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Common Catenary

3.1 INTRODUCTION

When a string (rope, wire or cable) is suspended from two points not in the same vertical line and carries a certain distributed load, then it assumes a curved form. The shape of the curve depends on the manner in which the load is distributed. Ropes of suspension bridges, telegraph cables, etc. are examples of the suspended strings.

In the present chapter we shall consider the equilibrium of perfectly flexible strings. All those strings which offer no resistance on bending at any point are called **flexible strings.** In a perfectly flexible string, the resultant action, across any section of the string, consists of a single force whose line of action is along the tangent to the curve formed by the string. A chain whose links are quite small and smooth can also be regarded as a flexible string. The normal section of the string or chain is taken to be so small that it may be regarded as a curved line. If the chain is formed of links joined together the links of the chain should be very small and smooth.

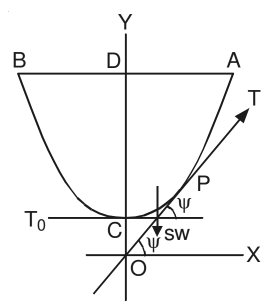
In practice two types of suspended strings are mostly found : **the catenary** which carries a uniformly distributed load along its length due to its own weight and the parabolic strings which carry a uniformly distributed load along the horizontal. These will be considered in the following discussion.

3.2. THE CATENARY

When a uniform flexible string or chain hangs freely under gravity between two points which are not in the same vertical line, then the curve, in which it hangs, is called a catenary.

When the weight per unit length of the suspended flexible string or chain is constant, then the catenary is called a uniform or common **catenary.**

3.2.1. Intrinsic Equation of Common Catenary



Let *ABC* be the uniform chain hanging freely from the points *A* and *B* which are not in a vertical line. *C* is the lowest point of the common catenary.

Let *P* be any point in the portion *CA* of the chain such that arc *CP* = *s*, measured along the curve. Then the weight of the portion *CP* of the chain = sw, where w is the weight per unit length of the chain. Also this weight sw acts vertically downwards through the C. G. of the arc *CP*.

The portion *CP* of the chain is in equilibrium under the action of the following forces :

(i) Tension T_0 at the lowest point *C* acting horizontally.

(ii) Tension *T* at the point *P* along the tangent at *P* making an angle Ψ with *x*-axis.

and (iii) the weight $s\omega$ of the portion *CP* acting vertically downwards.

Resolving these forces horizontally and vertically, we get

$$\mathbf{T}\cos\Psi = \mathbf{T}_0 \qquad \text{...(i)}$$

and

$$\mathbf{T}\sin\Psi = w\mathbf{s} \qquad \text{...(ii)}$$

Dividing (ii) by (i) we get $\psi = ws/T_0$ ...(iii)

Let tension T_0 at the lowest point *C* be taken equal to the weight of a length *c* of the chain *i.e.* $T_0 = wc$ (say).

Then from (iii) we get $\tan\Psi = \frac{ws}{wc} = \frac{s}{c}$ or **$s = c\tan\Psi$**

is the required intrinsic equation of the catenary, where c is called the **parameter** of the catenary.

From above we conclude that

(i) the horizontal component of tension at any point of the chain (i.e. $T\cos\Psi$) is equal to the tension T_0 at the lowest point which is constant and equal to wc.

and **(ii) the vertical component of tension at any point of the chain (i.e. $T\sin\Psi$)** is equal to the weight of the length of the chain measured from the lowest point to the point under consideration.

3.2.2. Cartesian Equation of the Common Catenary

We can prove as in § 3.2.1 that the intrinsic equation of the catenary is

$$s = c\tan\Psi \quad \text{....(i)}$$

or $$s = c\,.\frac{dy}{dx}$$

Differentiating both sides with respect to x, we get

$$\frac{ds}{dx} = c\frac{d^2y}{dx^2} \text{ or } \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} = c\frac{d^2y}{dx^2} \quad \text{...(ii)}$$

Put $$\frac{dy}{dx} = z \text{ then } \frac{dz}{dx} = \frac{d^2y}{dx^2}$$

$\therefore$ From (ii) $$\sqrt{(1+z^2)} = c\frac{dz}{dx} \text{ or } dx = \frac{c\,dz}{\sqrt{(1+z^2)}} \quad \text{...(iii)}$$

Integrating $x = c\sin h^{-1} z + A$,

where A is constant of integration.

Now choose the vertical line through the lowest point C as y-axis then at C, $x=0$ and $z = dy/dx = 0$, as the tangent at C is parallel to x-axis.

$\therefore$ From (iii) we get $0 = c\sin h^{-1} 0 + A$ or $A = 0$

$\therefore$ $x = c\sin h^{-1}z$ or $z = \sin h\,(x/c)$

or $dy/dx = \sin h\,(x/c)$ or $dy = \sin h\,(x/c)dx$

Integrating, $$y = c\cos h\,(x/c) + B, \quad \text{...(iv)}$$

where B is constant of integration.

Now choose the origin O at a depth c below the lowest point C on y-axis and let the horizontal line through O be taken as x-axis. Then the coordinates of the lowest point C are $(0, c)$

At C, from (iv) we get

$$c = c\cos h\,0 + B \quad \text{or} \quad c = c + B \text{ or } B = 0.$$

Hence **$y = c\cosh(x/c)$**,

is the required cartesian equation of the common catenary.

3.2.3. Some Definitions

(i) Vertex. The lowest point C is called the vertex of the common catenary.

(ii) Directrix. The axis of x which is a horizontal line at depth c below the lowest point C of the catenary is called its directrix.

(iii) Axis. The axis of y which is the vertical line through the lowest point C of the catenary is called its axis.

(iv) Span. If the two points of suspension A and B are in a horizontal line, then AB is called the span. If A be (x, y) then span $AB = 2x$.

(v) Sag. The distance CD is called the sag, where D is the mid-point of AB. If the ordinate of A be y, then

$$\text{sag} = DC = OD - OC = y - c$$

(vi) Parameter. The quantity c is called the parameter.

3.2.4. Properties of the Common Catenary

We know that the intrinsic and Cartesian equations of the catenary are

$$s = c \tan \psi \quad \text{....(i)}$$

and

$$y = c \cosh (x/c) \quad \text{...(ii)}$$

(a) Relation between s and x.

From (ii) we get $dy/dx = \sinh (x/c)$

$\therefore$ From (i) $s = c \tan \psi = c\,(dy/dx)$, $\because$ $dy/dx = \tan \Psi$

or $\mathbf{s = c \sinh (x/c)}$ ...(iii)

(b) Relation between s and y.

From (ii) and (iii) we get

$$y^2 - s^2 = c^2 [\cosh^2 (x/c) - \sinh^2 (x/c)] = c^2$$

or $\mathbf{y^2 = c^2 + s^2}$ ...(iv)

(c) Relation between y and ψ.

From (iv) we get

$$y^2 = c^2 + s^2 = c^2 + c^2 \tan^2 \Psi, \quad \text{from (i)}$$

or $y^2 = c^2 \sec^2 \psi$ $\therefore$ $\mathbf{y = c \sec \psi}$ ...(v)

(d) Relation between x and ψ.

From (i) $s = c \tan \psi$

Differentiating $\dfrac{ds}{d\psi} = c \sec^2 \psi$ or $\dfrac{ds}{dx} \cdot \dfrac{ds}{d\psi} = c \sec^2 \psi$

or $\dfrac{dx}{d\psi} = c \sec^2 \psi . \dfrac{dx}{ds} = c \sec^2 \psi . \cos \psi$, $\because$ $\dfrac{dx}{ds} = \cos \psi$

or $dx = c \sec \psi \, d\psi$

Integrating, $x = c \log (\sec \psi + \tan \psi)$, where A is constant of integration.

At the lowest point C, $x = 0$ and $\psi = 0$, $\therefore$ $A = 0$

Hence $\mathbf{x = c \log (\sec \psi + \tan \psi)}$

$$= c \log \left(\frac{y}{c} + \frac{s}{c} \right), \text{from (i) and (v)}$$

or $\mathbf{x = c \log [(y + s)/c]}$ ...(vi)

or $e^{x/c} = (y + s)/c$ or $y + s = ce^{x/c}$ ...(vii)

Also from (iv) and (vii) $x = c \log [\{\sqrt{(s^2 + c^2)} + s\}/c]$

(e) Relation between tension T and ordinate y.

In § 3.2.1 we have proved

$$T \cos \psi = T_0 \quad \text{or} \quad T = T_0 \sec \Psi$$

or $T = wc \sec \psi$, $\because$ $T_0 = wc$

$= wy$, from (v)

Hence $\mathbf{T} = wy$ ...(viii)

or tension at any point of the catenary varies as its height above the directrix.

EXAMPLES

1. *A given length 2s of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is*

$$\sqrt{(b^2 - s^2)} \log \{(b + s)/(b - s)\}$$

Sol. The span will be greatest when the terminal tension *i.e.* tension at the points of suspension is greatest.

Also maximum tension = $w\,b$, where w is the weight per unit length of the chain.

$\therefore$ If T be the tension at one of the points of suspension, then

$$T = bw \qquad \text{...(i)}$$

Also if ψ_1 be the inclination of the tangent at the point of suspension, then from "$T \sin\psi = ws$" we get

$$bw \sin\psi_1 = ws$$

or $$s\text{in } \psi_1 = s/b$$

$$\therefore \quad \tan\psi_1 = \frac{s}{\sqrt{(b^2 - s^2)}} \text{ and } \sec\psi_1 = \frac{b}{\sqrt{(b^2 - s^2)}} \qquad \text{...(ii)}$$

Also from "$s = c\tan\psi$" we get

$$s = cs/\sqrt{(b^2 - s^2)} \quad \text{or} \quad c = \sqrt{(b^2 - s^2)} \qquad \text{...(iii)}$$

$\therefore$ The required span $= 2x = 2c \log(\tan\psi_1 + \sec\psi_1)$

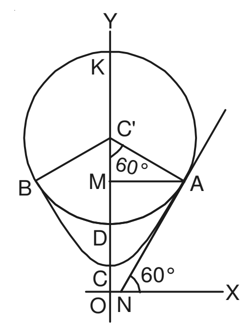
$$= 2\sqrt{(b^2 - s^2)} \log\left[\frac{s}{\sqrt{(b^2 - s^2)}} + \frac{b}{\sqrt{(b^2 - s^2)}}\right], \text{ from (ii), (iii)}$$

$$= 2\sqrt{(b^2 - s^2)} \log\left[\frac{(b+s)}{\sqrt{(b^2 - s^2)}}\right] = 2\sqrt{(b^2 - s^2)} \log\left[\sqrt{\left(\frac{b+s}{b-s}\right)}\right]$$

$$= 2\sqrt{(b^2 - s^2)} \cdot \frac{1}{2} \log[(b+s)/(b-s)] = \sqrt{(b^2 - s^2)}\,[\log[(b+s)/(b-s)]$$

2. *Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is*

$$a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right].$$



Sol. $AKBCA$ is the endless chain of which the portion $AKB = \frac{2}{3}$ of the circumference of the circle

$$= \frac{2}{3}[2\pi a] = \frac{4}{3}\pi a \qquad \text{...(i)}$$

The portion ACB is in the form of a catenary.

Also the arc ADB of the circle with centre $C' = \frac{1}{3}$ of the circumference of the circle.

Hence $\angle AC'B = \frac{1}{3}(2\pi) = 120°$,

so $\angle AC'D = 60°$.

Then if AN be the tangent to the catenary or the circle at A, then

$$\angle ANx = 60°.$$

Also AM, the length of perpendicular from A to $C'D$

$$= a \sin 60°$$

From "$x = c\log[\tan\psi + \sec\psi]$" at A we have

$$AM = c\log(\tan 60° + \sec 60°) \text{ or } \frac{a\sqrt{3}}{2} = c\log(\sqrt{3} + 2)$$

or $$c = a\sqrt{3}/\{2 \log(2+\sqrt{3})\} \qquad \text{...(ii)}$$

Now the portion ACB of the chain $= 2 \times \text{arc } CA$

$$= 2c \tan 60°, \text{ from "} s = c \tan\psi \text{"}$$

$$= 2.\left[\frac{a\sqrt{3}}{2\log(2+\sqrt{3})}\right]\sqrt{3}, \text{ from (ii)}$$

$$= 3a/\log(2+\sqrt{3}) \qquad \text{...(iii)}$$

$\therefore$ The length of the endless chain

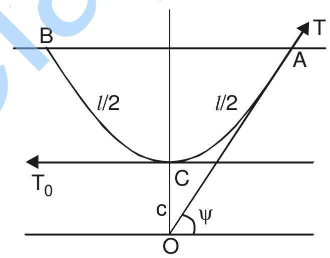
$$= \text{arc } AKB \text{ of the circle} + \text{arc } ACB \text{ of catenary}$$

$$= \frac{4\pi a}{3} + \frac{3a}{\log(2+\sqrt{3})}, \text{ from (i) and (iii)}$$

$$= a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right].$$

3. *A uniform chain, of length l, is to be suspended from two points A and B, in the same horizontal line so that either terminal tension is n times at the lowest point. Show that the span AB must be*

$$\frac{1}{\sqrt{(n^2-1)}}\log[n+\sqrt{(n^2-1)}].$$



Sol. We are given that

$$T = n \times T_0$$

i.e., $wc \sec\psi = n \times wc$ *i.e.,* $\sec\psi = n$

$\therefore$ $\tan\psi = \sqrt{(n^2-1)}$

From $s = c\tan\psi$, we have

$$\frac{1}{2}l = c\tan\psi = c\sqrt{(n^2-1)}$$

$\therefore$ $$c = \frac{l}{2\sqrt{(n^2-1)}}$$

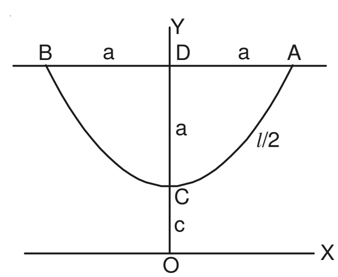
$\therefore$ Required span $= 2x = 2c\log(\sec\psi + \tan\psi)$

$$= \frac{2l}{2\sqrt{(n^2-1)}}\log[n+\sqrt{(n^2-1)}]$$

$$= \frac{l}{\sqrt{(n^2-1)}}\log[n+\sqrt{(n^2-1)}].$$

4. *A heavy uniform chain of length 2l is suspended by its ends which are on the same horizantal level. The distance apart 2a of the ends is such that its lowest point of the chain is at a distance a vertically below the ends. Prove that if c be the distance of the lowest point from the directrix of the catenary. then*

$$\frac{2a^2}{l^2-a^2} = \log\frac{l+a}{l-a} \text{ and } \tanh\frac{a}{c} = \frac{2al}{l^2+a^2}.$$



Sol. Let A and B be the end of the chain such that $AB = 2a$ and c is the lowest point of the chain such that $CD = a$.

Then, by $y^2 = c^2 + s^2$ at A, we get

$$(c+a)^2 = c^2 + l^2$$

or $$a^2 + 2ac = l^2 \Rightarrow c = (l^2 - a^2)/2a$$

Also, we know that

$$x = c\log(\tan\psi + \sec\psi)$$

or $$a = \frac{l^2 - a^2}{2a} \log \left[\left(\frac{s}{c} \right) + \left(\frac{y}{c} \right) \right]$$

$$= \frac{l^2 - a^2}{2a} \log \left[\frac{a + c + l}{(l^2 - a^2)/2a} \right]$$

or $$\frac{2a^2}{l^2 - a^2} = \log \left[\frac{a + \left\{ \frac{l^2 - a^2}{2a} + l \right\}}{(l^2 - a^2)/2a} \right]$$

or $$\frac{2a^2}{l^2 - a^2} = \log \left[\frac{a^2 + l^2 + 2al}{l^2 - a^2} \right] = \log \left[\frac{(a + l)^2}{l^2 - a^2} \right]$$

or $$\frac{2a^2}{l^2 - a^2} = \log \frac{l + a}{l - a}.$$

Also, we know that

$$y = c \cos h \left(\frac{x}{c} \right) \text{ and } s = c \sin h \left(\frac{x}{c} \right)$$

Dividing, we get $\frac{s}{y} = \tan h \left(\frac{x}{c} \right)$ and $s = l$ at A

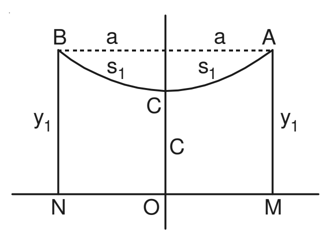
$\therefore$ At A, $$\tan h \frac{a}{c} = \frac{l}{a + c} \qquad \because \text{ from (i), } y_1 = a + c, x_1 = a$$

$$\tanh \frac{a}{c} = \frac{l}{a + \{(l^2 - a^2)/2a\}}$$

$$= \frac{2al}{(l^2 + a^2)}.$$

5. *A heavy string hangs over two fixed small smooth pegs. The two ends of the string are free and the central portion hangs in a catenary. If the two pegs are on the same level and distant 2a apart, show that equilibrium is impossible unless the length of the string is at least 2ae.*

Sol. A and B are the given pege such that $AB = 2a$. The tension in the string remains unaltered due to its passing round the pegs. Let y_1 be the ordinate of the point A and l_1 be the portion of the string hanging vertically from A. Let w be weight per unit length of the string. We know that tension at any point of the catenary is given by "$T = wy$"



$\therefore$ Tension at A due to the catenary $= wy_1$ ...(i)

Also for the portion l_1 hanging vertically at A, we have tension at $A =$ weight of portion l_1 of the chain $= wl_1$ ...(ii)

$\therefore$ From (i) and (ii) we have $wl_1 = wy_1$ or $l_1 = y_1$

Hence the free end of the hanging portion will be on the directrix of the catenary.

Similarly we have at B.

Let arc $CA = s_1$, where C is the lowest point of catenary.

$\therefore$ Total length say l of the string is given by

$$l = 2 \text{ (arc } CB + \text{hanging portion } AM)$$

$$= 2(s_1 + y_1) = 2[c \sin h(a/c) + c \cos h(a/c)].$$

since $y = c \cosh(x/c)$ and $s = c \sinh(x/c)$ and $x = a$ at A, then

$l = 2c[\sinh(a/c) + \cosh(a/c)] = 2c.\frac{1}{2}[(e^{a/c} - e^{-a/c}) + (e^{a/c} + e^{-a/c})]$

or $\qquad l = 2c\, e^{a/c}$ ...(iii)

If l is minimum then $\frac{dl}{dc} = 0$ and $\frac{d^2l}{dc^2}$ = positive.

From (iii) $dl/dc = 0$ given $2[e^{a/c} + c.e^{a/c}.(-a/c^2)] = 0$

or $\qquad (1 - a/c) = 0$ or $c = a$.

Also we can prove that for $c = a; d^2l/dc^2$ = positive

Hence the required value of $l = 2ae^{a/a}$, putting $c = a$ in (iii)

$= 2ae$

6. *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is*

$\mu \log\left[\frac{1 + \sqrt{(1 + \mu^2)}}{\mu}\right]$, *where* μ *is the coefficient of friction.*

OR

If the ends of a uniform inextensible string of length l hanging freely under gravity slide on a fixed rough horizontal rod whose coefficient of friction is μ, *show that at the most they can rest at a distance* $\mu l \log\left[\frac{1 + \sqrt{(1 + \mu^2)}}{\mu}\right]$.

