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UPSC CSE 2023 Mathematics Paper 2 – Solutions

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1a) **Let G be a group of order 10 and G' be a group of order 6. Examine whether there exists a homomorphism of G onto G'.**

order of group homomorphically mapped onto another group must divide the order of domain group

$o(G)=10 \quad o(G')=6$

6 does not divide 10 $\quad 6 \nmid 10$

So No Homomorphism of G onto G'

Method-2 : FTH

If G is group and H is normal subgroup of G then $\frac{G}{H}$ is isomorphic to subgroup of $\frac{G}{H}$

G' is not a normal subgroup of G,

G' is not a normal subgroup of G, G' Cannot be isomorphic to subgroup of G

$\Rightarrow$ G' Cannot be isomorphic to subgroup of G

$\Rightarrow$ No Homomorphism of G onto G'

1b) Express the ideal $4Z + 6Z$ as a principal ideal in the integral domain Z.

If $nZ+mZ = dZ$, then d is greatest common divisor of n & m

$4Z+6Z=dZ$

$d = GCD(4,6)$

$= 2$

$4Z+6Z=2Z$

$4, 6$

$2^2 \quad 2\times 3$

2

1c) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1.3\cdot5\ldots(2n-1)}{2.4.6\ldots(2n)} \cdot \frac{x^{2n+1}}{(2n+1)}, x > 0$$

$$U_n = \frac{1.3.5..(2n-1)}{2.4.6..\ 2n} \cdot \frac{x^{2n+1}}{2n+1}$$

$$U_{n+1} = \frac{1.3.5..(2n-1)}{2.4.6..2n} \cdot \left[\frac{(2n+1)}{(2n+2)}\right] \frac{x^{2n+3}}{2n+3}$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)\left(\frac{2n+3}{2n+1}\right) \times \frac{1}{x^2}$$

$$\underset{x\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \left(\frac{2+\frac{2}{n}}{2+1/n}\right)\left(\frac{2+3/n}{2+1/n}\right) \times \frac{1}{x^2} = \frac{1}{x^{\cancel{2n}}} \quad n\to\infty$$

$= \frac{1}{x^2}$ → Converge for $\frac{1}{x^2} > 1$ $\quad 1 < \frac{1}{x^2}$

$x^2 < 1$

$x < 1$

→ Diverge for $\frac{1}{x^2} < 1$

$1 < x^2$

ie $1 < x$

Check for $x = 1$ — **Rabee Test**

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+3) - (2n+1)^2}{(2n+1)^2}$$

$$= \frac{(4n^2+10n+6) - (4n^2+4n+1)}{(2n+1)^2} = \frac{6n+5}{(2n+1)^2}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2+5n}{(2n+1)^2} \quad \underset{n\to\infty}{Lt} \frac{6+5/n}{(2+1/n)^2} = \frac{6}{4} = \frac{3}{2} > 1$$

⇒ Converge for $x = 1$

Converge for $x \leq 1$

Diverge for $x > 1$

D.Alembert Test

$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = l$ → Converge $l > 1$

Div $l < 1$

Rabee Test

$\underset{n\to\infty}{Lt}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

1d) State the sufficient conditions for a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be analytic in its domain. Hence, show that $f(z) = \log z$ is analytic in its domain and find $\frac{df}{dz}$.

→ Sufficient Condition is f(z) satisfy Cauchy Reiman Condition

ie $u_x = v_y$

$u_y = -v_x$

$z = re^{i\theta}$

$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

$$\to f(z) = \log z = \log(re^{i\theta})$$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

$$= u + iv$$

$u = \frac{1}{2}\log(x^2+y^2)$ $\quad u_x = \frac{1}{2(x^2+y^2)} 2x = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{2}\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$

$v = \tan^{-1}\frac{y}{x}$ $\quad v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$

$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$

clearly $u_x = v_y = \frac{x}{x^2+y^2}$ $\quad u_y = -v_x = \frac{y}{x^2+y^2}$

Satisfy Cauchy Rieman ⇒ Analytic

$$\to \frac{df}{dz} = \frac{d}{dz}\log z = \frac{1}{z}$$

1e) **A person requires 24,24 and 20 units of chemicals A, B and C respectively for his garden. Product P contains 2,4 and 1 units of chemicals A, B and C respectively per jar and product Q contains 2,1 and 5 units of chemicals A, B and C respectively per jar. If a jar of P costs ₹30 and a jar of Q costs ₹50, then how many jars of each should be purchased in order to minimize the cost and meet the requirements?**

$\text{Min } Z = 30\cancel{x} + 50y$

$2x + 2y \geqslant 24$ $\qquad x, y \geqslant 0$

$4x + y \geqslant 24$

$x + 5y \geqslant 20$

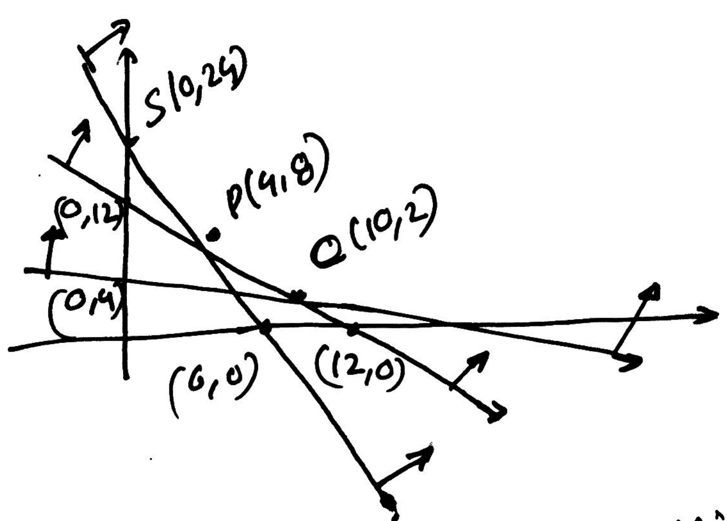
$2x + 2y = 24 \Rightarrow (x, 0) = (12, 0) \quad (0, y) = (0, 12)$

$4x + y = 24 \Rightarrow (x, 0) = (6, 0) \quad (0, y) = (0, 24)$

$x + 5y = 20 \Rightarrow (x, 0) = (20, 0) \quad (0, y) = (0, 4)$

Intersection $2x + 2y = 24$ & $4x + y = 24$ $\quad (x, y) = (4, 8)$

$2x + 2y = 24$ & $x + 5y = 20$ $\quad (x, y) = (1, 0)$



$\text{Min } Z = 30x + 50y$

at $S = 0 + 1200 = 1200$

$P \equiv 120 + 400 = 520$

$Q \equiv 300 + 100 = 400$

$R \equiv 600 + 0 = 600$

Min at Q which is 40

Purchase of P → 10 Jars

Q → 2 Jars

2a) Prove that a non-commutative group of order $2p$, where p is an odd prime, must have a subgroup of order p.

