.

From the definition of curl **F** as a limit. We have

$$\mathbf{n}_r . \text{curl}\, \mathbf{F}\, (\xi_r, \eta_r, \zeta_r)\, \Delta S_r z = \int_{C_r} \mathbf{F} . d\,\mathbf{r} + \in_r \Delta S_r,$$

where $\in_r$ tends to zero as ΔS_r tends to zero. Addition of these equations for r = 1, 2, 3, ..., m gives

$$\sum_{r=1}^{m} \mathbf{n}_r . \text{curl}\, \mathbf{F}\, (\xi_r, \eta_r, \zeta_r)\, \Delta S_r = \sum_{r=1}^{m} \int_{C_r} \mathbf{F} . d\mathbf{r} + \sum_{r=1}^{m} \in_r \Delta S_r$$

Now $\sum_{r=1}^{m} \in_r \Delta S_r \le S\,(\max \in_r)$, where S is the total area of the surface and hence this term tends to zero as m tends to infinity in such a way that each ΔS_r shrinks to a point.

Further, the contribution of the circulation from the two adjacent boundary curves is zero as they are traversed in opposite directions. Hence in the limit, we have

$$\int_S \mathbf{n} . \text{curl}\, \mathbf{F}\, dS = \int_A \mathbf{F} . d\,\mathbf{r} \quad ...(1)$$

where **n** is the vector field of positive unit normals to the surface S. We have thus Stoke's theorem :

$$\int_C \mathbf{F} . d\,\mathbf{r} = \int_S (\nabla \times F) . d\mathbf{S} \quad ...(2)$$

5.4 STOKES THEOREM IN CARTESIAN FORM

Let F_1, F_2, F_3 be the components of vector point function **F** so that $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ and an unit outward drawn normal be $\mathbf{r} = \mathbf{i}\, l + \mathbf{j}\, m + \mathbf{k}\, n$ where l, m, n are direction cosines.

Again $\quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}, \quad$ or $\quad d\,\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

$$\therefore \quad \int_C \mathbf{F} . d\,\mathbf{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz \quad ...(i)$$

Now,
$$\text{curl}\, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$\therefore \quad \mathbf{n} . \text{curl}\, \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) l + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) m + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) n$$

$$\therefore \quad \mathbf{n} . \text{curl}\, \mathbf{F}\, dS = \Sigma \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) l\, dS$$

$$= \Sigma \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\, dz$$

$$\because \quad l\, dS = \cos\alpha . dS = dy\, dz$$

Now by Stoke's Theorem

$$\int_C \mathbf{F} . d\,\mathbf{r} = \int_S \mathbf{n} . \text{curl}\, \mathbf{F}\, dS \quad \text{or} \quad \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \int_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\, dz + \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_3}{\partial x}\right) dz dz + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy\right]$$

This is cartesian equivalent of Stokes Theorem.

EXAMPLES

1. *Verify Stokes Theorem for* $\mathbf{F} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\mathbf{k}$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

Sol. We have the Stoke's Theorem as

$$\int_C \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Clearly, C the boundary of the upper half of the sphere is a circle $x^2 + y^2 = 1$ in the xy plane whose parametric equations be taken as

$$x = \cos t, \quad y = \sin t$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int (2x - y)\, dx - yz^2\, dy - y^2 z\, dz$$

Put $z = 0$

$$= \int_0^{2\pi} (2x - y) \frac{dx}{dt}\, dt$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt$$

$$= \int_0^{2\pi} (-2\cos t \sin t + \sin^2 t)\, dt$$

$$= \left[-\cos^2 t\right]_0^{2\pi} + 4\int_0^{\pi/2} \sin^2 t\, dt$$

$$= 0 + 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi. \qquad \text{...(i)}$$

Again

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \mathbf{i}(-2yz + 2yz) + j(0 - 0) + \mathbf{k}(0 + 1) = \mathbf{k}$$

$$\therefore \quad \text{curl } \mathbf{F} \cdot \mathbf{n} = \mathbf{k} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{k}$$

$$\therefore \quad \int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_S \mathbf{n} \cdot \mathbf{k}\, dS$$

$$= \int\int_R \mathbf{n} \cdot \mathbf{k} \frac{dx\, dy}{\mathbf{n} \cdot \mathbf{k}}$$

where R is the projection of S and xy-plane.

$$\therefore \quad \int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_{x=-1}^{1} \int_{y=-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} dx\, dy$$

$$= 4\int_0^1 \int_0^{\sqrt{(1-x^2)}} dx\, dy = 4\int_0^1 \sqrt{(1-x^2)}\, dx$$

$$= 4\left[\frac{4}{x}\sqrt{(1-x)^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \cdot \pi \qquad \text{...(iii)}$$

From (i) and (ii), we get $\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot dr$

Hence the Stokes Theorem.

2. *Evaluate by Stokes theorem* $\int_C (e^x\, dx + 2y\, dy - dz)$ *where C is the curve* $x^2 + y^2 = 4,\ z = 2.$

Sol. We have $\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$

$$= \int_C (e^x\,dx + 2y\,dy - dz)$$

where $\mathbf{F} = e^x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$, $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$

By Stokes Theorem $\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{S_1} \mathbf{n}\,.\,\text{curl}\,\mathbf{F}\,dS$ ...(i)

Where S_1 the surface whose boundary C is given by the circle $x^2 + y^2 = 4$, $z = 2$, *i.e.*, a circle with centre (0, 0, 2) and radius 2. Clearly $\mathbf{n} = \mathbf{k}$. Now,

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0$$

$\therefore$ $\mathbf{n}\,.\,\text{curl}\,\mathbf{F} = 0$

Hence, from (i) $\int_C \mathbf{F}\,.\,d\mathbf{r} = 0$.

3. *Verify Stoke's Theorem for* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$ *taken round the rectangle bounded by* $x = \pm a$, $y = 0$, $y = b$.

Sol. Clearly

$$\mathbf{F}\,.\,d\mathbf{r} = (x^2 + y^2)\,dx - 2xy\,dy \quad \text{...(i)}$$

$$\therefore \quad \int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{AD} \mathbf{F}\,.\,d\mathbf{r} + \int_{DC} \mathbf{F}\,.\,d\mathbf{r} + \int_{CB} \mathbf{F}\,.\,d\mathbf{r} + \int_{BA} \mathbf{F}\,.\,d\mathbf{r}$$

$$= I_1 + I_2 + I_3 + I_4$$

For I_1, $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\therefore \quad I_1 = \int [(x^2 + y^2)\,dx - 2xy\,dy]$$

$$= \int_a^{-a} (x^2 + b^2)\,dx + 0 \quad \because \; y = b$$

$$= \left[\frac{1}{3}x^3 + b^2 x\right]_a^{-a} = -\left(\frac{2}{3}a^3 + 2b^2 a\right)$$

Similarly, $I_2 = -ab^2$, $I_3 = \frac{2}{3}a^3$ and $I_4 = -ab^2$

$$\therefore \quad \int_C \mathbf{F}\,.\,d\mathbf{r} = -\frac{2}{3}a^3 - 2b^2 a - ab^2 + \frac{2}{3}a^3 - ab^2 = -4ab^2 \quad \text{...(ii)}$$

Again, we have curl $\mathbf{F} = -4y\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$

$$\therefore \quad \mathbf{n}\,.\,\text{curl}\,\mathbf{F} = \mathbf{k}\,.\,(-4y\,\mathbf{k}) = -4y.$$

$$dS = \frac{dx\,dy}{\mathbf{n}\,.\,\mathbf{k}} = \frac{dx\,dy}{\mathbf{k}\,.\,\mathbf{k}} = dx\,dy$$

$$\therefore \quad \int_S \mathbf{n}\,.\,\text{curl}\,\mathbf{F}\,dS = \int_{-a}^{a}\int_0^b -4x\,dx\,dy$$

$$= -4\int_{-a}^{0}\left[\frac{1}{2}y^2\,dx\right]_0^b$$

$$= -2b^2\,[x]_{-a}^{a} = -4ab^2 \quad \text{...(iii)}$$

Hence from (ii) and (iii), we get

$$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_S \mathbf{n}\,.\,\text{curl}\,\mathbf{F}\,dS = -4ab^2$$

EXERCISES

1. Prove that $\int_C \mathbf{r} \cdot d\mathbf{r} = 0$
2. Verify Stokes Theorem for the function $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$.
3. Prove that $\int_C \mathbf{r} \times d\mathbf{r} = 2\int_S \mathbf{n}\, d\mathbf{r}$ where symbols have their usual meanings.
4. Verify Stokes Theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$ integrated round the square in the plane $z = 0$ whose sides are along the lines $x = 0$, $x = a$, $y = 0$, $y = a$.
5. Verify Stokes Theorem for the vector field defined by $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ in the rectangular region in the xy-plane bounded by lines $x = 0, x = a, y = 0, y = b$.
6. Verify Stokes Theorem where $\mathbf{F} = (y - z)\,\mathbf{i} + yz\,\mathbf{j} - xz\,\mathbf{k}$ and S is given by $x = 0, y = 0, z = 0, x = 1, y = 1, z = 1$.
7. Verify Stokes Theorem given that $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j} + z\mathbf{k}$ and C is the circle $x^2 + y^2 = 1$ in the xy-plane and S the plane area bounded by C.
8. Verify the Stoke's Theorem for the function $\mathbf{F} = y\mathbf{i} + z\mathbf{j}$ over the plane surface $2x + 2y + z = 2$ lying in the first octant.
9. Verify the Stoke's theorem for the function $\mathbf{F} = x^2y^2\mathbf{i} + 2xy\,\mathbf{j}$ when the integration is taken around the volume enclosed by the rectangle $x = \pm a$, $y = 0$, $y = b$.

❑❑❑

GEOMETRY

(a) Two Dimensional

General equation of second degree. Tracing of conics. System of conics. Confocal conics. Polar equation of a conic.

Vector Calculus

9.0 INTRODUCTION

In Science and Engineering we often deal with the analysis of forces and velocities and other quantities which are vectors. These vectors are not constants but vary with position and time. Hence, they are functions of one or more variables.

Vector Calculus extends the concepts of differential calculus and integral calculus of real functions in an interval to vector functions and thus enabling us to analyse problems over curves and surfaces in three dimension. Vector Calculus finds applications in a wide variety of fields such as fluid flow, heat flow, solid mechanics, electrostatics etc.

In Vector Calculus we deal mainly with two kinds of functions, **scalar point functions** and **vector point functions and their fields**.

9.1 SCALAR AND VECTOR POINT FUNCTIONS

Definition 9.1 If to each point P($\vec{r}$) (the point P with position vector $\vec{r}$) of a region R in space there is a unique scalar or real number denoted by $\phi(\vec{r})$, then **ϕ is called a scalar point function in R**. The region R is called a **scalar field**.

Definition 9.2 If to each point P($\vec{r}$) of a region R in space there is a unique vector denoted by $\vec{F}(\vec{r})$, then **$\vec{F}$ is called a vector point function in R**. The region R is called a **vector field**.

Note

1. In applications, the domain of definition of point functions may be points in a region of space, points on a surface or points on a curve.
2. If we introduce cartesian coordinate system, then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ or

   $\vec{r} = (x, y, z)$ and instead of $\vec{F}(\vec{r})$ and $\phi(\vec{r})$ we can write

   $\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ or

   $\vec{F}(x, y, z) = F_1(x, y, z)\vec{i} + F_2(x, y, z)\vec{j} + F_3(x, y, z)\vec{k}$

   and $\phi(\vec{r})$ as $\phi(x, y, z)$
3. A vector or scalar field that has a geometrical or physical meaning should depend only on the points P where it is defined but not on the particular choice of the cartesian coordinates. In otherwords, the scalar and vector fields have the property of invariance under a transformation of space coordinates.

Examples of scalar field

1. Temperature T within a body is scalar field, namely temperature field.
2. When an iron bar is heated at one end, the temperature at various points will attain a steady state and the temperature will depend only on the position.

3. The pressure of air in earth's atmosphere is a scalar field called pressure field.
4. $\phi(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ defines a scalar field.

Examples of vector field

1. The velocity of a moving fluid at any instant is a vector point function and defines a vector field.
2. Earth's magnetic field is a vector field.
3. Gravitational force on a particle in space defines a vector field.
4. $\vec{F}(x, y, z) = x^2\vec{i} - y^2\vec{j} + z\vec{k}$ defines a vector field.

Note Vector and scalar functions may also depend on time or on other parameters.

Definition 9.3 Derivative of a Vector Function

A vector function $\vec{f}(t)$ is said to be differentiable at a point t, if $\lim_{\Delta t \to 0} \dfrac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$ exists.

Then it is denoted by $\dfrac{d\vec{f}}{dt}$ or $\vec{f}'$ and is called the derivative of the vector function $\vec{f}$ at t.

Note

1. If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ then $\vec{f}(t)$ is differentiable at t if and only if its components $f_1(t)$, $f_2(t), f_3(t)$ are differentiable at t and $\dfrac{d\vec{f}(t)}{dt} = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$
2. If the derivative of $\dfrac{d\vec{f}}{dt}$ w.r.to t exists, it is denoted by $\dfrac{d^2\vec{f}}{dt^2}$. Similarly, we denote higher derivatives.
3. If $\vec{c}$ is a constant vector, then $\dfrac{d\vec{c}}{dt} = \vec{0}$.

   For $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ and $\dfrac{d\vec{c}}{dt} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$.

9.1.1 Geometrical Meaning of Derivative

Let $\vec{r}(t)$ be the position vector of a point P with respect to the origin O.

As t varies continuously over a time interval P traces the curve C. Thus, the vector function $\vec{r}(t)$ represents a curve C in space.

Let $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ be the position vectors of neighbouring points P and Q on the curve C.

Then
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}$$

$\therefore \dfrac{\Delta\vec{r}}{\Delta t}$ is along the chord PQ.

If $\lim_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t}$ exists, it is denoted by $\dfrac{d\vec{r}}{dt}$ and $\dfrac{d\vec{r}}{dt}$ is in the directing of the tangent at P to the curve.

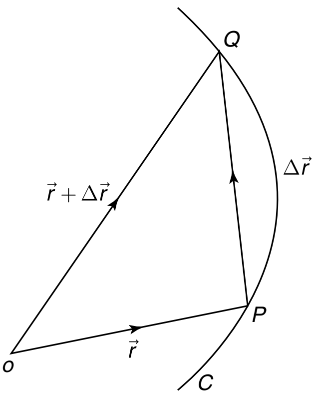


Fig. 9.1

If $\frac{d\vec{r}}{dt} \neq 0$, then $\frac{d\vec{r}}{dt}$ or $\vec{r}'(t)$ is called a tangent vector to the curve C at P.

The unit tangent vector at P is $= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \hat{u}(t)$.

Both $\vec{r}'(t)$ and $\hat{u}(t)$ are in the direction of increasing t. Hence, their sense depends on the orientation of the curve C.

9.2 DIFFERENTIATION FORMULAE

If $\vec{f}$ and $\vec{g}$ are differentiable vector functions of t and ϕ is a scalar function of t then

1. $\frac{d}{dt}(\vec{f} \pm \vec{g}) = \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}$

2. $\frac{d}{dt}(\phi\vec{f}) = \phi\frac{d\vec{f}}{dt} + \frac{d\phi}{dt}\vec{f}$

3. $\frac{d}{dt}(\vec{f} \cdot \vec{g}) = \vec{f} \cdot \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{g}$

4. $\frac{d}{dt}(\vec{f} \times \vec{g}) = \vec{f} \times \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \times \vec{g}$

5. $\frac{d}{dt}(\vec{f} \cdot \vec{g} \times \vec{h}) = \frac{d\vec{f}}{dt} \cdot \vec{g} \times \vec{h} + \vec{f} \cdot \frac{d\vec{g}}{dt} \times \vec{h} + \vec{f} \cdot \vec{g} \times \frac{d\vec{h}}{dt}$.

Note If $\vec{f}$ is a continuous function of a scalar s and s is a continuous function of t, then $\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{ds}\frac{ds}{dt}$.

6. Let $\vec{f}(t)$ be a vector function. $\vec{f}(t)$ changes if its magnitude is changed or its direction is changed or both magnitude and direction are changed. We shall find conditions under which a vector function will remain constant in magnitude or in direction.

(i) Let $\vec{f}(t)$ be a vector of constant length k.

Then $\qquad \vec{f} \cdot \vec{f} = |\vec{f}|^2 = k^2$

Differentiating w.r.to t, we get

$$\frac{d\vec{f}}{dt} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad 2\vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad \vec{f} \cdot \frac{d\vec{f}}{dt} = 0$$

$$\therefore \quad \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} = \text{is} \perp \text{to } \vec{f} \cdot$$

(ii) Let $\vec{f}(t)$ be a vector function with constant direction and let $\vec{a}$ be the unit vector in that direction

Then $\vec{f}(t) = \phi\vec{a}$, where $\qquad \phi = |\vec{f}|$

$$\therefore \quad \frac{d\vec{f}}{dt} = \frac{d\phi}{dt}\vec{a} + \phi\frac{d\vec{a}}{dt}.$$

But $\vec{a}$ is a constant vector, since its direction is fixed and magnitude is 1. $\therefore \frac{d\vec{a}}{dt} = \vec{0}$

$$\therefore \quad \frac{d\vec{f}}{dt} = \frac{d\phi}{dt}\vec{a}$$

Now $$\vec{f} \times \frac{d\vec{f}}{dt} = \phi\vec{a} \times \frac{d\phi}{dt}\vec{a} = \phi\frac{d\phi}{dt}\vec{a} \times \vec{a} = \vec{0} \qquad (\because \quad \vec{a} \times \vec{a} = \vec{0})$$

$$\therefore \quad \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} \text{ is parallel to } \vec{f}.$$

9.3 LEVEL SURFACES

Let ϕ be a continuous scalar point function defined in a region R in space. Then the set of all points satisfying the equation $\phi(x, y, z) = C$, where C is a constant, determines a surface which is called a **level surface** of ϕ. At every point on a level surface the function ϕ takes the same value C. If C is an arbitrary constant, the for different values of C, we get different level surfaces of ϕ.

No two level surfaces intersect. For, if $\phi = C_1$ and $\phi = C_2$ be two level surfaces of ϕ intersecting at a point P. Then $\phi(P) = C_1$ and $\phi(P) = C_2$ and so ϕ has two values at P which contradicts the uniqueness of value of the function ϕ. So, $\phi = C_1$ and $\phi = C_2$ do not intersect.

Thus, **only one level surface of ϕ passes through a given point**

For example, if $\phi(x, y, z)$ represents the temperature of (x, y, z) in a region R of space, then the level surfaces of equal temperature are called **isothermal surfaces**.

9.4 GRADIENT OF A SCALAR POINT FUNCTION OR GRADIENT OF A SCALAR FIELD

9.4.1 Vector Differential Operator

The symbolic vector $\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$ is called **Hamiltonian operator** or **vector differential operator** and is denoted by ∇ (read as del or nabla).

$$\therefore \qquad \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

It is also known as del operator. This operator can be applied on a scalar point function $\phi(x, y, z)$ or a vector point function $\vec{F}(x, y, z)$ which are differentiable functions. This gives rise to three field quantities namely gradient of a scalar, divergence of a vector and curl of a vector function.

Definition 9.4 Gradient

If $\phi(x, y, z)$ is a scalar point function continuously differentiable in a given region R of space, then the gradient of ϕ is defined by $\nabla\phi = \vec{i}\dfrac{\partial \phi}{\partial x} + \vec{j}\dfrac{\partial \phi}{\partial y} + \vec{k}\dfrac{\partial \phi}{\partial z}$.

It is abbreviated as grad ϕ. Thus, grad $\phi = \nabla\phi$.

Note Since $\nabla\phi$ is a vector, the gradient of a scalar point function is always a vector point function. Thus, $\nabla\phi$ defines a vector field.

Gradient is of great practical importance because some of the vector fields in applications can be obtained from scalar fields and scalar fields are easy to handle.

9.4.2 Geometrical Meaning of $\nabla\phi$

Let $\phi(x, y, z)$ be a scalar point function. Let $\phi(x, y, z) = C$ be a level surface of ϕ. Let P be a point on this surface with position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Then the differential $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ is tangent to the surface at P.

$$\text{Now} \qquad \nabla\phi \cdot d\vec{r} = \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right) \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi = 0 \qquad [\because \phi = C]$$

$\therefore$ $\nabla\phi$ is normal to the surface $\phi(x, y, z) = C$ at P.

So, a unit normal to the surface at P is $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

There is another unit normal in the opposite direction $= -\dfrac{\nabla\phi}{|\nabla\phi|}$.

9.4.3 Directional Derivative

The directional derivative of a scalar point function ϕ in a given direction $\vec{a}$ is the rate of change of ϕ in that direction. It is given by the component of $\nabla\phi$ in the direction of $\vec{a}$

$\therefore$ the directional derivative $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$.

Since $\nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\nabla\phi||\vec{a}|}{|\vec{a}|}\cos\theta$, where θ is the angle between $\nabla\phi$ and $\vec{a}$.

$$= |\nabla\phi|\cos\theta$$

So, the directional derivative at a given point is maximum if $\cos\theta$ is maximum.
i.e., $\cos\theta = 1 \Rightarrow \quad \theta = 0$.

$\therefore$ the maximum directional derivative at a point is in the direction of $\nabla\phi$ and the maximum directional derivative is $|\nabla\phi|$.

Note

1. The directional derivative is minimum when $\cos\theta = -1 \Rightarrow \theta = \pi$
   $\therefore$ the minimum directional derivative is $-|\nabla\phi|$
2. In fact, the vector $\nabla\phi$ is in the direction in which ϕ increases rapidly.
   i.e., outward normal and $-\nabla\phi$ points in the direction in which ϕ decreases rapidly.

9.4.4 Equation of Tangent Plane and Normal to the Surface

(i) Equation of tangent plane

Let A be a given point on the surface $\phi(x, y, z) = C$.
Let $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ be the position vector of A.
Let P be any point on the tangent plane to the surface at the point A and let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of P.
Then $\nabla\phi$ at A is normal to the surface and $\vec{r} - \vec{r}_0$ lies on the tangent plane at A.
$\therefore$ the equation of the tangent plane at the point A is $(\vec{r} - \vec{r}_0) . \nabla\phi = 0$

Note The cartesian equation of the plane at the point $A(x_0, y_0, z_0)$ is

$$(x - x_0)\frac{\partial \phi}{\partial x} + (y - y_0)\frac{\partial \phi}{\partial y} + (z - z_0)\frac{\partial \phi}{\partial z} = 0$$

where the partial derivatives are evaluated at the point (x_0, y_0, z_0).

(ii) Equation of the normal at the point A

Let A be a given point on the surface $\phi(x, y, z) = C$ and let $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ be the position vector of A.

Let $\vec{r}$ be the position vector of any point P on the normal at the point A. Then $\vec{r} - \vec{r}_0$ is parallel to the normal at the point A.

$\therefore$ the equation of the normal at the point A is $(\vec{r} - \vec{r}_0) \times \nabla\phi = 0$.

The cartesian equation of the normal at the point A is

$$\frac{x - x_0}{\frac{\partial \phi}{\partial x}} = \frac{y - y_0}{\frac{\partial \phi}{\partial y}} = \frac{z - z_0}{\frac{\partial \phi}{\partial z}},$$

where the partial derivatives are evaluated at (x_0, y_0, z_0).

9.4.5 Angle between Two Surfaces at a Common Point

We know that the angle between two planes is the angle between their normals.

We define angle between two surfaces at a point of intersection P is the angle between their tangent planes at P and hence, the angle between their normals at P.

The angle between two surfaces $f(x, y, z) = C_1$ and $g(x, y, z) = C_2$ at a common point P is the angle between their normals at the point P.

The normal at P to the surface $f(x, y, z) = C_1$ is ∇f.

The normal at P to the surface $g(x, y, z) = C_2$ is ∇g.

If θ is the angle between the normals at the point P, then $\cos\theta = \dfrac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$

(i) If $\theta = \dfrac{\pi}{2}$, then the normals are perpendicular and $\cos\theta = 0 \Rightarrow \dfrac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|} = 0 \Rightarrow \nabla f \cdot \nabla g = 0$

$\therefore$ if two surfaces are orthogonal at the point P then $\nabla f \cdot \nabla g = 0$

Conversely, if $\nabla f \cdot \nabla g = 0$, then $\theta = \dfrac{\pi}{2}$ That is they are orthogonal.

(ii) If $\theta = 0$, the normals at the common point coincide.

$\therefore$ the two tangent planes coincide and the surfaces touch at the common point.

9.4.6 Properties of Gradients

If f and g are scalar point functions which are differentiable, then

1. $\nabla C = 0$, where C is constant.
2. $\nabla(Cf) = C\nabla f$, where C is a constant.
3. $\nabla(f \pm g) = \nabla f \pm \nabla g$
4. $\nabla(fg) = f\nabla g + g\nabla f$
5. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ if $g \neq 0$

1. $\nabla C = 0$, C is constant.

Proof We know $\nabla\phi = \vec{i}\dfrac{\partial \phi}{\partial x} + \vec{j}\dfrac{\partial \phi}{\partial y} + \vec{k}\dfrac{\partial \phi}{\partial z}$ (1)

$$= \sum \vec{i} \frac{\partial \phi}{\partial x}$$

$\therefore$ $$\nabla C = \sum \vec{i} \frac{\partial C}{\partial x} = 0 \qquad \left[\because \text{C is a constant } \frac{\partial C}{\partial x} = 0, \frac{\partial C}{\partial y} = 0, \frac{\partial C}{\partial z} = 0\right] \blacksquare$$

2. $\nabla C\phi = C\nabla\phi$

Proof We have $$\nabla C\phi = \sum \vec{i} \frac{\partial}{\partial x}(C\phi) = C\sum \vec{i} \frac{\partial \phi}{\partial x} = C\nabla\phi \qquad \text{[using (1)]} \blacksquare$$

3. $\nabla(f \pm g) = \nabla f \pm \nabla g$

Proof We have $$\nabla(f \pm g) = \sum \vec{i} \frac{\partial}{\partial x}(f \pm g) \qquad \text{[using (1)]}$$

$$= \sum \left[\vec{i} \frac{\partial f}{\partial x} \pm \vec{i} \frac{\partial g}{\partial x}\right] = \sum \vec{i} \frac{\partial f}{\partial x} \pm \sum \vec{i} \frac{\partial g}{\partial x} = \nabla f \pm \nabla g$$

$\therefore$ $$\nabla(f \pm g) = \nabla f \pm \nabla g \qquad \blacksquare$$

4. $\nabla(fg) = f\nabla g + g\nabla f$

Proof We have $$\nabla(fg) = \sum \vec{i} \frac{\partial}{\partial x}(fg)$$

$$= \sum \vec{i} \left[f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}\right]$$

$$= \sum \vec{i} \left(f \frac{\partial g}{\partial x}\right) + \sum \vec{i} \left(g \frac{\partial f}{\partial x}\right)$$

$$= f \sum \vec{i} \frac{\partial g}{\partial x} + g \sum \vec{i} \frac{\partial f}{\partial x} = f\nabla g + g\nabla f$$

$\therefore$ $$\nabla(fg) = f\nabla g + g\nabla f \qquad \blacksquare$$

5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Proof We have $$\nabla\left(\frac{f}{g}\right) = \sum \vec{i} \frac{\partial}{\partial x}\left(\frac{f}{g}\right)$$

$$= \sum \vec{i} \left[\frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2}\right]$$

$$= \frac{1}{g^2}\left[g \sum \vec{i} \frac{\partial f}{\partial x} - f \sum \vec{i} \frac{\partial g}{\partial x}\right] = \frac{g\nabla f - f\nabla g}{g^2}$$

$\therefore$ $$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \qquad \blacksquare$$

WORKED EXAMPLES

EXAMPLE 1

Find grad ϕ for the following functions.

(i) $\phi(x, y, z) = 3x^2y - y^3z^2$ at the point (1, −2, 1)

(ii) $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$ at the point (1, 2, 1).

Solution.

(i) Given $\phi(x, y, z) = 3x^2y - y^3z^2$

We know $$\text{grad}\,\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r. to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 6xy, \quad \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2, \quad \frac{\partial\phi}{\partial z} = -2y^3z$$

At the point (1, −2, 1), $$\frac{\partial\phi}{\partial x} = 6\cdot 1(-2) = -12$$

$$\frac{\partial\phi}{\partial y} = 3\cdot 1^2 - 3\cdot(-2)^2 1^2 = 3 - 12 = -9$$

$$\frac{\partial\phi}{\partial z} = -2\cdot(-2)^3\cdot 1 = 16$$

$\therefore$ **at the point (1, −2, 1),** $\nabla\phi = -12\vec{i} - 9\vec{j} + 16\vec{k}$.

(ii) Given $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$

We know $$\text{grad}\,\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get,

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2 + y^2 + z^2}\cdot 2x, \quad \frac{\partial\phi}{\partial y} = \frac{1}{x^2 + y^2 + z^2}\cdot 2y, \quad \frac{\partial\phi}{\partial z} = \frac{1}{x^2 + y^2 + z^2}\cdot 2z$$

At the point (1, 2, 1), $$\frac{\partial\phi}{\partial x} = \frac{2\cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{\partial\phi}{\partial y} = \frac{2\cdot 2}{1^2 + 2^2 + 1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial\phi}{\partial z} = \frac{2\cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ **at the point (1, 2, 1),** $\text{grad}\,\phi = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k} = \frac{1}{3}[\vec{i} + 2\vec{j} + \vec{k}]$.

EXAMPLE 2

Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point (1, −2, −1) in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution.

Given $\phi(x, y, z) = x^2yz + 4xz^2$

We know $\text{grad } \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$$

At the point (1, −2, −1), $\frac{\partial\phi}{\partial x} = 2\cdot 1(-2)(-1) + 4(-1)^2 = 8$

$$\frac{\partial\phi}{\partial y} = 1^2\cdot(-1) = -1$$

$$\frac{\partial\phi}{\partial z} = 1^2(-2) + 8\cdot 1(-1) = -2 - 8 = -10$$

$\therefore$ **at the point (1, −2, −1),** $\nabla\phi = 8\vec{i} - \vec{j} - 10\vec{k}$

Given direction is $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$\therefore$ the directional derivative of ϕ at the point (1, −2, −1) in the direction of $\vec{a}$ is

$$\nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k})\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{\sqrt{4+1+4}} = \frac{16+1+20}{\sqrt{9}} = \frac{37}{3}$$

EXAMPLE 3

If $\vec{r} + x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ prove that (i) $\nabla r = \frac{\vec{r}}{r}$, (ii) $\nabla r^n = nr^{n-2}\vec{r}$, (iii) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ (iv) $\nabla(\log r) = \frac{\vec{r}}{r^2}$.

Solution.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$ (1)

(i) $\nabla r = \frac{\vec{r}}{r}$

We know $\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

Differentiating (1) partially w.r.to x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \nabla r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} = \frac{1}{r}[x\vec{i} + y\vec{j} + z\vec{k}] = \frac{\vec{r}}{r}$$

(ii) $\nabla r^n = nr^{n-2}\vec{r}$

We know
$$\nabla r^n = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$
$$= \vec{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \vec{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \vec{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$
$$= n\,r^{n-1}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = \frac{n\,r^{n-1}}{r}[x\vec{i} + y\vec{j} + z\vec{k}] = nr^{n-2}\vec{r}$$

(iii) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

We know,
$$\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$= \vec{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$
$$= -\frac{1}{r^2}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{\vec{r}}{r^3}$$

(iv) $\nabla(\log r) = \dfrac{\vec{r}}{r^2}$

We know, $\nabla(\log r) = \vec{i}\dfrac{\partial}{\partial x}(\log r) + \vec{j}\dfrac{\partial}{\partial y}(\log r) + \vec{k}\dfrac{\partial}{\partial z}(\log r)$
$$= \vec{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right) = \frac{1}{r}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = \frac{\vec{r}}{r^2}$$

EXAMPLE 4

Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at the point (1, –1, 3).

Solution.

Given $\phi = 2yz + z^2$

We know $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 2z, \quad \frac{\partial\phi}{\partial z} = 2y + 2z$$

At the point (1, −1, 3), $\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 2(3) = 6, \quad \frac{\partial \phi}{\partial z} = 2(-1) + 2 \cdot 3 = 4$

∴ **at the point (1, −1, 3),** $\nabla \phi = 6\vec{j} + 4\vec{k}$

Given direction is $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

∴ the directional derivative of ϕ at the point (1, −1, 3) in the direction of $\vec{a}$ is

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = (6\vec{j} + 4\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{1+4+4}} = \frac{12+8}{\sqrt{9}} = \frac{20}{3}$$

EXAMPLE 5

Find the directional derivative of $x^3 + y^3 + z^3$ at the point (1, −1, 2) in the direction of $\vec{i} + 2\vec{j} + \vec{k}$.

Solution.

Given $\phi(x, y, z) = x^3 + y^3 + z^3$

We know $\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$

Now differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial \phi}{\partial x} = 3x^2, \quad \frac{\partial \phi}{\partial y} = 3y^2, \quad \frac{\partial \phi}{\partial z} = 3z^2$$

At the point (1, –1, 2), $\frac{\partial \phi}{\partial x} = 3 \cdot 1^2 = 3, \quad \frac{\partial \phi}{\partial y} = 3(-1)^2 = 3, \quad \frac{\partial \phi}{\partial z} = 3 \cdot 2^2 = 12$

∴ **at the point (1, −1, 2),** $\nabla \phi = 3\vec{i} + 3\vec{j} + 12\vec{k}$

Given direction is $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$

∴ the directional derivative of ϕ at the point (1, –1, 2) in the direction of $\vec{a}$ is

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = (3\vec{i} + 3\vec{j} + 12\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + \vec{k})}{\sqrt{1+4+1}} = \frac{3+6+12}{\sqrt{6}} = \frac{21}{\sqrt{6}} = 21\frac{\sqrt{6}}{6} = \frac{7\sqrt{6}}{2}$$

EXAMPLE 6

Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).

Solution.

The given surface is $x^3 + y^3 + 3xyz = 3$, which is taken as $\phi = C$

∴ $\phi = x^3 + y^3 + 3xyz$

We know that $\nabla \phi$ is normal to the surface.

So, unit normal to the surface is $\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$

Now $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively,

we get, $$\frac{\partial\phi}{\partial x} = 3x^2 + 3yz,\quad \frac{\partial\phi}{\partial y} = 3y^2 + 3xz,\quad \frac{\partial\phi}{\partial z} = 3xy$$

At the point (1, 2, −1), $$\frac{\partial\phi}{\partial x} = 3\cdot1^2 + 3\cdot2(-1) = -3$$

$$\frac{\partial\phi}{\partial y} = 3\cdot2^2 + 3\cdot1(-1) = 9 \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 3\cdot1\cdot2 = 6$$

$\therefore$ **at the point (1, 2, −1),** $\nabla\phi = -3\vec{i} + 9\vec{j} + 6\vec{k}$

$\therefore$ unit normal to the given surface at the point (1, 2, −1) is

$$\vec{n} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{9+81+36}} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{126}}$$

Note If the surface equation is written as $x^3 + y^3 + 3xyz - 3 = 0$, then we take $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$. Here $C = 0$.

EXAMPLE 7

Find a unit normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

Solution.

Given $$\phi(x, y, z) = x^2y + 2xz^2$$

We know, $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2,\quad \frac{\partial\phi}{\partial y} = x^2,\quad \frac{\partial\phi}{\partial z} = 4xz$$

At the point (1, 0, 2), $$\frac{\partial\phi}{\partial x} = 2\cdot1\cdot0 + 2\cdot2^2 = 8,\quad \frac{\partial\phi}{\partial y} = 1^2 = 1,\quad \frac{\partial\phi}{\partial z} = 4\cdot1\cdot2 = 8$$

$\therefore$ **at the point (1, 0, 2),** $\nabla\phi = 8\vec{i} + \vec{j} + 8\vec{k}$

$\therefore$ unit normal vector to the given surface at the point (1, 0, 2) is

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{64+1+64}} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}$$

EXAMPLE 8

Find the maximum value of the directional derivative of $\phi = x^3yz$ at the point (1, 4, 1).

Solution.

Given $$\phi = x^3yz$$

We know, $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

The directional derivative is maximum in the direction of $\nabla\phi$ and the maximum value $= |\nabla\phi|$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2yz, \quad \frac{\partial\phi}{\partial y} = x^3z, \quad \frac{\partial\phi}{\partial z} = x^3y$$

At the point (1, 4, 1), $\quad \frac{\partial\phi}{\partial x} = 3\cdot1\cdot4\cdot1 = 12, \quad \frac{\partial\phi}{\partial y} = 1^3\cdot1 = 1 \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 1^3\cdot4 = 4$

$\therefore$ **at the point (1, 4, 1),** $\quad \nabla\phi = 12\vec{i} + \vec{j} + 4\vec{k}$

Maximum value of the directional derivative $= |\nabla\phi| = |12\vec{i} + \vec{j} + 4\vec{k}| = \sqrt{144+1+16} = \sqrt{161}$

EXAMPLE 9

In what direction from the point (1, 1, −2), is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum? Also find the maximum directional derivative.

Solution.

Given $\quad \phi = x^2 - 2y^2 + 4z^2$

We know that the directional derivative is maximum in the direction of $\nabla\phi$. The maximum value $= |\nabla\phi|$

We have $\quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -4y, \quad \frac{\partial\phi}{\partial z} = 8z$$

At the point (1, 1, −2), $\quad \frac{\partial\phi}{\partial x} = 2\cdot1 = 2, \quad \frac{\partial\phi}{\partial y} = -4\cdot1 = -4, \quad \frac{\partial\phi}{\partial z} = 8(-2) = -16$

$\therefore$ **at the point (1, 1, −2),** $\quad \nabla\phi = 2\vec{i} - 4\vec{j} - 16\vec{k} = 2[\vec{i} - 2\vec{j} - 8\vec{k}]$

$\therefore$ the directional derivative is maximum in the direction of $2(\vec{i} - 2\vec{j} - 8\vec{k})$

$$\text{Maximum value} = |\nabla\phi| = |2(\vec{i} - 2\vec{j} - 8\vec{k})| = 2\sqrt{1+4+64} = 2\sqrt{69}$$

EXAMPLE 10

Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point (2, −1, 2).

Solution.

The given surfaces are

$x^2 + y^2 + z^2 = 9 \quad (1) \quad$ and $\quad x^2 + y^2 - z = 3 \quad (2)$

$P(2, -1, 2)$ is a common point of (1) and (2)

Let $\quad f = x^2 + y^2 + z^2$ and $g = x^2 + y^2 - z$

Now, $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

Differentiating f partially w.r.to x, y, z respectively we get,

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial z} = 2z$$

At the point (2, −1, 2), $\frac{\partial f}{\partial x} = 2\cdot 2 = 4, \quad \frac{\partial f}{\partial y} = 2(-1) = -2, \quad \frac{\partial f}{\partial z} = 2(+2) = +4$

$\therefore$ **at the point (2, −1, 2),** $\nabla f = 4\vec{i} - 2\vec{j} + 4\vec{k}$

Now $\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$

Differentiating g partially w.r.to x, y, z respectively, we get

$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = 2y, \qquad \frac{\partial g}{\partial z} = -1$$

at the point (2, −1, 2), $\frac{\partial g}{\partial x} = 2\cdot 2 = 4, \quad \frac{\partial g}{\partial y} = 2(-1) = -2, \quad \frac{\partial g}{\partial z} = -1$

$\therefore$ **at the point (2, −1, 2),** $\nabla g = 4\vec{i} - 2\vec{j} - \vec{k}$

If $\boldsymbol{\theta}$ is the angle between the surfaces (1) and (2) at (2, –1, 2), then

$$\cos\boldsymbol{\theta} = \frac{\nabla f\cdot\nabla g}{|\nabla f||\nabla g|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k})}{\sqrt{16+4+16}}\cdot\frac{(4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+1}} = \frac{16+4-4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \qquad \boldsymbol{\theta} = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

EXAMPLE 11

Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at the point (1, −1, 2).

Solution.

The given surfaces are

$$5x^2 - 2yz - 9x = 0 \quad (1) \quad \text{and} \quad 4x^2y + z^3 - 4 = 0 \quad (2)$$

Let $f = 5x^2 - 2yz - 9x$ and $g = 4x^2y + z^3 - 4$

To prove (1) and (2) cut orthogonally at the point (1, –1, 2),

i.e., to prove $\nabla f\cdot\nabla g = 0$

Now $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 10x - 9, \quad \frac{\partial f}{\partial y} = -2z \quad \text{and} \quad \frac{\partial f}{\partial z} = -2y$$

$$\therefore \qquad \nabla f = (10x - 9)\vec{i} - 2z\vec{j} - 2y\vec{k}$$

and $$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$ $$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

At the point (1, −1, 2), $$\nabla f = (10-9)\vec{i} - 2\cdot 2\vec{j} - 2(-1)\vec{k} = \vec{i} - 4\vec{j} + 2\vec{k}$$

and $$\nabla g = 8\cdot 1\cdot(-1)\vec{i} + 4\cdot 1^2\vec{j} + 3\cdot 2^2\vec{k} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$ $$\nabla f\cdot\nabla g = (\vec{i} - 4\vec{j} + 2\vec{k})\cdot(-8\vec{i} + 4\vec{j} + 12\vec{k}) = -8 - 16 + 24 = 0$$

Hence, the two surfaces cut orthogonally at the point (1, −1, 2).

EXAMPLE 12

Find a and b if the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point (1, −1, 2).

Solution.

The given surfaces are

$$ax^2 - byz - (a+2)x = 0 \quad (1) \qquad \text{and} \qquad 4x^2y + z^3 - 4 = 0 \quad (2)$$

Let $f = ax^2 - byz - (a+2)x$ and $g = 4x^2y + z^3 - 4$

Given the surfaces (1) and (2) cut orthogonally at the point (1, −1, 2).

$\therefore$ $$\nabla f\cdot\nabla g = 0 \quad (3)$$

Now $$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2ax - a - 2, \quad \frac{\partial f}{\partial y} = -bz \quad \text{and} \quad \frac{\partial f}{\partial z} = -by$$

$\therefore$ $$\nabla f = (2ax - a - 2)\vec{i} - bz\vec{j} - by\vec{k}$$

and $$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$ $$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

At the point (1, −1, 2), $$\nabla f = (2a - a - 2)\vec{i} - b\cdot 2\vec{j} - b(-1)\vec{k}$$

$\Rightarrow$ $$\nabla f = (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and $$\nabla g = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$ $$\nabla f\cdot\nabla g = ((a-2)\vec{i} - 2b\vec{j} + b\vec{k})\cdot(-8\vec{i} + 4\vec{j} + 12\vec{k})$$
$$= -8(a-2) - 8b + 12b = -8a + 4b + 16$$

From (3), $$\nabla f\cdot\nabla g = 0 \Rightarrow -8a + 4b + 16 = 0 \Rightarrow 2a - b = 4 \quad (4)$$

Since (1, −1, 2) is a point on the surface $f = 0$, we get

$$a + 2b - (a+2) = 0 \Rightarrow 2b = 2 \Rightarrow b = 1$$

$\therefore \quad (4) \Rightarrow \qquad 2a = 4 + b = 4 + 1 = 5 \quad \Rightarrow \quad a = \frac{5}{2}$

$\therefore \qquad a = \frac{5}{2},\ b = 1$

EXAMPLE 13

Find the angle between the normals to the surface $xy = z^2$ at the points (1, 4, 2) and (–3, –3, 3).

Solution.

The given surface is $\quad xy - z^2 = 0$

$\therefore \quad \phi = xy - z^2$

We know $\nabla\phi$ is normal to the surface at the point (x, y, z)

Let $\vec{n}_1, \vec{n}_2$, be the normals to the surface at the points (1, 4, 2) and (–3, –3, 3) respectively.

$\therefore \quad \vec{n}_1 = \nabla\phi$ at the point (1, 4, 2)

and $\quad \vec{n}_2 = \nabla\phi$ at the point (–3, – 3, 3)

Now $\quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y,\quad \frac{\partial\phi}{\partial y} = x \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -2z$$

$\therefore \quad \nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$

At the point (1, 4, 2), $\quad \nabla\phi = 4\vec{i} + \vec{j} - 4\vec{k} \quad \therefore \quad \vec{n}_1 = 4\vec{i} + \vec{j} - 4\vec{k}$

At the point (–3, –3, 3), $\quad \nabla\phi = -3\vec{i} - 3\vec{j} - 6\vec{k} \quad \therefore \quad \vec{n}_2 = -3\vec{i} - 3\vec{j} - 6\vec{k}$

If θ is the angle between the normals, then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(4\vec{i} + \vec{j} - 4\vec{k})\cdot(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{16+1+16}\ \sqrt{9+9+36}}$$

$$= \frac{-12 - 3 + 24}{\sqrt{33}\ \sqrt{54}} = \frac{9}{\sqrt{33}\ \sqrt{54}} = \frac{1}{\sqrt{22}}$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

EXAMPLE 14

Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at the point (–1, 2, 1).

Solution.

Given $\quad \phi = xy^2 + yz^3$

$\therefore \quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$

At the point (2, –1, 1), $\quad \nabla\phi = (-1)^2\vec{i} + (-4+1)\vec{j} + 3(-1)1^2\vec{k} = \vec{i} - 3\vec{j} - 3\vec{k}$

The directional derivative of ϕ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at the point (–1, 2, 1) is required.

Let $f = x\log z - y^2 + 4$

$\therefore$ $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = \log z\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}$

At the point (−1, 2, 1), $\nabla f = \log 1\vec{i} - 4\vec{j} + \left(\frac{-1}{1}\right)\vec{k} = 0\vec{i} - 4\vec{j} - \vec{k} = -4\vec{j} - \vec{k}$

$\therefore$ $\vec{a} = -4\vec{j} - \vec{k}$

Required directional derivative is $= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \frac{(-4\vec{j} - \vec{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

EXAMPLE 15

If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, then find ϕ if $\phi(1, -2, 2) = 4$.

Solution.

Given $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ (1)

But $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ (2)

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, from (1) and (2), we get

$\frac{\partial\phi}{\partial x} = 2xyz^3$ (3) $\quad \frac{\partial\phi}{\partial y} = x^2z^3$ (4) $\quad \frac{\partial\phi}{\partial z} = 3x^2yz^2$ (5)

Integrating (3) partially w.r.to x, we get

$\phi = x^2yz^3 + f_1(y, z)$ (6)

Integrating (4) partially w.r.to y, we get,

$\phi = x^2z^3y + f_2(x, z)$ (7)

Integrating (5) partially w.r.to z, we get,

$\phi = x^2yz^3 + f_3(x, y)$ (8)

From (6), (7), (8), ϕ is obtained by adding all the terms and an arbitrary constant C, but omitting $f_1(y, z), f_2(x, z), f_3(x, y)$ and choosing only one of the repeated terms.

Thus, $\phi = x^2yz^3 + C$

Given $\phi(1, -2, 2) = 4$

$\therefore$ $1 \times (-2) \times 8 + C = 4 \Rightarrow C = 4 + 16 = 20$

$\therefore$ $\phi = x^2yz^3 + 20$

EXAMPLE 16

Find the equation of the tangent plane and the equation of the normal to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point (3, 2, 1).

Solution.

The given surface is $x^2 - 4y^2 + 3z^2 + 4 = 0$

Let $\phi = x^2 - 4y^2 + 3z^2 + 4$

$\therefore$ $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = 2x\vec{i} - 8y\vec{j} + 6z\vec{k}$

At the point (3, 2, 1), $\nabla\phi = 6\vec{i} - 16\vec{j} + 6\vec{k}$

We know that the equation of the tangent plane at the point (x_0, y_0, z_0) is

$$(x - x_0)\frac{\partial\phi}{\partial x} + (y - y_0)\frac{\partial\phi}{\partial y} + (z - z_0)\frac{\partial\phi}{\partial z} = 0$$

Now $\dfrac{\partial\phi}{\partial x} = 2x, \quad \dfrac{\partial\phi}{\partial y} = -8y \quad \text{and} \quad \dfrac{\partial\phi}{\partial z} = 6z$

Here $(x_0, y_0, z_0) = (3, 2, 1)$ $\therefore$ $\dfrac{\partial\phi}{\partial x} = 6, \quad \dfrac{\partial\phi}{\partial y} = -16 \quad \text{and} \quad \dfrac{\partial\phi}{\partial z} = 6$

$\therefore$ the equation of the tangent plane at the point (3, 2, 1) is

$$(x - 3)6 + (y - 2)(-16) + (z - 1)6 = 0$$

$\Rightarrow$ $3(x - 3) - 8(y - 2) + 3(z - 1) = 0$ [dividing by 2]

$\Rightarrow$ $3x - 8y + 3z - 9 + 16 - 3 = 0$

$\Rightarrow$ $3x - 8y + 3z + 4 = 0$

The equation of the normal at the point (x_0, y_0, z_0) is

$$\frac{x - x_0}{\frac{\partial\phi}{\partial x}} = \frac{y - y_0}{\frac{\partial\phi}{\partial y}} = \frac{z - z_0}{\frac{\partial\phi}{\partial z}}$$

The equation of the normal at the point (3, 2, 1) is

$$\frac{x-3}{6} = \frac{y-2}{-16} = \frac{z-1}{6} \Rightarrow \frac{x-3}{3} = \frac{y-2}{-8} = \frac{z-1}{3}.$$

EXAMPLE 17

If the directional derivative of $\phi(x, y, z) = a(x + y) + b(y + z) + c(z + x)$ has maximum value 12 at the point (1, 2, 1) in the direction parallel to the line $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-1}{3}$, find the value of a, b, c.

Solution.

Given $\phi = a(x + y) + b(y + z) + c(z + x)$

$\therefore$ $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$\Rightarrow$ $\nabla\phi = (a + c)\vec{i} + (a + b)\vec{j} + (b + c)\vec{k}$

We know that the directional derivative is maximum in the direction of $\nabla\phi$.

But given it is maximum in the direction parallel to the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-1}{3}$.

$$\therefore \qquad \frac{a+c}{1}=\frac{a+b}{2}=\frac{b+c}{3}=K$$

$$\Rightarrow \qquad a+c=K \quad (1) \qquad a+b=2K \quad (2) \qquad b+c=3K \quad (3)$$

Adding we get,

$$a+c+a+b+b+c=K+2K+3K$$

$$\Rightarrow \qquad 2(a+b+c)=6K \quad \Rightarrow \quad a+b+c=3K \quad (4)$$

Using (3), (4) $\Rightarrow$ $a+3K=3K \quad \Rightarrow \quad a=0$

From (1), $0+c=K \quad \Rightarrow \quad c=K$

From (2), $0+b=2K \quad \Rightarrow \quad b=2K$

Given the maximum value of directional derivative = 12

$$\Rightarrow \qquad |\nabla\phi|=12$$

$$\Rightarrow \qquad \sqrt{(a+c)^2+(a+b)^2+(b+c)^2}=12$$

$$\Rightarrow \qquad (a+c)^2+(a+b)^2+(b+c)^2=144$$

$$\Rightarrow \qquad K^2+4K^2+9K^2=144$$

$$\Rightarrow \qquad 14K^2=144 \quad \Rightarrow \quad K^2=\frac{144}{14} \quad \Rightarrow \quad K=\pm\frac{12}{\sqrt{14}}$$

$$\therefore \qquad a=0,\ b=\pm\frac{24}{\sqrt{14}},\ c=\pm\frac{12}{\sqrt{14}}$$

EXAMPLE 18

If $\vec{u}=x+y+z$, $\vec{v}=x^2+y^2+z^2$, $\vec{w}=xy+yz+zx$, then show that the vectors ∇u, ∇v, ∇w are coplanar.

Solution.

Given $\qquad u=x+y+z, \quad v=x^2+y^2+z^2, \quad w=xy+yz+zx$

Now,

$$\nabla u=\vec{i}\frac{\partial u}{\partial x}+\vec{j}\frac{\partial u}{\partial y}+\vec{k}\frac{\partial u}{\partial z}=\vec{i}+\vec{j}+\vec{k}$$

$$\nabla v=\vec{i}\frac{\partial v}{\partial x}+\vec{j}\frac{\partial v}{\partial y}+\vec{k}\frac{\partial v}{\partial z}=2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla w=\vec{i}\frac{\partial w}{\partial x}+\vec{j}\frac{\partial w}{\partial y}+\vec{k}\frac{\partial w}{\partial z}=(y+z)\vec{i}+(z+x)\vec{j}+(x+y)\vec{k}$$

We know that three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if their scalar triple product $\vec{a} \cdot \vec{b} \times \vec{c} = 0$.

$\therefore$ $\nabla u, \nabla v, \nabla w$ are coplanar, if $\nabla u \cdot \nabla v \times \nabla w = 0$

Now $\nabla u \cdot \nabla v \times \nabla w = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \quad R_2 \to R_2 + R_3$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0 \qquad [\text{since } R_1 = R_2]$$

$\therefore$ the vectors $\nabla u, \nabla v, \nabla w$ are coplanar.

EXERCISE 9.1

1. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find $\nabla\phi$ at the point (1, –2, –1).
2. If $\phi = 2xz - y^2$ find grad ϕ at the point (1, 3, 2).
3. Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at the point (1, 1, 1) in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$.
4. Find the directional derivative of $xyz - xy^2z^2$ at the point (1, 2, –1) in the direction of the vector $\vec{i} - \vec{j} - 3\vec{k}$.
5. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where $Q = (5, 0, 4)$.
6. Find the unit normal vector to the surface
   (i) $x^2 + 2y^2 + z^2 = 7$ at the point (1, –1, 2). (ii) $x^2 + y^2 - z^2 = 1$ at the point (1, 1, 1).
   (iii) $x^2 + y^2 - z = 1$ at the point (1, 1, 1). (iv) $x^2 + y^2 = z$ at the point (1, 2, 5).
7. Find the angle between the surfaces $x^2 + y + z = 2$ and $x\log z = y^2 - 1$ at the point (1, 1, 1).
8. Find the angle between the surfaces $2yz + z^2 = 3$ and $x^2 + y^2 + z^2 = 3$ at the point (1, 1, 1).
9. Find the angle between the surfaces $xyz = 4$ and $x^2 + y^2 + z^2 = 9$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.
10. Find the equation of the tangent plane and normal line to the surface $xz^2 + x^2y - z + 1 = 0$ at the point (1, –3, 2).
11. Find the equation of the tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, –1, 2).
12. Find the equation of the tangent plane and normal line to the surface $2z - x^2 = 0$ at the point $P(2, 0, 2)$.
13. Find ϕ if
    (i) $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (8z^3 - 3x^2yz^2)\vec{k}$

(ii) $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ if $\phi(1, -2, 2) = 4$

(iii) $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

(iv) $\nabla\phi = (2xyz + x)\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

(v) $\nabla\phi = (y + \sin z)\vec{i} + x\vec{j} + x\cos z\vec{k}$.

14. Find the angle between the normals to the intersecting surfaces $xy - z^2 - 1 = 0$ and $y^2 - 3z - 1 = 0$ at the point (1, 1, 0).
15. Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).
16. Find the values of a and b so that the surfaces $ax^3 - by^2z = (a + 3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at the point (2, –1, –3).
17. The temperature at any point in space is given by $T = xy + yz + zx$. Find the direction in which the temperature changes most rapidly from the point (1, 1, 1) and determine the maximum rate of change.
18. In what direction is the directional derivative of the function $\phi = x^2 - 2y^2 + 4z^2$ from the point (1, 1, –1) is maximum and what is its value?
19. Find the maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point (1, 1, –4).
20. Find $\nabla\phi$ at the point (1, 1, 1) if $\phi(x, y, z) = x^2y + y^2x + z^2$.
21. Find the directional derivative of $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point (1, 1, –1) in the direction $2\vec{i} - \vec{j} - \vec{k}$.
22. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2\vec{i} + 3\vec{j} + 6\vec{k}$ at the point (3, 1, 2).
23. Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at (1, 2, 3) in the direction of $2\vec{i} + \vec{j} - \vec{k}$.
24. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $P(1, -2, -1)$ in the direction of PQ, where Q is (3, –3, –3).
25. Find a unit normal to the surface $xy^3z^2 = 4$ at the point (–1, –1, 2).
26. In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
27. What is the greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3)?
28. Find the angle between the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at the point (4, –3, 2).
29. Find ϕ if $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$.

ANSWERS TO EXERCISE 9.1

1. $-6\vec{i} - 9\vec{j} - 4\vec{k}$ 2. $4\vec{i} - 6\vec{j} + 2\vec{k}$ 3. $\frac{19}{3}$ 4. $\frac{29}{\sqrt{11}}$ 5. $\frac{28}{\sqrt{21}}$

6. (i) $\frac{\vec{i} - 2\vec{j} + 2\vec{k}}{3}$ (ii) $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ (iii) $\frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$ (iv) $\frac{2\vec{i} + 4\vec{j} - 5\vec{k}}{3\sqrt{5}}$

7. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$ 8. $\cos^{-1}\sqrt{\frac{3}{5}}$ 9. $\cos^{-1}\sqrt{\frac{2}{3}}$

10. $2x - y - 3z + 1 = 0, \dfrac{x-1}{-2} = \dfrac{y+3}{1} = \dfrac{z-2}{3}$

11. $7x - 3y + 8z - 26 = 0, \dfrac{x-1}{7} = \dfrac{y+1}{-3} = \dfrac{z-2}{3}$

12. $2x - z = 2; \dfrac{x-2}{-2} = \dfrac{y}{0} = \dfrac{z-2}{1}$

13. (i) $\phi = xy^2 - x^2yz^3 + 3y + 2z^4 + c$ (ii) $\phi = x^2yz^3 + 20$

(iii) $\phi = 3x^2y + xz^3 - yz + c$ (iv) $\phi = x^2yz + \dfrac{x^2}{2} + c$ (v) $\phi = xy + x\sin z + c$

14. $\cos^{-1}\left(\dfrac{2}{\sqrt{26}}\right)$

15. $\cos^{-1}\dfrac{13}{3\sqrt{22}}$

16. $a = -\dfrac{7}{3}, b = \dfrac{64}{9}$

17. $\vec{i} + \vec{j} + \vec{k}, 2\sqrt{3}$

18. $2\vec{i} - 4\vec{j} - 8\vec{k}, 2\sqrt{21}$

19. 1652

20. $\nabla\phi = 3\vec{i} + 3\vec{j} + 2\vec{k}$

21. $\dfrac{16}{\sqrt{6}}$

22. $\dfrac{45}{7}$

23. $\dfrac{86}{\sqrt{6}}$

24. $\dfrac{37}{3}$

25. $-\dfrac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$

26. $96\sqrt{19}$

27. 9

28. $\theta = \cos^{-1}\left(\sqrt{\dfrac{19}{29}}\right)$

29. $\phi = 3x^2y + xz^3 - yz + c$

9.5 DIVERGENCE OF A VECTOR POINT FUNCTION OR DIVERGENCE OF A VECTOR FIELD

Definition 9.5 If $\vec{F}(x, y, z)$ be a vector point function continuously differentiable in a region R of space, then **the divergence of** $\vec{F}$ is defined by

$$\nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as div $\vec{F}$ and thus, div $\vec{F} = \nabla \cdot \vec{F}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then $\quad \nabla \cdot \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

If $\vec{F}$ is a constant vector, then $\nabla \cdot \vec{F} = 0$ and conversely if $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ is a constant vector.