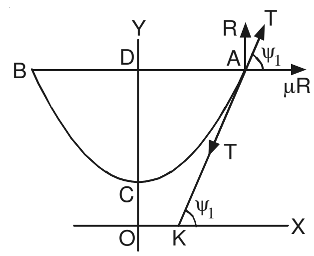
Sol. ACB is the uniform string with C as the lowest point. If the span AB is maximum, then the end links are in limiting equilibrium.

$\therefore$ The forces acting on the link at A are :–

(i) Normal reaction R acting perpendicular to AB,

(ii) force of friction μR acting along the rod in the direction as shown in the figure and,

(iii) the tension T along the tangent in the direction AK. Let ψ_1 be the inclination of the tangent at A to the horizontal. Then resolving these forces acting on the link at a horizontally and vertically we have



$T\cos\psi_1 = \mu R$ and $T\sin\psi_1 = R$

Dividing $\qquad \tan\psi_1 = \frac{R}{\mu R} = \frac{1}{\mu}$ ...(i)

$\therefore \quad \sec\psi_1 = \sqrt{(1 + \tan^2\psi_1)} = \sqrt{\left(1 + \frac{1}{\mu^2}\right)} = \sqrt{(\mu^2 + 1)}/\mu$ ...(ii)

Also from "$s = c\tan\psi$" at A we get

$\frac{l}{2} = c\tan\psi_1 = c/\mu$, from (i) or $c = \frac{\mu l}{2}$ ...(iii)

$\therefore \qquad$ required value of span $AB = 2 \times DA$

$= 2c\log(\tan\psi_1 + \sec\psi_1) \qquad \because x = c\log(\tan\psi + \sec\psi)$

$= 2\left(\frac{1}{2}\mu l\right)\log\left[\frac{1}{\mu} + \frac{\sqrt{(\mu^2 + 1)}}{\mu}\right]$, from (i), (ii) and (iii)

$= \mu l \log\left[\frac{1 + \sqrt{(\mu^2 + 1)}}{\mu}\right]$

Also the ratio of max. span to the length of the chain

$$= \frac{\text{max. span AB}}{l} = \frac{\mu l \log\{1+\sqrt{(\mu^2+1)}\}/\mu}{l}, \text{ proved above}$$

$$= \mu \log\left[\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right]$$

EXERCISES

1. If T be the tension at any point P of a catenary, and T_0 that at the lowest point C, prove that $T^2 - T_0^2 = W^2$, W being the weight of the arc CP of the catenary.

2. The extremities of a heavy string of length $2l$ and weight $2\omega l$ are attached to two small rings which can slide on a fixed horizontal wire. Each of these rings is acted on by a horizontal force equal to $l\omega$. Show that the distance apart of the ring is $2l \log(1+\sqrt{2})$.

3. The extremities A and B of a heavy uniform string of length l are attached to two small rings which can slide on a smooth horizontal wire. Each of these rings are acted on by a horizontal force $F = \frac{1}{2} wl$, where w is the weight per unit length of the string. Find the span.

[**Ans.** $l \log(1+\sqrt{2})$]

4. A rope of length $2l$ feet is suspended between two points at the same level and the lowest point of the rope is b feet below the points of suspension. Show that the horizonatal compoment of tension is $(w/2b)(l^2 - b^2)$, w being the weight of the rope per foot of its length.

5. If α and β be the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary and l the length of the portion, show that the height of one extremity above the other is $l \sin\frac{1}{2}(\alpha+\beta)/\cos\frac{1}{2}(\alpha-\beta)$.

6. A unifrom chain of length l is suspended from two points in the same horizontal line. If the tension at the highest point is twice that at the lowest point, show that the span is $\left(\frac{l}{\sqrt{3}}\right)\log(2+\sqrt{3})$.

7. If a chain is suspended from two points A, B on the same level, and depth of the middle point below AB is (l/n), where $2l$ is the length of the chain, show that the horizontal span AB is equal to $l\left(n-\frac{1}{n}\right)\log\left(\frac{1+n}{n-1}\right)$.

8. Prove that for the catenary $y = c \cos h \frac{x}{a}$ the perpendicular dropped from the foot of the ordinate upon tangent is of constant length.

9. The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles ϕ to the vertical. Prove that the sag in the middle is $\frac{1}{2} l \tan\frac{1}{2}\phi$.

10. A uniform chain of length l is suspended from two points in the same horizontal line. If the tension at the heighest point is twice that at the lowest point, show that the span is

$$l\left[n - \sqrt{\left(n^2 - \frac{1}{4}\right)}\right].$$

11. A heavy uniform string of length l, suspended from a fixed point A and its other end B is pulled by a horizontal force equal to the weight of a length a of the string. Show that the horizontal and vertical distances between A and B are $a \sin h^{-1}(l/a)$ and $\{\sqrt{(l^2+a^2)}-a\}$ respectively.

12. A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end A attached by light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles of 60° and 30° respectively with the horizontal. Prove that the ratio of these lengths is $(\sqrt{3}-1):1$.

13. A heavy uniform string hangs over two smooth pegs in the same horizontal line. If the length of each portion which hangs freely is n times the length between the pegs, prove that the whole length of the string is to the distance between the pegs as $k : \log k$, where $k=\sqrt{[(2n+1)/(2n-1)]}$.

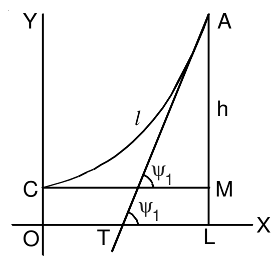
14. The end links of a uniform chain slide along a fixed rough horizontal rod. If the angle of friction be λ, prove that the ratio of the maximum span to the length of the chain is $\log \cot \frac{1}{2}\lambda : \cot \lambda$.

SOME MORE EXAMPLES

1. *A box-kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite, the inclination of the wire to the ground is* $2\tan^{-1}(h/l)$ *and its tensions there and at the ground are*

$$w.\frac{l^2+h^2}{2h} \text{ and } w.\frac{l^2-h^2}{2h},$$

where w is the weight of the wire per unit of length.



Sol. A is the kite and C is the hand. $\therefore CA\,(=l)$ is the wire paid out. C is the vertex of the catenary and CM, the horizontal line through C, represents the level of the ground. Let ψ_1 be the inclination of the tangent at A to the horizon. The ordinate of A is $(h+c)$.

$\therefore$ From $y^2=c^2+s^2$ at A, we get

$$(c+h)^2=c^2+l^2 \text{ or } h^2+2ch=l^2 \text{ or } c=\frac{l^2-h^2}{2h} \quad \text{...(i)}$$

$\therefore$ Tension at the ground = tension at C, the lowest point

$$= wc = w[(l^2-h^2)/2h], \text{ from (i)}$$

And tension at $A = wy = w(c+h) = w\left[\left(\frac{l^2-h^2}{2h}\right)+h\right]$, from (i)

$$= w[(l^2+h^2)/2h]$$

Also from $s=c\tan\psi$ at A we get

$$l=c\tan\psi_1 \text{ or } \tan\psi_1=\frac{l}{c}=\frac{l}{(l^2+h^2)/2h}, \text{ from (i)}$$

or $$\tan\psi_1=\frac{2hl}{(l^2-h^2)}=\frac{2(h/l)}{1-(h^2/l^2)}, \text{ dividing by } l^2$$

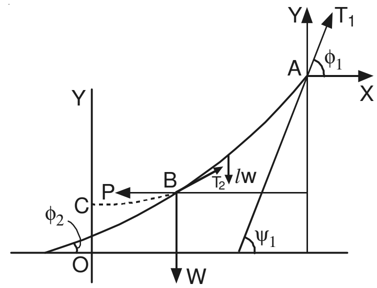
$$=\frac{2\tan\theta}{1-\tan^2\theta}, \text{ where } \tan\theta=(h/l)$$

or $$\tan\psi_1=\tan 2\theta, \text{ where } \theta=\tan^{-1}=(h/l)$$

or $$\psi_1=2\theta=2\tan^{-1}(h/l)$$

2. *A weight W is suspended from a fixed point by a uniform string of length l and weight per unit of length. It is drawn aside by a horizontal force P. Show that in the position of equilibrium, the distance of W from the vertical through the fixed point is*

$$\frac{P}{w}\left[\sinh^{-1}\left(\frac{W+lw}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right].$$



Sol. Let AB be the string of length l, suspended from the fixed point A and weight W is hanging from B. This weight W is drawn aside by a horizontal force P acting at B. Let T_1 and T_2 be the tensions at A and B respectively.

Since at B, the tension T_2 balances the force P and weight W, $\therefore$ the angle ψ_2, the inclination of the tangent at B to the horizon, is given by

$$\tan\psi_2 = W/P \quad \text{...(i)}$$

Let X and Y be the horizontal and vertical components of the tension T_1 at A. Then for equilibrium of the string AB, resolving the forces acting on the string AB horizontally and vertically we have

$$X = P,\ Y = W + lw \quad \text{...(ii)}$$

$\therefore$ If ψ_1 be the inclination of the tangent at A to the horizon, then

$$\tan\psi_1 = \frac{Y}{X} = \frac{W+lw}{P} \quad \text{...(iii)}$$

Let C be the vertex of the catenary AB. Let x_1 and x_2 be the horizontal distances of A and B from C. Let arc $CB = s$

Then from $s = c\tan\psi$ at B, we get

$$s = c\tan\psi_1 = c\,(W/P),\ \text{from (i)} \quad \text{...(iv)}$$

And from $s = c\sin h\,(x/c)$ at B, we get

$$s = c\sin h\,(x_2/c) \text{ or } x_2 = c\sin h^{-1}(s/c)$$

or $$x_2 = c\sin h^{-1}(W/P),\ \text{from (iv)} \quad \text{...(v)}$$

Also the horizontal component of tension at any point is constant and equal to ωc.

$\therefore$ at B we get $P = wc$ or $c = P/w$ ...(vi)

$\therefore$ From (v) we have $x_2 = (P/w)\sin h^{-1}(W/P)$ ...(vii)

Similarly with the help of (iii) we can show that

$$x_1 = (P/w)\sin h^{-1}[(W+lw)/P] \quad \text{...(viii)}$$

Hence the required horizontal distance $= BD = x_1 - x_2$

$$= \frac{P}{w}\left[\sin h^{-1}\left(\frac{W+lw}{P}\right)-\sin h^{-1}\left(\frac{W}{P}\right)\right], \quad \text{from (vii) and (viii)}$$

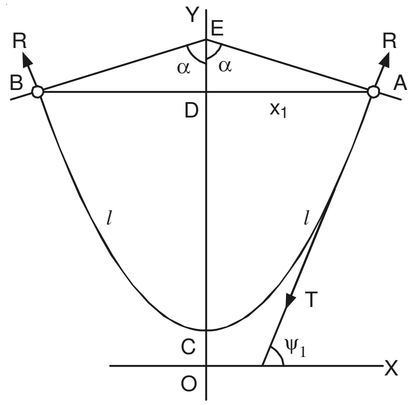
3. *The end links of a uniform chain of length 2l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles* α *to the vertical. Prove that the sag in the middle is* $l\tan\frac{1}{2}\alpha$ *and the distance between the links is* $2l\cot\alpha\sin h^{-1}(\tan\alpha)$.

Sol. ABC is the chain of length $2l$ with C as the vertex of the catenary.

$\therefore$ arc $CA = l$

Also the link at A is in equilibrium under the action of two forces viz, the normal reaction R acting at right angles to the rod and the tension T in the chain at A acting along the tangent at A to the catenary.

Hence the tangent at A to the catenary will be at right angles to the rod AE *i.e.* inclined at angle



to the horizon *i.e.* $\psi_1 = \alpha$, where ψ_1 is the inclination of the tangent at A to the horizon.

$\therefore$ Sag in the middle $= DC = y_1 - c$, where y_1 is the ordinate of A

$= c \sec\alpha - c$, because $y_1 = c\sec\alpha$, at A

$= c\,(1-\cos\alpha)/\cos\alpha$ ...(i)

Also from $s = c\tan\psi$, at A, we get $l = c\tan\alpha$

or $c = l\cot\alpha$ ...(ii)

$\therefore$ from (i) sag in the middle $= l\cot\alpha\,(1-\cos\alpha)/\cos\alpha$

$$= l\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = l\tan\left(\frac{1}{2}\alpha\right)$$

Let x_1 be the abscissa of A *i.e.* $DA = x_1$

Also from $s = c\sin h\,(x/c)$ at A we get

$l = c\sin h\,(x_1/c) = l\cot\alpha\sin h\,(x_1/l\cot\alpha)$, from (ii)

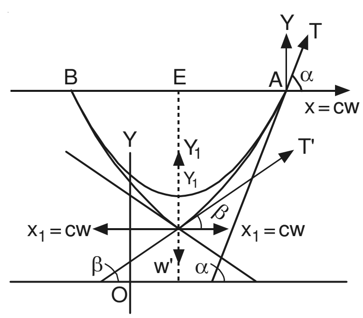
or $\tan\alpha = \sin h\,(x_1/l\cot\alpha)$ or $x_1 = l\cot\alpha\sin h^{-1}(\tan\alpha)$

$\therefore$ the distance between the links $= AB = 2.AD$

$= 2x_1 = 2l\cot\alpha\sin h^{-1}(\tan\alpha)$.

4. *A uniform string of weight W is suspended from two points at the same level and a weight W' is attached to its lowest point. If α and β are now the inclinations to the horizontal of the tangents at the lowest and the highest points, prove that*

$$\frac{\tan\alpha}{\tan\beta} = 1 + \frac{W}{W'}.$$



Sol. Before the attachment of W', the string is hanging in the shape of catenary AVB.

Let the weight be now suspended from the point V, which descends to V'.

Then AV' and BV' are the parts of two equal catenaries. Let the vertex of the catenary AV' be C and let the parameter of this catenary be c. Let T and T' be the tensions at A and V' respectively and let the resolved parts of be X, Y and X_1, Y_1 in the horizontal and vertical directions respectively.

Considering the equlibrium of the whole system, we have in the vertical direction,

$Y + Y$ = **total downward force** $= W + W'$ or $Y = \frac{1}{2}(W + W')$ ...(i)

Also the direction of T makes an angle α wich the horizontal, therefore

$$\tan\alpha = Y / X \qquad \text{...(ii)}$$

Considering the equilibrium of the point, we have in the vertical direction,

$$Y_1 + Y_1 = W' \text{ or } Y_1 = \frac{1}{2}W' \qquad \text{...(iii)}$$

Also the direction of T' makes an angle β with the horizontal,

$\therefore$ $$\tan\beta = Y_1 / X_1 \qquad \text{...(iv)}$$

Also the horizontal component of tension at each point is the same and equal to the tension at the lowest point.

$\therefore$ $$X_1 = X = wc,$$

where w is the weight per unit length of chain.

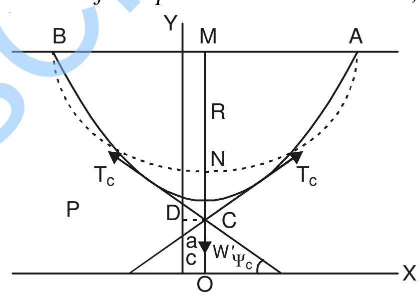
From (ii) and (iv) $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{(Y/X)}{(Y_1/X_1)} = \dfrac{Y}{Y_1}$, $\because$ from (v) $X = X_1$

or $$\frac{\tan\alpha}{\tan\beta} = \frac{\frac{1}{2}(W+W')}{\frac{1}{2}W'}, \text{ from (i), and (ii)}$$

or $$\frac{\tan\alpha}{\tan\beta} = 1 + \frac{W}{W'}.$$

5. *A uniform chain of lengt l and weight W, hangs between two fixed points at the same level, and weight W' is attached at the middle point. If k be the sag in the middle prove that pull on either support is*

$$\frac{k}{2l}W + \frac{l}{4k}W' + \frac{l}{8k}W.$$



Sol. Let a chain of lenght l and weight W be suspended from points A and B at the same level. The chain hangs freely in a catenary ANB. When a weight W' is attached at the middle point N of the chain then it will descend downwards to C, and two positions AC and BC of the chain each of length $\frac{1}{2}l$ will be the parts of two equal catenaries. Let D be the vertex and OX the directrix of the new catenary of which AC is an arc.

weight per unit length of the chain = $w = W / l$

Let T_C, T_C be the tensions at the point C in the chain AC and CB acting along the tangents at C, then resolving the forces at C vertically, we have

$$2T_C \sin\psi_C = W'$$

But from $T\sin\psi = wa$, we have

$$T_C \sin\psi_C = wa, \text{ where are } DC = a.$$

$\therefore$ From (i), we have

$$2wa = W' \text{ and } a = \frac{W'}{2w} = \frac{W'l}{2W} \qquad \text{...(ii)}$$

Let y_A and y_C be the ordinates of the points A and C respectively, then at A,

$$s = s_A = \text{arc } DC = \text{arc } DC + \text{arc } CA = a + \frac{1}{2}l$$

and $\quad y = y_A$

at C, $\quad s = s_C = \text{arc } DC = a \quad$ and $y = y_C$

Also, we have $\quad y_C + k = y_A$

or $\quad y_C = y_A - k$

Now, from $y^2 = c^2 + s^2$, we get

$$y_A{}^2 = c^2 + \left(a + \frac{1}{2}l\right)^2 \text{ and } y_C{}^2 = c^2 + a^2$$

Subtracting, we get

$$y_A^2 - y_C^2 = al + \frac{1}{4}l^2$$

or $$y_A^2 - (y_A - k)^2 = al + \frac{1}{4}l^2$$

or $$2ky_A - k^2 = al + \frac{1}{4}l^2$$

$$= \frac{W'l^2}{2W} + \frac{l^2}{4} \text{ [from (ii)]}$$

or $$y_A = \frac{k}{2} + \frac{l^2W'}{4kW} + \frac{l^2}{8k}$$

Hence the pull at either point of support A or B

$$= wy_A = \frac{W}{l}\left[\frac{k}{2} + \frac{l^2W'}{4kW} + \frac{l^2}{8k}\right]$$

$$= \frac{k}{2l}W + \frac{l}{4k}W' + \frac{l}{8k}W.$$

6. *A heavy string of uniform density and thickness is suspended from two given points in the same horizontal plane. A weight an nth that of the string is attached to its lowest point. Show that if θ and ϕ be the inclination to the vertical of the tangents at the highest and lowest points of the string then*

$$\tan\phi = (1+n)\tan\theta.$$

Sol. Refer the Figure of Example 4

Let W be the weight $2l$ the length of the string suspended from two points A and B in the same horizontal line. Then the weight attached at middle point C of the string is W/n. Also, weight per unit length w of the string is

$$w = \frac{W}{2l}$$

If ψ_C and ψ_A be the angles of inclination of the tangents at C and A to the horizontal respectively, then

$$\psi_C = \frac{1}{2}\pi - \phi, \quad \psi_A = \frac{1}{2}\pi - \theta$$

For the equilibrium of the point C, resolving the forces acting on it vertically, we have

$$2T_C \sin\psi_C = W/n \quad \text{...(i)}$$

where T_C is the tension at C in each of the string AC and BC.