Solution. Suppose G has two distinct subgroups H and K, where

. $o(H) = o(K) = p, H \neq K$.

By Lagrange's Theorem, $o(H \cap K)$, divides $o(H) = p$

$\Rightarrow$ $o(H \cap K) = 1$ or p, since p is prime.

If $o(H \cap K) = p$, then $o(H \cap K) = o(H)$ or $H \cap K = H$

$(\because H \cap K \subseteq H)$

$\Rightarrow H = K$, which is a contradiction. Thus $o(H \cap K) = 1$.

We have

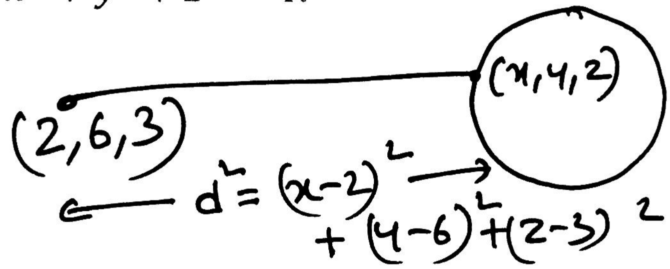
$$o(HK) = \frac{o(H)o(K)}{o(H \cap K)} = \frac{p \cdot p}{1} = p^2 > o(G) = 2p,$$

since $p > 2$ and p is prime.

But $o(HK) > o(G)$ is impossible. Hence the result.

SuccessClap Questn Bank
Algebra - Cosets Qn-19

2b) Using the method of Lagrange's multipliers, find the minimum and maximum distances of the point $P(2,6,3)$ from the sphere $x^2 + y^2 + z^2 = 4$.



$x^2+y^2+z^2=4$

Find Max/Min of $d^2 = (x-2)^2+(y-6)^2+(z-3)^2$

Subject to $x^2+y^2+z^2=4$

Lagrange $\Rightarrow$ $\left.\begin{array}{l}(x-2)dx+(y-6)dy+(z-3)dz \\ x\,dx+y\,dy+z\,dz\end{array}\right]$

$$\left.\begin{array}{ll}(x-2)+\lambda x=0 & \Rightarrow x=\frac{2}{1+\lambda} \\ (y-6)+\lambda y=0 & y=\frac{6}{1+\lambda} \\ (z-3)+\lambda z=0 & z=\frac{3}{1+\lambda}\end{array}\right]$$ Put in $x^2+y^2+z^2=4$

$$\frac{4+36+9}{(1+\lambda)^2}=4 \Rightarrow (\lambda+1)^2=\frac{49}{4} \Rightarrow \lambda+1=\pm\frac{7}{2}$$

$\lambda=\frac{7}{2}-1=\frac{5}{2}$

$x=\frac{2}{1+\lambda}=\frac{2}{1+5/2}=\frac{4}{7}$

$y=\frac{6}{1+\lambda}=\frac{6}{1+5/2}=\frac{12}{7}$

$z=\frac{3}{1+\lambda}=\frac{3}{1+5/2}=\frac{6}{7}$

$d^2=(x-2)^2+(y-6)^2+(z-3)^2$

$=\left(\frac{4}{7}-2\right)^2+\left(\frac{12}{7}-6\right)^2+\left(\frac{6}{7}-3\right)^2$

$=\frac{10^2+30^2+15^2}{7^2}=\frac{1225}{7^2}=\frac{35^2}{7^2}$

$d=\frac{35}{7}=5$

Max $d=7$

Min $d=5$

$\lambda=-\frac{7}{2}-1=-9/2$

$x=\frac{2}{1+\lambda}=\frac{2}{1-9/2}=-\frac{4}{7}$

$y=\frac{6}{1-9/2}=-\frac{12}{7}$

$z=\frac{3}{1-9/2}=-6/7$

$d^2=\left(\frac{4}{7}+2\right)^2+\left(\frac{12}{7}+6\right)^2+\left(\frac{6}{7}+3\right)^2$

$=\frac{18^2+54^2+27^2}{7^2}$

$=\frac{3969}{7^2}=\frac{7^2\times 9^2}{7^2}$

$d=7$

2c) Evaluate using $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ contour integration

$\boxed{\text{Method 1}}$

$$z = e^{i\theta} \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$5+4\cos\theta = 5 + \frac{2(z^2+1)}{z} = \frac{2z^2+5z+2}{z}$$

$$\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + \frac{1}{e^{2i\theta}}\right) = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) = \frac{z^4+1}{2z^2}$$

$$\left.\begin{aligned} z = e^{i\theta} \quad dz &= i e^{i\theta} d\theta \\ &= i z\, d\theta \end{aligned}\right] \Rightarrow d\theta = \frac{dz}{iz}$$

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \int_c \frac{z^4+1}{2z^2} \frac{z}{(2z^2+5z+2)} \times \frac{dz}{iz}$$

$$= \int_c \frac{z^4+1}{i4z^2\left(z^2+\frac{5z}{2}+1\right)}\, dz$$

$$z^2+\frac{5z}{2}+1 = (z-\alpha)(z-\beta)$$

Clearly $\alpha\beta = \frac{1}{1} = 1$

$$= (z-\alpha)\left(z-\frac{1}{\alpha}\right)$$

$$= \frac{1}{4i}\int_c f(z)\, dz$$

$$f(z) = \frac{z^4+1}{z^2(z-\alpha)\left(z-\frac{1}{\alpha}\right)} \qquad \text{Poles } z = 0, \alpha, 1/\alpha$$

$$\text{Res}(z=0) = \frac{1}{1!} \mathop{Lt}_{z\to 0} \frac{d}{dz}\left((z-0)^2 f(z)\right] = \mathop{Lt}_{z\to 0} \frac{d}{dz}\left[\frac{z^4+1}{\cancel{z^2}(z-\alpha)\left(z-\frac{1}{\alpha}\right)}\right]$$

$$= \mathop{Lt}_{z\to 0} \frac{(z-\alpha)\left(z-\frac{1}{\alpha}\right)4z^3 - (z^4+1)\left[z-\alpha)+\left(z-\frac{1}{\alpha}\right)\right]}{(z-\alpha)^2\left(z-\frac{1}{\alpha}\right)^2}$$

$$= \frac{\alpha+1/\alpha}{\alpha^2\left(\frac{1}{\alpha}\right)^2} = \alpha + \frac{1}{\alpha}$$

$$\text{Res}(z=\alpha) = \mathop{Lt}_{z\to\infty} (z-\alpha) f(z) = \mathop{Lt}_{z\to\alpha} \frac{z^4+1}{z^2\left(z-\frac{1}{\alpha}\right)} = \frac{\alpha^4+1}{\alpha^2\left(\alpha-\frac{1}{\alpha}\right)}$$

$$\alpha+\frac{1}{\alpha} + \frac{\alpha^4+1}{\alpha^2\left(\alpha-\frac{1}{\alpha}\right)} = \frac{\alpha^2\left(\alpha^2-\frac{1}{\alpha^2}\right)+\alpha^4+1}{\alpha^2\left(\alpha-\frac{1}{\alpha}\right)}$$