Note (i) From the definition it is clear that div $\vec{F}$ is a scalar point function. So, the divergence of a vector field is a scalar point function. The notation $\nabla \cdot \vec{F}$ is not a scalar product in the usual sense, since $\nabla \cdot \vec{F} \neq \vec{F} \cdot \nabla$. In fact $\vec{F} \cdot \nabla = F_1\dfrac{\partial}{\partial x} + F_2\dfrac{\partial}{\partial y} + F_3\dfrac{\partial}{\partial z}$ is a scalar operator.

9.5.1 Physical Interpretation of Divergence

Physical interpretation of divergence applied to a vector field is that it gives approximately the 'loss' of the physical quantity at a given point per unit volume per unit time.

(i) If $\vec{v}(x, y, z)$ is the moving fluid at a point (x, y, z), then the 'loss' of the fluid per unit volume per unit time at the point is given by div $\vec{v}$. Thus, divergence gives a measure of the outward flux per unit volume of the flow at (x, y, z).

If there is no 'loss' of fluid anywhere, then div $\vec{v} = 0$ and the fluid is said to be incompressible.

(ii) If $\vec{v}$ represents an electric flux, div $\vec{v}$ is the amount of electric flux which diverges per unit volume in unit time.

(iii) If $\vec{v}$ represents the heat flux, div $\vec{v}$ is the rate at which heat is issuing from a point per unit volume.

Definition 9.6 Solenoidal Vector

If div $\vec{F} = 0$ everywhere in a region R, **then** $\vec{F}$ **is called a solenoidal vector point function** and ***R*** **is called a solenoidal field.**

9.6 CURL OF A VECTOR POINT FUNCTION OR CURL OF A VECTOR FIELD

Definition 9.7 If $\vec{F}(x, y, z)$ be a vector point function continuously differentiable in a region R, then **the curl of** $\vec{F}$ is defined by

$$\nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as curl $\vec{F}$

Thus, $$\text{curl } \vec{F} = \nabla \times \vec{F}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\begin{aligned} \text{curl } \vec{F} &= \nabla \times \vec{F} \\ &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \\ &= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \vec{j}\left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] \end{aligned}$$

This is symbolically written as $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $\vec{F}$ is a constant vector, then curl $\vec{F} = \vec{0}$

9.6.1 Physical Meaning of Curl $\vec{F}$

If $\vec{F}$ represents the linear velocity of the point P of a rigid body that rotates about a fixed axis (e.g., top) with constant angular velocity $\vec{\omega}$, then curl $\vec{F}$ at P is equal to $2\vec{\omega}$.

If the body is not rotating, then $\vec{\omega} = \vec{0}$ $\quad\therefore\quad$ Curl $\vec{F} = \vec{0}$

Definition 9.8 Irrotational Vector Field

Let $\vec{F}(x, y, z)$ be a vector point function. If curl $\vec{F} = \vec{0}$ at all points in a region R, then $\vec{F}$ is said to be an **irrotational vector in *R***. The vector field R is called an **irrotational vector field**.

Definition 9.9 Conservative Vector Field

A vector field $\vec{F}$ is said to be **conservative** if there exists a scalar function ϕ such that $\vec{F} = \nabla\phi$

Note

1. In a conservative vector field $\vec{F} = \nabla\phi$

$$\therefore \quad \nabla \times \vec{F} = \nabla \times \nabla\phi = \vec{0} \quad \Rightarrow \quad \vec{F} \text{ is irrotational.}$$

2. This scalar function ϕ is called the **scalar potential of $\vec{F}$.**
Only irrotational vectors will have scalar potential ϕ.

WORKED EXAMPLES

EXAMPLE 1

Prove that $\nabla \times \nabla\phi = 0$, where ϕ is a scalar point function.

Solution.

We have $$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}, \quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad \nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= 0 \qquad \left[\text{Assuming } \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}, \frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$\therefore$ $\nabla\phi$ is always an irrotational vector.

EXAMPLE 2

Find the divergence and curl of the vector $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (2, –1, 1).

Solution.

Given $$\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$$

$$\therefore \quad \text{div } \vec{v} = \nabla\cdot\vec{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

At the point (2, −1, 1), $\nabla\cdot\vec{v} = (-1)\cdot 1 + 3\cdot 4 + 2\cdot 2\cdot 1 - (-1)^2 = -1 + 12 + 4 - 1 = 14$

and $$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \vec{i}[0 - 2yz - 0] - \vec{j}[z^2 - 0 - xy] + \vec{k}[6xy - xz]$$

$$= -2yz\vec{i} - (z^2 - xy)\vec{j} + (6xy - xz)\vec{k}$$

At the point (2, −1, 1),

$$\nabla\times\vec{v} = -2(-1)\cdot 1\vec{i} - (1^2 - 2(-1))\vec{j} + [6\cdot 2(-1) - 1\cdot 2]\vec{k} = 2\vec{i} - 3\vec{j} - 14\vec{k}$$

EXAMPLE 3

Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution.

Given $\qquad \vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

$\vec{F}$ is irrotational if curl $\vec{F} = \vec{0}$

$$\text{Now curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \vec{i}[-1 + 1] - \vec{j}[3z^2 - 3z^2] + \vec{k}[6x - 6x] = \vec{0}.$$

∴ $\quad \vec{F}$ is irrotational vector.

EXAMPLE 4

Prove that (i) div $\vec{r} = 3$, (ii) curl $\vec{r} = \vec{0}$ where $\vec{r}$ is the position vector of a point (x, y, z) in space.

Solution.

Given $\vec{r}$ is the position vector of a point (x, y, z) in space.

∴ $\qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

(i) $\text{div } \vec{r} = \nabla\cdot\vec{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

$$\text{(ii) Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] - \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \vec{i}[0 - 0] - \vec{j}[0 - 0] + \vec{k}[0 - 0] = \vec{0}$$

∴ $\quad \vec{r}$ is an irrotational vector.

EXAMPLE 5

Find the value of a if the vector
$\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$ is solenoidal.

Solution.

Given $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$
is solenoidal.

$$\therefore \quad \nabla \cdot \vec{F} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}(2x^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 0$$

$$\Rightarrow \quad 4xy + 2xy + axy = 0$$

$$\Rightarrow \quad 6xy + axy = 0$$

$$\Rightarrow \quad xy(6+a) = 0 \quad \Rightarrow \quad (6+a) = 0 \quad \Rightarrow \quad a = -6 \qquad [\because x \neq 0,\ y \neq 0]$$

EXAMPLE 6

Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is irrotational and solenoidal.

Solution.

Given $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$.
We have to prove $\vec{F}$ is irrotational and solenoidal.
i.e., to prove $\nabla \times \vec{F} = \vec{0}$ and $\nabla \cdot \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \vec{i}(3x - 3x) - \vec{j}[3y - 2z - (-2z + 3y)] + \vec{k}[3z + 2y - (2y + 3z)] = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x + (-2x + 2) = 0$$

$\therefore$ $\vec{F}$ is solenoidal.

EXAMPLE 7

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$, prove that $r^n\vec{r}$ is solenoidal if $n = -3$ and irrotational for all values of n.

Solution.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore \quad r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad r^2 = x^2 + y^2 + z^2 \qquad (1)$

$$r^n\vec{r} = r^n(x\vec{i} + y\vec{j} + z\vec{k}) = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\therefore \quad \text{div}\,(r^n\vec{r}) = \nabla \cdot (r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \qquad (2)$$

But $\quad \dfrac{\partial}{\partial x}(r^n x) = r^n + x \cdot nr^{n-1}\dfrac{\partial r}{\partial x}, \qquad \dfrac{\partial}{\partial y}(r^n y) = r^n + y \cdot nr^{n-1}\dfrac{\partial r}{\partial y}$

and $\quad \dfrac{\partial}{\partial z}(r^n z) = r^n + z \cdot nr^{n-1}\dfrac{\partial r}{\partial z}$

We have, $\quad r^2 = x^2 + y^2 + z^2, \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \frac{\partial}{\partial x}(r^n x) = r^n + nxr^{n-1}\cdot\frac{x}{r} = r^n + nx^2 r^{n-2}$$

$$\frac{\partial}{\partial y}(r^n y) = r^n + nyr^{n-1}\cdot\frac{y}{r} = r^n + ny^2 r^{n-2}$$

and $\quad \dfrac{\partial}{\partial z}(r^n z) = r^n + nzr^{n-1}\cdot\dfrac{z}{r} = r^n + nz^2 r^{n-2}$

Substitute in (2).

$$\therefore \ \text{div}\,(r^n\vec{r}) = r^n + nx^2r^{n-2} + r^n + ny^2r^{n-2} + r^n + nz^2r^{n-2}$$

$$= 3r^n + nr^{n-2}(x^2+y^2+z^2) = 3r^n + nr^{n-2}\cdot r^2 = 3r^n + nr^n = (n+3)r^n$$

If $n = -3$, then div $(r^n\vec{r}) = 0 \quad \therefore \ r^n\vec{r}$ is solenoidal if $n = -3$

$$\text{Now } \nabla \times r^n\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$= \vec{i}\left(nzr^{n-1}\frac{\partial r}{\partial y} - nyr^{n-1}\frac{\partial r}{\partial z}\right) - \vec{j}\left(nzr^{n-1}\frac{\partial r}{\partial x} - nxr^{n-1}\frac{\partial r}{\partial z}\right) + \vec{k}\left(nyr^{n-1}\frac{\partial r}{\partial x} - nxr^{n-1}\frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left(nzr^{n-1}\frac{y}{r} - nyr^{n-1}\frac{z}{r}\right) - \vec{j}\left(nzr^{n-1}\cdot\frac{x}{r} - nxr^{n-1}\frac{z}{r}\right) + \vec{k}\left(nyr^{n-1}\cdot\frac{x}{r} - nxr^{n-1}\cdot\frac{y}{r}\right)$$

$$= \vec{i}(nr^{n-2}yz - nr^{n-2}yz) - \vec{j}(nr^{n-2}xz - nr^{n-2}xz) + \vec{k}(nr^{n-2}xy - nr^{n-2}xy) = \vec{0}$$

$\therefore \ \nabla \times (r^n\vec{r}) = \vec{0}$ for all values of n.

Hence, $r^n\vec{r}$ is irrotational for all values of n.

EXAMPLE 8

Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

Solution.

Given $\quad \vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$

Now
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix}$$
$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y \cos x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

Hence, there exist a scalar function ϕ such that $\vec{F} = \nabla\phi$

$$\Rightarrow \quad (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \quad (1) \qquad \frac{\partial\phi}{\partial y} = 2y\sin x - 4 \quad (2) \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 3xz^2 \quad (3)$$

Integrating (1) w.r.to x, $\quad \phi = y^2 \sin x + z^3 x + f_1(y, z) \quad (4)$

Integrating (2) w.r.to y, $\quad \phi = y^2 \sin x - 4y + f_2(x, z) \quad (5)$

Integrating (3) w.r.to z, $\quad \phi = xz^3 + f_3(x, y) \quad (6)$

From (4), (5), (6), $\phi = y^2 \sin x + xz^3 - 4y + c$ is the scalar potential, where c is an arbitrary constant.

EXAMPLE 9

(i) Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.
(ii) Find a, b, c if $(x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$ is irrotational.

Solution.

(i) Let $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$

Given $\vec{F}$ is solenoidal.

$$\therefore \quad \nabla \cdot \vec{F} = 0$$
$$\Rightarrow \quad \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$
$$\Rightarrow \quad 3 + a + 2 = 0 \Rightarrow a = -5$$

(ii) Let $\vec{F} = (x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$

Given $\vec{F}$ is irrotational.

$$\therefore \quad \nabla \times \vec{F} = \vec{0} \quad \Rightarrow \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\left[\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right] = 0$$

$$\Rightarrow \quad \vec{i}(c+1) - \vec{j}(-1-a) + \vec{k}(b-1) = \vec{0}$$
$$\Rightarrow \quad (c+1)\vec{i} + (1+a)\vec{j} + (b-1)\vec{k} = \vec{0}$$
$$\therefore \quad c+1=0,\ 1+a=0,\ b-1=0$$
$$\therefore \quad a=-1, \quad b=1 \text{ and } c=-1$$

EXAMPLE 10

Determine $f(r)$ so that the vector $f(r)\,\vec{r}$ is both solenoidal and irrotational.

Solution.
If $\vec{r}$ is not specified, it will always represent the position vector of any point (x, y, z).

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{and} \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \quad \therefore \quad r^2 = x^2+y^2+z^2 \qquad (1)$$

$$\therefore \quad f(r)\vec{r} = f(r)(x\vec{i} + y\vec{j} + z\vec{k}) = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}$$

Given $f(r)\,\vec{r}$ is solenoidal.

$$\therefore \quad \nabla\cdot(f(r)\vec{r}) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) = 0 \qquad (2)$$

But
$$\frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\frac{\partial r}{\partial x}$$
$$\frac{\partial}{\partial y}(f(r)y) = f(r) + yf'(r)\frac{\partial r}{\partial y}$$
and
$$\frac{\partial}{\partial z}(f(r)z) = f(r) + zf'(r)\frac{\partial r}{\partial z}$$

Differentiating (1) we get, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\cdot\frac{x}{r} = f(r) + \frac{x^2}{r}f'(r)$$

Similarly,
$$\frac{\partial}{\partial y}(f(r)y) = f(r) + \frac{y^2}{r}f'(r)$$
and
$$\frac{\partial}{\partial z}(f(r)z) = f(r) + \frac{z^2}{r}f'(r)$$

$$\therefore\ (2) \quad \Rightarrow \quad f(r) + \frac{x^2}{r}f'(r) + f(r) + \frac{y^2}{r}f'(r) + f(r) + \frac{z^2}{r}f'(r) = 0$$

$$\Rightarrow \quad 3f(r) + \frac{f'(r)}{r}(x^2+y^2+z^2) = 0$$

$$\Rightarrow \quad 3f(r) + \frac{f'(r)}{r}\cdot r^2 = 0$$

$$\Rightarrow \quad 3f(r) + rf'(r) = 0 \quad \Rightarrow \quad \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

[here r is real variable.]

Integrating w.r.to 'r', we get $\int \frac{f'(r)}{f(r)}\,dr = -3\int \frac{1}{r}\,dr$

$\Rightarrow$ $\log_e f(r) = -3\log_e r + \log c$

$\Rightarrow$ $\log_e f(r) = -\log_e r^3 + \log_e c = \log_e \frac{c}{r^3} \quad \Rightarrow \quad f(r) = \frac{c}{r^3}$

where c is the constant of integration.

Now $\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}(f(r)y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(f(r)z) - \frac{\partial}{\partial z}(f(r)x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x)\right]$$

$$= \sum \vec{i}\left[zf'(r)\cdot\frac{\partial r}{\partial y} - y\cdot f'(r)\cdot\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\cdot\frac{y}{r} - y\cdot f'(r)\cdot\frac{z}{r}\right] = \sum \vec{i} f'(r)\left[\frac{yz}{r} - \frac{yz}{r}\right] = \vec{0}$$

$\therefore$ $f(r)\vec{r}$ is irrotational for all $f(r)$ and it is solenoidal for $f(r) = \frac{c}{r^3}$, where c is arbitrary constant.

Hence, the required function is $f(r) = \frac{c}{r^3}$, for which $f(r)\vec{r}$ is both solenoidal and irrotational.

EXERCISE 9.2

1. If $\vec{F} = xy^2 + 2x^2yz\vec{j} - 3yz^2\vec{k}$, then find div $\vec{F}$ and curl $\vec{F}$ at (1, 1, –1).
2. If $F = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ then find curl curl $\vec{F}$.
3. Find div $\vec{F}$ and curl $\vec{F}$ at (1, 1, 1)
   if $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$.
4. Show that the following vectors are solenoidal.
   (i) $\vec{F} = (2+3y)\vec{i} + (x-2z)\vec{j} + x\vec{k}$
   (ii) $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$
   (iii) $\vec{F} = 3x^2y\vec{i} - 4xy^2\vec{j} + 2xyz\vec{k}$
5. Find the value of a if $\vec{F} = ay^4z^2\vec{i} + 4x^3z^2\vec{j} + 5x^2y^2\vec{k}$ is solenoidal.
6. If the vector $3x\vec{i} + (x+y)\vec{j} - az\vec{k}$ is solenoidal, then find a.
7. Show that the following vectors are irrotational.
   (i) $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$
   (ii) $\vec{F} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x - y)\vec{k}$
   (iii) $\vec{F} = (4xy - z^2)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$

8. Find the value of a if $\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ is irrotational.

9. If $\vec{F} = (ax^2 + 2y^2 + 1)\vec{i} + (4xy + by^2 z - 3)\vec{j} + (c - y^3)\vec{k}$ is irrotational, then find the values of a, b, c.

10. Show that $F = (2x + 3y + z^2)\vec{i} + (3x + 2y + z)\vec{j} + (y + 2zx)\vec{k}$ is irrotational and hence, find its scalar potential.

11. Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

12. Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational, find its scalar potential.

13. Find the div $\vec{\mathrm{F}}$ and curl $\vec{\mathrm{F}}$, where $\vec{\mathrm{F}} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

14. If $\vec{v} = \vec{w} \times \vec{r}$, prove that $\vec{w} = \frac{1}{2}\text{curl}\ \vec{v}$, where $\vec{w}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z).

15. If $\vec{r}$ is the position vector of a point (x, y, z) in space and $\vec{A}$ is a constant vector, prove that $\vec{A} \times \vec{r}$ is solenoidal.

16. Prove that the vector $\vec{\mathrm{F}} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2z)\vec{k}$ is solenoidal.

17. Show that $\vec{v} = xyz^2\vec{u}$ is solenoidal, where

$$\vec{u} = (2x^2 + 8xy^2 z)\vec{i} + (3x^3 y - 3xy)\vec{j} - (4y^2 z^2 + 2x^3 z)\vec{k}.$$

ANSWERS TO EXERCISE 9.2

1. $5; -5\vec{i} - 6\vec{k}$
2. $2[z\vec{i} + x\vec{j} + y\vec{k}]$
3. $6; -2\vec{i} + 2\vec{k}$
5. a can be any real number
6. 4
8. 2
9. $a = 3, b = -3, c = 2$
10. $\phi = x^2 + y^2 + 3xy + yz + z^2x + c$
11. $\phi = y^2 \sin x + xz^3 - 4y + c$
12. $\phi = 3x^2y + xz^3 - yz + c$
13. div $\vec{\mathrm{F}} = b(x + y + z)$ Curl $\vec{\mathrm{F}} = \vec{\mathrm{O}}$

9.7 VECTOR IDENTITIES

We shall list the vector identities into two categories.

(i) ∇ operator applied once to point functions.
(ii) ∇ operator applied twice to point functions.

TYPE 1.

If f and g are scalar point functions we have already proved the following results.

1. $\nabla c = 0$, where c is a constant.
2. $\nabla(c\phi) = c\nabla\phi$, where c is constant.
3. $\nabla(f \pm g) = \nabla f \pm \nabla g$
4. $\nabla(fg) = f\nabla g + g\nabla f$
5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{f^2}$

6. If $\vec{F}$ and $\vec{G}$ are vector point functions, then $\nabla\cdot(\vec{F}+\vec{G}) = \nabla\cdot\vec{F}+\nabla\cdot\vec{G}$.

Proof

$$\begin{aligned}\nabla\cdot(\vec{F}+\vec{G}) &= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{F}+\vec{G})\\ &= \vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}+\frac{\partial\vec{G}}{\partial x}\right)+\vec{j}\cdot\left(\frac{\partial\vec{F}}{\partial y}+\frac{\partial\vec{G}}{\partial y}\right)+\vec{k}\cdot\left(\frac{\partial\vec{F}}{\partial z}+\frac{\partial\vec{G}}{\partial z}\right)\\ &= \left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}+\vec{j}\cdot\frac{\partial\vec{F}}{\partial y}+\vec{k}\cdot\frac{\partial\vec{F}}{\partial z}\right)+\left(\vec{i}\cdot\frac{\partial\vec{G}}{\partial x}+\vec{j}\cdot\frac{\partial\vec{G}}{\partial y}+\vec{k}\cdot\frac{\partial\vec{G}}{\partial z}\right)\\ &= \nabla\cdot\vec{F}+\nabla\cdot\vec{G}\end{aligned}$$

Similarly, $\nabla\cdot(\vec{F}-\vec{G}) = \nabla\cdot\vec{F}-\nabla\cdot\vec{G}$ ■

7. If f is a scalar point function and $\vec{G}$ is a vector point function, then $\nabla\cdot(f\vec{G}) = \nabla f\cdot\vec{G}+f(\nabla\cdot\vec{G})$

Proof Let $\vec{G} = G_1\vec{i}+G_2\vec{j}+G_3\vec{k}$, then $f\vec{G} = fG_1\vec{i}+fG_2\vec{j}+fG_3\vec{k}$

$$\therefore \quad \nabla\cdot(f\vec{G}) = \frac{\partial}{\partial x}(fG_1)+\frac{\partial}{\partial y}(fG_2)+\frac{\partial}{\partial z}(fG_3)$$

$$= f\frac{\partial G_1}{\partial x}+\frac{\partial f}{\partial x}G_1+f\frac{\partial G_2}{\partial y}+\frac{\partial f}{\partial y}G_2+f\frac{\partial G_3}{\partial z}+\frac{\partial f}{\partial z}G_3$$

$$= \frac{\partial f}{\partial x}G_1+\frac{\partial f}{\partial y}G_2+\frac{\partial f}{\partial z}G_3+f\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)$$

$$\therefore \quad \nabla\cdot(f\vec{G}) = \nabla f\cdot\vec{G}+f(\nabla\cdot\vec{G})$$ ■

8. If f is a scalar point function and $\vec{G}$ is a vector point function, then $\nabla\times(f\vec{G}) = \nabla f\times\vec{G}+f(\nabla\times G)$

Proof Let $\vec{G} = G_1\vec{i}+G_2\vec{j}+G_3\vec{k}$ $\therefore$ $f\vec{G} = fG_1\vec{i}+fG_2\vec{j}+fG_3\vec{k}$

$$\text{Now } \nabla\times(f\vec{G}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ fG_1 & fG_2 & fG_3\end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(fG_3)-\frac{\partial}{\partial z}(fG_2)\right]-\vec{j}\left[\frac{\partial}{\partial x}(fG_3)-\frac{\partial}{\partial z}(fG_1)\right]+\vec{k}\left[\frac{\partial}{\partial x}(fG_2)-\frac{\partial}{\partial y}(fG_1)\right]$$

$$= \vec{i}\left[f\frac{\partial G_3}{\partial y}+G_3\frac{\partial f}{\partial y}-f\frac{\partial G_2}{\partial z}-G_2\frac{\partial f}{\partial z}\right]-\vec{j}\left[f\frac{\partial G_3}{\partial x}+G_3\frac{\partial f}{\partial x}-f\frac{\partial G_1}{\partial z}-G_1\frac{\partial f}{\partial z}\right]$$

$$+\vec{k}\left[f\frac{\partial G_2}{\partial x}+G_2\frac{\partial f}{\partial x}-f\frac{\partial G_1}{\partial y}-G_1\frac{\partial f}{\partial y}\right]$$

$$= f\left[\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z}\right)\vec{i}-\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)\vec{j}+\left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)\vec{k}\right]$$

$$+\left(\frac{\partial f}{\partial y}G_3-\frac{\partial f}{\partial z}G_2\right)\vec{i}-\left(\frac{\partial f}{\partial x}G_3-\frac{\partial f}{\partial z}G_1\right)\vec{j}+\left(\frac{\partial f}{\partial x}G_2-\frac{\partial f}{\partial y}G_1\right)\vec{k}$$

$$= f\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ G_1 & G_2 & G_3\end{vmatrix}+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ G_1 & G_2 & G_3\end{vmatrix}$$

$\therefore \quad \nabla\times(f\vec{G}) = f(\nabla\times\vec{G})+(\nabla f)\times\vec{G}.$ ■

9. If $\vec{F}$ and $\vec{G}$ are vector point functions, then

$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$$

Proof We know that $\nabla f = \vec{i}\frac{\partial f}{\partial x}+\vec{j}\frac{\partial f}{\partial y}+\vec{k}\frac{\partial f}{\partial z} = \sum \vec{i}\frac{\partial f}{\partial x}$

$\therefore$
$$\nabla(\vec{F}\cdot\vec{G}) = \sum \vec{i}\frac{\partial}{\partial x}(\vec{F}\cdot\vec{G})$$
$$= \sum \vec{i}\left[\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}+\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right]$$
$$= \sum\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i}+\sum\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{i} \qquad (1)$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\therefore \quad (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b}-\vec{a}\times(\vec{b}\times\vec{c})$

$\therefore$
$$\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i} = (\vec{F}\cdot\vec{i})\left(\frac{\partial \vec{G}}{\partial x}\right)-\vec{F}\times\left(\frac{\partial \vec{G}}{\partial x}\times\vec{i}\right)$$
$$= (\vec{F}\cdot\vec{i})\left(\frac{\partial \vec{G}}{\partial x}\right)+\vec{F}\times\left(\vec{i}\times\frac{\partial \vec{G}}{\partial x}\right)$$

$\therefore$
$$\sum\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i} = \left(\vec{F}\cdot\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{G}+\vec{F}\times\sum\left(\vec{i}\times\frac{\partial \vec{G}}{\partial x}\right)$$
$$= (\vec{F}\cdot\nabla)\vec{G}+\vec{F}\times(\nabla\times\vec{G}) \qquad (2)$$

Interchanging $\vec{F}$ and $\vec{G}$, we get

$$\sum\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{i} = (\vec{G}\cdot\nabla)\vec{F}+\vec{G}\times(\nabla\times\vec{F}) \qquad (3)$$

Substituting (2) and (3) in (1) we get

$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G}+\vec{F}\times(\nabla\times\vec{G})+(G\cdot\nabla)\vec{F}+\vec{G}\times(\nabla\times\vec{F})$$

$\therefore$
$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$$ ■

10. If $\vec{F}$ and $\vec{G}$ are vector point functions then

$$\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$$

i.e., div $\vec{F}\times\vec{G} = \vec{G}\cdot$Curl $\vec{F} - \vec{F}\cdot$ Curl $\vec{G}$.

Proof

$$\nabla\cdot(\vec{F}\times\vec{G}) = \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum \vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$= \sum \vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum \vec{i}\cdot\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

In a scalar triple product $\cdot$ and $\times$ can be interchanged.

$\therefore$ we get $\nabla\cdot(\vec{F}\times\vec{G}) = \sum\left(\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right)\cdot\vec{G} - \sum\left(\vec{i}\times\frac{\partial\vec{G}}{\partial x}\right)\cdot\vec{F}$

$\Rightarrow$ $\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - (\nabla\times\vec{G})\cdot\vec{F}$ ■

11. If $\vec{F}$ and $\vec{G}$ are vector product functions, then

$$\nabla\times(\vec{F}\times\vec{G}) = \vec{F}(\nabla\cdot\vec{G}) - \vec{G}(\nabla\cdot\vec{F}) + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$$

Proof

$$\nabla\times(\vec{F}\times\vec{G}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$\Rightarrow$ $$\nabla\times(\vec{F}\times\vec{G}) = \sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum \vec{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) \quad (1)$$

We know $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\therefore$ $$\sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) = \sum\left[(\vec{i}\cdot\vec{G})\frac{\partial\vec{F}}{\partial x} - \left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}\right]$$

$$= \sum(\vec{G}\cdot\vec{i})\frac{\partial\vec{F}}{\partial x} - \sum\left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}$$ ■

$$= \vec{G}\cdot\left(\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{F} - \left(\sum\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}$$

$\Rightarrow$ $$\sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) = (\vec{G}\cdot\nabla)\vec{F} - (\nabla\cdot\vec{F})\vec{G} \quad (2)$$

Similarly, $\sum \vec{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) = \sum\left[\left(\vec{i}\cdot\frac{\partial\vec{G}}{\partial x}\right)\vec{F} - \sum(\vec{i}\cdot\vec{F})\frac{\partial\vec{G}}{\partial x}\right]$

$$= \sum\left(\vec{i}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{F} - \sum(\vec{i}\cdot\vec{F})\frac{\partial \vec{G}}{\partial x}$$

$$= \left(\sum\vec{i}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{F} - \sum\left(\vec{F}\cdot\vec{i}\frac{\partial}{\partial x}\right)\vec{G}$$

$$= (\nabla\cdot\vec{G})\vec{F} - \vec{F}\cdot\left(\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{G}$$

$$\Rightarrow \qquad \sum\vec{i}\times\left(\vec{F}\times\frac{\partial \vec{G}}{\partial x}\right) = (\nabla\cdot\vec{G})\vec{F} - (\vec{F}\cdot\nabla)\vec{G} \qquad (3)$$

Substituting (2) and (3) in (1), we get

$$\nabla\times(\vec{F}\times\vec{G}) = (\vec{G}\cdot\nabla)\vec{F} - (\nabla\cdot\vec{F})\vec{G} + (\nabla\cdot\vec{G})\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$$

$$\therefore \qquad \text{Curl } \vec{F}\times\vec{G} = \vec{F}(\text{div } \vec{G}) - \vec{G}(\text{div } \vec{F}) + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$$ ■

TYPE II – Identities – ∇ Applied Twice

1. **If f is scalar point function, then div grad $f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$**

Proof We know, grad $f = \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

$$\text{div (grad } f) = \nabla\cdot\nabla f$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\therefore \qquad \text{div (grad } f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ ■

Note $\nabla\cdot\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a scalar operator called the Laplacian operator.

2. **If $\vec{F}$ is a vector point function, then div curl $\vec{F} = 0$.**

Proof Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, where F_1, F_2, F_3 are scalar functions of x, y, z.

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\therefore \quad \text{div Curl } \vec{F} = \nabla\cdot\nabla\times\vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_1}{\partial z\partial y}$$

$$\Rightarrow \quad \text{div Curl } \vec{F} = 0 \qquad \left[\text{Since } \frac{\partial^2 F_3}{\partial x\partial y}=\frac{\partial^2 F_3}{\partial y\partial x}, \frac{\partial^2 F_2}{\partial x\partial z}=\frac{\partial^2 F_2}{\partial z\partial x}, \frac{\partial^2 F_1}{\partial y\partial z}=\frac{\partial^2 F_1}{\partial z\partial y}\right] \blacksquare$$

3. If $\vec{F}$ is a vector point function, then

curl (Curl $\vec{F}$) = $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$.

Proof Let $\vec{F} = F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$, where F_1, F_2, F_3 are scalar functions.

Then
$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-\vec{j}\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+\vec{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$\therefore \quad \text{Curl curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial^2 F_2}{\partial y\partial x}-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_3}{\partial z\partial x}\right]$$

$$= \sum \vec{i}\left\{\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}-\left(\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum \vec{i}\left\{\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial x\partial y}+\frac{\partial^2 F_3}{\partial x\partial z}-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum \vec{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum \vec{i}\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{F})-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right\}$$

$$= \sum \vec{i}\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{F}) - \nabla^2 F_1\right\}$$

$$= \left(\sum \vec{i}\frac{\partial}{\partial x}\right)(\nabla\cdot\vec{F}) - \nabla^2\left(\sum \vec{i}F_1\right)$$

$$\therefore \qquad \nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F} \qquad \blacksquare$$

WORKED EXAMPLES

EXAMPLE 1

Prove that $\nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$.

Solution.
We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \qquad \nabla\left(\frac{1}{r^n}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r^n}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r^n}\right)$$

$$= \vec{i}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{n}{r^{n+1}}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right] = -\frac{n}{r^{n+2}}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{n}{r^{n+2}}\vec{r}$$

$$\therefore \qquad \nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$$

Note We have $\nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$

If $n = 1, 2, 3, 4, \ldots$

Then $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}, \quad \nabla\left(\frac{1}{r^2}\right) = -\frac{2}{r^4}\vec{r}, \quad \nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\vec{r}, \quad \nabla\left(\frac{1}{r^4}\right) = -\frac{4}{r^6}\vec{r}$ and so on.

EXAMPLE 2

Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Solution.

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla(r^n) = \sum \vec{i}\frac{\partial}{\partial x}(r^n) = \sum \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\, nr^{n-1}\frac{x}{r} = nr^{n-2}\sum x\vec{i} = nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k}) = nr^{n-2}\vec{r} \quad \text{if } n \geq 3 \qquad (1)$$

$$\nabla^2(r^n) = \nabla\cdot(\nabla r^n) = \nabla\cdot(nr^{n-2}\vec{r})$$

$$= n[\nabla r^{n-2}\cdot\vec{r} + r^{n-2}(\nabla\cdot\vec{r})]$$

$$= n[(n-2)r^{n-4}\vec{r}\cdot\vec{r} + r^{n-2}3] \qquad \text{[using (1)]}$$

$$= n[(n-2)r^{n-4}r^2 + 3r^{n-2}] = nr^{n-2}[n-2+3] = n(n+1)r^{n-2}$$

Note We have $\nabla(r^n) = nr^{n-2}\vec{r}$
If $n = 1, 2, 3, 4, \ldots$

$$\nabla(r) = \frac{1}{r}\vec{r}, \quad \nabla(r^2) = 2\cdot r^{2-2}\vec{r} = 2\vec{r}, \quad \nabla(r^3) = 3r\vec{r}, \quad \nabla(r^4) = 4r^2\vec{r} \ \ldots \nabla(r^{n-2}) = (n-2)r^{n-4}\vec{r}, \text{ etc.}$$

EXAMPLE 3

Prove that $\nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4}$.

Solution.

We have $\nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\vec{r}, \quad \nabla\left(\frac{1}{r^4}\right) = \frac{-4}{r^6}\vec{r}$ and $\nabla\cdot\vec{r} = 3$

$$\therefore \quad \nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \nabla\cdot\left(r\frac{-3}{r^5}\vec{r}\right) = \nabla\cdot\left(\frac{-3}{r^4}\vec{r}\right)$$

$$= -3\left[\nabla\left(\frac{1}{r^4}\right)\cdot\vec{r} + \frac{1}{r^4}\nabla\cdot\vec{r}\right]$$

$$= -3\left[-\frac{4}{r^6}(\vec{r}\cdot\vec{r}) + \frac{3}{r^4}\right] = -3\left[-\frac{4}{r^6}r^2 + \frac{3}{r^4}\right] = -3\left[\frac{-4}{r^4} + \frac{3}{r^4}\right] = \frac{3}{r^4}$$

EXAMPLE 4

If ϕ and ψ satisfy Laplace equation, prove that the vector $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal.

Solution.
Given ϕ and ψ satisfy Laplace equation.

$$\therefore \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (1) \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (2)$$

To prove $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal, we have to prove $\text{div}(\phi\nabla\psi - \psi\nabla\phi) = 0$
Now $\quad \text{div}(\phi\nabla\psi - \psi\nabla\phi) = \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)$

$$= \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi)$$
$$= \nabla \phi \cdot \nabla \psi + \phi(\nabla \cdot \nabla \psi) - [\nabla \psi \cdot \nabla \phi + \psi(\nabla \cdot \nabla \phi)]$$
$$= \phi \nabla^2 \psi - \psi \nabla^2 \phi \qquad [\because\ \nabla\phi \cdot \nabla\psi = \nabla\psi \cdot \nabla\phi]$$
$$= 0 \qquad \text{[from (1) and (2)]}$$

$\therefore$ $(\phi \nabla \psi - \psi \nabla \phi)$ is solenoidal.

EXAMPLE 5

Show that $\nabla^2 f(r) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$.

Solution.

We have, $\nabla f(r) = f'(r)\dfrac{\vec{r}}{r}$

$$\therefore\ \nabla^2 f(r) = \nabla \cdot \nabla f(r) = \nabla \cdot f'(r)\frac{\vec{r}}{r} = \left(\nabla \frac{f'(r)}{r}\right)\cdot \vec{r} + \frac{f'(r)}{r}(\nabla \cdot \vec{r})$$
$$= \left(\nabla \frac{f'(r)}{r}\right)\cdot \vec{r} + \frac{3f'(r)}{r} \qquad [\because\ \nabla \cdot \vec{r} = 3]$$
$$= \left(\frac{r\nabla f'(r) - f'(r)\nabla r}{r^2}\right)\cdot \vec{r} + \frac{3f'(r)}{r} \qquad \left[\because\ \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}\right]$$
$$= \frac{\left(rf''(r)\dfrac{\vec{r}}{r} - f'(r)\dfrac{\vec{r}}{r}\right)\cdot \vec{r}}{r^2} + \frac{3f'(r)}{r} \qquad \left[\because\ \nabla f'(r) = f''(r)\frac{\vec{r}}{r} \text{ and } \nabla r = \frac{\vec{r}}{r}\right]$$
$$= \frac{[rf''(r) - f'(r)]\vec{r}\cdot\vec{r}}{r^3} + \frac{3f'(r)}{r}$$
$$= \frac{[rf''(r) - f'(r)]}{r^3}r^2 + \frac{3f'(r)}{r} \qquad [\because\ \vec{r}\cdot\vec{r} = r^2]$$
$$= \frac{rf''(r) - f'(r)}{r} + \frac{3f'(r)}{r} = f''(r) + \frac{2f'(r)}{r} = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

9.8 INTEGRATION OF VECTOR FUNCTIONS

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable t such that $\dfrac{d}{dt}\vec{F}(t) = \vec{f}(t)$. Then $\vec{F}(t)$ is called an indefinite integral of $\vec{f}(t)$ with respect to t and is written as $\int \vec{f}(t)dt = \vec{F}(t) + \vec{c}$, where $\vec{c}$ is an arbitrary constant vector independent of t and is called the constant of integration.

The definite integral of $\vec{f}(t)$ between the limits $t = t_1$ and $t = t_2$ is given by

$$\int_{t_1}^{t_2} \vec{f}(t)\,dt = \left[\vec{F}(t)\right]_{t_1}^{t_2} = \vec{F}(t_2) - \vec{F}(t_1).$$

As in the case of differentiation of vectors, in order to integrate a vector function, we integrate its components.

If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, then

$$\int \vec{f}(t)\,dt = \vec{i}\int f_1(t)\,dt + \vec{j}\int f_2(t)\,dt + \vec{k}\int f_3(t)\,dt$$

9.8.1 Line Integral

An integral evaluated over a curve C is called a line integral. We call C as the path of integration. We assume every path of integration of a line integral to be piecewise smooth consisting of finitely many smooth curves.

Definition 9.10 A line integral of a vector point function $\vec{F}(\vec{r})$ over a curve C, where $\vec{r}$ is the position vector of any point on C, is defined by $\int_C \vec{F}\cdot d\vec{r}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ and $\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$

Here F_1, F_2, F_3 are functions of $x, y, z,$ where x, y, z depend on a parameter $t \in [a, b]$, since $\vec{r}(t)$ is the equation of the curve C.

Then we can write $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt.$

If the path of integration C is a closed curve, we write $\oint_C$ instead of $\int_C$.

Note

1. Since $\frac{d\vec{r}}{dt}$ is a tangent vector to the curve C the line integral $\int_C \vec{F}\cdot d\vec{r}$ is also called the tangential line integral of $\vec{F}$ over C and **line integral is a scalar**.
2. Two other types of line integrals are also considered. $\int_C \vec{F}\times d\vec{r}$ and $\int_C \phi\, d\vec{r}$ are vectors.

WORKED EXAMPLES

EXAMPLE 1

If $\vec{F} = 3xy\,\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution.

Given $\vec{F} = 3xy\vec{i} - y^2\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$, where $\vec{r}$ is the position vector of any point (x, y) on $y = 2x^2$.

$\therefore \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

and $\quad \vec{F}\cdot d\vec{r} = (3xy\vec{i} - y^2\vec{j})\cdot(dx\vec{i} + dy\vec{j}) = 3xydx - y^2dy$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\,dx - y^2 dy)$$

Equation of C is $y = 2x^2 \quad \therefore \quad dy = 4x\,dx.$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3x\cdot 2x^2 dx - 4x^4\cdot 4x\,dx)$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = -\frac{7}{6}$$

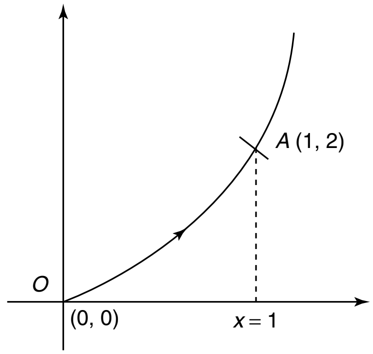


Fig. 9.2

EXAMPLE 2

If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the curve C given by $x = t, y = t^2, z = t^3$.

Solution.

Given $\quad \vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

and $\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

and $\quad \vec{F}\cdot d\vec{r} = \left[(3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right]\cdot\left[dx\vec{i} + dy\vec{j} + dz\vec{k}\right]$

$$= (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

Given $\quad x = t, \quad y = t^2, \quad z = t^3$ is the curve.

$\therefore \quad dx = dt, \quad dy = 2t\,dt, \quad dz = 3t^2\,dt$

When $\quad x = 0, t = 0$ and $x = 1, t = 1. \qquad$ Limits for t are $t = 0, t = 1$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3\cdot t^2 + 6\cdot t^2)dt - 14\cdot t^5\cdot 2t\,dt + 20t^7\cdot 3t^2\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1 = 3 - 4 + 6 = 5.$$

EXAMPLE 3

Evaluate the line integral $\int_C (y^2dx - x^2dy)$ around the triangle whose vertices are (1, 0), (0, 1), (–1, 0) in the positive sense.

Solution.

Given the path C consists of the sides of the $\Delta\,ABC$, where $A(-1, 0)$, $B(1, 0)$ and $C(0, 1)$.

Equation of AB is $y = 0$

Equation of BC is $\frac{y-0}{0-1} = \frac{x-1}{1-0} \quad \Rightarrow \quad y = -x + 1$

Equation of CA is $\frac{y-1}{1-0} = \frac{x-0}{0+1} \Rightarrow y = x+1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy) + \int_{CA} (y^2dx - x^2dy)$$

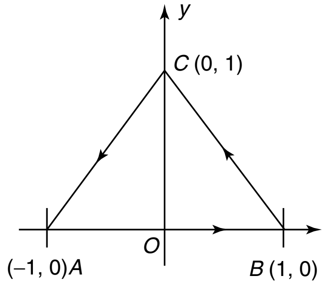


Fig. 9.3

On AB, $y = 0$, $\therefore$ $dy = 0$ and x varies from -1 to 1

$$\therefore \int_{AB} (y^2dx - x^2dy) = \int_{-1}^{1} 0\ dx = 0$$

On BC, $y = -x+1$ $\therefore$ $dy = -dx$ and From B to C, x varies from 1 to 0.

$$\therefore \int_{BC} (y^2dx - x^2dy) = \int_1^0 (-x+1)^2 dx - x^2(-dx) = \int_1^0 (x^2 - 2x + 1 + x^2)dx$$

$$= \int_1^0 (2x^2 - 2x + 1)dx$$

$$= \left[2\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_1^0 = 0 - \left(\frac{2}{3} - 1 + 1\right) = -\frac{2}{3}$$

On CA, $y = x+1$ $\therefore$ $dy = dx$ and From C to A, x varies from 0 to -1

$$\therefore \int_{CA} (y^2dx - x^2dy) = \int_0^{-1} (x+1)^2 dx - x^2dx = \int_0^{-1} (x^2 + 2x + 1 - x^2)dx$$

$$= \int_0^{-1} (2x+1)dx = \left[x^2 + x\right]_0^{-1} = 1 - 1 - 0 = 0$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = 0 + \left(-\frac{2}{3}\right) + 0 = -\frac{2}{3}$$

EXAMPLE 4

If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the straight line joining (0, 0, 0) to (1, 1, 1).

Solution.

Given $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\therefore$ $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$$\therefore \quad \vec{F}\cdot d\vec{r} = \left[(3x^2+6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}\right]\cdot\left[dx\vec{i} + dy\vec{j} + dz\vec{k}\right]$$

$$= (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$$

Equation of the line joining (0, 0, 0) to (1, 1, 1) is

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} \quad \Rightarrow \quad x = y = z = t, \text{ say}$$

$$\therefore \quad dx = dt, \quad dy = dt, \quad dz = dt$$

At the point (0, 0, 0), $t = 0$ and at the point (1, 1, 1), $t = 1$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2+6t)dt - 14t^2 dt + 20t^3 dt$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)dt$$

$$= \int_0^1 (20t^3 - 11t^2 + 6t)dt = \left[20\frac{t^4}{4} - 11\frac{t^3}{3} + 6\frac{t^2}{2}\right]_0^1 = 5 - \frac{11}{3} + 3 = \frac{13}{3}$$

Definition 9.11 Work Done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which is moved along arc AB then $\int_A^B \vec{F}\cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in displacing the particle from A to B.

Conservative force field

A line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in domain D if and only if $\vec{F} = \nabla\phi$ for some scalar function ϕ defined in D. Such a force field is called a conservative field.

In the conservative field the total work done by $\vec{F}$ from A to B is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r}$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_C d\phi = \int_A^B d\phi$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = [\phi]_A^B = \phi(B) - \phi(A)$$

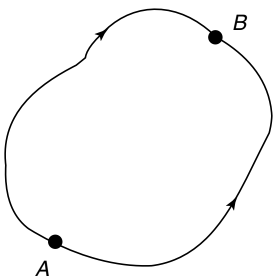


Fig. 9.4

So, in a conservative field the work done depends on the value of ϕ at the end points A and B of the path, but not on the path.

Note

1. ϕ is scalar potential.
2. If $\vec{F}$ is conservative, then $\quad \vec{F} = \nabla\phi \quad \Rightarrow \quad \nabla\times\vec{F} = \nabla\times\nabla\phi = \vec{0}$
   $\therefore \quad \vec{F}$ is irrotational.
3. If C is a simple closed curve and $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r} = 0$.

WORKED EXAMPLES

EXAMPLE 5

Show that $\vec{F} = (e^x z - 2xy)\vec{i} - (x^2 - 1)\vec{j} + (e^x + z)\vec{k}$ is a conservative field. Hence, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where the end points of C are (0, 1, –1) and (2, 3, 0).

Solution.

To prove that $\vec{F}$ is conservative, we have to prove $\nabla\times\vec{F} = \vec{0}$

Now
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x z - 2xy & 1-x^2 & e^x + z \end{vmatrix}$$
$$= \vec{i}[0] - \vec{j}(e^x - e^x) + \vec{k}(-2x + 2x) = \vec{0}$$

Hence, $\vec{F}$ is conservative. $\quad \therefore \quad \vec{F} = \nabla\phi$

$$\Rightarrow \quad (e^x z - 2xy)\vec{i} + (1-x^2)\vec{j} + (e^x + z)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = e^x z - 2xy \quad (1) \qquad \frac{\partial\phi}{\partial y} = 1 - x^2 \quad (2) \qquad \frac{\partial\phi}{\partial z} = e^x + z \quad (3)$$

Integrating (1) w. r. to x, $\quad \phi = ze^x - x^2y + f_1(y, z)$

Integrating (2) w. r. to y, $\quad \phi = (1 - x^2)y + f_2(x, z)$

Integrating (3) w. r. to z, $\quad \phi = e^x z + \dfrac{z^2}{2} + f_3(x, y)$

$$\therefore \quad \phi = ze^x - x^2y + y + \frac{z^2}{2} + C$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = [\phi]_{(0,1,-1)}^{(2,3,0)}$$

$$= [ze^x - x^2y + y + \frac{z^2}{2} + c]_{(0,1,-1)}^{(2,3,0)}$$

$$= \left[0 - 2^2\cdot 3 + 3 + C - \left(-1 - 0 + 1 + \frac{1}{2} + C\right)\right] = -12 + 3 - \frac{1}{2} = -\frac{19}{2}.$$

EXAMPLE 6

If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$, then check whether the integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C.

Solution.

Given $$\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$$

Now $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left\{\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right\} - \vec{j}\left\{\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2)\right\}$$

$$+ \vec{k}\left\{\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2)\right\}$$

$$= \vec{i}\{0 - 0\} - \vec{j}\{-6x^2z + 6x^2z\} + \vec{k}\{4x - 4x\} = \vec{0}$$

$\therefore$ $\vec{F}$ is conservative.

Hence, $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path C.

EXAMPLE 7

Show that $\vec{F} = (2xy + z^3)i + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative field. Find the scalar potential and work done in moving an object in this field from (1, –2, 1) to (3, 1, 4).

Solution.

Given $$\vec{F} = (2xy + z^3)i + x^2\vec{j} + 3xz^2\vec{k}$$

Now $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= \vec{i}[0 - 0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2x - 2x] = \vec{0}$$

$\therefore$ $\vec{F}$ is conservative.

So, there exists a scalar function ϕ such that $\vec{F} = \nabla\phi$.

$\Rightarrow \qquad (2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}=\vec{i}\dfrac{\partial\phi}{\partial x}+\vec{j}\dfrac{\partial\phi}{\partial y}+\vec{k}\dfrac{\partial\phi}{\partial z}$

$\therefore \qquad \dfrac{\partial\phi}{\partial x}=2xy+z^3 \quad (1) \qquad \dfrac{\partial\phi}{\partial y}=x^2 \quad (2) \qquad \dfrac{\partial\phi}{\partial z}=3xz^2 \quad (3)$

Integrating (1) partially w.r.to x, $\phi = x^2y + z^3x + f_1(y, z)$
Integrating (2) partially w.r.to y, $\phi = x^2y + f_2(x, z)$
Integrating (3) partially w.r.to z, $\phi = xz^3 + f_3(x, y)$
$\therefore$ $\phi = x^2y + xz^3 + C$
Since $\vec{F}$ is conservative, work done by the force $\vec{F}$ from (1, –2, 1) to (3, 1, 4) is equal to

$$[\phi]_{(1,-2,1)}^{(3,1,4)}=\left[x^2y+xz^3+C\right]_{(1,-2,1)}^{(3,1,4)}$$
$$=3^2\cdot 1+3\cdot 4^3+C-\left[(1^2(-2)+1\cdot 1^3)+C\right]=9+192+1=202 \text{ units.}$$

EXERCISE 9.3

1. Prove that if $\vec{F}=\phi\nabla\psi$, then $\vec{F}\cdot(\nabla\times\vec{F})=0$.
2. Prove that Curl (ϕ grad ϕ) = $\vec{0}$.
3. Show that $\nabla\cdot(\phi\nabla\psi-\psi\nabla\phi)=\phi\nabla^2\psi-\psi\nabla^2\phi$.
4. Prove that $\nabla\times(\phi\nabla\psi)=\nabla\phi\times\nabla\psi$.
5. Prove that $\nabla\times[f(r)\vec{r}]=\vec{0}$.
6. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$, along the straight line joining the points (1, –2, 1) and (3, 2, 4).
7. Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(2y+3)\vec{i}+xz\vec{j}+(yz-x)\vec{k}$ along the line joining the points (0, 0, 0) to (2, 1, 1).
8. Find the work done in moving a particle in the force field $\vec{F}=3x^2\vec{i}+(2xz-y)\vec{j}-z\vec{k}$ from $t=0$ to $t=1$ along the curve $x=2t^2$, $y=t$, $z=4t^3$.
9. Show that $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$ is conservative. Find its scalar potential and find the work done in moving a particle from (1, –2, 1) to (3, 1, 2).
10. Find the work done by the force $\vec{F}=-xy\vec{i}+y^2\vec{j}+z\vec{k}$ in moving a particle over a circular path $x^2+y^2=4$, $z=0$ from (2, 0, 0) to (0, 2, 0).
11. Find the work done when a force $\vec{F}=(x^2-y^2+x)\vec{i}-(2xy+y)\vec{j}$ moves a particle in the xy plane from (0, 0) to (1, 1) along the curve $y^2=x$. If the path is $y=x$, whether the work done is different or same. If it is same, state the reason.
12. Find the total work done in moving a particle in a force field given by $\vec{F}=3xy\vec{i}-5z\vec{j}+10x\vec{k}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

13. For the vector function $\vec{F} = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$ determine $\int_C \vec{F} \cdot d\vec{r}$ around the unit circle with centre at the origin in the xy plane.
14. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x - 3y)\vec{i} + (x - 2y)\vec{j}$ and C is the closed curve in the xy plane. $x = 2\cos t$, $y = 2\sin t$ and $t = 0$ to $t = 2\pi$.
15. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.
16. Prove that $\nabla \times (\nabla r^n) = \vec{0}$.
17. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$.
18. Show that the vector field $\vec{F}$, where
$\vec{F} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2xz)\vec{k}$, is conservative and find its scalar potential.

ANSWERS TO EXERCISE 9.3

6. 211 [*Hint*: $\vec{F}$ is conservative] 7. 5 8. $\frac{13}{6}$ 9. 34
10. $\frac{16}{3}$ 11. $\frac{-2}{3}, \frac{-2}{3}$, $\vec{F}$ is conservative. 12. 303
13. 0 14. 24π 17. $\frac{135}{4}$
18. $\phi = xy + xy^2 + yz + xz^2 + c$.

9.9 GREEN'S THEOREM IN A PLANE

Green's theorem gives a relation between a double integral over a region R in the xy plane and the line integral over a closed curve C enclosing the region R. It helps to evaluate line integral easily.

Statement of Green's theorem

If $P(x, y)$ and $Q(x, y)$ are continuous functions with continuous partial derivatives in a region R in the xy plane and on its boundary C which is a simple closed curve then

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C is described in the anticlockwise sense (which is the positive sense).

Green's theorem in a plane

Proof Let R be the region in the xy-plane bounded by the simple closed curve C traced in the anticlockwise sense, which is the positive sense. We assume any line parallel to the axes meet the curve in not more than two points. The curve C consists of two arcs APB and BQA as in figure.

Let $y = f_1(x)$ and $y = f_2(x)$ be the equations of these arcs.

Clearly, $f_1(x) \le f_2(x)$ in $[a, b]$

Now, $$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$

$$= \int_a^b \left[P(x, y)\right]_{f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b \left[P((x, f_2(x)) - P((x, f_1(x))\right] dx$$

$$= \int_a^b P((x, f_2(x))\,dx - \int_a^b P((x, f_1(x))\,dx$$

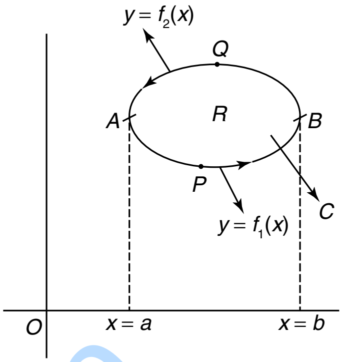


Fig. 9.5

However, $\int_a^b P(x, f_2(x))\,dx$ is numerically equal to the line integral $\int_{AQB} P(x, y)\,dx$ taken along the curve AQB.

But the positive sense is BQA (anticlockwise)

$\therefore$ $$\int_a^b P(x, f_2(x))\,dx = -\int_{BQA} P(x, y)\,dx$$

Similarly, $$\int_a^b P(x, f_1(x))\,dx = \int_{APQ} P(x, y)\,dx$$

$\therefore$ $$\iint_R \frac{\partial P}{\partial y}\,dy = -\int_{APB} P(x, y)dx - \int_{BQA} P(x, y)dx$$

$$= -\left\{\int_{APB} P(x, y)dx + \int_{BQA} P(x, y)dx\right\} = -\oint_C P(x, y)\,dx$$

$\Rightarrow$ $$\int_C P(x, y)dx = -\iint_R \frac{\partial P}{\partial y}dx\,dy \qquad (1)$$

Now, we regard the curve C as constituted of the arcs QAP and PBQ.

Let their equations be $x = \phi_1(y)$ and $x = \phi_2(y)$

Then $\phi_1(y) \le \phi_2(y)$ in $[c, d]$

$$\iint_R \frac{\partial Q}{\partial x}dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=\phi_1(y)}^{x=\phi_2(y)} \frac{\partial Q}{\partial x}\,dx\right] dy$$

$$= \int_c^d [Q(x,y)]_{x=\phi_1(y)}^{x=\phi_2(y)}\, dy$$

$$= \int_c^d [Q(\phi_2(y), y) - \phi(\phi_1(y), y)]\, dy$$

$$= \int_c^d Q(\phi_2(y), y)\, dy - \int_c^d Q(\phi_2(y), y)\, dy$$

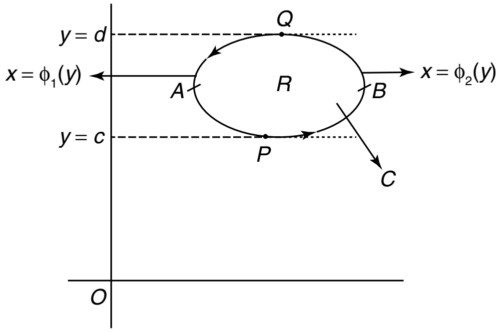


Fig. 9.6

But, $\int_c^d Q(\phi_2(y), y)\, dy$ is the line integral $\int_{PBQ} Q(x,y)\, dy$

and $\int_c^d Q(\phi_2(y), y)\, dy$ is the line integral $\int_{PAQ} Q(x,y)\, dy$

However, the positive sense of arc is QAP.

$\therefore \quad \int_c^d Q(\phi_2(y), y)\, dy = -\int_{QAP} Q(x,y)\, dy$

$\therefore \quad \iint_R \frac{\partial Q}{\partial x} dx\, dy = \int_{PBQ} Q(x,y) dy + \int_{QAP} Q(x,y) dy = \int_C Q(x,y) dy$

$\therefore \quad \int_C Q(x,y) dy = \iint_R \frac{\partial Q}{\partial x} dx\, dy \qquad (2)$

Adding the equations (1) and (2), we get

$$\int_C P(x,y) dx + \int_C Q(x,y) dy = -\iint_R \frac{\partial P}{\partial y} dx\, dy + \iint_R \frac{\partial Q}{\partial x} dx\, dy$$

$$\Rightarrow \quad \oint_C P\, dx + Q\, dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\, dy$$

Note We have proved the theorem by taking a simple closed region. The theorem is also valid in a region which can be divided into regions enclosed by simple closed curves.