Also, we have

$$T_C = wc\sec\psi_C$$

or $$T_C\cos\psi_C = wc = \frac{W}{2l}c \quad \text{...(ii)}$$

Dividing (i) by (ii), we have

$$2\tan\psi_C = \frac{2l}{nc}$$

$$\tan\left(\frac{\pi}{2} - \phi\right) = \frac{l}{nc} \quad \text{or} \quad \cot\phi = \frac{l}{nc}$$

$\therefore$ $$c = \left(\frac{l}{n}\right)\tan\phi$$

Now, from $s = c\tan\psi$, we get

$$\text{arc } DC = c\tan\psi_C \text{ and } \text{arc } DC = c\tan\psi_A$$

where D is the vertex of the new catenary whose an arc is AC.

Subtracting we get

$$\text{arc } DA - \text{arc } DC = c\ (\tan \psi_A - \tan \psi_C)$$

$\therefore$ $$\text{arc } CA = c\left[\tan\left(\frac{\pi}{2} - \theta\right) - \tan\left(\frac{\pi}{2} - \phi\right)\right]$$

or $$l = \left(\frac{l}{n}\right) \tan\phi\,[\cot\theta - \cot\phi]$$

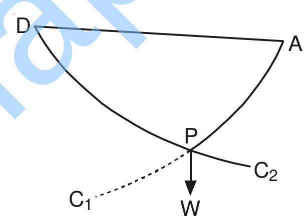
or $$n = \tan\phi \cot\theta - 1$$

or $$(1 + n) = \tan\phi \cot\theta$$

or $$\tan\phi = (1 + n)\tan\theta.$$

7. *A heavy uniform string is suspended from two points A and B in the same horizontal line, and to any point P of the string a heavy particle is attached. Prove that the two portions of the string are parts of equal catenaries. If θ and ϕ be the angles the tangent at P makes with the horizontal, α, β, those made by the tangents at A and B, show that $\dfrac{\tan\theta + \tan\phi}{\tan\alpha + \tan\beta}$ is constant for all positions of P.*

Sol. Let W be the weight attached at P and w be the weight per unit length of string. Let the whole length of the string be l and C_1, C_2 be the vertices of the catenaries AP and BP respectively.



The point P is in equilibrium under the action of the following forces :–

the weight W at P acting vertically downwards and the tensions T_1 and T_2 at P in the portions PA and PB making angles θ and ϕ with the horizon.

Resolving these forces horizontally and vertically, we get

$$T_1 \cos\theta = T_2 \cos\phi \quad \text{...(i)}$$

and $$T_1 \sin\theta + T_2 \sin\phi = W \quad \text{...(ii)}$$

Also if c_1 and c_2 be the parameters of the catenaries PA and PB, then

$$T_1 \cos\theta = wc_1 \text{ and } T_2 \cos\phi = wc_2 \quad \text{...(iii)}$$

$\therefore$ from (i) and (iii) we get $wc_1 - wc_2$

or $c_1 = c_2$ *i.e.* the portions PA and PB are parts of equal catenaries.

Also from (ii) and (iii) we get

$wc_1 \tan\theta + wc_2 \tan\phi = W$, putting the values of T_1 and T_2 in (ii)

or $$wc\,(\tan\theta + \tan\phi) = W, \quad \because c_1 = c_2 = c \text{ (say)}$$

or $$W = wc\,(\tan\theta + \tan\phi) \quad \text{...(iv)}$$

Also from $s = c\tan\psi$ we get

$$\text{arc } PA = \text{arc } CA = \text{arc } CP = c\tan\alpha - c\tan\theta$$

and $$\text{arc } PB = \text{arc } C_2B - \text{arc } C_2P = c\tan\beta - c\tan\phi$$

Adding arc PA + arc $PB = c\,[\tan\alpha - \tan\beta - \tan\theta + \tan\beta - \tan\phi]$

or $$l = c\,[\tan\alpha + \tan\beta - \tan\theta - \tan\phi], \quad \text{...(v)}$$

l being the length of the string.

Dividing (v) by (iv) we get

$$\frac{l}{W} = \frac{(\tan\alpha + \tan\beta) - (\tan\theta + \tan\phi)}{w\,(\tan\theta + \tan\phi)}$$

or $$\frac{lw}{W} = \left(\frac{\tan\alpha + \tan\beta}{\tan\theta + \tan\phi}\right) - 1$$

or $$\frac{\tan\alpha + \tan\beta}{\tan\theta + \tan\phi} = \frac{lw}{W} + 1 = \frac{lw + W}{W} = \text{constant.}$$

8. *If for the catenary $y = c \cosh(x/c)$, the normal at any point P meets the directrix at Q, show that $PQ = \rho = c\sec^2\psi$.*

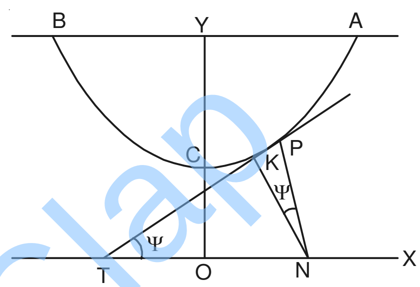
Sol. The intrinsic equation of the given catenary is

$$s = c\tan\psi.$$

$\therefore$ The tangent at any point $P(x, y)$ on it makes a angle ψ with x-axis or the directrix of the catenary and the normal makes and angle ψ with the ordinate *PN* at *P*. Draw the normal at *P* and let this normal meet x-axis at *Q*.

Then,

$$PQ = PN\sec\psi$$
$$= y\sec\psi$$
$$= (c\sec\psi).\sec\psi$$
$$= c\sec^2\psi$$



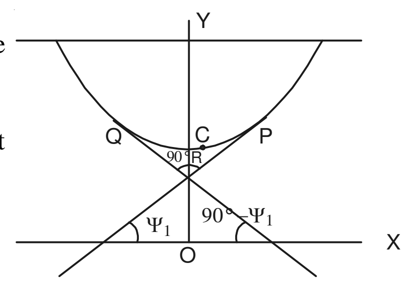
Also,

$$\rho = \frac{ds}{d\psi}$$
$$= \frac{d}{d\psi}(c\tan\psi) = c\sec^2\psi$$

$\therefore$ $$PQ = c\sec^2\psi = \rho.$$

9. *If the tangents at the points P and Q of a catenary are at right angles, prove that the tension at the middle points of the arc PQ is equal to the weight of a length of the string that equal half the arc PQ.*

Sol. Since the angle at P and *Q* is a right angle, therefore *P* and *Q* are on either side of *C*, the vertex of the catenary.



Let *R* be the middle point of arc *PQ* and y_1 be the ordinate of *R*. Then

Tension at $R = wy_1$ ...(i)

Let arc $CP = s_1$ and arc $CQ = s_2$. Also let the tangent at *P* be inclined at an angle ψ_1 to the horizon.

at *P* $s_1 = c\tan\psi_1$

and at *Q* $s_2 = c\tan(90 - \psi_1) = c\cot\psi_1$

Also, arc *CR* = arc *CP* – arc *RP* = $s_1 - \frac{1}{2}(s_1 + s_2)$

or $$\text{arc } CR = \frac{1}{2}(s_1 - s_2) = \frac{1}{2}(\tan\psi_1 - \cot\psi_1)$$

From $y^2 = c^2 + s^2$ at *P*, we have

$$y_1^2 = c^2 + (\text{arc } CR)^2 = c^2 + \frac{1}{4}c^2(\tan\psi_1 - \cot\psi_1)^2$$
$$= \left(\frac{1}{4}c^2\right)[4 + \tan^2\psi_1 + \cot^2\psi_1 - 2]$$
$$= \left(\frac{1}{4}c^2\right)(\tan\psi_1 + \cot\psi_1)^2 = \frac{1}{4}(c\tan\psi_1 + c\cot\psi_1)^2$$
$$= \frac{1}{4}(s_1 + s_2)^2$$

or $$y_1 = \frac{1}{2}(s_1 + s_2) = \frac{1}{2}\text{ arc } PQ$$

$\therefore$ From (i) tension at $R = w.\frac{1}{2}$ arc PQ

$= $ weight of a lenght of string that equals $\frac{1}{2}$ arc PQ.

EXERCISES

1. A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be x times the weight of the chain, show that its greatest possible distance from n is $(2l/\lambda)\log[\lambda + \sqrt{(1+\lambda^2)}]$, where $(1/\lambda) = \mu(2n+1)$ and is the coefficient of friction.

2. One end of a uniform chain ABC of length l is stretched to a fixed point A and a height h above a rough table. The portion BC is straight and rests on the table in a vertical plane through A. If the end C be free, prove that in the limiting equilibrium the length of the hanging portion is given by the equation

$$s^2 + 2\mu hs = h^2 + 2\mu hl$$

3. A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by $2c \sin h\,(a/2c) = \sqrt{(l^2 - b^2)}$. Prove also that, if the tensions at A and B are T_1 and T_2 respectively, then

$$T_1 - T_2 = W\sqrt{\left[1 + \left\{4c^2/(l^2 - b^2)\right\}\right]} \text{ and } T_1 - T_2 = bW/l,$$

where is the weight of the chain.

4. A and B are two points in the same horizontal line distance $2a$ apart. AO, OB are two equal heavy strings tied together at O and carrying a weight at O. If l is the length of each string, d the depth of O below AB, show that the parameter c of the catenary in which either string hangs is given by

$$l^2 - d^2 = 2c^2\,[\cos h\,(a/c) - 1]$$

5. A uniform chain of length $2l$ and weight is suspended from two points in the same horizontal line. A load is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h. Show that the terminal tension is $\frac{1}{2}w\left[(2l^2 + h^2)/lh\right]$.

6. An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs.

7. A uniform chain of length l rests in a straight line on a rough horizontal table. One end of the chain is raised to a height h above the table and the chain is on the point of motion. Show that the length of the straight part on the table is $l + \mu h - \{(\mu^2+1)\,h^2 + 2\mu lh\}$. where μ is the coefficient of friction.

8. A string of length l is attached to a fixed point A, and the other end B is pulled with a force wa incliived at an angle α to the horizon, w being the weight per unit length of the string. show that the vertical distance of A above B is $\sqrt{(l^2 + a^2 + 2la\sin\alpha)} - a$.

9. A uniform inextensible string of length l and weight wl, carries at one end B, a particle of weight W which is placed on a smooth plane inclined at 30° to the horizontal. The other end of the string is attached to a point A, situated at a height h above the horizontal through B

and in the vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the tangent at B to the catenary lying in the inclined plane if

$$\frac{W}{w}=\frac{(l-h)(l+h)}{\left(h-\frac{1}{2}l\right)}$$

10. A uniform heavy chain is fastened at its extremities to two rings of equal weight, which slide on smooth rods intersecting in a vertical plane, and inclined at the same angle α to the vertical. Find the condition that the tension at the lowest point may be equal to half the weight of the chain, and in that case, show that the vertical distance of the rings from the point of intersection of the rods is $l \cot \alpha \log (\sqrt{2}+1)$.

11. A uniform chain of length $2l$ and weight W is suspended from two points, A and B in the same horizontal line. A load P is now suspended from the middle point V of the chain and the depth of the point below AB is found to be h.

Show that each terminal tension $=\frac{1}{2}[P(l/h)+W\{(h^2+l^2)/2hl\}]$

12. One extremity of a uniform string is attached to a fixed point and the string rests partly on a smooth inclined plane. Prove that the directrix of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.

If α is the inclination of the plane, β the inclination of the tangent at the fixed extremity and l the whole length of the string, prove that the length of the portion in contact with the plane is

$$l \cos\beta/\cos\alpha \cos(\beta-\alpha).$$

13. Find the parametric equations of a common catenary.

[**Hint :** Show that $x=c\log(\sec\psi+\tan\psi)$, $y=c\sec\psi$]

3.3. APPROXIMATIONS OF THE COMMON CATENARY

(i) The equation of the catenary is

$$y=c\cosh(x/c)=\frac{1}{2}c\,(e^{x/c}+e^{-x/c})$$

$$=\frac{1}{2}c\left[\left\{1+(x/c)+\frac{1}{2!}(x^2/c^2)+\ldots\right\}+1-(x/c)+\frac{1}{2!}(x^2/c^2)-\ldots\right]$$

$$=c\left[1+\frac{1}{2!}(x/c)^2+\frac{1}{4!}(x/c)^4+\ldots\right]$$

Now, if x/c is small, then neglecting powers of x/c higher than second the above equation reduces to

$$y=c\,[1+(x^2/2c^2)]=c+\frac{x^2}{2c}.$$

This shows that as long as x is small compared to c, the curve coincides very nearly with a parabola of latus rectum $2c$ or $2T_0/w$.

This is so when a light cord is tightly stretched between two points, because in this case w is small while T_0 is large, and c being equal to T_0/w is also large. Examples of such a case are electric transmission wires and telegraph wires attached between poles.

(ii) When x is large, $e^{-x/c}$ becomes very small, hence

$$y = c \cosh (x/c) = \frac{1}{2} c\,(e^{x/c} + e^{-x/c})$$

behaves as $y = \frac{1}{2} c e^{x/c}$

which is an exponential curve.

Thus for very large values of x, *i.e.*, at points far removed from the lowest point, catenary behaves as an exponential curve.

3.3.1 Sag of Tightly Stretched Wires

Now we shall consider tightly stretched strings which appear nearly to be a straight line. For instance, telegraph wires are so tightly stretched that they appear to be horizontal.

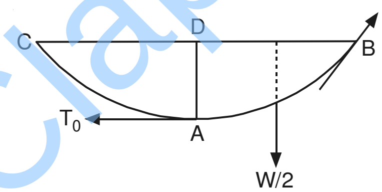
Let B, C be two points between which a wire is stretched in the shape of a catenary. Let A be the lowest point of this catenary. Let W be the weight, l the length of the wire; and span $BC = a$ and sag $AD = k$ of the catenary. Also let T_0 be the horizontal tension at A.

Now consider the equilibrium of the portion AB of the wire. The portion is in equilibrium under the action of the following forces—

(i) Tension T_0 acting horizontally at A.

(ii) Tension T at B, acting along the tangent at B.

(iii) Weight $\frac{1}{2} W$ of this portion acting at its centre of gravity which is approximately at a distance $\frac{1}{2} l$ from B.



Taking moments of these forces about B, we get

$$\frac{1}{2} W.\left(\frac{1}{4} l\right) = T_0 k$$

or
$$T_0 = lW/8k \qquad \text{...(i)}$$

Also the radius of curvature of the catenary $\rho = c \sec^2 \psi$ which shows that c will be large if ρ is large, *i.e.*, if the curve is flat near the vertex. Hence for telegraph wire x/c is very small.

Also we can calculate the extra length of the wire required on account of the sag by considering the relation

$$s = c \sinh (x/c) \qquad \text{...(ii)}$$

for a catenary,

or
$$s = c\left[(x/c) + \frac{1}{3!}(x/c)^3 + \ldots\right]$$

neglecting higher powers of x/c and retaining only two terms in the expression of (x/c). Thus

$$s = x + \frac{1}{6}\frac{x^3}{c^2}$$

or
$$s - x = \frac{1}{6}\frac{x^3}{c^2} = \frac{x^3 w^2}{6T_0} \qquad (\text{by } cw = T_0)$$

Now, putting $x = \frac{1}{2} a$, where the span $BC = a$, we have

$$s - \frac{1}{2} a = \frac{a^3 w^2}{48 T_0^2} \text{ or } 2s - a = \frac{a^3 w^3}{24 T_0^2}$$

Hence, the total increase in the length of the wire due to sag

$$= \text{arc } BAC - BK = 2s - a = \frac{a^3 w^2}{24 T_0^2}$$

EXAMPLES

Ex 1. *A telegraph wire stretched between two poles at distance a feet apart sags n feet in the middle, prove that the tension at the ends is approximately* $w[(a^2/8n)+(7n/6)]$.

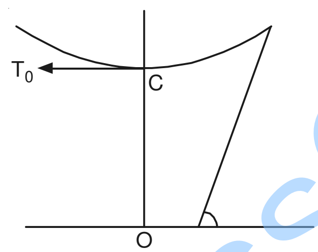
Sol. We know sag = $y - c$

$\therefore$ At A, $n = c\cosh(a/2c) - c$, $\because y = c\cosh(x/c)$ and $x = \frac{1}{2}a$ at A

$$= c[\cosh(a/2c) - 1] = c\left[1 + \frac{1}{2!}\left(\frac{a}{2c}\right)^2 + \frac{1}{4!}\left(\frac{a}{2c}\right)^4 + \dots - 1\right]$$

or $$n = c\left[\left(\frac{a^2}{8c^2}\right) + \left(\frac{a^4}{24 \times 16c^4}\right) + \dots\right]$$

or $$n = \frac{a^2}{8c} + \frac{a^4}{24 \times 16c^3} + \dots \qquad \text{...(i)}$$



Since it is a telegraph wire, therefore c is very large in this case and hence to a first approximation

$$n = a^2/8c \text{ or } c = a^2/8n. \qquad \text{...(ii)}$$

Substituting this value of c in (i) we get

$$n = \frac{a^2}{8c} + \frac{a^4}{24 \times 16(a^2/8n)^3} \text{ approx.}$$

or $$\frac{a^2}{8n} = n - \frac{(8n)^3}{24 \times 16a^2} = n - \frac{4n^3}{3a^2} = n\left(1 - \frac{4n^2}{3a^2}\right)$$

or $$c = \frac{a^2}{8n}\left(1 - \frac{4n^2}{3a^2}\right)^{-1} = \frac{a^2}{8n}\left[1 + \frac{4n^2}{3a^2}\right], \text{ approx.}$$

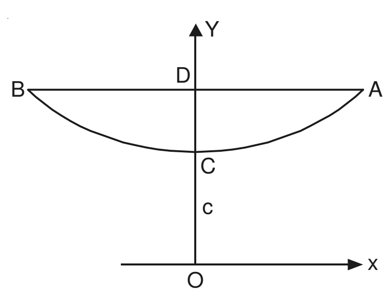
or $$c = \frac{a^2}{8n} + \frac{n}{6} \qquad \text{...(iii)}$$

$\therefore$ Tension at where $A = wy_1$, where y_1 is the ordinate of A

$$= w(n + c), \because y_1 = DO = DC + CO = n + c$$

$$= w\left[n + \frac{a^2}{8n} + \frac{n}{6}\right] = w\left[\frac{a^2}{8n} + \frac{7n}{6}\right].$$

2. *A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of the same height, so will produce the least possible tension at the posts. Show that* $l = (d/\lambda)\sinh\lambda$, *where* λ *is given by the equation* $\lambda\tanh\lambda = l$.



Sol. Here spen $AB = d$ so that $DA = \frac{1}{2}d = DB$, where D is midpoint of AB.