$$= \frac{\alpha^4 - 1 + \alpha^4 + 1}{\frac{\alpha^2(\alpha^2-1)}{\alpha}} = \frac{2\alpha^4}{\alpha(\alpha^2-1)} = \frac{2\alpha^3}{\alpha^2-1}$$

$$I = 2\pi i \,(\text{Sum of residue})$$

$$= 2\pi i \times \frac{1}{4i}\left[\frac{2\alpha^3}{\alpha^2-1}\right]$$

$$= \pi \frac{(-1/8)}{-3/4} = \frac{\pi}{6} \quad (\text{Ans})$$

$z^2 + \frac{5}{2}z + 1 = 0$

$\alpha = \dfrac{-5/2 \pm \sqrt{\frac{25}{4} - \ldots}}{2}$

$= \dfrac{-5/2 \pm 3/2}{2}$

$-\frac{1}{2}, -2$

$-1/2$ lie inside

$\alpha = -1/2$

$\frac{1}{4} - 1 = -\frac{3}{4}$

$\boxed{\text{Method-2}}$

$$I = \int_0^{2\pi} \frac{\text{Real } e^{2iz}}{5 + 4\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= R.P \oint \frac{z^2 dz}{2i\left(z^2 + \frac{5}{2}z + 1\right)}$$

Real $e^{2iz} = z^2$

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha} \frac{(z-\alpha)\, z^2}{2i\,(z-\alpha)\left(z - \frac{1}{\alpha}\right)}$$

$$= \frac{\alpha^2}{2i\left(\alpha - \frac{1}{\alpha}\right)}$$

$$= \frac{1}{4.2i\left(\frac{3}{2}\right)} = \frac{1}{12i}$$

$\alpha = -\frac{1}{2}$

$-\frac{1}{2} + 2 = \frac{3}{2}$

$\alpha^2 = \frac{1}{4}$

$$I = 2\pi i \,(\text{Sum of residue}) = 2\pi i \times \frac{1}{12i} = \frac{\pi}{6}$$

3a) **Prove that $x^2 + 1$ is an irreducible polynomial in $Z_3[x]$. Further show that the quotient ring $\frac{Z_3[x]}{\langle x^2+1 \rangle}$ is a field of 9 elements.**

$Z_3 \rightarrow \{0, 1, 2\}$

$f(x) = x^2 + 1 \rightarrow f(0) = 0^2 + 1 = 1 \neq 0$

$f(1) = 1^2 + 1 = 2 \neq 0$

$f(2) = 2^2 + 1 = 4 + 1 = \underline{5 = 2} \neq 0$

No soln $\Rightarrow$ Not irreducible in Z_3

* Check for Z_5 (Example only : Not asked)

$f(2) = 2^2 + 1 = 5 = 0$ in Z_5 Soln exist

$f(3) = 3^2 + 1 = 10 = 0$ in Z_5 Soln exist

$x^2 + 1 = (x+2)(x+3)$ in Z_5

Thm: Let F be a field and p(x) be an irreducible polynomial over F. Then $\frac{F[x]}{\langle p(x) \rangle}$ is a field

* Z_3 is field : Why? Z_p is field iff p is prime

(x^2+1) irreducible in $Z_3[x]$

$\Rightarrow \frac{Z_3[x]}{\langle x^2+1 \rangle}$ is field

Gallian Thm 17.5 Corollary 1

$\rightarrow \frac{Z_3[x]}{\langle x^2+1 \rangle} = \{ax + b + \langle x^2+1 \rangle \; a, b \in Z_3\}$

This field has $3 \times 3 = 9$ elements

Gallian Example 11 Page 319

3b) Prove that $u(x,y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its conjugate harmonic function $v(x,y)$ and express the corresponding analytic function $f(z)$ in terms of z.

SuccessClap : Question Bank Analytic Function

Qn-46

$$u = e^x(x\cos y - y\sin y)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$u_x = v_y$

$u_y = -v_x$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= e^x(x\sin y + \sin y + y\cos y)dx + e^x(x\cos y - y\sin y + \cos y)dy$$

$$v = \int_{y=\text{constant}} e^x(x\sin y + \sin y + y\cos y)dx + \int_{\text{remove } x \text{ terms}} (\quad)dy + c$$

"0"

$$= \int_{y=\text{const}} \underline{e^x(x\sin y + \sin y)} + (\underbrace{e^x y\cos y}_{\text{const}})dx$$

$$e^x(x\sin y) \quad + e^x y\cos y$$

$$= e^x(x\sin y + y\cos y)$$

$$\int e^x(f(x) + f'(x)) = e^x f(x)$$

$$f(z) = u + iv$$

$$= e^x[x\cos y - y\sin y + ix\sin y + iy\cos y] + c$$

$$= e^x(x+iy)(\underline{\cos y + i\sin y}) + c$$

$$e^{iy} = \cos y + i\sin y$$

$$= e^{(x+iy)}(x+iy) + c$$

$$= e^z \cdot z + C$$

3c) **Solve the following linear programming problem by Big M method**

Minimize $Z = 2x_1 + 3x_2$

subject to

$$x_1 + x_2 \geq 9$$
$$x_1 + 2x_2 \geq 15$$
$$2x_1 - 3x_2 \leq 9$$
$$x_1, x_2 \geq 0$$

Is the optimal solution unique? Justify your answer.

$\text{Max } Z^* = \text{Min}(-Z) = -2x_1 - 3x_2$

subject to $x_1 + x_2 \geq 9$

$x_1 + 2x_2 \geq 15$

$2x_1 - 3x_2 \leq 9 \qquad x_1, x_2 \geq 0$

$\text{Max } Z^* = -2x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

subject $x_1 + x_2 - S_1 + 0S_2 + 0S_3 + A_1 + 0A_2 = 9$

$x_1 + 2x_2 + 0S_1 - S_2 + 0S_3 + 0A_1 + A_2 = 15$

$2x_1 - 3x_2 + 0S_1 + 0S_2 + S_3 + 0A_1 + 0A_2 = 9$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

IBFS $(x_1, x_2, S_1, S_2, S_3, A_1, A_2) = (0, 0, 0, 0, 9, 9, 15)$

	c_j	-2	-3	0	0	0	-M	-M		
C_B	Basic	x_1	x_2	S_1	S_2	S_3	A_1	A_2	b	θ
-M	A_1	1	1	-1	0	0	1	0	9	9
-M	A_2	1	2	0	-1	0	0	1	15	15/2 →
0	S_3	2	-3	0	0	1	0	0	9	-3
$Z_j = \Sigma C_B a_{ij}$		-2M	-3M	M	M	0	-M	-M	-24M	
$C_j = C_j - Z_j$		-2+2M	-3+3M ↑	-M	-M	0	0	0		