Corollary Area of the region R bounded by C is $= \iint_R dxdy = \frac{1}{2}\oint_C (xdy - ydx)$

Proof In Green's theorem, take $P = -y$ and $Q = x$. $\therefore \frac{\partial P}{\partial y} = -1$ and $\frac{\partial Q}{\partial x} = 1$

Then $\oint_C (-ydx + xdy) = \iint_R (1+1) dxdy = 2\iint_R dxdy$

$\therefore \quad \frac{1}{2}\oint_C (xdy - ydx) = \iint_R dxdy$ ■

9.9.1 Vector Form of Green's Theorem

Let $\vec{F} = P\vec{i} + Q\vec{j}$ and $\vec{r} = x\vec{i} + y\vec{j}$

$\therefore$ $d\vec{r} = dx\vec{i} + dy\vec{j}$ and $\vec{F}.d\vec{r} = P\,dx + Q\,dx$

Now,
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \vec{i}\left[0 - \frac{\partial Q}{\partial z}\right] - \vec{j}\left[0 - \frac{\partial P}{\partial z}\right] + \vec{k}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \qquad \left[\because \frac{\partial Q}{\partial z} = 0; \frac{\partial P}{\partial z} = 0\right]$$

$$\therefore \quad \nabla \times \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\therefore$ Green's theorem becomes $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \vec{k}\, d\,R,$ where $d\,R = dx\,dy$

WORKED EXAMPLES

EXAMPLE 1

Using Green's theorem evaluate $\int_C [(x^2 - y^2)dx + 2xy\,dy]$, where C is the closed curve of the region bounded by $y^2 = x$ and $x^2 = y$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

The given line integral is $\int_C [(x^2 - y^2)dx + 2xydy]$

Here $P = x^2 - y^2$ and $Q = 2xy$

$$\therefore \quad \frac{\partial P}{\partial y} = -2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y + 2y = 4y$$

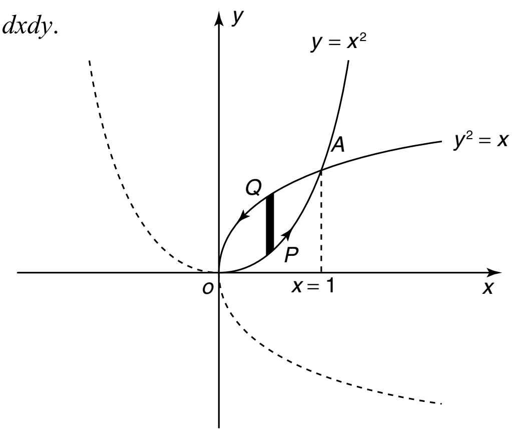


Fig. 9.7

$$\therefore \int_C (x^2 - y^2)\,dx + 2xydy = \iint_R 4\,ydxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 4\,ydydx$$

$$= 4\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 2\int_0^1 (x - x^4)dx = 2\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{5}$$

EXAMPLE 2

Evaluate $\int_C [(\sin x - y)dx - \cos x dy]$, where C is the triangle with vertices $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Given line integral is $\int_C [(\sin x - y)dx - \cos x dy]$

Here $\qquad P = \sin x - y$ and $Q = -\cos x$

$\therefore \qquad \frac{\partial P}{\partial y} = -1$ and $\frac{\partial Q}{\partial x} = \sin x$

$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin x + 1$

$$\int_C [(\sin x - y)dx - \cos x dy] = \iint_R (\sin x + 1) dxdy$$

Equation of OB is $\frac{y-0}{0-1} = \frac{x-0}{0-\frac{\pi}{2}} \quad \Rightarrow \quad y = \frac{2x}{\pi}$

Equation of AB is $\qquad x = \frac{\pi}{2}$

In this region R, x varies from $\frac{\pi y}{2}$ to $\frac{\pi}{2}$ and y varies from 0 to 1.

$$\therefore \int_C [(\sin x - y)dx - \cos x dy] = \int_0^1 \left[\int_{\pi y/2}^{\pi/2} (\sin x + 1)dx\right] dy$$

$$= \int_0^1 [-\cos x + x]_{\pi y/2}^{\pi/2}\, dy$$

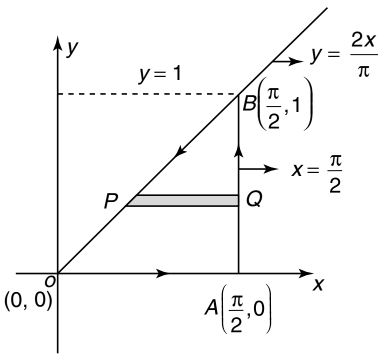


Fig. 9.8

$$= \int_0^1 \left[\left(-\cos\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(-\cos\frac{\pi y}{2} + \frac{\pi y}{2}\right)\right] dy$$

$$= \int_0^1 \left(\frac{\pi}{2} + \cos\frac{\pi y}{2} - \frac{\pi y}{2}\right) dy$$

$$= \left[\frac{\pi}{2} y + \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} - \frac{\pi}{2}\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2} + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2} + \frac{2}{\pi} - \frac{\pi}{4} = \frac{2}{\pi} + \frac{\pi}{4}$$

EXAMPLE 3

Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$, C being the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

The given line integral is $\int_C e^{-x}(\sin y dx + \cos y dy)$

Here $\quad P = e^{-x}\sin y \quad$ and $\quad Q = e^{-x}\cos y$

$$\therefore \quad \frac{\partial P}{\partial y} = e^{-x}\cos y \quad \text{and} \quad \frac{\partial Q}{\partial x} = -e^{-x}\cos y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$$

$$\therefore \oint_C e^{-x}(\sin y\,dx + \cos y\,dy) = \iint_R -2e^{-x}\cos y\,dxdy$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\; dxdy$$

$$= -2\left[\int_0^{\frac{\pi}{2}} \cos y\; dy\right]\left[\int_0^{\pi} e^{-x} dx\right]$$

$$= -2[\sin y]_0^{\pi/2}\left[\frac{e^{-x}}{-1}\right]_0^{\pi} = 2\left(\sin\frac{\pi}{2}\right)(e^{-\pi} - e^0) = 2(e^{-\pi} - 1)$$

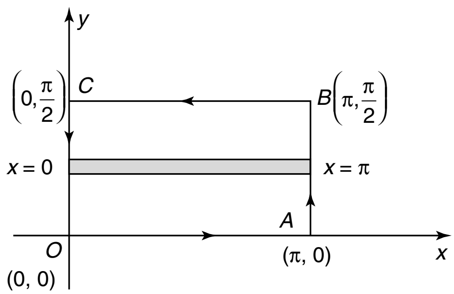


Fig. 9.9

EXAMPLE 4

Find the area bounded between the curves $y^2 = 4x$ and $x^2 = 4y$ using Green's theorem.

Solution.
We know, by Green's theorem the area bounded by a simple closed curve C is

$$\frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Here C consists of the curves C_1 and C_2.

$$\therefore \qquad \text{area} = \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx\right] = \frac{1}{2}[I_1 + I_2]$$

On C_1: $\quad x^2 = 4y$

$$\therefore \qquad 2x\,dx = 4\,dy \quad \Rightarrow \quad dy = \frac{1}{2}x\,dx$$

and x varies from 0 to 4.

$$\therefore \qquad I_1 = \int_{C_1} x\,dy - y\,dx$$

$$= \int_0^4 x\cdot\frac{1}{2}x\,dx - \frac{x^2}{4}\,dx$$

$$= \int_0^4 \left(\frac{x^2}{2} - \frac{x^2}{4}\right)dx = \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{64}{4\cdot 3} = \frac{16}{3}$$

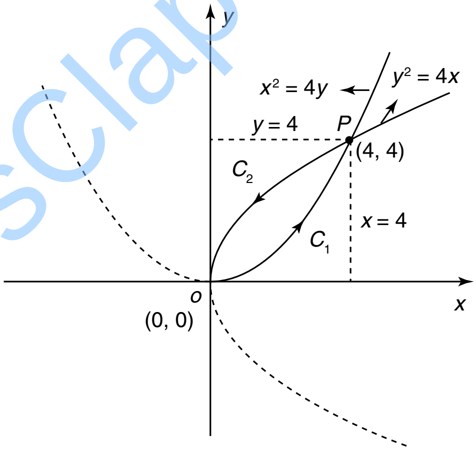


Fig. 9.10

On C_2: $y^2 = 4x \quad \therefore \quad 2y\,dy = 4\,dx \quad \Rightarrow \quad dx = \frac{1}{2}y\,dy$
and y varies from 4 to 0.

$$\therefore \qquad I_2 = \int_{C_2} x\,dy - y\,dx$$

$$= \int_4^0 \frac{y^2}{4}\,dy - y\cdot\frac{1}{2}y\,dy$$

$$= \int_4^0 \left(\frac{y^2}{4} - \frac{y^2}{2}\right)dy = \int_4^0 -\frac{y^2}{4}\,dy = \frac{1}{4}\int_0^4 y^2\,dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3}$$

$$\therefore \qquad \text{area} = \frac{1}{2}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{16}{3}$$

EXAMPLE 5

Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

The given integral is $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

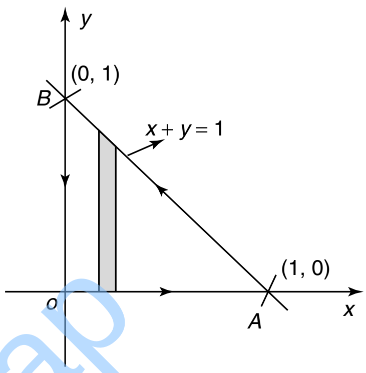


Fig. 9.11

Here $\quad P = 3x^2 - 8y^2 \quad$ and $\quad Q = 4y - 6xy$

$\therefore \quad \frac{\partial P}{\partial y} = -16y \quad$ and $\quad \frac{\partial Q}{\partial x} = -6y$

$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y + 16y = 10y$

$\therefore \quad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} 10y\, dy\, dx$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{-5}{3}[0-1] = \frac{5}{3}$$

$$\Rightarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \frac{5}{3} \quad (1)$$

We shall now compute the line integral $\int_C Pdx + Qdy$

Now $\quad \int_C Pdx + Qdy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$$= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$+ \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy = I_1 + I_2 + I_3$$

On OA: $y = 0 \quad \therefore \; dy = 0$ and x varies from 0 to 1.

$\therefore \quad I_1 = \int_0^1 3x^2 dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1$

On AB: $x + y = 1 \quad \Rightarrow \quad y = 1 - x \quad \therefore \; dy = -dx$ and x varies 1 to 0.

$\therefore \quad I_2 = \int_1^0 (3x^2 - 8(1-x)^2)dx + [4(1-x) - 6x(1-x)](-dx)$

$$= \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6(x - x^2)]dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - 4\frac{(1-x)^2}{-2} + 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right]_1^0$$

$$= \left[0 + \frac{8}{3} + 2 + 0 - \left\{1 + 6\left(\frac{1}{2} - \frac{1}{3}\right)\right\}\right] = \frac{8}{3} + 2 - 1 - 1 = \frac{8}{3}$$

On BO: $x = 0 \quad \therefore \quad dx = 0$ and y varies from 1 to 0

$$\therefore \qquad I_3 = \int_1^0 4y\,dy = 2\left[y^2\right]_1^0 = -2$$

$$\therefore \qquad \int Pdx + Qdy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \qquad (2)$$

(1) and (2) give the same value. Hence, Green's theorem is verified.

EXAMPLE 6

Verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2dy$, where C is the boundary, of the area between $y = x^2$ and $y = x$.

Solution.
Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

The given line integral is

$$\int_C (xy + y^2)\,dx + x^2dy$$

Here $\qquad P = xy + y^2 \quad$ and $\quad Q = x^2$

$$\therefore \qquad \frac{\partial P}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2x$$

$$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x - 2y = x - 2y$$

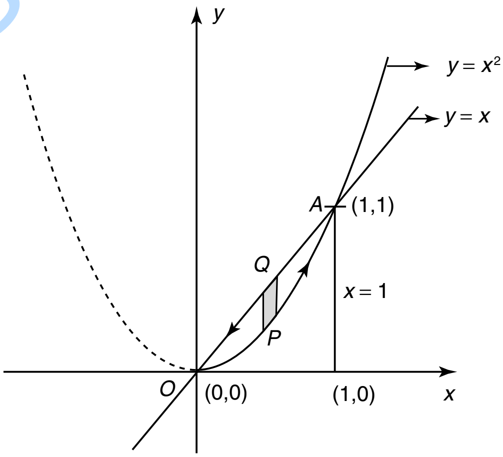


Fig. 9.12

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx = \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - (x^3 - x^4)]\,dx = \int_0^1 [(x^4 - x^3)]\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = -\frac{1}{20} \qquad (1)$$

We shall now compute the line integral $\int_C Pdx + Qdy$

Now $\quad \int_C Pdx + Qdy = \int_C (xy + y^2)\,dx + x^2 dy$

$$= \int_{C_1} (xy + y^2)\,dx + x^2 dy + \int_{C_2} (xy + y^2)\,dx + x^2 dy = I_1 + I_2$$

On C_1: $y = x^2$, $\therefore$ $dy = 2x\,dx$ and x varies from 0 to 1.

$$I_1 = \int_0^1 (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

On C_2: $y = x$, $\therefore$ $dy = dx$ and x varies from 1 to 0.

$$\therefore \quad I_2 = \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx = \int_0^1 3x^2\,dx = 3\left[\frac{x^3}{3}\right]_1^0 = -1$$

$$\therefore \quad \int_C Pdx + Qdy = \frac{19}{20} - 1 = -\frac{1}{20} \qquad (2)$$

(1) and (2) give the same value.
Hence, Green's theorem is verified.

9.10 SURFACE INTEGRALS

Suppose a surface is bounded by a simple closed curve C, then we can regard the surface as having two sides separated by C. One of which is arbitrarily chosen as the positive side and the other is the negative side. If the surface is a closed surface, then the outerside is taken as the positive side and the inner side is the negative side. A unit normal at any point of the positive side of the surface is denoted by $\vec{n}$ and is called the outward drawn normal and its direction is considered positive.

Any integral which is evaluated over a surface is called a surface integral.

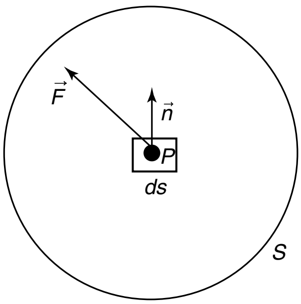


Fig. 9.13

Definition 9.12 Surface Integral

Let S be a surface of finite area which is smooth or piece-wise smooth (e.g. a sphere is a smooth surface and a cube is a piecewise smooth surface). Let $\vec{F}(x, y, z)$ be a vector point function defined at each point of S. Let P be any point on the surface and let $\vec{n}$ be the outward unit normal at P. Then the surface integral of $\vec{F}$ over S is defined as $\iint_S \vec{F} \cdot \vec{n}\, dS$

If we associate a vector $d\vec{S}$ (called vector area) with the differential of surface area dS such that $|d\vec{S}| = dS$ and direction of $d\vec{S}$ is $\vec{n}$, then

$$d\vec{S} = \vec{n}\, dS$$

$\therefore \iint_S \vec{F}\cdot\vec{n}\, dS$ can also be written as $\iint_S \vec{F}\cdot\overrightarrow{dS}$

Note

1. In physical application the integral $\iint_S \vec{F}\cdot\overrightarrow{dS}$ is called the normal flux of $\vec{F}$ through the surface S, because this integral is a measure of the volume emerging from S per unit time.

9.10.1 Evaluation of Surface Integral

To evaluate a surface integral over a surface it is usually expressed as a double integral over the orthogonal projection of S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point.

Let R be the orthogonal projection of S on the xy plane.

Then the element surface dS is projected to an element area $dx\, dy$ in the xy plane as in fig.

$\therefore$ $dx\, dy = dS \cos\theta$, where θ is the angle between the planes of dS and xy–plane.

Let $\vec{n}$ be the unit normal to dS and $\vec{k}$ is the unit normal to the xy–plane.

Since angle between the planes is equal to the angle between the normals,

θ is the angle between the normals $\vec{n}$ and $\vec{k}$.

$$\therefore \qquad \cos\theta = \frac{\vec{n}\cdot\vec{k}}{|\vec{n}||\vec{k}|}$$

$$= \vec{n}\cdot\vec{k} \text{ [Since } |\vec{n}| = 1, |\vec{k}| = 1]$$

We take the acute angle between the normals and

So, we take $|\vec{n}\cdot\vec{k}|$

$$\therefore \qquad dx\, dy = dS|\vec{n}\cdot\vec{k}| \quad \Rightarrow \quad dS = \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$$

Hence, $\displaystyle\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$

Similarly, taking the projection on the yz and zx planes, we get

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dy\, dz}{|\vec{n}\cdot\vec{i}|}$$

and
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dz\, dx}{|\vec{n}\cdot\vec{j}|}$$

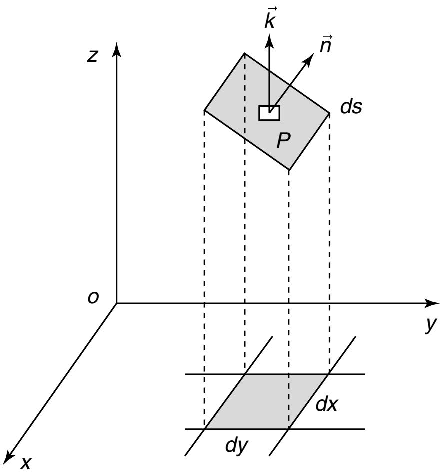


Fig. 9.14

Corollary

The surface area $\displaystyle\iint_S dS = \iint_R \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|} = \iint_{R_1} \frac{dy\, dz}{|\vec{n}\cdot\vec{i}|} = \iint_{R_2} \frac{dz\, dx}{|\vec{n}\cdot\vec{j}|}$

9.11 VOLUME INTEGRAL

Any integral which is evaluated over a volume bounded by a surface is called a volume integral.

If V is the volume bounded by a surface S, then

$\iiint_V \phi(x, y, z)dV$ and $\iiint_V \vec{F}\, dV$ are called volume integrals.

If we divide V into rectangular blocks by drawing planes parallel to the coordinate planes, then

$$dV = dx\, dy\, dz.$$

$$\therefore \qquad \iiint_V \phi\, dV = \iiint_V \phi(x, y, z)\, dxdydz$$

If
$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

then
$$\iiint_V \vec{F}dV = \vec{i}\iiint_V F_1 dxdydz + \vec{j}\iiint_V F_2 dxdydz + \vec{k}\iiint_V F_3 dxdydz$$

WORKED EXAMPLES

EXAMPLE 1

Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$ if $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and S is the surface of the plane $3x + 2y + 6z = 6$ contained in the first octant.

Solution.

Given $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and the surface $3x + 2y + 6z = 6$.

Let
$$\phi = 3x + 2y + 6z$$

Let R be the projection of S in the xy plane.

$\therefore$ R is the $\Delta\, AOB$

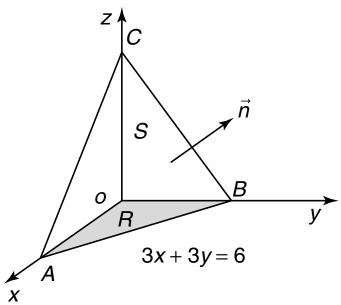


Fig. 9.15

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\, \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$$

where $\vec{n}$ is unit normal to S and $\vec{k}$ is the unit normal to xy–plane.

Normal to the surface is $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

$\therefore$ unit normal is $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9+4+36}} = \dfrac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$

$$\therefore \qquad \vec{F}\cdot\vec{n} = (4y\vec{i} + 18z\vec{j} - x\vec{k})\cdot\frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$$

$$= \frac{1}{7}(12y + 36z - 6x) = \frac{6}{7}(2y + 6z - x)$$

$$\vec{n}\cdot\vec{k} = \frac{1}{7}(3\vec{i}+2\vec{j}+6\vec{k})\cdot\vec{k} = \frac{6}{7}$$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \frac{6}{7}(2y+6z-x)\frac{dx\,dy}{\frac{6}{7}} = \iint_R (2y+6z-x)dx\,dy$$

We have $3x + 2y + 6z = 6$

$\Rightarrow \qquad 6z = 6 - 3x - 2y$

$\therefore \qquad 2y + 6z - x = 2y + 6 - 3x - 2y - x = 6 - 4x$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R (6-4x)\, dx\, dy$$

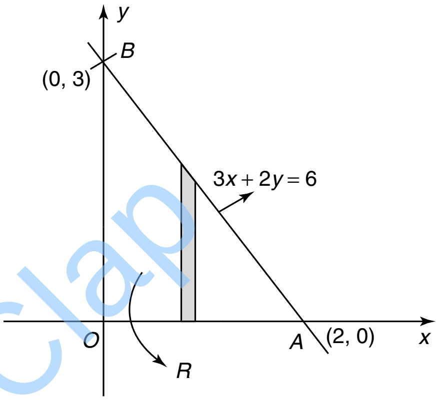


Fig. 9.16

The plane $3x + 2y + 6z = 6$ meets the xy–plane $z = 0$ in line AB.

$\therefore$ the equation of AB is $3x + 2y = 6$

$\therefore$ the point A is (2, 0) and the point B is (0, 3)

Now $\qquad 3x + 2y = 6 \quad \Rightarrow \quad y = \dfrac{6-3x}{2}$

$\therefore$ In R, x varies from 0 to 2 and y varies from 0 to $\dfrac{6-3x}{2}$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\, dS = \int_0^2 \int_0^{\frac{6-3x}{2}} (6-4x)\, dy\, dx = 2\int_0^2 \int_0^{\frac{6-3x}{2}} (3-2x)\, dy\, dx$$

$$= 2\int_0^2 \left[(3-2x)\,y\right]_0^{\frac{6-3x}{2}} dx$$

$$= 2\int_0^2 (3-2x)\frac{(6-3x)}{2}\, dx$$

$$= 3\int_0^2 (3-2x)(2-x)\, dx$$

$$= 3\int_0^2 (6-7x+2x^2)\, dx$$

$$= 3\left[6x - \frac{7x^2}{2} + 2\frac{x^3}{3}\right]_0^2$$

$$= 3\left[6\times 2 - 7\times\frac{4}{2} + 2\times\frac{8}{3}\right] = 3\left[12 - 14 + \frac{16}{3}\right] = -6 + 16 = 10$$

EXAMPLE 2

Evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$ if $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is part of the surface $x^2 + y^2 + z^2 = 1$, which lies in the first octant.

Solution.

Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and the surface is $x^2 + y^2 + z^2 = 1$

Let $$\phi = x^2 + y^2 + z^2$$

The normal to the surface is $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

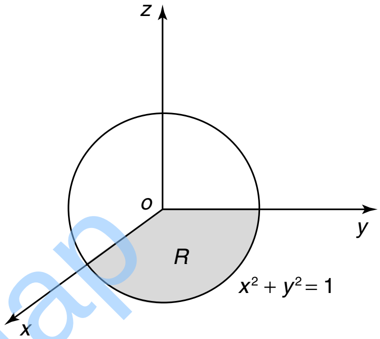


Fig. 9.17

$\therefore$ unit normal is $$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\vec{i} + y\vec{j} + z\vec{k} \qquad [\because x^2 + y^2 + z^2 = 1]$$

$\therefore$ $$\vec{F}\cdot\vec{n} = (yz\vec{i} + zx\vec{j} + xy\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k})$$
$$= xyz + xyz + xyz = 3xyz$$

The projection of the surface of the sphere in the first octant into the xy plane is R, which is the quadrant of the circle $x^2 + y^2 = 1$, $z = 0$, $x \geq 0$, $y \geq 0$ and $\vec{k}$ is the unit normal to R.

$\therefore$ $$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot\vec{n}\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$
$$= \iint_R 3xyz\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

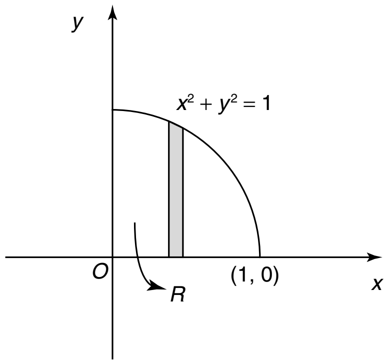


Fig. 9.18

But $$\vec{n}\cdot\vec{k} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{k} = z$$

$\therefore$ $$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R 3xyz\,\frac{1}{z}dx\,dy$$
$$= \iint_R 3xy\,dx\,dy$$
$$= \int_0^1\int_0^{\sqrt{1-x^2}} 3xy\,dx\,dy$$
$$= 3\int_0^1\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} x\,dx = \frac{3}{2}\int_0^1 x(1-x^2)dx$$
$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}$$

EXAMPLE 3

Evaluate $\iint \vec{F}\cdot\vec{n}\,dS$, where $\vec{F}=4xz\,\vec{i}-y^2\vec{j}+yz\vec{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution.

Given $\quad \vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\vec{k}$

S is the surface of the cube, which is piecewise smooth surface consisting of six smooth surfaces.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{ABEF} \vec{F}\cdot\vec{n}\,dS + \iint_{OCDG} \vec{F}\cdot\vec{n}\,dS + \iint_{BCDE} \vec{F}\cdot\vec{n}\,dS + \iint_{OAFG} \vec{F}\cdot\vec{n}\,dS + \iint_{OABC} \vec{F}\cdot\vec{n}\,dS + \iint_{DEFG} \vec{F}\cdot\vec{n}\,dS$$

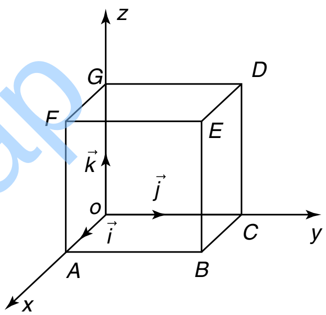


Fig. 9.19

On the face *ABEF*: $x = 1, \quad \vec{n} = \vec{i}$

$\therefore \quad \vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{i} = 4xz = 4z$

and $\quad dS = \dfrac{dy\,dz}{|\vec{n}\cdot\vec{i}|} = \dfrac{dy\,dz}{|\vec{i}\cdot\vec{i}|} = dy\,dz$

$$\therefore \iint_{ABEF} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 4z\,dz\,dy = 4\cdot[y]_0^1\left[\frac{z^2}{2}\right]_0^1 = 4\cdot 1\cdot\frac{1}{2} = 2$$

On the face *OCDG*: $x = 0, \quad \vec{n} = -\vec{i}$

$$\vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i}) = -4xz = 0$$

$$\therefore \iint_{OCDG} \vec{F}\cdot\vec{n}\,dS = 0$$

On the face *BCDE*: $y = 1, \quad \vec{n} = \vec{j}$

$\therefore \quad dS = \dfrac{dx\,dz}{|\vec{n}\cdot\vec{j}|} = \dfrac{dx\,dz}{|\vec{j}\cdot\vec{j}|} = dx\,dz$

and $\quad \vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j} = -y^2 = -1$

$$\therefore \iint_{BCDE} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 (-1)dx\,dz = -[x]_0^1[z]_0^1 = -1$$

On the face *OAFG*: $y = 0, \quad \vec{n} = -\vec{j}$

$\therefore \quad \vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j}) = y^2 = 0$

$$\therefore \iint_{OAFG} \vec{F}\cdot\vec{n}\,dS = 0$$

On the face *DEFG*: $z = 1, \quad \vec{n} = \vec{k}$

$\therefore \qquad \vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k} = yz = y$

and $\qquad dS = \dfrac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \dfrac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = dx\,dy$

$$\therefore \qquad \iint_{DEFG} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = 1\times\frac{1}{2} = \frac{1}{2}$$

On the face *OABC*: $z = 0, \qquad \vec{n} = -\vec{k}$

$$\vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k}) = -yz = 0$$

$$\therefore \qquad \iint_{OABC} \vec{F}\cdot\vec{n}\,dS = 0$$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = 2 + (-1) + \frac{1}{2} = \frac{3}{2}.$$

9.12 GAUSS DIVERGENCE THEOREM

The divergence theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss divergence theorem

Let V be the volume bounded by a closed surface S. If a vector function $\vec{F}$ is continuous and has continuous partial derivatives inside and on S, then the surface integral of $\vec{F}$ over S is equal to the volume integral of divergence of $\vec{F}$ taken throughout V.

i.e.,
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{F}\,dV$$

If $\vec{n}$ is the outward normal to the surface $d\vec{S} = \vec{n}\,dS$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

Proof Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\therefore \qquad \vec{F}\cdot\vec{n} = F_1(\vec{i}\cdot\vec{n}) + F_2(\vec{j}\cdot\vec{n}) + F_3(\vec{k}\cdot\vec{n})$$

and $\quad \vec{F}\cdot\vec{n}\,dS = F_1(\vec{i}\cdot\vec{n})dS + F_2(\vec{j}\cdot\vec{n})dS + F_3(\vec{k}\cdot\vec{n})dS$

$$= F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$$

But $\qquad \nabla\cdot\vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

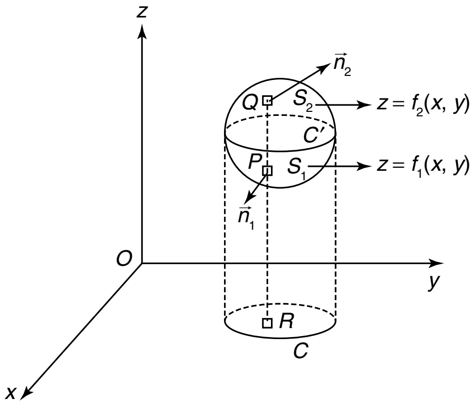


Fig. 9.20

Hence, Gauss theorem in Cartesian form is

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) \equiv \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

We shall assume that S is a closed surface such that any line drawn parallel to coordinate axes cuts S in almost two points. The lines drawn parallel to Z-axis touching the surface S determine the curve C' on it and intersect the xy-plane along the curve C. Now, the curve C' divides the surface S into two parts S_1 and S_2.

S_1 and S_2 are called the lower and upper surfaces.

Let $z = f_1(x, y)$ and $z = f_2(x, y)$ be the equations of S_1 and S_2, respectively.

The projection of S on the xy-plane is the region R bounded by C.

Now consider the triple integral $\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz$ over the volume V enclosed by S.

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\right] dx\, dy$$

$$= \iint_R [F_3(x, y, z)]_{z=f_1(x,y)}^{z=f_2(x,y)} dx\, dy$$

$$= \iint_R \left[F_3(x, y, f_2(x, y)) - F_3(x, y, f_1(x, y))\right] dx\, dy$$

$$\Rightarrow \qquad \iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_R F_3(x, y, f_2(x, y))\, dx\, dy - \iint_R F_3(x, y, f_1(x, y))\, dx\, dy \qquad (1)$$

Let a line parallel to the z-axis meet S_1 at the point P and S_2 at the point Q. Let dS_1 and dS_2 be element surface at P and Q, respectively and their projections in the xy-plane be $dx\, dy$.

Let $\vec{n}_1$ be the outward unit normal at P to S_1 and $\vec{n}_2$ be the outward unit normal at Q to S_2.

Let the angle between $\vec{n}_2$ and $\vec{k}$ be γ_2 and γ_2 is acute, since $\vec{k}$ is unit vector in the direction of the positive z-axis.

Then
$$dx\, dy = \cos\gamma_2\, dS_2 = \vec{k}\cdot\vec{n}_2\, d\, S_2$$

Let the angle between $\vec{n}_1$ and $\vec{k}$ be γ_1 and it is obtuse. [$\because \vec{k}$ is upward and $\vec{n}_1$ is downward]

$\therefore$
$$dx\, dy = -\cos\gamma_1\, dS_1 = -\vec{k}\cdot\vec{n}_1\, d\, S_1$$

Hence,
$$\iint_R F_3(x, y\, f_2(x, y))\, dx\, dy = \iint_{S_2} F_3 \vec{k}\cdot\vec{n}_2\; dS_2$$

and
$$\iint_R F_3(x, y\, f_1(x, y))\, dx\, dy = -\iint_{S_1} F_3 \vec{k}\cdot\vec{n}_1\; dS_1$$

Substituting in (1), we get

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_{S_2} F_3 \vec{k}\cdot\vec{n}_2\; dS_2 + \iint_{S_1} F_3 \vec{k}\cdot\vec{n}_1\; dS_1$$

$\Rightarrow$
$$\iiint_V \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_S F_3\vec{k}\cdot\vec{n}\;dS \tag{2}$$

Similarly, projecting S on the yz- and zx-planes, we get

$$\iiint_V \frac{\partial F_2}{\partial y} dx\,dy\,dz = \iint_S F_2\vec{j}\cdot\vec{n}\;dS \tag{3}$$

and
$$\iiint_V \frac{\partial F_1}{\partial x} dx\,dy\,dz = \iint_S F_1\vec{i}\cdot\vec{n}\;dS \tag{4}$$

Adding equations (2), (3) and (4), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1\vec{i}+F_2\vec{j}+F_3\vec{k})\cdot\vec{n}\;dS$$

$\Rightarrow$
$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS$$

9.12.1 Results Derived from Gauss Divergence Theorem

The following results are immediate consequence of Gauss divergence theorem:

(1) $\iint_S \phi\,\vec{n}\,dS = \iiint_V \nabla\,\phi\,dV$ (2) $\iint_S \vec{F}\times\vec{n}\,dS = -\iiint_V \nabla\times\vec{F}\,dV$

where ϕ is the scalar point function defined in the region V enclosed by the closed surface S.

Solution.

(1) $\iint_S \phi\,\vec{n}\,dS = \iiint_V \nabla\,\phi\,dV$.

Gauss divergence theorem is

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS \tag{1}$$

Let $\vec{F} = \phi\vec{a}$, where $\vec{a}$ is an arbitrary constant vector.

$\therefore$ (1) becomes $\iiint_V \left(\nabla\cdot\phi\;\vec{a}\right)dS = \iint_S \phi\;\vec{a}\cdot\vec{n}\,dS$ (2)

Now, $\nabla\cdot\phi\vec{a} = \nabla\phi\cdot\vec{a} + \phi(\nabla\cdot\vec{a}) = \nabla\phi\cdot\vec{a}$ $[\because\ \nabla\cdot\vec{a}=0]$

$\therefore$
$$\iiint_V \left(\nabla\cdot\phi\vec{a}\right)dV = \iiint_V \left(\nabla\phi\cdot\vec{a}\right)dV \tag{3}$$

and
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \phi\vec{a}\cdot\vec{n}\,dS = \iint_S \phi\;\vec{n}\,dS\cdot\vec{a} \tag{4}$$

$\therefore$ Using (3) and (4) in (2), we get

$$\iiint_V \nabla\phi\cdot\vec{a}\,dV = \iint_S \phi\,\vec{n}\,dS\cdot\vec{a}$$

$\Rightarrow$ $$\vec{a}\cdot\iiint_V \nabla\phi\, dV = \vec{a}\cdot\iint_S \phi\vec{n}\, dS$$

$\Rightarrow$ $$\iiint_V \nabla\phi\, dV = \iint_s \phi\ \vec{n}\, dS \qquad [\because \vec{a} \text{ is arbitrary}]$$

2. $\iint_S \vec{F}\times\vec{n}\; dS = -\iiint_V \nabla\times\vec{F}\, dV$

Gauss divergence theorem is $\iiint_V \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, dS$ (1)

Let $\vec{F} = \vec{a}\times\vec{F}$, where a is an arbitrary constant vector.

$\therefore$ $$\nabla\cdot\vec{F} = \nabla\cdot(\vec{a}\times\vec{F}) = \vec{F}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{F}) = -\vec{a}\cdot(\nabla\times\vec{F}) \qquad [\because \nabla\times\vec{a}=0]$$

and $$\vec{F}\cdot\vec{n} = \vec{a}\times\vec{F}\cdot\vec{n} = \vec{a}\cdot(\vec{F}\times\vec{n})$$

$\therefore$ (1) becomes $-\iiint_V \vec{a}\cdot(\nabla\times\vec{F})\, dV = \iint_S (\vec{a}\cdot\vec{F}\times\vec{n})\, dS$

$\Rightarrow$ $$-\vec{a}\cdot\iiint_V \nabla\times\vec{F}\, dV = \vec{a}\cdot\iint_S \vec{F}\times\vec{n}\, dS$$

$\Rightarrow$ $$-\iiint_V \nabla\times\vec{F}\, dV = \iint_S \vec{F}\times\vec{n}\, dS$$

$\Rightarrow$ $$\iint_S \vec{F}\times\vec{n}\, dS = -\iiint_V \nabla\times\vec{F}\, dV.$$

If S is closed surface, then prove that Eq.

(1) $\iint_S d\mathbf{S} = \iiint_V \nabla\cdot\vec{n}\, dV$ (2) $\iint_S d\mathbf{S} = 0$ (3) $\iint_S \vec{r}\times\vec{n}\, dS = 0$

(4) $\iiint_V (\nabla\times\vec{n})\, dV = 0$ (5) $\iint_S \frac{\vec{r}}{r^3}\cdot\vec{n}\, dS = 0$ (6) $\iint_S r^4\vec{n}\; dS = 4\iiint_V r^4\vec{r}\; dV$

(7) $\iint_S f(r)\vec{r}\times\vec{n}\, dS = 0$ (8) $\iint_S (\nabla r^2\cdot\vec{n})\, dS = 6V$ (9) $\iint_S (\nabla\times\vec{r})\cdot\vec{n}\, dS = 0$

Solution.

(i) To prove $\iint_S d\mathbf{S} = \iiint_V \nabla\cdot\vec{n}\, dV$.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV$ (1)

Let $\vec{F} = \vec{n}$ $\therefore$ $\nabla\cdot\vec{F} = \nabla\cdot\vec{n}$ and $\vec{F}\cdot\vec{n} = \vec{n}\cdot\vec{n} = 1$

$\therefore$ (1) becomes $$\iint_S dS = \iiint_V \nabla\cdot\vec{n}\, dV.$$

(2) To prove $\iint_S d\mathbf{S} = 0$.

We have $\iiint_V \nabla\phi\, dV = \iint_S \phi\ \vec{n}\, dS$ (1)

Let $\phi = 1.$

$\therefore \quad \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(1) = 0 \quad \therefore \iiint_V \nabla\phi\, dV = 0$

$\therefore \quad \iint_S \vec{n}\, dS = 0 \quad \Rightarrow \quad \iint_S dS = 0$ [using (1)]

(3) To prove $\iint_S \vec{r} \times \vec{n}\, dS = 0.$

We have $\iint_S \vec{F} \times \vec{n}\, dS \equiv -\iiint_V \nabla \times \vec{F}\, dV$

Let $\vec{F} = \vec{r}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = 0$$

$\therefore \quad \iiint_V \nabla \times \vec{F}\, dv = 0 \quad \therefore \iint_S \vec{F} \times \vec{n}\, dS = 0 \Rightarrow \iint_S \vec{r} \times \vec{n}\, dS = 0$ [using (1)]

(4) To prove $\iint_V \nabla \times \vec{n}\, dV = 0.$

We have $\iint_S \vec{F} \times \vec{n}\, dS = -\iiint_V \nabla \times \vec{F}\, dV$ (1)

Let $\vec{F} = \vec{n} \quad \therefore \quad \vec{F} \times \vec{n} = \vec{n} \times \vec{n} = 0$

$\therefore \quad \iint_S \vec{F} \times \vec{n}\, dS = 0 \quad \therefore \quad \iiint_V \nabla \times \vec{F}\, dV = 0 \quad \Rightarrow \quad \iiint_V \nabla \times \vec{n}\, dV = 0$ [using (1)]

(5) To prove $\iint_S \frac{\vec{r}}{r^3} \cdot \vec{n}\, dS = 0.$

Gauss divergence theorem is $\iint_S \vec{F}.\vec{n}\, dS = \iiint_V \Delta \cdot \vec{F}\, dS$ (1)

Let $\vec{F} = \frac{\vec{r}}{r^3} \quad \therefore \quad \nabla \cdot \vec{F} = \nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}(\nabla \cdot \vec{r}) + \nabla\left(\frac{1}{r^3}\right) \cdot \vec{r}$

$\therefore \quad \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

Now $r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \qquad \nabla\left(\frac{1}{r^3}\right) = \nabla(r^{-3}) = \vec{i}\,\frac{\partial}{\partial x}(\vec{r}^3) + \vec{j}\,\frac{\partial}{\partial y}(\vec{r}^3) + \vec{k}\,\frac{\partial}{\partial z}(\vec{r}^3)$$

$$= \vec{i}\,(-3)\vec{r}^4\,\frac{\partial r}{\partial x} + \vec{j}\,(-3)\vec{r}^4\,\frac{\partial r}{\partial y} + \vec{k}\,(-3)\vec{r}^4\,\frac{\partial r}{\partial z}$$

$$= -\frac{3}{r^4}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = -\frac{3}{r^5}[x\vec{i} + y\vec{j} + z\vec{k}] = -\frac{3}{r^4}\vec{r}$$

$$\therefore \qquad \nabla\cdot\vec{F} = \nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = \frac{3}{r^3} - \frac{3}{r^5}(\vec{r}\cdot\vec{r}) = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$$\therefore \qquad \iiint_V \nabla\cdot\vec{F}\,dV = 0 \;\Rightarrow\; \iint_S \vec{F}\cdot\vec{n}\,dS = 0 \qquad \text{[using (1)]}$$

$$\therefore \qquad \iint_S \frac{r}{r^3}\cdot\vec{n}\,dS = \iiint_V \left(\nabla\cdot\frac{\vec{r}}{r^3}\right)dS = 0.$$

(6) To prove $\iint_S r^4\vec{n}\,dS = 4\iiint_V r^2\vec{r}\,dV$.

We have $\iiint_V \nabla\phi\,dV = \iint_S \phi\,\vec{n}\,dS$

Let $\phi = r^4 \quad \therefore \quad \nabla\phi = \vec{i}\,\frac{\partial}{\partial x}(r^4) + \vec{j}\,\frac{\partial}{\partial y}(r^4) + \vec{k}\,\frac{\partial}{\partial z}(r^4)$

$$= 4r^3\cdot\frac{x}{r}\vec{i} + 4r^3\cdot\frac{y}{r}\vec{j} + 4r^3\cdot\frac{z}{r}\vec{k} = 4r^2[x\vec{i} + y\vec{j} + z\vec{k}] = 4r^2\vec{r}$$

$\therefore$ (1) becomes, $\iiint_V 4r^2\vec{r}\,dV = \iint_S r^4\,\vec{n}\,dS \;\Rightarrow\; 4\iiint_V r^2\vec{r}\,dV = \iint_S r^4\,\vec{n}\,dS$

(7) To prove: $\iint_S f(r)\vec{r}\times\vec{n}\,dS = 0$.

We have $\iint_S \vec{F}\times\vec{n}\,dS = -\iiint_V \nabla\times\vec{F}\,dV \qquad (1)$

Let $\vec{F} = f(r)\vec{r}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \qquad \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x)x & f(x)y & f(x)z \end{vmatrix} = \sum \vec{i}\left[\frac{\partial}{\partial y}f(r)z - \frac{\partial}{\partial z}f(r)y\right]$$

Now $\left[\frac{\partial}{\partial y}f(r)z - \frac{\partial}{\partial z}f(r)y\right] = \left[f(r)\frac{\partial z}{\partial y} + zf'(r)\frac{\partial r}{\partial y}\right] - \left[f(r)\frac{\partial y}{\partial z} + y\,f'(r)\frac{\partial r}{\partial z}\right]$

$$= 0 + z\,f'(r)\frac{y}{r} - 0 - yf'(r)\cdot\frac{z}{r} = \frac{f'(r)}{r}[yz - yz] = 0$$

$$\therefore \qquad \vec{i}\left[\frac{\partial}{\partial y}(f(r)z)-\frac{\partial}{\partial z}(f(r)y)\right]=0$$

Similarly, $\vec{j}\left[\frac{\partial}{\partial x}(f(r)z)-\frac{\partial}{\partial z}(f(r)y)\right]=0$ and $\vec{k}\left[\frac{\partial}{\partial x}(f(r)y)-\frac{\partial}{\partial z}(f(r)x)\right]=0$

$$\therefore \qquad \nabla\times\vec{F}=0 \quad \therefore \quad \iiint_V \nabla\times\vec{F}\,dV=0$$

$\therefore$ (1) becomes $\quad \iint_S \vec{F}\times\vec{n}\,dS=0 \quad \Rightarrow \quad \iint_S f(r)\vec{r}\times\vec{n}\,dS=0$

(8) To prove $\iint_S (\nabla r^2\cdot\vec{n})\,dS=6V$.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS=\iiint_V \nabla\cdot\vec{F}\,dV$ (1)

Let $\vec{F}=\nabla r^2$ and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k} \quad \Rightarrow \quad r^2=x^2+y^2+z^2$

$$\therefore \qquad \nabla\cdot\vec{F}=\nabla\cdot\nabla r^2$$

Now, $\quad \nabla r^2=\vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$

$$=2x\vec{i}+2y\vec{j}+2z\vec{k}=2[x\vec{i}+y\vec{j}+z\vec{k}]=2\vec{r}$$

$$\therefore \qquad \nabla\cdot\nabla r^2=\nabla\cdot 2\vec{r}=2\nabla\cdot\vec{r}$$

But $\quad \nabla\cdot\vec{r}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(x\vec{i}+y\vec{j}+z\vec{k})=1+1+1=3$

$$\therefore \qquad \nabla\cdot\nabla r^2=2\cdot 3=6 \quad \therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV=\iiint_V 6\,dV=6V$$

$\therefore$ (1) becomes $\iint_S \vec{F}\cdot\hat{n}\,dS=6V \quad \Rightarrow \quad \iint_S \nabla r^2\cdot\vec{n}\,dS=6V$

WORKED EXAMPLES

EXAMPLE 1

Let V be the region bounded by a closed surface S. Let f and g be scalar point functions that together with their derivatives in any directions are uniformly continuous within the region V. Then

$$\iiint_V (f\nabla^2 g-g\nabla^2 f)\,dV=\iint_S (f\nabla g-g\nabla f)\cdot\vec{n}\,dS.$$

Solution.

Gauss divergence theorem is

$$\iiint_V \nabla\cdot\vec{F}\,dV=\iint_S \vec{F}\cdot\vec{n}\,dS$$

Put $\vec{F} = f\nabla g \quad \therefore \quad \nabla\cdot\vec{F} = \nabla\cdot(f\nabla g) = f(\nabla\cdot\nabla g) + \nabla f\cdot\nabla g = f\nabla^2 g + \nabla f\cdot\nabla g$

and $\vec{F}\cdot\vec{n} = (f\nabla g)\cdot\vec{n}$

$\therefore$ by divergence theorem becomes

$$\iiint_V (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV = \iint_S (f\nabla g\cdot\vec{n})\,dS \qquad (1)$$

Interchanging f and g, we get

$$\iiint_V (g\nabla^2 f + \nabla g\cdot\nabla f)\,dV = \iint_S (g\nabla f\cdot\vec{n})\,dS \qquad (2)$$

(1) − (2) $\Rightarrow$
$$\iiint_V (f\nabla^2 g - g\nabla^2 f)\,dV = \iint_S (f\nabla g - g\nabla f)\cdot\vec{n}\,dS \qquad (3)$$

Note This result is known as **Green's theorem**.
Equation (1) is called Green's first identity and equation (3) is called Green's second identity.

EXAMPLE 2

Prove that $\iiint_V \frac{1}{r^2}\,dV = \iint_S \frac{\vec{r}}{r^2}\cdot\vec{n}\,dS.$

Solution.

Gauss divergence theorem is

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS \qquad (1)$$

Put $\vec{F} = \frac{\vec{r}}{r^2} = r^{-2}\vec{r}$. Then $\nabla\cdot\vec{F} = \nabla\cdot(r^{-2}\vec{r}) = (\nabla\cdot\vec{r})r^{-2} + \nabla r^{-2}\cdot\vec{r}$

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

and $r^2 = x^2 + y^2 + z^2$

$\therefore$
$$2r\frac{\partial r}{\partial x} = 2x \;\Rightarrow\; \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^{-2} = \vec{i}\frac{\partial}{\partial x}(r^{-2}) + \vec{j}\frac{\partial}{\partial y}(r^{-2}) + \vec{k}\frac{\partial}{\partial z}(r^{-2})$$

$$= \vec{i}(-2)r^{-3}\frac{\partial r}{\partial x} + \vec{j}(-2)r^{-3}\frac{\partial r}{\partial y} + \vec{k}(-2)r^{-3}\frac{\partial r}{\partial z}$$

$$= -2r^{-3}\frac{x}{r}\vec{i} - 2r^{-3}\frac{y}{r}\vec{j} - 2r^{-3}\frac{z}{r}\vec{k} = \frac{-2}{r^4}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{2\vec{r}}{r^4}$$

$\therefore$
$$\nabla\cdot\vec{F} = 3r^{-2} + \left(\frac{-2}{r^4}\vec{r}\cdot\vec{r}\right) = \frac{3}{r^2} - \frac{2}{r^4}\times r^2 = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$\therefore$ (1) becomes
$$\iiint_V \frac{1}{r^2}\,dV = \iint_S \frac{\vec{r}}{r^2}\cdot\vec{n}\,dS$$

EXAMPLE 3

Using divergence theorem, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$.

Solution.
Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

$$\therefore \quad \iint \vec{F}\cdot\vec{n}\,dS = \int_0^2\int_0^2\int_0^2 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 (4z - y)\left[x\right]_0^2 dy\,dz$$

$$= \int_0^2\int_0^2 (4z - y)2\,dy\,dz$$

$$= 2\int_0^2 \left[4zy - \frac{y^2}{2}\right]_0^2 dz$$

$$= 2\int_0^2 \left(4z\cdot 2 - \frac{4}{2}\right)dz = 2\cdot\int_0^2 (8z - 2)\,dz$$

$$= 2\left[\frac{8z^2}{2} - 2z\right]_0^2 = 2\left[8\cdot\frac{4}{2} - 2\cdot 2\right] = 2[16 - 4] = 2\times 12 = 24.$$

EXAMPLE 4

Using Gauss divergence theorem, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

Solution.
Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2)$$

$$\therefore \quad \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V 3(x^2 + y^2 + z^2)\,dx\,dy\,dz$$

We shall evaluate this triple integral by using spherical polar coordinates.
$x = r \sin\theta \cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

then $$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\right| dr\,d\theta\,d\phi = r^2 \sin\theta\, dr\,d\theta\,d\phi$$

and $x^2 + y^2 + z^2 = r^2$
Here r varies from 0 to a, θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\therefore \quad \iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3r^4 \sin\theta\, dr\,d\theta\,d\phi$$

$$= 3\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^4\,dr$$

$$= 3[\phi]_0^{2\pi}[-\cos\theta]_0^{\pi}\left[\frac{r^5}{5}\right]_0^a$$

$$= 3\cdot 2\pi(-\cos\pi + \cos 0)\cdot\frac{a^5}{5}$$

$$= 6\pi\cdot 2\cdot\frac{a^5}{5} = \frac{12\pi}{5}a^5$$

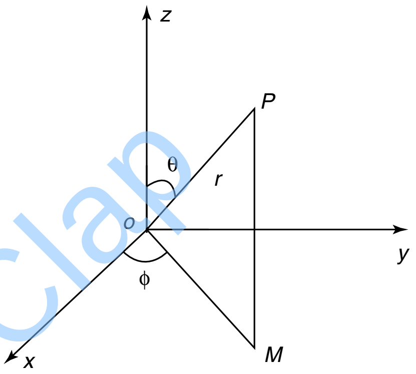


Fig. 9.21

Note We have $\operatorname{div}\vec{F} = 3(x^2 + y^2 + z^2)$. Since the equation of the surface is $x^2 + y^2 + z^2 = a^2$, we cannot take $\operatorname{div}\vec{F} = 3a^2$ because $\vec{F}$ is defined in the volume inside and on S.
But $x^2 + y^2 + z^2 = a^2$ is true only for points on S.

EXAMPLE 5

Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0$, $x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution.
Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz \qquad [\because\ dV = dx\,dy\,dz]$$

$$= \int_0^1\int_0^1 (4z - y)[x]_0^1\,dy\,dz = \int_0^1\int_0^1 [4z - y]\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2} \right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2} \right] dz = \left[4\frac{z^2}{2} - \frac{1}{2}z \right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow \qquad \iiint_V \nabla \cdot \vec{F}\, dv = \frac{3}{2} \qquad (1)$$

We shall now evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$

Here the surface S consists of the six faces of the cube.

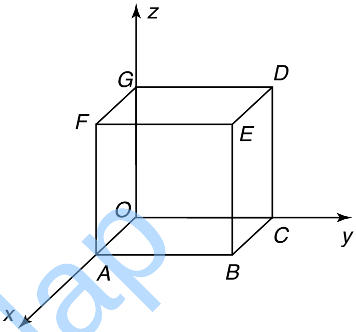


Fig. 9.22

$$\therefore \qquad \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \vec{F} \cdot \vec{n}\, dS + \iint_{S_2} \vec{F} \cdot \vec{n}\, dS + \iint_{S_3} \vec{F} \cdot \vec{n}\, dS + \iint_{S_4} \vec{F} \cdot \vec{n}\, dS + \iint_{S_5} \vec{F} \cdot \vec{n}\, dS + \iint_{S_6} \vec{F} \cdot \vec{n}\, dS$$

We shall simplify the computation and put it in the form of a table.

Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	dS
S_1 = ABEF	$x = 1$	$\vec{i}$	$4xz = 4z$	$dy\, dz$
S_2 = OCDG	$x = 0$	$-\vec{i}$	$-4xz = 0$	$dy\, dz$
S_3 = BCDE	$y = 1$	$\vec{j}$	$-y^2 = -1$	$dx\, dz$
S_4 = OAFG	$y = 0$	$-\vec{j}$	$y^2 = 0$	$dx\, dz$
S_5 = DEFG	$z = 1$	$\vec{k}$	$yz = y$	$dx\, dy$
S_6 = OABC	$z = 0$	$-\vec{k}$	$-yz = 0$	$dx\, dy$

$$\therefore \qquad \iint_{S_1} \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 4z\, dy\, dz = 4[y]_0^1 \left[\frac{z^2}{2} \right]_0^1 = 4 \cdot 1 \cdot \frac{1}{2} = 2$$

$$\iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_{S_2} 0\, dy\, dz = 0$$

$$\iint_{S_3} \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 -1\, dx\, dz = -[x]_0^1 [z]_0^1 = -1$$

$$\iint_{S_4} \vec{F} \cdot \vec{n}\, dS = \iint_{S_4} 0\, dx\, dz = 0$$

$$\iint_{S_5} \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 y\, dx\, dy = [x]_0^1 \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}$$

and
$$\iint_{S_6} \vec{F} \cdot \vec{n}\, dS = \iint_{S_6} 0\, dx\, dy = 0$$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = 2+0+(-1)+0+\frac{1}{2}+0=\frac{3}{2} \qquad (2)$$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 6

Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by the planes $x = \pm 1$, $y = \pm 1, z = \pm 1$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $\quad \vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\therefore \qquad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(z)+\frac{\partial}{\partial z}(yz) = 2x+0+y = 2x+y$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x+y)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[x^2+yx\right]_{-1}^{1}dy\,dz = \int_{-1}^{1}\int_{-1}^{1}\left[1+y-(1-y)\right]dy\,dz = \int_{-1}^{1}\int_{-1}^{1} 2y\,dy\,dz = 0$$

$$\left[\int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd function, Here } y \text{ is odd function}\right]$$

$$\Rightarrow \quad \iiint_V \nabla\cdot\vec{F}\,dV = 0 \qquad (1)$$

We shall now compute $\iint_S \vec{F}\cdot\vec{n}\,dS$

S is the surface consisting of the six faces of the cube.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS + \iint_{S_2}\vec{F}\cdot\vec{n}\,dS + \iint_{S_3}\vec{F}\cdot\vec{n}\,dS + \iint_{S_4}\vec{F}\cdot\vec{n}\,dS + \iint_{S_5}\vec{F}\cdot\vec{n}\,dS + \iint_{S_6}\vec{F}\cdot\vec{n}\,dS$$

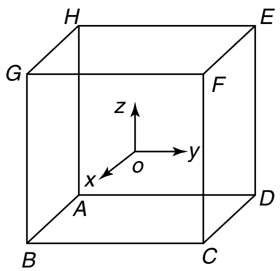


Fig. 9.23

We shall simplify the computations and put it in the form of a table.

Faces	Equation	Outward normal $\vec{n}$	$\vec{F}\cdot\vec{n}$	dS
$S_1 = $ BCFG	$x=1$	$\vec{i}$	$x^2 = 1$	$dy\,dz$
$S_2 = $ ADEH	$x=-1$	$-\vec{i}$	$-x^2 = -1$	$dy\,dz$
$S_3 = $ CDEF	$y=1$	$\vec{j}$	z	$dz\,dx$
$S_4 = $ ABGH	$y=-1$	$-\vec{j}$	$-z$	$dz\,dx$
$S_5 = $ EFGH	$z=1$	$\vec{k}$	$yz = y$	$dx\,dy$
$S_6 = $ ABCD	$z=-1$	$-\vec{k}$	$-yz = y$	$dx\,dy$

$\therefore \quad \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = [y]_{-1}^{1}[z]_{-1}^{1} = (1+1)(1+1) = 4$

$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} -1dy\,dz = -[y]_{-1}^{1}[z]_{-1}^{1} = -[1+1][1+1] = -4$

$\iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} zdz\,dx = 0$ $\quad [\because z$ is odd function$]$

$\iint_{S_4} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} -zdz\,dx = -\int_{-1}^{1}\int_{-1}^{1} zdz\,dx = 0$

$\iint_{S_5} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy = 0$ $\quad [\because y$ is odd function$]$

$\iint_{S_6} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy = 0$

$\therefore \quad \iint_{S} \vec{F}\cdot\vec{n}\,dS = 4-4+0+0+0+0 = 0 \qquad (2)$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 7

Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface of the region, bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0$, $z = 3$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \text{div}\,\vec{F}dV$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k} \quad \therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$

$= 4 - 4y + 2z$

and z varies from 0 to 3,

Also given $x^2 + y^2 = 4$

$\Rightarrow \quad y^2 = 4 - x^2 \quad \Rightarrow \quad y = \pm\sqrt{4-x^2}$

and $\quad y = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad \iiint_V \nabla\cdot\vec{F}dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dz\,dy\,dx$

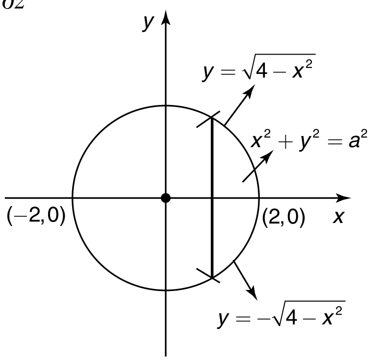


Fig. 9.24

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-4y)z + 2\frac{z^2}{2} \right]_0^3 dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4-4y)\cdot 3 + 9]\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [21-12y]\,dy\,dx$$

$$= \int_{-2}^{2} \left[21y - 12\frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} [21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6(4-x^2-(4-x^2)]\,dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\,dx$$

$$= 84\int_{0}^{2} \sqrt{4-x^2}\,dx \qquad \left[\because \sqrt{4-x^2} \text{ is even function} \right]$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2 = 84\,[0 + 2\sin^{-1}1 - 0] = 84\cdot 2\,\frac{\pi}{2} = 84\pi$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = 84\,\pi \qquad (1)$$

We shall now compute the surface integral $\iint_S \vec{F}\cdot\vec{n}\,dS$.

S consists of the bottom surface S_1, top surface S_2 and the curved surface S_3 of the cylinder.

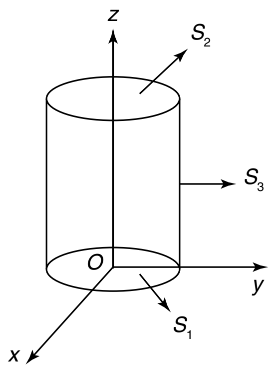


Fig. 9.25

On S_1: Equation is $z = 0$, $\vec{n} = -\vec{k}$

$\therefore \qquad \vec{F}\cdot\vec{n} = -z^2 = 0 \quad \Rightarrow \quad \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = 0$

On S_2: Equation is $z = 3$, $\vec{n} = \vec{k}$

$\therefore \qquad \vec{F}\cdot\vec{n} = z^2 = 9, \quad dS = \dfrac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \dfrac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = dx\,dy$

$\therefore \qquad \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} 9\,dx\,dy = 9\iint_{S_2} dx\,dy$

$= 9$ (area of the circle S_2) $= 9\,\pi\,2^2 = 36\pi$.