Also arc $CA = \frac{1}{2}l$ = arc CB, where C is the lowest point of the catenary.
Let T be the tension at the points *i.e.* at A or B

Then $T = wy = wc \cos h\,(d/2c)$,

$\because \quad y = c \cos h\,(x/c)$ and $x = \frac{1}{2}d$ at A

or $\quad T = wc \cos h\,(d/2c)$ ...(i)

If T is minimum, then

$$\frac{dT}{dc} = 0 \text{ and } \frac{d^2T}{dc^2} = \text{positive}.$$

From (i), $\frac{dT}{dc} = 0$ gives $w[\cos h\,(d/2c) - c \sin h\,(d/2c)\,.(d/2c^2)] = 0$

or $\cos h\left(\frac{d}{2c}\right) - \left(\frac{d}{2c}\right)\sin h\left(\frac{d}{2c}\right) = 0$ or $\left(\frac{d}{2c}\right)\tan h\left(\frac{d}{2c}\right) = 1$

or $\quad \lambda \tan h\,\lambda = 1$, where $\lambda = (d/2c)$ ...(ii)

Also $\frac{d^2T}{dc^2} = w\left[\left(-\frac{d}{2c^2}\right)\sin h\left(\frac{d}{2c}\right) - \left(\frac{d}{2c}\right)\left(-\frac{d}{2c^2}\right)\cos h\left(\frac{d}{2c}\right) + \left(\frac{d}{2c^2}\right)\sin h\left(\frac{d}{2c}\right)\right]$

$$= \frac{wd^2}{4c^3}\cos h\,(d/2c) = \text{positive}.$$

Hence T is minimum when λ tan h $= \lambda = 1$, where $\lambda = (d/2c)$

Also from $s = c \sin h\,(x/c)$ at A, we have

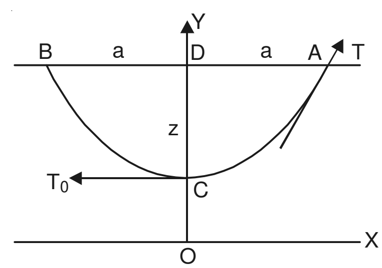
$$\left(\frac{1}{2}l\right) = c \sin h\,(d/2c) \text{ at } A, \because \text{ at } A,\ s = \frac{1}{2}l \text{ and } x = \frac{1}{2}d$$

or $\quad l = 2c \sin h\,(d/2c) = d(2c/d)\sin h\,(d/2c)$

$= d(1/\lambda)\sin h\,\lambda, \quad \because$ from (ii) $\lambda = d/2c$

Hence $l = (d/\lambda)\sin h\,\lambda$ where $\lambda \tan h\,\lambda = 1$.

3. *A uniform chain is hung up from two points at the same level and distant 2a apart. If z is the sag at the middle, show that* $z = c\,[\cos h\,(a/c) - 1]$. *If z is small compared to a, show that* $2cz = a^2$ *nearly.*



Sol. Let ABC be the uniform chain, C the lowest piont, *i.e.,* the vertex of the catenary formed by the chain and OX its directrix.

We have, $\quad y = c \cos h\left(\frac{x}{c}\right)$

At A, $\quad x = \frac{1}{2}AB = a$.

Therefore if $y = y_A$ for the point A, then

$$y_A = c \cos h\left(\frac{a}{c}\right)$$

$\therefore$ Sag at the middle,

$$z = \text{OD} - \text{OC} = y_A - c = c \cos h\left(\frac{a}{c}\right) - c$$

$$= c\left[\cos h\left(\frac{a}{c}\right) - 1\right]$$

$$= c\left[1 + \frac{1}{2!}\left(\frac{a}{c}\right)^2 + \frac{1}{4!}\left(\frac{a}{c}\right)^4 + ... - 1\right]$$

$$= c\left[\frac{1}{2!}\frac{a^2}{c^2} + \frac{1}{4!}\frac{a^4}{c^4} + ...\right]$$

$$= \frac{1}{2!}\frac{a^2}{c} + \frac{1}{4!}\frac{a^4}{c^3} + ...$$

If z is small compared to a, then c, must be large. Therefore neglecting the higher powers of $1/c$ in the above expansion, we get

$$z = \frac{1}{2!}\frac{a^2}{c}$$

or $$2cz = a^2.$$

EXERCISES

1. A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation k above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $8k^2/3l$.

2. If the length of a uniform chain suspended between points at the same level is adjusted so that the tension at the points of support is a minimum for that particular span $2d$, show that the equation to determine c is $\cot h\,(d/c) = d/c$.

3. A uniform chain, of length $2l$, has its ends attached to two points in the same horizontal line at a distance $2a$ apart. If l is only a little greater than a, show that the tension of the chain is approximately equal to the weight of a length $\sqrt{\{a^3/6(l-a)\}}$ of the chain and that the sag or depression of the lowest point of the chain below its ends is nearly.

$$\frac{1}{2}\sqrt{\{6a(l-a)\}}$$

❒❒❒

Virtual Work

2.1 INTRODUCTION

Statics is that branch of mechanics which deals with the conditions under which bodies remain at rest relative to surrounding objects. In such circumstance bodies are said to be in a **state of equilibrium**, *i.e.*, they are acted upon by forces which balance one another. If the forces do not balance, the bodies will not remain at rest but will change their positions relative to surrounding objects, *i.e.*, they will be in motion. That branch of mechanics which deals with the conditions under which bodies are in motion is called **Dynamics.**

To simplify the treatment of the subject we shall assume that the bodies are **rigid**, *i.e.*, the distance between each pair of particles composing the body remains invariable. Actually bodies do not possess this property since they become slightly strained under the action of forces and are capable of compression, extension or distortion, but changes in their dimensions are for many purposes negligible.

Bodies may be regarded as a conglomeration of particles held together by forces of cohesion. There is no limit to the number of particles forming a body. A **particle** may be considered as a portion of matter indefinitely small in size so that the distances between its different parts are negligible. Thus it can be regarded as a mathematical point with a definite mass.

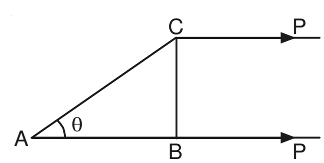
We sometimes speak of a body as a 'particle'. When we do so, we do not take into consideration its actual dimensions, we simply represent its position by a mathematical point. In problems of Astronomy, even the Earth and the Sun are sometimes treated as particles, their dimensions are neglected when compared with their mutual distance.

A body does not change its position, *i.e.*, its state of rest or of motion, of its own accord. If it changes its position, there must be some external effort or action on the body which compels it to change its position. That effort or action is known as force. Newton has defined force **as an action exerted on a body in order to change its state of rest or uniform motion in a straight line.**

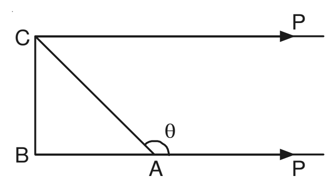
If two or more forces act upon a body and are so arranged that the body remains at rest, the forces are said to be in **equilibrium.**

2.2. WORK

A force is said to do work when its point of application moves in the direction of the force.



If a particle A is acted upon by a constant force P acting parallel to AB and the particle is moved from A to C, and if CB drawn perpendicular to AB, then the product $P \times$ (distance AB) is called the **work done** by the force on the particle in changing its position.



The distance AC is called displacement of the particle and AB is the projection of this **displacement** on the direction of the force.

If the direction of the force AP makes with the direction of the displacement AC an obtuse angle, the projection AB is in opposite direction to the force P, the work done by P is negative.

Hence the **work done by a force when its point of application undergoes a displacement is the product of the force and the projection of the displacement on the line of action of the force.**

In the figure $\angle CAB = \theta,$ then the work done by P

$$= P \times AB = P \times AC \,.\, \frac{AB}{AC} = P \times AC \cos\theta$$
$$= AC \times P\cos\theta = AC \times (\text{resolved part of } P \text{ along } AC)$$

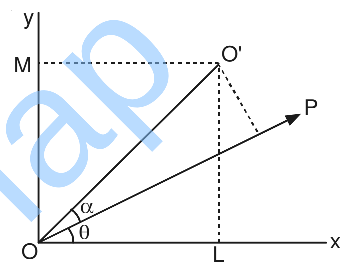
Hence the **work done by a force is the product of the displacement of its point of application and the resolved part of the force in the direction of the displacement.**

Work done by a couple : The work done by a couple, when it is rotated about an axis perpendicular to its own plane, is equal to the moment of couple multiplied by the angle of rotation.

If G be the moment of the couple and α be the angle of rotation, the work done by the couple $= G.\alpha$.

2.2.1 Theorem

Theorem 1. The work done by a force in moving a particle from one position to another is equal to the sum of the works done by its components.



Let the components of the force P acting at the particle O in two directions Ox and Oy at right angles to one another be X and Y respectively, and let P makes an angle θ with Ox, then

$$X = P\cos\theta \text{ and } Y = P\sin\theta$$

Let the point of application of P be moved from O to O'. Draw $O'N$ perpendicular to the line of action of P and let $\angle NOO' = \alpha$.

The sum of the works done by X and

$$Y = X.OL + Y.OM$$
$$= P\cos\theta . OO' \cos(\theta+\alpha) + P\sin\theta \,.\, OO' \sin(\theta+\alpha)$$
$$= P\,OO' [\cos\theta\cos(\theta+\alpha) + \sin\theta\sin(\theta+\alpha)]$$
$$= P.OO' \cos(\theta+\alpha-\theta) = P.OO'\cos\alpha = P.ON$$
$$= \text{work done by } P.$$

Theorem 2. When any number of forces act upon a particle and the particle undergoes a small displacement which is consistent with geometrical conditions under which the system exists, the algebraic sum of the works done by the several forces is equal to the work done by the resultant.

If the particle is moved from O to O', the algebraic sum of the works done by the several forces

$= OO' \times$ algebraic sum of the resolved parts of the several forces along OO'

$= OO' \times$ resolved part of the resultant along OO'

$=$ work done by the resultant.

2.3. VIRTUAL WORK AND PRINCIPLE OF VIRTUAL WORK

When a particle is in equilibrium under the action of a number of coplanar forces, we have no motion of the particle under these forces and consequently there is no actual displacement. But we often find it convenient for the sake of argument to suppose a particle displaced from one position O to another position O', although the particle may have no tendency to move in this direction, or, perhaps, in any direction. In this case OO' is called the **Virtual displacement** of the particle and the work which would have been done by a force acting on the particle during the displacement is called **Virtual work.**

The work 'Virtual' implies that the displacement is not actual but only hypothetical or virtual, so that the work would be done if the displacement were actually made. Hence the result of the previous article may be stated thus :

If a system of forces acting on a particle be in equilibrium and the particle undergoes a virtual displacement consistent with the geometrical conditions of the system, the algebraic sum of the virtual works is zero.

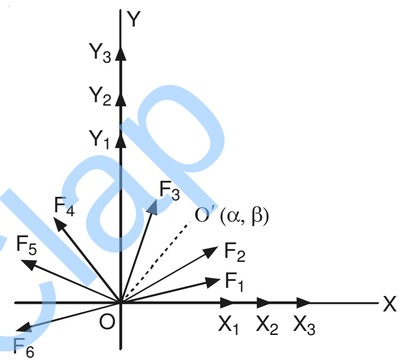
This property is called the **Principle of Virtual Work.**

It must be remembered that the virtual displacements are to be regarded as **small quantities of the first order of smallness**, and if the algebraic sum of the works done in terms of the displacement is of second or higher order of smallness, the forces will be considered as doing no work.

2.3.1. Principle of Virtual Work for a System of Coplanar Forces

Acting on a Particle

Statement : *The necessary and sufficient condition that a particle acted upon by a number of coplanar forces be in equilibrium is that sum of the virtual works done by the forces in any small virtual displacement consistent with the geometrical conditions of the system is zero.*



Proof : Let any number of forces $F_1, F_2, F_3, ...$ act on a particle at O. Let the resolved parts of these forces be $X_1, X_2, X_3, ...$ and $Y_1, Y_2, Y_3, ...$ in two mutually perpendicular directions Ox and Oy respectively. Then the sum of these resolved parts in the direction Ox and Oy are ΣX_1, *i.e.,* $X_1 + X_2 + X_3 + ...$ and ΣY_1, *i.e.,* $Y_1 + Y_2 + Y_3 + ...$

Necessary condition. Let the particle at O be given a virtual displacement OO' and let the coordinates of O' be (α, β) referred to Ox and Oy. Then the projections of OO' along Ox and Oy are α and β respectively.

Now the virtual work done by the force

$F_1 =$ the algebraic sum of virtual works done by its resolved parts X_1 and Y_1

$= X_1 \alpha + Y_1 \beta.$

Similarly the virtual works done by the forces $F_2, F_3, ...$ are

$(X_2 . \alpha + Y_2 . \beta)$; $(X_3 . \alpha + Y_3 . \beta), ...$

$\therefore$ The algebraic sum of the virtual works done by all the forces

$$= (X_1\alpha + Y_1\beta) + (X_2\alpha + Y_2\beta) + (X_3\alpha + Y_3\beta) + ...$$
$$= \alpha\,(X_1 + X_2 + X_3 + ...) + \beta\,(Y_1 + Y_2 + Y_3 + ...)$$
$$= \alpha\,(\Sigma X_1) + \beta\,(\Sigma Y_1) \quad ...(i)$$

But as the system is in equilibrium so $\Sigma X_1 = 0$ and $\Sigma Y_1 = 0$

$\therefore$ From (i) the algebraic sum of the virtual works done by all the forces = 0.

Hence the necessary condition for equilibrium is that the algebraic sum of the virtual works done by the different forces, when the particle is given a small virtual displacement consistent with the geometrical conditions of the system is zero.

Sufficient condition (converse of principle of virtual work).

If the algebraic sum of the virtual works done by the system of forces acting on a particle be zero for any arbitrary small virtual displacement, the particle is in equilibrium.

When OO' is the small displacement, where O' is (α, β), from (i) above we have

$$\alpha\,(\Sigma Y_1) + \beta\,(\Sigma Y_1) = 0. \quad ...(ii)$$

Let a new small displacement OP be given where P is (α', β), then

$$\alpha'\,(\Sigma X_1) + \beta\,(\Sigma X_1) = 0. \quad ...(iii)$$

Subtracting (iii) from (ii) we get $(\alpha - \alpha')\,(\Sigma X_1) = 0$

But $\alpha - \alpha' \neq 0$, so $\Sigma X_1 = 0$

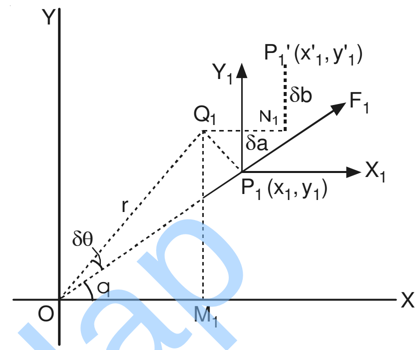
Similarly giving another small displacement OQ where Q is (α, β') we can prove that $\Sigma Y_1 = 0$.

Hence $\Sigma X_1 = 0$ and $\Sigma Y_1 = 0$, *i.e.,* the particle is in equilibrium.

2.3.2 Principle of Virtual Work for a System of Coplanar Forces Acting at Different Points of a Rigid Body

Statement. *If a system of coplanar forces acting upon a rigid body, keep the body in equilibrium, and if the body be given a slight virtual displacement, consistent with the geometrical conditions of the system, then the algebraic sum of the virtual works done by the forces is zero. Conversely if this algebraic sum be zero, then the forces are in equilibrium.*

Proof. Take an arbitrary point O in the plane in which the forces are acting. Through O draw straight lines Ox and Oy mutually at right angles.



Let P_1 be the point of application of a force F_1 acting upon the body. Let the coordinates of P_1 referred to the axes Ox and Oy be (x_1, y_1) and its polar coordinates referred to the pole O be (r, θ).

Then $\qquad x_1 = r\cos\theta,$

$\qquad y_1 = r\sin\theta, \qquad$ ...(i)

where $OP = r$ and $\angle POX = \theta$.

Now give the body a slight displacement, such that P_1 takes the position P_1'. It is done by rotating the body through a small angle $\delta\theta$ and then moving it parallel to the axes through small distances δa and δb.

Let (x_1', y_1') be the coordinates of P_1', then

$$x_1' = OM_1 + Q_1N_1 = r\cos(\theta + \delta\theta) + \delta a$$

$$= r\,L\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta) + \delta a$$

$$= r\cos\theta - \delta\theta . r\sin\theta + \delta a, \qquad \because \cos\delta\theta = 1 \text{ and } \sin\delta\theta = \delta\theta$$

or $\quad x_1' = x_1 - \delta\theta .\, y_1 + \delta a, \qquad$ from (i) $\qquad$ ...(ii)

and $\quad y_1' = P_1'N_1 + Q_1M_1 = \delta b + r\sin(\theta + \delta\theta) = \delta b + r(\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta)$

$$= \delta b + r\sin\theta + \delta\theta .\, r\cos\theta, \qquad \because \cos\delta\theta = 1,\ \sin\delta\theta = \delta\theta$$

or $\quad y_1' = \delta b + y_1 + x_1 .\, \delta\theta, \qquad$ from (i) $\qquad$ ...(iii)

$\therefore$ Displacement of P_1 along the x-axis

$$= (x_1' - x_1) = \delta a - y_1\,\delta\theta, \qquad \text{from (ii)} \qquad ...(iv)$$

And displacement of P_1 along the y-axis

$$= (y_1' - y_1) = \delta b + x_1 .\delta\theta, \qquad \text{from (iii)} \qquad ...(v)$$

Let X_1 and Y_1 be the resolved parts of the forces F_1 in the directions of Ox and Oy respectively.

The virtual work done by the force F_1

= sum of the virtual works done by the resolved parts X_1 and Y_1

= X_1 . displacement of P_1 along $Ox + Y_1$. displacement of P_1 along Oy

$= X_1(\delta a - y_1\delta\theta) + Y_1(\delta b + x_1\delta\theta)$, from (iv) and (v)

$= X_1 .\, \delta a + Y_1 .\, \delta b + (x_1Y_1 - y_1X_1)\,\delta\theta.$

Similarly we can find the virtual works done by other forces $F_2, F_3, \ldots$ acting upon the body (for all these $\delta a, \delta b,$ and $\delta\theta$ will remain the same) as

$$X_2\,\delta a + Y_2 .\, \delta b + (x_2Y_2 - y_2X_2)\,\delta\theta;$$

$$X_3\,\delta a + Y_3 .\, \delta b + (x_3Y_3 - y_3X_3)\,\delta\theta; \ldots$$

$\therefore$ The algebraic sum of the virtual works done by the system of forces acting upon the rigid body

$$= [X_1\delta a + Y_1\delta b + (x_1Y_1 - y_1X_1)\,\delta\theta] + [X_2\delta a + Y_2\delta b + (x_2Y_2 - y_2X_2)\,\delta\theta] + ...$$
$$= (X_1 + X_2 + X_3 + ...)\,\delta a + (Y_1 + Y_2 + Y_3 + ...)\,\delta b$$
$$+ [(x_1Y_1 - y_1X_1) + (x_2Y_2 - y_2X_2) + ...]\,\delta\theta$$
$$= (\Sigma X_1)\,\delta a + (\Sigma Y_1)\,\delta b + [\Sigma(x_1Y_1 - y_1X_1)]\,\delta\theta \qquad ...(A)$$

Now if the system of forces acting upon the rigid body be in equilibrium then we know that

$$\Sigma X_1 = 0,\ \ \Sigma Y_1 = 0\ \text{ and }\ \Sigma\,(x_1Y_1 - y_1X_1) = 0$$

$\therefore$ If the body be in equilibrium under the action of a system of forces, then if a slight virtual displacement be given consistent with the geometrical condition, then the algebraic sum of the virtual works done by the forces is zero, from (A) with the help of above relations. (This is also known as the necessary condition).