C_j		-2	-3	0	0	0	$-M$	$-M$		
C_B	Basic	x_1	x_2	S_1	S_2	S_3	A_1	A_2	b	θ
$-M$	A_1	$1/2$	0	-1	$1/2$	0	1	$-1/2$	$3/2$	$3 \rightarrow$
-3	x_2	$1/2$	1	0	$-1/2$	0	0	$1/2$	$15/2$	15
0	S_3	$57/2$	0	0	$-3/2$	1	0	$3/2$	$63/2$	9
$Z_j=\sum C_B a_{ij}$		$-\frac{M}{2}-1/2$	-3	$-M$	$-\frac{M}{2}+3/2$	0	$-M$	$-\frac{M}{2}-3/2$	$\frac{-3M-45}{2}$	
$C_j=c_j-Z_j$		$-1/2+\frac{M}{2}$ ↑	0	M	$\frac{M}{2}-3/2$	0	0	$-\frac{M}{2}+3/2$		
-2	x_1	1	0	-2	1	0	2	-1	3	
-3	x_2	0	1	1	-1	0	-1	1	6	
0	S_3	0	0	7	-5	1	-7	2	21	
$Z_j=\sum C_B a_{ij}$		-2	-3	1	1	0	-1	-1	-24	
$C_j=c_j-Z_j$		0	0	-1	-1	0	$-M+1$	$-M+1$		

All $C_j \leq 0$

$\text{Max } Z^* = -24$

$\text{Min } Z = 24$

4a) Prove that the oscillation of a real-valued bounded function f defined on $[a,b]$ is the supremum of the set $\{|f(x_1)-f(x_2)|: x_1, x_2 \in [a,b]\}$.

Let $M = \sup_{x \in [a,b]} f(x)$ $\quad m = \inf_{x\in[a,b]} f(x)$, f is bounded

In $[a,b]$ Oscillation $= M-m$ (By Definition)

To prove $M-m = \sup\{|f(x_1)-f(x_2)|\}$ $x_1, x_2 \in [a,b]$

Let $\varepsilon > 0$, $x_1, x_2 \in [a,b]$

$$M-\frac{\varepsilon}{2} < f(x_1) \leq M \quad — ①$$

$$m \leq f(x_2) < m+\frac{\varepsilon}{2} \quad — ②$$

(−1) Multi

$$-m-\frac{\varepsilon}{2} < -f(x_2) \leq -m \quad — ③$$

① + ③

$$M-m-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} < f(x_1)-f(x_2) < M-m$$

$$M-m-\varepsilon < f(x_1)-f(x_2) < M-m$$

$$\Rightarrow |f(x_1)-f(x_2)| \leq M-m$$

$\downarrow$ Sup value (Max value) is $M-m$ as its $\leq M-m$

$$\sup\{|f(x_1)-f(x_2)|\} = M-m$$

4b) Classify the singular point $z = 0$ of the function $f(z) = \frac{e^z}{z-\sin z}$ and obtain the principal part of its Laurent series expansion.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!}$$

$$z - \sin z = \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \frac{z^9}{9!} = \frac{z^3}{6}\left[1 - \left(\frac{6}{5!}z^2 - \frac{6}{7!}z^4 + \frac{6}{9!}z^6\right)\right]$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}$$

$$\frac{e^z}{z-\sin z} = \frac{\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} \ldots\right]}{(z^3/6)\left[1 - \left(\frac{6}{5!}z^2 - \frac{6}{7!}z^4 + \frac{6}{9}z^6 \ldots\right)\right]}^{-1}$$

$$= \left(\frac{6}{z^3}\right)\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}\right)\left(1 + \left(\frac{6}{5!}z^2 - \frac{6}{7!}z^4 + \frac{6}{9}z^6 + \ldots\right) + \left(\frac{6}{5!}z^2 - \frac{6}{7!}z^4 + \ldots\right)^2\right)$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3$$

$$= \left(\frac{6}{z^3}\right)\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots\right) + \left(\frac{6}{5!}z^2 + \frac{6}{5!}z^3 + \frac{6z^4}{5!2!} + \frac{6}{5!3!}z^5 + \frac{6z^6}{5!4!}\right.$$

square

$$\left(-\frac{6}{7!}z^4 - \frac{6}{7!}z^5 - \frac{6z^6}{7!2!} - \frac{6}{7!3!}z^7 + \ldots\right) + \left(\left(\frac{6}{5!}z^2\right)^2 + \ldots\right)$$

$$= \left(\frac{6}{z^3}\right)\left[1 + z + z^2\left(\frac{1}{2!} + \frac{6}{5!}\right) + z^3\left(\frac{1}{3!} + \frac{6}{5!}\right) + z^4\left(\frac{1}{4!} + \frac{6}{5!2!} - \frac{6}{7!} + \left(\frac{6}{5!}\right)^2\right) + z^5 + \ldots\right.$$

$$= \frac{6}{z^3} + \frac{6}{z^2} + \left(\frac{11}{20}\right)\left(\frac{1}{z}\right) + \left(\frac{23}{60}\right) + (\ )z + (\ )z^2 + \ldots$$

$$\frac{1}{2!} + \frac{6}{5!} = \frac{1}{2} + \frac{6}{5\times4\times3\times2} = \frac{1}{2} + \frac{1}{20} = \frac{11}{20}$$

$z=0$ is isolated pole singularity of order 3

$$\frac{1}{3} + \frac{1}{20} = \frac{23}{}$$

4c) **A department head has 5 subordinates and 5 jobs to be performed. The time (in hours) that each subordinate will take to perform each job is given in the matrix below:**

		Jobs				
		A	B	C	D	E
	I	4	9	4	12	4
	II	15	11	20	5	8
Subordinates	III	17	7	15	12	18
	IV	9	13	11	9	14
	V	6	11	12	9	14

How should the jobs be assigned, one to each subordinate, so as to minimize the total time? Also, obtain the total minimum time to perform all the jobs if the subordinate IV cannot be assigned job C.

	A	B	C	D	E
I	4	9	4	12	4
II	15	11	20	5	8
III	17	7	15	12	18
IV	9	13	11	9	14
V	6	11	12	9	14

Subtract min element →

	A	B	C	D	E
I	0	5	0	8	0
II	10	6	15	0	3
III	10	0	8	5	11
IV	0	4	2	0	5
V	0	5	6	3	8

	A	B	C	D	E
I	0	5	0	8	0
II	10	6	15	0	3
III	10	0	8	5	11
IV	0	4	2	0	5
V	0	5	6	3	8

	A	B	C	D	E
I	2	5	0	10	0
II	10	4	13	0	1
III	12	0	8	7	11
IV	0	2	0	0	3
V	0	3	4	3	6

Cost Min = 4+5+7+11+6 = 33

(II)