On S_3: Equation of the cylinder is $x^2 + y^2 = 4$

Let $\phi = x^2 + y^2$

$$\therefore \quad \nabla\phi = \vec{i}\frac{\partial}{\partial x}\phi + \vec{j}\frac{\partial}{\partial y}\phi + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}\,2x + 2y\vec{j} + 0k = 2(x\vec{i} + y\vec{j})$$

$$\therefore \text{ the normal} \quad \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{4}} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

$$\therefore \quad \vec{F}\cdot\vec{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\frac{1}{2}(x\vec{i} + y\vec{j}) = 2x^2 - y^3$$

Since S_3 is the surface of a cylinder $x^2 + y^2 = 4$, we use cylindrical polar coordinates to evaluate $\iint_{S_3} \vec{F}\cdot\vec{n}\,dS$

$\therefore\ x = 2\cos\theta,\quad y = 2\sin\theta,\quad z = z \quad \therefore \quad dS = 2\,d\theta\,dz$

θ varies from 0 to 2π and z varies from 0 to 3

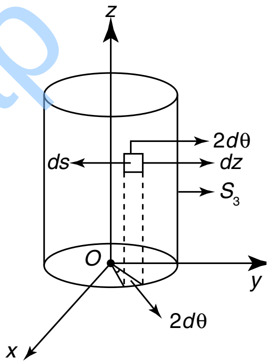


Fig. 9.26

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \int_0^3\int_0^{2\pi} (2\cdot 4\cos^2\theta - 8\sin^3\theta)\,2d\theta\,dz$$

$$= 16\int_0^3\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta\,dz$$

$$= 16\int_0^3\int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}(3\sin\theta - \sin 3\theta)\right] d\theta\,dz$$

$$= 16\int_0^3 \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right]_0^{2\pi} dz$$

$$= 16\int_0^3 \left\{\frac{1}{2}\left[2\pi + \frac{\sin 4\pi}{2} - 0\right] - \frac{1}{4}\left[-3\cos 2\pi + \frac{\cos 6\pi}{3} - \left(-3\cos 0 + \frac{\cos 0}{3}\right)\right]\right\} dz$$

$$= 16\int_0^3 \left(\pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12}\right) dz$$

$$= 16\pi\int_0^3 dz = 16\pi[z]_0^3 = 16\pi\times 3 = 48\pi$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = 36\pi + 48\pi = 84\pi \qquad (2)$$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 8

Verify Gauss divergence theorem for $\vec{F} = a(x + y)\vec{i} + a(y - x)\vec{j} + z^2\vec{k}$ over the region bounded by the upper hemisphere $x^2 + y^2 + z^2 = a^2$ and the plane $z = 0$.

Solution.

Gauss divergence theorem is

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$$

Given $\qquad \vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$

$$\therefore \qquad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(a(y-x)) + \frac{\partial}{\partial z}(z^2) = a + a + 2z = 2(a+z)$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}dV = \iiint_V 2(a+z)\,dV$$

$$= 2a\iiint_V dV + 2\iiint_V z\,dV$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 2a\frac{2\pi}{3}a^3 + \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a^2-x^2-y^2)\,dy\,dx \qquad \left[\because V = \frac{2}{3}\pi a^3\right]$$

$$= \frac{4\pi a^4}{3} + \int_{-a}^{a} 2\int_0^{\sqrt{a^2-x^2}}(a^2-x^2-y^2)\,dy\,dx \qquad [\because a^2-x^2-y^2 \text{ is even in } y]$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)\sqrt{a^2-x^2} - \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)^{3/2} - \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

$$= \frac{4\pi a^4}{3} + 2\cdot\frac{2}{3}\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{4\pi a^4}{3} + \frac{4}{3}\times 2\int_0^a (a^2-x^2)^{3/2}\,dx = \frac{4\pi a^4}{3} + \frac{8}{3}I \qquad [\because (a^2-x^2)^{3/2} \text{ is even}]$$

where $\qquad I = \int_0^a (a^2-x^2)^{3/2}dx$

Put $\qquad x = a\sin\theta \quad \therefore \quad dx = a\cos\theta\,d\theta$

When $\quad x = 0, \quad \sin\theta = 0 \Rightarrow \theta = 0$ and when $x = a, \quad \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^3 \cos^3 \theta \cdot a \cos \theta d\theta$$

$$= a^4 \int_0^{\pi/2} \cos^4 \theta \, d\theta = a^4 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2} = a^4 \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^4}{16}$$

$$\iiint_V \nabla \cdot \vec{F} = \frac{4\pi a^4}{3} + \frac{8}{3} \cdot \frac{3\pi a^4}{16} = \frac{(8+3)}{6} \pi a^4 = \frac{11}{6} \pi a^4 \qquad (1)$$

Now we shall compute the double integral $\iint_S \vec{F} \cdot \vec{n} \, dS$

S consists of S_1 and S_2

$$\therefore \qquad \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_{S_1} \vec{F} \cdot \vec{n} \, dS_1 + \iint_{S_2} \vec{F} \cdot \vec{n} \, dS_2$$

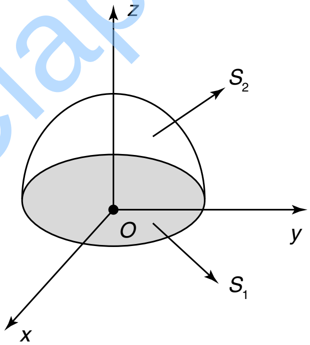


Fig. 9.27

On S_1: $z = 0$, $\vec{n} = -\vec{k}$

$$\therefore \quad \vec{F} \cdot \vec{n} = (a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}) \cdot (-\vec{k}) = -z^2 = 0$$

$$\therefore \quad \iint_{S_1} \vec{F} \cdot \vec{n} \, dS = 0$$

On S_2: $x^2 + y^2 + z^2 = a^2$

Let $\qquad \phi = x^2 + y^2 + z^2$

$$\therefore \qquad \nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$= 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \qquad \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a} \text{ and } \vec{n} \cdot \vec{k} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a} \cdot \vec{k} = \frac{z}{a}$$

$$\vec{F} \cdot \vec{n} = [a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}] \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$= (x+y)x + (y-x)y + \frac{z^3}{a} = x^2 + y^2 + \frac{z^3}{a}$$

$\iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot \vec{n} \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|}$, where R is the projection of S_2 on the xy–plane.

$$\therefore \quad \iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iint_R \left(x^2 + y^2 + \frac{z^3}{a} \right) \frac{dx \, dy}{\frac{z}{a}}$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + z^2 \right) dx \, dy$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + [a^2 - x^2 - y^2] \right) dx \, dy$$

Changing to polar coordinate, we have

$$x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2 + y^2 \quad \text{and} \quad dx\,dy = r\,dr\,d\theta$$

$$\therefore \quad \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \int_0^a\int_0^{2\pi}\left\{\frac{ar^2}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\theta$$

$$= \int_0^a\int_0^{2\pi}\left\{\frac{-a(a^2-r^2)+a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\theta$$

$$= \int_0^a\int_0^{2\pi}\left\{-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^a \left\{-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr$$

$$= [\theta]_0^{2\pi}\int_0^a (-a\sqrt{a^2-r^2})r + a^3(a^2-r^2)^{-1/2} r + (a^2-r^2)\,r]\,dr$$

$$= 2\pi\left\{\int_0^a + \frac{a}{2}(a^2-r^2)(-2r)dr - \frac{a^3}{2}\int_0^a (a^2-r^2)^{-1/2}(-2r)dr + \int_0^a (a^2 r - r^3)dr\right\}$$

$$= 2\pi\left\{\frac{a}{2}\left[\frac{(a^2-r^2)^{3/2}}{\frac{3}{2}}\right]_0^a - \frac{a^3}{2}\left[\frac{(a^2-r^2)^{1/2}}{\frac{1}{2}}\right]_0^a + \left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a\right\}$$

$$= 2\pi\left[\frac{a}{3}(0-a^3) - a^3(0-a) + \frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= 2\pi\left[-\frac{a^4}{3} + a^4 + \frac{a^4}{4}\right] = 2\pi \times \frac{11a^4}{12} = \frac{11\pi a^4}{6}$$

$$\therefore \quad \iint_S \vec{F}\cdot\vec{n}\,dS = 0 + \frac{11\pi a^4}{6} = \frac{11\pi a^4}{6} \qquad (2)$$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 9

Evaluate $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ over the surface $z = 0, z = h, x^2 + y^2 = a^2$.

Solution.

We know Gauss divergence theorem in cartesian form is

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

Given surface integral is $\iint_S x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy$

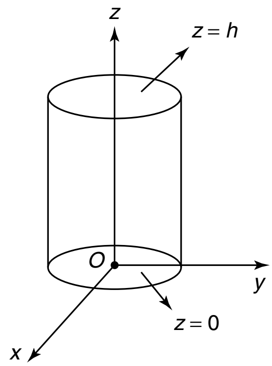


Fig. 9.28

Here $F_1 = x^3, \quad F_2 = x^2 y, \quad F_3 = x^2 z$

$$\therefore \quad \frac{\partial F_1}{\partial x} = 3x^2, \quad \frac{\partial F_2}{\partial y} = x^2, \quad \frac{\partial F_3}{\partial z} = x^2$$

$$\therefore \quad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \iiint_V 5x^2\, dx\, dy\, dz$$

$$= 5\int_{z=0}^{h}\int_{y=-a}^{a}\int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x^2\, dx\, dy\, dz$$

$$= 5\int_{0}^{h}\int_{y=-a}^{a}\left[2\int_{0}^{\sqrt{a^2-y^2}} x^2\, dx\right] dy\, dz \qquad [\because x^2 \text{ is even}]$$

$$= 10\int_{z=0}^{h}\int_{y=-a}^{a}\left[\frac{x^3}{3}\right]_0^{\sqrt{a^2-y^2}} dydz$$

$$= \frac{10}{3}\int_{z=0}^{h}\int_{y=-a}^{a}(a^2-y^2)^{3/2}\, dydz$$

$$= \frac{10}{3}\int_0^h dz\left[2\int_0^a (a^2-y^2)^{3/2}\, dy\right]$$

$$= \frac{20}{3}[z]_0^h\int_0^a (a^2-y^2)^{3/2}\, dy = \frac{20}{3}h\int_0^a (a^2-y^2)^{3/2}\, dy = \frac{20h}{3}\times I$$

where $$I = \int_0^a (a^2-y^2)^{3/2}\, dy$$

Put $y = a\sin\theta \quad \therefore dy = a\cos\theta\, d\theta$

When $y = 0, \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = 0$ and when $y = a, \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}(a^2 - a^2\sin^2\theta)^{3/2} a\cos\theta d\theta = a^4\int_0^{\pi/2}\cos^3\theta\cos\theta d\theta$$

$$= a^4\int_0^{\pi/2}\cos^4\theta d\theta = a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^4}{16}$$

$$\therefore \quad \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \frac{20}{3}h\times\frac{3\pi a^4}{16} = \frac{5}{4}\pi a^4 h$$

9.13 STOKE'S THEOREM

Stoke's theorem gives a relation between **line integral** and **surface integral**.

Theorem 9.1 If S is an open surface bounded by a simple closed curve C and if $\vec{F}$ is continuous having continuous partial derivatives in S and on C, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot \vec{n}\, dS$,

where C is traversed in the positive direction.

Proof Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r}$ be the position vector of any point P on S.

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \quad F\cdot d\vec{r} = \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k}) = F_1dx + F_2dy + F_3dz$$

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \oint_C (F_1dx + F_2dy + F_3dz)$$

Let $z = f(x, y)$ be the equation of the surface S enclosed by the curve C.

Any line parallel to Z-axis intersects the surface in at most one point. The positive direction of the normal $\vec{n}$ is that it makes an acute angle with the positive Z-axis (or $\vec{k}$).

The projection of S on the xy-plane is a region R enclosed by C'.

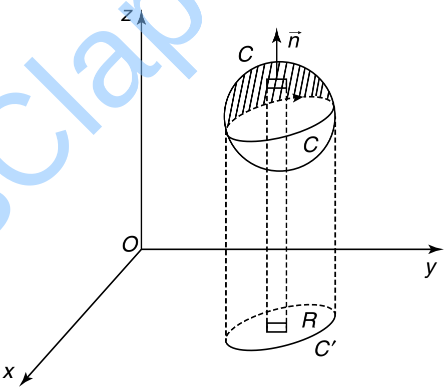


Fig. 9.29

Now,
$$\oint_C F_1dx = \oint_C F_1(x,y,z)dx$$
$$= \oint_{C'} F_1\left((x,y,f(x,y)\right)dx = \oint_{C'} P(x,y)dx$$

where $\quad P(x,y) = F_1\left(x,y\, f(x,y)\right)$

By Green's theorem,

$$\oint_{C'} P(x,y)dx = \iint_R -\frac{\partial P}{\partial y}dx\,dy \qquad [\because Q = 0 \text{ here}]$$

But $\quad P(x,y) = F_1\left(x,y\, f(x,y)\right)$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y} \qquad [\because P(x,y) = F_1(x,y,z) \text{ and } z = f(x,y)] \qquad (1)$$

$$\therefore \quad \oint_{C'} P(x,y)dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y}\right)dx\,dy \qquad (2)$$

Now $\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \iint_S \nabla\times(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot\vec{n}\,dS$

Consider $\iint_S (\nabla \times F_1\vec{i}) \cdot \vec{n}\, dS$

But $$\nabla \times F_1\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}\left(0 - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(0 - \frac{\partial F_1}{\partial y}\right) = \frac{\partial F_1}{\partial z}\vec{j} - \frac{\partial F_1}{\partial y}\vec{k}$$

$\therefore$ $$(\nabla \times F_1\vec{i}) \cdot \vec{n} = \frac{\partial F_1}{\partial z}\vec{j} \cdot \vec{n} - \frac{\partial F_1}{\partial y}\vec{k} \cdot \vec{n} \quad (3)$$

We have $$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + f(x, y)\vec{k} \quad [\text{since } z = f(x, y)]$$

$\therefore$ $$\frac{\partial \vec{r}}{\partial y} = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$$

But $\frac{\partial \vec{r}}{\partial y}$ is a tangent vector to S at P, and hence, $\frac{\partial \vec{r}}{\partial y}$ is $\perp$ to n. $\therefore$ $\frac{\partial \vec{r}}{\partial y} \cdot \vec{n} = 0$

Substituting in (4), we get $\vec{j} \cdot \vec{n} + \frac{\partial f}{\partial y}\vec{k} \cdot \vec{n} = 0 \quad \Rightarrow \quad \vec{j} \cdot \vec{n} = -\frac{\partial f}{\partial y}\vec{k} \cdot \vec{n}$

$\therefore$ (3) $\Rightarrow$ $$\nabla \times F_1\vec{i} \cdot \vec{n} = \frac{\partial F_1}{\partial z}\left(-\frac{\partial f}{\partial y}\vec{k} \cdot \vec{n}\right) - \frac{\partial F_1}{\partial y}\vec{k} \cdot \vec{n} = -\left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)\vec{k} \cdot \vec{n}$$

$\therefore$ $$\iint_S (\nabla \times F_1\vec{i}) \cdot \vec{n}\, dS = -\iint_S \left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)(\vec{k} \cdot \vec{n})\, dS$$

$\Rightarrow$ $$\iint_S (\nabla \times F_1\vec{i}) \cdot \vec{n}\, dS = -\iint_R \left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right) dx\, dy \quad (5)$$

From (2) and (5), we get

$$\oint_{C'} F_1\, dx = \iint_S \nabla \times F_1\vec{i} \cdot \vec{n}\, dS$$

Similarly, $$\oint_{C'} F_2\, dy = \iint_S (\nabla \times F_2\vec{j}) \cdot \vec{n}\, dS \quad (6)$$

and $$\oint_{C'} F_3\, dz = \iint_S (\nabla \times F_3\vec{k}) \cdot \vec{n}\, dS \quad (7)$$

Adding (5), (6), and (7), we get

$$\oint_{C'} F_1 dx + F_2 dy + F_3 dz = \iint_S \nabla \times \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \cdot \vec{n}\, dS$$

$\Rightarrow$ $$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS$$

Note

If S is the region R in the xy–plane, bounded by the simple closed curve C, then $\vec{n} = \vec{k}$ is the outward unit normal.

$\therefore$ Stoke's theorem in the plane is $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{Curl } \vec{F} \cdot \vec{k}\, dR$,

which is Green's theorem.

Cartesian form of Stoke's theorem

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

and $\vec{F} \cdot d\vec{r} = F_1\, dx + F_2\, dy + F_3\, dz$

$\therefore$ the cartesian form of Stoke's theorem is $\oint_C (F_1\, dx + F_2\, dy + F_3\, dz)$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dzdx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy\right]$$

Note

If $\vec{F} = P\vec{i} + Q\vec{j}$ and $\vec{r} = x\vec{i} + y\vec{j}$, then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$ and $\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$\therefore$ $\text{Curl } \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\therefore$ Stokes theorem in the plane is $\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

which is Green's thorem.

WORKED EXAMPLES

EXAMPLE 1

Prove that $\oint_C \vec{r} \cdot d\vec{r} = 0$, where C is the simple closed curve.

Solution.

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on C. $\therefore$ $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\cdot$

Stokes theorem is $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, dS$

Here $\vec{F} = \vec{r}$.

$$\therefore \qquad \text{Curl } \vec{F} = \text{Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) + (0-0) = \vec{0}$$

$$\therefore \qquad \oint_C \vec{r}\cdot d\vec{r} = 0$$

EXAMPLE 2

If A is solenoidal, then prove that $\iint_S \nabla^2 \vec{A}\cdot\vec{n} dS = -\oint_C \text{Curl } \vec{A}\cdot d\vec{r}$.

Solution.

Given $\vec{A}$ is solenoidal. $\quad \therefore \quad \nabla\cdot\vec{A} = 0$

We know $\qquad \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A}$

Stoke's theorem is $\qquad \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$

Putting $\vec{F} = \nabla\times\vec{A}$, we get $\nabla\times\vec{F} = -\nabla^2\vec{A}$

$$\therefore \qquad \iint_S -\nabla^2\vec{A}\cdot\vec{n}\, dS = \oint_C \nabla\times\vec{A}\cdot d\vec{r}$$

$$\Rightarrow \qquad \iint_S \nabla^2\vec{A}\cdot\vec{n}\, dS = -\oint_C \text{Curl } \vec{A}\cdot d\vec{r}$$

EXAMPLE 3

Prove that $\oint_C \phi\, d\vec{r} = -\iint_S \nabla\phi\times\vec{n}\, dS$.

Solution.

Stoke's theorem is $\qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS$

Put $\vec{F} = \phi\, \vec{a}$, where $\vec{a}$ an arbitrary constant vector.

$$\therefore \qquad \oint_C (\phi\vec{a})\cdot d\vec{r} = \iint_S \nabla\times\phi\vec{a}\cdot\vec{n}\, dS$$

We know curl $\qquad \phi\vec{a} = \nabla\times\phi\vec{a} = \nabla\phi\times\vec{a} + \phi\nabla\times\vec{a} = \nabla\phi\times\vec{a} \qquad [\because \nabla\times\vec{a} = \vec{0}]$

$$\therefore \qquad \oint_C (\phi\vec{a})\cdot d\vec{r} = \iint_S (\nabla\phi\times\vec{a})\cdot\vec{n}\, dS$$

$\Rightarrow \quad \oint_C \phi \vec{a} \cdot d\vec{r} = -\iint_S (\vec{a} \times \nabla \phi) \cdot \vec{n}\, dS$

$\Rightarrow \quad \vec{a} \cdot \left(\oint_C \phi\, d\vec{r} \right) = -\iint_S \vec{a} \cdot (\nabla \phi \times \vec{n})\, dS$ [Interchanging dot and cross]

$\Rightarrow \quad \vec{a} \cdot \left(\oint_C \phi\, d\vec{r} \right) = -\vec{a} \cdot \iint_S \nabla \phi \times \vec{n}\, dS = \vec{a} \cdot \left(-\iint_S \nabla \phi \times \vec{n}\, dS \right)$

$\therefore \quad \oint_C \phi\, d\vec{r} = -\iint_S \nabla \phi \times \vec{n}\, dS$ [$\because$ $\vec{a}$ is arbitrary]

EXAMPLE 4

If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, then show that $\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = 0$.

Solution.

Suppose the sphere is cut by a plane into two parts S_1 and S_2 and let C be the curve binding these two parts.

Then $\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS$

By Stoke's theorem, $\iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$

and $\iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS = -\oint_C \vec{F} \cdot d\vec{r}$, because for S_2

the positive sense of the curve C is the opposite direction of C in S_1

$\therefore \quad \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} = 0$

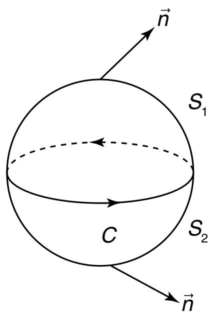


Fig. 9.30

EXAMPLE 5

Evaluate $\int_C (xy\,dx + xy^2 dy)$ by Stoke's theorem, where C is the square in the xy–plane with vertices (1, 0), (−1, 0), (0, 1), (0, −1).

Solution.

Stoke's theorem is $\oint_C \vec{F} \cdot dr = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS$

Given $\int_C (xy\,dx + xy^2 dy)$ and $\vec{r} = x\vec{i} + y\vec{j}$ $\quad \therefore \quad d\vec{r} = dx\vec{i} + dy\vec{j}$.

Here $\vec{F} \cdot d\vec{r} = xy\,dx + xy^2 dy \quad \therefore \quad \vec{F} = xy\,\vec{i} + xy^2\,\vec{j}$

$$\therefore \quad \text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y^2 - x)$$

$$\Rightarrow \quad \text{Curl } \vec{F} = (y^2 - x)\vec{k}$$

Also given C is the square in the xy plane with vertices (1, 0), (–1, 0), (0, 1), (0, –1).

$$\therefore \quad \vec{n} = \vec{k} \text{ and } dS = dx\, dy$$

$$\therefore \quad \text{Curl } \vec{F} \cdot \vec{n} = (y^2 - x)\vec{k} \cdot \vec{k} = y^2 - x$$

$$\therefore \quad \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_R (y^2 - x)\, dx\, dy$$

where R is the region inside the square.

That is $\displaystyle \int_C xy\, dx + xy^2 dy = \iint_R (y^2 - x)\, dx\, dy$

We shall now evaluate this double integral.

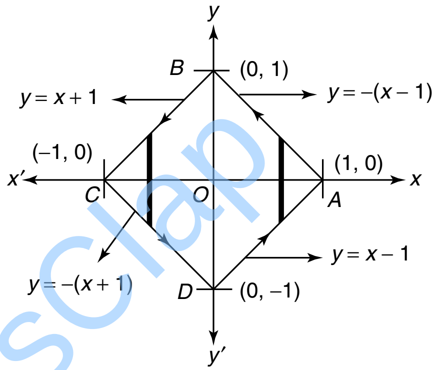


Fig. 9.31

Equation of AB in intercept form is

$$\frac{x}{1} + \frac{y}{1} = 1 \quad \Rightarrow \quad x + y = 1 \quad \Rightarrow \quad y = -x + 1 \quad \Rightarrow \quad y = -(x-1)$$

Equation of BC is $\displaystyle \frac{x}{-1} + \frac{y}{1} = 1 \quad \Rightarrow \quad y - x = 1 \quad \Rightarrow \quad y = x + 1$

Equation of CD is $\displaystyle \frac{x}{-1} + \frac{y}{-1} = 1 \quad \Rightarrow \quad x + y = -1 \quad \Rightarrow \quad y = -(x + 1)$

Equation of AD is $\displaystyle \frac{x}{1} + \frac{y}{-1} = 1 \quad \Rightarrow \quad y - x = -1 \quad \Rightarrow \quad y = x - 1$

$$\therefore \int_C (xy\, dx + xy^2 dy) = \int_{-1}^{0} \int_{-(x+1)}^{x+1} (y^2 - x)\, dy dx + \int_0^1 \int_{x-1}^{-(x-1)} (y^2 - x)\, dy dx$$

$$= \int_{-1}^{0} \left[\frac{y^3}{3} - xy\right]_{-(x+1)}^{x+1} dx + \int_0^1 \left[\frac{y^3}{3} - xy\right]_{x-1}^{-(x-1)} dx$$

$$= \int_{-1}^{0} \frac{1}{3}\{[(x+1)^3 - (-(x+1))^3] - x[x + 1 - (-(x+1))]\}\, dx$$

$$+ \int_0^1 \left\{\frac{1}{3}[(-(x-1))^3 - (x-1)^3] - x[-(x-1) - (x-1)]\right\} dx$$

$$= \int_{-1}^{0}\left\{\frac{1}{3}[(x+1)^3+(x+1)^3]-x[(x+1)+(x+1)]\right\}dx$$

$$+\int_{0}^{1}\left\{-\frac{1}{3}[(x-1)^3+(x-1)^3]+x[x-1+x-1]\right\}dx$$

$$= \int_{-1}^{0}\left[\frac{2}{3}(x+1)^3-2x(x+1)\right]dx+\int_{0}^{1}\left[-\frac{2}{3}(x-1)^3+2x(x-1)\right]dx$$

$$= \left[\frac{2}{3}\frac{(x+1)^4}{4}-2\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\right]_{-1}^{0}+\left[-\frac{2}{3}\frac{(x-1)^4}{4}+2\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\right]_{0}^{1}$$

$$= \frac{2}{3}\left(\frac{1}{4}\right)-2\left\{0-\left[\frac{1}{3}(-1)^3+\frac{(-1)^2}{2}\right]\right\}-\frac{2}{3}\left[0-\left(\frac{1}{4}\right)\right]+2\left[\frac{1}{3}-\frac{1}{2}\right]$$

$$= \frac{1}{6}-\frac{2}{3}+1+\frac{1}{6}+\frac{2}{3}-1=\frac{2}{6}=\frac{1}{3}$$

EXAMPLE 6

Evaluate $\int_C [(x+y)dx+(2x-z)dy+(y+z)dz]$ where C is the boundary of the triangle with the vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6), using Stoke's theorem.

Solution.

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS,$$

where S is the surface of the triangle ABC bounded by the curve C, consisting of the sides of the triangle in the figure.

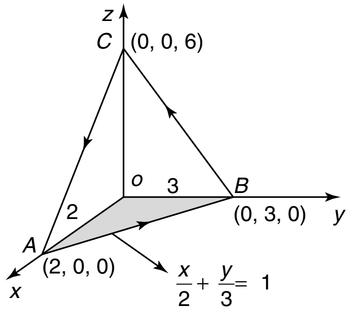


Fig. 9.32

Given $\vec{F}\cdot d\vec{r} = (x+y)dx+(2x-z)dy+(y+z)dz$

Here $\vec{F} = (x+y)\vec{i}+(2x-z)\vec{j}+(y+z)\vec{k}$

$$\therefore \quad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y+z)-\frac{\partial}{\partial z}(2x-z)\right]-\vec{j}\left[\frac{\partial}{\partial x}(y+z)-\frac{\partial}{\partial z}(x+y)\right]+\vec{k}\left[\frac{\partial}{\partial x}(2x-z)-\frac{\partial}{\partial y}(x+y)\right]$$

$$= \vec{i}[1-(-1)]-\vec{j}[0-0]+\vec{k}(2-1)] = 2\vec{i}+\vec{k}$$

Equation of the plane ABC is $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$ [intercept form]

$\therefore \qquad \phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} \quad , \quad \frac{\partial \phi}{\partial x} = \frac{1}{2} \quad , \quad \frac{\partial \phi}{\partial y} = \frac{1}{3} \quad , \quad \frac{\partial \phi}{\partial z} = \frac{1}{6}$

$\therefore \qquad \nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} = \frac{1}{2}\vec{i} + \frac{1}{3}\vec{j} + \frac{1}{6}\vec{k} = \frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})$

$\therefore \qquad \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})}{\frac{1}{6}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k})$

$\therefore \qquad \text{Curl } \vec{F} \cdot \vec{n} = (2\vec{i} + \vec{k}) \cdot \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k}) = \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}}$

$\therefore \qquad \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_S \frac{7}{\sqrt{14}} dS = \frac{7}{\sqrt{14}} \iint_R \frac{dx\,dy}{|\vec{n} \cdot \vec{k}|}$

where R is the orthogonal projection of S on the xy–plane.

But $\qquad \vec{n} \cdot \vec{k} = \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k}) \cdot \vec{k} = \frac{1}{\sqrt{14}}$

$\therefore \qquad \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \frac{7}{\sqrt{14}} \iint_R \frac{dx\,dy}{\frac{1}{\sqrt{14}}}$

$$= 7\iint_R dx\,dy = 7 \times \text{Area of } \Delta OAB = 7 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 21$$

$\therefore \qquad \oint_C [(x+y)dx + (2x-z)dy + (y+z)dz] = 21.$

EXAMPLE 7

Using Stoke's theorem, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0), (1, 1, 0).

Solution.

Given $\qquad \vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS$$

Now $\qquad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2)\right]\vec{i} - \left[\frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2)\right]\vec{j}$$
$$+ \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]\vec{k}$$
$$= (0)\,\vec{i} - [-1]\,\vec{j} + [2x - 2y]\,\vec{k} = \vec{j} + 2\,(x-y)\,\vec{k}.$$

Given C is the boundary of the triangle formed by the points (0, 0, 0), (1, 0, 0) and (1, 1, 0) which lie in the xy–plane. $\quad\therefore\quad \vec{n} = \vec{k}$

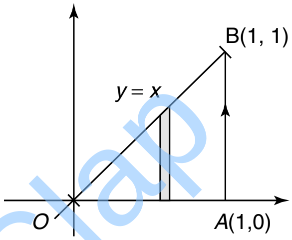


Fig. 9.33

$\therefore \qquad \text{curl}\,\vec{F}\cdot\vec{n} = 2\,(x-y)$

$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S 2\,(x-y)\,dx\,dy$

Equation of OB is $y = x$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = 2\int_0^1\int_0^x (x-y)\,dy\,dx$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2} - 0\right] dx = 2\int_0^1 \frac{x^2}{2}\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

EXAMPLE 8

Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$, where S is the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0$ and $z = 2$ above the xy–plane.

Solution.

Given $\qquad \vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}.$

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$$

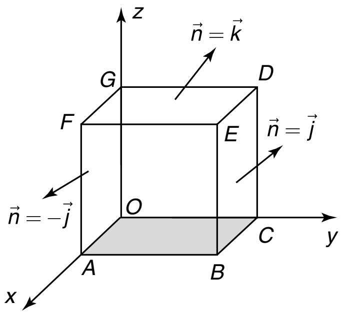


Fig. 9.34

Now $\qquad \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right]$$
$$= \vec{i}[(0-y)] - \vec{j}[-z-(-1)] + \vec{k}(0-1) = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

We shall compute $\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS$.

Given S is the open surface consisting of 5 faces of the cube except the face $OABC$.

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n}\, dS$$
$$+ \iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	dS
S_1 = ABEF	$x = 2$	$\vec{i}$	$-y$	$dy\,dz$
S_2 = OCDG	$x = 0$	$-\vec{i}$	y	$dy\,dz$
S_3 = BCDE	$y = 2$	$\vec{j}$	$z-1$	$dx\,dz$
S_4 = OAFG	$y = 0$	$-\vec{j}$	$-(z-1)$	$dx\,dz$
S_5 = DEFG	$z = 2$	$\vec{k}$	-1	$dx\,dy$

$\therefore$
$$\iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -y\, dy\, dz = \int_0^2 dz \cdot \int_0^2 (-y)\, dy = [z]_0^2 \left[\frac{-y^2}{2}\right]_0^2 = 2(-2) = -4$$

$$\iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 y\, dy\, dz = \int_0^2 dz \int_0^2 y\, dy = [z]_0^2 \left[\frac{y^2}{2}\right]_0^2 = 2 \cdot 2 = 4$$

$$\iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 (z-1)\, dz\, dx = \int_0^2 dx \cdot \int_0^2 (z-1)\, dz = [x]_0^2 \cdot \left[\frac{(z-1)}{2}\right]_0^2$$
$$= 2 \cdot \frac{1}{2}\left\{(2-1)^2 - (-1)^2\right\} = 1 - 1 = 0$$

$$\iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -(z-1)\, dz\, dx = 0 \qquad \text{[as above]}$$

and
$$\iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -1\, dx\, dy = -[x]_0^2 [y]_0^2 = -4$$

$\therefore$
$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = -4 + 4 + 0 + 0 - 4 = -4 \qquad (1)$$

We shall now compute the line integral over the simple closed curve C bounding the surface consisting of the edges OA, AB, BC and CO in $z = 0$ plane

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

Now
$$\vec{F}\cdot d\vec{r} = \left[(y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}\right]\cdot\left[dx\vec{i} + dy\vec{j} + dz\vec{k}\right]$$
$$= (y-z+2)dx + (yz+4)dy - xzdz$$

$$\Rightarrow \quad \vec{F}\cdot d\vec{r} = (y+2)dx + 4dy \qquad [\because z = 0]$$

On OA: $y = 0$ $\quad\therefore\ dy = 0$ and $\vec{F}\cdot d\vec{r} = 2dx$ and x varies from 0 to 2

$$\therefore \quad \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 2dx = 2[x]_0^2 = 4$$

On AB: $x = 2$ $\quad\therefore\ dx = 0$ and $\vec{F}\cdot d\vec{r} = 4dy$ and y varies from 0 to 2

$$\therefore \quad \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4dy = 4[y]_0^2 = 8$$

On BC: $y = 2$ $\quad\therefore\ dy = 0$ and $\vec{F}\cdot d\vec{r} = 4dx$ and x varies from 2 to 0

$$\therefore \quad \int_{BC} \vec{F}\cdot d\vec{r} = \int_2^0 4dx = 4[x]_2^0 = 4(-2) = -8$$

On CO: $x = 0$ $\quad\therefore\ dx = 0$, $\vec{F}\cdot d\vec{r} = 4dy$ and y varies from 2 to 0

$$\therefore \quad \int_{CO} \vec{F}\cdot d\vec{r} = \int_2^0 4dy = 4[y]_2^0 = -8$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \qquad (2)$$

From (1) and (2), $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\ \vec{F}\cdot\vec{n}\ dS$

Hence, Stoke's theorem is verified.

EXAMPLE 9

Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution.
Given $\quad \vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\ \vec{F}\cdot\vec{n}\ dS$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) = 4y\vec{k}$$

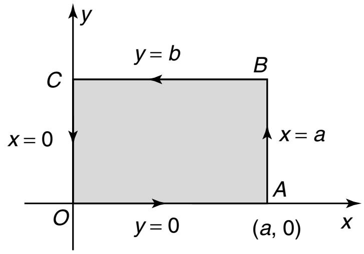


Fig. 9.35

Since the surface is a rectangle in the xy–plane, normal $\vec{n} = \vec{k}$

$$\therefore \quad \text{Curl } \vec{F} \cdot \vec{n} = 4y\vec{k} \cdot \vec{k} = 4y$$

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n} \, dS = \int_0^a \int_0^b 4y \, dx \, dy$$

$$\Rightarrow \quad \iint_S \text{Curl } \vec{F} \cdot \vec{n} \, dS = \int_0^a dx \int_0^b 4y \, dy = [x]_0^a \, 4\left[\frac{y^2}{2}\right]_0^b = 2ab^2 \qquad (1)$$

We shall now compute the line integral.

$$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

Now $\qquad \vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xydy$

On OA: $y = 0$ $\qquad \therefore \; dy = 0$ and $\vec{F} \cdot d\vec{r} = x^2 dx$ and x varies from 0 to a

$$\therefore \quad \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a$ $\qquad \therefore \; dx = 0$ and $\vec{F} \cdot d\vec{r} = 2aydy$ and y varies from 0 to b

$$\therefore \quad \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2aydy = 2a\left[\frac{y^2}{2}\right]_0^b = ab^2$$

On BC: $y = b$ $\qquad \therefore \; dy = 0$ and $\vec{F} \cdot d\vec{r} = (x^2 - b^2)dx$ and x varies from a to 0

$$\therefore \quad \int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 - b^2)dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = 0 - \left(\frac{a^3}{3} - b^2 a\right) = ab^2 - \frac{a^3}{3}$$

On CO: $x = 0$ $\qquad \therefore \; dx = 0$ and $\vec{F} \cdot d\vec{r} = 0$

$$\therefore \quad \int_{CO} \vec{F} \cdot d\vec{r} = 0$$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2 \qquad (2)$$

From (1) and (2), $\qquad \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n} \, dS$

Hence, Stoke's theorem is verified.

Note Stoke's theorem in the plane is Green's theorem. This is indeed Green's theorem verification.

EXAMPLE 10

Verify Stoke's theorem for the vector field $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over the upper half surface $x^2 + y^2 + z^2 = 1$, bounded by its projections on the xy–plane.

Solution.

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, dS$$

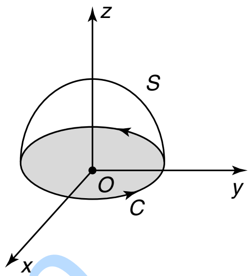


Fig. 9.36

Given $\quad \vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

$$\therefore \qquad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x - y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x - y)\right]$$

$$= \vec{i}[-2yz + 2yz] - \vec{j}[0 - 0] + \vec{k}[0 - (-1)] = \vec{k}$$

$$\therefore \qquad \vec{F}\cdot\vec{n} = \vec{k}\cdot\vec{n}$$

The surface is the upper hemisphere $x^2 + y^2 + z^2 = 1$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S \vec{k}\cdot\vec{n}\, dS = \iint_R \vec{k}\cdot\vec{n}\, \frac{dxdy}{|\vec{k}\cdot\vec{n}|},$$

where R is the projection of S on the xy–plane.

$\therefore$ R is the circle $x^2 + y^2 = 1$ in the xy–plane.

$$\therefore \qquad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_R dx\, dy$$

$$\Rightarrow \qquad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \text{area of the circle} = \pi \cdot 1^2 = \pi \qquad (1)$$

Now C is the circle $x^2 + y^2 = 1$ in the $z = 0$ plane.

Parametric equations are $\quad x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$

$$\therefore \qquad \int_C \vec{F}\cdot d\vec{r} = \int_C [(2x - y)dx - yz^2dy - y^2zdz] = \oint_C (2x - y)dx \qquad [\because\ z = 0]$$

Now $\qquad x = \cos\theta \quad \Rightarrow \quad dx = -\sin\theta\, d\theta$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$\Rightarrow \quad \oint_C \vec{F}\cdot d\vec{r} = \frac{1}{2}\left[(\cos 4\pi - \cos 0) + 2\pi - \frac{\sin 4\pi}{2} - 0\right] = \frac{1}{2}[1 - 1 + 2\pi] = \pi \qquad (2)$$

From (1) and (2), $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS$

Hence, Stoke's theorem is verified.

EXAMPLE 11

Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0$ and $y = b$.

Solution.

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS$$

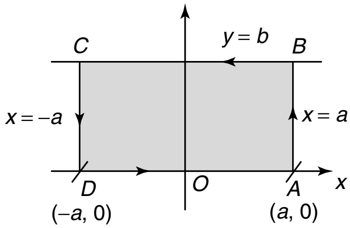


Fig. 9.37

Given $\qquad \vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$$\therefore \qquad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0 - 0] - \vec{j}(0 - 0) + \vec{k}(-2y - 2y) = -4y\vec{k}$$

Since S is the rectangular surface, $\vec{n} = \vec{k}$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S -4y\vec{k}\cdot\vec{k}\, dx\, dy$$

$$= -4\int_0^b \int_{-a}^{a} y\, dx\, dy = -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^{a} = -2b^2 \cdot 2a = -4ab^2$$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = -4ab^2 \qquad (1)$$

We shall now compute the line integral $\oint_C \vec{F}\cdot d\vec{r}$.

Now $\qquad \vec{F}\cdot d\vec{r} = [(x^2 + y^2)\vec{i} - 2xy\vec{j}]\cdot[dx\vec{i} + dy\vec{j}] = (x^2 + y^2)dx - 2xy\, dy$

$\therefore$ $$\oint_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

On AB: $x = a$ $\therefore$ $dx = 0$ and $\vec{F}\cdot d\vec{r} = -2ay\,dy$ and y varies from 0 to b

$\therefore$ $$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b (-2a)y\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

On BC: $y = b$ $\therefore$ $dy = 0$ and $\vec{F}\cdot d\vec{r} = (x^2 + b^2)dx$ and x varies from a to $-a$

$\therefore$ $$\int_{BC} F\cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^{-a}$$

$$= \frac{1}{3}(-a^3 - a^3) + b^2(-a - a) = \frac{-2}{3}a^3 - 2ab^2$$

On CD: $x = -a$ $\therefore$ $dx = 0$ and $\vec{F}\cdot d\vec{r} = 2aydy$ and y varies from b to 0

$\therefore$ $$\int_{CD} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = a(0 - b^2) = -ab^2$$

On DA: $y = 0$ $\therefore$ $dy = 0$ and $\vec{F}\cdot d\vec{r} = x^2 dx$ and x varies from $-a$ to a

$\therefore$ $$\int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx = 2\int_0^a x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^a = \frac{2}{3}a^3$$

$\therefore$ $$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2}{3}a^3 - 2ab^2 - ab^2 + \frac{2}{3}a^3 = -4ab^2$$

From (1) and (2), $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\,dS$

Hence, Stoke's theorem is verified.

EXAMPLE 12

Verify stokes theorem for $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$, where S is the open surface of the cube formed by the planes $x = -a, x = a, y = -a, y = a, z = -a, z = a$ in which $z = -a$ is cut open.

Solution.

Stoke's theorem is $\oint \vec{F}.\,d\vec{r} = \iint_S \text{curl } \vec{F}.\vec{n}\,ds$

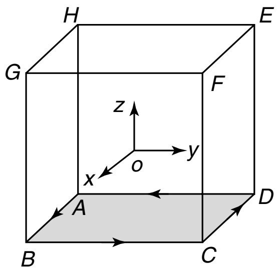


Fig. 9.38

Given $$\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$$

$\therefore$ $$\text{Curl F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2 y) - \frac{\partial}{\partial z}(z^2 x)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial z}(y^2 z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(z^2 x) - \frac{\partial}{\partial y}(y^2 z)\right]$$

$$= (x^2 - 2zx)\vec{i} + (y^2 - 2xy)\vec{j} + (z^2 - 2yz)\vec{k}$$

We shall now compute $\iint_S curl\ \vec{F}.\ \vec{n}\ dS$

Given S is the open surface consisting of the five faces of the cube except face $ABCD$

$$\therefore \iint_S \text{curl } \vec{F}.\vec{n}dS = \iint_{S_1} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_2} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_3} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_4} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_5} \text{curl } \vec{F}.\vec{n}dS$$

Face	Equation	Normal $\vec{n}$	Curl $\vec{F}.\vec{n}$	dS
$S_1 = BCFG$	$x = a$	$\vec{i}$	$a^2 - 2az$	$dy\ dz$
$S_2 = ADEH$	$x = -a$	$-\vec{i}$	$-(a^2 + 2az)$	$dy\ dz$
$S_3 = CDEF$	$y = a$	$\vec{j}$	$a^2 - 2ax$	$dz\ dx$
$S_4 = ABGH$	$y = -a$	$-\vec{j}$	$-(a^2 + 2ax)$	$dz\ dx$
$S_5 = EFGH$	$z = a$	$\vec{k}$	$a^2 - 2ay$	$dx\ dy$

$$\iint_{S_1} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a}(a^2 - 2az)dydz$$

$$= \left[\int_{-a}^{a} dy\right]\left[\int_{-a}^{a} a^2 - 2az\, dydz\right] = [y]_{-a}^{a}\left[a^2 z - 2a\frac{z^2}{2}\right]_{-a}^{a} = [a+a]\left[a^2(a+a) - a(a^2 - a^2)\right] = 4a^4$$

$$\iint_{S_2} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az)dydz = -\left[\int_{-a}^{a} dy\right]\left[\int_{-a}^{a} a^2 + 2az\right]dz$$

$$= -[y]_{-a}^{a}\left[a^2 z + 2a\frac{z^2}{2}\right]_{-a}^{a} = -[a+a]\left[a^2(a+a) + a(a^2 - a^2)\right] = -4a^4$$

Similarly, $\iint_{S_3} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a}(a^2 - 2ax)dzdx = 4a^4$

$$\iint_{S_4} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2ax)dzdx = -4a^4$$

and $\iint_{S_5} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a}(a^2 - 2ay)\,dxdy = 4a^4$

$$\therefore \quad \iint_S \text{curl } \vec{F}.\vec{n}dS = 4a^4 - 4a^4 + 4a^4 - 4a^4 + 4a^4 = 4a^4 \qquad (1)$$

We shall now compute the line integral over the simple closed curve C consisting of the edges AB, BC, CD, DA. Here $z = -a$, $dz = 0$

$$\therefore \quad \vec{F}.d\vec{r} = y^2 zdx + z^2 xdy + x^2 ydz = -ay^2 dx + a^2 xdy$$

On AB: $y = -a \therefore dy = 0$

$\vec{F}.d\vec{r} = -a^3 dx$ and x varies from $-a$ to a.

$$\therefore \quad \int_{AB} \vec{F}.d\vec{r} = \int_{-a}^{a} -a^3 dx = -a^3 [x]_{-a}^{a} = -a^3 .2a = -2a^4$$

On BC: $x = a \therefore dx = 0$, $\vec{F}.d\vec{r} = a^3 dy$ and y varies from $-a$ to a.

$$\therefore \quad \int_{BC} \vec{F}.d\vec{r} = \int_{-a}^{a} a^3\, dy = a^3 [r]_{-a}^{a} = a^3 .2\, a = 2\, a^4$$

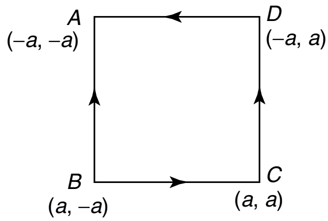


Fig. 9.39

On CD: $y = a \therefore dy = 0$, $\vec{F}.d\vec{r} = -a^3 dx$ and x varies from a to $-a$

$$\therefore \quad \int_{CD} \vec{F}.d\vec{r} = \int_{a}^{-a} -a^3 dx = -a^3 [x]_{a}^{-a} = -a^3(-2a) = 2a^4$$

On DA: $x = -a \therefore dx = 0$, $\vec{F}.d\vec{r} = -a^3 dy$ and y varies from a to $-a$.

$$\therefore \quad \int_{DA} \vec{F}.d\vec{r} = \int_{a}^{-a} -a^3 dy = -a^3 [y]_{a}^{-a} = -a^2(-2a) = 2a^4 \qquad (2)$$

$$\therefore \quad \oint_{c} \vec{F}\cdot d\vec{r} = -2\,a^4 + 2\,a^4 + 2\,a^4 + 2\,a^4 = 4\,a^4 \qquad (3)$$

From (1) and (2), we get $\iint_S \text{curl } \vec{F}.\vec{n} ds = \oint_C \vec{F}.d\vec{r}$

Hence, Stoke's theorem is verified.

EXERCISE 9.4

1. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = 12x^2 y\vec{i} - 3yz\vec{j} + 2z\vec{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.
2. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = (2x^2 - 3z)\vec{i} + 2y\vec{j} - 4xz\vec{k}$, where S is the surface of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.
3. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the curved surface of the cylinder $x^2 + y^2 = 1$ included in the first octant between the planes $z = 0$ and $z = 2$.
4. If $\vec{F} = xy^2\vec{i} - yz^2\vec{j} + zx^2\vec{k}$, find $\iint_S \vec{F}\cdot\vec{n}\, dS$ over the sphere $x^2 + y^2 + z^2 = 1$.
5. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

6. Evaluate $\oint_C (x^2+xy)dx+(x^2+y^2)dy$, where C is the square formed by the lines $y=\pm 1$, $x=\pm 1$, by Green's theorem.
7. Using Green's theorem evaluate $\oint_C (x^2+y)dx - xy^2\,dy$ taken around the square whose vertices are (0, 0), (1, 0), (1, 1) (0, 1)
8. Using Green's theorem find the value of $\int_C (xy-x^2)dx + x^2y\,dy$ along the closed curve C formed by $y=0$, $x=1$ and $y=x$.
9. Verify Green theorem for $\int_C (15x^2-4y^2)dx+(2y-3x)dy$, where C is the curve enclosing the area bounded by $y=x^2$, $x=y^2$
10. Verify Green theorem in the plane for $\int_C (3x^2-8y^3)dx+(4y-6xy)dy$, where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$.
11. Using Green's theorem find the area of $x^{2/3}+y^{2/3}=a^{2/3}$.
    [**Hint:** Area $=\frac{1}{2}\int_C (xdy-ydx)$, C is the boundary of the curve]
12. Using Green's theorem in xy plane find the area of the region in the xy plane bounded by $y^3=x^2$ and $y=x$.
13. Using Green's theorem evaluate $\int_C (2x^2-y^2)dx+(x^2+y^2)dy$, where C is the boundary of the area in the xy plane bounded by x–axis and the semi circle $x^2+y^2=1$ in the upper half of the plane.
14. Verify Gauss divergence theorem for $\vec{F}=x^2\vec{i}+z\vec{i}+yz\vec{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.
15. Verify Gauss divergence theorem for $\vec{F}=(x^3-yz)\vec{i}-2x^2y\vec{j}+2\vec{k}$ over the parallelopiped bounded by the planes $x=0$, $x=1$, $y=0$, $y=2$, $y=2$, $z=0$, $z=3$.
16. Verify Gauss divergence theorem for $\vec{F}=x^2\vec{i}+z\vec{j}+yz\vec{k}$ over a unit cube.
17. Verify Gauss divergence theorem for $\vec{F}=(x^3-yz)\vec{i}-zx^2y\vec{j}+2\vec{k}$ over the cube $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.
18. Verify the divergence theorem for $\vec{F}=2xy\vec{i}+yz^2\vec{j}+xz\vec{k}$, where S is the rectangular parallelopiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$, $z=3$.
19. Using divergence theorem show that
    $$\iint_S x^2dy+y^2dzdy+2z(xy-x-y)dxdy=\frac{1}{2}$$, where S is the surface of the cube $x=y=z=0$, $y=z=1$.
20. Use divergence theorem to evaluate $\iint_S (2xy\vec{i}+yz^2\vec{j}+xz\vec{k}).d\vec{S}$, where S is the surface of the region bounded by $x=y=z=0$, $y=3$, $x+2z=6$.
21. Prove that $\iint_S [x(y-z)\vec{i}+y(z-x)\vec{j}+z(x-y)\vec{k}]\cdot d\vec{S}=0$, where S is any closed surface.

22. Verify Stoke's theorem for $\vec{F} = 2z\vec{i} + x\vec{j} + y^2\vec{k}$, where S is the surface of the paraboloid $z = 4 - x^2 - y^2$ and C is the simple closed curve in the xy plane.
23. Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C its boundary.
24. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j} + xyz\vec{k}$ over the surface of the box bounded by the planes $x = 0, y = 0, x = a, y = b, z = c$ above the xy plane.
25. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by $x = 0, x = a, y = 0, y = b$.
26. Verify Stoke's theorem for $\vec{F} = -y^3\vec{i} + x^3\vec{j}$ and the closed curve C is the boundary of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
27. If ϕ is scalar point function, use Stoke's theorem to prove curl (grad ϕ) = 0.
28. Evaluate $\iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS$, where S is the surface $x^2 + y^2 + z^2 = a^2$ above the xy–plane and $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$.
29. Evaluate $\int_C yz\,dx + zx\,dy + xy\,dz$, where C is the curve $x^2 + y^2 = 1, z = y^2$.
30. Evaluate $\iint \nabla \times \vec{F} \cdot \vec{n}\, dS$ for $\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$ over the surface of the cylinder $x^2 + y^2 = 4$, bounded by the plane $z = 9$ and open at the end $z = 0$.
31. Find the area of a circle of radius a using Green's theorem.
32. Using Green's theorem evaluate $\oint_C [(2xy - x^2)dx + (x^2 + y^2)dy]$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$
33. Verify Green's theorem in a plane for the integral $\int_C (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 4$.
34. Verify Green's theorem in the plane for $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where C is the square with vertices (0, 0) (2, 0), (2, 2), (2, 0).
35. Evaluate $\iiint_V \nabla \cdot \vec{F}\, dV$ if $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the volume of the region enclosed by the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
36. If S is any closed surface enclosing volume V and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ prove that $\iint_S \vec{F} \cdot \vec{n}\, dS = (a + b + c)V$
37. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped bounded by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.
38. Verify Stoke's theorem for $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$ where S is the open surface of the cube formed by the planes $x = -a, x = a, y = -a, y = a, z = -a, z = a$ in which $z = -a$ is cut open.

39. Evaluate $\iint_S \text{Curl } \vec{F}\cdot\vec{n}\,dS$, where $\vec{F} = (y-z)\vec{i} + yz\,\vec{j} - xz\,\vec{k}$ and S is the open surface bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ above the xy plane.

ANSWERS TO EXERCISE 9.4

1. $\frac{49}{120}$	2. $\frac{16}{3}$	3. 3	4. $\frac{4}{3}\pi$	5. $\frac{3}{2}$	6. 0	7. $-\frac{4}{3}$
8. $-\frac{1}{12}$	11. $\frac{3}{8}\pi a^3$	12. $\frac{1}{10}$	13. $\frac{4}{3}$	20. $\frac{351}{2}$	28. 0	29. 0
30. 8π	31. πa^2	32. 0	35. 3	36. $(a + b + c)\,V$	39. -1	

SHORT ANSWER QUESTIONS

1. **If $\vec{r} = x\vec{i} + y\,\vec{j} + z\,\vec{k}$ and $|\vec{r}| = r$, then find ∇r.**
2. **Find grad ϕ at the point $(1, -2, -1)$, where $\phi = 3x^2y - y^3z^2$.**
3. **What is the greatest rate of increase of $\phi = xyz^2$ at the point $(1, 0, 3)$?**
4. **Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point $(1, -1, 2)$.**
5. **Find the directional derivative of $\phi = xyz$ at $(1, 1, 1)$ in the direction of $\vec{i} + \vec{j} + \vec{k}$**
6. **The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at the point $(4, 4, 2)$ desires to fly in such a direction that it gets cooled faster. Find the direction in which it should fly.**
7. **Find the normal derivative of $\phi = x^3 - y^3 + z$ at the point $(1, 1, 1)$.**
8. **Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1, 2)$.**
9. **Find the equation of the tangent plane to the surface $x^2 + y^2 - z = 0$ at the point $(2, -1, 5)$.**
10. **If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, find div (curl $\vec{F}$).**
11. **Prove that $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} - (6xy + 2x^2y^2)\vec{k}$ is solenoidal.**
12. **Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoial.**
13. **If ϕ is a scalar point function, prove that $\nabla\phi$ is solenoidal and irrotational if ϕ is a solution of Laplace equation.**
14. **Find the values of a, b, c if $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.**
15. **If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.**
16. **Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2x + y)\vec{j}$ moves a particle from the origin to the point $(1, 1)$ along $y^2 = x$.**
17. **Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.**
18. **Evaluate $\int_C (x^2 - xy)dx + (x^2 + y^2)dy$, where C is the square formed by the lines $y = \pm 1, x = \pm 1$ using Green's theorem.**

19. Using Stoke's theorem prove that curl (grad ϕ) = 0.

20. If S any closed surface show that $\iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = 0$.

OBJECTIVE TYPE QUESTIONS.

A. Fill up the blanks

1. $\nabla\left(\frac{1}{r}\right) =$ ________

2. If $\phi(x, y, z) = x^2y + xy^2 + z^2$, then $\nabla\phi$ at (1, 1, 1) is = ________.

3. The directional derivative of $\phi = x^3 + y^3 + z^3$ at (1, −1, 2) in the direction of $\vec{i} + 2\vec{j} + \vec{k}$ is = ________.

4. The unit normal to the surface $xy^2z^3 = 1$ at the point (1, 1, 1) is = ________.

5. The greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3) is = ________.

6. Equation of the normal to the surface $x^2 + y^2 + z^2 = 25$ at the point (1, 0, 3) is = ________.

7. If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, then $curl\ \vec{F}$ = ________.

8. If $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal, then value of a is = ________.

9. If $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ and the curve c is the line joining the points (1, −2, 1) and (3, 2, 4), then $\int_C \vec{F} \cdot d\vec{r}$ = ________.

10. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the region bounded by the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$, then $\iiint_v \nabla \cdot \vec{F} dv$ = ________.

B. Choose the correct answer

1. If $\phi = x^2 + y^2 + z^2 - 8$, then grad ϕ at (2, 0, 2) is

   (a) $\vec{i} + 4\vec{k}$ (b) $\vec{i} + \vec{j} + \vec{k}$ (c) $4\vec{i} + \vec{k}$ (d) $4\vec{i} + 4\vec{j} + 4\vec{k}$

2. $div\left(\frac{\vec{r}}{r}\right)$ is equal to

   (a) $\frac{1}{r}$ (b) $\frac{2}{r}$ (c) $\frac{3}{r}$ (d) $\frac{4}{r}$

3. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $curl\ \vec{r}$ is equal to

   (a) $\vec{o}$ (b) $\vec{i}$ (c) $\vec{j}$ (d) $\vec{k}$

4. If $\phi = x^2 - y^2$, then $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ is equal to

(a) 0 (b) 2 (c) -2 (d) 1

5. If $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xy^2 - y)\vec{k}$, then ϕ is equal to

(a) $xz - yz + c$ (b) $3x^2y + xz^3$ (c) $xz^3 - yz + c$ (d) $3x^2y - yz + c$

6. The unit normal at (1, 2, 5) on $x^2 + y^2 = z$ is

(a) $\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$ (b) $\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$ (c) $\frac{\vec{i} + 4\vec{j} - 5\vec{k}}{\sqrt{42}}$ (d) $\frac{2\vec{i} + 4\vec{j} - 5\vec{k}}{3\sqrt{5}}$

7. The equation of the tangent plane to the surface at (2, 0, 2) is

(a) $x - y - z = 0$ (b) $2x - z = 2$ (c) $3x + y - 2z = 2$ (d) None of these

8. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, then $\int_c \vec{F}\cdot\vec{dr}$, where c is the segment on $y = x$ from (0, 0) to (1, 1) is

(a) $-\frac{7}{6}$ (b) $\frac{7}{12}$ (c) $\frac{7}{6}$ (d) $-\frac{7}{12}$

9. Find the work done when the force $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ displaces a particle from the points corresponding to $x = 1$ to $x = 2$ along $y = x^3$

(a) 24 (b) 64 (c) -84 (d) 94

10. Using Green's theorem in the plane, evaluate $\int_c (2x - y)dx + (x + y)dy$, where c is the circle $x^{2+} + y^2 = 4$ in the plane

(a) 2π (b) 4π (c) -4π (d) 8π

ANSWERS

A. Fill up the blanks

1. $-\frac{\vec{r}}{r^3}$ 2. $3\vec{i} + 3\vec{j} + 2\vec{k}$ 3. $\frac{7\sqrt{6}}{2}$ 4. $\frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{14}}$ 5. 9

6. $\frac{x-4}{4} = \frac{y}{0} = \frac{z-3}{3}$ 7. $\vec{o}$ 8. -5 9. 21 10. 3

B. Choose the correct answer

1. (c) 2. (c) 3. (a) 4. (a) 5. (b) 6. (d) 7. (b) 8. (b) 9. (d) 10. (d)

and $\dfrac{\partial}{\partial z}\left(\dfrac{f}{g}\right)=\dfrac{g\dfrac{\partial f}{\partial z}-f\dfrac{\partial g}{\partial z}}{g^2}$.

$$\therefore \quad \nabla\left(\frac{f}{g}\right)=\frac{1}{g^2}\left\{\mathbf{i}\left(g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}\right)+\mathbf{j}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}\right)+\mathbf{k}\left(g\frac{\partial f}{\partial z}-f\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)-f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\,\nabla f-f\,\nabla g\right\}.$$

SOLVED EXAMPLES

Ex. 1. *If* $\mathbf{A}=x^2yz\,\mathbf{i}-2xz^3\,\mathbf{j}+xz^2\,\mathbf{k}$, $\mathbf{B}=2z\,\mathbf{i}+y\,\mathbf{j}-x^2\,\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$ *at* $(1, 0, -2)$. **[Kanpur 1975, 79]**

Solution. We have $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x^2yz & -2xz^3 & xz^2\\ 2z & y & -x^2\end{vmatrix}$

$=(2x^3z^3-xyz^2)\,\mathbf{i}+(2xz^3+x^4yz)\,\mathbf{j}+(x^2y^2z+4xz^4)\,\mathbf{k}.$

$\therefore \quad \dfrac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})=-xz^2\,\mathbf{i}+x^4z\,\mathbf{j}+2x^2yz\,\mathbf{k}.$

Again $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})=\dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\}$

$=-z^2\,\mathbf{i}+4x^3z\,\mathbf{j}+4xyz\,\mathbf{k}.$ ...(1)

Putting $x=1$, $y=0$ and $z=-2$ in (1), we get the required derivative at the point $(1, 0, -2)=-4\mathbf{i}-8\mathbf{j}$.