Converse of the Principle of Virtual Work (sufficient condition).

Statement : *If there be a system of forces acting on different points of a rigid body which is given a small arbitrary virtual displacement consistent with the geometrical conditions and the algebraic sum of the virtual works done by the forces is zero, then the forces are in equilibrium.*

Since the sum of the virtual works done by the forces is zero, therefore from (A) above we get

$$\delta a.\Sigma X_1 + \delta b.\Sigma Y_1 + \delta\theta\,\Sigma\,(x_1Y_1 - y_1X_1) = 0 \qquad ...(B)$$

(**Note :** This equation is known as the equation of virtual work).

Now let us choose a displacement in which the body moves only a small distance δa parallel to Ox. Then putting $\delta b = 0$ and $\delta\theta = 0$ in (B) we get

$$\delta a.\Sigma X_1 = 0\ \text{ or }\ \Sigma X_1 = 0 \qquad ...(vi)$$

Similarly let us choose a displacement in which the body moves only a small distance δb parallel to Oy. Then putting $\delta a = 0$ and $\delta\theta = 0$ in (B) we get

$$\delta b.\Sigma Y_1 = 0\ \text{ or }\ \Sigma Y_1 = 0 \qquad ...(vii)$$

Finally let us choose a displacement in which the body only rotates about O through a small angle $\delta\theta$, then putting $\delta a = 0$ and $\delta b = 0$ in (B) we get

$$\delta\theta.\Sigma\,(x_1Y_1 - y_1X_1) = 0\ \text{ or }\ \Sigma(x_1Y_1 - y_1X_1) = 0\,. \qquad ...(viii)$$

Hence the general conditions of equilibrium for a system of coplanar forces is fully satisfied, *i.e.*, $\Sigma X_1 = 0; \Sigma Y_1 = 0$ and $\Sigma\,(x_1Y_1 - y_1X_1) = 0$.

Hence the forces acting upon the rigid body are in equilibrium.

2.3.3. Forces which can be omitted in forming the equation of virtual work

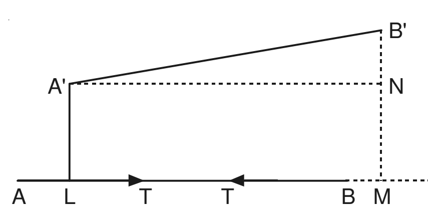
The advantage of the principle of virtual work is mainly in the fact that certain forces can be avoided in writing the equations of virtual work.

The following are some of the important forces which do not appear in the equations of virtual work.

(a) The tension of an inextensible string.

Let T be the tension in the string AB. Let this string AB be displaced to the position $A'B'$, such that $A'B'$ is inclined at an angle θ to AB, where θ is a small angle and

$$A'B' = AB.$$



From A' and B' draw $A'L$ and $B'M$ perpendicular to AB, produced if necessary. From A' draw $A'N$ perpendicular to $B'M$.

Now tension T in the string AB is equivalent to a force T at A in the direction AB and a force T at B in the direction BA.

$\therefore$ The virtual work done by the tension when string is displaced from the position AB to $A'B'$.

= sum of the virtual works done by T at A and T at B

$$= T.AL - T.BM = T\,(AB - LB) - T\,(LM - LB)$$
$$= T\,(AB - LM) = T\,(AB - A'N)$$

$$= T\,(AB - A'B' \cos\theta) = T.AB\,(1 - \cos\theta), \quad \text{as } A'B' = AB$$

$$= T.AB\left[1 - \left(1 - \frac{\theta^2}{2\,!} + \ldots\ldots\right)\right]$$

$$= 0, \qquad \because \theta \text{ being small its higher powers can be neglected.}$$

(b) The thrust in a light rod.

This is similar to that of tension in an inextensible string, the only difference being that here we have T at A in the direction BA and T at B in the direction AB, *i.e.*, the directions of T (as shown by arrow marks in fig. as case (a) above) are reversed.

Hence the virtual work done by the thrust in the light rod when it is displaced from the position AB to $A'B'$

$$= T.BM - T.AL = 0, \text{ , as in case (a) above.}$$

(c) Reaction *R* of any smooth surface with which the body is in contact.

Let A be the point of contact of the body with the smooth surface then the reaction R acts at A normal to the surface. Now if the point A moves to a neighbouring point A', then AA' is perpendicular to the reaction R and therefore the virtual work done by R is zero.

(d) The mutual pressure between two bodies in contact.

Since the action and reaction are equal and opposite, therefore the virtual work done by one is balanced by the other.

(e) The reaction at a fixed point or a fixed axis about which the body rotates.

In this case the point of application does not undergo any displacement (as it is fixed) hence no virtual work is done.

(f) Reaction at a point of contact with fixed surface on which the body rolls without sliding.

In this case the point of contact of the body with the surface is momentarily at rest and so it does not undergo any displacement, hence the work done by the normal reaction or the force of friction at the point of contact is zero.

2.4. TO FIND THE TENSION OF A STRING OR THRUST IN A ROD

In many statical problems we are required to find tension of a string or thrust in a rod whereas in case (a) and (b) of last article we have shown that these forces do no work if the system be given a small vitrual displacement consistent with the geometrical conditions of the problem. In order to get over this difficulty, we shall proceed in the following way.

Let T be the tension in the string AB. The condition of equilibrium of surrounding bodies is not altered if we replace the string AB by a force T acting at A along AB and a force T acting at B along BA. Let $AB = l$ and let the string AB be displaced to the position $A'B'$, where $A'B' = l + \delta l$.

Then the **work done by the tension**

$$T = T\,[AB - A'B' \cos\theta] = T\,[AB - A'B'], \;\; \theta \text{ being small}$$

$$= T\,[l - (l + \delta l)] = -T.\delta l,$$

where δl is the increase in the length of the string.

Similarly in the case of the rod AB, the **work done by the thrust T = T** δl, since the directions of T would be reversed in this case.

2.4.1 Method of Procedure for Solving Problems

(a) Draw the figure and mark the forces as given.

(b) Replace the string by two forces T and T acting inwards and if AB be of length l, then the virtual work done by tension T of the string $= -T.\delta l$.

(c) Replace the rod by two forces T and T acting outwards and if AB be of length l, then the virtual work done by thrust T of the rod $= T.\delta l$.

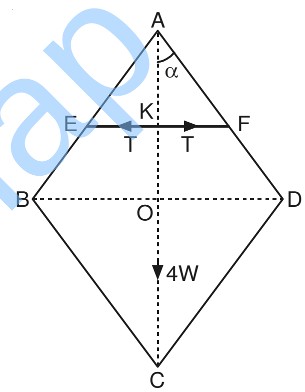
(d) Measure the distances of the points of application of various forces from a fixed point or a fixed line. If z be the distance of point of application of a force F from a fixed point, then after small virtual displacement this distance will be increased by an amount δz and thus the work done by the force F is $F\,\delta z$. *The sign of this work done is positive if z has been measured in the direction of F and is negative if measured in a direction opposite to F.*

PROBLEMS INVOLVING SIMPLE FRAMEWORKS

EXAMPLES

1. *Four equal heavy uniform rods are freely jointed so as to form a rhombus which is freely suspended by one angular point, and the middle points of the two upper rods are connected by a light rod so that the rhombus cannot collapse. Prove that the tension of this light rod is 4W, where W is the weight of each rod and is the angle of the rhombus at the point of suspension.*

Sol. The total weight $4W$ of the four rods acts at O, the point of intersection of the diagonals of the rhombus. Let T be the thrust in the rod EF, joining the mid-points of AB and AD. Remove the rod EF and retain its effect, *i.e.*, replace the rod EF by a force T at E in the direction FE and a force T at F in the direction EF.



Here the point of suspension A is a fixed point. Let the distances be measured from the fixed point A. Let each rod be of length $2a$.

Then $AO = 2a\cos\alpha$ and

$EF = 2\,EK = 2a\sin\alpha = l$ (say)

Give the system a small virtual displacement such that α becomes $\alpha + \delta\alpha$ and l becomes $l + \delta l$, then the equation of virtual work is

$$4W\delta\,(AO) + T\delta l = 0$$

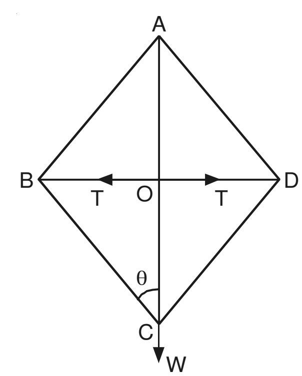
or $\quad 4W\delta(2a\cos\alpha) + T\delta\,(2a\sin\alpha) = 0, \quad \because\ l = 2a\sin\alpha$

or $\quad -8Wa\sin\alpha.\delta\alpha + T\,2a\cos\alpha\,\delta\alpha = 0$

or $\quad T = 4W\tan\alpha, \qquad (\because\ \delta\alpha \neq 0).$

2. *Five weightless rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. If a weight W be attached to C and the system be suspended from A, show that there is a thrust in BD equal to* $W\sqrt{3}$.

Sol. Since the system is suspended from A and a weight W is attached at C, therefore AC will be vertical. Let T be the thrust in the rod BD. Replace the rod BD by a force T and B in the direction DB and a force T at D in the direction BD.



Let the distances be measured from the fixed point A. Let each rod be of length $2a$ and inclined to the vertical at an angle θ and

$$AC = 2AO = 2\,(2a\cos\theta) = 4a\cos\theta$$

and $\quad BD = 2BO = 2\,(2a\sin\theta) = 4a\sin\theta = l$ (say)

Give the system a small virtual displacement such that θ becomes $\theta + \delta\theta$ and l becomes $l + \delta l$. The equation of virtual work will be

$$W\,\delta(AC) + T\,\delta l = 0$$

or $\quad W\,\delta(4a\cos\theta) + T\,\delta\,(4a\sin\theta) = 0$

or $\quad -4aW\sin\theta\,\delta\theta + 4aT\cos\theta\,\delta\theta = 0$

or $\quad T = W\tan\theta \qquad (\because\ \delta\theta \neq 0) \qquad$ ...(i)

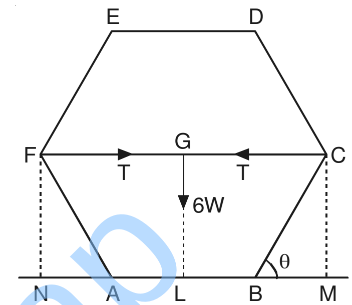
Now, in the position of equilibrium, ABD is an equilateral triangle

$\because\ AB = BD = AD$ is given.

Therefore, $\angle BAD = 2\theta = 60°$ or $\theta = 30°$

$\therefore$ $T = W \tan 30° = W/\sqrt{3}.$

3. *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is* $W/\sqrt{3}$.



Sol. Let T be the tension in the string FC. Replace the string FC by a force T at F in the direction FC and a force T in the direction CF. The total weight $6W$ of the rods acts at G, the centre of the hexagon. Let each side of the hexagon be $2a$ and each slant side be inclined to the horizon at an angle θ.

Then $GL = CM = a \sin \theta$

and $FC = NM = AB + 2BM$

or $FC = 2a + 2(2a \cos \theta) = 2a + 4a \cos \theta = l$ (say)

Let the distances be measured in the upward direction from the rod AB which is fixed. Give the system a small virtual displacement such that θ becomes $\theta + \delta\theta$ and l becomes $l + \delta l$, then the equation of virtual work is

$$-6W\delta(GL) - \delta l = 0$$

or $6W\delta(2a \sin \theta) + T\delta(2a + 4a \cos \theta) = 0,$

Substituting the values of GL and l

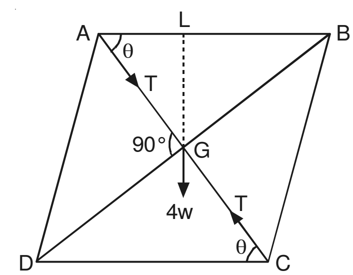
or $12aW \cos\theta - 4aT \sin \theta = 0,$ $\because \delta\theta \neq 0$

or $T = 3W \cot \theta$ ....(i)

In the position of equilibrium $\theta = 60°$, therefore from (i) we have

$$T = 3W \cot 60° = 3W(1/\sqrt{3}) = W\sqrt{3}.$$

4. *A string of length a forms the shorter diagonal of a rhombus of four uniform rods, each of length b and weight w which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension in the string is* $2w(2b^2 - a^2)/[b\sqrt{(4b^2 - a^2)}]$.



Sol. The total weight $4w$ of the rods acts at G, the centre of gravity of the rhombus. Let T be the tension in the string AC, such that $AC = a$. Replace the string AC by a force T at A and a force T at C in the directions as marked in the figure. Let $\angle ACD = \theta$, then $\angle BAC = \theta$. As AB is fixed, distances will be measured from AB in the downward direction.

Now $AG = AB \cos \theta = b \cos \theta$

$\therefore$ $AC = 2.AG = 2b \cos \theta$ ...(i)

Also $GL = AG.\sin \theta = b \cos \theta \sin \theta = \frac{1}{2} b \sin 2\theta$ ...(ii)

Give the system a slight virtual displacement such that θ changes into $\theta + \delta\theta$, then the equation of virtual work is

$$4w\delta(GL) - T\delta(AC) = 0 \quad \text{or} \quad 4w\delta(\tfrac{1}{2} b \sin 2\theta) - T\delta(2b \cos \theta) = 0$$

Substituting the values of GL and AC from (i) and (ii).

or $4w.b \cos 2\theta + T.\delta\, 2b \sin \theta = 0,$ $\because \delta\theta \neq 0$

or $T = -2w \cos 2\theta / \sin \theta = -\dfrac{2w(2\cos^2\theta - 1)}{\sqrt{(1-\cos^2\theta)}}$ ...(iii)

In the position of equilibrium $AC = a$ given

$\therefore$ from (i) $2b \cos \theta = a$ or $\cos \theta = \dfrac{a}{2b}$

$\therefore$ from (iii) $T = -2w[2(a/2b)^2 - 1]\Big/\sqrt{\{1-(a/2b)^2\}}$

$$= -2w(a^2 - 2b^2)\Big/b\sqrt{(4b^2 - a^2)} = 2w(2b^2 - a^2)/b\sqrt{(4b^2 - a^2)}.$$

5. *Four light rods are jointed together to form a quadrilateral OABC. The lengths are such that OA = OC = a and AB = BC = b. The framework hangs in a vertical plane with OA and OC resting in contact with two smooth pegs distant l apart and in the same horizontal level. A weight W hangs at B. If θ, ϕ are the inclinations of OA and AB respectively to the horizontal, prove that these values are given by the equations*

$$a\cos\theta = b\cos\phi \text{ and } \frac{1}{2}l\sec^2\theta\sin\phi = a\sin(\theta+\phi)$$

Sol. P and Q are the two smooth pegs.

$$\therefore \quad PQ = l \text{ (given)}.$$

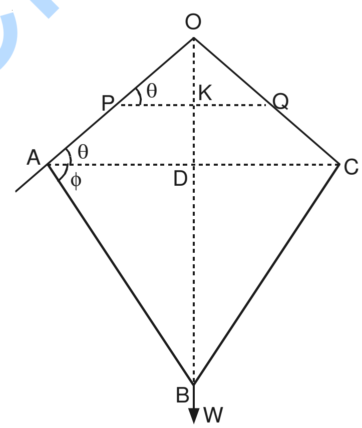
Let K be the mid-point of PQ, then $PK = QK = \frac{1}{2}l$

In ΔOAD, $AD = OA\cos\theta = a\cos\theta$.

In ΔABD, $AD = AB\cos\phi = b\cos\phi$.

Equating these two values of AD we get $a\cos\theta = b\cos\phi$

or $$a\sin\theta\,\delta\theta = b\sin\phi\,\delta\phi \quad \text{...(i)}$$



Here the distances are being measured downwards from the fixed line PQ.

Now in ΔOPK,

$$OK = PK\tan\theta = \frac{1}{2}l\tan\theta$$

$$\therefore \quad KB = BD + DK$$

$$= BD + (OD - OK)$$

$$= b\sin\phi + a\sin\theta - \frac{1}{2}l\tan\theta$$

Give the system a small virtual displacement, such that θ changes into $\theta + \delta\theta$ and ϕ changes into $\phi + \delta\phi$, then the equation of virtual work is $W\delta(KB) = 0$

or $$\delta(KB) = 0$$

or $$\delta(b\sin\phi + a\sin\theta - \frac{1}{2}l\tan\theta) = 0$$

or $$b\cos\phi\,\delta\phi + a\cos\theta\,\delta\theta - \frac{1}{2}l\sec^2\theta\,\delta\theta = 0$$

or $$(\frac{1}{2}l\sec^2\theta - a\cos\theta)\,\delta\theta = b\cos\phi\,\delta\phi \quad \text{...(ii)}$$

Dividing (ii) by (i) we get

$$\frac{\frac{1}{2}l\sec^2\theta - a\cos\theta}{a\sin\theta} = \frac{b\cos\phi}{b\sin\phi} = \frac{\cos\phi}{\sin\phi}$$

or $$\frac{1}{2}l\sec^2\theta\sin\phi - a\cos\theta\sin\phi = a\sin\theta\cos\phi$$

or $$\frac{1}{2}l\sec^2\theta\sin\phi = a(\sin\theta\cos\phi + \cos\theta\sin\phi) = a\sin(\theta+\phi).$$

EXERCISES

1. Four rods of equal weight w form a rhombus, with smooth hinges at the joints. The frame is suspended by the point A and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping them inclined at an angle α to AC. Show that thrust in the stiffening rod is

$$(2W + 4w)\tan\alpha.$$

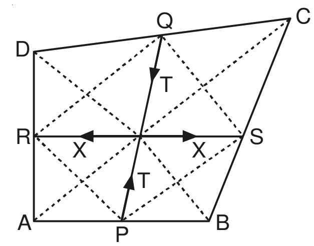
2. A square framework formed of uniform heavy rods of equal weight W jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod. [**Ans.** $4W$]
3. Four equal uniform rods each of weight w, are freely jointed to form a rhombus $ABCD$. The framework is suspended freely from A and a weight W is attached to each of the joints B, C and D. If two horizontal forces each of magnitude P acts at B and D keep the angle BAD equal to 120°, prove that
$$P = (W + w)2\sqrt{3}.$$
4. A regular hexagon $ABCDEF$ composed of six equal heavy rods jointed together, and two opposite angles C and F are connected by a string, which is horizontal, AB being in contact with a horizontal plane. A weight W' is placed at the middle point of DE. If W be the weight of each rod, show that the tension of the string is $(3W + W')/\sqrt{3}$.
5. Six equal rods, each of weight W are jointed to form a hexagon $ABCDEF$. The upper rod AF is kept in a horizontal position and the system lies in a vertical plane. The middle points of the rods AB and EF are connected by a light rod and is of such length that the rods AB and EF are inclined at an angle of 45° to the horizontal. Find the thrust of the rod. [**Ans.** $6W$]
6. Six equal bars are freely jointed at their extremities forming a regular hexagon $ABCDEF$, which is kept in shape by vertical strings joining the middle points of BC, CD and AF, EF respectively; the side AB being horizontal and uppermost. Prove that tension of each string is three times the weight of a bar.
7. Four uniform rods are jointed to form a rectangle $ABCD$, AB is fixed in a vertical position with A uppermost, and the rectangle is kept in shape by a string joining AC. Find the Tension of the string, given $AB = 2a$, $BC = 2b$, and w = weight of the rectangle.
8. A quadrilateral $ABCD$, formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that $\angle ABC$ is a right angle. Apply the principle of virtual work to show that the tension of the string is $(W + W')\sin^2\theta + W'$; where W is the weight of an upper rod and W' of a lower rod and $\angle BAD = 2\theta$.
9. Six equal light rods are jointed to form a regular hexagon $ABCDEF$ which is suspended at A and F so that AF is horizontal. A light rod BE also keeps the frame from collapsing and is of such a length that the rods ending in the points B, E are inclined at angles of 45° to the vertical. Equal weights are suspended from B, C, D and E. Find the stress in BE. [**Ans.** $3W$]
10. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon. The rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string. Prove that its tension is $3W$.
11. $ABCD$ is a quadrilateral formed of four uniform freely jointed rods, of which $AB = AD$ and each of weight W, and $BC = CD$ and each of weight W'. A string joins A to C. It is freely suspended from A. If $\angle BAD = 2\theta$ and $\angle BCD = 2\phi$, show that the tension in the string is
$$\frac{W\tan\theta + W'(2\tan\theta + \tan\phi)}{\tan\theta + \tan\phi}.$$

2.4.2 SOME MORE PROBLEMS ON FRAME-WORKS

EXAMPLES

1. *A smoothly jointed framework of light rods forms quadrilateral ABCD. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other opposite pair by a light rod in a state of thrust X ; show that* $\dfrac{T}{PQ} = \dfrac{X}{RS}$.