	A	B	C	D	E
I	0	5	0	8	0
II	10	6	15	0	3
III	10	0	8	5	11
IV	0	4	M	0	5
V	0	5	6	3	8

	A	B	C	D	E
I	3	5	0	11	0
II	10	3	12	0	0
III	13	0	8	8	11
IV	0	1	M	0	2
V	0	2	3	3	5

	A	B	C	D	E
I	3	5	0	11	0
II	10	3	12	0	0
III	13	0	8	8	11
IV	0	1	M	0	2
V	0	2	3	3	5

A→V B→III C→I
D→IV E→II
Min Cost = 4+8+7+9+6
= 34

5a) **By eliminating the arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$, form partial differential equation.**

$$z = f(x^2-y) + g(x^2+y)$$

$$\frac{\partial z}{\partial x} = 2xf'(x^2-y) + 2xg'(x^2+y)$$

$$\frac{\partial z}{\partial y} = -f'(x^2-y) + \cancel{2xg''}\; g'(x^2+y)$$

$$\frac{\partial^2 z}{\partial x^2} = 2f'(x^2-y) + 2g'(x^2+y) + 4x^2f''(x^2-y) + 4x^2g''(x^2+y)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y)$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 2f'(x^2-y) + 2g'(x^2+y) + 4x^2f''(x^2-y) + 4x^2g''(x^2-y) - 2f'(x^2-y) - 2g'(x^2+y)$$

$$= 4x^2\left[f''(x^2-y) + g''(x^2+y)\right]$$

$$= 4x^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2\frac{\partial^2 z}{\partial y^2}$$

5b) Given $\frac{dy}{dx} = \frac{y^2 - x}{y^2 + x}$ with initial condition $y = 1$ at $x = 0$. Find the value of y for $x = 0.4$ by Euler's method, correct to 4 decimal places, taking step length $h = 0.1$.

$y' = \frac{y^2 - x}{y^2 + x}$ $\quad y(0) = 1 \quad h = 0.1$

$y_1 = y_0 + h f(x_0, y_0) = 1 + (0.1) f(0, 1) = 1.1$

$y_2 = y_1 + h f(x_1, y_1) = 1.1847$

$y_3 = y_2 + h f(x_2, y_2) = 1.2598$

$y_4 = y_3 + h f(x_3, y_3) = 1.328$

$y(0.4) = 1.328$

(ii) $(7AB \cdot 432)_{16} - (5CA \cdot D61)_{16}$

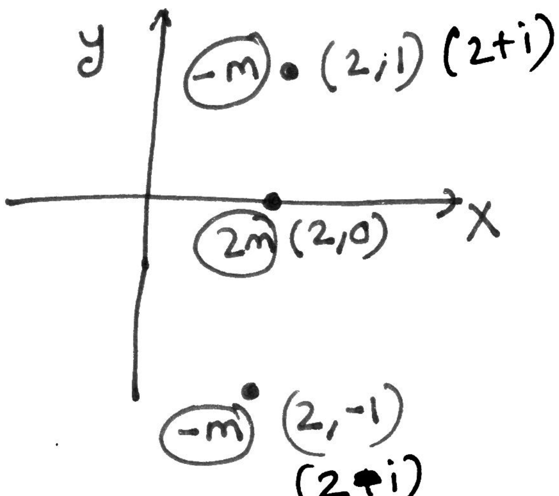
$(7AB.432)_{16} = (11110101011.010000110010)_2$

$(5CA.D61)_{16} = (10111001010.110101100001)_2$

$$
\begin{array}{r}
1111010\,1011.\,010000110010 \\
-\;1011100\,1010.\,\cancel{1}10101100000 \\
\hline
00111100000.\,011011010001
\end{array}
$$

$= (1E0.6D1)_{16}$

5e) **In a fluid motion, there is a source of strength $2m$ placed at $z = 2$ and two sinks of strength m are placed at $z = 2 + i$ and $z = 2 - i$. Find the streamlines.**



$w = -m\log(z-a)$

Formula

$$w = -2m\log(z-2) + m\log[z-(2+i)] + m\log[z-(2-i)]$$

$z = x+iy$

$$w = -2m\log[(x-2)+iy] + m\log[(x-2)+i(y-1)] + m\log[(x-2)+i(y+1)]$$

$= \varphi + i\psi$

$$\psi = -2m\tan^{-1}\frac{y}{x-2} + \left(\tan^{-1}\frac{y-1}{x-2}\right)m + \tan^{-1}\left(\frac{y+1}{x-2}\right)m$$

$$\frac{\psi}{m} = -2\tan^{-1}C' + \tan^{-1}A + \tan^{-1}B$$

$A = \frac{y-1}{x-2}$

$B = \frac{y+1}{x-2}$

$C' = \frac{y}{x-2}$

$$\boxed{2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} \qquad \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB} \qquad \tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}}$$

$$2\tan^{-1}C' = \tan^{-1}\frac{2C'}{1-C'^2} = \tan^{-1}C$$

$C = \frac{2C'}{1-C'^2}$

$$\frac{\psi}{m} = (\tan^{-1}A + \tan^{-1}B) - \tan^{-1}C = \tan^{-1}\left(\frac{A+B}{1-AB}\right) - \tan^{-1}C$$

$$= \tan^{-1}\frac{\left(\frac{A+B}{1-AB}\right) - C}{1+\left(\frac{A+B}{1-AB}\right)C}$$

Streamlines $\Rightarrow \psi = \text{Constant} \Rightarrow \frac{\psi}{m} = \text{Const} \Rightarrow \tan\frac{\psi}{m} = \text{Const}$

ie
$$\frac{\left(\frac{A+B}{1-AB}\right) - C}{1+\left(\frac{A+B}{1-AB}\right)C} = \lambda$$

Final Eqn

$$\boxed{\frac{A+B+ABC-C}{1-AB+AC+BC} = \lambda}$$

$A = \frac{y-1}{x-2}$

$B = \frac{y+1}{x-2}$

$C = 2C'$

Its Not possible to reduce into eqn. Streamlines | Simplify Not possible

$C' = y/(x-2)$

6a) **Find the surface passing through the two lines** $z = x = 0$ **and** $z - 1 = x - y = 0$. **and satisfying** $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$

$(D^2 - 4DD' + 4D'^2)z = 0$

$\cancel{(D-2D')^2 z=0}$ $(D-2D')^2 z = 0$

$z = \varphi_1(y+2x) + x\varphi_2(y+2x)$

Pass thru $z = x = 0$ $\Rightarrow 0 = \varphi_1(y)$

$\Rightarrow \varphi_1(y+2x) = 0$

$\Rightarrow z = x\varphi_2(y+2x)$

↳ Pass through $z - 1 = x - y = 0$

ie $z = 1$ $x = y$

$1 = x\varphi_2(3x)$

$3 = 3x\varphi_2(3x)$

$\Rightarrow \varphi_2(y+2x) = \frac{3}{y+2x}$

$z = \frac{x3}{y+2x}$ $\qquad (y+2x)z = 3x$

SuccessClap : Question Bank : PDE - Homogeneous
Qn-22

6b) Solve the system of linear equations

$$7x_1 - x_2 + 2x_3 = 11$$
$$2x_1 + 8x_2 - x_3 = 9$$
$$x_1 - 2x_2 + 9x_3 = 7$$

correct up to 4 significant figures by the Gauss-Seidel iterative method. Take initially guessed solution as $x_1 = x_2 = x_3 \equiv 0$.