Ex. 2. *If* $f(x, y, z)=3x^2y-y^3z^2$, *find grad* f *at the point* $(1, -2, -1)$. **[Agra 1978]**

Solution. We have

grad $f=\nabla f=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$

$=\mathbf{i}\dfrac{\partial}{\partial x}(3x^2y-y^3z^2)+\mathbf{j}\dfrac{\partial}{\partial y}(3x^2y-y^3z^2)+\mathbf{k}\dfrac{\partial}{\partial z}(3x^2y-y^3z^2)$

$=\mathbf{i}\,(6xy)+\mathbf{j}\,(3x^2-3y^2z^2)+\mathbf{k}\,(-2y^3z)$

$=6xy\,\mathbf{i}+(3x^2-3y^2z^2)\,\mathbf{j}-2y^3z\mathbf{k}.$

Putting $x=1$, $y=-2$, $z=-1$, we get

$$\nabla f=6(1)(-2)\,\mathbf{i}+\{3(1)^2-3(-2)^2(-1)^2(-1)^2\}\,\mathbf{j}-2(-2)^3(-1)\,\mathbf{k}$$

$$=-12\mathbf{i}-9\mathbf{j}-16\mathbf{k}.$$

Ex. 3. *If* $r=|\mathbf{r}|$ *where* $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, *prove that*

(*i*) $\nabla f(r)=f'(r)\nabla r$, (*ii*) $\nabla r=\frac{1}{r}\mathbf{r}$, **[Rohilkhand 1981]**

(*iii*) $\nabla f(r)\times\mathbf{r}=\mathbf{0}$, (*iv*) $\nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}$, **[Kanpur 1976]**

(*v*) $\nabla \log|\mathbf{r}|=\frac{\mathbf{r}}{r^2}$,

(*vi*) $\nabla r^n=nr^{n-2}\mathbf{r}$.

[Kanpur 1970; Rohilkhand 76; B.H.U. 70]

Solution. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

$$\therefore \quad r^2=x^2+y^2+z^2.$$

(i) $\nabla f(r)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f(r)$

$=\mathbf{i}\frac{\partial}{\partial x}f(r)+\mathbf{j}\frac{\partial}{\partial y}f(r)+\mathbf{k}\frac{\partial}{\partial z}f(r)$

$=\mathbf{i}f'(r)\frac{\partial r}{\partial x}+\mathbf{j}f'(r)\frac{\partial r}{\partial y}+\mathbf{k}f'(r)\frac{\partial r}{\partial z}$

$=f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)=f'(r)\nabla r.$

(ii) We have $\nabla r=\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}$.

Now $r^2=x^2+y^2+z^2$; $\therefore$ $2r\frac{\partial r}{\partial x}=2x$ *i.e.* $\frac{\partial r}{\partial x}=\frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ $\nabla r=\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}=\frac{1}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r}\mathbf{r}=\hat{\mathbf{r}}.$

(iii) We have as in part (i), $\nabla f(r)=f'(r)\nabla r$.

But as in part (ii) $\nabla r=\frac{1}{r}\mathbf{r}$.

$$\therefore \quad \nabla f(r)=f'(r)\frac{1}{r}\mathbf{r}.$$

$\therefore$ $\nabla f(r)\times\mathbf{r}=\left\{f'(r)\frac{1}{r}\mathbf{r}\right\}\times\mathbf{r}=\left\{\frac{1}{r}f'(r)\right\}(\mathbf{r}\times\mathbf{r})$

$=\mathbf{0}$, since $\mathbf{r}\times\mathbf{r}=\mathbf{0}$.

(iv) We have $\nabla\left(\frac{1}{r}\right)=\mathbf{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\mathbf{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\mathbf{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$=\mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$=-\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}\right)$$

$$=-\frac{1}{r^2}\left(\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}\right) \text{ [see part (ii)]}$$

$$=-\frac{1}{r^3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-\frac{1}{r^3}\mathbf{r}.$$

(v) We have $\nabla \log |\mathbf{r}|=\nabla \log r$

$$=\mathbf{i}\frac{\partial}{\partial x}\log r+\mathbf{j}\frac{\partial}{\partial y}\log r+\mathbf{k}\frac{\partial}{\partial z}\log r$$

$$=\frac{1}{r}\frac{\partial r}{\partial x}\mathbf{i}+\frac{1}{r}\frac{\partial r}{\partial y}\mathbf{j}+\frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k}=\frac{1}{r}\left(\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}\right)$$

$$=\frac{1}{r^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r^2}\mathbf{r}.$$

(vi) We have $\nabla r^n=\mathbf{i}\dfrac{\partial}{\partial x}r^n+\mathbf{j}\dfrac{\partial}{\partial y}r^n+\mathbf{k}\dfrac{\partial}{\partial z}r^n$

$$=\mathbf{i}\,nr^{n-1}\frac{\partial r}{\partial x}+\mathbf{j}\,nr^{n-1}\frac{\partial r}{\partial y}+\mathbf{k}\,nr^{n-1}\frac{\partial r}{\partial z}=nr^{n-1}\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)$$

$$=nr^{n-1}\nabla r$$

$$=nr^{n-1}\frac{1}{r}\mathbf{r} \qquad \left[\because \quad \nabla r=\frac{\mathbf{r}}{r} \text{ as in part (ii)}\right]$$

$$=nr^{n-2}\mathbf{r}.$$

Ex. 4. *Prove that* $f(u)\nabla u=\nabla\int f(u)\,du$.

Solution. We have $\nabla\int f(u)\,du$

$$=\Sigma\,\mathbf{i}\frac{\partial}{\partial x}\left\{\int f(u)\,du\right\} \qquad \text{[by def. of gradient]}$$

$$=\Sigma\mathbf{i}\left\{\frac{d}{du}\int f(u)\,du\right\}\frac{\partial u}{\partial x}=\Sigma\mathbf{i}\,f(u)\frac{\partial u}{\partial x}=f(u)\,\Sigma\,\mathbf{i}\frac{\partial u}{\partial x}=f(u)\nabla u.$$

Ex. 5. *Show that*

(*i*) *grad* $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, (*ii*) *grad* $[\mathbf{r},\mathbf{a},\mathbf{b}]=\mathbf{a}\times\mathbf{b}$,

where **a** *and* **b** *are constant vectors.* **[Rohilkhand 1981; Bombay 70]**

Solution. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then a_1, a_2, a_3 are constants. Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$\therefore \quad \mathbf{r}\cdot\mathbf{a}=a_1x+a_2y+a_3z.$

$\therefore \quad$ grad $(\mathbf{r}\cdot\mathbf{a})=\nabla(\mathbf{r}\cdot\mathbf{a})=\nabla(a_1x+a_2y+a_3z)$

$$=\mathbf{i}\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\mathbf{j}\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\mathbf{k}\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)$$

$$=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$$

(ii) grad $[\mathbf{r},\mathbf{a},\mathbf{b}]=$ grad $\{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})\}$, where $\mathbf{a}\times\mathbf{b}$ is a constant vector

$=\mathbf{a}\times\mathbf{b}$ as in part (i).

Ex. 6. (*i*) *Interpret the symbol* $\mathbf{a}\cdot\nabla$.

(*ii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\,\phi$.

(*iii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\mathbf{a}$.

Solution. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then

$$\mathbf{a}\cdot\nabla=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)$$

$$=a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

Thus the symbol $\mathbf{a}\cdot\nabla$ stands for the operator

$$a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

(ii) $$(\mathbf{a}\cdot\nabla)\,\phi=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\cdot\nabla\phi=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=a_1\frac{\partial\phi}{\partial x}+a_2\frac{\partial\phi}{\partial y}+a_3\frac{\partial\phi}{\partial z}.$$

Hence $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\phi$.

(iii) $$(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\mathbf{r}$$

$$=a_1\frac{\partial\mathbf{r}}{\partial x}+a_2\frac{\partial\mathbf{r}}{\partial y}+a_3\frac{\partial\mathbf{r}}{\partial z}.$$

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore$ $\dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \dfrac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \dfrac{\partial\mathbf{r}}{\partial z}=\mathbf{k}$.

$\therefore$ $(\mathbf{a}\cdot\nabla)\,\mathbf{r}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}$.

Exercises

1. If $\mathbf{f}=(2x^2y-x^4)\,\mathbf{i}+(e^{xy}-y\sin x)\,\mathbf{j}+x^2\cos y\,\mathbf{k}$, verify that
$$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x}=\frac{\partial^2\mathbf{f}}{\partial x\,\partial y}.$$
[Agra 1978]

2. If $\phi(x,y,z)=x^2y+y^2x+z^2$, find $\nabla\phi$ at the point (1, 1, 1). [Agra 1979]

Ans. $3\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

$\left[\text{Note that } \nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.\right]$

3. Find grad f, where f is given by
$f=x^3-y^3+xz^2$, at the point (1, −1, 2). [Agra 1977]

Ans. $7\mathbf{i}-3\mathbf{j}+4\mathbf{k}$.

4. If $u=x+y+z$, $v=x^2+y^2+z^2$, $w=yz+zx+xy$, prove that
$$(\text{grad } u)\cdot[(\text{grad } v)\times(\text{grad } w)]=0.$$ **[Kolhapur 1978]**

5. If $\mathbf{F}=\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\mathbf{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\mathbf{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\mathbf{k}$,
prove that
(i) $\mathbf{F}=\mathbf{r}\times\nabla f$, (ii) $\mathbf{F}\cdot\mathbf{r}=0$, (iii) $\mathbf{F}\cdot\nabla f=0$.

6. If $\phi=(3r^2-4r^{1/2}+6r^{-1/3})$, show that
$$\nabla\phi=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

7. Prove that $\nabla\phi\cdot d\,\mathbf{r}=d\phi.$

8. ρ and p are two scalar point functions such that ρ is a function of p; show that $\nabla\rho=\dfrac{d\rho}{dp}\nabla p.$

9. Prove that $\mathbf{A}\cdot\left(\nabla\dfrac{1}{r}\right)=-\dfrac{\mathbf{A}\cdot\mathbf{r}}{r^3}.$

10. Prove that $\nabla r^{-3}=-3r^{-5}\,\mathbf{r}.$ **[Agra 1974]**

§ 5. **Level Surfaces.** Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type
$$f(x, y, z)=c, \text{ (arbitrary constant)}$$
constitute a family of surfaces in three dimensional space. The surfaces of this family are called *level surfaces.* Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called *iso-f surfaces*.

Theorem 1. *Let $f(x, y, z)$ be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof. Let (x_1, y_1, z_1) be any point of the region R. Then the level surface $f(x, y, z)=f(x_1, y_1, z_1)$ passes through this point.

Now suppose the level surfaces $f(x, y, z)=c_1$ and $f(x, y, z)=c_2$ pass through the point (x_1, y_1, z_1). Then
$$f(x_1, y_1, z_1)=c_1 \text{ and } f(x_1, y_1, z_1)=c_2.$$
Since $f(x, y, z)$ has a unique value at (x_1, y_1, z_1) therefore we have
$$c_1=c_2.$$
Hence only one level surface passes through the point
$$(x_1, y_1, z_1).$$

Theorem 2. *∇f is a vector normal to the surface $f(x, y, z)=c$ where c is a constant.* **[Agra 1968; Kerala 75]**

Proof. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the level surface $f(x, y, z)=c$. Let
$$Q\,(x+\delta x,\ y+\delta y,\ z+\delta z)$$

be a neighbouring point on this surface. Then the position vector of $Q = \mathbf{r} + \delta\mathbf{r} = (x+\delta x)\,\mathbf{i} + (y+\delta y)\,\mathbf{j} + (z+\delta z)\,\mathbf{k}$.

$\therefore \quad \vec{PQ} = (\mathbf{r}+\delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k}$.

As $Q \to P$, the line PQ tends to tangent at P to the level surface. Therefore $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

$$= \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = \nabla f \cdot d\mathbf{r}.$$

Since $f(x, y, z) = \text{constant}$, therefore $df = 0$.

$\therefore \quad \nabla f \cdot d\mathbf{r} = 0$ so that ∇f is a vector perpendicular to $d\mathbf{r}$ and therefore to the tangent plane at P to the surface

$$f(x, y, z) = c.$$

Hence ∇f is a vector normal to the surface $f(x, y, z) = c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z) = c$ passing through that point

§ 6. Directional Derivative of a scalar point function.

[Agra 1972; Kolhapur 73; Bombay 70]

Definition. *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\displaystyle\lim_{Q \to P} \frac{f(Q) - f(P)}{PQ}$, *if it exists, is called the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of directional derivative. Let P be the point (x, y, z) and let Q be the point $(x+\delta x, y+\delta y, z+\delta z)$. Suppose $PQ = \delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If $\delta f = f(x+\delta x, y+\delta y, z+\delta z) - f(x, y, z) = f(Q) - f(P)$, then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{\mathbf{a}}$. Now the directional derivative of f at P in the

Normal at P. Let $\mathbf{R}=X\,\mathbf{i}+Y\,\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q\,(X, Y, Z)$ on the normal at P to the surface. The vector $\vec{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \quad (\mathbf{R}-\mathbf{r})\times\nabla f=\mathbf{0} \qquad ...(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form. The vectors

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k} \text{ and } \nabla f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

will be parallel if

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}=p\left(\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of i, j, k, we get

$$X-x=p\frac{\partial f}{\partial x},\; Y-y=p\frac{\partial f}{\partial y},\; Z-z=p\frac{\partial f}{\partial z}$$

or

$$\frac{X-x}{\frac{\partial f}{\partial x}}=\frac{Y-y}{\frac{\partial f}{\partial y}}=\frac{Z-z}{\frac{\partial f}{\partial z}}$$

are the equations of the normal at P.

SOLVED EXAMPLES

Ex. 1. *Find a unit normal vector to the level surface $x^2y+2xz=4$ at the point $(2, -2, 3)$.*

Solution. The equation of the level surface is

$$f(x, y, z)\equiv x^2y+2xz=4.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\nabla\,(x^2y+2xz)=(2xy+2z)\,\mathbf{i}+x^2\,\mathbf{j}+2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

$\therefore$ $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{|-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}|}=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{\sqrt{(4+16+16)}}=-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}.$$

The vector $-(-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k})$ *i.e.*, $\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Ex. 2. *Find the directional derivatives of a scalar point function f in the direction of coordinate axes.*

Solution. The grad f at any point (x, y, z) is the vector

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

The directional derivative of f in the direction of $\mathbf{i}$

$$=\text{grad } f\cdot\mathbf{i}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f}{\partial x}.$$

Similarly the directional derivatives of f in the directions of $\mathbf{j}$ and $\mathbf{k}$ are $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Ex. 3. *Find the directional derivative of* $f(x, y, z)=x^2yz+4xz^2$ *at the point* $(1, -2, -1)$ *in the direction of the vector* $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$. **[Allahabad 1978]**

Solution. We have $f(x, y, z)=x^2yz+4xz^2$.

$\therefore$ grad $f=(2xyz+4z^2)\,\mathbf{i}+x^2z\,\mathbf{j}+(x^2y+8xz)\,\mathbf{k}$

$=8\mathbf{i}-\mathbf{j}-10\mathbf{k}$ at the point $(1, -2, -1)$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$,

then $$\hat{\mathbf{a}}=\frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{(4+1+4)}}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}.$$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\cdot\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k})=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Ex. 4. *Find the directional derivative of*

$$f(x, y, z)=x^2-2y^2+4z^2$$

at the point $(1, 1, -1)$ *in the direction of* $2\mathbf{i}+\mathbf{j}-\mathbf{k}$ **[Agra 1979]**

Ans. $8/\sqrt{6}$.

Ex. 5. *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P(1, 2, 3)$ *in the direction of the line* PQ *where* Q *is the point* $(5, 0, 4)$. **[Agra 1980]**

Solution. Here grad $f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$

$=2x\,\mathbf{i}-2y\,\mathbf{j}+4z\,\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ at the point $(1, 2, 3)$.

Also $\overrightarrow{PQ}=$ position vector of $Q-$ position vector of P

$=(5\mathbf{i}+0\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$,

then $$\hat{\mathbf{a}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}.$$

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{\mathbf{a}}=(2\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot\left\{\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Ex. 6. *In what direction from the point* (1, 1, −1) *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum ? Also find the value of this maximum directional derivative.*

Solution. We have grad $f=2x\mathbf{i}-4y\mathbf{j}+8z\mathbf{k}$

$$=2\mathbf{i}-4\mathbf{j}-8\mathbf{k} \text{ at the point } (1, 1, -1).$$

The directional derivative of f is a maximum in the direction of grad $f=2\mathbf{i}-4\mathbf{j}-8\mathbf{k}$.

The maximum value of this directional derivative

$=|\text{grad } f|=|2\mathbf{i}-4\mathbf{j}-8\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}$.

Ex. 7. *For the function* $f=y/(x^2+y^2)$, *find the value of the directional derivative making an angle 30° with the positive x-axis at the point* (0, 1).

Solution. We have grad $f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$

$$=\frac{-2xy}{(x^2+y^2)^2}\mathbf{i}+\frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j}=-\mathbf{j} \text{ at the point } (0, 1).$$

If $\hat{\mathbf{a}}$ is a unit vector along the line which makes an angle 30° with the positive x-axis, then

$$\hat{\mathbf{a}}=\cos 30^\circ\,\mathbf{i}+\sin 30^\circ\,\mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{i}+\tfrac{1}{2}\mathbf{j}.$$

$\therefore$ the required directional derivative is

$$=\text{grad } f\cdot\hat{\mathbf{a}}=(-\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)=-\tfrac{1}{2}.$$

Ex. 8. *What is the greatest rate of increase of* $u=xyz^2$ *at the point* (1, 0, 3) ? **[Agra 1968]**

Solution. We have $\nabla u=yz^2\,\mathbf{i}+xz^2\,\mathbf{j}+2xyz\,\mathbf{k}$.

$\therefore$ at the point (1, 0, 3), we have

$$\nabla u=0\,\mathbf{i}+9\,\mathbf{j}+0\,\mathbf{k}=9\,\mathbf{j}.$$

The greatest rate of increase of u at the point (1, 0, 3)

$=$the maximum value of $\dfrac{du}{ds}$ at the point (1, 0, 3)

$=|\nabla u|$, at the point (1, 0, 3)

$=|9\mathbf{j}|=9$.

Ex. 9. *Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.*

Solution. Let $f(x, y, z)$ be a scalar point function and let $\mathbf{a}$ be a unit vector along a tangent line to the level surface $f(x, y, z)=c$.

We know that ∇f is a normal vector at any point of the surface $f(x, y, z)=c$. Therefore the vectors ∇f and $\mathbf{a}$ are perpendicular.

Now the directional derivative of f in the direction of $\mathbf{a}$

$$=\mathbf{a}\cdot\nabla f=0.$$

Ex. 10. *Find the equations of the tangent plane and normal to the surface* $2xz^2-3xy-4x=7$ *at the point* $(1, -1, 2)$.

Solution. The equation of the surface is

$$f(x, y, z)\equiv 2xz^2-3xy-4x=7.$$

We have $\text{grad } f=(2z^2-3y+4)\,\mathbf{i}-3x\,\mathbf{j}+4xz\,\mathbf{k}$

$$=7\mathbf{i}-3\mathbf{j}+8\mathbf{k}, \text{ at the point } (1, -1, 2).$$

$\therefore$ $7\mathbf{i}-3\mathbf{j}+8\mathbf{k}$ is a vector along the normal to the surface at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$ is $=\mathbf{r}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, then the vector $\mathbf{R}-\mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot\text{grad } f=0,$$

i.e. $$\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$$

i.e. $$\{(X-1)\,\mathbf{i}+(Y+1)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$$

i.e. $$7(X-1)-3(Y+1)+8(Z-2)=0.$$

The equations of the normal to the surface at the point $(1, -1, 2)$ are

$$\frac{X-1}{\frac{\partial f}{\partial x}}=\frac{Y+1}{\frac{\partial f}{\partial y}}=\frac{Z-2}{\frac{\partial f}{\partial z}}, \text{ i.e. } \frac{X-1}{7}=\frac{Y+1}{-3}=\frac{Z-2}{8}.$$

Ex. 11. *Find the equations of the tangent plane and normal to the surface* $xyz=4$ *at the point* $(1, 2, 2)$. **[Agra 1970]**

Solution. The equation of the surface is

$$f(x, y, z)\equiv xyz-4=0.$$

We have $\text{grad } f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$

$$=4\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \text{ at the point } (1, 2, 2).$$

$\therefore$ $4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a vector along the normal to the surface at the point (1, 2, 2).

The position vector of the point (1, 2, 2) is $=\mathbf{r}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, 2, 2), the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot \text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-\ )\,\mathbf{k}\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $4\,(X-1)+2\,(Y-2)+2\,(Z-2)=0,$

i.e. $4X+2Y+2Z=12,$ *i.e*, $2X+Y+Z=6.$

The equations of the normal to the surface at the point (1, 2, 2) are

$$\frac{X-1}{\frac{\partial f}{\partial x}}=\frac{Y-2}{\frac{\partial f}{\partial y}}=\frac{Z-2}{\frac{\partial f}{\partial z}},$$

i.e. $\dfrac{X-1}{4}=\dfrac{Y-2}{2}=\dfrac{Z-2}{2}$, *i.e.* $\dfrac{X-1}{2}=\dfrac{Y-2}{1}=\dfrac{Z-2}{1}$.

Ex. 12. *Given the curve* $x^2+y^2+z^2=1$, $x+y+z=1$ *(intersection of two surfaces), find the equations of the tangent line at the point* (1, 0, 0). **[Agra 1969]**

Solution. A normal to $x^2+y^2+z^2=1$ at (1, 0, 0) is

$$\text{grad } f_1=\text{grad }(x^2+y^2+z^2)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=2\mathbf{i}.$$

A normal to $x+y+z=1$ at (1, 0, 0) is

$$\text{grad } f_2=\text{grad }(x+y+z)=1\ \mathbf{i}+1\ \mathbf{j}+1\ \mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

The tangent line at the point (1, 0, 0) is perpendicular to both these normals. Therefore it is parallel to the vector

$$(\text{grad } f_1)\times(\text{grad } f_2).$$

Now $(\text{grad } f_1)\times(\text{grad } f_2)=2\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$

$$=2\mathbf{i}\times\mathbf{j}+2\mathbf{i}\times\mathbf{k}=2\mathbf{k}-2\mathbf{j}=0\mathbf{i}-2\mathbf{j}+2\mathbf{k}.$$

Now to find the equations of the line through the point (1, 0, 0) and parallel to the vector $0\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

The required equations are

$$\frac{X-1}{0}=\frac{Y-0}{-2}=\frac{Z-0}{2}$$

i.e. $X=1,\ \dfrac{Y}{-1}=\dfrac{Z}{1}$.

Ex. 13. *Find the angle between the surfaces* $x^2+y^2+z^2=9$, *and* $z=x^2+y^2-3$ *at the point* (2, −1, 2). **(Kanpur 1978, 80)**

Solution. Angle between two surfaces at a point is the angle between the normals to the surfaces at the point.

Let $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$.

Then grad $f_1=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ and grad $f_2=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1=$grad f_1 at the point $(2, -1, 2)$ and $\mathbf{n}_2=$grad f_2 at the point $(2, -1, 2)$. Then

$$\mathbf{n}_1=4\mathbf{i}-2\mathbf{j}+4\mathbf{k} \text{ and } \mathbf{n}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}.$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point $(2, -1, 2)$. If θ is the angle between these vectors then
$$\mathbf{n}_1\cdot\mathbf{n}_2=|\mathbf{n}_1|\,|\mathbf{n}_2|\cos\theta$$
or
$$16+4-4=\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}\cos\theta.$$
$$\therefore \quad \cos\theta=\frac{16}{6\sqrt{(21)}} \text{ or } \theta=\cos^{-1}\frac{8}{3\sqrt{(21)}}.$$

Exercises

1. Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point $(2, 0, 0)$.

   Ans. $4\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\frac{1}{3\sqrt{2}}(4\mathbf{i}+\mathbf{j}-\mathbf{k})$.

2. Find the unit vector normal to the surface $x^2-y^2+z=2$ at the point $(1, -1, 2)$. **Ans.** $\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$.

3. Find the unit normal to the surface $z=x^2+y^2$ at the point $(-1, -2, 5)$. **(Kanpur 1975, 79)**

   Ans. $\left(\frac{1}{\sqrt{21}}\right)(2\mathbf{i}+4\mathbf{j}+\mathbf{k})$.

4. Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point $(1, 1, 1)$. **(Allahabad 1979)**

   Ans $\left(\frac{1}{\sqrt{11}}\right)(\mathbf{i}-3\mathbf{j}-\mathbf{k})$.

5. Find the directional derivative of $\phi=xy+yz+zx$ in the direction of vector $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ at $(1, 2, 0)$. **Ans.** 10/3.

6. Find the directional derivative of $\phi(x, y, z)=x^2yz+4xz^2$ at the point $(1, -2, 1)$ in the direction $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

   (Poona 1970; Allahabad 78) **Ans.** $-13/3$.

7. Find the directional derivative of the function
   $$f=xy+yz+zx$$
   in the direction of the vector $2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ at the point $(3, 1, 2)$.

   (Rohilkhand 1980, 81; Agra 75)

   Ans. 45/7.

8. Find the directional derivatives of $\phi=xyz$ at the point $(2, 2, 2)$, in the directions

(i) **i**, (ii) **j**, (iii) **i**+**j**+**k**.

Ans. (i) 4, (ii) 4, (iii) $4\sqrt{3}$.

9. Find the greatest value of the directional derivative of the function $2x^2-y-z^4$ at the point (2, −1, 1). **Ans.** 9.

10. Find the maximum value of the directional derivatives of $\phi=x^2yz$ at the point (1, 4, 1). **(Bombay 1970)**

Ans. 9.

11. Find the equation of the tangent plane to the surface $yz-zx+xy+5=0$, at the point (1, −1, 2).

Ans. $3x-3y+2z=10$.

12. Find the equations of the tangent plane and normal to the surface $x^2+y^2+z^2=25$ at the point (4, 0, 3).

Ans. $4x+3z=25$; $\dfrac{x-4}{4}=\dfrac{y}{0}=\dfrac{z-3}{3}$.

13. Find the equations of the tangent plane and normal to the surface $z=x^2+y^2$ at the point (2, −1, 5).

Ans. $4x-2y-z=5$; $\dfrac{x-2}{4}=\dfrac{y+1}{-2}=\dfrac{z-5}{-1}$.

14. Find the angle of intersection at (4, −3, 2) of spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$.

Ans. $\cos^{-1}\sqrt{(19/29)}$.

15. If **F** and f are point functions, show that the components of the former tangential and normal to the level surface

$f=0$ are $\dfrac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}$ and $\dfrac{(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$.

Solution. The unit normal vector to the surface $f=0$ is

$$=\frac{\nabla f}{|\nabla f|}.$$

∴ The magnitude of the component of **F** along the normal

$$=\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}.$$

∴ the component of **F** along the normal

$$=\left\{\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}\right\}\frac{\nabla f}{|\nabla f|}=\frac{(\mathbf{F}\cdot\nabla f)}{|\nabla f|^2}\nabla f=\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f.$$

Consequently the tangential component of **F** is

$$=\mathbf{F}-\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f=\frac{(\nabla f\cdot\nabla f)\,\mathbf{F}-(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$$

$$=\frac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}\quad[\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}].$$

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (x^2 y \, \mathbf{i} - 2xz \, \mathbf{j} + 2yz \, \mathbf{k})$$

$$= \frac{\partial}{\partial x} (x^2 y) + \frac{\partial}{\partial y} (-2xz) + \frac{\partial}{\partial z} (2yz) = 2xy + 0 + 2y = 2y \, (x+1).$$

(ii) We have $\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y} (2yz) - \frac{\partial}{\partial z} (-2xz) \right] \mathbf{i} - \left[\frac{\partial}{\partial x} (2yz) - \frac{\partial}{\partial z} (x^2 y) \right] \mathbf{j} + \frac{\partial}{\partial x} \left[(-2xz) - \frac{\partial}{\partial y} (x^2 z) \right] \mathbf{k}$$

$$= (2z + 2x) \, \mathbf{i} - 0 \, \mathbf{j} + (-2z - x^2) \, \mathbf{k} = (2x + 2z) \, \mathbf{i} - (x^2 + 2z) \, \mathbf{k}.$$

(iii) We have $\text{curl curl } \mathbf{f} = \nabla \times (\nabla \times \mathbf{f})$

$$= \nabla \times [(2x + 2z) \, \mathbf{i} - (x^2 + 2z) \, \mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y} (-x^2 - 2z) \right] \mathbf{i} - \left[\frac{\partial}{\partial x} (-x^2 - 2z) - \frac{\partial}{\partial z} (2x + 2z) \right] \mathbf{j} + \left[0 - \frac{\partial}{\partial y} (2x + 2z) \right] \mathbf{k}$$

$$= 0 \, \mathbf{i} - (-2x - 2) \, \mathbf{j} + (0 - 0) \, \mathbf{k} = (2x + 2) \, \mathbf{j}.$$

Ex. 4. *Determine the constant a so that the vector* $\mathbf{V} = (x + 3y) \, \mathbf{i} + (y - 2z) \, \mathbf{j} + (x + az) \, \mathbf{k}$ *is solenoidal.* [Kanpur 1978]

Solution. A vector $\mathbf{V}$ is said to be solenoidal if $\text{div } \mathbf{V} = 0$.

We have $\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\partial}{\partial x} (x + 3y) + \frac{\partial}{\partial y} (y - 2z) + \frac{\partial}{\partial z} (x + az)$

$$= 1 + 1 + a = 2 + a.$$

Now $\text{div } \mathbf{V} = 0$ if $2 + a = 0$ *i.e.* if $a = -2$.

Ex. 5. *Show that the vector* $\mathbf{V} = (\sin y + z) \, \mathbf{i} + (x \cos y - z) \, \mathbf{j} + (x - y) \, \mathbf{k}$ *is irrotational.*

Solution. A vector $\mathbf{V}$ is said to be irrotational if $\text{curl } \mathbf{V} = \mathbf{0}$.
We have $\text{curl } \mathbf{V} = \nabla \times \mathbf{V}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y+z & x\cos y-z & x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]\mathbf{k}$$

$$=(-1+1)\,\mathbf{i}-(1-1)\,\mathbf{j}+(\cos y-\cos y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ **V** is irrotational.

Ex. 6. *If* **V** *is a constant vector, show that*

(*i*) *div* $\mathbf{V}=0$, (*ii*) *curl* $\mathbf{V}=\mathbf{0}$.

Solution. (i) We have $\text{div}\,\mathbf{V}=\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\cdot\mathbf{0}+\mathbf{j}\cdot\mathbf{0}+\mathbf{k}\cdot\mathbf{0}=0.$$

(ii) We have $\text{curl}\,\mathbf{V}=\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\times\mathbf{0}+\mathbf{j}\times\mathbf{0}+\mathbf{k}\times\mathbf{0}=\mathbf{0}.$$

Ex. 7. *If* **a** *is a constant vector, find*

(*i*) *div* $(\mathbf{r}\times\mathbf{a})$, **[Rohilkhand 1980, 81]**

(*ii*) *curl* $(\mathbf{r}\times\mathbf{a})$. **[Rohilkhand 1981]**

Solution. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have $\mathbf{r}\times\mathbf{a}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$=(a_3y-a_2z)\,\mathbf{i}+(a_1z-a_3x)\,\mathbf{j}+(a_2x-a_1y)\,\mathbf{k}.$$

(i) $\text{div}\,(\mathbf{r}\times\mathbf{a})=\frac{\partial}{\partial x}(a_3y-a_2z)+\frac{\partial}{\partial y}(a_1z-a_3x)+\frac{\partial}{\partial z}(a_2x-a_1y)$

$$=0+0+0=0.$$

(ii) $\text{curl}\,(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y-a_2z & a_1z-a_3x & a_2x-a_1y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_1z-a_3x)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_3y-a_2z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(a_1z-a_3x)-\frac{\partial}{\partial y}(a_3y-a_2z)\right]\mathbf{k}$$

$$=-2a_1\mathbf{i}-2a_2\mathbf{j}-2a_3\mathbf{k}=-2(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=-2\mathbf{a}.$$

Ex. 8. *If* $\mathbf{V}=e^{xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, *find curl* $\mathbf{V}$.

[Meerut 1969; Agra 70]

Solution. We have curl $\mathbf{V}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & e^{xyz} & e^{xyz}\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(e^{xyz})-\frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(e^{xyz})-\frac{\partial}{\partial x}(e^{xyz})\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(e^{xyz})-\frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$=e^{xyz}(xz-xy)\,\mathbf{i}+e^{xyz}(xy-yz)\,\mathbf{j}+e^{xyz}(yz-xz)\,\mathbf{k}.$$

Ex. 9. *Evaluate div* $\mathbf{f}$ *where*

$\mathbf{f}=2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k}.$ **[Kanpur 1970]**

Solution. We have

$$\text{div } \mathbf{f}=\nabla\cdot\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k})$$

$$=\frac{\partial}{\partial x}(2x^2z)+\frac{\partial}{\partial y}(-xy^2z)+\frac{\partial}{\partial z}(3y^2x)$$

$$=4xz-2xyz+0=2xz(2-y).$$

Ex. 10. *Show that* $\nabla^2(x/r^3)=0.$

Solution. $\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right).$

Now $\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right\}=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{x}{r}\right\}\left[\because\ r^2=x^2+y^2+z^2 \text{ gives } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$=-\frac{3}{r^4}\frac{x}{r}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}$$

Again $\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right\}=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}$

$$=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\} \qquad \left[\because \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right)=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}.$$

Similarly $\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{3x}{r^5}+\frac{5xz^2}{r^7}.$

Therefore adding we get

$$\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}-\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}(x^2+y^2+z^2)=-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0.$$

Exercises

1. If $\mathbf{f}=xy^2\mathbf{i}+2x^2yz\mathbf{j}-3yz^2\mathbf{k}$, find div **f** and curl **f**. What are their values at the point $(1, -1, 1)$? **[Agra 1979]**

   Ans. y^2+2x^2z-6yz; $-(3z^2+2x^2y)\,\mathbf{i}+(4xyz-2xy)\,\mathbf{k}$.
   At the point $(1, -1, 1)$, div $\mathbf{f}=9$ and curl $\mathbf{f}=-\mathbf{i}-2\mathbf{k}$.

2. If $\mathbf{f}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, find div **f** and curl **f**.

   Ans. $-2x-2y-2z$; $2(y-z)\,\mathbf{i}+2(z-x)\,\mathbf{j}+2(x-y)\,\mathbf{k}$.

3. If $\mathbf{F}=x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k}$, find div **F**, curl **F** at $(1, -1, 1)$.

   Ans. div $\mathbf{F}=-3$, curl $\mathbf{F}=-6\mathbf{i}+2\mathbf{j}$.

4. Find div **f** and curl **f** where
   $\mathbf{f}=\text{grad}\,(x^3+y^3+z^3-3xyz)$.

   Ans. div $\mathbf{f}=6(x+y+z)$; curl $\mathbf{f}=\mathbf{0}$.

5. Find the divergence and curl of the vector
   $\mathbf{f}=(x^2-y^2)\,\mathbf{i}+2xy\mathbf{j}+(y^2-xy)\,\mathbf{k}$. **[Agra 1977]**

   Ans. div $\mathbf{f}=4x$, curl $\mathbf{f}=(2y-x)\,\mathbf{i}+y\mathbf{j}$.

6. Given $\phi=2x^3y^2z^4$, find div (grad ϕ).

   Ans. $12xy^2z^4+4x^3z^4+24x^3y^2z^2$.

7. If $u=x^2-y^2+4z$, show that $\nabla^2u=0$.

8. If $u=3x^2y$ and $v=xz^2-2y$, then find
   grad [(grad u)$\cdot$(grad v)].

   Ans. $(6yz^2-4x)\,\mathbf{i}+6xz^2\,\mathbf{j}+12xyz\,\mathbf{k}$.

9. If $\mathbf{f}=(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}$, prove that
   $\mathbf{f}\cdot\text{curl}\,\mathbf{f}=0$. **[Kanpur 1980; Agra 78, 80]**

10. If $\mathbf{f}=f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k}$, show that
    $\nabla\cdot\mathbf{f}=\nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k}$,

$$\nabla \times \mathbf{f} = \nabla f_1 \times \mathbf{i} + \nabla f_2 \times \mathbf{j} + \nabla f_3 \times \mathbf{k}.$$

11. Find the constants a, b, c so that the vector
$\mathbf{F} = (x+2y+az)\,\mathbf{i} + (bx-3y-z)\,\mathbf{j} + (4x+cy+2z)\,\mathbf{k}$
is irrotational.

Ans. $a=4,\ b=2,\ c=-1$.

§ 11. Important Vector Identities.

1. *Prove that div* $(\mathbf{A}+\mathbf{B}) =$ *div* $\mathbf{A} +$ *div* $\mathbf{B}$

or $$\nabla \cdot (\mathbf{A}+\mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}.$$

Proof. We have

$$\text{div}\,(\mathbf{A}+\mathbf{B}) = \nabla \cdot (\mathbf{A}+\mathbf{B}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (\mathbf{A}+\mathbf{B})$$

$$= \mathbf{i} \cdot \frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B}) + \mathbf{j} \cdot \frac{\partial}{\partial y}(\mathbf{A}+\mathbf{B}) + \mathbf{k} \cdot \frac{\partial}{\partial z}(\mathbf{A}+\mathbf{B})$$

$$= \mathbf{i} \cdot \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x} \right) + \mathbf{j} \cdot \left(\frac{\partial \mathbf{A}}{\partial y} + \frac{\partial \mathbf{B}}{\partial y} \right) + \mathbf{k} \cdot \left(\frac{\partial \mathbf{A}}{\partial z} + \frac{\partial \mathbf{B}}{\partial z} \right)$$

$$= \left(\mathbf{i} \cdot \frac{\partial \mathbf{A}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{A}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{A}}{\partial z} \right) + \left(\mathbf{i} \cdot \frac{\partial \mathbf{B}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{B}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{B}}{\partial z} \right)$$

$$= \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} = \text{div}\,\mathbf{A} + \text{div}\,\mathbf{B}.$$

2. *Prove that curl* $(\mathbf{A}+\mathbf{B}) =$ *curl* $\mathbf{A} +$ *curl* $\mathbf{B}$

or $$\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}.$$

Proof. We have curl $(\mathbf{A}+\mathbf{B}) = \nabla \times (\mathbf{A}+\mathbf{B})$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \times (\mathbf{A}+\mathbf{B}) = \Sigma\, \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B}) = \Sigma\, \mathbf{i} \times \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x} \right)$$

$$= \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} + \Sigma\, \mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} = \text{curl}\,\mathbf{A} + \text{curl}\,\mathbf{B}.$$

3. *If* $\mathbf{A}$ *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

$$\textit{div}\,(\phi \mathbf{A}) = (\textit{grad}\,\phi) \cdot \mathbf{A} + \phi\,\textit{div}\,\mathbf{A}$$

or $$\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi\,(\nabla \cdot \mathbf{A}).$$

[Meerut B. Sc. Physics 1983; Venkateswara 74; Rohilkhand 80; Agra 71, 74; Bombay 70]

Proof. We have

$$\text{div}\,(\phi \mathbf{A}) = \nabla \cdot (\phi \mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) \cdot (\phi \mathbf{A})$$

$$= \mathbf{i} \cdot \frac{\partial}{\partial x}(\phi \mathbf{A}) + \mathbf{j} \cdot \frac{\partial}{\partial y}(\phi \mathbf{A}) + \mathbf{k} \cdot \frac{\partial}{\partial z}(\phi \mathbf{A})$$

$$= \Sigma \left\{ \mathbf{i} \cdot \frac{\partial}{\partial x}(\phi \mathbf{A}) \right\} = \Sigma \left\{ \mathbf{i} \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{A} + \phi \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\cdot\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note $\mathbf{a}\cdot(m\mathbf{b})=(m\mathbf{a})\cdot\mathbf{b}=m(\mathbf{a}\cdot\mathbf{b})$]

$$=\left\{\Sigma\frac{\partial\phi}{\partial x}\mathbf{i}\right\}\cdot\mathbf{A}+\phi\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A}).$$

4. *Prove that curl* $(\phi\mathbf{A})=(\text{grad }\phi)\times\mathbf{A}+\phi$ *curl* $\mathbf{A}$

or $\nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi(\nabla\times\mathbf{A})$.

[Agra 1968; Meerut 67, 68, 72; Bombay 68; Kanpur 76; Punjab 63]

Proof. We have

$$\text{curl }(\phi\mathbf{A})=\nabla\times(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\phi\mathbf{A})$$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\phi\mathbf{A})\right\}=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\times\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note that $\mathbf{a}\times(m\mathbf{b})=(m\mathbf{a})\times\mathbf{b}=m(\mathbf{a}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\right\}\times\mathbf{A}+\phi\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\times\mathbf{A}+\phi(\nabla\times\mathbf{A}).$$

5. *Prove that div* $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot$*curl* $\mathbf{A}-\mathbf{A}\cdot$*curl* $\mathbf{B}$

or $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$.

[Agra 1970, Punjab 66, Banaras 68, Calicut 74, Allahabad 76, 78, 79, Meerut 72]

Proof. We have

$$\text{div }(\mathbf{A}\times\mathbf{B})=\Sigma\left\{\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}+\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\cdot\mathbf{B}\right\}-\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{A}\right)\right\}$$

[Note $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\cdot\mathbf{B}-\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\cdot\mathbf{A}\right\}=(\text{curl }\mathbf{A})\cdot\mathbf{B}-\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\cdot\mathbf{A}$$

$$=(\text{curl }\mathbf{A})\cdot\mathbf{B}-(\text{curl }\mathbf{B})\cdot\mathbf{A}=\mathbf{B}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\text{curl }\mathbf{B}.$$

6. *Prove that*

$$\text{curl } (\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}-\mathbf{B}\text{ div }\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\text{ div }\mathbf{B}.$$

[Agra 1972, 74, Allahabad 77, Punjab 61]

Proof. We have curl $(\mathbf{A}\times\mathbf{B})=\nabla\times(\mathbf{A}\times\mathbf{B})$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}-(\mathbf{i}\cdot\mathbf{A})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{i}\cdot\mathbf{B})\frac{\partial\mathbf{A}}{\partial x}-\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}\right\}-\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{B}\cdot\mathbf{i})\frac{\partial\mathbf{A}}{\partial x}\right\}-\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\mathbf{A}-\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\left\{\mathbf{B}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{A}-\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\mathbf{B}$$

$$=(\text{div }\mathbf{B})\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\text{div }\mathbf{A})\,\mathbf{B}.$$

7. *Prove that*

$$\text{grad }(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}+(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{B}\times\text{curl }\mathbf{A}+\mathbf{A}\times\text{curl }\mathbf{B}.$$

[Allahabad 1980, 82, Rohilkhand 78, Punjab 67, Banaras 68]

Proof. We have

$$\text{grad }(\mathbf{A}\cdot\mathbf{B})=\nabla(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\cdot\mathbf{B}\right)$$

$$=\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}+\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}. \quad \ldots(1)$$

Now we know that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$.

$$\therefore\quad(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

$$\therefore\quad\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}=(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}-\mathbf{A}\times\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{i}\right)$$

$$=(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}+\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right).$$

Thus $$\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\mathbf{A}\times\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B}). \quad \ldots(2)$$

Similarly $$\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}=(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}). \quad \ldots(3)$$

Putting the values from (2) and (3) in (1), we get

$$\text{grad }(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}).$$

Note. If we put **A** in place of **B**, then

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{A})=2\,(\mathbf{A}\cdot\nabla)\,\mathbf{A}+2\mathbf{A}\times(\nabla\times\mathbf{A})$$

or
$$\tfrac{1}{2}\,\text{grad}\,\mathbf{A}^2=(\mathbf{A}\cdot\nabla)\,\mathbf{A}+\mathbf{A}\times\text{curl}\,\mathbf{A}.$$

8. *Prove that div grad* $\phi=\nabla^2\phi$

i.e. $\nabla\cdot(\nabla\phi)=\nabla^2\phi.$ **[Rohilkhand 1981; Agra 70]**

Proof. We have

$$\nabla\cdot(\nabla\phi)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero*

i.e. $\nabla\times(\nabla\phi)=\mathbf{0}$, *i.e.* *curl grad* $\phi=\mathbf{0}$.

[Rohilkhand 1981; Agra 74; Delhi 64; Banaras 70; Meerut 72; Kerala 74; Venkateswara 74; Kanpur 70]

Proof. We have grad $\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.$

$\therefore$ curl grad $\phi=\nabla\times\text{grad}\,\phi$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10. *Prove that div curl* $\mathbf{A}=0$, *i.e.*, $\nabla\cdot(\nabla\times\mathbf{A})=0.$

[Agra 1970, Kerala 74; Kolhapur 73; Bombay 68]

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}.$

Then curl $\mathbf{A}=\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

Now div curl $\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$=\frac{\partial^2 A_3}{\partial x\,\partial y}-\frac{\partial^2 A_2}{\partial x\,\partial z}+\frac{\partial^2 A_1}{\partial y\,\partial z}-\frac{\partial^2 A_3}{\partial y\,\partial x}+\frac{\partial^2 A_2}{\partial z\,\partial x}-\frac{\partial^2 A_1}{\partial z\,\partial y}$$

$=0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

11 *Prove that*

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$. **[Meerut B.Sc. Physics 1983; Allahabad 81; Agra 71]**

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then
$$\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore\quad \nabla\times(\nabla\times\mathbf{A})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y\,\partial x}+\frac{\partial^2 A_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})-(\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})\right\}\mathbf{i}\right]-\nabla^2\Sigma A_1\mathbf{i}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}.$$

Solved Examples

Ex. 1. *Prove that grad $f(u)=f'(u)$ grad u.*

Solution. We have

$$\text{grad } f(u)=\mathbf{i}\frac{\partial}{\partial x}f(u)+\mathbf{j}\frac{\partial}{\partial y}f(u)+\mathbf{k}\frac{\partial}{\partial z}f(u)$$

$$=\mathbf{i}\, f'(u)\frac{\partial u}{\partial x}+\mathbf{j}\, f'(u)\frac{\partial u}{\partial y}+\mathbf{k}\, f'(u)\frac{\partial u}{\partial z}$$

$$=f'(u)\left[\mathbf{i}\frac{\partial u}{\partial x}+\mathbf{j}\frac{\partial u}{\partial y}+\mathbf{k}\frac{\partial u}{\partial z}\right]=f'(u)\text{ grad } u.$$

Ex. 2. *Taking* $\mathbf{F}=x^2y\,\mathbf{i}+xz\,\mathbf{j}+2yz\,\mathbf{k}$ *verify that div curl* $\mathbf{F}=0$.

[Agra 1968]

Solution. We have Curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & xz & 2yz\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(x^2y)\right]\mathbf{k}$$

$$=(2z-x)\,\mathbf{i}-0\,\mathbf{j}+(z-x^2)\,\mathbf{k}=(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}.$$

Now div curl $\mathbf{F}=\text{div}\,[(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}]$

$$=\frac{\partial}{\partial x}(2z-x)+\frac{\partial}{\partial z}(z-x^2)=-1+1=0.$$

Ex. 3. *Find* $\nabla\phi$ *and* $|\nabla\phi|$ *when*

$$\phi=(x^2+y^2+z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}$$

Solution. Let $r^2=x^2+y^2+z^2$. Then we can write $\phi=r^2e^{-r}$.

Now $\nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.$

We have $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}=[2re^{-r}-r^2e^{-r}]\dfrac{\partial r}{\partial x}.$

But $r^2=x^2+y^2+z^2.$

Therefore $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

So $\dfrac{\partial\phi}{\partial x}=re^{-r}(2-r)\dfrac{x}{r}=(2-r)\,e^{-r}\,x.$

Similarly $\dfrac{\partial\phi}{\partial y}=(2-r)\,e^{-r}\,y$ and $\dfrac{\partial\phi}{\partial z}=(2-r)\,e^{-r}\,z.$

Therefore $\nabla\phi=(2-r)\,e^{-r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(2-r)\,e^{-r}\,\mathbf{r}.$

Also $|\nabla\phi|=|(2-r)\,e^{-r}\,\mathbf{r}|=(2-r)\,e^{-r}\,|\mathbf{r}|=(2-r)\,e^{-r}\,r.$

Ex. 4. *Prove that div* $(r^n\,\mathbf{r})=(n+3)\,r^n$.

[Meerut 1971; Rohilkhand 78; Agra 76]

Solution. We have

$$\text{div}\,(\phi\mathbf{A})=\phi\,(\text{div}\,\mathbf{A})+\mathbf{A}\cdot\text{grad}\,\phi.$$

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=r^n$ in this identity, we get

$$\text{div}\,(r^n\,\mathbf{r})=r^n\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^n$$
$$=3r^n+\mathbf{r}\cdot(nr^{n-1}\,\text{grad}\,r)$$
$$[\because\ \text{div}\,\mathbf{r}=3 \text{ and grad } f(u)=f'(u)\text{ grad } u]$$
$$=3r^n+\mathbf{r}\cdot\left[nr^{n-1}\frac{1}{r}\,\mathbf{r}\right] \qquad \left[\because\ \text{grad}\,r=\hat{\mathbf{r}}=\frac{1}{r}\,\mathbf{r}\right]$$
$$=3r^n+nr^{n-2}\,(\mathbf{r}\cdot\mathbf{r})=3r^n+nr^{n-2}\,r^2=(n+3)\,r^n.$$

Ex. 5. *Prove that* $\nabla^2\,(r^n\,\mathbf{r})=n\,(n+3)\,r^{n-2}\,\mathbf{r}$. **[Agra 1970]**

Solution. We have $\nabla^2\,(r^n\,\mathbf{r})=\nabla\,[\nabla\cdot(r^n\,\mathbf{r})]=\text{grad}\,[\text{div}\,(r^n\mathbf{r})]$

$$=\text{grad}\,[(\text{grad}\,r^n)\cdot\mathbf{r}+r^n\,\text{div}\,\mathbf{r}]$$
$$=\text{grad}\,[(nr^{n-2}\,\mathbf{r})\cdot\mathbf{r}+3r^n]=\text{grad}\,[nr^{n-2}\,\mathbf{r}^2+3r^n]$$
$$=\text{grad}\,[nr^{n-2}\,r^2+3r^n]=\text{grad}\,[(n+3)\,r^n]$$
$$=(n+3)\,\text{grad}\,r^n=(n+3)\,nr^{n-2}\,\mathbf{r}=n\,(n+3)\,r^{n-2}\,\mathbf{r}.$$

Ex. 6. *Prove that div* $\left(\dfrac{\mathbf{r}}{r^3}\right)=0.$ **[Banaras 1970]**

Solution. We have $\text{div}\left(\dfrac{1}{r^3}\,\mathbf{r}\right)=\text{div}\,(r^{-3}\,\mathbf{r})$

$$=r^{-3}\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^{-3}=3r^{-3}+\mathbf{r}\cdot(-3r^{-4}\,\text{grad}\,r)$$
$$=3r^{-3}+\mathbf{r}\cdot\left(-3r^{-4}\,\frac{1}{r}\,\mathbf{r}\right)$$
$$=3r^{-3}-3r^{-5}\,(\mathbf{r}\cdot\mathbf{r})=3r^{-3}-3r^{-5}\,r^2=3r^{-3}-3r^{-3}=0.$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Ex. 7. *Prove that div* $\hat{\mathbf{r}}=2/r$. **[Kanpur 1979]**

Solution. $\text{div}\,(\hat{\mathbf{r}})=\text{div}\left(\dfrac{1}{r}\,\mathbf{r}\right)$. Now proceed as in Ex. 4.

Alternative Method.

$$\text{div}\,\hat{\mathbf{r}}=\text{div}\left(\frac{1}{r}\,\mathbf{r}\right)=\text{div}\left[\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right]$$
$$=\text{div}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$
$$=\left(\frac{1}{r}-\frac{x}{r^2}\frac{\partial r}{\partial x}\right)+\left(\frac{1}{r}-\frac{y}{r^2}\frac{\partial r}{\partial y}\right)+\left(\frac{1}{r}-\frac{z}{r^2}\frac{\partial r}{\partial z}\right).$$

Now $r^2=x^2+y^2+z^2$. $\therefore$ $2r\dfrac{\partial r}{\partial x}=2x$ *i.e.* $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

Similarly $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$.

$$\therefore\quad \text{div}\,\hat{\mathbf{r}}=\frac{3}{r}-\left(\frac{x}{r^2}\,\frac{x}{r}+\frac{y}{r^2}\,\frac{y}{r}+\frac{z}{r^2}\,\frac{z}{r}\right)$$

$$=\frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}.$$

Ex. 8. *Prove that the vector $f(r)\,\mathbf{r}$ is irrotational.*

[Agra 1974; Kanpur 1975]

Solution. The vector $f(r)\,\mathbf{r}$ will be irrotational if

$$\text{curl}\,[f(r)\,\mathbf{r}]=\mathbf{0}.$$

We know that Curl $(\phi\mathbf{A})=(\text{grad}\,\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Putting $\phi=f(r)$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

Curl $[f(r)\,\mathbf{r}]=[\text{grad}\,f(r)]\times\mathbf{r}+f(r)$ curl $\mathbf{r}$

$=[f'(r)\,\text{grad}\,r]\times\mathbf{r}+f(r)\,\mathbf{0}$ $\quad[\because$ curl $\mathbf{r}=\mathbf{0}]$

$=\left[f'(r)\,\frac{1}{r}\,\mathbf{r}\right]\times\mathbf{r}=f'(r)\,\frac{1}{r}\,(\mathbf{r}\times\mathbf{r})=\mathbf{0}$, since $\mathbf{r}\times\mathbf{r}=\mathbf{0}$.

$\therefore$ The vector $f(r)\,\mathbf{r}$ is irrotational.

Ex. 9. (a) *Prove that* $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$.

Solution. We know that if ϕ is a scalar function then

$$\nabla^2\phi=\nabla\cdot(\nabla\phi).$$

$\therefore$ $\nabla^2 f(r)=\nabla\cdot\{\nabla f(r)\}=\text{div}\,\{\text{grad}\,f(r)\}$

$=\text{div}\,\{f'(r)\,\text{grad}\,r\}=\text{div}\left\{\frac{1}{r}f'(r)\,\mathbf{r}\right\}$

$=\frac{1}{r}f'(r)\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\left\{\frac{1}{r}f'(r)\right\}$

$=\frac{3}{r}f'(r)+\mathbf{r}\cdot\left[\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}\text{grad}\,r\right]$

$=\frac{3}{r}f'(r)+\mathbf{r}\cdot\left[\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\frac{1}{r}\,\mathbf{r}\right]$

$=\frac{3}{r}f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\right](\mathbf{r}\cdot\mathbf{r})$

$=\frac{3}{r}f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\right]r^2$

$=\frac{3}{r}f'(r)-\frac{1}{r}f'(r)+f''(r)=f''(r)+\frac{2}{r}f'(r).$

Ex. 9. (b) *If* $\nabla^2 f(r)=0$, *show that*

$$f(r)=\frac{c_1}{r}+c_2,$$

where $r^2=x^2+y^2+z^2$ *and* c_1, c_2 *are arbitrary constants.*

[Bombay 1969]

Solution. As shown in the preceding example, if

$r^2=x^2+y^2+z^2$, then $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$.

$\therefore$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{2}{r} f'(r)=0 \quad \text{or} \quad \frac{f''(r)}{f'(r)}=-\frac{2}{r}.$$

Integrating with respect to r, we get

$$\log f'(r)=-2 \log r+\log c, \text{ where } c \text{ is a constant}$$

$$=\log \frac{c}{r^2}.$$

$$\therefore \quad f'(r)=\frac{c}{r^2}.$$

Again integrating,

$$f(r)=-\frac{c}{r}+c_2 \text{ where } c_2 \text{ is a constant}$$

$$=\frac{c_1}{r}+c_2, \text{ replacing } -c \text{ by } c_1.$$

Ex. 10. *Prove that* $\nabla^2\left(\frac{1}{r}\right)=0$.

[Agra 1976; Rohilkhand 81; Kanpur 76]

Solution. We have

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla \frac{1}{r}\right)=\text{div}\left(\text{grad } \frac{1}{r}\right)$$

$$=\text{div}\left(-\frac{1}{r^2} \text{ grad } r\right)=\text{div}\left(-\frac{1}{r^2} \frac{1}{r} \mathbf{r}\right)=\text{div}\left(-\frac{1}{r^3} \mathbf{r}\right)$$

$$=\left(-\frac{1}{r^3}\right) \text{div } \mathbf{r}+\mathbf{r}\cdot\text{grad}\left(-\frac{1}{r^3}\right)=-\frac{3}{r^3}+\mathbf{r}\cdot\left[\frac{d}{dr}\left(-\frac{1}{r^3}\right) \text{grad } r\right]$$

$$=-\frac{3}{r^3}+\mathbf{r}\cdot\left(\frac{3}{r^4} \frac{1}{r} \mathbf{r}\right)=-\frac{3}{r^3}+\frac{3}{r^5}(\mathbf{r}\cdot\mathbf{r})=-\frac{3}{r^3}+\frac{3}{r^5} r^2=0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Ex. 11. *Prove that div grad* $r^n=n(n+1) r^{n-2}$,

i.e., $\nabla^2 r^n=n(n+1) r^{n-2}$.

[Kanpur 1978, 80; Rohilkhand 81; Agra 69; Calicut 75]

Solution. We have $\nabla^2 r^n=\nabla\cdot(\nabla r^n)=\text{div}(\text{grad } r^n)$

$$=\text{div}(nr^{n-1} \text{ grad } r)=\text{div}\left(nr^{n-1} \frac{1}{r} \mathbf{r}\right)=\text{div}(nr^{n-2}\mathbf{r})$$

$$=(nr^{n-2}) \text{ div } \mathbf{r}+\mathbf{r}\cdot(\text{grad } nr^{n-2})$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2) r^{n-3} \text{ grad } r]$$

$$=3nr^{n-2}+\mathbf{r}\cdot\left[n(n-2) r^{n-3} \frac{1}{r} \mathbf{r}\right]$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2) r^{n-4} \mathbf{r}]=3nr^{n-2}+n(n-2) r^{n-4}(\mathbf{r}\cdot\mathbf{r})$$

$$=3nr^{n-2}+n(n-2) r^{n-4} r^2=nr^{n-2}(3+n-2)=n(n+1) r^{n-2}.$$

Note. If $n=-1$, then $\nabla^2(r^{-1})=(-1)(-1+1) r^{-3}=0$.