Sol. Let $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Let tension T in the string PQ be replaced by a force T at P and a force T at Q in the directions as marked in the figure. Similarly replace the thrust



X in the rod RS by a force X at R and a force X at S as marked in the figure.

Let $PQ = l$ and $RS = x$. Give the system a slight virtual displacement such that l changes to $l + \delta l$ and x changes to $x + \delta x$, then the equation of virtual work is

$$-T\,\delta l + X\,\delta x = 0$$

or $$T/X = \delta x/\delta l \qquad \text{...(i)}$$

Again join O, the point of intersection of PQ and RS, with A, B, C and D. Now OP is the median of ΔAOB,

$\therefore\ OA^2 + OB^2 = 2.OP^2 + 2.AP^2$, by plane geometry

$$= 2\left(\frac{1}{2}PQ\right)^2 + 2\left(\frac{1}{2}AB\right)^2 = \frac{1}{2}PQ^2 + \frac{1}{2}AB^2 \qquad \text{...(ii)}$$

Similarly in ΔBOC, OS is the median,

$$\therefore\quad OB^2 + OC^2 + 2.OS^2 + 2.BS^2 = 2\left(\frac{1}{2}RS\right)^2 + 2\left(\frac{1}{2}BC\right)^2 = \frac{1}{2}RS^2 + \frac{1}{2}BC^2 \qquad \text{...(iii)}$$

Subtracting (iii) from (ii) we get

$$2\,(OA^2 - OC^2) = PQ^2 + AB^2 - RS^2 - BC^2 \qquad \text{...(iv)}$$

Similarly from ΔAOD and ΔCOD we can have

$$OA^2 + OD^2 = \frac{1}{2}RS^2 + \frac{1}{2}AD^2 \quad \text{and} \quad OC^2 + OD^2 = \frac{1}{2}PQ^2 + \frac{1}{2}CD^2$$

Substracting as before, we get

$$2\,(OA^2 - CO^2) = RS^2 + AD^2 - PQ^2 - CD^2 \qquad \text{...(v)}$$

Equating the values of $2\,(OA^2 - OC^2)$ from (iv) and (v) we get

$$PQ^2 + AB^2 - RS^2 - BC^2 = RS^2 + AD^2 - PQ^2 - CD^2$$

or $$2\,(PQ^2 - RS^2) = AD^2 + BC^2 - AB^2 - DC^2$$

or $2l^2 - 2x^2 = d^2 + b^2 - a^2 - c^2 =$ constant as a, b, c, d are the lengths of the rods

Differentiating, $4l\,dl - 4x\ \delta x = 0$ or $\delta x/\delta l = l/x$ ...(vi)

$\therefore$ From (i) and (vi) we get $\dfrac{T}{X} = \dfrac{l}{x}$ or $\dfrac{T}{l} = \dfrac{X}{x}$ or $\dfrac{T}{PQ} = \dfrac{X}{RS}$

2. *The middle points of opposite sides of jointed quendrilateral are connected by light rods of lengths l and l'. If T and T' be the tensions in these rods, prove that*

$$\frac{T}{l} + \frac{T'}{l'} = 0$$

Sol. In this example we can proceed exactly as in the last example remembering that the thrust X of the last example has been replaced by tension T and x has been replaced by l'.

$\therefore$ The equation of virtual work in this case is

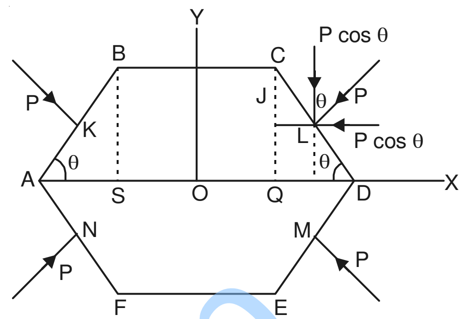
$$-T.\delta l - T'\delta l' = 0 \quad \text{or} \quad \frac{T}{T'} = -\frac{\delta l'}{\delta l}$$

From results (vi) of the last example and $x = l'$, $\dfrac{\delta l'}{\delta l} = \dfrac{l}{l'}$

$$\therefore \qquad \frac{T}{T'} = -\frac{l}{l'} \quad \text{or} \quad \frac{T}{l} + \frac{T'}{l'} = 0.$$

3. *ABCDEF is a regular hexagon formed of light rods smoothly jointed at their ends with a diagonal rod AD. Four equal forces P act inwards at the middle points of AB, CD, DE, FA at right angles to the respective sides. Find the stress in the diagonal AD and state whether it is tension or a thrust.*

Sol. Let K, L, M and N be the mid-points of the sides AB, CD, DE and FA of the hexagon. The given forces are acting at these points. Let each side of the hexagon be $2a$. Take the centre O of the hexagon as origin and the axes are shown in the figure. Referred to these axes let (x, y) be the coordinates of L, then we have



$$x = OR = OQ + JL = a + a\cos\theta \quad ...(i)$$

and $$y = LR = LD \sin\theta = a \sin\theta \quad ...(ii)$$

Also resolved parts of P at L are $P\sin\theta$ and $P\cos\theta$ towards the negative sides of the axes.

Let the system be given a small virtual displacement such that θ changes into $\theta + \delta\theta$, x changes into $x + \delta x$ etc., but the direction of the axes remain unaltered. Due to this displacement x and y coordinates of L will increase, *i.e.*, δx and δy are positive whereas the resolved parts of the force P at L are negative, so the work done by these resolved parts will be negative.

Also, the virtual works done by the force P at L

= sum of the virtual work done by its resolved parts

$= - P\sin\theta\,\delta x - P\cos\theta\,\delta y$

$= - P\sin\theta . \delta(a + a\cos\theta) - P\cos\theta . \delta(a\sin\theta)$ from (i) and (ii)

$= Pa\sin^2\theta . \delta\theta - Pa\cos^2\theta\,\delta\theta = - Pa\cos 2\theta . \delta\theta$

These forces P at K, M and N also do the same work.

$\therefore$ Total virtual work $= -4Pa\cos 2\theta . \delta\theta$.

Also, let T be the thrust in the rod AD then the equation of the virtual work is

$$-4Pa\cos 2\theta . \delta\theta + T\,\delta(AD) = 0,$$

where $AD = AS + SQ + QD = 2a\cos\theta + 2a + 2a\cos\theta$

$\therefore$ $$-Pa\cos 2\theta . \delta\theta + T\,\delta(2a + 4a\cos\theta) = 0$$

or $$-P\cos 2\theta - T\sin\theta = 0 \quad \because \delta\theta \neq 0$$

or $$T = -(P\cos 2\theta)/\sin\theta.$$

In the position of equilibrium $\theta = 60°$, so, we get

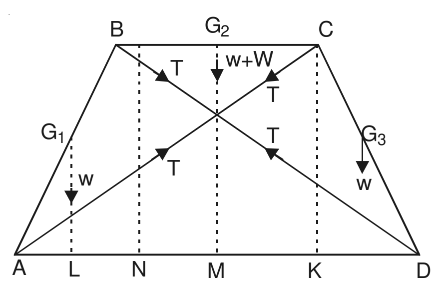
$$T = -\frac{P\cos 120°}{\sin 60°} = -\frac{P(-1/2)}{(\sqrt{3}/2)} = \frac{P}{\sqrt{3}},$$

which being positive, hence there is thrust in the rod AB.

4. *Three uniform rods AB, BC, CD each of weight w are freely jointed together at B and C, and rest in a vertical plane. A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework so that AB and CD are each inclined at an angle* α *to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then tension of each string will be of magnitude*

$$\left(w + \frac{1}{2}W\right)\cos\alpha \operatorname{cosec}\frac{1}{2}\alpha.$$

Sol. Let G_1, G_2 and G_3 be the mid-points of the rods AB, BC and CD respectively. The weights at G_1, G_2 and G_3 are $w, (w + W)$ and w respectively. G_1L and G_2M are perpendiculars drawn from G_1 and G_2 to AD. Let $AB = 2a$ then $AG_1 = a$.



$\therefore \quad G_1L = AG_1 \sin\alpha = a \sin\alpha$

and $\quad G_2M = BN = AB \sin\alpha = 2a \sin\alpha$

Also, $\quad AD = AN + NK + KD = 2a\cos\alpha + 2a + 2a\cos\alpha$

$$= 2a + 4a\cos\alpha.$$

From ΔABD, we get

$$BD^2 = AB^2 + AD^2 - 2AB.\cos\alpha$$

$$= (2a)^2 + (2a + 4a\cos\alpha)^2 - 2.2a\,(2a + 4a\cos\alpha)\cos\alpha$$

$$= 8a^2 + 8a^2\cos\alpha = 8a^2(1 + \cos\alpha)$$

$$= 16a^2\cos^2\frac{\alpha}{2}$$

$\therefore \quad BD = 4a\cos\dfrac{\alpha}{2} = AC.$

Let T be the tension in the strings AC and BD. Let the distances be measured from AD upwards, since AD is fixed.

Give the system a small virtual displacement such that α changes to $\alpha + \delta\alpha$, then the equation of the virtual work is

$$-2w\,\delta(G_1L) - (w + W)\,\delta(G_2M) - T\delta(AC) - T\,\delta(BD) = 0$$

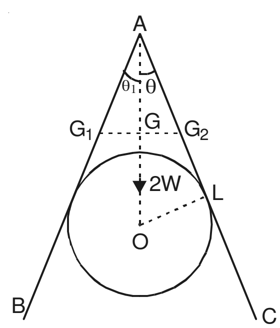
or $\quad -2w\,(\delta(a\sin\alpha) - (w + W)\,\delta(2a\sin\alpha) - 2T\,(2a\cos\dfrac{\alpha}{2}) = 0$

or $\quad -w\cos\alpha - (w + W)\cos\alpha) + T.\,2\sin\dfrac{\alpha}{2} = 0 \qquad \because\ \delta\alpha \neq 0$

or $\quad T = \left(w + \dfrac{1}{2}W\right)\cos\alpha\,\mathrm{cosec}\,\dfrac{\alpha}{2}.$

5. *Two equal uniform rods AB and AC, each of length 2b, are freely joined at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then* $b\sin^3\theta = a\cos\theta$.

Sol. Let G_1 and G_2 be the centres of gravity of the rods AB and AC. Then $AG_1 = \frac{1}{2}AB = b$. Let G be the mid-point of G_1G_2. Then if W be the weight of each rod, total weight $2W$ of the rods acts at G.



Also from ΔAOL, we get $AO = a\,\mathrm{cosec}\,\theta$.

Also from ΔAGG_1, we get

$$AG = AG_1\cos\theta = b\cos\theta$$

$\therefore \quad OG = AO - AG = a\,\mathrm{cosec}\,\theta - b\cos\theta \qquad ...(i)$

Give the system a small virtual displacement such that θ changes to $\theta + \delta\theta$, then as $2W$ is the only force here which appears in the equation of virtual work, so the equation of virtual work is

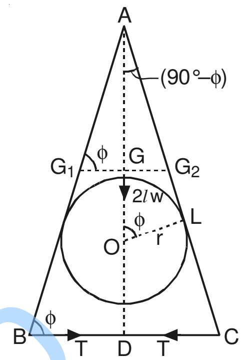
$$-2W\,\delta(OG) = 0 \quad \text{or} \quad \delta(OG) = 0, \qquad \because\ W \neq 0$$

or $\quad \delta(a\,\mathrm{cosec}\,\theta - b\cos\theta) = 0$

or $\quad -a\,\mathrm{cosec}\,\theta\cot\theta + b\sin\theta = 0, \qquad \because\ \delta\theta \neq 0$

or $\quad a\cos\theta = b\sin^3\theta.$

6. *Two rods each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination* ϕ *to the horizontal. Find the tension in the string and if the string be slack, show that* ϕ *satisfies the equation* $\tan^3\phi + \tan\phi = l/2r$.



Sol. As the peg is fixed so its centre O is also fixed. Let G_1 and G_2 be the centres of gravity of the rods AB and AC. Then $AG_1 = \frac{1}{2}l$. Let G be the mid-point of G_1G_2. Then the total weight $2lw$ of the rods is acting at G. From O draw OL perpendicular to AC. Then in ΔAOL we get $AO = r\sec\phi$.

Also from $\angle AGG_1$

$$AG = AG_1 \sin\phi = \left(\frac{1}{2}l\right)\sin\phi \qquad \text{...(i)}$$

$$\therefore \quad OG = AO - AG = r\sec\phi - \frac{1}{2}l\sin\phi$$

and $\quad BC = 2.BD = 2.AB\cos\phi = 2l\cos\phi \qquad$ ...(ii)

Let T be the tension in the string BC. Replace the string BC by a force T at B and a force T at C in the directions BC and CB respectively. Let the distances be measured upwards from the fixed point O.

Give the system a small virtual displacement such that ϕ changes into $\phi + \delta\phi$, then the equation of virtual work is

$$-2lw.\delta(OG) - T\,\delta(BC) = 0$$

or $\quad -2lw\,\delta(r\sec\phi - \frac{1}{2l}\sin\phi) - T\,\delta\,(2l\cos\phi) = 0, \qquad$ from (i) and (ii)

or $\quad -w\,(r\sec\phi\tan\phi - \frac{1}{2}l\cos\phi) + T\sin\phi = 0, \qquad \because\ \delta\phi \neq 0$

or $\quad T = w\,(r\sec^2\phi - \frac{1}{2}l\cot\phi)$

Now if the string be slack, then $T = 0$

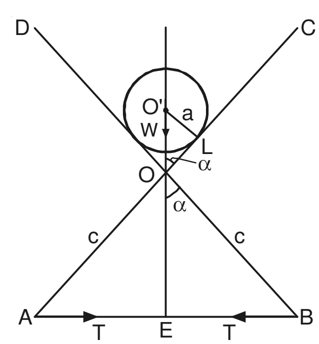
i.e. $\quad w\,(r\sec^2\phi - \frac{1}{2}l\cot\phi) = 0$

or $\quad 2r\,(1 + \tan^2\phi) = l\cot\phi = l/\tan\phi, \qquad \because\ w \neq 0$

or $\quad \tan\phi + \tan^3\phi = l/2r$

7. *Two light rods AOC and BOD are smoothly hinged connected by a string of length* $2c\sin\alpha$. *The rods rest in a vertical plane, with ends A and B on a smooth horizontal table. A smooth circular disc of radius a and weight W is placed on the rods above O with its plane vertical so that the rods are tangents to the disc. Prove that the tension of the string is*

$$\frac{1}{2}W\,\{(a/c)\,\mathrm{cosec}^2\,\alpha + \tan\alpha\}.$$



Sol. Let O' be the centre of the circular disc of weight W. $O'O$ is the vertical line through O' meeting AB at its middle point E.

Given $OA = OB = c$

and $\quad AB = 2c\sin\alpha$, *i.e.*, $BE = c\sin\alpha$.

$\therefore$ From triangle BOE, it is evident that $\angle BOE = \alpha$.

From O' draw $O'L$ perpendicular to AC.

Then $O'O = a\,\mathrm{cosec}\,\alpha$, from $\Delta O'OL$.

$\therefore \quad O'E = O'O + OE$

$= a \text{ cosec } \alpha + OB \cos \alpha$
$= a \text{ cosec } \alpha + c \cos \alpha.$

Let T be the tension in the string AB. Replace the string AB by a force T at A and force T at B in the directions AB and BA respectively. Let the distances be measured upwards from the fixed horizontal level AB.

Give the system a small virtual displacement such that α changes into $\alpha + \delta\alpha$, then the equation of virtual work is

$$-W\,\delta(O'E) - T\,\delta(AB) = 0$$

or $$-W\,\delta\,(a \text{ cosec } \alpha + c \cos \alpha) - T\,\delta(2c \sin \alpha) = 0$$

or $$W\,(a \text{ cosec } \alpha + c \cos \alpha) - T\,\delta\,(2c \sin \alpha) = 0 \quad \because \delta\alpha \neq 0$$

or $$T = \frac{1}{2} W \left[(a/c) \text{ cosec}^2 \alpha + \tan \alpha \right].$$

EXERCISES

1. Four rods are jointed to form a parallelogram, the opposite points are joined of strings forming diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.
2. A parallelogram $ABCD$ is formed of four rods jointed together. Equal and opposite forces P are applied inwards at A and C and a string connects B and D. Prove that the tension in the string is $P\,(BD/AC)$.
3. Four equal uniform bars, each of weight W, are joined together so as to form a rhombus. This is suspended vertically from one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing. Show that if 2θ be the angle at the fixed joint in the figure of equilibrium, then $\dfrac{\sin^2 \theta}{\cos \theta} = \dfrac{Pr}{4\,(P + 2\omega)\,a}$, where r is the radius of the sphere and $2a$ is the length of each bar.
4. Two equal uniform rods AB and AC each of length a and weight ω are freely jointed at A and placed in a vertical plane across a smooth horizontal cylinder of radius c. The ends B and C are connected together by a taut string of length l which is horizontal and not in contact with the cylinder. Show that the tension in the string is
$$\omega \left[\frac{4ac}{l^2} - \frac{1}{2\sqrt{(4a^2 - l^2)}} \right].$$
5. A rod AB is moveable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that horizontal force necessary to keep the ring at rest in $\dfrac{W \cos \alpha \cos \beta}{2 \sin (\alpha + \beta)}$, where W is the weight of the rod and α, β the inclinations of the rod and the string to the horizontal.
6. Six equal heavy rods, freely hinged at the ends, form a regular hexagon $ABCDEF$, which hung up by the corner A is kept from altering its shape by two light rods BF and CE. Prove that the thrust in these rods are $\frac{1}{2}(5\sqrt{3})\,W$ and $\frac{1}{2}\sqrt{3}W$, where W is the weight of each rod.
7. Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane. B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is $W/(\tan B + \tan C)$, where W is the total weight of the rods AB and AC.