$7x - y + 2z = 12$

$2x + 8y - z = 9$

$x - 2y + 9z = 7$

$x_{k+1} = \frac{1}{7}(11 + y_k - 2z_k)$

$y_{k+1} = \frac{1}{8}(9 - 2x_{k+1} - z_k)$

$z_{k+1} = \frac{1}{9}(7 - x_{k+1} + 2y_{k+1})$

Inital $(x, y, z) = 0, 0, 0)$

$\left.\begin{array}{l} x_1 = 1.5714 \\ y_1 = 0.7321 \end{array}\right]$ It-1

$z_1 = 0.7659$

$\left.\begin{array}{l} x_2 = 1.4572 \\ y_2 = 0.8564 \end{array}\right]$ It-2

$\cancel{z_2} = 0.8062$

$\left.\begin{array}{l} x_3 = 1.4634 \\ y_3 = 0.8599 \\ z_3 = 0.8063 \end{array}\right\}$ It-3

$\left.\begin{array}{l} x_4 = 1.4639 \\ y_4 = 0.8598 \\ z_4 = 0.8062 \end{array}\right]$ It-4

$x = 1.46$

$y = 0.86$

$z = 0.81$

6c) A mechanical system with 2 degrees of freedom has the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m(w_1^2x^2 + w_2^2y^2) + kxy$$

where m, w_1, w_2, k are constants. Find the parameter θ so that under the transformation $x = q_1\cos\theta - q_2\sin\theta$, $y = q_1\sin\theta + q_2\cos\theta$ the Lagrangian in terms of q_1, q_2 will not contain the product term q_1q_2. Find the Lagrange's equations w.r.t. q_1 and q_2 independent of parameter θ.

To Solve this question, you need PATIENCE, Lot of PATIENCE

$$x = q_1\cos\theta - q_2\sin\theta$$

$$\dot{x} = \dot{q}_1\cos\theta - q_1\sin\theta\,\dot{\theta} - \dot{q}_2\sin\theta - q_2\cos\theta\,\dot{\theta}$$

$$= \cos\theta(\dot{q}_1 - q_2\dot{\theta}) - \sin\theta(q_1\dot{\theta} + \dot{q}_2)$$

$$\dot{x}^2 = \cos^2\theta(\dot{q}_1 - q_2\dot{\theta})^2 + \sin^2\theta(q_1\dot{\theta} + \dot{q}_2)^2 - 2\sin\theta\cos\theta(\dot{q}_1 - q_2\dot{\theta})(q_1\dot{\theta} + \dot{q}_2)$$

$$y = q_1\sin\theta + q_2\cos\theta$$

$$\dot{y} = \dot{q}_1\sin\theta + q_1\cos\theta\,\dot{\theta} + \dot{q}_2\cos\theta - q_2\sin\theta\,\dot{\theta}$$

$$= \cos\theta(q_1\dot{\theta} + \dot{q}_2) + \sin\theta(\dot{q}_1 - q_2\dot{\theta})$$

$$\dot{y}^2 = \cos^2\theta(q_1\dot{\theta} + \dot{q}_2)^2 + \sin^2\theta(\dot{q}_1 - q_2\dot{\theta})^2 + 2\sin\theta\cos\theta(q_1\dot{\theta} + \dot{q}_2)(\dot{q}_1 - q_2\dot{\theta})$$

$$\dot{x}^2 + \dot{y}^2 = \cos^2\theta\left[(\dot{q}_1 - q_2\dot{\theta})^2 + (q_1\dot{\theta} + \dot{q}_2)^2\right] + \sin^2\theta\left[(q_1\dot{\theta} + \dot{q}_2)^2 + (\dot{q}_1 - q_2\dot{\theta})^2\right]$$

$()\times 2\sin\theta\cos\theta \rightarrow$ gets Cancelled

$$\dot{x}^2+\dot{y}^2 = \cos^2\theta\left[\dot{q}_1^2+q_2^2\dot{\theta}^2-2\dot{q}_1 q_2\dot{\theta} + q_2^2\dot{\theta}^2+\dot{q}_2^2+2q_1\dot{\theta}\dot{q}_2\right] + \sin^2\theta\left[q_1^2\dot{\theta}^2+\dot{q}_2^2+2q_1\dot{q}_2\dot{\theta} + \dot{q}_1^2+q_2^2\dot{\theta}^2-2\dot{q}_1 q_2\dot{\theta}\right]$$

Consider $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$ → we see there is <u>no</u> $(q_1 \cdot q_2)$ term (product of q_1 & q_2)

$$x^2 = q_1^2\cos^2\theta+q_2^2\sin^2\theta-2q_1q_2\sin\theta\cos\theta$$

$$y^2 = q_1^2\sin^2\theta+q_2^2\cos^2\theta+2q_1q_2\sin\theta\cos\theta$$

In term $\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2)$

↳ Then $(q_1\cdot q_2)$ term is $\frac{1}{2}m(-\omega_1^2 2\sin\theta\cos\theta) + \frac{1}{2}m(\omega_2^2 2\sin\theta\cos\theta)$

$$= \frac{m}{2}\left[\omega_2^2-\omega_1^2\right]\sin 2\theta$$

In $-\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2)$ → $-\frac{m}{2}\left[\omega_2^2-\omega_1^2\right]\sin 2\theta$

→ $xy = (q_1\cos\theta-q_2\sin\theta)(q_1\sin\theta+q_2\cos\theta)$

$$= q_1^2\sin\theta\cos\theta+q_1q_2\cos^2\theta-q_1q_2\sin^2\theta - q_2^2\sin\theta\cos\theta$$

The $(q_1\cdot q_2)$ term is $\cos^2\theta-\sin^2\theta=\cos 2\theta$

In kxy it is $k\cos 2\theta$

In Lagrange $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2)$
$+ kxy$

the $(q_1 \cdot q_2)$ terms ae

$$-\frac{m}{2}\left[\omega_2^2 - \omega_1^2\right]\sin 2\theta + k\cos 2\theta$$

To make L independent of ~~$(q_1 \times q_2)$~~ $(q_1 \times q_2)$
↳ Epuate to 0

$$-\frac{m}{2}\left[\omega_2^2 - \omega_1^2\right]\sin 2\theta + k\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2K}{m(\omega_2^2 - \omega_1^2)}$$