Ex. 12. *Prove that* $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$ **[Meerut 1972; Bombay 70]**

Solution. We have $\nabla^2(\phi\psi)=\nabla\cdot[\nabla(\phi\psi)]$

$=\nabla\cdot[\phi(\nabla\psi)+\psi(\nabla\phi)]=\nabla\cdot[\phi(\nabla\psi)]+\nabla\cdot[\psi(\nabla\phi)]$

$=\phi\nabla\cdot(\nabla\psi)+(\nabla\psi)\cdot(\nabla\phi)+\psi\nabla\cdot(\nabla\phi)+(\nabla\psi)\cdot(\nabla\phi)$

$=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$

Ex. 13. *Prove that div* $(\nabla\phi\times\nabla\psi)=0.$

Solution. We know that

$$\text{div}(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{Curl}\,\mathbf{A}-\mathbf{A}\cdot\text{Curl}\,\mathbf{B}.$$

$\therefore$ $\text{div}(\nabla\phi\times\nabla\psi)=(\nabla\psi)\cdot(\text{Curl}\,\nabla\phi)-(\nabla\phi)\cdot(\text{Curl}\,\nabla\psi)$

$=(\nabla\psi)\cdot\mathbf{0}-(\nabla\phi)\cdot\mathbf{0}$ $\quad[\because$ curl grad $\phi=\mathbf{0}]$

$=0.$

Ex. 14. *If* **A** *and* **B** *are irrotational, prove that* $\mathbf{A}\times\mathbf{B}$ *is solenoidal.* **[Bombay 1970; Kanpur 77, 79]**

Solution. If **A** and **B** are irrotational, then

$$\text{curl}\,\mathbf{A}=\mathbf{0},\ \text{curl}\,\mathbf{B}=\mathbf{0}.$$

Now $\text{div}(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}=\mathbf{B}\cdot\mathbf{0}-\mathbf{A}\cdot\mathbf{0}=0.$

Since div $(\mathbf{A}\times\mathbf{B})$ is zero, therefore $\mathbf{A}\times\mathbf{B}$ is solenoidal.

Ex. 15. *Prove that curl* $(\phi\ \text{grad}\ \phi)=\mathbf{0}.$

Solution. We know that

$$\text{curl}(\phi\mathbf{A})=\text{grad}\,\phi\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$$

Putting grad ϕ in place of **A**, we get

$$\text{curl}(\phi\,\text{grad}\,\phi)=\text{grad}\,\phi\times\text{grad}\,\phi+\phi\,\text{curl}\,\text{grad}\,\phi$$
$$=\mathbf{0}+\phi\mathbf{0}.$$

Here grad $\phi\times$ grad $\phi=\mathbf{0}$, since it is the cross product of two equal vectors. Also curl grad $\phi=\mathbf{0}$.

$\therefore$ curl $(\phi$ grad $\phi)=\mathbf{0}+\mathbf{0}=\mathbf{0}.$

Ex. 16. *If f and g are two scalar point functions, prove that* *div* $(f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g.$ **[Meerut 1972]**

Solution. We have $\nabla g=\dfrac{\partial g}{\partial x}\mathbf{i}+\dfrac{\partial g}{\partial y}\mathbf{j}+\dfrac{\partial g}{\partial z}\mathbf{k}.$

Therefore $f\nabla g=f\dfrac{\partial g}{\partial x}\mathbf{i}+f\dfrac{\partial g}{\partial y}\mathbf{j}+f\dfrac{\partial g}{\partial z}\mathbf{k}.$

So $\text{div}(f\nabla g)=\dfrac{\partial}{\partial x}\left(f\dfrac{\partial g}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(f\dfrac{\partial g}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(f\dfrac{\partial g}{\partial z}\right)$

$=f\left(\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}+\dfrac{\partial^2 g}{\partial z^2}\right)+\left(\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x}+\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial y}+\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial z}\right)$

$$=f\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)g$$
$$+\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right)$$
$$=f\,\nabla^2 g+\nabla f\cdot\nabla g.$$

Ex. 17. *A vector function* **f** *is the product of a scalar function and the gradient of a scalar function. Show that*

$$\mathbf{f}\cdot curl\ \mathbf{f}=0.$$ **[Kerala 1975]**

Solution. Let $\mathbf{f}=\psi$ grad ϕ, where ψ and ϕ are scalar functions. We have curl $\mathbf{f}=$cur (ψ grad ϕ).

We know that curl $(\phi\mathbf{A})=(\text{grad }\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

$\therefore$ curl (ψ grad ϕ)$=$(grad ψ)$\times$(grad ϕ)$+\psi$ (curl grad ϕ)

$=$(grad ψ)$\times$(grad ϕ) [$\because$ curl grad $\phi=\mathbf{0}$]

Now $\mathbf{f}\cdot$curl $\mathbf{f}=(\psi$ grad $\phi)\cdot\{(\text{grad }\psi)\times(\text{grad }\phi)\}$

$=[\psi$ grad ϕ, grad ψ, grad $\phi]=\psi$ [grad ϕ, grad ψ, grad ϕ]

$=0$, since the value of a scalar triple product is zero if two vectors are equal.

Ex. 18. *Prove that* $\nabla\cdot\left\{r\nabla\left(\frac{1}{r^3}\right)\right\}=\frac{3}{r^4}$

or, *div* $[r$ *grad* $r^{-3}]=3r^{-4}$.

Solution. We have $\nabla\left(\frac{1}{r^3}\right)=\text{grad } r^{-3}$

$$=\frac{\partial}{\partial x}(r^{-3})\,\mathbf{i}+\frac{\partial}{\partial y}(r^{-3})\,\mathbf{j}+\frac{\partial}{\partial z}(r^{-3})\,\mathbf{k}.$$

Now $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{\partial r}{\partial x}$. But $r^2=x^2+y^2+z^2$.

Therefore $2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$.

So $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{x}{r}=-3r^{-5}x$.

Similarly $\frac{\partial}{\partial y}(r^{-3})=-3r^{-5}y$ and $\frac{\partial}{\partial z}(r^{-3})=-3r^{-5}z$.

Therefore $\nabla\left(\frac{1}{r^3}\right)=-3r^{-5}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

$$\therefore\quad r\nabla\left(\frac{1}{r^3}\right)=-3r^{-4}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

$$\therefore\quad \nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=\frac{\partial}{\partial x}(-3r^{-4}x)+\frac{\partial}{\partial y}(-3r^{-4}y)+\frac{\partial}{\partial z}(-3r^{-4}z).$$

Now $\frac{\partial}{\partial x}(-3r^{-4}x)=12\,r^{-5}\frac{\partial r}{\partial x}x-3r^{-4}$

$$=12\,r^{-5}\,\frac{x}{r}\,x-3r^{-4}=12r^{-6}\,x^2-3r^{-4}.$$

Similarly $\dfrac{\partial}{\partial y}(-3r^{-4}\,y)=12r^{-6}\,y^2-3r^{-4}$

and $\dfrac{\partial}{\partial z}(-3r^{-4}\,z)=12r^{-6}\,z^2-3r^{-4}.$

Hence $\nabla\cdot\left(r\nabla\dfrac{1}{r^3}\right)=12r^{-6}\,(x^2+y^2+z^2)-9r^{-4}$

$$=12r^{-6}\,r^2-9r^{-4}=12r^{-4}-9r^{-4}=3r^{-4}.$$

Ex. 19. *Prove that* $\mathbf{a}\cdot\left(\nabla\dfrac{1}{r}\right)=-\dfrac{\mathbf{a}\cdot\mathbf{r}}{r^3}$.

Solution. We have

$$\text{grad}\,\frac{1}{r}=-\frac{1}{r^2}\,\text{grad}\,r=-\frac{1}{r^2}\,\frac{1}{r}\,\mathbf{r}=-\frac{1}{r^3}\,\mathbf{r}.$$

$$\therefore\quad \mathbf{a}\cdot\left(\nabla\frac{1}{r}\right)=\mathbf{a}\cdot\left(-\frac{1}{r^3}\,\mathbf{r}\right)=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

Ex. 20. *Prove that*

$$\mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\frac{1}{r}\right)=\frac{3\,(\mathbf{a}\cdot\mathbf{r})\,(\mathbf{b}\cdot\mathbf{r})}{r^5}-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}$$

where **a** *and* **b** *are constant vectors.*

Solution. As shown in the last example, we have

$$\mathbf{a}\cdot\nabla\,\frac{1}{r}=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

$$\therefore\quad \mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\frac{1}{r}\right)=\mathbf{b}\cdot\nabla\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)=\mathbf{b}\cdot\Sigma\mathbf{i}\,\frac{\partial}{\partial x}\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\,\frac{\partial}{\partial x}\,(\mathbf{a}\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r})\,\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\left(\mathbf{a}\cdot\frac{\partial\mathbf{r}}{\partial x}\right)+3\,(\mathbf{a}\cdot\mathbf{r})\,r^{-4}\,\frac{\partial r}{\partial x}\right\}$$

[$\because$ **a** is a constant vector]

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{\mathbf{a}\cdot\mathbf{i}}{r^3}+\frac{3x}{r^5}\,(\mathbf{a}\cdot\mathbf{r})\right\}\quad\left[\because\ \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i}\ \text{and}\ \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\mathbf{b}\cdot\Sigma\left\{-\frac{1}{r^3}\,(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+\frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{r})\,x\mathbf{i}\right\}$$

$$=\mathbf{b}\cdot\left\{-\frac{1}{r^3}\,\mathbf{a}+\frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right\}\quad[\because\ \Sigma(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}=\mathbf{a},\ \text{and}\ \Sigma x\mathbf{i}=\mathbf{r}]$$

$$=-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}+\frac{3\,(\mathbf{a}\cdot\mathbf{r})\,(\mathbf{b}\cdot\mathbf{r})}{r^5}.$$

Ex. 21. *Prove that div* $(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot$*curl* **A**. [Rohilkhand 1979]

Solution. We know that

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}.$$

$\therefore$ div $(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot$curl $\mathbf{A}-\mathbf{A}\cdot$curl $\mathbf{r}$
$=\mathbf{r}\cdot$curl $\mathbf{A}-\mathbf{A}\cdot\mathbf{0}$ [$\because$ curl $\mathbf{r}=\mathbf{0}$]
$=\mathbf{r}\cdot$curl $\mathbf{A}$.

Ex. 22. *If* **a** *is a constant vector, prove that*

div $\{r^n(\mathbf{a}\times\mathbf{r})\}=0$. **[Allahabad 1980; Rohilkhand 77]**

Solution. We have

div $(\phi\mathbf{A})=\phi$ div $\mathbf{A}+\mathbf{A}\cdot$grad ϕ.

$\therefore$ div $\{r^n(\mathbf{a}\times\mathbf{r})\}=r^n$ div $(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot$grad r^n

$=r^n$ div $(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot(nr^{n-1}$ grad $r)$

$=r^n$ $(\mathbf{r}\cdot$curl $\mathbf{a}-\mathbf{a}\cdot$curl $\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\left(nr^{n-1}\dfrac{1}{r}\mathbf{r}\right)$

$=r^n(\mathbf{r}\cdot\mathbf{0}-\mathbf{a}\cdot\mathbf{0})+nr^{n-2}(\mathbf{a}\times\mathbf{r})\cdot\mathbf{r}$

[$\because$ curl of constant vector is zero and curl $\mathbf{r}=\mathbf{0}$]

$=nr^{n-2}[\mathbf{a},\mathbf{r},\mathbf{r}]$

$=0$, since a scalar triple product having two equal vectors is zero.

Ex. 23. *Prove that*

$$\nabla\cdot(U\nabla V-V\nabla U)=U\nabla^2V-V\nabla^2U.$$

[Meerut 1969; Bombay 69; Agra 70]

Solution. We have $\nabla\cdot(U\nabla V-V\nabla U)$
$=\nabla\cdot(U\nabla V)-\nabla\cdot(V\nabla U)$.

Now $\nabla\cdot(U\nabla V)=U\{\nabla\cdot(\nabla V)\}+(\nabla U)\cdot(\nabla V)$
$=U\nabla^2V+(\nabla U)\cdot(\nabla V)$.

Interchanging U and V, we get

$\nabla\cdot(V\nabla U)=V\nabla^2U+(\nabla V)\cdot(\nabla U)$.

$\therefore$ $\nabla\cdot(U\nabla V-V\nabla U)$

$=[U\nabla^2V+(\nabla U)\cdot(\nabla V)]-[V\nabla^2U+(\nabla V)\cdot(\nabla U)]$

$=U\nabla^2V-V\nabla^2U$.

Ex. 24. *If* **a** *and* **b** *are constant vectors, prove that*

(*i*) *div* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=-2\mathbf{b}\cdot\mathbf{a}$, **[Rohilkhand 1979]**

(*ii*) *curl* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}$. **[Rohilkhand 1979]**

Solution. (i) We have $(\mathbf{r}\times\mathbf{a})\times\mathbf{b}=(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}$.

$\therefore$ div $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=$div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$
$=$div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-$div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$ ...(1)

But div $(\phi\mathbf{A})=\phi$ div $\mathbf{A}+\mathbf{A}\cdot$grad ϕ.

Taking $\phi=\mathbf{b}\cdot\mathbf{r}$ and $\mathbf{A}=\mathbf{a}$, we get

div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=(\mathbf{b}\cdot\mathbf{r})$ div $\mathbf{a}+\mathbf{a}\cdot$grad $(\mathbf{b}\cdot\mathbf{r})$.

Since **a** is a constant vector, therefore div $\mathbf{a}=0$.

Also let $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

Then $\mathbf{b}\cdot\mathbf{r}=(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$=b_1x+b_2y+b_3z$ where b_1, b_2, b_3 are constants.

$\therefore$ grad $(\mathbf{b}\cdot\mathbf{r})=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}=\mathbf{b}$.

$\therefore$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=\mathbf{a}\cdot\mathbf{b}$. ...(2)

Again div $[(\mathbf{b}\cdot\mathbf{a})]\,\mathbf{r}=(\mathbf{b}\cdot\mathbf{a})$ div $\mathbf{r}+\mathbf{r}\cdot$grad $(\mathbf{b}\cdot\mathbf{a})$.

But div $\mathbf{r}=3$. Also grad $(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}$ because $\mathbf{b}\cdot\mathbf{a}$ is constant.

$\therefore$ div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=3\,(\mathbf{b}\cdot\mathbf{a})$. ...(3)

Substituting the values from (2) and (3) in (1), we get

div $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=(\mathbf{a}\cdot\mathbf{b})-3\,(\mathbf{b}\cdot\mathbf{a})=-2\mathbf{b}\cdot\mathbf{a}$.

(ii) Curl $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=$curl $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$=$curl $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-$curl $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$.

But curl $(\phi\mathbf{A})=$grad $\phi\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

$\therefore$ curl $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=[$grad $(\mathbf{b}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{b}\cdot\mathbf{r})$ curl $\mathbf{a}$

$=\mathbf{b}\times\mathbf{a}$ [$\because$ curl $\mathbf{a}=\mathbf{0}$ and grad $(\mathbf{b}\cdot\mathbf{r})=\mathbf{b}$]

Also curl $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=[$grad $(\mathbf{b}\cdot\mathbf{a})]\times\mathbf{r}+(\mathbf{b}\cdot\mathbf{a})$ curl $\mathbf{r}$

$=\mathbf{0}$ [$\because$ grad $(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}$ and curl $\mathbf{r}=\mathbf{0}$]

$\therefore$ curl $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}-\mathbf{0}=\mathbf{b}\times\mathbf{a}$.

Ex. 25. *If* $\mathbf{a}$ *is a constant vector, prove that*

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}\,(\mathbf{a}\cdot\mathbf{r})$$

Solution. We have

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}.$$

Now $\dfrac{\partial}{\partial x}\left(\dfrac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\dfrac{3}{r^4}\dfrac{\partial r}{\partial x}(\mathbf{a}\times\mathbf{r})+\dfrac{1}{r^3}\left(\mathbf{a}\times\dfrac{\partial\mathbf{r}}{\partial x}\right)+\dfrac{1}{r^3}\left(\dfrac{\partial\mathbf{a}}{\partial x}\times\mathbf{r}\right)$ ...(1)

Now $\dfrac{\partial\mathbf{a}}{\partial x}=\mathbf{0}$ because $\mathbf{a}$ is a constant vector.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore$ $\dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i}$.

Further $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

$\therefore$ (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{x}{r}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i}).$$

$$\therefore\quad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}\Big[(\mathbf{i}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{r}\Big]+\frac{1}{r^3}\Big[(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\mathbf{i}\Big]$$

$$=-\frac{3x}{r^5}\,x\mathbf{a}+\frac{3x}{r^5}\,a_1\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}$$

[$\because$ $\mathbf{i}\cdot\mathbf{r}=x$ and $\mathbf{i}\cdot\mathbf{a}=a_1$ if $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$]

$$=-\frac{3x^2}{r^5}\,\mathbf{a}+\frac{3}{r^5}\,a_1x\,\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}.$$

$$\therefore\quad \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$=\left\{-\frac{3}{r^5}\,\Sigma x^2\right\}\mathbf{a}+\left\{\frac{3}{r^5}\,\Sigma\,a_1x\right\}\mathbf{r}+\frac{3}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,\Sigma a_1\,\mathbf{i}$$

$$=-\frac{3}{r^5}\,r^2\,\mathbf{a}+\frac{3}{r^5}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}+\frac{3}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,\mathbf{a}$$

[$\because$ $\Sigma x^2=r^2$, $\Sigma a_1x=\mathbf{r}\cdot\mathbf{a}$, $\Sigma a_1\mathbf{i}=\mathbf{a}$]

$$=-\frac{\mathbf{a}}{r^3}+\frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}.$$

Ex. 26. *Prove that* $\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2 f).$ **[Agra 1971]**

Solution. We have

$$\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\operatorname{div}\left\{\frac{f(r)}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right\}$$

$$=\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\,x\right\}+\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}\,y\right\}+\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}\,z\right\}\qquad\ldots(1)$$

Now $\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\,x\right\}=\frac{f(r)}{r}+x\,\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$

$$=\frac{f(r)}{r}+x\left\{\frac{f'(r)}{r}-\frac{1}{r^2}\,f(r)\right\}\frac{x}{r}=\frac{f(r)}{r}+\frac{x^2}{r^2}\,f'(r)-\frac{x^2}{r^3}\,f(r).$$

Similarly $\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}\,y\right\}=\frac{f(r)}{r}+\frac{y^2}{r^2}\,f'(r)-\frac{y^2}{r^3}\,f(r)$

and $\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}\,z\right\}=\frac{f(r)}{r}+\frac{z^2}{r^2}\,f'(r)-\frac{z^2}{r^3}\,f(r).$

Putting these values in (1), we get

$$\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{3}{r}\,f(r)+\frac{r^2}{r^2}\,f'(r)-\frac{r^2}{r^3}\,f(r)$$

$$=\frac{2}{r}\,f(r)+f'(r)=\frac{1}{r^2}\left[2rf(r)+r^2f'(r)\right]=\frac{1}{r^2}\,\frac{d}{dr}\left[r^2f(r)\right].$$

Exercises

1. Verify that curl grad $f=0$, where

$f=x^2y+2xy+z^2.$ [Agra 1973]

2. Prove that
curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-\text{curl}\,(\phi\nabla\psi)$. **[Bombay 1969]**
3. Show that curl $(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=0$, where $\mathbf{a}$ is a constant vector.
[**Hint.** Use identity 4. Note that $\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}$, if $\mathbf{a}$ is a constant vector.]
4. If $\mathbf{a}$ is a constant vector, then prove that
(i) $\nabla(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+\mathbf{a}\times\text{curl}\,\mathbf{u}$,
(ii) $\nabla\cdot(\mathbf{a}\times\mathbf{u})=-\mathbf{a}\cdot\text{curl}\,\mathbf{u}$,
(iii) $\nabla\times(\mathbf{a}\times\mathbf{u})=\mathbf{a}\,\text{div}\,\mathbf{u}-(\mathbf{a}\cdot\nabla)\,\mathbf{u}$.
5. Prove that $\mathbf{a}\cdot\{\nabla(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})\}=\text{div}\,\mathbf{v}$, where $\mathbf{a}$ is a constant *unit* vector.
6. Given that $\rho\mathbf{F}=\nabla p$ where ρ, p, $\mathbf{F}$ are point functions, prove that $\mathbf{F}\cdot\text{curl}\,\mathbf{F}=0$. **[Kerala 1975]**
7. Show that curl $\mathbf{a}\phi(r)=\dfrac{1}{r}\phi'(r)\,\mathbf{r}\times\mathbf{a}$, where $\mathbf{a}$ is a constant vector.
8. Prove that curl $(\mathbf{a}\times\mathbf{r})\,r^n=(n+2)\,r^n\,\mathbf{a}-n\,r^{n-2}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}$. **[Rohilkhand 1977]**
9. Prove that curl grad $r^n=0$.
10. If $\mathbf{r}$ is the position vector of the point (x, y, z) show that curl $(r^n\mathbf{r})=0$, where r is the module of $\mathbf{r}$. **[Kanpur 1978]**
11. Prove that $r^n\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n+3=0$. **[Agra 1976; Rohilkhand 78]**
12. If $\mathbf{u}=(1/r)\,\mathbf{r}$, show that $\nabla\times\mathbf{u}=\mathbf{0}$. **[Kanpur 1979]**
13. If $\nabla^2 f(r)=0$, show that $f(r)=c_1\log r+c_2$ where $r^2=x^2+y^2$ and c_1, c_2 are arbitrary constants. **[Poona 1970]**

[**Hint.** First show that if $r^2=x^2+y^2$, then

$$\nabla^2 f(r)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(r)=\frac{f'(r)}{r}+f''(r).\Big]$$

14. If $\mathbf{u}=(1/r)\,\mathbf{r}$ find grad (div $\mathbf{u}$). **[Kanpur 1976]**
Ans. $(-2/r^3)\,\mathbf{r}$.
15. Prove that $\frac{1}{2}\nabla\mathbf{a}^2=(\mathbf{a}\cdot\nabla)\,\mathbf{a}+\mathbf{a}\times\text{curl}\,\mathbf{a}$.
16. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=\mathbf{a}\cdot\mathbf{b}.$$

17. Prove that $\nabla^2\left[\nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right)\right]=2r^{-4}$.
18. Evaluate div $\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}$, where $\mathbf{a}$ is a constant vector. **[Kanpur 1976]**
Ans. $2\mathbf{a}^2$.

§ 12. Invariance.

Theorem 1. *Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator* ∇ *remains invariant.*

Proof. Let O be the fixed origin. Let Ox, Oy, Oz be one system of rectangular axes and Ox', Oy', Oz' be the other system of rectangular axes. Take **i**, **j**, **k** as unit vectors along Ox, Oy, Oz and **i**$'$, **j**$'$, **k**$'$ as unit vectors along Ox', Oy', Oz'. Let P be any point in space whose co-ordinates are (x, y, z) or (x', y', z') with respect to the two systems of axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the lines Ox', Oy', Oz' with respect to the co-ordinate axes Ox, Oy, Oz.

The scheme of transformation will be as follows ;

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \qquad \ldots(1)$$

Also we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where **i**, **j**, **k** are unit vectors along co-ordinate axes. Therefore

$$\left.\begin{aligned} \mathbf{i}' &= l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k} \\ \mathbf{j}' &= l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k} \\ \mathbf{k}' &= l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k} \end{aligned}\right\} \qquad \ldots(2)$$

If V is any function (vector or scalar) of x, y, z, then

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial V}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial V}{\partial z'}\frac{\partial z'}{\partial x}.$$

$$\therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

But from (1), $\dfrac{\partial x'}{\partial x} = l_1, \ \dfrac{\partial y'}{\partial x} = l_2, \ \dfrac{\partial z'}{\partial x} = l_3.$

$$\left.\begin{aligned} \therefore \quad \frac{\partial}{\partial x} &\equiv l_1 \frac{\partial}{\partial x'} + l_2 \frac{\partial}{\partial y'} + l_3 \frac{\partial}{\partial z'} \\ \text{Similarly} \quad \frac{\partial}{\partial y} &\equiv m_1 \frac{\partial}{\partial x'} + m_2 \frac{\partial}{\partial y'} + m_3 \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial z} &\equiv n_1 \frac{\partial}{\partial x'} + n_2 \frac{\partial}{\partial y'} + n_3 \frac{\partial}{\partial z'} \end{aligned}\right\} \qquad \ldots(3)$$

Multiplying the equations (3) by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\equiv\mathbf{i}'\frac{\partial}{\partial x'}+\mathbf{j}'\frac{\partial}{\partial y'}+\mathbf{k}'\frac{\partial}{\partial z'}.$$

Theorem 2. *If $\phi(x, y, z)$ is a scalar invariant with respect to a rotation of axes, then grad ϕ is a vector invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorem 1 and obtain the equations (1) and (2).

Now suppose the function $\phi(x, y, z)$ becomes $\phi'(x', y', z')$ after rotation of axes. Then by hypothesis $\phi(x, y, z)=\phi'(x', y', z')$.

By chain rule of differentiation, we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1$, $\dfrac{\partial y'}{\partial x}=l_2$, $\dfrac{\partial z'}{\partial x}=l_3$.

$$\left.\begin{aligned}\therefore\quad \frac{\partial\phi}{\partial x}&=l_1\frac{\partial\phi'}{\partial x'}+l_2\frac{\partial\phi'}{\partial y'}+l_3\frac{\partial\phi'}{\partial z'}\\ \text{Similarly}\quad \frac{\partial\phi}{\partial y}&=m_1\frac{\partial\phi'}{\partial x'}+m_2\frac{\partial\phi'}{\partial y'}+m_3\frac{\partial\phi'}{\partial z'}\\ \text{and}\quad \frac{\partial\phi}{\partial z}&=n_1\frac{\partial\phi'}{\partial x'}+n_2\frac{\partial\phi'}{\partial y'}+n_3\frac{\partial\phi'}{\partial z'}\end{aligned}\right\}\quad\ldots(3)$$

Multiplying these equations by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}=\mathbf{i}'\frac{\partial\phi'}{\partial x'}+\mathbf{j}'\frac{\partial\phi'}{\partial y'}+\mathbf{k}'\frac{\partial\phi'}{\partial z'}$$

or
$$\text{grad }\phi=\text{grad }\phi'.$$

Theorem 3. *If $\mathbf{V}(x, y, z)$ is a vector function invariant with respect to a rotation of axes, then div $\mathbf{V}$ is a scalar invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorems 1 and 2.

Now suppose the function $\mathbf{V}(x, y, z)$ becomes $\mathbf{V}'(x', y', z')$ after rotation of axes. Then by hypothesis

$$\mathbf{V}(x, y, z)=\mathbf{V}'(x', y', z').$$

By chain rule of differentiation, we have

$$\frac{\partial\mathbf{V}}{\partial x}=\frac{\partial\mathbf{V}'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1$, $\dfrac{\partial y'}{\partial x}=l_2$, $\dfrac{\partial z'}{\partial x}=l_3$.

$$\left.\begin{aligned}\therefore \quad \frac{\partial \mathbf{V}}{\partial x} &= l_1 \frac{\partial \mathbf{V}'}{\partial x'} + l_2 \frac{\partial \mathbf{V}'}{\partial y'} + l_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{Similarly} \quad \frac{\partial \mathbf{V}}{\partial y} &= m_1 \frac{\partial \mathbf{V}'}{\partial x'} + m_2 \frac{\partial \mathbf{V}'}{\partial y'} + m_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{and} \quad \frac{\partial \mathbf{V}}{\partial z} &= n_1 \frac{\partial \mathbf{V}'}{\partial x'} + n_2 \frac{\partial \mathbf{V}'}{\partial y'} + n_3 \frac{\partial \mathbf{V}'}{\partial z'}\end{aligned}\right\} \quad \ldots(3)$$

Taking dot product of these three equations by **i, j, k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{div } \mathbf{V} = \text{div } \mathbf{V}'.$$

Theorem 4. *If* **V** (x, y, z) *is a vector function invariant under a rotation of axes, then curl* **V** *is a vector invariant under this rotation.* **[Punjab 1966]**

Proof. Proceed exactly in the same manner as in theorem 3.

In place of taking dot product of equations (3), take cross product. We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{curl } \mathbf{V} = \text{curl } \mathbf{V}'.$$

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinate axes, then $dV = dx\,dy\,dz$ and the above volume integral becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz.$$

If $\mathbf{F}(x, y, z)$ is a vector function, then

$$\iiint_V \mathbf{F}\, dV$$

is also an example of a volume integral.

SOLVED EXAMPLES

Ex. 1. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j}$ *and curve* C *is the arc of the parabola* $y = x^2$ *in the* x-y *plane from* $(0, 0)$ *to* $(1, 1)$.

Solution. We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Let $x = t$; then $y = t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} = t\mathbf{i} + t^2\mathbf{j}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = t^2\mathbf{i} + t^6\mathbf{j}$.

At the point $(0, 0)$, $t = x = 0$. At the point $(1, 1)$, $t = 1$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 (t^2\mathbf{i} + t^6\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})\, dt$$

$$= \int_0^1 (t^2 + 2t^7)\, dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Method 2. In the xy-plane we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

$$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

Therefore $\mathbf{F} \cdot d\mathbf{r} = (x^2\mathbf{i} + y^3\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) = x^2\,dx + y^3\,dy$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2\,dx + y^3\,dy).$$

Now along the curve C, $y = x^2$. Therefore $dy = 2x\,dx$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^1 [x^2\,dx + x^6(2x)\,dx]$$

$$= \int_0^1 (x^2 + 2x^7)\,dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Ex. 2. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + xy\mathbf{j}$ *and curve* C *is the arc of the curve* $y = x^3$ *from* $(0, 0)$ *to* $(2, 8)$.

Solution. The curve C is the curve $y=x^3$ from (0, 0) to (2, 8). Let $x=t$, then $y=t^3$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then $\mathbf{r}(t)=x\mathbf{i}+y\mathbf{j}=t\mathbf{i}+t^3\mathbf{j}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\mathbf{i}+3t^2\mathbf{j}.$$

Also in terms of t, $\mathbf{F}=(t^2-t^6)\,\mathbf{i}+t^4\mathbf{j}$.

At the point (0, 0), $t=x=0$. At the point (2, 8), $t=2$.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt=\int_0^2 [(t^2-t^6)\,\mathbf{i}+t^4\mathbf{j}]\cdot(\mathbf{i}+3t^2\mathbf{j})\,dt$$

$$=\int_0^2 [(t^2-t^6)+3t^6]\,dt=\int_0^2 [t^2+2t^6]\,dt$$

$$=\left[\frac{t^3}{3}+\frac{2t^7}{7}\right]_0^2=\left[\frac{8}{3}+\frac{256}{7}\right]=\frac{824}{21}.$$

Ex. 3. *If* $\mathbf{F}=3xy\,\mathbf{i}-y^2\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* C *is the curve in the xy-plane,* $y=2x^2$, *from* (0, 0) *to* (1, 2).

[**Kanpur 1978; Agra 76**]

Solution. The parametric equations of the parabola $y=2x^2$ can be taken as

$$x=t,\ y=2t^2.$$

At the point (0, 0), $x=0$ and so $t=0$. Again at the point (1, 2), $x=1$ and so $t=1$.

Now $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (3xy\,\mathbf{i}-y^2\mathbf{j})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

[$\because$ $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, so that $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}$]

$$=\int_C (3xy\,dx-y^2\,dy)=\int_{t=0}^1 \left(3xy\frac{dx}{dt}-y^2\frac{dy}{dt}\right) dt$$

$$=\int_0^1 (3.t.2t^2.1-4t^4.4t)\,dt$$

[$\because$ $x=t$, $y=2t^2$ so that $dx/dt=1$ and $dy/dt=4t$]

$$=\int_0^1 (6t^3-16t^5)\,dt=\left[6.\frac{t^4}{4}-16.\frac{t^6}{6}\right]_0^1$$

$$=\frac{6}{4}-\frac{16}{6}=\frac{3}{2}-\frac{8}{3}=-\frac{7}{6}.$$

Ex. 4. *Find the work done when a force*

$$\mathbf{F}=(x^2-y^2+x)\,\mathbf{i}-(2xy+y)\,\mathbf{j}$$

moves a particle in xy-plane from (0, 0) *to* (1, 1) *along the parabola* $y^2=x$.

[**Kanpur 1980**]

Solution. Let C denote the arc of the parabola $y^2=x$ from the point (0, 0) to the point (1, 1). The parametric equations of the parabola $y^2=x$ can be taken as $x=t^2$, $y=t$. At the point (0, 0), $t=0$ and at the point (1, 1), $t=1$. The required work done

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \{(x^2-y^2+x)\,\mathbf{i}-(2xy+y)\,\mathbf{j}\}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$=\int_C [(x^2-y^2+x)\,dx-(2xy+y)\,dy]$$

$$=\int_{t=0}^{1}\left[(x^2-y^2+x)\frac{dx}{dt}-(2xy+y)\frac{dy}{dt}\right]dt$$

$$=\int_0^1 [(t^4-t^2+t^2).2t-(2t^3+t).1]\,dt$$

$$=\int_0^1 [2t^5-2t^3-t]\,dt=\left[2.\frac{t^6}{6}-2.\frac{t^4}{4}-\frac{t^2}{2}\right]_0^1$$

$$=\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}=-\tfrac{2}{3}.$$

Ex. 5. *Evaluate* $\int (x\,dy-y\,dx)$ *around the circle* $x^2+y^2=1$.

Solution. Let C denote the circle $x^2+y^2=1$. The parametric equations of this circle are $x=\cos t$, $y=\sin t$.

To integrate around the circle C we should vary t from 0 to 2π.

$$\therefore \quad \oint_C (x\,dy-y\,dx)=\int_0^{2\pi}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

$$=\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,dt=\int_0^{2\pi} dt=2\pi.$$

Ex. 6. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\mathbf{F}=\mathbf{i}\cos y-\mathbf{j}x\sin y$

and C *is the curve* $y=\sqrt{(1-x^2)}$ *in xy-plane from* (1, 0) *to* (0, 1).

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C (\mathbf{i}\cos y-\mathbf{j}x\sin y)\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy)=\int_C (\cos y\,dx-x\sin y\,dy)$$

$$=\int_C d(x\cos y)=\Big[x\cos y\Big]_{(1,0)}^{(0,1)}=0-1=-1.$$

Ex. 7. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=xy\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and curve* C *is the arc of* $y=x^2-4$ *from* (2, 0) *to* (4, 12).

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C [xy\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\mathbf{i}+dy\mathbf{j})$$

$$=\int_C [xy\,dx+(x^2+y^2)\,dy]=\int_C xy\,dx+\int_C (x^2+y^2)\,dy.$$

Along C, $y=x^2-4$ and $x^2=y+4$.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{x=2}^{4} x\,(x^2-4)\,dx+\int_{y=0}^{12} (y+4+y^2)\,dy$$

$$=\left[\frac{x^4}{4}-2x^2\right]_2^4+\left[\frac{y^2}{2}+4y+\frac{y^3}{3}\right]_0^{12}=732.$$

Ex. 8. *Evaluate* $\int_C xy^3\,ds$, *where* C *is the segment of the line* $y=2x$ *in the xy-plane from* $(-1, -2)$ *to* $(1, 2)$.

Solution. The parametric form of the curve C can be taken as
$$\mathbf{r}(t)=t\mathbf{i}+2t\mathbf{j}\ (-1\leqslant t\leqslant 1).$$

We have $\frac{d\mathbf{r}}{dt}=\mathbf{i}+2\mathbf{j}$.

Now $\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}$.

$$\therefore\quad \left|\frac{d\mathbf{r}}{dt}\right|=\left|\frac{d\mathbf{r}}{ds}\right|\frac{ds}{dt}=\frac{ds}{dt},\ \text{because}\ \frac{d\mathbf{r}}{ds}\ \text{is unit vector.}$$

$$\therefore\quad \frac{ds}{dt}=|\mathbf{i}+2\mathbf{j}|=\sqrt{5}.$$

$$\therefore\quad \int_C xy^3\,ds=\int_C\left(xy^3\frac{ds}{dt}\right)dt=\int_{-1}^{1} t\,(2t)^3\sqrt{5}\,dt$$

$$=8\sqrt{5}\int_{-1}^{1} t^4\,dt=\frac{16}{\sqrt{5}}.$$

Ex. 9. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+zx\mathbf{k}$ *and curve* C *is* $\mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$, t *varying from* -1 *to* $+1$.

Solution. Along the curve C,
$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}.$$

$$\therefore\quad x=t,\ y=t^2,\ z=t^3\ \text{and}\ \frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}.$$

$\therefore$ Along the curve C, we have
$$\mathbf{F}=(t\times t^2)\,\mathbf{i}+(t^2\times t^3)\,\mathbf{j}+(t^3\times t)\,\mathbf{k}=t^3\mathbf{i}+t^5\mathbf{j}+t^4\mathbf{k}.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{-1}^{1}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$

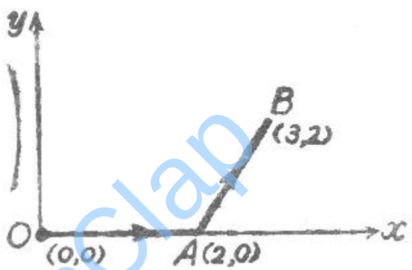
$$=\int_{-1}^{1}(t^3\mathbf{i}+t^5\mathbf{j}+t^4\mathbf{k})\cdot(\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k})\,dt=\int_{-1}^{1}(t^3+2t^6+3t^6)\,dt$$

$$=\int_{-1}^{1}(t^3+5t^6)\,dt=\int_{-1}^{1}t^3\,dt+5\int_{-1}^{1}t^6\,dt$$

$$=0+5\,(2)\int_0^1 t^6\,dt=10\left[\frac{t^7}{7}\right]_0^1=\frac{10}{7}.$$

Ex. 10. *If* $\mathbf{F}=(2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is the curve in the xy-plane consisting of the straight lines from* (0, 0) *to* (2, 0) *and then to* (3, 2).

Solution. The path of integration C has been shown in the figure. It consists of the straight lines OA and AB.



We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C [(2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$=\int_C [(2x+y)\,dx+(3y-x)\,dy].$$

Now along the straight line OA, $y=0$, $dy=0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y-0=\frac{2-0}{3-2}(x-2)\text{ i.e., } y=2x-4.$$

$\therefore$ along AB, $y=2x-4$, $dy=2dx$ and x varies from 2 to 3.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^2[(2x+0)\,dx+0]+\int_2^3[(2x+2x-4)\,dx+(6x-12-x)\,2dx]$$

$$=\left[x^2\right]_0^2+\int_2^3(14x-28)\,dx=4+14\int_2^3(x-2)\,dx$$

$$=4+14\left[\frac{(x-2)^2}{2}\right]_2^3=4+7=11.$$

Ex. 11. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F}=(x^2+y^2)\,\mathbf{i}-2xy\mathbf{j}$, *curve C is the rectangle in the xy-plane bounded by* $y=0$, $x=a$, $y=b$, $x=0$.

[Meerut 1981; Kanpur 79]

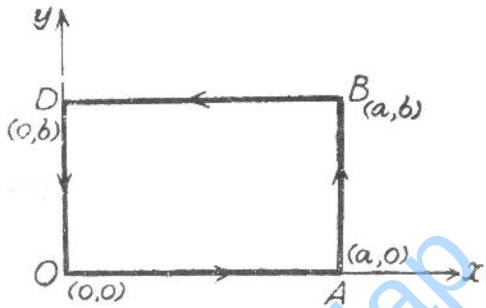
Solution. In the x-y plane $z=0$. Therefore

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}\text{ and } d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}.$$

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(x^2+y^2)\,\mathbf{i} - 2xy\mathbf{j}]\cdot(dx\mathbf{i}+dy\mathbf{j})$

$$= \int_C [(x^2+y^2)\,dx - 2xy\,dy]$$



Now on OA, $y=0$, $dy=0$ and x varies from 0 to a,
on AB, $x=a$, $dx=0$ and y varies from 0 to b,
on BD, $y=b$, $dy=0$ and x varies from a to 0,
on DO, $x=0$, $dx=0$ and y varies from b to 0.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^0 + 0 = -2ab^2.$$

Ex. 12. *Find the total work done in moving a particle in a force field given by* $\mathbf{F} = 3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k}$ *along the curve* $x=t^2+1$, $y=2t^2$, $z=t^3$ *from* $t=1$ *to* $t=2$. **[Kanpur 1978]**

Solution. Let C denote the arc of the given curve from $t=1$ to $t=2$. Then the total work done

$$= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_1^2 \left(3xy\frac{dx}{dt} - 5z\frac{dy}{dt} + 10x\frac{dz}{dt}\right) dt$$

$$= \int_1^2 [3(t^2+1)(2t)^2(2t) - (5t^3)(4t) + 10(t^2+1)(3t^2)]\,dt$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303.$$

Ex. 13. *Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field* **F** *is given by*

$$\mathbf{F}=(2x-y+2z)\,\mathbf{i}+(x+y-z)\,\mathbf{j}+(3x-2y-5z)\,\mathbf{k}.$$

[Ka ·ar 1979]

Solution. In the xy-plane, we have $z=0$. Therefore

$$\mathbf{F}=(2x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}+(3x-2y)\,\mathbf{k}.$$

The circle C is given by $x^2+y^2=4$ or $x=2\cos t$, $y=2\sin t$.

$$\therefore\quad \mathbf{r}=x\mathbf{i}+y\mathbf{j}=2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}.$$

$$\therefore\quad \frac{d\mathbf{r}}{dt}=-2\sin t\,\mathbf{i}+2\cos t\,\mathbf{j}.$$

Also $\mathbf{F}=(4\cos t-2\sin t)\,\mathbf{i}+(2\cos t+2\sin t)\,\mathbf{j}$
$+(6\cos t-4\sin t)\,\mathbf{k}.$

In moving round the circle once t will vary from 0 to 2π.

The required work done is $=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$

$$=\int_0^{2\pi}[-2\sin t\,(4\cos t-2\sin t)+2\cos t\,(2\cos t+2\sin t)]\,dt$$

$$=\int_0^{2\pi}[4(\sin^2 t+\cos^2 t)-4\sin t\cos t]\,dt$$

$$=\int_0^{2\pi}(4-4\sin t\cos t)\,dt=\Big[4t-2\sin 2t\Big]_0^{2\pi}=8\pi.$$

Ex. 14. *If* $\mathbf{F}=(3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is a straight line joining* (0, 0, 0) *to* (1, 1, 1).

[Meerut B. Sc. Physics 1983]

Solution. The equations of the straight line joining (0, 0, 0) and (1, 1, 1) are

$$\frac{x-0}{1-0}=\frac{y-0}{1-0}=\frac{z-0}{1-0}=t \text{ (say)}.$$

Then along C, $x=t$, $y=t$, $z=t$.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}$. $\therefore$ $d\mathbf{r}=(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt$.

Also along C, $\mathbf{F}=(3t^2+6t)\,\mathbf{i}-14t^2\,\mathbf{j}+20t^3\,\mathbf{k}$.

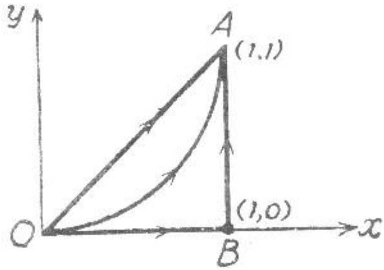
At (0, 0, 0), $t=0$ and at (1, 1, 1), $t=1$.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{t=0}^{1}[(3t^2+6t)-14t^2+20t^3]\,dt=\frac{13}{3}.$$

Ex. 15. *If* $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *from* (0, 0) *to* (1, 1) *along the following paths C :*

(*a*) *the parabola* $y=x^2$, [Agra 1973]
(*b*) *the straight lines from* (0, 0) *to* (1, 0) *and then to* (1, 1).
(*c*) *the straight line joining* (0, 0) *and* (1, 1).

Solution. The three paths of integration have been shown in the figure. We have



$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_C (y\mathbf{i}-x\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})$$

$$=\int_C (y\, dx-x\, dy).$$

(a) C is the arc of parabola $y=x^2$ from (0, 0) to (1, 1).
Here $dy=2xdx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 [x^2\, dx-x\,(2x)\, dx]=\int_0^1 -x^2dx=-\tfrac{1}{3}.$$

(b) C is the curve consisting of straight lines OB and BA.
Along OB, $y=0$, $dy=0$ and x varies from 0 to 1.
Along BA, $x=1$, $dx=0$ and y varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 0\, dx+\int_0^1 -1\, dy=-1.$$

(c) C is the straight line OA. The equation of OA is

$$y-0=\frac{1-0}{1-0}(x-0) \text{ i.e. } y=x.$$

$\therefore$ $dy=dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 (x\,dx-x\,dx)=0.$$

Note. We observe here that **F** is a vector field such that its line integral depends not only on the end points but also on the geometric shape of the path of integration. We shall discuss this topic in depth in the latter portion of this chapter.

Ex. 16. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and* C *is the portion of the curve* $\mathbf{r}=a\cos t\,\mathbf{i}+b\sin t\,\mathbf{j}+ct\,\mathbf{k}$, *from* $t=0$ *to* $t=\pi/2$. [Agra 1975]

Solution. Along the curve C,

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=a\cos t\,\mathbf{i}+b\sin t\,\mathbf{j}+ct\,\mathbf{k}.$$

$$\therefore \quad x=a\cos t,\ y=b\sin t,\ z=ct.$$

Now $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$= \int_C (yz\,dx + zx\,dy + xy\,dz) = \int_C d\,(xyz)$

$= \Big[xyz \Big]_{t=0}^{t=\pi/2} = \Big[(a\cos t).(b\sin t).(ct) \Big]_0^{\pi/2}$

$= abc \Big[t\cos t\sin t \Big]_0^{\pi/2} = abc\,(0-0) = 0.$

Ex 17. *Evaluate*

$$\int_C \{(2xy^3 - y^2\cos x)\,dx + (1 - 2y\sin x + 3x^2y^2)\,dy\}$$

where C is the arc of the parabola $2x = \pi y^2$ *from* $(0, 0)$ *to* $(\frac{1}{2}\pi, 1)$.

[Meerut 1977]

Solution. We know that $M dx + N dy$ is an exact differential if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Here $M = 2xy^3 - y^2\cos x$; $\therefore$ $\dfrac{\partial M}{\partial y} = 6xy^2 - 2y\cos x$.

Also $N = 1 - 2y\sin x + 3x^2y^2$; $\therefore$ $\dfrac{\partial N}{\partial x} = -2y\cos x + 6xy^2$.

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Therefore $M dx + N dy$ is an exact differential.

Let $\phi\,(x, y)$ be such that

$$d\phi = (2xy^3 - y^2\cos x)\,dx + (1 - 2y\sin x + 3x^2y^2)\,dy.$$

Then $\dfrac{\partial\phi}{\partial x}\,dx + \dfrac{\partial\phi}{\partial y}\,dy = (2xy^3 - y^2\cos x)\,dx + (1 - 2y\sin x + 3x^2y^2)\,dy.$

$\therefore$ $\dfrac{\partial\phi}{\partial x} = 2xy^3 - y^2\cos x$ which gives $\phi = x^2y^3 - y^2\sin x + f_1(y)$ ...(1)

Also $\dfrac{\partial\phi}{\partial y} = (1 - 2y\sin x + 3x^2y^2)$ which gives $\phi = y - y^2\sin x + x^2y^3 + f_2(x).$ ...(2)

The values of ϕ given by (1) and (2) agree if we take $f_1(y) = y$ and $f_2(x) = 0$. Then $\phi = y - y^2\sin x - x^2y^3$.

$\therefore$ The given integral

$= \int_C d\phi = \int_C d\,(y - y^2\sin x + x^2y^3)$

$= \Big[y - y^2\sin x + x^2y^3 \Big]_{(0,\,0)}^{(\pi/2,\,1)}$

$= \Big[\Big\{ 1 - 1\times\sin\dfrac{\pi}{2} + \dfrac{\pi^2}{4}\times 1 \Big\} - 0 \Big] = \dfrac{\pi^2}{4}.$

Ex. 18. *Find the circulation of* **F** *round the curve C where*

$$\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$$

and C is the circle $x^2+y^2=1,\ z=0.$

Solution. By definition, the circulation of **F** along the curve C is

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$=\oint_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})=\oint_C (y\,dx+z\,dy+x\,dz)$$

$$=\oint_C y\,dx \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi} \sin\theta\,(-\sin\theta)\,d\theta \qquad [\because \text{ on } C,\ x=\cos\theta,\ y=\sin\theta]$$

$$=-\int_0^{2\pi} \sin^2\theta\,d\theta=-\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$=-\tfrac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi}=-\pi.$$

Ex. 19. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and S is that part of the surface of the sphere* $x^2+y^2+z^2=1$ *which lies in the first octant.* **[Agra 1974; Kanpur 79; Meerut 84 (P)]**

Solution. A vector normal to the surface S is given by

$$\nabla(x^2+y^2+z^2)=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

Therefore $\mathbf{n}$ = a unit normal to any point (x, y, z) of S

$$=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

since $x^2+y^2+z^2=1$ on the surface S.

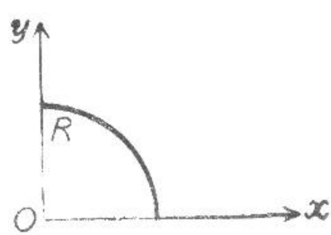
We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1,\ z=0.$

We have

$$\mathbf{F}\cdot\mathbf{n}=(yz\mathbf{i}+zx\mathbf{j}+xy\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$
$$=3xyz.$$

Also $\mathbf{n}\cdot\mathbf{k}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{k}=z.$

$$\therefore\quad |\mathbf{n}\cdot\mathbf{k}|=z.$$



Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$

$$=\iint_R \frac{3xyz}{z}\, dx\, dy = 3\iint_R xy\, dx\, dy$$

$$=3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\, r\, d\theta\, dr,$$ on changing to polars

$$=3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\, d\theta = \tfrac{3}{4}\left(\tfrac{1}{2}\right)=\tfrac{3}{8}.$$

Ex. 20. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, *where* $\mathbf{F}=z\mathbf{i}+x\mathbf{j}-3y^2z\mathbf{k}$ *and* S *is the surface of the cylinder* $x^2+y^2=16$ *included in the first octant between* $z=0$ *and* $z=5$.

Solution. A vector normal to the surface S is given by

$$\nabla(x^2+y^2)=2x\mathbf{i}+2y\mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$=\frac{2x\mathbf{i}+2y\mathbf{j}}{\sqrt{(4x^2+4y^2)}}=\frac{x\mathbf{i}+y\mathbf{j}}{4},$$ since $x^2+y^2=16$, on the surface S.

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iint_R \mathbf{F}\cdot\mathbf{n}\frac{dx\, dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now $\mathbf{F}\cdot\mathbf{n}=(z\mathbf{i}+x\mathbf{j}-3y^2z\mathbf{k})\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}}{4}\right)=\frac{1}{4}(xz+xy)$,

$$\mathbf{n}\cdot\mathbf{j}=\left(\frac{x\mathbf{i}+y\mathbf{j}}{4}\right)\cdot\mathbf{j}=\frac{y}{4}.$$

Therefore the required surface integral is

$$=\iint_R \frac{xz+xy}{4}\,\frac{dx\, dz}{y/4}$$

$$=\int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{(16-x^2)}}+x\right)dx\, dz,$$ since $y=\sqrt{(16-x^2)}$ on S

$$=\int_0^5 (4z+8)\, dz=90.$$

Ex. 21. *Evaluate* $\iiint_V \phi\, dV$, *where* $\phi=45x^2y$ *and* V *is the closed region bounded by the planes* $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Solution. We have

$$\iiint_V \phi\, dV=\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\, dx\, dy\, dz$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\Big[\, z\, \Big]_0^{8-4x-2y} dx\, dy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)\,dx\,dy$$

$$=45\int_{x=0}^{2}\left[x^2(8-4x)\frac{y^2}{2}-2x^2\frac{y^3}{3}\right]_0^{4-2x} dx$$

$$=45\int_0^2 \frac{x^2}{3}(4-2x)^3\,dx=128.$$

Ex. 22. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$,

where $\mathbf{F}=(x+y^2)\,\mathbf{i}-2x\mathbf{j}+2yz\mathbf{k}$ *and* S *is the surface of the plane* $2x+y+2z=6$ *in the first octant.* **[Kanpur 1970]**

Solution. A vector normal to the surface S is given by

$$\nabla\,(2x+y+2z)=2\mathbf{i}+\mathbf{j}+2\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$=\frac{2\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{(4+1+4)}}=(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$.

We have $\mathbf{F}\cdot\mathbf{n}=[(x+y^2)\,\mathbf{i}-2x\,\mathbf{j}+2yz\,\mathbf{k}]\cdot(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})$

$$=\tfrac{2}{3}(x+y^2)-\tfrac{2}{3}x+\tfrac{4}{3}yz=\tfrac{2}{3}y^2+\tfrac{4}{3}yz.$$

Also $\mathbf{n}\cdot\mathbf{k}=(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})\cdot\mathbf{k}=\tfrac{2}{3}.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R [\tfrac{2}{3}y^2+\tfrac{4}{3}yz]\cdot\tfrac{3}{2}dx\,dy$

$$=\iint_R (y^2+2yz)\,dx\,dy$$

$$=\iint_R \left[y^2+2y\left(\frac{6-2x-y}{2}\right)\right]dx\,dy,$$ using the fact that

$z=\dfrac{6-2x-y}{2}$ from the equation of S

$$=\iint_R (y^2+6y-2xy-y^2)\,dx\,dy=2\iint_R y\,(3-x)\,dx\,dy$$

$$=2\int_{y=0}^{6}\int_{x=0}^{(6-y)/2} y\,(3-x)\,dx\,dy.$$

[Note that R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$. To evaluate the double integral over R, keep y fixed and integrate with respect to x from $x=0$ to $x=\dfrac{6-y}{2}$; then integrate with respect to y from $y=0$ to $y=6$. In this way R is completely covered].

$$=2\int_{y=0}^{6} y\left[3x-\frac{x^2}{2}\right]_{x=0}^{(6-y)/2} dy$$

$$=2\int_{0}^{6} y\left[\frac{3(6-y)}{2}-\frac{(6-y)^2}{8}\right] dy$$

$$=2\int_{0}^{6} y\left[9-\frac{3y}{2}-\frac{36}{8}+\frac{12y}{8}-\frac{y^2}{8}\right] dy$$

$$=2\int_{0}^{6} y\left[\frac{36}{8}-\frac{y^2}{8}\right] dy=\int_{0}^{6}\left[9y-\frac{y^3}{4}\right] dy$$

$$=\left[9\frac{y^2}{2}-\frac{y^4}{16}\right]_0^6=\left[9.\frac{36}{2}-\frac{36\times 36}{16}\right]=[162-81]=81.$$

Ex. 23. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=y\,\mathbf{i}+2x\,\mathbf{j}-z\mathbf{k}$ *and* S *is the surface of the plane* $2x+y=6$ *in the first octant cut off by the plane* $z=4$.

Solution. A vector normal to the surface S is given by

$$\nabla(2x+y)=2\mathbf{i}+\mathbf{j}.$$

Therefore $\mathbf{n}=$a unit normal vector at any point (x, y, z) of S

$$=\frac{2\mathbf{i}+\mathbf{j}}{\sqrt{(4+1)}}=\frac{1}{\sqrt{5}}(2\mathbf{i}+\mathbf{j}).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to xy-plane.

Now $\mathbf{F}\cdot\mathbf{n}=(y\mathbf{i}+2x\mathbf{j}-z\mathbf{k})\cdot\left(\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}\right)=\frac{2}{\sqrt{5}}y+\frac{2}{\sqrt{5}}x.$

Also $$\mathbf{n}\cdot\mathbf{j}=\frac{1}{\sqrt{5}}(2\mathbf{i}+\mathbf{j})\cdot\mathbf{j}=\frac{1}{\sqrt{5}}.$$

$\therefore$ the required surface integral is

$$=\iint_R\left(\frac{2}{\sqrt{5}}y+\frac{2}{\sqrt{5}}x\right)\cdot\sqrt{5}\,dx\,dz=\iint_R 2(y+x)\,dx\,dz$$

$$=2\iint_R [6-2x+x]\,dx\,dz,\text{ since } y=6-2x \text{ on } S$$

$$=2\iint_R (6-x)\,dx\,dz=2\int_{z=0}^{4}\int_{x=0}^{3}(6-x)\,dx\,dz$$

$$=2\int_{x=0}^{3}(6-x)\Big[z\Big]_0^4 dx=8\left[6x-\frac{x^2}{2}\right]_0^3=8\left[18-\frac{9}{2}\right]=108.$$

Exercises

1. Find $\int_C \mathbf{t} \cdot d\mathbf{r}$

where $\mathbf{t}$ is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin.

Hint. For any curve, $\dfrac{d\mathbf{r}}{ds}$ = unit tangent vector = $\mathbf{t}$.

$$\therefore \int_C \mathbf{t} \cdot d\mathbf{r} = \int_C \mathbf{t} \cdot \frac{d\mathbf{r}}{ds}\, ds = \int_C \mathbf{t} \cdot \mathbf{t}\, ds = \int_C ds$$

$= \int_0^{2\pi} ds$, since along the unit circle C, s goes from 0 to 2π

$= 2\pi$.

2. If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14\,yz\mathbf{j} + 20xz^2\,\mathbf{k}$, then evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ from (0, 0, 0) to (1, 1, 1) along the curve

$x = t,\ y = t^2,\ z = t^3$. **Ans.** 5.

3. Integrate the function $\mathbf{F} = x^2\,\mathbf{i} - xy\,\mathbf{j}$ from the point (0, 0) to (1, 1) along the parabola $y^2 = x$. **[Rohilkhand 1978]**
Ans. $-\frac{1}{12}$.

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is $x^2y^2\,\mathbf{i} + y\mathbf{j}$ and C is $y^2 = 4x$ in the xy-plane from (0, 0) to (4, 4). **[Agra 1978; Kanpur 77]**
Ans. 264.

5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where,

$\mathbf{F} = c\,[-3a \sin^2 t \cos t\,\mathbf{i} + a\,(2 \sin t - 3 \sin^3 t)\,\mathbf{j} + b \sin 2t\mathbf{k}]$
and C is given by $\mathbf{r} = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + bt\mathbf{k}$
from $t = \pi/4$ to $\pi/2$. **[Delhi 1970]**

Ans. $\frac{1}{2}c\,(a^2 + b^2)$.

$\left[\textbf{Hint.} \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\pi/4}^{\pi/2} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt\right]$.

6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\mathbf{k}$ and C is the arc of the curve $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = 2\pi$.
Ans. 3π. **[Agra 1974, 77]**

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = xy\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}$$

and C is the x-axis from $x = 2$ to $x = 4$ and the line $x = 4$ from $y = 0$ to $y = 12$. **Ans.** 768.

8. Find the work done in moving a particle in a force field

$$\mathbf{F}=3x^2\,\mathbf{i}+(2xz-y)\,\mathbf{j}+z\mathbf{k}$$

along the line joining (0, 0, 0) to (2, 1, 3). **[Delhi 1969]**

Ans. 16.

9. Calculate $\int_C [(x^2+y^2)\,\mathbf{i}+(x^2-y^2)\,\mathbf{j}]\cdot d\mathbf{r}$

where C is the curve :

(i) $y^2=x$ joining (0, 0) to (1, 1).

(ii) $x^7=y$ joining (0, 0) to (1, 1).