8. Find equal uniform rods, freely jointed at their ends, form a regular pentagon *ABCDE* and *BE* is jointed by a weigthless bar. The system is suspended from *A* in a vertical plane. Prove that the thrust in *BE* is $W \cot(\pi/10)$, where *W* is the weight of each rod.

9. A regular pentagon *ABCDE* formed of equal uniform rods each of weight *W* is suspended from the point *A* and is maintained in shape by a light rod joining the middle points of *BC* and *DE*. Prove that the stress in the light rod is $2W \cot(\pi/10)$.

10. A step-ladder has a pair of legs which are jointed by a hinge at the top, and are connected by a cord attached at one third of the distance from lower end to the top. If the weight of the each leg be W_1 and acts at their middle points and if a man of weight *W* is two-thirds the way up the ladder, show that the tension in the cord is $\frac{1}{2}\left(W + \frac{3}{2}W_1\right)\tan\alpha$, α being the inclination of each leg to the vertical.

11. Four uniform rods each of length *a* and weight *w*, are smoothly jointed together to form a rhombus *ABCD*. The rhombus is kept in shape by a light strut *BD*, and hung from *A*. A uniform circular disc of radius *r* and weight w_1 is placed inside the triangle *BCD* with its plane vertical and its smooth rim in contact with *BC* and *DC;* its highest point in below *BD*. If the $\angle BAC$ is θ, show that the thrust in *BD* is

$$(2w + w_1)\tan\theta - \frac{r\,w_1}{2a\sin^2\theta}.$$

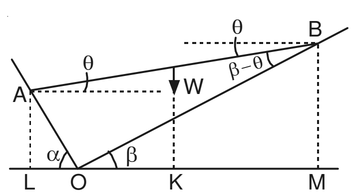
12. A freely jointed framework is formed of five equal uniform rods each of weight *W*. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string. If the two upper and two lower rods make angle θ and ϕ respectively with the vertical, prove that the tension of the string is to the weight of the rod as

$$(4\tan\theta + 2\tan\phi) : (\tan\theta + \tan\phi)$$

2.4.3. (e) Problems on bodies resting on inclined planes or pegs.

EXAMPLES

1. *A heavy uniform rod of length 2a, rests with its ends in contact with two smooth inclined planes of inclination* α *and* β *to the horizon. If* θ *be the inclination of the rod to the horizon, prove by the principle of virtual work that* $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$.



Sol. The weight *W* of the rod *AB* acts at *G*, the middle point of *AB* which is resting on the inclined planes given by *OA* and *OB* in the diagram.

From *A*, *B* and *G* draw *AL*, *BM* and *GK* perpendiculars to the horizontal plane through *O*.

Give the system a small virtual displacement such that θ changes to $\theta + \delta\theta$ whereas α and β remain unaltered, taking the horizontal plane through *O* upwards fixed, we have the equation of virtual work as

$$-W\delta(GN) = 0 \quad \text{or} \quad \delta(GN) = 0 \qquad \text{...(i)}$$

remembering that the other forces viz. reactions at *A* and *B* of the inclined planes do not appear in the equation of virtual work.

Now $\quad GK = \frac{1}{2}(AL + BM)$ or $GK = \frac{1}{2}[OA\sin\alpha + OB\sin\beta]$

Now in triangle *AOB*, $\quad \frac{OA}{\sin(\beta-\theta)} = \frac{OB}{\sin(\alpha+\theta)} = \frac{AB}{\sin[\pi-(\alpha+\beta)]}$

or $\quad OA = \frac{2a\sin(\beta-\theta)}{\sin(\alpha+\beta)}$ and $OB = \frac{2a\sin(\alpha+\theta)}{\sin(\alpha+\beta)}$, $\because AB = 2a$ (given)

$\therefore$ Substituting these values of OA and OB in (ii) we get

$$GK = \frac{1}{2} \cdot \frac{2a}{\sin(\alpha+\beta)} [\sin(\beta-\theta)\sin\alpha + \sin(\alpha+\beta)\sin\beta]$$

Hence from (i)

$$\delta\left[\frac{a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha + \sin(\alpha+\theta)\sin\beta\}\right] = 0$$

or $\quad -\cos(\beta-\theta)\sin\alpha + \cos(\alpha+\theta)\sin\beta = 0, \qquad \because \delta\theta \neq 0$

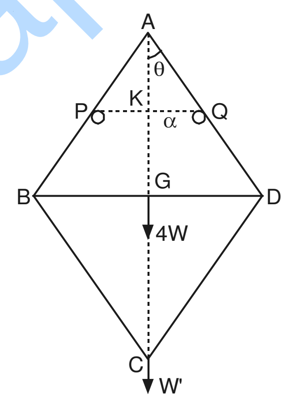
or $\quad -\cos\beta\cos\theta\sin\alpha - \sin\beta\sin\theta\sin\alpha + \cos\alpha\cos\theta\sin\beta - \sin\alpha\sin\theta\sin\beta = 0$

or $\quad 2\sin\alpha\sin\beta\sin\theta = \cos\theta(\sin\beta\cos\alpha - \cos\beta\sin\alpha)$

or $\quad \tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta).$

2. *A rhombus is formed of rods each of weight W and length l with smooth joints. It rests symmetrically with its upper sides in contact with two smooth pegs at the same level and at a distance 2a apart. A weight W' is hung at the lowest point. If the sides of rhombus make an angle θ with the vertical, prove that*

$$\sin^3\theta = \frac{a(4W+W')}{l(4W+2W')}.$$



Sol. The total weight $4W$ of the rods is acting at G, the centre of the rhombus. The distances are measured downwards from the line PQ, which being the line joining the pegs P and Q is fixed. Let K be the mid-point of PQ. Then $KQ = a$.

$\therefore$ In triangle AKG we get $\quad AK = a\cot\theta$

Also $AG = AD\cos\theta = l\cos\theta = GC.$

$\therefore \quad KC = KG + GC = 2l\cos\theta - a\cot\theta,$

and $KC = KG + GC = 2l\cos\theta - a\cot\theta,$

Give the system a small virtual displacement, such that θ changes into $\theta+\delta\theta$, then the equation of virtual work is

$$4W\delta(KG) + W'\delta(KC) = 0$$

or $\quad 4W\delta(l\cos\theta - a\cot\theta) + W'\delta(2l\cos\theta - a\cot\theta) = 0$

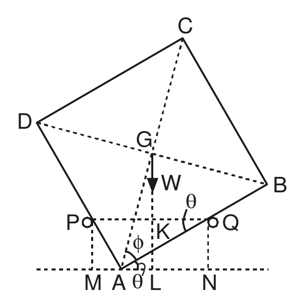
or $\quad 4W\delta(-l\sin\theta + a\,\text{cosec}^2\theta) + W'(-2l\sin\theta + a\,\text{cosec}^2\theta) = 0 \qquad \because \delta\theta \neq 0$

or $\quad (4W+2W')\,l\sin\theta = (4W+W')\,a\,\text{cosec}^2\theta$

or $\quad \sin^3\theta = a(4W+W')/[l(4W+2W')]$

3. *A square of side 2a is placed with its plane vertical between two smooth pegs, which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is*

$$\textit{either}\ \frac{1}{4}\pi\ \textit{or}\ \frac{1}{2}\sin^{-1}\{(a^2-c^2)/c^2\}.$$



Sol. The weight W (say) of the square is acting at its centre G. The line joining the pegs P and Q is the fixed line from which the distances are being measured upwards. Let BA be inclined to horizon at an angle θ.

Let the system be given a small virtual displacement such that θ changes into $\theta+\delta\theta$, then the equation of virtual work is $-W\delta(GK) = 0$ or $\delta(GK) = 0$

The reactions at P and Q do not appear in the equation of virtual work.

Now in ΔPAQ, $\quad AQ = PQ\cos\theta = c\cos\theta$

$\therefore$ In ΔAQN, $QN = AQ \sin\theta = c\cos\theta\sin\theta$ ...(ii)

Also $\angle GAB = 45°$, therefore in ΔGAL; we get

$$GL = AG\sin(45° + \theta) = AB\cos 45°\sin(45° + \theta)$$
$$= 2a\,(1/\sqrt{2})\sin(45° + \theta) = a\sqrt{2}\sin(45° + \theta)$$

$\therefore$ $GK = GL - KL = GL - QN$ (see figure)

$$= a\sqrt{2}\sin(45° + \theta) - c\cos\theta\sin\theta, \text{ from (ii)}$$
$$= a\sqrt{2}\,[\sin 45°\cos\theta + \cos 45°\sin\theta] - c\cos\theta\sin\theta$$
$$= a(\cos\theta + \sin\theta) - \frac{1}{2}c\sin 2\theta$$

$\therefore$ from (i) $\delta\left[a(\cos\theta - \sin\theta) - \frac{1}{2}c\sin 2\theta\right] = 0$

or $a(-\sin\theta + \cos\theta) - c\cos 2\theta = 0$

or $a(\cos\theta - \sin\theta) - c(\cos^2\theta - \sin^2\theta) = 0$

or $(\cos\theta - \sin\theta)[a - c(\cos\theta + \sin\theta)] = 0$, which gives

either $\cos\theta - \sin\theta = 0$, *i.e.*, $\tan\theta = 1$, *i.e.*, $\theta = \frac{1}{4}\pi$

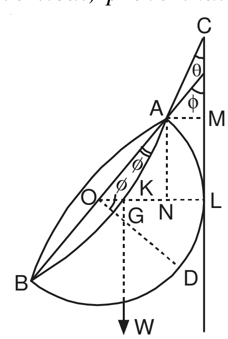
or $a - c(\cos\theta + \sin\theta) = 0$ or $a = c(\cos\theta + \sin\theta)$

or $a^2 = c^2(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta)$, squaring both sides.

or $a^2 = c^2 + c^2\sin 2\theta$ or $\sin 2\theta = (a^2 - c^2)/c^2$

or $\theta = \frac{1}{2}\sin^{-1}[(a^2 - c^2)/c^2]$

4. *A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If* θ, ϕ *are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that* $\tan\phi = \frac{3}{8} + \tan\theta$.



Sol. AC is the string of length l (say). O is the centre of the base of the hemisphere of radius r (say), OD is the axis of the hemisphere on which its centre of gravity G lies, such that $OG = \frac{3}{8}r$. The weight W of the hemisphere acts at G. The distances are measured downwards from the fixed displacement such that θ changes into $\theta + \delta\theta$ and ϕ changes into $\phi + \delta\phi$ whereas the length of the string remains unaltered ($\therefore$ its tension will not do any work) then the equation of virtual work is

$W\delta$ (depth of G below C) $= 0$

i.e. $W\delta(CL + KG) = 0$

or $\delta(CL + KG) = 0$ $\because W \neq 0$ ...(i)

Now $CM = CA\cos\theta = l\cos\theta$; $ML = AN = OA\cos\phi = r\cos\phi$

$\therefore$ $CL = CM + ML = l\cos\theta + r\cos\phi$ and $KG = OG\sin\phi = \frac{3}{8}r\sin\phi$

$\therefore$ From (i) we get

$$\delta\left(l\cos\theta + r\cos\phi + \frac{3}{8}r\sin\phi\right) = 0$$

or $-l\sin\theta\,\delta\theta - r\sin\phi\,\delta\phi + \frac{3}{8}r\cos\phi\,\delta\phi = 0$

or $l\sin\theta\,\delta\theta = \left(\frac{3}{8}\cos\phi - \sin\phi\right)r\,\delta\phi$ ...(ii)

Also OL = radius of the hemisphere $= r$

Again $OL = ON + NL = ON + AM$

$$= r \sin \phi + l \sin \theta$$

Differentiating, $0 = r \cos \phi \, \delta\phi + l \cos \theta \, \delta\theta$,
r, being constant.

or $$l \cos \theta \, \delta\theta = - r \cos \phi \, \delta\phi \qquad ...(iii)$$

Dividing (ii) by (iii) we get

$$\tan \theta = -\frac{3}{8} + \tan \phi$$

or $$\tan \phi = \frac{3}{8} + \tan \theta.$$

EXERCISES

1. A uniform beam of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show by the principle of virtual work that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1} (b/a)^{1/3}$.
2. Four equal rods, each of length $2a$ and weight W, are freely jointed to form a square $ABCD$ which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at distance $2b$ apart on the same level. Find the stress in the rod BC. [**Ans.** $2(a - b\sqrt{2})W/a$]
3. $ABCD$ is a rhombus formed with four rods each of length l and weight w jointed by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD. B and D are in a horizontal line and jointed by a string. If the distance of the pegs apart is $2d$ and the angle A is 2α, show that the tension in the string is
$$\left[\left(\frac{d}{2l}\right)(W + 4w) \operatorname{cosec}^3 \alpha - (W + 2w)\right] \tan \alpha.$$
4. A rhombus $ABCD$, formed of four weightless rods each of length a freely jointed at their extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant $2c$ apart and in contact with AB and AD. Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If 2α be the angle of the rhombus at A. Find the tension of the string. $\left[\textbf{Ans.} \frac{1}{2} W \left\{\left(\frac{3c}{a}\right) \operatorname{cosec}^3 \alpha - 5\right\} \tan \alpha\right]$.
5. A rhomboidal framework $ABCD$ is formed of equal light rods of length a smoothly jointed together. It rests in a vertical plane with diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on highest point A, will produce in BD a thrust of magnitude $W(2a^2c - b^2)/[b^2 - (4a^2 - b^2)^{1/2}]$.
6. An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2 \alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.
7. $ABCD$ is a rhombus formed with four rods each of length l and negligible weight jointed by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD. B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is 2α, show that the tension in the string is
$$W \tan \alpha \left(\frac{c}{2l} \operatorname{cosec}^3 \alpha - l\right)$$

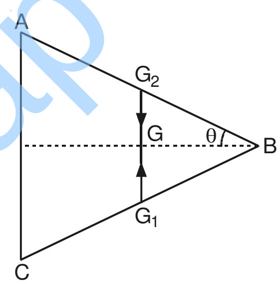
8. Weights w_1, w_2 are fastened to a light inextensible string BC at the points B, C, the end A being fixed. Prove that if a horizontal force P is applied at C and in equilibrium, AB, BC are inclined at angles θ, ϕ to the vertical, then

$$P = (w_1 + w_2) \tan \theta = w_3 \tan \phi$$

2.4.4 PROBLEMS ON ELASTIC STRINGS AND ENDLESS STRINGS

EXAMPLES

1. *One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are jointed by an elastic string, of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is* $2 \sin^{-1}\left(\frac{3}{4}\right)$.



Sol. Let G_1 and G_2 be the mid-points of the rods AB and BC. Let T be the tension in the string G_1G_2. Replace this string by a force T at G_1 and a force T at G_2 in the directions G_1G_2 and G_2G_1 respectively. Let G be the mid-point of G_1G_2, then the total weight $2W$ of the rods is acting at G, where W is the weight of each rod.

Let each rod be inclined to the horizon at an angle θ. The point A being fixed, the distances are measured from A downwards.

Given $AB = 2a = BC$, so $AL = AB \sin\theta \equiv 2a \sin\theta$

and $G_1G = BG_1 \sin\theta = a \sin\theta \quad \therefore \quad G_1G_2 = 2GG_1 = 2a \sin\theta$

Hence $AL = G_1G_2 = 2a \sin\theta$

Give the system a small virtual displacement such that θ changes into $\theta + \delta\theta$, then the equation of virtual work is

$$-T\delta(G_1G_2) + 2W\delta(AL) = 0$$

or $\quad -T + 2W = 0, \quad \because$ from (i) $\quad G_1G_2 = AL$ and $\delta\theta \neq 0$

or $\quad T = 2W \quad$ ...(i)

Also by Hooke's Law $T = \lambda\left(\dfrac{l-a}{a}\right) = 4W\left[\dfrac{G_1G_2 - a}{a}\right], \quad \because \lambda = 4W$

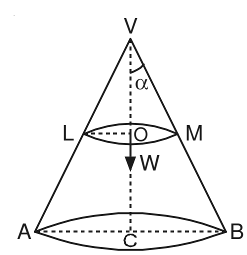
i.e. $\quad T = 4W\left[\dfrac{2a\sin\theta - a}{a}\right], \quad \because G_1G_2 = 2a\sin\theta$ from (ii)

or $\quad 2W = 4W(2\sin\theta - 1), \quad$ from (ii)

or $\quad 1 = 2(2\sin\theta - 1) \quad$ or $\quad \sin\theta = \dfrac{3}{4}$

$\therefore$ Required angle $ABC = 2\theta = 2\sin^{-1}\left(\dfrac{3}{4}\right)$.

2. *An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be* 2α.



Sol. V is the vertex and C the centre of the base of the cone. O is the centre and r the radius of the endless chain LM of weight W (say) resting in the form of a circular band round the cone. Since V is fixed so the disatnces are measured from V downwards. Now in ΔVOL, $VO = r \cot\alpha$ and the length of the string $= 2\pi r$.

Let the string be given a small virtual displacement such that r changes into $r + \delta r$, then the virtual work done by the tension T in the string is

$-T\delta l$, where l is the length of the string and the work done by the weight W of the string is $W\delta(VO)$, therefore the equation of the virtual work is

$$W\delta(VO) - T\delta(2\pi r) = 0, \quad \because 2\pi r \text{ is the length of the string.}$$

i.e. $$W\delta(r\cot\alpha) - T\delta(2\pi r) = 0, \quad \because VO = r\cot\alpha$$

or $$T = (W\cot\alpha)/(2\pi), \quad \because \delta r \neq 0.$$ **Ans.**

EXERCISES

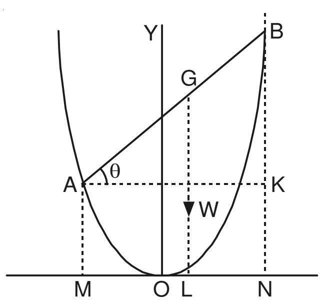
1. Four equal jointed rods, each of length a, are hung from an angular point which is connected by an elastic string with the opposite point. If the rods hang in the form of a square and if the modulus of elasticity of the string be equal to the weight of the rod, show that unstretched length of the string is $\dfrac{a\sqrt{2}}{3}$.
2. A heavy elastic string, whose natural length is $2a$, is placed round a smooth cone whose axis is vertical and whose semivertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is $a[1+(W\cot\alpha)/2\pi\lambda]$.
3. A smooth cone of weight W stands inverted in a circular hole with its axis vertical. A string is wrapped twice round the cone just above the hole and pulled tight. What must be the tension in the string so that it will just raise the cone ?
$\left[\textbf{Ans. } \dfrac{(W\cot\alpha)}{4\pi}\right]$.
4. A smooth sphere of radius r and weight W rests in a horizontal circular hole of radius a. A string is wrapped once round the sphere dust above the hole and then pulled tight. What must be tension in the string so that it will just raise the sphere.
$\left[\textbf{Ans. } \dfrac{Wa}{2\pi\sqrt{(r^2-a^2)}}\right]$.
5. A heavy elastic circular rod of natural length $2\pi a$ and weight W is placed round a smooth circular cone in a symmetrical position, whose axis is vertical and semivertical angle is 45°. Prove that in the position of equilibrium the tension of the cord is $(W/2\pi)$ and the vertical depth of the plane of the cord from the vertex of the cone is a $[1+W/2\pi\lambda]$ where λ is the modulus of elasticity of the cord.