7a) **(i) Find the conjunctive normal form (CNF) of the following Boolean function:** $f(x, y, z, t) = x \cdot y \cdot z + \bar{x} \cdot y \cdot (t + \bar{z})$

$$f = xyz + \bar{x}y(t+\bar{z})$$

$$= xyz + \bar{x}yt + xy\bar{z}$$

$$= xyz(t+\bar{t}) + \bar{x}yt(z+\bar{z}) + xy\bar{z}(t+\bar{t})$$

$$= \underset{(8+4+2+1=15)}{xyzt} + \underset{(14)}{xyz\bar{t}} + \underset{(7)}{\bar{x}yzt} + \underset{(5)}{\bar{x}y\bar{z}t}$$

$$+ \underset{(13)}{xy\bar{z}t} + \underset{(12)}{xy\bar{z}\bar{t}}$$

$$\boxed{\begin{array}{cccc} 2^3 & 2^2 & 2^1 & 2^0 \\ 8 & 4 & 2 & 1 \end{array}}$$

$$= \sum_m (5,7,12,13,14,15)$$

$$\downarrow$$

$$f' = \left[\sum_m (5,7,12,13,14,15)\right]'$$

$$\downarrow$$

$$f = \prod_M (0,1,2,3,4,6,8,9,10,11)$$

$$= (x+y+z+t)(x+y+z+t')(x+y+z'+t)$$
$$(x+y+z'+t')(x+y'+z+t)$$
$$(x+y'+z'+t)(x'+y+z+t)$$
$$(x'+y+z+t')(x'+y+z'+t)$$
$$(x'+y+z'+t')$$

(ii) Express the Boolean function $f(x, y, z) = x + \overline{(\bar{x} \cdot \bar{y} + \bar{x} \cdot z)} + z$ in disjunctive normal form (DNF) and construct the truth table for the function.

$$f(x,y,z) = x + z + \overline{(\bar{x} \cdot \bar{y} + \bar{x} \cdot z)}$$

$$(x+y) \cdot (x+\bar{z})$$

$x + xy +$ [illegible] $+ y\bar{z} + x\bar{z}$

$$= x + z + xy + x\bar{z} + y\bar{z}$$

$$= x(y+\bar{y})(z+\bar{z}) + (x+\bar{x})(y+\bar{y})z + xy(z+\bar{z}) + x\bar{z}(y+\bar{y}) + (x+\bar{x})(y\bar{z})$$

$$= x[yz + y\bar{z} + \bar{y}z + \bar{y}\bar{z}] + z[xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}] + xyz + xy\bar{z} + xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z})$$

$$= \underset{7}{xyz} + \underset{6}{xy\bar{z}} + \underset{5}{x\bar{y}z} + \underset{4}{x\bar{y}\bar{z}} + \underset{7}{xyz} + \underset{5}{x\bar{y}z} + \underset{3}{\bar{x}yz} + \underset{0}{\bar{x}\bar{y}\bar{z}} + \underset{6}{xy\bar{z}} + \underset{4}{x\bar{y}\bar{z}} + \underset{6}{xy\bar{z}} + \underset{2}{\bar{x}y\bar{z}} + \underset{7}{xyz} + \underset{6}{xy\bar{z}}$$

7 6 5 4 3 2 0

$$= xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

Truth Table

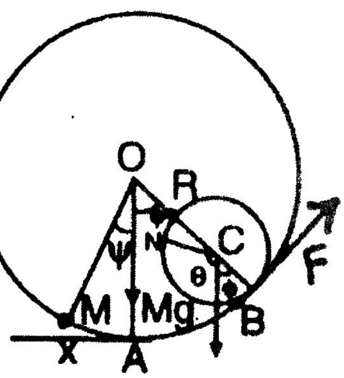
	x	y	z	$f(x,y,z)$
0	0	0	0	1
2	0	1	0	1
3	0	1	1	1
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

7b) A perfectly rough ball is at rest within a hollow cylindrical roller. The roller is drawn along a level path with uniform velocity V. Let a and b be the radii of the ball and the roller respectively. If $V^2 > \frac{27}{7} g(b-a)$, then show that the ball will roll completely round the inside of the rotter.

Sol. Let O be the centre of the roller and C the centre of the spherical ball moving inside the cylindrical roller.

Let CN be the radius of the ball which was vertical when it was in its lowest position.

When the roller has moved through a distance x, let CN have turned through an angle θ. The line joining the centre makes an angle ϕ with the vertical and the ball has turned through an angle θ.



As there is no sliding,

$$\text{arc } BM = \text{arc } BN$$

i.e. $b(\phi + \psi) = a(\theta + \phi)$
or $(b-a)\phi = a\theta - b\psi$
Again the velocity of the roller is constant
i.e. $\dot{x} = b\dot{\psi} = V$.
Then $\dot{x} = b\ddot{\psi} = 0$.
Let R and F be the normal reaction and friction.

As C describes a circle of radius $(b-a)$ about O, so accelerations along CO and perpendicular to CO are $(b-a)\dot{\phi}^2$ and $(b-a)\ddot{\phi}$ respectively.

Thus equation of motion are $m(b-a)\dot{\phi}^2 = Rmg\cos\phi$,

$m(b-a)\ddot{\phi} = F - mg\sin\phi$;

and $m\frac{2a^2}{5}\ddot{\theta} = -F\cdot a$,

Eliminating F between above two equations , we get

$$(b-a)\ddot{\phi} = -\frac{2a}{5}\ddot{\theta} - g\sin\phi \text{ or } (b-a)\ddot{\phi} + \frac{2}{5}(b-a)\ddot{\phi} = -g\sin\phi$$

[since $(b-a)\ddot{\phi} = a\ddot{\theta} - a\ddot{\psi} = a\ddot{\theta}$ by virtue of (2)]

or $\frac{7}{5}(b-a)\ddot{\phi} = -g\sin\phi$.

Integrating it, we get $\frac{7}{5}(b-a)\dot{\phi}^2 = 2g\cos\phi + A$.

Initially the velocity of the C.G. is $\dot{x} + (b-a)\dot{\phi} = 0$,

i.e. $(b-a)\dot{\phi} = -\dot{x} = -V$. $\therefore\ A = \frac{7V^2}{5(b-a)} - 2g$.

Hence the equation (7) gives

$$\frac{7}{5}(b-a)\dot{\phi}^2 = -2g(1-\cos\phi) + \frac{7V^2}{5(b-a)}.$$

Substituting for $\dot{\phi}^2$ from (8) in (3), we get

$$\frac{R}{m} = g\cos\phi + \frac{V^2}{b-a} - \frac{10}{7}(-\cos\phi) = \frac{1}{7}\left(17g\cos\phi - 10g + \frac{7V^2}{b-a}\right)$$

The necessary condition that the ball should roll completely round the fixed cylinder is that R is positive when $\phi = \pi$, and if R is positive in this position, when it will be positive in all positions.