(iii) consisting of two straight lines joining (0, 0) to (1, 0) and (1, 0) to (1, 1).

(iv) consisting of three straight lines joining (0, 0) to (2, −2), (2, −2) to (0, −1) and (0, −1) to (1, 1).

Ans. (i) $\frac{7}{15}$, (ii) $\frac{38}{45}$, (iii) 1, (iv) $-\frac{7}{3}$.

10. Find the circulation of **F** round the curve C, where

$$\mathbf{F}=\mathbf{i}e^x \sin y+\mathbf{j}\,e^x \cos y$$

and C is the rectangle whose vertices are

$(0, 0), (1, 0), (1, \tfrac{1}{2}\pi), (0, \tfrac{1}{2}\pi)$. **Ans.** 0.

11. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F}=18z\,\mathbf{i}-12\,\mathbf{j}+3y\,\mathbf{k}$ and S is the surface of the plane

$2x+3y+6z=12$ in the first octant. **Ans.** 24.

12. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$, where $\mathbf{A}=xy\,\mathbf{i}-x^2\,\mathbf{j}+(x+z)\,\mathbf{k}$, S is the portion of the plane $2x+2y+z=6$ included in the first octant and **n** is a unit normal to S. **[Meerut 1974]**

13. If $\mathbf{F}=2y\,\mathbf{i}-z\,\mathbf{j}+x^2\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4$ and $z=6$, then evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. **Ans.** 132.

[**Hint.** $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dy\,dz}{|\mathbf{n}\cdot\mathbf{i}|}$, where R is the projection of S on the yz-plane].

14. If $\mathbf{F}=(2x^2-3z)\,\mathbf{i}-2xy\,\mathbf{j}-4x\,\mathbf{k}$, then evaluate $\iiint_V \nabla\cdot\mathbf{F}\,dV$ where V is the closed region bounded by the planes

$x=0,\ y=0,\ z=0$ and $2x+2y+z=4$. **[Kanpur 1976]**

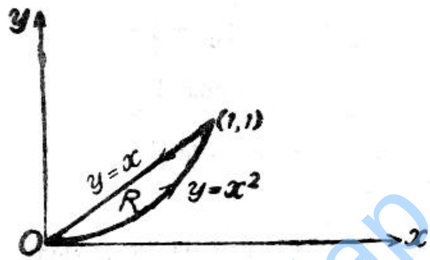
Ans. $\frac{8}{3}$.

SOLVED EXAMPLES

Ex. 1. *Verify Green's theorem in the plane for*

$\oint_C (xy+y^2)\, dx+x^2 dy$ *where C is the closed curve of the region bounded by* $y=x$ *and* $y=x^2$.

Solution. By Green's theorem in plane, we have



$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C (M dx+N dy).$$

Here $M=xy+y^2$, $N=x^2$.

The curves $y=x$ and $y=x^2$ intersect at (0, 0) and (1, 1). The positive direction in traversing C is as shown in the figure.

We have $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy$

$$=\iint_R \left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(xy+y^2)\right] dx\, dy$$

$$=\iint_R (2x-x-2y)\, dx\, dy=\iint_R (x-2y)\, dx\, dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\, dy\, dx=\int_{x=0}^{1}\left[xy-y^2\right]_{y=x^2}^{x} dx$$

$$=\int_0^1 [x^2-x^2-x^3+x^4]\, dx=\int_0^1 (x^4-x^3)\, dx$$

$$=\left[\frac{x^5}{5}-\frac{x^4}{4}\right]_0^1=\frac{1}{5}-\frac{1}{4}=-\frac{1}{20}.$$

Now let us evaluate the line integral along C. Along $y=x^2$, $dy=2x\, dx$. Therefore along $y=x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2)+x^4\}\, dx+x^2\,(2x)\, dx]$$

$$=\int_0^1 (3x^3+x^4)\, dx=\frac{19}{20}.$$

Along $y=x$, $dy=dx$. Therefore along $y=x$, the line integral equals

$$\int_1^0 [\{(x)(x)+x^2\}\, dx+x^2\, dx]=\int_1^0 3x^4\, dx=-1.$$

Therefore the required line integral $=\frac{19}{20}-1=-\frac{1}{20}$. Hence the theorem is verified.

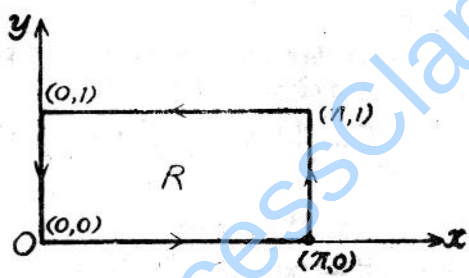
Ex. 2. *Evaluate by Green's theorem*

$$\oint_C (x^2-\cosh y)\, dx+(y+\sin x)\, dy,$$

where C is the rectangle with vertices (0, 0), (π, 0), (π, 1), (0, 1).

Solution. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C (Mdx+Ndy).$$



Here $M=x^2-\cosh y,\ N=y+\sin x$.

$$\therefore \frac{\partial N}{\partial x}=\cos x,\ \frac{\partial M}{\partial y}=-\sinh y.$$

Hence the given line integral is equal to

$$\iint_R (\cos x+\sinh y)\, dx\, dy=\int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x+\sinh y)\, dy\, dx$$

$$=\int_{x=0}^{\pi}\Big[y\cos x+\cosh y\Big]_{y=0}^{1} dx=\int_{x=0}^{\pi} [\cos x+\cosh 1-1]\, dx$$

$$=\Big[\sin x+x\cosh 1-x\Big]_0^{\pi}=\pi(\cosh 1-1).$$

Ex. 3. *Evaluate by Green's theorem*

$$\oint_C (\cos x\sin y-xy)\, dx+\sin x\cos y\, dy,$$

where C is the circle $x^2+y^2=1$.

Solution. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C (M\, dx+N\, dy).$$

Here $M=\cos x \sin y-xy,\ N=\sin x \cos y.$

$$\therefore \quad \frac{\partial M}{\partial y}=\cos x \cos y-x,\ \frac{\partial N}{\partial x}=\cos x \cos y.$$

Hence the given line integral is equal to

$$\iint_R x\,dx\,dy=\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta\, r\,d\theta\,dr, \text{ changing to polars}$$

$$=\int_{\theta=0}^{2\pi}\left[\frac{r^3}{3}\right]_0^1 \cos\theta\,d\theta=\tfrac{1}{3}\left[\sin\theta\right]_0^{2\pi}=\tfrac{1}{3}(0)=0.$$

Ex. 4. *Show that the area bounded by a simple closed curve C is given by* $\frac{1}{2}\oint (x\,dy-y\,dx)$. *Hence find the area of the ellipse* $x=a\cos\theta,\ y=b\sin\theta.$ [Agra 1974]

Solution. By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\oint_C M\,dx+N\,dy.$$

Putting $M=-y,\ N=x$, we get

$$\oint_C (x\,dy-y\,dx)=\iint_R \left[\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right] dx\,dy$$

$$=2\iint_R dx\,dy=2A, \text{ where } A \text{ is the area bounded by } C.$$

Hence $$A=\tfrac{1}{2}\oint_C (x\,dy-y\,dx).$$

The area of the ellipse $=\tfrac{1}{2}\oint_C (x\,dy-y\,dx)$

$$=\tfrac{1}{2}\int_{\theta=0}^{2\pi}\left(a\cos\theta\,\frac{dy}{d\theta}-b\sin\theta\,\frac{dx}{d\theta}\right)d\theta$$

$$=\tfrac{1}{2}\int_0^{2\pi}(ab\cos^2\theta+ab\sin^2\theta)\,d\theta=\tfrac{1}{2}ab\int_0^{2\pi}d\theta=\pi ab.$$

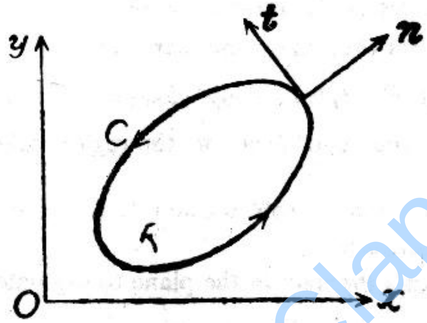
Ex. 5. *Introducing* $\mathbf{A}=N\mathbf{i}-M\mathbf{j}$, *show that the formula in Green's theorem may be written as*

$$\iint_R \text{div}\,\mathbf{A}\,dx\,dy=\oint_C \mathbf{A}\cdot\mathbf{n}\,ds,$$

where **n** *is the outward unit normal vector to C and s is the arc length of C.*

Solution. We have $\mathbf{A}=N\mathbf{i}-M\mathbf{j}$.

$$\therefore \quad \text{div } \mathbf{A}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}.$$



$$\therefore \iint_R \text{div } \mathbf{A}\, dx\, dy=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy$$

$$=\oint_C (M\, dx+N\, dy), \text{ by Green's theorem.}$$

Now $M\, dx+N\, dy=(M\mathbf{i}+N\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})=(M\mathbf{i}+N\mathbf{j})\cdot d\mathbf{r}$

$$=\left\{(M\mathbf{i}+N\mathbf{j})\cdot \frac{d\mathbf{r}}{ds}\right\} ds.$$

Now if **t** is a unit tangent vector to C, then $\mathbf{t}=\frac{d\mathbf{r}}{ds}$. Also if **k** is a unit vector perpendicular to xy-plane, then $\mathbf{t}=\mathbf{k}\times\mathbf{n}$.

$$\therefore \quad M\, dx+N\, dy=[(M\mathbf{i}+N\mathbf{j})\cdot\mathbf{t}]\, ds=[(M\mathbf{i}+N\mathbf{j})\cdot(\mathbf{k}\times\mathbf{n})]\, ds$$
$$=[(M\mathbf{i}+N\mathbf{j})\times\mathbf{k}]\cdot\mathbf{n}\, ds=(M\mathbf{i}\times\mathbf{k}+N\mathbf{j}\times\mathbf{k})\cdot\mathbf{n}\, ds$$
$$=(N\mathbf{i}-M\mathbf{j})\cdot\mathbf{n}\, ds=\mathbf{A}\cdot\mathbf{n}\, ds.$$

Hence the result.

Note. Putting $\mathbf{A}=\nabla\phi$ in the above result, we get

$$\iint_R \text{div } (\nabla\phi)\, dx\, dy=\oint_C (\nabla\phi)\cdot\mathbf{n}\, ds$$

or
$$\iint \nabla^2\phi\, dx\, dy=\oint \frac{\partial\phi}{\partial n}\, ds, \text{ since } \nabla\phi=\frac{\partial\phi}{\partial n}\,\mathbf{n}.$$

Exercises

1. Verify Green's theorem in the plane for

$$\int_C (2xy-x^2)\, dx+(x^2+y^2)\, dy,$$

where C is the boundary of the region enclosed by $y=x^2$ and $y^2=x$ described in the positive sense. [Meerut 1973]

2. Verify Green's theorem in the plane for

$$\int_C [(3x^2-8y^2)\, dx+(4y-6xy)\, dy)],$$

where C is the boundary of the region defined by $y=\sqrt{x}$, $y=x^2$.

[Hint. Proceed as in solved example 1. Here each integral will come out to be $\frac{3}{2}$].

3. Apply Green's theorem in the plane to evaluate

$$\int_C \{(y-\sin x)\, dx+\cos x\, dy\},$$

where C is the triangle enclosed by the lines

$$y=0,\ x=\pi,\ \pi y=2x.$$ [Agra 1973]

Ans. $-\dfrac{\pi}{4}-\dfrac{2}{\pi}$.

$\Bigg[$**Hint.** Here $M=y-\sin x$, $N=\cos x$. Therefore the given

integral$=\displaystyle\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\int_{x=0}^{\pi/2}\int_{y=0}^{(2/\pi)x} (-\sin x-1)\, dx\, dy\Bigg]$

4. Evaluate by Green's theorem in plane

$$\int_C (e^{-x} \sin y\, dx+e^{-x} \cos y\, dy),$$

where C is the rectangle with vertices

$(0, 0)$, $(\pi, 0)$, $(\pi, \frac{1}{2}\pi)$, $(0, \frac{1}{2}\pi)$. Ans. $2(e^{-\pi}-1)$.

5. If $\mathbf{F}=(x^2-y^2)\,\mathbf{i}+2xy\mathbf{j}$ and $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, find the value of $\int \mathbf{F}\cdot d\mathbf{r}$ around the rectangular boundary $x=0$, $x=a$, $y=0$, $y=b$. [Gauhati 1973]

Ans. $2ab^2$.

6. Verify Green's theorem in the plane for

$$\int_C (x^2-xy^3)\, dx+(y^2-2xy)\, dy,$$

where C is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$. [Meerut 1974]

7. Apply Green's theorem in the plane to evaluate

$$\int_C [(2x^2 - y^2)\, dx + (x^2+y^2)\, dy],$$

where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y=(1-x^2)^{1/2}$. **Ans.** 4/3.

[**Hint.** By Green's theorem the given integral

$$= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{(1-x^2)}} (2x+2y)\, dx\, dy \Big].$$

8. If C is the simple closed curve in the xy-plane not enclosing the origin, show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0, \text{ where } \mathbf{F} = \frac{-\mathbf{i}y + \mathbf{j}x}{x^2+y^2}.$$

§ 7. The Divergence theorem of Gauss.

Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\mathbf{F}(x, y, z)$ is a vector function of position which is continuous and has continuous first partial derivatives in V. Then

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outwards drawn unit normal vector to S.

[Kanpur 1977, 79; Agra 72; Allahabad 80; Rohilkhand 80; Madras 83; Kerala 75; Meerut B. Sc. Physics 83]

Since $\mathbf{F} \cdot \mathbf{n}$ is the normal component of vector $\mathbf{F}$, therefore divergence theorem may also be stated as follows :

The surface integral of the normal component of a vector $\mathbf{F}$ taken over a closed surface is equal to the integral of the divergence of $\mathbf{F}$ taken over the volume enclosed by the surface.

Cartesian equivalent of Divergence Theorem.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then $\nabla \cdot \mathbf{F} = \text{div}\, \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

If α, β, γ are the angles which outward drawn unit normal $\mathbf{n}$ makes with positive directions of x, y, z-axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\mathbf{n}$ and we have

$$\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}.$$

$$\therefore \quad \mathbf{F} \cdot \mathbf{n} = (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot (\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$$

$$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma.$$

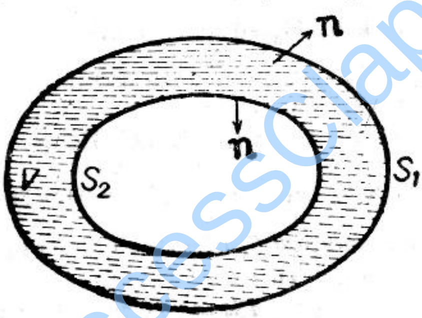
Therefore the divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

or
$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

The proof of the theorem can now be extended to a region V which can be subdivided into finitely many special regions of the above type by drawing auxiliary surfaces. In this case we apply the theorem to each sub-region and then add the results. The sum of the volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxiliary surfaces cancel in pairs, while the sum of the remaining surface integrals is equal to the surface integral over the whole boundary S of V.

Note. The divergence theorem is applicable for a region Y if it is bounded by two closed surfaces S_1 and S_2 one of which lies



within the other. Here outward drawn normals will have the directions as shown in the figure.

§ 8. Some deductions from divergence theorem.

1. Green's theorem. *Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\, \nabla^2\psi - \psi\, \nabla^2\phi)\, dV = \iint_S (\phi\, \nabla\psi - \psi\, \nabla\phi) \cdot \mathbf{n}\, dS.$$

[Agra 1971, Gauhati 72; M. U. 1979; Indore 1979]

Proof. By divergence theorem, we have

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Putting $\mathbf{F} = \phi\, \nabla\psi$, we get

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\phi\, \nabla\psi)$$
$$= \phi\, (\nabla \cdot \nabla\psi) + (\nabla\phi) \cdot (\nabla\psi) = \phi\, \nabla^2\psi + (\nabla\phi) \cdot (\nabla\psi).$$

Also $\quad \mathbf{F} \cdot \mathbf{n} = (\phi\, \nabla\psi) \cdot \mathbf{n}.$

$\therefore$ divergence theorem gives

$$\iiint_V [\phi\nabla^2\psi+(\nabla\phi)\cdot(\nabla\psi)]\, dV = \iint_S (\phi\nabla\psi)\cdot\mathbf{n}\, dS \quad ...(1)$$

[Meerut 1970]

This is called *Green's first identity or theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2\phi+(\nabla\psi)\cdot(\nabla\phi)]\, dV = \iint_S [\psi\nabla\phi]\cdot\mathbf{n}\, dS \quad ...(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\, dV=\iint_S (\phi\nabla\psi-\psi\nabla\phi)\cdot\mathbf{n}\, dS \quad ...(3)$$

This is called *Green's second identity or Green's theorem in symmetrical form.*

Since $\nabla\psi=\dfrac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi=\dfrac{\partial\phi}{\partial n}\mathbf{n}$, therefore

$$(\phi\nabla\psi-\psi\nabla\phi)\cdot\mathbf{n}=\left(\phi\frac{\partial\psi}{\partial n}\mathbf{n}-\psi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}$$

$$=\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\, dV=\iint_S \left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right) dS.$$

[Meerut 1972, 80]

Note. Harmonic function. *If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi=0$, then ϕ is called harmonic function. If ϕ and ψ* are both harmonic functions, then $\nabla^2\phi=0$, $\nabla^2\psi=0$. Hence from Green's second identity, we get

$$\iint_S \left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right) dS=0.$$

2. *Prove that* $\displaystyle\iiint_V \nabla\phi\, dV=\iint_S \phi\mathbf{n}\, dS.$

[Agra 1972; Allahabad 77]

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\, dV=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Taking $\mathbf{F}=\phi\mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla \cdot (\phi \mathbf{C})\, dV = \iint_S (\phi \mathbf{C}) \cdot \mathbf{n}\, dS \qquad ...(1)$$

Now $\nabla \cdot (\phi \mathbf{C}) = (\nabla \phi) \cdot \mathbf{C} + \phi (\nabla \cdot \mathbf{C})$

$= (\nabla \phi) \cdot \mathbf{C}$, since $\nabla \cdot \mathbf{C} = 0$.

Also $(\phi \mathbf{C}) \cdot \mathbf{n} = \mathbf{C} \cdot (\phi \mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C} \cdot (\nabla \phi)\, dV = \iint_S \mathbf{C} \cdot (\phi \mathbf{n})\, dS$$

or $$\mathbf{C} \cdot \iiint_V \nabla \phi\, dV = \mathbf{C} \cdot \iint_S \phi \mathbf{n}\, dS$$

or $$\mathbf{C} \cdot \left[\iiint_V \nabla \phi\, dV - \iint_S \phi \mathbf{n}\, dS \right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla \phi\, dV = \iint_S \phi \mathbf{n}\, dS.$$

3. *Prove that* $\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$

[Gauhati 1971, 74]

Proof. In divergence theorem taking $\mathbf{F} = \mathbf{B} \times \mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla \cdot (\mathbf{B} \times \mathbf{C})\, dV = \iint_S (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{n}\, dS. \qquad ...(1)$$

Now $\nabla \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot \text{curl } \mathbf{B} - \mathbf{B} \cdot \text{curl } \mathbf{C}$

$= \mathbf{C} \cdot \text{curl } \mathbf{B}$, since curl $\mathbf{C} = \mathbf{0}$.

Also $(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{n} = [\mathbf{B}, \mathbf{C}, \mathbf{n}] = [\mathbf{C}, \mathbf{n}, \mathbf{B}] = \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C} \cdot \text{curl } \mathbf{B})\, dV = \iint_S \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \cdot \iiint_V (\nabla \times \mathbf{B})\, dV = \mathbf{C} \cdot \iint_S (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \cdot \left[\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS \right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Hence we must have

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS = \mathbf{0}$$

or $$\iiint_V (\nabla \times \mathbf{B})\, dV = \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

SOLVED EXAMPLES

Ex. 1. *For any closed surface S, prove that*

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = 0.$$

Solution. By divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V (\text{div curl } \mathbf{F}) \, dV, \text{ where } V \text{ is the volume enclosed by } S$$

$$= 0, \text{ since div curl } \mathbf{F} = 0.$$

Ex. 2. *Evaluate* $\iint_S \mathbf{r} \cdot \mathbf{n} \, dS$, *where S is a closed surface.*

[Madras 1983; Rohilkhand 76; Allahabad 75]

Solution. By the divergence theorem, we have

$$\iint_S \mathbf{r} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{r} \, dV$$

$$= \iiint_V 3 \, dV, \text{ since } \nabla \cdot \mathbf{r} = \text{div } \mathbf{r} = 3$$

$$= 3V, \text{ where } V \text{ is the volume enclosed by } S.$$

Ex. 3. *If* $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$, *a, b, c are constants, show that* $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \frac{4}{3} \pi (a+b+c)$, *where S is the surface of a unit sphere.*

[Kerala 1974; Agra 80; Rohilkhand 77; Allahabad 80, 82]

Solution. By the divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V (\nabla \cdot \mathbf{F}) \, dV,$$

where V is the volume enclosed by S

$$= \iiint_V [\nabla \cdot (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k})] \, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right] dV$$

$$= \iiint_V (a+b+c) \, dV = (a+b+c) V = (a+b+c) \frac{4}{3} \pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3} \pi (1)^3$ *i.e.*, $\frac{4}{3} \pi$.

Ex. 4. *If* $\mathbf{n}$ *is the unit outward drawn normal to any closed surface S, show that* $\iiint_V \text{div } \mathbf{n} \, dV = S$.

Solution. We have by the divergence theorem,

$$\iiint_V \text{div } \mathbf{n}\, dV = \iint_S \mathbf{n}\cdot\mathbf{n}\, dS = \iint_S dS = S.$$

Ex. 5. *Prove that*

$$\iiint_V \nabla\phi\cdot\mathbf{A}\, dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\, dS - \iiint_V \phi\nabla\cdot\mathbf{A}\, dV.$$

Solution. By divergence theorem, we have

$$\iiint_V \nabla\cdot(\phi\mathbf{A})\, dV = \iint_S (\phi\mathbf{A})\cdot\mathbf{n}\, dS. \qquad \text{...(1)}$$

Now $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})$.

Also $(\phi\mathbf{A})\cdot\mathbf{n} = \phi(\mathbf{A}\cdot\mathbf{n})$.

Hence (1) gives

$$\iiint_V [(\nabla\phi)\cdot\mathbf{A} + \phi(\nabla\cdot\mathbf{A})]\, dV = \iint_S \phi\,\mathbf{A}\cdot\mathbf{n}\, dS$$

or $$\iiint_V (\nabla\phi)\cdot\mathbf{A}\, dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\, dS - \iiint_V \phi\,\nabla\cdot\mathbf{A}\, dV.$$

Ex. 6. *Prove that* $\int_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = 0.$

Solution. We have $\int_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = \int_S (\nabla\phi\times\nabla\psi)\cdot\mathbf{n}\, dS$

$= \int_V \nabla\cdot(\nabla\phi\times\nabla\psi)\, dV$, by divergence theorem

$= 0$ $\quad [\because \nabla\cdot(\nabla\phi\times\nabla\psi) = 0.$ See Ex. 13 page 65]

Ex. 7. *Prove that*

$$\int_V \nabla\phi\cdot\text{curl } \mathbf{F}\, dV = \int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S}.$$

Solution. We have $\int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S} = \int_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\, dS$

$= \int_V \nabla\cdot(\mathbf{F}\times\nabla\phi)\, dV$, by divergence theorem applied to the vector function $\mathbf{F}\times\nabla\phi$

$= \int_V (\nabla\phi\cdot\text{curl } \mathbf{F} - \mathbf{F}\cdot\text{curl } \nabla\phi)\, dV$

[By vector identity 5 on page 57]

$= \int_V \nabla\phi\cdot\text{curl } \mathbf{F}\, dV.$ $\quad [\because \text{curl } \nabla\phi = 0]$

Ex. 8. *Prove that* $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS.$

Solution. $\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS = \iint_S \left(\frac{\mathbf{r}}{r^2}\right)\cdot\mathbf{n}\, dS$

$$=\iiint_V \nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right) dV, \text{ by divergence theorem.}$$

Now $\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)=\frac{1}{r^2}(\nabla\cdot\mathbf{r})+\mathbf{r}\cdot\nabla\left(\frac{1}{r^2}\right)$

$$=\frac{3}{r^2}+\mathbf{r}\cdot\left(-\frac{2}{r^3}\nabla r\right)=\frac{3}{r^2}-\frac{2}{r^3}\left(\mathbf{r}\cdot\frac{\mathbf{r}}{r}\right)=\frac{3}{r^2}-\frac{2}{r^4}r^2=\frac{1}{r^2}.$$

Hence $\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2} dS=\iiint_V \frac{dV}{r^2}.$

Ex. 9. *If* $\mathbf{F}=\nabla\phi$ *and* $\nabla^2\phi=0$, *show that for a closed surface S*

$$\iiint_V \mathbf{F}^2\, dV=\iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS.$$ **[Rohilkhand 1978, 79]**

Solution. By divergence theorem, we have

$$\iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V [\nabla\cdot(\phi\mathbf{F})]\, dV.$$

Now $\nabla\cdot(\phi\mathbf{F})=(\nabla\phi\cdot\mathbf{F})+\phi(\nabla\cdot\mathbf{F})=\mathbf{F}\cdot\mathbf{F}+\phi(\nabla\cdot\nabla\phi)$

$$=\mathbf{F}^2+\phi\nabla^2\phi=\mathbf{F}^2, \text{ since } \nabla^2\phi=0.$$

Hence $\iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V \mathbf{F}^2\, dV.$

Ex. 10. *If* $\mathbf{F}=\nabla\phi$, $\nabla^2\phi=-4\pi\rho$, *show that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=-4\pi\iiint_V \rho\, dV.$$

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V (\nabla\cdot\mathbf{F})\, dV.$$

Now $\nabla\cdot\mathbf{F}=\nabla\cdot(\nabla\phi)=\nabla^2\phi=-4\pi\rho.$

$$\therefore \iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V (-4\pi\rho)\, dV=-4\pi\iiint_V \rho\, dV.$$

Ex 11. *If* $\mathbf{C}=\frac{1}{2}\nabla\times\mathbf{B}$, $\mathbf{B}=\nabla\times\mathbf{A}$, *show that*

$$\frac{1}{2}\iiint_V \mathbf{B}^2\, dV=\frac{1}{2}\iint_S \mathbf{A}\times\mathbf{F}\cdot\mathbf{n}\, dS+\iiint_V \mathbf{A}\cdot\mathbf{C}\, dV.$$

Solution. We have by divergence theorem

$$\frac{1}{2}\iint_S (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\, dS=\frac{1}{2}\iiint_V \nabla\cdot(\mathbf{A}\times\mathbf{B})\, dV.$$

Now $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\text{curl }\mathbf{B}$

$$=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})=\mathbf{B}\cdot\mathbf{B}-\mathbf{A}\cdot(2\mathbf{C})=\mathbf{B}^2-2(\mathbf{A}\cdot\mathbf{C}).$$

Hence $\frac{1}{2}\iint_S (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\, dS=\frac{1}{2}\iiint_V [\mathbf{B}^2-2(\mathbf{A}\cdot\mathbf{C})]\, dV$

$$=\frac{1}{2}\iiint_V \mathbf{B}^2\, dV-\iiint_V \mathbf{A}\cdot\mathbf{C}\, dV$$

or $\frac{1}{2}\iiint_V \mathbf{B}^2\, dV = \frac{1}{2}\iint_S \mathbf{A}\times\mathbf{B}\cdot\mathbf{n}\, dS + \iiint_V \mathbf{A}\cdot\mathbf{C}\, dV.$

Ex. 12. *If ϕ is harmonic in V, then*

$$\iint_S \frac{\partial \phi}{\partial n}\, dS = 0$$

where S is the surface enclosing V. [Meerut 1972]

Solution. We have $\iint_S \frac{\partial \phi}{\partial n}\, dS = \iint_S \left(\frac{\partial \phi}{\partial n}\hat{\mathbf{n}}\right)\cdot\mathbf{n}\, dS$

$= \iint_S (\nabla\phi)\cdot\mathbf{n}\, dS$

$= \iiint_V \nabla\cdot(\nabla\phi)\, dV$, by divergence theorem

$= \iiint_V \nabla^2\phi\, dV$

$= 0$, since $\nabla^2\phi = 0$ in V because ϕ is harmonic in V.

Ex. 13. *If ϕ is harmonic in V, then*

$$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iiint_V |\nabla\phi|^2\, dV.$$

[Meerut 1969, Agra 70]

Solution. We have

$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iint_S \left(\phi \frac{\partial \phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}\, dS = \iint_S (\phi\nabla\phi)\cdot\mathbf{n}\, dS$

$= \iiint_V \nabla\cdot(\phi\nabla\phi)\, dV$, by divergence theorem

$= \iiint_V [(\nabla\phi\cdot\nabla\phi) + \phi(\nabla\cdot\nabla\phi)]\, dV$

$= \iiint_V [(\nabla\phi)^2 + \phi\nabla^2\phi]\, dV$

$= \iiint_V |\nabla\phi|^2\, dV$, since $\nabla^2\phi = 0$ and $(\nabla\phi)^2 = |\nabla\phi|^2$.

Ex. 14. *If ϕ is harmonic in V and $\frac{\partial \phi}{\partial n} = 0$ on S, then ϕ is constant in V.*

Solution. Since ϕ is harmonic in V, therefore as in exercise 13, we have

$$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iiint_V |\nabla\phi|^2\, dV.$$

But $\frac{\partial \phi}{\partial n} = 0$ on S. Therefore $\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = 0.$

$\therefore \quad \iiint_V |\nabla\phi|^2 \, dV = 0.$

$\therefore \quad |\nabla\phi|^2 = 0$ in V.

$\therefore \quad \nabla\phi = 0$ in V.

$\therefore \quad \phi =$ constant in V.

Ex. 15. *If ϕ and ψ are harmonic in V and $\frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial n}$ on S, then $\phi = \psi + c$ in V, where c is a constant.*

Solution. We have, $\nabla^2\phi = 0$, $\nabla^2\psi = 0$ in V.

$\therefore \quad \nabla^2(\phi-\psi) = \nabla^2\phi - \nabla^2\psi = 0$ in V.

Therefore $\phi - \psi$ is harmonic in V.

Again on S, $\frac{\partial}{\partial n}(\phi-\psi) = \frac{\partial\phi}{\partial n} - \frac{\partial\psi}{\partial n} = 0.$

Thus $\phi-\psi$ is harmonic in V and on S we have

$$\frac{\partial}{\partial n}(\phi-\psi) = 0.$$

Hence as in exercise 14, we have

$\phi - \psi = c$, where c is a constant

or $\quad \phi = \psi + c$.

Ex. 16. *If div* **F** *denotes the divergence of a vector field* **F** *at a point P, show that*

$$\text{div } \mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F}\cdot\mathbf{n} \, dS}{\delta V}$$

where δV is the volume enclosed by the surface δS and the limit is obtained by shrinking δV to the point P.

Solution. We have by the divergence theorem,

$$\iiint_{\delta V} \text{div } \mathbf{F} \, dV = \iint_{\delta S} \mathbf{F}\cdot\mathbf{n} \, dS. \qquad \ldots(1)$$

By the mean value theorem of integral calculus, the left hand side can be written as

$$\overline{\text{div } \mathbf{F}} \iiint_{\delta V} dV = \overline{\text{div } \mathbf{F}} \, \delta V,$$

where $\overline{\text{div } \mathbf{F}}$ is some value intermediate between the maximum and minimum of div **F** throughout δV. Therefore (1) gives

$$\overline{\text{div } \mathbf{F}} \, \delta V = \iint_{\delta S} \mathbf{F}\cdot\mathbf{n} \, dS$$

or
$$\overline{\text{div } \mathbf{F}} = \frac{\iint_{\delta S} \mathbf{F} \cdot \mathbf{n} \, dS}{\delta V}.$$

Taking the limit as $\delta V \to 0$ such that P is always interior to δV, $\overline{\text{div } \mathbf{F}}$ approaches the value div $\mathbf{F}$ at point P. Hence, we get
$$\text{div } \mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F} \cdot \mathbf{n} \, dS}{\delta V}.$$

Ex. 17. *Show that* $\iint_S \mathbf{n} \, dS = 0$ *for any closed surface* S.

Solution. Let $\mathbf{C}$ be any arbitrary constant vector. Then
$$\mathbf{C} \cdot \iint_S \mathbf{n} \, dS = \iint_S \mathbf{C} \cdot \mathbf{n} \, dS$$
$$= \iiint_V (\nabla \cdot \mathbf{C}) \, dV, \text{ by divergence theorem}$$
$$= 0, \text{ since div } \mathbf{C} = 0.$$

Thus $\mathbf{C} \cdot \iint_S \mathbf{n} \, dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n} \, dS = 0$.

Ex. 18. *Prove that* $\iint_S \mathbf{r} \times \mathbf{n} \, dS = 0$ *for any closed surface* S.

Solution. Let $\mathbf{C}$ be any arbitrary constant vector. Then
$$\mathbf{C} \cdot \iint_S \mathbf{r} \times \mathbf{n} \, dS = \iint_S \mathbf{C} \cdot [(\mathbf{r} \times \mathbf{n})] \, dS = \iint_S (\mathbf{C} \times \mathbf{r}) \cdot \mathbf{n} \, dS$$
$$= \iiint_V [\nabla \cdot (\mathbf{C} \times \mathbf{r})] \, dV, \text{ by divergence theorem}$$
$$= \iiint_V [\mathbf{r} \cdot \text{curl } \mathbf{C} - \mathbf{C} \cdot \text{curl } \mathbf{r}] \, dV$$
$$= 0, \text{ since curl } \mathbf{C} = 0 \text{ and } \mathbf{r} = 0.$$

Thus $\mathbf{C} \cdot \iint_S \mathbf{r} \times \mathbf{n} \, dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r} \times \mathbf{n} \, dS = 0$.

Ex. 19. *Prove that* $\iint_S (\nabla \phi) \times \mathbf{n} \, dS = 0$ *for a closed surface* S.

Solution. Let $\mathbf{C}$ be an arbitrary constant vector. Then

$$\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\, dS=\iint_S \mathbf{C}\cdot[(\nabla\phi)\times\mathbf{n}]\, dS$$

$$=\iint_S [\mathbf{C}\times\nabla\phi]\cdot\mathbf{n}\, dS$$

$$=\iiint_V [\nabla\cdot(\mathbf{C}\times\nabla\phi)]\, dV, \text{ by div. theorem}$$

$$=\iiint_V [\nabla\phi\cdot\text{curl }\mathbf{C}-\mathbf{C}\cdot\text{curl }\nabla\phi]\, dV$$

$$=0, \text{ since curl }\mathbf{C}=\mathbf{0}\text{ and curl }\nabla\phi=\mathbf{0}.$$

Thus $\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\, dS=0$, where $\mathbf{C}$ is an arbitrary vector.

Hence we must have $\iint_S (\nabla\phi)\times\mathbf{n}\, dS=\mathbf{0}$.

Ex. 20. *Prove that* $\iint \mathbf{n}\times(\mathbf{a}\times\mathbf{r})\, dS=2V\mathbf{a}$,

where $\mathbf{a}$ *is a constant vector and* V *is the volume enclosed by the closed surface* S.

Solution. We know that

$$\iiint_V \nabla\times\mathbf{B}\,dV=\iint_S \mathbf{n}\times\mathbf{B}\, dS. \qquad \text{[see page 110]}$$

Putting $\mathbf{B}=\mathbf{a}\times\mathbf{r}$, we get

$$\iint_S \mathbf{n}\times(\mathbf{a}\times\mathbf{r})\, dS=\iiint_V \nabla\times(\mathbf{a}\times\mathbf{r})\, dV$$

$$=\iiint_V \text{curl }(\mathbf{a}\times\mathbf{r})\, dV$$

$$=\iiint_V 2\mathbf{a}\, dV, \text{ since curl }(\mathbf{a}\times\mathbf{r})=2\mathbf{a}$$

$$=2\mathbf{a}\iiint_V dV=2\mathbf{a}V.$$

Ex. 21. *A vector* $\mathbf{B}$ *is always normal to a given closed surface* S. *Show that* $\iiint_V \text{curl }\mathbf{B}\, dV=\mathbf{0}$, *where* V *is the region bounded by* S.

Solution. We know that

$$\iiint_V \text{curl }\mathbf{B}\, dV=\iint_S \mathbf{n}\times\mathbf{B}\, dS.$$

Since $\mathbf{B}$ is normal to S, therefore $\mathbf{B}$ is parallel to $\mathbf{n}$. Therefore $\mathbf{n}\times\mathbf{B}=\mathbf{0}$.

$$\therefore \quad \iint_S \mathbf{n}\times\mathbf{B}\, dS=\mathbf{0}.$$

$$\therefore \quad \iiint_V \text{curl } \mathbf{B} \, dV = \mathbf{0}.$$

Ex. 22. *Express* $\int_V \{(\text{grad } \rho)\cdot\mathbf{v}+\rho \text{ div } \mathbf{v}\}\, dV$, *as a surface integral.* [Gauhati 1972, 77]

Solution. We know that

$$\text{div } (\rho\mathbf{v})=(\text{grad } \rho)\cdot\mathbf{v}+\rho \text{ div } \mathbf{v}.$$

[See vector identity 3 on page 56].

$$\therefore \quad \int_V \{(\text{grad } \rho)\cdot\mathbf{v}+\rho \text{ div } \mathbf{v}\}\, dV=\int_V \text{div } (\rho\mathbf{v})\, dV$$

$$=\int_V \nabla\cdot(\rho\mathbf{v})\, dV$$

$$=\int_S (\rho\mathbf{v})\cdot\mathbf{n}\, dS, \text{ by Gauss divergence theorem}$$

$$=\int_S \rho\, (\mathbf{v}\cdot\mathbf{n})\, dS.$$

Ex. 23. *Using the divergence theorem, show that the volume V of a region T bounded by a surface S is*

$$V=\iint_S x\, dy\, dz=\iint_S y\, dz\, dx=\iint_S z\, dx\, dy$$

$$=\tfrac{1}{3}\iint_S (x\, dy\, dz+y\, dz\, dx+z\, dx\, dy).$$

Solution. By divergence theorem, we have

$$\iint_S x\, dy\, dz=\iiint_V \left(\frac{\partial}{\partial x}(x)\right) dV=\iiint_V dV=V$$

$$\iint_S y\, dz\, dx=\iiint_V \left[\frac{\partial}{\partial y}(y)\right] dV=\iiint_V dV=V$$

$$\iint_S z\, dx\, dy=\iiint_V \left[\frac{\partial}{\partial z}(z)\right] dV=\iiint_V dV=V.$$

Adding these results, we get

$$3V=\iint_S (x\, dy\, dz+y\, dz\, dx+z\, dx\, dy)$$

or

$$V=\tfrac{1}{3}\iint_S (x\, dy\, dz+y\, dz\, dx+z\, dx\, dy).$$

Ex. 24. *Verify divergence theorem for*

$$\mathbf{F}=(x^2-yz)\,\mathbf{i}+(y^2-zx)\,\mathbf{j}+(z^2-xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped

$$0\leqslant x\leqslant a,\ 0\leqslant y\leqslant b,\ 0\leqslant z\leqslant c.$$ [Meerut 1976]

Solution. We have div $\mathbf{F}=\nabla\cdot\mathbf{F}$

$$=\frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy)=2x+2y+2z.$$

$$\therefore \quad \text{volume integral}=\iiint_V \nabla\cdot\mathbf{F}\,dV=\iiint_V 2(x+y+z)\,dV$$

$$=2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(x+y+z)\,dx\,dy\,dz$$

$$=2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{x^2}{2}+yx+zx\right]_{x=0}^{a}dy\,dz$$

$$=2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2}+ay+az\right]dy\,dz=2\int_{z=0}^{c}\left[\frac{a^2}{2}y+a\frac{y^2}{2}+azy\right]_{y=0}^{b}dz$$

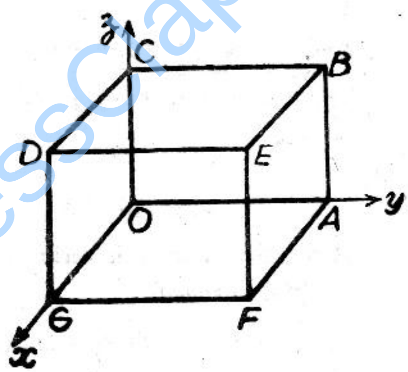
$$=2\int_{z=0}^{c}\left[\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right]dz=2\left[\frac{a^2b}{2}z+\frac{ab^2}{2}z+ab\frac{z^2}{2}\right]_0^c$$

$$=[a^2bc+ab^2c+abc^2]=abc(a+b+c).$$

Surface Integral. We shall now calculate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

over the six faces of the rectangular parallelopiped.



Over the face *DEFG*,

$\mathbf{n}=\mathbf{i},\ x=a.$

Therefore

$$\iint_{DEFG}\mathbf{F}\cdot\mathbf{n}\,dS$$

$$=\int_{z=0}^{c}\int_{y=0}^{b}[(a^2-yz)\,\mathbf{i}+(y^2-za)\,\mathbf{j}+(z^2-ay)\,\mathbf{k}]\cdot\mathbf{i}\,dy\,dz$$

$$=\int_{z=0}^{c}\int_{y=0}^{b}(a^2-yz)\,dy\,dz=\int_{z=0}^{c}\left[a^2y-z\frac{y^2}{2}\right]_{y=0}^{b}dz$$

$$=\int_{z=0}^{c}\left[a^2b-\frac{zb^2}{2}\right]dz=\left[a^2bz-\frac{z^2}{4}b^2\right]_0^c$$

$$=a^2bc-\frac{c^2b^2}{4}.$$

Over the face *ABCO*, $\mathbf{n}=-\mathbf{i},\ x=0.$ Therefore

$$\iint_{ABCO}\mathbf{F}\cdot\mathbf{n}\,dS=\iint[(0-yz)\,\mathbf{i}+\ldots+\ldots]\cdot(-\mathbf{i})\,dy\,dz$$

$$=\int_{z=0}^{c}\int_{y=0}^{b}yz\,dy\,dz=\int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_{y=0}^{b}dz=\int_{z=0}^{c}\frac{b^2}{2}z\,dz=\frac{b^2c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n}=\mathbf{j}$, $y=b$. Therefore

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{z=0}^{c}\int_{x=0}^{a} [(x^2-bz)\,\mathbf{i}+(b^2-zx)\,\mathbf{j} +(z^2-bx)\,\mathbf{k}]\cdot\mathbf{j}\, dx\, dz$$

$$=\int_{z=0}^{c}\int_{x=0}^{a} (b^2-zx)\, dx\, dz=b^2 ca-\frac{a^2c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n}=-\mathbf{j}$, $y=0$. Therefore

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{z=0}^{c}\int_{x=0}^{a} zx\, dx\, dz=\frac{c^2a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n}=\mathbf{k}$, $z=c$. Therefore

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{y=0}^{b}\int_{x=0}^{a} (c^2-xy)\, dx\, dy=c^2ab-\frac{a^2b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n}=-\mathbf{k}$, $z=0$. Therefore

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{y=0}^{b}\int_{x=0}^{a} xy\, dx\, dy=\frac{a^2b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_{S} \mathbf{F}\cdot\mathbf{n}\, dS=\left(a^2bc-\frac{c^2b^2}{4}+\frac{c^2b^2}{4}\right)+\left(b^2ca-\frac{a^2c^2}{4}+\frac{a^2c^2}{4}\right) +\left(c^2ab-\frac{a^2b^2}{4}+\frac{a^2b^2}{4}\right)$$

$$=abc\,(a+b+c).$$

Hence the theorem is verified.

Ex. 25. *Evaluate*

$$\iint_S x^2\, dy\, dz+y^2\, dz\, dx+2z\,(xy-x-y)\, dx\, dy$$

where S is the surface of the cube

$$0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1,\ 0\leqslant z\leqslant 1.$$ **[Meerut 1968]**

Solution. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}\{2z\,(xy-x-y)\}\right] dV$$

$$=\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} [2x+2y+2xy-2x-2y]\, dx\, dy\, dz$$

$$=2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xy\, dx\, dy\, dz=2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}\,y\right]_{x=0}^{1} dy\, dz$$

$$=2\int_{z=0}^{1}\int_{y=0}^{1}\frac{y}{2}\, dy\, dz=\int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz$$

$$=\int_{z=0}^{1}\tfrac{1}{2}\, dz=\tfrac{1}{2}\Big[\,z\,\Big]_0^1=\tfrac{1}{2}.$$

Ex. 26. *By transforming to a triple integral evaluate*

$$I=\iint_S (x^3\, dy\, dz+x^2y\, dz\, dx+x^2z\, dx\, dy)$$

where S is the closed surface bounded by the planes $z=0$, $z=b$ *and the cylinder* $x^2+y^2=a^2$. [Meerut 1969, 80]

Solution. By divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial z}(x^2z)\right] dV$$

$$=\int_{z=0}^{b}\int_{y=-a}^{a}\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}}(3x^2+x^2+x^2)\, dx\, dy\, dz$$

$$=4\times 5\int_{z=0}^{b}\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}} x^2\, dx\, dy\, dz$$

$$=20\int_{z=0}^{b}\int_{y=0}^{a}\left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy\, dz=\frac{20}{3}\int_{z=0}^{b}\int_{y=0}^{a}(a^2-y^2)^{3/2} dy dz$$

$$=\frac{20}{3}\int_{y=0}^{a}\left[(a^2-y^2)^{3/2}\, z\right]_{z=0}^{b} dy=\frac{20}{3}\int_{y=0}^{a} b\,(a^2-y^2)^{3/2}\, dy.$$

Put $y=a \sin t$ so that $dy=a \cos t\, dt$.

$$\therefore \quad I=\frac{20}{3}\, b\int_0^{\pi/2} a^3 \cos^3 t\,(a \cos t)\, dt$$

$$=\frac{20}{3}\, a^4 b\int_0^{\pi/2}\cos^4 t\, dt=\frac{20}{3}\, a^4 b\,\frac{3}{4.2}\,\frac{\pi}{2}=\frac{5}{4}\,\pi a^4 b.$$

Ex. 27. *Apply Gauss's divergence theorem to evaluate*

$$\iint_S [(x^3-yz)\, dy\, dz-2x^2y\, dz\, dx+z\, dx\, dy]$$

over the surface of a cube bounded by the coordinate planes and the planes $x=y=z=a$.

Solution. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy)$$

$$=\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

Here $F_1=x^3-yz$, $F_2=-2x^2y$, $F_3=z$.

$$\therefore \quad \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=3x^2-2x^2+1=x^2+1.$$

$\therefore$ the given surface integral is equal to the volume integral

$$\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2+1)\, dx\, dy\, dz$$

$$=\int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dy\, dz$$

$$=\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{3}+a\right)dy\,dz=a^2\left(\frac{a^3}{3}+a\right).$$

Ex. 28. *If* $\mathbf{F}=x\mathbf{i}-y\mathbf{j}+(z^2-1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *where S is the closed surface bounded by the planes* $z=0$, $z=1$ *and the cylinder* $x^2+y^2=4$. [Kanpur 1978, 80]

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{F}\,dV.$$

Here $\text{div}\,\mathbf{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(z^2-1)$

$$=1-1+2z=2z.$$

$$\therefore \iiint_V \text{div}\,\mathbf{F}\,dV=\int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} 2z\,dx\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}\Big[2zx\Big]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}4z\sqrt{(4-y^2)}\,dy\,dz=\int_{y=-2}^{2}\left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1}dy$$

$$=2\int_{y=-2}^{2}\sqrt{(4-y^2)}\,dy=4\int_0^2\sqrt{(4-y^2)}\,dy$$

$$=4\left[\frac{y}{2}\sqrt{(4-y^2)}+2\sin^{-1}\frac{y}{2}\right]_0^2=4\,[2\sin^{-1}1]=4(2)\frac{\pi}{2}=4\pi.$$

Ex. 29. *Find* $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$,

where $\mathbf{A}=(2x+3z)\mathbf{i}-(xz+y)\mathbf{j}+(y^2+2z)\mathbf{k}$

and S is the surface of the sphere having centre at (3, −1, 2) *and radius* 3. [Meerut 1974]

Solution. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{A}\,dV.$$

Now $\text{div}\,\mathbf{A}=\frac{\partial}{\partial x}(2x+3z)+\frac{\partial}{\partial y}\{-(xz+y)\}+\frac{\partial}{\partial z}(y^2+2z)$

$$=2-1+2=3.$$

$$\therefore \iint_S \mathbf{A}\cdot\mathbf{n}\,dS=\iiint_V 3\,dV=3\iiint_V dV=3V.$$

But V is the volume of a sphere of radius 3. Therefore $V=\frac{4}{3}\pi(3)^3=36\pi$.

$$\therefore \iint_S \mathbf{A}\cdot\mathbf{n}\,dS=3V=3\times 36\pi=108\pi.$$

Ex. 30. *Apply divergence theorem to evaluate*

$$\iint_S [(x+z)\, dy\, dz+(y+z)\, dz\, dx+(x+y)\, dx\, dy]$$

where S is the surface of the sphere $x^2+y^2+z^2=4$.

Solution. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x+z)+\frac{\partial}{\partial y}(y+z)+\frac{\partial}{\partial z}(x+y)\right] dV$$

$=\iiint_V 2dV=2\iiint_V dV=2V$, where V is the volume of the sphere $x^2+y^2+z^2=4$

$$=2\left[\frac{4}{3}\pi(2)^3\right]=\frac{64}{3}\pi.$$

Ex. 31. *If S is any closed surface enclosing a volume V and* $\mathbf{F}=x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k}$, *prove that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=6V.$$

[Kanpur 1979; Rohilkhand 80; Agra 78]

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V \text{div}\,\mathbf{F}\, dV=\iiint_V \text{div}\,(x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k})\, dV$$

$$=\iiint_V \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(2y)+\frac{\partial}{\partial z}(3z)\right] dV$$

$$=\iiint_V (1+2+3)\, dV=6\iiint_V dV=6V.$$

Ex. 32. *Evaluate*

$$\iint_S (y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\cdot\mathbf{n}\, dS$$

where S is the part of the sphere $x^2+y^2+z^2=1$ *above the xy-plane and bounded by this plane.* **[Agra 1969; Bombay 66]**

Solution. By divergence theorem, we have

$$\iint_S (y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\cdot\mathbf{n}\, dS$$

$=\iiint_V \text{div}\,(y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\, dV,$ where V is the volume enclosed by S

$$=\iiint_V \left[\frac{\partial}{\partial x}(y^2z^2)+\frac{\partial}{\partial y}(z^2x^2)+\frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$=\iiint_V 2zy^2\, dV=2\iiint_V zy^2\, dV.$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate this triple integral. In polars $dV=(dr)\,(r d\theta)\,(r\sin\theta\, d\phi)=r^2\sin\theta\, dr\, d\theta\, d\phi$. Also $z=r\cos\theta$, $y=r\sin\theta\sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π. The triple integral is

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)\,(r^2\sin^2\theta\sin^2\phi)\,r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$=2\cdot\frac{1}{6}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\sin^3\theta\cos\theta\sin^2\phi\, d\theta\, d\phi,$$

on integrating with respect to r.

[Note that the order of integration is immaterial because the limits of r, θ and ϕ are all constants].

$$=\frac{1}{3}\cdot\frac{2}{4\cdot 2}\int_0^{2\pi}\sin^2\phi\, d\phi, \text{ on integrating with respect to } \theta$$

$$=\frac{1}{12}\cdot 4\int_0^{\pi/2}\sin^2\phi\, d\phi=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{12}.$$

Ex. 33. *By converting the surface integral into a volume integral evaluate*

$$\iint_S (x^3\, dy\, dz+y^3\, dz\, dx+z^3\, dx\, dy),$$

where S is the surface of the sphere $x^2+y^2+z^2=1$. [Bombay 1970]

Solution. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy)$$

$$=\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\, dy\, dz,$$

where V is the volume enclosed by S.

Here $F_1=x^3$, $F_2=y^3$, $F_3=z^3$.

$$\therefore\quad \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=3\,(x^2+y^2+z^2).$$

$\therefore$ the given surface integral

$$=\iiint_V 3\,(x^2+y^2+z^2)\, dx\, dy\, dz$$

$$=3\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2r^2\sin\theta\, dr\, d\theta\, d\phi,$$

changing to polar spherical coordinates

$$=3\times 2\pi\times 2\times\frac{1}{5}=\frac{12\pi}{5}.$$

Ex. 34. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2=x^2+y^2$ *and the plane* $z=4$, *if*

$$\mathbf{F}=4xz\,\mathbf{i}+xyz^2\,\mathbf{j}+3z\,\mathbf{k}.$$

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V \text{div}\,\mathbf{F}\, dV,$$

where V is the volume enclosed by S.

Here $\text{div}\,\mathbf{F}=\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}(xyz^2)+\frac{\partial}{\partial z}(3z)=4z+xz^2+3.$

Also V is the region bounded by the surfaces

$$z=0,\ z=4,\ z^2=x^2+y^2.$$

Therefore $\iiint_V \text{div}\,\mathbf{F}\, dV=\iiint_V (4z+xz^2+3)\, dx\, dy\, dz$

$$=\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} (4z+xz^2+3)\, dx\, dy\, dz$$

$$=2\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=0}^{\sqrt{(z^2-y^2)}} (4z+3)\, dx\, dy\, dz,$$

since $\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} x\, dx=0$

$$=2\int_{z=0}^{4}\int_{y=-z}^{z} (4z+3)\sqrt{(z^2-y^2)}\, dy\, dz,$$

on integrating with respect to x

$$=4\int_{z=0}^{4}\int_{y=0}^{z} (4z+3)\sqrt{(z^2-y^2)}\, dy\, dz$$

$$=4\int_{z=0}^{4} (4z+3)\left[\frac{y}{2}\sqrt{(z^2-y^2)}+\frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$=4\int_0^4 (4z+3)\left[\frac{z^2}{2}\sin^{-1}1\right] dz=\pi\int_0^4 (4z^3+3z^2)\, dz$$

$$=\pi\left[z^4+z^3\right]_0^4=\pi(256+64)=320\pi.$$

Ex. 35. *Show that* $\iint_S (x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\cdot\mathbf{n}\, dS$ *vanishes where* S *denotes the surface of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution. We have by divergence theorem

$$\iint_S (x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\cdot\mathbf{n}\, dS$$

$$=\iiint_V \operatorname{div}(x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\,dV, \text{ where } V \text{ is the volume enclosed by } S$$

$$=\iiint_V (2x+2y+2z)\,dx\,dy\,dz$$

$$=2\int_{z=-c}^{c}\int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}}\int_{x=-a\sqrt{\{1-y^2/b^2-z^2/c^2\}}}^{a\sqrt{\{1-(y^2/b^2)-(z^2/c^2)\}}}(x+y+z)\,dx\,dy\,dz$$

$$=4\int_{z=-c}^{c}\int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}}(y+z)\sqrt{\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)}\,dy\,dz,$$

on integrating with respect to x

$$\left[\text{Note that } \int_{-a}^{a} f(x)\,dx=0 \text{ if } f(-x)=-f(x) \text{ and } \int_{-a}^{a} f(x)\,dx =2\int_0^a f(x)\,dx \text{ if } f(-x)=f(x)\right]$$

$$=8\int_{z=-c}^{c}\int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} z\sqrt{\left(1-\frac{z^2}{c^2}-\frac{y^2}{b^2}\right)}\,dy\,dz$$

$$=8\int_{z=-c}^{c}\int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} \frac{z}{b}\sqrt{\left\{b^2\left(1-\frac{z^2}{c^2}\right)-y^2\right\}}\,dy\,dz$$

$$=\frac{8}{b}\int_{z=-c}^{c} z\left[\frac{y}{2}\sqrt{\left\{b^2\left(1-\frac{z^2}{c^2}\right)-y^2\right\}} +\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{y}{b\sqrt{\{1-(z^2/c^2)\}}}\right]_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}}dz$$

$$=\frac{8}{b}\int_{z=-c}^{c} z\left[\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}1\right]dz=\frac{8}{b}\int_{z=-c}^{c} z\,\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\frac{\pi}{2}dz=0$$

Ex. 36. *If* $\mathbf{F}=(x^2+y-4)\,\mathbf{i}+3xy\,\mathbf{j}+(2xz+z^2)\,\mathbf{k}$, *evaluate* $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ *where S is the surface of the sphere* $x^2+y^2+z^2=16$ *above the xy-plane.*

Solution. The surface $x^2+y^2+z^2=16$ meets the plane $z=0$ in a circle C given by $x^2+y^2=16$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the region bounded by S'.

If $\mathbf{n}$ denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n}=-\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

Now by an application of Gauss divergence theorem, we have

$$\iint_{S'} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=0$$

[See Ex. 1 page 111]

or $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS + \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = 0$

[$\because$ S' consists of S and S_1]

or $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS - \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k} \, dS = 0$ [$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

or $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k} \, dS.$

Now $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$= 0\mathbf{i} - z\,\mathbf{j} + (3y-1)\,\mathbf{k} = -z\,\mathbf{j} + (3y-1)\,\mathbf{k}.$

$\therefore$ $\text{curl } \mathbf{F} \cdot \mathbf{k} = \{-z\,\mathbf{j} + (3y-1)\,\mathbf{k}\} \cdot \mathbf{k} = 3y-1.$

$\therefore$ $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} (3y-1) \, dS$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (3r \sin \theta - 1) \, r \, d\theta \, dr,$ changing to polars

[Note that S_1 is a circle in xy plane with centre origin and radius 4]

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} 3r^2 \sin \theta \, d\theta \, dr - \int_{\theta=0}^{2\pi} \int_{r=0}^{4} r \, d\theta \, dr$

$= 0 - \int_{\theta=0}^{2\pi} \left[\frac{r^2}{2} \right]_0^4 d\theta$ $\left[\because \int_{\theta=0}^{2\pi} \sin \theta \, d\theta = 0 \right]$

$= -8 \left[\theta \right]_0^{2\pi} = -16\pi.$

Ex. 37. *Evaluate* $\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dS$, *where* $\mathbf{A} = (x-z)\,\mathbf{i} + (x^3+yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$ *and S is the surface of the cone* $z = 2 - \sqrt{(x^2+y^2)}$ *above the xy-plane.* [Meerut 1974]

Solution. The surface $z = 2 - \sqrt{(x^2+y^2)}$ meets the xy-plane in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. By application of divergence theorem, we have

$$\iint_{S'} \text{curl } \mathbf{A} \cdot \mathbf{n} \, dS = 0$$ [See Ex. 1 page 111]

or $\iint_S \text{curl } \mathbf{A} \cdot \mathbf{n} \, dS + \iint_{S_1} \text{curl } \mathbf{A} \cdot \mathbf{n} \, dS = 0$

or $\iint_S \text{curl } \mathbf{A} \cdot \mathbf{n} \, dS = \iint_{S_1} \text{curl } \mathbf{A} \cdot \mathbf{k} \, dS$ [$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

Now $\text{curl } \mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x-z & x^3+yz & -3xy^2 \end{vmatrix}$

$=\mathbf{i}\,(-6xy-y)+\mathbf{j}\,(-1+3y^2)+\mathbf{k}\,(3x^2-0).$

$\therefore$ $\text{curl } \mathbf{A}\cdot\mathbf{k}=3x^2.$

$\therefore$ $\iint_S \text{curl } \mathbf{A}\cdot\mathbf{n}\, dS=\iint_{S_1} 3x^2\, dS$

$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\cos^2\theta\; r\, d\theta\, dr$, changing to polars

$=3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\cos^2\theta\, d\theta\, dr=3\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2\cos^2\theta\, d\theta$

$=12\int_0^{2\pi}\cos^2\theta\, d\theta$

$=12\times 4\int_0^{\pi/2}\cos^2\theta\, d\theta=48\times\frac{1}{2}\times\frac{\pi}{2}=12\pi.$

Ex. 38. *Evaluate* $\iint_S (ax^2+by^2+cz^2)\, dS$ *over the sphere* $x^2+y^2+z^2=1$ *using the divergence theorem.*

Solution. Let us first put the integral

$$\iint_S (ax^2+by^2+cz^2)\, dS \text{ in the form}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is unit normal vector to S.

The normal vector to $\phi(x, y, z)\equiv x^2+y^2+z^2-1=0$ is

$$=\nabla\phi=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

$\therefore$ $\mathbf{n}=\dfrac{\nabla\phi}{|\nabla\phi|}=\dfrac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{[4\,(x^2+y^2+z^2)]}}$

$=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ $\quad [\because\ x^2+y^2+z^2=1, \text{ on } S]$

Now we are to choose $\mathbf{F}$ such that

$$\mathbf{F}\cdot\mathbf{n}=\mathbf{F}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=ax^2+by^2+cz^2.$$

Obviously $\mathbf{F}=ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}.$

Now $\iint_S (ax^2+by^2+cz^2)\, dS$

$=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F}=ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$

$=\iiint_V \text{div } \mathbf{F}\, dV$, by divergence theorem

$$=\iiint_V (a+b+c)\,dV \qquad [\because \quad \text{div } \mathbf{F}=a+b+c]$$

$$=(a+b+c)\iiint_V dV=(a+b+c)\,V$$

$=(a+b+c)\,\frac{4}{3}\pi$, since the volume V enclosed by the sphere S of unit radius is $\frac{4}{3}\pi$.

Ex. 39. Gauss's theorem. *Let S be a closed surface and let* $\mathbf{r}$ *denote the position vector of any point* (x, y, z) *measured from an origin O. Then*

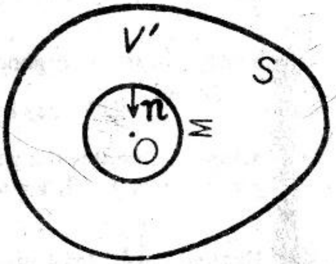
$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\,dS$$

is equal to (*i*) *zero if O lies outside S;* (*ii*) 4π *if O lies inside S.*

Proof. (*i*) *When origin O is outside S.* In this case $\mathbf{F}=\frac{\mathbf{r}}{r^3}$ is continuously differentiable throughout the region V enclosed by S. Hence by divergence theorem, we have

$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\left(\frac{\mathbf{r}}{r^3}\right)dV=0, \text{ since div}\left(\frac{\mathbf{r}}{r^3}\right)=0.$$

(*ii*) *When origin O is inside S.* In this case divergence theorem cannot be applied to the region V enclosed by S since $\mathbf{F}=\frac{\mathbf{r}}{r^3}$ has a point of discontinuity at the origin. To remove this difficulty let us enclose the origin by a small sphere Σ of radius ϵ.



The function $\mathbf{F}$ is continuously differentiable at the points of the region V' enclosed between S and Σ. Therefore applying divergence theorem for this region V', we have

$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\,dS=\iint_\Sigma \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\,d\Sigma$$

$$=\iint_{V'} \text{div}\left(\frac{\mathbf{r}}{r^3}\right)dV'=0, \text{ since div}\left(\frac{\mathbf{r}}{r^3}\right)=0.$$

$$\therefore \quad \iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\,dS=-\iint_\Sigma \left(\frac{\mathbf{r}}{r^3}\right)\cdot\mathbf{n}\,d\Sigma.$$

Now on the sphere Σ, the outward drawn normal $\mathbf{n}$ is directed towards the centre. Therefore on Σ, we have

$$\mathbf{n}=-\frac{\mathbf{r}}{\epsilon}.$$

$$\therefore \quad -\iint_\Sigma \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\, d\Sigma = -\iint_\Sigma \frac{\mathbf{r}}{\epsilon^3}\cdot\left(-\frac{\mathbf{r}}{\epsilon}\right) d\Sigma, \text{ since on } \Sigma,\ r=\epsilon$$

$$=\iint_\Sigma \frac{r^2}{\epsilon^4}\, d\Sigma=\iint_\Sigma \frac{\epsilon^2}{\epsilon^4}\, d\Sigma=\frac{1}{\epsilon^2}\iint_\Sigma d\Sigma=\frac{1}{\epsilon^2}\, 4\pi\epsilon^2=4\pi.$$

Hence $\displaystyle\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\, dS=4\pi.$

Exercises

1. Verify divergence theorem for $\mathbf{F}=4xz\mathbf{i}-y^2\mathbf{j}+yz\mathbf{k}$ taken over the cube bounded by

$$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1.$$

[**Hint.** Proceed as in Ex. 24. Here we shall have

$$\iiint_V \nabla\cdot\mathbf{F}\, dV=\frac{3}{2}.$$

The six surface integrals will come out to be 2, 0, -1, 0, $\frac{1}{2}$ and 0. Their sum is $=\frac{3}{2}$.

Hence the theorem is verified].

2. Evaluate, by Green's theorem in space (*i.e.*, Gauss divergence theorem), the integral

$$\iint_S 4xz\,dy\,dz-y^2dz\,dx+yz\,dx\,dy,$$

where S is the surface of the cube bounded by

$x=0,\ y=0,\ z=0,\ x=1,\ y=1,\ z=1.$ **[Meerut 1974; Kanpur 77]**

Ans. $\frac{3}{2}$.

3. Verify Gauss divergence theorem to show that

$$\iint_S \{(x^3-yz)\,\mathbf{i}-2x^2y\mathbf{j}+2\mathbf{k}\}\cdot\mathbf{n}\, dS=\tfrac{1}{3}a^5,$$

where S denotes the surface of the cube bounded by the planes $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a.$

[Rohilkhand 1979; Agra 77]

4. Evaluate $\displaystyle\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS$ where S denotes the surface of the cube bounded by the planes $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$ by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly.

[Agra 1979]

[**Hint.** Here $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V \text{div}\,\mathbf{F}\, dV$$

$$=\iiint_V 3\,dV=3V=3a^3, \text{ as } V=a^3=\text{the volume of the cube}].$$

5. Evaluate by divergence theorem the integral

$$\iint_S xz^2\, dy\, dz+(x^2y-z^3)\, dz\, dx+(2xy+y^2z)\, dx\, dy,$$

where S is the entire surface of the hemispherical region bounded by $z=\sqrt{(a^2-x^2-y^2)}$ and $z=0$. **[Meerut 1974]**

[Hint. Proceed as in Ex. 32]. **Ans.** $\dfrac{2\pi a^5}{5}$.

6. By using Gauss divergence theorem, evaluate

$$\iint_S (x\mathbf{i}+y\mathbf{j}+z^2\mathbf{k})\cdot\mathbf{n}\, dS$$

where S is the closed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1$. **[Agra 1973]**

[Hint. Proceed as in Ex. 34]. **Ans.** $7\pi/6$.

7. Use divergence theorem to find $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ for the vector $\mathbf{F}=x\mathbf{i}-y\mathbf{j}+2z\mathbf{k}$ over the sphere $x^2+y^2+(z-1)^2=1$. **Ans.** $8\pi/3$.

8. If $\mathbf{F}=ax\mathbf{i}+by\mathbf{j}+cz\mathbf{k}$, where a, b, c are constants, show that

$$\iint_S (\mathbf{n}\cdot\mathbf{F})\, dS=\frac{4\pi}{3}(a+b+c),$$

S being the surface of the sphere $(x-1)^2+(y-2)^2+(z-3)^2=1$. **[Gauhati 1971]**

9. Use divergence theorem to evaluate

$$\iint_S [x\, dy\, dz+y\, dz\, dx+z\, dx\, dy],$$

where S is the surface $x^2+y^2+z^2=1$. **Ans.** 4π.

19. Verify the divergence theorem for

$$\mathbf{F}=4x\mathbf{i}-2y^2\mathbf{j}+z^2\mathbf{k}$$

taken over the region bounded by the surfaces

$x^2+y^2=4$, $z=0$, $z=3$. **[Allahabad 1978]**

[Hint. Show that each of the two integrals is $=84\pi$].

11. Verify divergence theorem for

$$\mathbf{F}=2x^2y\mathbf{i}-y^2\mathbf{j}+4xz^2\mathbf{k}$$

taken over the region in the first octant bounded by

$y^2+z^2=9$ and $x=2$. **[Kanpur 1976]**

[Hint. Show that each of the two integrals is $=180$].

12. Verify divergence theorem for the function $\mathbf{F}=y\mathbf{i}+x\mathbf{j}+z^2\mathbf{k}$ over the cylindrical region bounded by

$x^2+y^2=a^2$, $z=0$ and $z=h$. **[Kanpur 1975; Allahabad 79]**

13. If $\mathbf{F}=y\mathbf{i}+(x-2xz)\mathbf{j}-xy\mathbf{k}$, evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **[Kanpur 1980]**

[**Hint.** Proceed as in Ex. 36. Here (curl $\mathbf{F}$)$\cdot\mathbf{k}=-z=0$ over the surface S_1 bounded by the circle $x^2+y^2=a^2$, $z=0$]. **Ans.** 0.

14. Evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, dS$, where
$\mathbf{A}=[xye^z+\log(z+1)-\sin x]\,\mathbf{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **Ans.** 0.

15. Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$, where
$\mathbf{F}=(x^2+y-4)\,\mathbf{i}+3xy\mathbf{j}+(2xz+z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z=4-(x^2+y^2)$ above the xy-plane. **Ans.** -4π.

16. Compute

(i) $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\, dS$, and

(ii) $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2+by^2+cz^2=1$.

Ans. (i) $\dfrac{4\pi(a+b+c)}{3\sqrt{(abc)}}$, (ii) $\dfrac{4\pi}{\sqrt{(abc)}}$.

17. Evaluate $\iint_S (x^2+y^2)\, dS$, where S is the surface of the cone $z^2=3(x^2+y^2)$ bounded by $z=0$ and $z=3$. **Ans.** 9π.

18. Prove that

$$\int_V \mathbf{f}\cdot\text{curl}\,\mathbf{F}\, dV=\int_S \mathbf{F}\times\mathbf{f}\cdot d\mathbf{S}+\int_V \mathbf{F}\cdot\text{curl}\,\mathbf{f}\, dV.$$

[**Hint.** Apply divergence theorem for the vector function $\mathbf{F}\times\mathbf{f}$].

19. Let $\mathbf{r}$ denote the position vector of any point (x, y, z) measured from an origin O and let $r=|\mathbf{r}|$.

Evaluate $\iint_S \dfrac{\mathbf{r}}{r^3}\cdot\mathbf{n}\, dS$ where S is the sphere $x^2+y^2+z^2=a^2$.

Ans. 4π. **[Calicut 1975]**

§ 9. **Stoke's Theorem.** *Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let* $\mathbf{F}(x, y, z)$ *be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then*

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS=\iint_S (\text{curl}\,\mathbf{F})\cdot d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn

SOLVED EXAMPLES

Ex. 1. *Prove that* $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0$.

Solution. By Stoke's theorem

$$\oint_C \mathbf{r} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{r}) \cdot \mathbf{n}\, dS = 0, \text{ since curl } \mathbf{r} = \mathbf{0}.$$

Ex. 2. *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}$.

Solution. By Stoke's theorem, we have

$$\oint_C \nabla (\phi \psi) \cdot d\mathbf{r} = \iint_S [\text{curl grad } (\phi \psi)] \cdot \mathbf{n}\, dS$$

$$= 0, \text{ since curl grad } (\phi \psi) = \mathbf{0}.$$

But $\nabla (\phi \psi) = \phi \nabla \psi + \psi \nabla \phi$.

$$\therefore \quad \oint_C (\phi \nabla \psi + \psi \nabla \phi) \cdot d\mathbf{r} = 0$$

or

$$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}.$$

Ex. 3. (a) *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n}\, dS$.

Solution. By Stoke's theorem, we have

$$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi \nabla \psi)] \cdot \mathbf{n}\, dS$$

$$= \iint_S [\nabla \phi \times \nabla \psi + \phi \text{ curl grad } \psi] \cdot \mathbf{n}\, dS$$

$$= \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n}\, dS, \text{ since curl grad } \psi = \mathbf{0}.$$

Ex. 3. (b) *Show that* $\int_C \phi \nabla \phi \cdot d\mathbf{r} = 0$, *C being a closed curve.*

Solution. Applying Stoke's theorem to the vector function $\phi \nabla \phi$, we have

$$\int_C (\phi \nabla \phi) \cdot d\mathbf{r} = \iint_S [\text{curl } (\phi \nabla \phi)] \cdot \mathbf{n}\, dS$$

$$=\iint_S [\phi \text{ curl } \nabla\phi + \nabla\phi \times \nabla\phi] \cdot \mathbf{n}\, dS$$

$$=\iint_S 0 \cdot \mathbf{n}\, dS \quad [\because \text{ curl } \nabla\phi = 0 \text{ and } \nabla\phi \times \nabla\phi = 0]$$

$$=0.$$

Ex. 4. *Prove that* $\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$

[Kanpur 1977]

Solution. Let $\mathbf{A}$ be any arbitrary constant vector. Let $\mathbf{F} = \phi\mathbf{A}$. Applying Stoke's theorem for $\mathbf{F}$, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \{\nabla \times (\phi\mathbf{A})\} \cdot \mathbf{n}\, dS = \iint_S [\nabla\phi \times \mathbf{A} + \phi \text{ curl } \mathbf{A}] \cdot d\mathbf{S}$$

$$= \iint_S (\nabla\phi \times \mathbf{A}) \cdot d\mathbf{S}, \text{ since curl } \mathbf{A} = 0.$$

$$\therefore \quad \oint_C (\phi\mathbf{A}) \cdot d\mathbf{r} = \iint_S \mathbf{A} \cdot (d\mathbf{S} \times \nabla\phi)$$

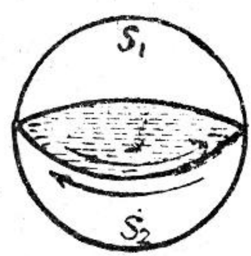
or $\mathbf{A} \cdot \oint_C \phi\, d\mathbf{r} = \mathbf{A} \cdot \iint_S d\mathbf{S} \times \nabla\phi$ or $\mathbf{A} \cdot \left[\oint_C \phi\, d\mathbf{r} - \iint_S d\mathbf{S} \times \nabla\phi\right] = 0.$

Since $\mathbf{A}$ is an arbitrary vector, therefore we must have

$$\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$$

Ex. 5. *By Stoke's theorem prove that div curl* $\mathbf{F} = 0$.

Solution. Let V be any volume enclosed by a closed surfaces. Then by divergence theorem



$$\iiint_V \nabla \cdot (\text{curl } \mathbf{F})\, dV = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Divide the surface S into two portions S_1 and S_2 by a closed curve C. Then

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS_1 + \iint_{S_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS_2. \quad \ldots(1)$$

By Stoke's theorem right hand side of (1) is

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}-\oint_C \mathbf{F}\cdot d\mathbf{r}=0.$$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \quad \iiint_V \nabla\cdot(\text{curl } \mathbf{F})\, dV=0.$$

Now this equation is true for all volume elements V. Therefore we have $\nabla\cdot(\text{curl } \mathbf{F})=0$

or $\qquad$ div curl $\mathbf{F}=0$.

Ex. 6. *By Stoke's theorem prove that curl grad* $\phi=0$.

Solution. Let S be any surface enclosed by a simple closed curve C. Then by stoke's theorem, we have

$$\iint_S (\text{curl grad } \phi)\cdot\mathbf{n}\, dS=\oint_C \text{grad } \phi\cdot d\mathbf{r}.$$

$$\text{Now grad } \phi\cdot d\mathbf{r}=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi.$$

$$\therefore \quad \oint_C \text{grad } \phi\cdot d\mathbf{r}=\oint_C d\phi=\Big[\phi\Big]_A^A, \text{ where } A \text{ is any point on } C$$

$$=0.$$

Therefore we have $\iint_S (\text{curl grad } \phi)\cdot\mathbf{n}\, dS=0.$

Now this equation is true for all surface elements S.

Therefore we have, curl grad $\phi=0$.

Ex. 7. *Verify Stoke's theorem for* $\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ *where* S *is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and* C *is its boundary.* [Bombay 1970; Meerut 81; Agra 79; Rohilkhand 77]

Solution. The boundary C of S is a circle in the xy plane of radius unity and centre origin. The equations of the curve C are $x^2+y^2=1$, $z=0$. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C (y\,dx+z\,dy+x\,dz)=\oint_C y\,dx, \text{ since on } C,\ z=0 \text{ and } dz=0$$

$$=\int_0^{2\pi} \sin t \frac{dx}{dt}\, dt=\int_0^{2\pi} -\sin^2 t\, dt$$

$$=-\tfrac{1}{2}\int_0^{2\pi}(1-\cos 2t)\, dt=-\tfrac{1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$=-\pi. \qquad \ldots(1)$$

Now let us evaluate $\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS$. We have curl **F**

$$=\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=-\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS=\iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\, dS \text{ [See Ex. 36 Page 126]}$$

$$=\iint_{S_1}(-\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot\mathbf{k}\, dS=\iint_{S_1}(-1)\, dS=-\iint_{S_1} dS=-S_1.$$

But S_1=area of a circle of radius $1=\pi(1)^2=\pi$.

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS=-\pi. \qquad \ldots(2)$$

Hence from (1) and (2), the theorem is verified.

Ex. 8. *Verify Stoke's theorem for* $\mathbf{F}=(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}$, *where S is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and C is its boundary.*

[**Kanpur 1970; Rohilkhand 78; Allahabad 78; Agra 73, 76, 80**]

Solution. The boundary C of S is a circle in the xy plane of radius unity and centre origin. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C[(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}]\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C[(2x-y)\, dx-yz^2\, dy-y^2z\, dz]$$

$$=\oint_C (2x-y)\, dx, \text{ since } z=0 \text{ and } dz=0$$

$$=\int_0^{2\pi}(2\cos t-\sin t)\frac{dx}{dt}\, dt=-\int_0^{2\pi}(2\cos t-\sin t)\sin t\, dt$$

$$=-\int_0^{2\pi}[\sin 2t-\tfrac{1}{2}(1-\cos 2t)]\,dt=-\left[-\frac{\cos 2t}{2}-\tfrac{1}{2}t+\tfrac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$=-[(-\tfrac{1}{2}+\tfrac{1}{2})-\tfrac{1}{2}(\pi-0)+\tfrac{1}{4}(0-0)]=\pi. \quad \text{...(1)}$$

Also $(\nabla\times\mathbf{F})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}$

$$=(-2yz+2yz)\,\mathbf{i}-(0-0)\,\mathbf{j}+(0+1)\,\mathbf{k}=\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\iint_{S_1}(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dS$$

[by an application of divergence theorem, see Ex. 36, page 126]

$$=\iint_{S_1}\mathbf{k}\cdot\mathbf{k}\,dS=\iint_{S_1}dS=S_1=\pi. \quad \text{...(2)}$$

Hence from (1) and (2), the theorem is verified.

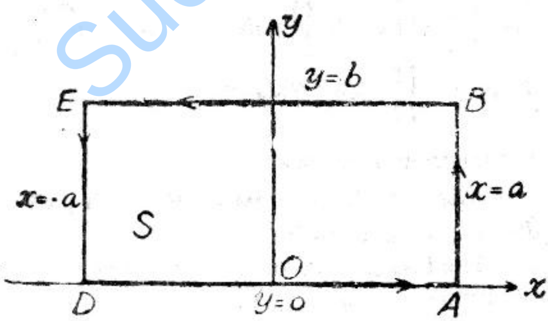
Ex. 9. *Verify Stoke's theorem for*

$$\mathbf{F}=(x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}$$

taken round the rectangle bounded by

$$x=\pm a,\ y=0,\ y=b. \quad \text{[Meerut 1967]}$$

Solution. We have



curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y^2 & -2xy & 0\end{vmatrix}$

$$=(-2y-2y)\,\mathbf{k}=-4y\mathbf{k}.$$

Also $\mathbf{n}=\mathbf{k}$.

$$\therefore \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y\mathbf{k})\cdot\mathbf{k}\, dx\, dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\, dy = -4\int_{y=0}^{b}\Big[xy\Big]_{x=-a}^{a} dy$$

$$= -4\int_{y=0}^{b} 2ay\, dy = -4\Big[ay^2\Big]_0^b = -4ab^2.$$

Also $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C [(x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

$$= \oint_C [(x^2+y^2)\, dx - 2xy\, dy]$$

$$= \int_{DA} [(x^2+y^2)\, dx - 2xy\, dy] + \int_{AB} + \int_{BE} + \int_{ED}.$$

Along DA, $y=0$ and $dy=0$. Along AB, $x=a$ and $dx=0$.
Along BE, $y=b$ and $dy=0$. Along ED, $x=-a$ and $dx=0$.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=-a}^{a} x^2\, dx + \int_{y=0}^{b} -2ay\, dy$$

$$+ \int_{x=a}^{-a} (x^2+b^2)\, dx + \int_{y=b}^{0} 2ay\, dy$$

$$= \int_{-a}^{a} x^2\, dx - \int_{-a}^{a} (x^2+b^2)\, dx - 4a\int_0^b y\, dy$$

$$= -\int_{-a}^{a} b^2\, dx - 4a\int_0^b y\, dy = -2ab^2 - 4a\Big[\frac{y^2}{2}\Big]_0^b = -4ab^2.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS.$

Hence the theorem is verified.

Ex. 10. *Verify Stoke's theorem for* $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j}$, *where* S *is the circular disc* $x^2+y^2 \leqslant 1$, $z=0$.

Solution. The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0 \leqslant t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (-y^3\,\mathbf{i} + x^3\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (-y^3\, dx + x^3\, dy) = \int_{t=0}^{2\pi} \Big\{-y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt}\Big\} dt$$

$$=\int_0^{2\pi} [-\sin^3 t\,(-\sin t)+\cos^3 t\,(\cos t)]\,dt$$

$$=\int_0^{2\pi} (\cos^4 t+\sin^4 t)\,dt=4\int_0^{\pi/2} (\cos^4 t+\sin^4 t)\,dt$$

$$=4\left\{\frac{3.1}{4.2}\,\frac{\pi}{2}+\frac{3.1}{4.2}\,\frac{\pi}{2}\right\}=\frac{3\pi}{2}.$$

Also $\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}=(3x^2+3y^2)\,\mathbf{k}.$

Here $\mathbf{n}=\mathbf{k}$ because the surface S is the xy-plane.

$\therefore\quad (\nabla\times\mathbf{F})\cdot\mathbf{n}=(3x^2+3y^2)\,\mathbf{k}\cdot\mathbf{k}=3\,(x^2+y^2).$

$$\therefore\quad \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=3\iint_S (x^2+y^2)\,dS$$

$$=3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2 r\,d\theta\,dr, \text{ changing to polars}$$

$$=\frac{3}{4}\int_0^{2\pi} d\theta=\frac{3}{4}\,(2\pi)=\frac{3\pi}{2}.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\frac{3\pi}{2}.$

Hence the theorem is verified.

Ex. 11. *Evaluate by Stoke's theorem*

$$\oint_C (e^x\,dx+2y\,dy-dz)$$

where C is the curve $x^2+y^2=4,\ z=2.$ **[Meerut 1969; Agra 72]**

Solution. $\oint_C (e^x\,dx+2y\,dy-dz)$

$$=\oint_C (e^x\mathbf{i}+2y\mathbf{j}-\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{F}=e^x\mathbf{i}+2y\mathbf{j}-\mathbf{k}.$$

Now curl $\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS$$

$$=0, \text{ since curl } \mathbf{F}=0.$$

Ex. 12. *Evaluate by Stoke's theorem*

$$\oint_C (yz\, dx+xz\, dy+xy\, dz)$$

where C is the curve $x^2+y^2=1,\ z=y^2$. **[Kanpur 1980]**

Solution. Here $\mathbf{F}=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$.

$$\therefore \quad \text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$=(x-x)\,\mathbf{i}-(y-y)\,\mathbf{j}+(z-z)\,\mathbf{k}=0.$$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS$$

$$=0, \text{ since curl } \mathbf{F}=0.$$

Ex. 13. *Evaluate* $\oint_C (xy\, dx+xy^2\, dy)$ *by Stoke's theorem where C is the square in the xy-plane with vertices* $(1, 0), (-1, 0), (0, 1), (0, -1)$.

Solution. Here $\mathbf{F}=xy\mathbf{i}+xy^2\mathbf{j}$.

$$\therefore \quad \text{curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}=(y^2-x)\,\mathbf{k}.$$

Also $\mathbf{n}=\mathbf{k}$.

$\therefore$ curl $\mathbf{F}\cdot\mathbf{n}=(y^2-x)\,\mathbf{k}\cdot\mathbf{k}=y^2-x.$

$\therefore$ The given line integral

$$=\iint_S (y^2-x)\, dS$$

$$=\int_{y=-1}^{1}\int_{x=-1}^{1} (y^2-x)\, dx\, dy=\int_{y=-1}^{1}\left[y^2x-\frac{x^2}{2}\right]_{x=-1}^{1} dy$$

$$=\int_{y=-1}^{1} 2y^2 dy=2\left[\frac{y^3}{3}\right]_{-1}^{1}=\frac{4}{3}.$$

Ex. 14. *Evaluate* $\oint_C \mathbf{F}\cdot d\mathbf{r}$ *by Stoke's theorem where* $\mathbf{F}=y^2\mathbf{i}+x^2\mathbf{j}-(x+z)\,\mathbf{k}$ *and C is the boundary of the triangle with vertices at* (0, 0, 0), (1, 0, 0), (1, 1, 0).

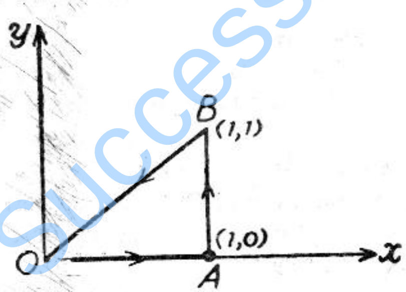
Solution. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}=0\mathbf{i}+\mathbf{j}+2(x-y)\,\mathbf{k}.$$

Also we note that z-coordinate of each vertex of the triangle is zero. Therefore the triangle lies in the x-y plane. So $\mathbf{n}=\mathbf{k}$.

$\therefore$ Curl $\mathbf{F}\cdot\mathbf{n}=[\mathbf{j}+2(x-y)\,\mathbf{k}]\cdot\mathbf{k}=2(x-y)$.

In the figure, we have only considered the x-y plane.



The equation of the line OB is $y=x$.

By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\, dS$$

$$=\int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\,dx\,dy=2\int_{x=0}^{1}\left[xy-\frac{y^2}{2}\right]_{y=0}^{x} dx$$

$$=2\int_{x=0}^{1}\left[x^2-\frac{x^2}{2}\right] dx=2\int_0^1 \frac{x^2}{2}\, dx=\int_0^1 x^2\, dx=\frac{1}{3}.$$

Ex. 15. *Evaluate by Stoke's theorem*

$$\oint_C (\sin z\, dx-\cos x\, dy+\sin y\, dz)$$

where C is the boundary of the rectangle

$0 \leqslant x \leqslant \pi, 0 \leqslant y \leqslant 1, z=3.$

Solution. Here $\mathbf{F}=\sin z\,\mathbf{i}-\cos x\mathbf{j}+\sin y\mathbf{k}$.

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}=\cos y\mathbf{i}+\cos z\mathbf{j}+\sin x\mathbf{k}.$$

Since the rectangle lies in the plane $z=3$, therefore $\mathbf{n}=\mathbf{k}$.

$\therefore$ $\text{curl } \mathbf{F}\cdot\mathbf{n}=(\cos y\mathbf{i}+\cos z\mathbf{j}+\sin x\mathbf{k})\cdot\mathbf{k}=\sin x.$

By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\, dS$$

$$=\int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\, dx\, dy=\int_{x=0}^{\pi} \sin x\, dx=2.$$

Ex. 16. *Apply Stoke's theorem to prove that*

$$\int_C (y dx+z dy+x dz)=-2\sqrt{2}\pi a^2$$

where C is the curve given by

$$x^2+y^2+z^2-2ax-2ay=0,\quad x+y=2a$$

and begins at the point $(2a, 0, 0)$ *and goes at first below the z-plane.*

(Agra 1969; Meerut 82)

Solution. The centre of the sphere $x^2+y^2+z^2-2ax-2ay=0$ is the point $(a, a, 0)$. Since the plane $x+y=2a$ passes through the point $(a, a, 0)$, therefore the circle C is great circle of this sphere.

$\therefore$ Radius of the circle C

$$=\text{radius of the sphere}=\sqrt{(a^2+a^2)}=a\sqrt{2}.$$

$$\text{Now } \int_C (y dx+z dy+x dz)=\int_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot d\mathbf{r}$$

$$=\iint_S [\text{curl } (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})]\cdot\mathbf{n}\, dS,$$

where S is any surface of which circle C is boundary [Stoke's theorem].

$$\text{Now curl } (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$=-\mathbf{i}-\mathbf{j}-\mathbf{k}=-(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle C. Then a vector normal to S is grad $(x+y)=\mathbf{i}+\mathbf{j}$.

$\therefore$ $\mathbf{n}$ = unit normal to $S=\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$.

$$\therefore \int_C (y\,dx+z\,dy+x\,dz)=\iint_S -(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right)dS$$

$$=-\frac{2}{\sqrt{2}}\iint_S dS=-\frac{2}{\sqrt{2}}(\text{area of the circle of radius } a\sqrt{2})$$

$$=-\sqrt{2}\,(2\pi a^2).$$

Ex. 17. *Use Stoke's theorem to evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=y\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\mathbf{k}$ *and S is the surface of the sphere* $x^2+y^2+z^2=a^2$ *above the xy-plane.*

Solution. The boundary C of the surface S is the circle $x^2+y^2=a^2$, $z=0$. Suppose $x=a\cos t$, $y=a\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [y\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}]\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$$

$$=\int_C [y\,dx+(x-2xz)\,dy-xy\,dz]$$

$$=\int_C (y\,dx+x\,dy) \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi}\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right)dt$$

$$=\int_0^{2\pi}[a\sin t\,(-a\sin t)+a\cos t\,(a\cos t)]\,dt$$

$$=a^2\int_0^{2\pi}(\cos^2 t-\sin^2 t)\,dt=a^2\int_0^{2\pi}\cos 2t\,dt=a^2\left[\frac{\sin 2t}{2}\right]_0^{2\pi}=0.$$

Ex. 18. *Evaluate the surface integral* $\iint_S$ *curl* $\mathbf{F}\cdot\mathbf{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z=1-x^2-y^2$ *for which* $z\geqslant 0$, *and*
$\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$. **(Bombay 1970)**

Solution. The boundary C of the surface S is the circle $x^2+y^2=1$, $z=0$. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)=\int_C y\,dx+z\,dy+x\,dz$$

$$=\int_C y\,dx \quad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi} y\frac{dx}{dt}\,dt=\int_0^{2\pi}\sin t\,(-\sin t)\,dt=-\int_0^{2\pi}\sin^2 t\,dt$$

$$=-4\int_0^{\pi/2}\sin^2 t\,dt=-4\times\tfrac{1}{2}\times\frac{\pi}{2}=-\pi.$$

Ex. 19. *If* $\mathbf{F}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, *evaluate* $\iint$ *curl* $\mathbf{F}\cdot\mathbf{n}\,dS$ *taken over the portion of the surface* $x^2+y^2+z^2-2ax+az=0$ *above the plane* $z=0$, *and verify Stoke's theorem.*

Solution. The surface $x^2+y^2+z^2-2ax+az=0$ meets the plane $z=0$ in the circle C given by $x^2+y^2-2ax=0$, $z=0$. The polar equation of the circle C lying in the xy-plane is $r=2a\cos\theta$, $0\leqslant\theta<\pi$. Also the equation $x^2+y^2-2ax=0$ can be written as $(x-a)^2+y^2=a^2$. Therefore the parametric equations of the circle C can be taken as

$$x=a+a\cos t,\ y=a\sin t,\ z=0,\ 0\leqslant t<2\pi.$$

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS.$$

Now curl $\mathbf{F}\cdot\mathbf{k}=$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}\cdot\mathbf{k}$$

$$=\left[\frac{\partial}{\partial x}(z^2+x^2-y^2)-\frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\mathbf{k}\cdot\mathbf{k}$$

$[\because\ \mathbf{i}\cdot\mathbf{k}=\mathbf{j}\cdot\mathbf{k}=0]$

$$=2(x-y).$$

$$\therefore\ \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS=\iint_{S_1}2(x-y)\,dS$$

$$=2\int_{\theta=0}^{\pi}\int_{r=0}^{2a\cos\theta}(r\cos\theta-r\sin\theta)\,r\,d\theta\,dr,$$

changing to polars

$$=2\int_{\theta=0}^{\pi}(\cos\theta-\sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta$$

$$=2\times\frac{8a^3}{3}\int_0^{\pi}(\cos\theta-\sin\theta)\cos^3\theta\,d\theta$$

$$=\frac{16a^3}{3}\int_0^{\pi}(\cos^4\theta-\cos^3\theta\sin\theta)\,d\theta$$

$$=\frac{16a^3}{3}\int_0^{\pi}\cos^4\theta\,d\theta \qquad \left[\because \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta=0\right]$$

$$=2\times\frac{16a^3}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=2\times\frac{16a^3}{3}\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}=2\pi a^3. \qquad \ldots(1)$$

Also $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (y^2+z^2-x^2)\,dx$

$$+(z^2+x^2-y^2)\,dy+(x^2+y^2-z^2)\,dz$$

$$=\int_C (y^2-x^2)\,dx+(x^2-y^2)\,dy \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi}(x^2-y^2)\left(\frac{dy}{dt}-\frac{dx}{dt}\right)dt$$

$$=\int_0^{2\pi}[(a+a\cos t)^2-a^2\sin^2 t]\,(a\cos t+a\sin t)\,dt$$

$$=a^3\int_0^{2\pi}(1+\cos^2 t+2\cos t-\sin^2 t)\,(\cos t+\sin t)\,dt$$

$$=a^3\int_0^{2\pi}2\cos^2 t\,dt, \text{ the other integrals vanish}$$

$$=2a^3\times4\int_0^{\pi/2}\cos^2 t\,dt=8a^3\times\tfrac{1}{2}\times\frac{\pi}{2}=2\pi a^3. \qquad \ldots(2)$$

Comparing (1) and (2), we see that

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Hence Stoke's theorem is verified.

Ex. 20. *Prove that a necessary and sufficient condition that* $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ *for every closed curve C lying in a simply connected region R is that* $\nabla\times\mathbf{F}=\mathbf{0}$ *identically.*

Solution. Sufficiency. Suppose R is simply connected and curl $\mathbf{F}=\mathbf{0}$ everywhere in R. Let C be any closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS=0.$$

Necessity. Suppose $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ for every closed path C and assume that $\nabla\times\mathbf{F}\neq 0$ at some point A.

Then taking $\nabla\times\mathbf{F}$ as continuous, there must exist a region with A as an interior point, where $\nabla\times\mathbf{F}\neq 0$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point is in the same direction as $\nabla\times\mathbf{F}$, *i.e.* $\nabla\times\mathbf{F}=\lambda\mathbf{n}$ where λ is a positive constant. Let C be the boundary of S. Then by Stoke's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS=\iint_S \lambda\mathbf{n}\cdot\mathbf{n}\, dS$$

$$=\lambda S>0.$$

This contradicts the hypothesis that $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ for every closed path C. Therefore we must have $\nabla\times\mathbf{F}=0$ everywhere in R.

Exercises

1. Verify Stoke's theorem for the function
$$\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$$
where curve is the unit circle in the xy-plane bounding the hemisphere $z=\sqrt{(1-x^2-y^2)}$.
[Agra 1975; Rohilkhand 81; Kanpur 78]
[**Hint.** Proceed as in Ex. 7 Page 139. Show that
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\pi=\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS\Big]$$

2. Verify Stoke's theorem for $\mathbf{A}=2y\mathbf{i}+3x\mathbf{j}-z^2\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=9$ and C is its boundary. **[Meerut 1975]**

3. Verify Stoke's theorem for the vector $\mathbf{q}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$ taken over the half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy-plane. **[Gauhati 1973]**

4. Verify Stoke's theorm for the function
$$\mathbf{F}=x^2\mathbf{i}+xy\mathbf{j}$$
integrated along the rectangle, in the plane $z=0$, whose sides are along the lines $x=0$, $y=0$, $x=a$ and $y=b$. **[Meerut 1976]**

5. Verify Stoke's theorem for a vector field defined by $\mathbf{F}=(x^2-y^2)\mathbf{i}+2xy\mathbf{j}$ in the rectangular region in the xy-plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.
[Kanpur 1975]

6. Verify Stoke's theorem for the function

$$\mathbf{F}=x^2\,\mathbf{i}+xy\,\mathbf{j},$$

integrated round the square, in the plane $z=0$, whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$. **[Bombay 1970]**

[**Hint.** Proceed as in Ex. 9 Page 141. Show that

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\tfrac{1}{2}a^3=\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS].$$

7. Verify Stoke's theorem for the function

$$\mathbf{F}(x, y, z)=xy\mathbf{i}+xy^2\mathbf{j}$$

integrated round the square with vertices (1, 0, 0), (1, 1, 0), (0, 1, 0) and (0, 0, 0),
where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along x-axis and y-axis respectively. **[Meerut 1979]**

8. Verify Stoke's theorem for the vector $\mathbf{A}=3y\mathbf{i}-xz\mathbf{j}+yz^2\mathbf{k}$, where S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$ and C is its boundary. **[Meerut 1973, 77]**

9. By converting into a line integral evaluate

$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, dS$, where $\mathbf{A}=(x-z)\,\mathbf{i}+(x^2+yz)\,\mathbf{j}-3xy^2\mathbf{k}$

and S is the surface of the cone $z=2-\sqrt{(x^2+y^2)}$ above the xy-plane. **Ans.** 12π. **[Meerut 1974]**

10. By converting into a line integral evaluate

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$$

where $\mathbf{F}=(x^2+y-4)\,\mathbf{i}+3xy\mathbf{j}+(2xz+z^2)\,\mathbf{k}$
and S is the surface of (i) the hemisphere $x^2+y^2+z^2=16$ above the xy-plane (ii) the paraboloid $z=4-(x^2+y^2)$ above the xy-plane. **Ans.** (i) -16π, (ii) -4π.

11. Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$, where
$\mathbf{F}=(y-z+2)\mathbf{i}+(yz+4)\mathbf{j}-xz\mathbf{k}$ and S is the surface of the cube $x=y=z=0$, $x=y=z=2$ above the xy-plane. **Ans.** -4.
[**Hint.** The curve C bounding the surface S is the square, say $OABC$, in the xy-plane given by $x=0$, $x=2$, $y=0$, $y=2$].

12. Show that

$$\iint_S \phi \text{ curl } \mathbf{F}\cdot d\mathbf{S}=\int_C \phi\, \mathbf{F}\cdot d\mathbf{r}-\iint_S (\text{grad } \phi\times\mathbf{F})\cdot d\mathbf{S}.$$

[**Hint.** Apply Stoke's theorem to the vector $\phi\mathbf{F}$].

13. If $\mathbf{f}=\nabla\phi$ and $\mathbf{g}=\nabla\psi$ are two vector point functions, such that

$$\nabla^2\phi=0,\ \nabla^2\psi=0$$

equations of C can be taken as $x=\cos t$, $y=\sin t$, $z=0$, $0 \leqslant t < 2\pi$.

We have $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int \left(-\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right)$

$$= \int_{t=0}^{2\pi} \left[-\frac{\sin t}{\cos^2 t + \sin^2 t} \frac{dx}{dt} + \frac{\cos t}{\cos^2 t + \sin^2 t} \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

Thus we see that $\oint_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

Definition. Irrotational vector field. *A vector field* **F** *is said to be irrotational if curl* $\mathbf{F} = 0$. **(Calicut 1975; Allahabad 79)**

We see that an irrotational field **F** is characterised by any one of the three conditions :

(i) $\mathbf{F} = \nabla \phi$,

(ii) $\nabla \times \mathbf{F} = 0$,

(iii) $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path.

Any one of these conditions implies the other two.

SOLVED EXAMPLES

Ex. 1. *Are the following forms exact ?*

(*i*) $xdx - ydy + zdz$. (*ii*) $e^y dx + e^x dy + e^z dz$.

(*iii*) $yzdx + xzdy + xydz$. (*iv*) $y^2 z^3 dx + 2xyz^3 dy + 3xy^2 z^2 dz$.

Solution. (i) We have

$$xdx - ydy + zdz = (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$
$$= \mathbf{F} \cdot d\mathbf{r}, \text{ where}$$
$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}.$$

We have Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$

$\therefore$ the given form is exact.

(ii) Here $\mathbf{F} = e^y \mathbf{i} + e^x \mathbf{j} + e^z \mathbf{k}$. We have

Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (e^x - e^y)\,\mathbf{k}.$

Since curl $\mathbf{F} \neq 0$, therefore the given form is not exact.

(iii) Here $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\,\mathbf{i} - (y-y)\,\mathbf{j} + (z-z)\,\mathbf{k} = \mathbf{0}.$$

Since curl $\mathbf{F} = 0$, therefore the given form is exact.

(iv) Here $\mathbf{F} = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}.$$

$\therefore$ the given form is exact.

Ex. 2. *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(*i*) $xdx - ydy - zdz$. (*ii*) $dx + zdy + ydz$

(*iii*) $\cos x\, dx - 2yz\, dy - y^2\, dz$. (*iv*) $(z^2 - 2xy)\, dx - x^2dy + 2xzdz$.

Solution. (i) Here $\mathbf{F} = x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & -z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$,

or $x\mathbf{i} - y\mathbf{j} - z\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = x \text{ whence } \phi = \frac{x^2}{2} + f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial \phi}{\partial y} = -y \text{ whence } \phi = -\frac{y^2}{2} + f_2(x, z) \qquad \ldots(2)$$

$$\frac{\partial \phi}{\partial z} = -z \text{ whence } \phi = -\frac{z^2}{2} + f_3(x, y). \qquad \ldots(3)$$

The constants of integration are functions of the variables not involved in the integration because the derivatives are partial.

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = -\frac{y^2+z^2}{2}, f_2(x, z) = \frac{x^2-z^2}{2}, f_3(x, y) = \frac{x^2-y^2}{2}.$$

$\therefore \quad \phi = \dfrac{x^2-y^2-z^2}{2}$ to which may be added any constant.

Hence $\phi = \dfrac{x^2-y^2-z^2}{2} + C$, where C is a constant.

(ii) Here $\mathbf{F} = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$

or $\mathbf{i} + z\mathbf{j} + y\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\dfrac{\partial \phi}{\partial x} = 1$ whence $\phi = x + f_1(y, z)$ ...(1)

$\dfrac{\partial \phi}{\partial y} = z$ whence $\phi = zy + f_2(x, z)$ ...(2)

$\dfrac{\partial \phi}{\partial z} = y$ whence $\phi = yz + f_3(x, y)$ ...(3)

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = zy, f_2(x, z) = x, f_3(x, y) = x.$$

$\therefore \quad \phi = x + yz$ to which may be added any constant.

$\therefore \quad \phi = x + yz + C.$

(iii) Here $\mathbf{F} = \cos x\mathbf{i} - 2yz\mathbf{j} - y^2\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$$= (-2y+2y)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$,

or $\cos x\mathbf{i} - 2yz\mathbf{j} - y^2\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\dfrac{\partial \phi}{\partial x} = \cos x$ whence $\phi = \sin x + f_1(y, z)$ ...(1)

$$\frac{\partial\phi}{\partial y}=-2yz \text{ whence } \phi=-y^2z+f_2(x, z) \quad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=-y^2 \text{ whence } \phi=-y^2z+f_3(x, y). \quad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y, z)=-y^2z$, $f_2(x, z)=\sin x$, $f_3(x, y)=\sin x$.

$\therefore$ $\phi=\sin x-y^2z$ to which may be added any constant.

$\therefore$ $\phi=\sin x-y^2z+C$.

(iv) Here $\mathbf{F}=(z^2-2xy)\,\mathbf{i}-x^2\mathbf{j}+2xz\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-2xy & -x^2 & 2xz \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F}=\nabla\phi$

or $(z^2-2xy)\,\mathbf{i}-x^2\mathbf{j}+2xz\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=z^2-2xy \text{ whence } \phi=z^2x-x^2y+f_1(y, z) \quad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=-x^2 \text{ whence } \phi=-x^2y+f_2(x, z) \quad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=2xz \text{ whence } \phi=xz^2+f_3(x, y). \quad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=0,\ f_2(x, z)=xz^2,\ f_3(x, y)=-x^2y.$$

$\therefore$ $\phi=z^2x-x^2y$ to which may be added any constant.

$\therefore$ $\phi=z^2x-x^2y+C$.

Ex. 3. *Show that* $\mathbf{F}=(2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\mathbf{k}$ *is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from*

$$(1, -2, 1) \textit{ to } (3, 1, 4).$$

Solution. The field $\mathbf{F}$ will be conservative if $\nabla\times\mathbf{F}=\mathbf{0}$.

We have

$$\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}=\mathbf{0}.$$

Therefore **F** is a conservative force field.

Let $\mathbf{F}=\nabla\phi$

or $(2xy+z^3)\,\mathbf{i}+x^2\,\mathbf{j}+3xz^2\,\mathbf{k}=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=2xy+z^3 \text{ whence } \phi=x^2y+z^3x+f_1(y,z) \quad ...(1)$$

$$\frac{\partial\phi}{\partial y}=x^2 \text{ whence } \phi=x^2y+f_2(x,z) \quad ...(2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \text{ whence } \phi=xz^3+f_3(x,y) \quad ...(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y,z)=0,\ f_2(x,z)=z^3x,\ f_3(x,y)=x^2y.$$

$\therefore$ $\phi=x^2y+xz^3$ to which may be added any constant.

$\therefore$ $\phi=x^2y+xz^3+C$.

$$\text{Work done}=\int_{(1,-2,1)}^{(3,1,4)} \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_{(1,-2,1)}^{(3,1,4)} d\phi=\Big[\ \phi\ \Big]_{(1,-2,1)}^{(3,1,4)}$$

$$=\Big[x^2y+xz^3\Big]_{(1,-2,1)}^{(3,1,4)}=202.$$

Ex. 4. *Show that the vector field* **F** *ginen by*

$$\mathbf{F}=(y+\sin z)\,\mathbf{i}+x\mathbf{j}+x\cos z\mathbf{k}$$

is conservative. Find its scalar potential.

Solution. We have

$$\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+\sin z & x & x\cos z \end{vmatrix}=\mathbf{0}.$$

$\therefore$ the vector field **F** is conservative.

Let $\mathbf{F}=\nabla\phi$

or $(y+\sin z)\,\mathbf{i}+x\mathbf{j}+x\cos z\mathbf{k}=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=y+\sin z \text{ whence } \phi=xy+x\sin z+f_1(y,z) \quad ...(1)$$

$$\frac{\partial\phi}{\partial y}=x \qquad \text{whence } \phi=xy+f_2(x,z) \quad ...(2)$$

$$\frac{\partial\phi}{\partial z}=x\cos z \text{ whence } \phi=x\sin z+f_3(x,y) \quad ...(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=0,\ f_2(x, z)=x \sin z,\ f_3(x, y)=xy.$$

$\therefore$ $\phi=xy+x \sin z$ to which may be added any constant.

$\therefore$ $\phi=xy+x \sin z+C$.

Ex. 5. *Evaluate*

$$\int_C 2xyz^2\, dx+(x^2z^2+z \cos yz)\, dy+(2x^2yz+y \cos yz)\, dz$$

where C is any path from (0, 0, 1) *to* (1, $\frac{1}{4}\pi$, 2). **(Meerut 1968)**

Solution. We have $\mathbf{F}=2xyz^2\,\mathbf{i}+(x^2z^2+z \cos yz)\,\mathbf{j}+(2x^2yz+y \cos yz)\,\mathbf{k}$.

$$\therefore \quad \nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2+z \cos yz & 2x^2yz+y \cos yz \end{vmatrix}$$

$$=(2x^2z+\cos yz-yz \sin yz-2x^2z-\cos yz+yz \sin yz)\,\mathbf{i}-(4xyz-4xyz)\,\mathbf{j}+(2xz^2-2xz^2)\,\mathbf{k}=0.$$

$\therefore$ the given line integral is independent of path in space.

Let $\mathbf{F}=\nabla\phi$. Then

$$\frac{\partial\phi}{\partial x}=2xyz^2 \text{ whence } \phi=x^2yz^2+f_1(y, z) \quad ...(1)$$

$$\frac{\partial\phi}{\partial y}=x^2z^2+z \cos yz \text{ whence } \phi=x^2z^2y+\sin yz+f_2(x, z) \quad ...(2)$$

$$\frac{\partial\phi}{\partial z}=2x^2yz+y \cos yz \text{ whence } \phi=x^2yz^2+\sin yz+f_3(x, y) \quad ...(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=\sin yz,\ f_2(x, z)=0,\ f_3(x, y)=0.$$

$\therefore$ $\phi=x^2yz^2+\sin yz$ to which may be added any constant.

The given line integral is therefore

$$\int_C d(x^2yz^2+\sin yz)=\Big[x^2yz^2+\sin yz\Big]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$=\pi+\sin \tfrac{1}{2}\pi=\pi+1.$$

Ex. 6. *Evaluate*

$$\int_C yz\,dx+(xz+1)\, dy+xy\, dz,$$

where C is any path from (1, 0, 0) *to* (2, 1, 4).

[Meerut 1969; Agra 72]

Solution. We have $\mathbf{F}=yz\mathbf{i}+(xz+1)\,\mathbf{j}+xy\mathbf{k}$.

$$\therefore \quad \nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+1 & xy \end{vmatrix}$$

$$=(x-x)\,\mathbf{i}-(y-y)\,\mathbf{j}+(z-z)\,\mathbf{k}=0.$$

$\therefore$ the differential form $yzdx+(xz+1)\,dy+xydz$ is exact and the given line integral is independent of path.

Let $\mathbf{F}=\nabla\phi$

or $\quad yz\mathbf{i}+(xz+1)\,\mathbf{j}+xy\mathbf{k}=\frac{\partial\phi}{\partial x}\,\mathbf{i}+\frac{\partial\phi}{\partial y}\,\mathbf{j}+\frac{\partial\phi}{\partial z}\,\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=yz \text{ whence } \phi=xyz+f_1\,(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=xz+1 \text{ whence } \phi=xyz+y+f_2\,(x, z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=xy \text{ whence } \phi=xyz+f_3\,(x, y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1\,(y, z)=y,\ f_2\,(x, z)=0,\ f_3\,(x, y)=y.$$

$\therefore$ $\phi=xyz+y$ to which may be added any constant.

The given line integral is therefore

$$=\int_{(1,\,0,\,0)}^{(2,\,1,\,4)} d\,(xyz+y)=\Big[\ xyz+y\ \Big]_{(1,\,0,\,0)}^{(2,\,1,\,4)}$$

$$=[8+1-0-0]=9.$$

Ex. 7. *Show that the form under the integral sign is exact and evaluate*

$$\int_{(0,\,2,\,1)}^{(2,\,0,\,1)} [ze^x\,dx+2yz\,dy+(e^x+y^2)\,dz].$$

Solution. Here $\mathbf{F}=ze^x\,\mathbf{i}+2yz\,\mathbf{j}+(e^x+y^2)\,\mathbf{k}$.

We have curl $\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & 2yz & e^x+y^2 \end{vmatrix}$

$$=(2y-2y)\,\mathbf{i}-(e^x-e^x)\,\mathbf{j}+0\mathbf{k}=0.$$

$\therefore$ the form under the integral sign is exact and consequently the line integral is independent of path in space.

Let $\mathbf{F}=\nabla\phi$

or $ze^x \mathbf{i}+2yz\,\mathbf{j}+(e^x+y^2)\,\mathbf{k}=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{j}+\frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x}=ze^x \text{ whence } \phi=ze^x+f_1(y, z) \qquad ...(1)$$

$$\frac{\partial \phi}{\partial y}=2yz \text{ whence } \phi=y^2z+f_2(x, z) \qquad ...(2)$$

$$\frac{\partial \phi}{\partial z}=e^x+y^2 \text{ whence } \phi=e^xz+y^2z+f_3(x, y) \qquad ...(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=y^2z,\ f_2(x, z)=e^xz,\ f_3(x, y)=0.$$

$\therefore$ $\phi=ze^x+y^2z$ to which may be added any constant. The given line integral is therefore

$$=\int_{(0,\,2,\,1)}^{(2,\,0,\,1)} d\,(ze^x+y^2z)=\left[ze^x+y^2z\right]_{(0,\,2,\,1)}^{(2,\,0,\,1)}$$

$$=[e^2+0-1-4]=e^2-5.$$

Ex. 8. *If* $\mathbf{F}=\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is the curve* $y=\sqrt{(1-x^2)}$ *in the x-y plane from* (1, 0) *to* (0, 1).

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (\cos y\,dx-x\sin y\,dy)$

$$=\int_1^0 \cos\sqrt{(1-x^2)}\,dx-\int_0^1 \sqrt{(1-y^2)}\sin y\,dy.$$

It is difficult to evaluate the integrals directly. However we observe that

$$\text{curl }\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & -x\sin y & 0 \end{vmatrix}$$

$$=0\,\mathbf{i}+0\,\mathbf{j}+(-\sin y+\sin y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ the given line integral is independent of path.

Let $\mathbf{F}=\nabla\phi$

or $\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{j}+\frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x}=\cos y \text{ whence } \phi=x\cos y+f_1(y, z) \qquad ...(1)$$

$$\frac{\partial \phi}{\partial y}=-x\sin y, \text{ whence } \phi=x\cos y+f_2(x, z) \qquad ...(2)$$

$$\frac{\partial \phi}{\partial z}=0 \text{ whence } \phi=f_3(x, y). \qquad ...(3)$$

From (1), (2), (3), we see that $\phi = x \cos y$.

$\therefore$ The given line integral is equal to

$$\int_{(1,\,0)}^{(0,\,1)} d\,(x \cos y) = \Big[x \cos y \Big]_{(1,\,0)}^{(0,\,1)} = [0 - 1 \cos 0] = -1.$$

Ex. 9. *Show that the vector field* **F** *given by*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ *such that* $\mathbf{F} = \nabla \phi$.

Solution. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= (-x + x)\,\mathbf{i} - (-y + y)\,\mathbf{j} + (-z + z)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ The vector field **F** is irrotational.

Let $\mathbf{F} = \nabla \phi$

or $(x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}.$

Then

$$\frac{\partial \phi}{\partial x} = x^2 - yz \text{ whence } \phi = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \text{...(1)}$$

$$\frac{\partial \phi}{\partial y} = y^2 - zx \text{ whence } \phi = \frac{y^3}{3} - xyz + f_2(x, z) \qquad \text{...(2)}$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \text{ whence } \phi = \frac{z^3}{3} - xyz + f_3(x, y). \qquad \text{...(3)}$$

(1), (2), (3) each represents ϕ. These agree

if we choose $f_1(y, z) = \dfrac{y^3}{3} + \dfrac{z^3}{3}$, $f_2(x, z) = \dfrac{x^3 + z^3}{3}$, $f_3(x, y) = \dfrac{x^3 + y^3}{3}$.

Therefore $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz + C.$

Exercises

1. Show that

$(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz$ is an exact differential of some function ϕ and find this function.

Ans. $\phi = y^2 z^3 \sin x - x^4 z + C.$

2. (i) Show that the vector field

$$\mathbf{F} = (2xy^2 + yz)\,\mathbf{i} + (2x^2 y + xz + 2yz^2)\,\mathbf{j} + (2y^2 z + xy)\,\mathbf{k}$$

is conservative.

(ii) Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is conservative and find ϕ such that $\mathbf{F} = \nabla \phi$. **[Kanpur 1980]**

Ans. $\phi = \frac{1}{2}(x^2 + y^2 + z^2) + C.$

3. Show that

$$\mathbf{F}=(\sin y+z)\,\mathbf{i}+(x\cos y-z)\,\mathbf{j}+(x-y)\,\mathbf{k}$$

is a conservative vector field and find a function ϕ such that

$\mathbf{F}=\nabla\phi.$ **[Bombay 1966]**

Ans. $\phi=x\sin y+xz-yz+C.$

4 Show that the vector field defined by

$$\mathbf{F}=(2xy-z^3)\,\mathbf{i}+(x^2+z)\,\mathbf{j}+(y-3xz^2)\,\mathbf{k}$$

is conservative, and find the scalar potential of **F**.

[Bombay 1970]

5. Show that the following vector functions **F** are irrotational and find the corresponding scalar ϕ such that

$$\mathbf{F}=\nabla\phi.$$

(i) $\mathbf{F}=x^3\,\mathbf{i}+y^3\,\mathbf{j}+z^3\,\mathbf{k}.$

(ii) $\mathbf{F}=(y\sin z-\sin x)\,\mathbf{i}+(x\sin z+2yz)\,\mathbf{j}+(xy\cos z+y^2)\,\mathbf{k}.$

(iii) $\mathbf{F}=(\sin y+z\cos x)\,\mathbf{i}+(x\cos y+\sin z)\,\mathbf{j}+(y\cos z+\sin x)\mathbf{k}.$

[Calcutta 1975]

Ans. (i) $\phi=\frac{1}{4}(x^4+y^4+z^4)+C.$

(ii) $\phi=xy\sin z+\cos x+y^2z+C.$

(iii) $\phi=x\sin y+y\sin z+z\sin x+C.$

6. Find a, b, c if $\mathbf{F}=(3x-3y+az)\,\mathbf{i}+(bx+2y-4z)\,\mathbf{j}+(2x+cy+z)\mathbf{k}$ is irrotational. **[Calicut 1974]**

Ans. $a=2,\ b=-3,\ c=-4.$

7. Show that

$$(2x\cos y+z\sin y)\,dx+(xz\cos y-x^2\sin y)\,dy+x\sin y\,dz=0$$

is an exact differential equation and hence solve it.

Ans. Solution is $x^2\cos y+xz\sin y=C.$

8. If **F** is irrotational in a simply connected region R, show that there exists a scalar field ϕ such that $\mathbf{F}=\text{grad}\,\phi.$

[Calicut 1975]

§ 11. Physical interpretation of divergence and curl.

[Meerut 1968]

Physical interpolation of divergence. *Suppose that there is a fluid motion whose velocity at any point is* $\mathbf{v}(x, y, z)$. *Then the loss of fluid per unit unit volume per time in a small parallelopiped having centre at* $P(x, y, z)$ *and edges parallel to the co-ordinate axes and having lengths* $\delta x, \delta y, \delta z$ *respectively, is given approximately by*

$$div\ \mathbf{v}=\nabla\cdot\mathbf{v}.$$

Le $\mathbf{v}=v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}.$

x . omponent of velocity $\mathbf{v}$ at $P=v_1(x, y, z).$

∴ x-component of $\mathbf{v}$ at centre of ' $AFED$ which is perp ndicular to x-axis and is nearer to or