Hence

$$\left\{\frac{7V^2}{b-a} - 10 + 17g\cos\phi\right\}_{\phi=\pi} > 0$$

$$\frac{7V^2}{b-a} > 27g \text{ or } V^2 > \frac{27g(b-a)}{7}.$$

7c) **Solve the partial differential equation**

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\ 0<x<L, t>0$$

subject to the conditions

$$u(0,t)=0,\ u(L,t)=0,\ t>0$$

$$u(x,0)=x, \left(\frac{\partial u}{\partial t}\right)_{t=0}=1,\ 0<x<L$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \qquad u(0,t)=0 \quad u(l,t)=0 \qquad u(x,0)=f(x) \quad \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x)$$

$$u(x,t)=\sum_{n=1}^{\infty}\left[C_n \cancel{\sin}\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l}$$

$$C_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad D_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$$c=a \qquad f(x)=x \quad g(x)=1 \qquad l=L \qquad \cancel{Kx=n\pi}$$

$$KL=n\pi$$

$$C_n=\frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

$$=\frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L$$

$$=\frac{2}{L}\left[-\frac{L^2}{n\pi}(-1)^n\right]=-\frac{2L}{n\pi}(-1)^n$$

$$D_n=\frac{2}{n\pi a}\int_0^L\sin\frac{n\pi x}{L}\,dx=\frac{2}{n\pi a}\left.-\cos\frac{n\pi}{L}\cdot\left(\frac{L}{n\pi}\right)\right|_0^L$$

$$=\frac{2L}{n^2\pi^2 a}[1+1]=\frac{4L}{n^2\pi^2 a}$$

$$u(x,t)=\sum\left[C_n\cos\frac{n\pi at}{L}+D_n\sin\frac{n\pi at}{L}\right]\sin\frac{n\pi x}{L}$$

$$C_n=-\frac{2L}{n\pi}(-1)^n \qquad D_n=\frac{4L}{n^2\pi^2 a}$$

8a) Reduce the partial differential equation

$$\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\left(1 + \frac{1}{x}\right) + \frac{z}{x} = 0$$

to canonical form.

Given $0 \cdot r - s + t + p - q(1 + 1/x) + (z/x) = 0$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 0,\ S = -1\ \ T = 1$.
Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.
The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$.
Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields
$dy/dx + 1 = 0$ or $dx + dy = 0$ Integrating it, $x + y = c, c$ being constant
Choose $u = x + y$ and $v = x$,

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow u$ and v are independent functions.
Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v},$$
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u},$$

we have $\partial/\partial y = \partial/\partial u$

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right), \text{ using (5)}$$

$$t = \partial^2 z/\partial u^2.$$

Eqn reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\, \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\hat{\partial}^2 z/\partial u\, \hat{\partial} - (\hat{\partial} z/\hat{\partial} v) + (1/v) \times (\hat{\partial} z/\partial u) - (z/v) = 0$, which is the required canonical

SuccessClap: Question Bank PDE - Canonical Qn=3

8b) Compute a root of the equation $\log_{10}(2x+1) - x^2 + 3 = 0$, in the interval [0, 3], by Regula-Falsi method, correct to 6 decimal places.

$f(x) = -x^2 + 3 + \log_{10}(2x+1)$

$f(2) = -0.30103 < 0$ $\quad$ $f(1) = 2.477 > 0$

$x_1 = 2$ $\quad$ $x_0 = 1$

$x_2 = x_0 - \dfrac{x_1 - x_2}{f(x_1) - f(x_0)} f(x_0)$

$= 1.891644$ — Itr-1

$f(x_2) = 0.10141$ (+ve)

$x_1 = 2 \longleftrightarrow x_0 = \cancel{0.} \; 1.891644$

$x_2 = 1.918948$ — Itr-2

$f(x_2) = 0.002294$ (+ve)

$x_1 = 2 \longleftrightarrow x_0 = 1.918948$

$x_2 = 1.91956$ — Itr-3

$f(x_2) = 0.000051$

$x_1 = 2$ $\quad$ $x_0 = 1.91$

Soln is $x_0 = 1.91956$

8c) Determine under what conditions the velocity field $u = c(x^2 - y^2), v = -2cxy, w = 0$ is a solution to the Navier-Stokes momentum equations. Assuming that the conditions are met, determine the resulting pressure distribution, when z ' is up and the external body forces are. $B_x = 0 = B_y, B_z = -g.$

Ans. Given: $u = a(x^2 - y^2)$ $\quad v = -2axy$

$w = 0$

Also, $\vec{g} = 0\hat{\imath} + 0\hat{\jmath} - g\hat{k}$

$$\frac{\partial u}{\partial x} = 2ax, \frac{\partial u}{\partial y} = -2ay$$

Therefore, $\frac{\partial v}{\partial x} = -2ay, \frac{\partial v}{\partial y} = -2ax$

$$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0, \frac{\partial w}{\partial t} = 0$$

As the velocity variables are not changing with respect to time, it can be considered that the flow is steady.

Our objective is to find p(x, y, z, t).

Since the flow is steady, so we will find p(x, y, z).

i.e., in the Navier-Stokes equations:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

i.e. $\rho \times 0 - \frac{\partial p}{\partial x} + \mu[2a + (-2a) + 0] = \rho[0 + a(x^2 - y^2)2ax - 2ay(-2ay) + 0]$

i.e. $-\frac{\partial p}{\partial x} = \rho[2a^2x(x^2 - y^2) + 4a^2xy^2]$

$= \rho[2a^2x^3 + 2a^2xy^2]$

$= 2a^2\rho x(x^2 + y^2)$

i.e. $\frac{\partial p}{\partial x} = -2a^2\rho x(x^2 + y^2)$

In y-direction:

$$0 - \frac{\partial p}{\partial y} + \mu[0] = \rho\left[\frac{\partial v}{\partial t} + a(x^2 - y^2)(-2ay) + (-2axy)(-2ax)\right]$$

i.e. $-\frac{\partial p}{\partial y} = \rho[-2a^2x^2y + 2a^2y^3 + 4a^2yx^2]$

i.e. $\frac{\partial p}{\partial y} = -2a^2\rho y(x^2 + y^2)$

In z-direction:

$$-\rho g - \frac{\partial p}{\partial z} + \mu[0] = \rho[0]$$

i.e. $\frac{\partial p}{\partial z} = -\rho g$

The vertical pressure gradient is hydrostatic.

$$p = \int \frac{\partial p}{\partial x} dx\Big|_{x,y}$$

The solution

$$= \int -2\rho a^2 x(x^2 + y^2)dx$$

$$= -2\rho a^2 \left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] + f_1(y, z)$$

$$\text{Again, } \frac{\partial p}{\partial y} = -2\rho a^2 x^2 y + \frac{\partial f_1}{\partial y}$$

$$= -2\rho a^2 y(x^2 + y^2)$$

$$\frac{\partial f_1}{\partial y} = -2\rho a^2 y^3$$

Comparing,

$$f_1 = \int \frac{\partial f_1}{\partial y} dy\Big|_z = -2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\therefore p = -2\rho a^2 \left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] - 2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\frac{\partial p}{\partial z} = 0 + \frac{\partial f_2}{\partial z} = -\rho g$$

$$f_2(z) = -\rho g z + c$$

where, c is a constant.

$$p(x, y, z) = -2\rho a^2 \left[\frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2}\right] - \rho g z + c$$