2.4.5. PROBLEMS INVOLVING CURVES

EXAMPLES

1. *A smooth parabolic wire is fixed with its axis vertical and vertex downwards and in it is placed a uniform rod of length 2l with its ends resting on the wire. Show that for equilibrium, the rod is either horizontal, or makes with the horizontal an angle* θ *given by* $\cos^2\theta = 2a/l$, $4a$, $4a$ *being latus rectum of the parabola.*

Sol. Let the equation of the parabola be $x^2 = 4ay$. AB is the given rod of length $2l$ placed in the parabolic wire. Let the coordinates of A be $(2a, at^2)$, then



as $AK = 2l\cos\theta$,

and $BK = 2l\sin\theta$, hence the

coordinates of B are $(2at + 2l\cos\theta,\ at^2 + 2l\sin\theta)$.

Also B is point on the parabola $x^2 = 4ay$, so we have

$$(2at + 2l\cos\theta)^2 = 4a(at^2 + 2l\sin\theta)$$

or $$4l^2\cos^2\theta + 8at\,l\cos\theta = 8al\sin\theta$$

or $$t = \tan\theta - \{(l\cos\theta)/2a\} \quad \text{...(i)}$$

Let W, the weight of the rod be acting at G. From G draw GL perpendicular to x-axis. Give the rod a small virtual displacement such that changes into $\theta + \delta\theta$, then the equation of virtual work is

$$-W\,\delta(GL) = 0 \text{ or } \delta\,(GL) = 0 \qquad \text{...(ii)}$$

Here the distances are measured upwards from x-axis which is a fixed line.

Now $\quad GL = \frac{1}{2}(AM + BN) = \frac{1}{2}[at^2 + (at^2 + 2l \sin\theta)]$

$$= at^2 + l\sin\theta = a\left[\tan\theta - \left(\frac{l}{2a}\right)\cos\theta\right]^2 + l\sin\theta, \qquad \text{from (i)}$$

or $$GL = a\tan^2\theta + \left(\frac{l^2}{4a}\right)\cos^2\theta$$

$\therefore$ From (ii), we get

$$\delta\left[a\tan^2\theta + \left(\frac{l^2}{4a}\right)\cos^2\theta\right] = 0$$

or $$2a\tan\theta\sec^2\theta - \left(\frac{l^2}{2a}\right)\cos\theta\sin\theta = 0$$

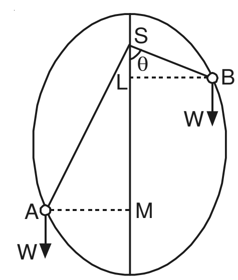
or $$\left[2a\sec^3\theta - \left(\frac{l^2}{2a}\right)\cos\theta\right]\sin\theta = 0$$

which gives either $\theta = 0$, *i.e.* $\theta = 0$, *i.e.* the rod is horizontal

or $$2a\sec^3\theta - \left(\frac{l^2}{2a}\right)\cos\theta = 0 \text{ or } \cos^4\theta = \frac{4a^2}{l^2}$$

or $\cos^2\theta = \frac{2a}{l}$, which gives the inclined position of rod.

2. *Two small smooth rings of equal weight slide on a fixed elliptic wire whose major axis is vertical. They are connected by a string which passes over a small smooth peg at the upper focus; show that the weights will be in equilibrium wherever they are placed.*



Sol. A and B are the given rings of equal weight W (say) connected by a string ASB passing over a small smooth peg at the focus S. Let the focus S be taken as pole, then the equation of the ellipse is

$$l/r = 1 - e\cos\theta \qquad \text{...(i)}$$

Let B be $(r - \theta)$, then $SB = r$,

so that $SA = a - r$, where a is the length of the string. From A and B draw AM and BL perpendiculars to the vertical line through S.

Then $SL = BS\cos\theta = (r - l)/e$, from (i).

Similarly $SM = (a - r - l)/e$

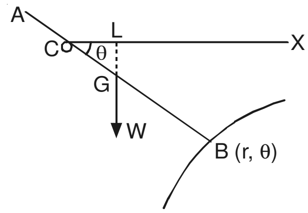
Here S being a fixed point, the distances are measured downwards from S. Give the system a small virtual displacement such that θ changes into $\theta + \delta\theta$, then the sum of the virtual work

$$= W\delta(SL) + W\delta(SM) = W\delta\{(r-l)/e\} + W\delta\{(a-r-l)/e\}$$
$$= (W/e)\,\delta r - (W/e)\,\delta r = 0, \text{ for all values of r}$$

i.e., the rings are in equilibrium wherever they may be placed.

3. *A heavy rod, of length 2l, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions, show that the curve is conchoid whose polar equation, with C as pole is* $r = l + (a/\sin\theta)$.

Sol. Let the polar coordinates of B referred to C as pole and the horizontal line through C as initial line be (r, θ). Let W the weight of the rod be acting at its middle point G. From G draw GL perpendicular to the initial line.



Then $BG = \frac{1}{2} AB = l$,

$\therefore \quad CG = CB - BG = r - l$

$\therefore \quad GL = CG \sin\theta = (r - l)\sin\theta$

Here C being fixed, the horizontal line through C is also fixed. Let the distance be measured downwards form this line. If the rod is slightly displaced (virtual displacement only) then only W does the virtual work, the reactions at B and C do not appear in the equation of virtual work, so the equation of virtual work is

$$W\,\delta(GL) = 0 \text{ or } \delta(GL) = 0$$

or $\quad \delta[(r - l)\sin\theta] = 0$, from (i)

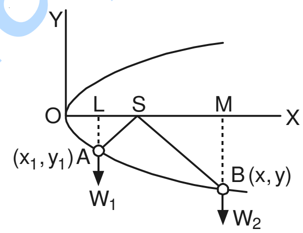
Integrating $(r - l)\sin\theta =$ constant $= a$, say

or $\quad r = (a/\sin\theta) + l$, which is the required equation of the locus of B *i.e.* the plane curve on which B rests.

4. *Two heavy rings slide on a smooth parabolic wire whose axis is horizontal and plane vertical and are connected by a string passing round a smooth peg at the focus. Prove that in the position of equilibrium their weights are proportional to the vertical depths below the axis.*

Sol. Let S be the focus of the parabola, whose equation is

$$y^2 = 4ax. \qquad \text{...(i)}$$



Let A and B be the two given rings of weights W_1 and W_2 (say) connected by the string ASB of length l (say).

Let the coordinates of A and B referred to the axes as shown in the figure be $(x_1, -y_1)$ and $(x_2, -y_2)$.

Then $AL = y_1$ and $BM = y_2$.

Also as $A\,(x_1, -y_1)$ and $B\,(x_2, -y_2)$ lie on (i), so we have

$$y_1^2 = 4ax_1 \text{ and } y_2^2 = 4ax_2 \qquad \text{...(ii)}$$

$$\left.\begin{array}{l}\text{Again focal distance } SA = a + x_1 \\ \text{and focal distance } SB = a + x_2\end{array}\right\}$$

$\therefore \quad$ Length of the string $= SA + SB$

or $\quad l = (a + x_1) + (a + x_2) = 2a + x_1 + x_2$

or $\quad l - 2a = \frac{y_1}{4a} + \frac{y_2}{4a'}$, $\quad$ from (ii)

or $\quad y_1^2 + y_2^2 = 4a(l - 2a) \quad$ or $\; y_1\delta y_1 + y_2\delta y_2 = 0$ or

$$y_1\delta y_1 = -\,y_2\delta y_2. \qquad \text{...(iii)}$$

Also if the system be given a small virtual displacement then measuring the distances from the fixed line $0x$, the equation of virtual work is

$$W_1\delta\,(AL) + W_2\delta(BM) = 0$$

or $\quad W_1\delta y_1 = -\,W_2\delta y_2 \qquad$ ...(iv)

Dividing (iv) by (iii) $W_1/y_1 = W_2/y_2$ or $W_1/W_2 = y_1/y_2$

i.e. weights of the rings are proportional to their depth below the axis.

EXERCISES

1. On a fixed circular wire (centre O and radius r) in a vertical plane slide two small rings A and B, each of weight W. The rings are jointed by a light inextensible string of length $2a\ (< 2r)$ on which slides a small smooth ring C of weight P. Prove that for equilibrium either both parts of the string are vertical or else P is at a distance from the centre of the wire equal to $\left\{\frac{W}{W + P}(r^2 - a^2)\right\}^{1/2}$.

2. A small heavy ring slides on a smooth wire whose plane is vertical, and is connected by a string over a small pulley in the plane of the curve with a weight which hangs freely. If the ring be in equilibrium in any position on the wire, show that the form of the latter is a conic whose focus is at the pulley.

3. A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any position, find the equation to the curve.

4. Two small rings of equal weights, slide on a smooth wire in the shape of a parabola whose axis is vertical and vertex upwards, and attract one another with a force which varies as the distance. If they can rest in any symmetrical position on the curve, show that they will rest in all symmetrical position.

5. Three equal and similar rods *AB*, *BC*, *DC* freely jointed at *B* and *C* have small weightless rings attached to them at *A* and *D*. The rings slide on a smooth parabolic wire, whose axis is vertical and vertex upwards and whose latus rectum is half the sum of the lengths of three rods. Prove that in the position of equilibrium the inclination of *AB* or *CD* to the vertical is given by

$$\cos\theta - \sin\theta + \sin 2\theta = 0.$$

6. A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.

$$\left[\textbf{Ans. } \frac{W_1}{W_2} = \cot\phi / \cot\theta\right].$$

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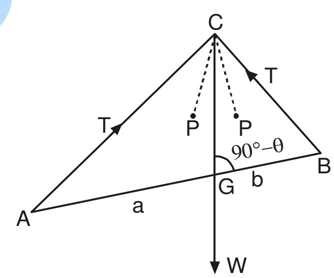
1. *Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C. AB is a heavy beam of weight W, whose centre of gravity is a feet from A and b feet from B, show that AB is inclined to the horizontal at on angle* $\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right]$.

Sol. Since each string supports a weight P at its free end, therefore tension in each of the two strings AC and BC is equal to P.



Now the rod AB is in equilibrium under three forces, namely.

(i) tension T at A along AC.

(ii) tension T at B along BC. and (iii) weight W through the centre of gravity G of the rod.

Since the directions of the two tensions meet at C, therefore the third force W also must pass through C, *i.e.*, the line CG is vertical.

The resultant of two equal tensions must bisect the angle between them, and also as this resultant balances W, therefore line of action of this resultant must be vertical. Hence CG is vertical and bisects the angle C.

i.e., $\angle ACG = \angle BCG = \frac{1}{2}C$,

Using Lami's Theorem, we have

$$\frac{T}{\sin\frac{1}{2}C} = \frac{W}{\sin C} \quad i.e., \quad \frac{P}{\sin\frac{1}{2}C} = \frac{W}{\sin C}$$

or $$\cos\frac{C}{2} = \frac{W}{2P}$$

$$\sin\left(90° - \frac{1}{2}C\right) = \frac{W}{2P} \quad \text{or} \quad 90° - \frac{1}{2}C = \sin^{-1}\left(\frac{W}{2P}\right)$$

Let θ be the angle that AB makes with the horizontal; then $\angle BGC = 90° - \theta$.

Using Trigonometrical Theorem, we have

$$(a+b)\cot(90° - \theta) = a\cot\tfrac{1}{2}C - b\cot\tfrac{1}{2}C$$

or $$\tan\theta = \frac{a-b}{a+b}\cot\frac{1}{2}C$$

i.e., $$\tan\theta = \frac{a-b}{a+b}\tan\left(90 - \frac{1}{2}C\right) = \frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right) \quad \text{from (1)}$$

or $$\theta = \tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

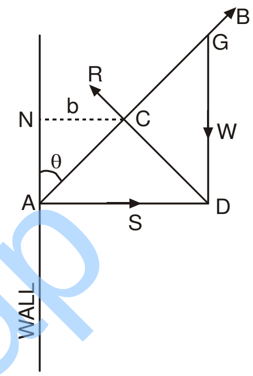
2. *A uniform beam, of length 2a rests in equilibrium against a smooth vertical wall and upon a peg at a distance b from the wall; show that the inclination of the beam to the vertical is* $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$.

Sol. The peg is at *C*. The distance of the peg from the wall is *b, i.e., CN = b.*

Let θ be inclination of the rod *AB* to the wall; then

$$AC = b \text{ cosec } \theta;$$

so $$CG = a - b \text{ cosec } \theta.$$



There are three forces on the rod *AB;* reaction *R* at the peg *C* is at right angle to the rod *AB,* the reaction *S* of the wall is in the horizontal direction; they meet at *D;* therefore the third force *W* (weight of the beam) also must pass through *D.*

Thus we see from figure that

$$\angle CAD = 90° - \theta, \ \angle CGD = \theta \quad \text{and} \quad \angle ACD = 90°.$$

For ΔAGD, using Trigonometrical Theorem, we have

$$(AC + CG) \cot 90° = CG \cot CAD - CA \cot AGD$$

i.e., $$0 = (a - b \text{ cosec } \theta) \cot (90° - \theta) - b \text{ cosec } \theta \cot \theta$$

or $$0 = (a - b \text{ cosec } \theta) \tan \theta - b \text{ cosec } \theta \cot \theta$$

i.e., $$a \tan \theta = b \text{ cosec } \theta \tan \theta + \text{cosec } \theta \cot \theta$$

$$= b\left(\frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta}\right) = \frac{b}{\cos\theta \sin^2\theta}$$

i.e., $$\sin^3\theta = \frac{b}{a} \quad \text{or} \quad \sin\theta = \left(\frac{b}{a}\right)^{1/3}$$

i.e., $$\theta = \sin^{-1} = \left(\frac{b}{a}\right)^{1/3}.$$

3. *A uniform rod rests on a fixed smooth sphere with its lower end pressing against a smooth vertical wall which touches the sphere. If* θ *is the angle which the rod makes with the vertical, when in equilibrium, prove that*

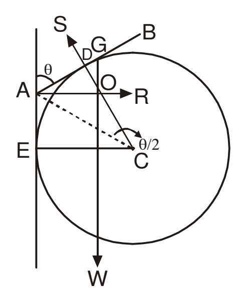
$$a = 2l \sin\frac{\theta}{2} \cos^3\frac{\theta}{2},$$

where l is the length of the rod, and a radius of the sphere.

Sol. The rod *AB* is in equilibrium under three forces *R, S* and *W* which meet in a point *O.* The centre of the sphere is *C;* join *AC.*

It can be easily seen that the $\angle DCE = \theta$; *AD* and *AE* are tangents to the circle from the same point *A;* therefore *AC* bisects the angle *DCE;* hence the $\angle ACD = \frac{\theta}{2}$.

Now from right-angled ΔACD, $AD = CD \tan\frac{\theta}{2} = a \tan\frac{\theta}{2}$



$\therefore$ $$DG = AG - AD = \frac{l}{2} - a\tan\frac{\theta}{2}.$$

Now using the trigonometrical theorem for the ΔOAG, we have

$$(AD + DG) \cot ADO = AD \cot AGO - DG \cot OAG$$

i.e., $$(AD + DG) \cot 90° = AD \cot\theta - DG \cot (90° - \theta)$$

or $$0 = AD \cot\theta - DG \tan\theta$$

i.e., $$AD \cot\theta = DG \tan\theta$$

or $$a\tan\frac{\theta}{2}\cot\theta = \left(\frac{l}{2} - a\tan\frac{\theta}{2}\right)\tan\theta$$

or $\quad a\,(\cot\theta + \tan\theta)\tan\dfrac{\theta}{2} = \dfrac{l}{2}\tan\theta \quad$ or $\quad a\,\dfrac{1}{\sin\theta\cos\theta}\,.\,\tan\dfrac{\theta}{2} = \dfrac{l}{2}\,.\,\dfrac{\sin\theta}{\cos\theta}$

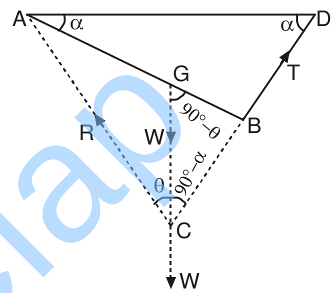
i.e., $\quad a = \dfrac{l}{2}\sin^2\theta\cot\dfrac{\theta}{2}$

i.e., $\quad a = \dfrac{l}{2}\,.\,4\sin^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2}\,.\cos\dfrac{\theta}{2}\,.\,\dfrac{1}{\sin\frac{1}{2}\theta} = 2l\sin\dfrac{\theta}{2}\cos^3\dfrac{\theta}{2}.$

4. *One end of a uniform rod is attached to a hinge and the other end supported by a string attached to the extremities of the rod, and the rod and the string are inclined at the same angle to the horizontal. If W be the weight of the rod, show that the action of the hinge is*

$$\tfrac{1}{4}W\,(8 + \mathrm{cosec}^2\,\alpha)^{1/2}$$

Find also the tension of the string.



Sol. *AB* is the rod and *BD* is the string.

There are three forces in equilibrium on the rod *AB*.

The directions of the tension *T* and the weight *W* meet at *C;* therefore the third force, that is, reaction at *A* must also pass through *C*. Let the direction of reaction *R* make and angle θ with the vertical.

Using the Trigonometric Theorem [formula $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$] for the ΔABC, we have

$(AG + GB)\cot(90° - \alpha) = AG\cot\theta - GB\cot(90° - \alpha)$

i.e., $\quad 2a\tan\alpha = a\cot\theta - a\tan\alpha$

or $\quad \cot\theta = 3\tan\alpha, \qquad \ldots(1)$

so that $\quad \cos\theta = \dfrac{3\tan\alpha}{[1 + 9\tan^2\alpha]^{1/2}} \quad$ and $\quad \sin\theta = \dfrac{1}{(1 + 9\tan^2\alpha)^{1/2}}$

Now using Lami's Theorem, we have

$$\frac{R}{\sin(90° - \alpha)} = \frac{T}{\sin\theta} = \frac{W}{\sin(90° - \alpha + \theta)}$$

or $$\frac{R}{\cos\alpha} = \frac{T}{\sin\theta} = \frac{W}{\cos(\alpha - \theta)}$$

$\therefore \quad R = W\dfrac{\cos\alpha}{\cos(\alpha - \theta)} = W\dfrac{\cos\alpha}{\cos\alpha\cos\theta + \sin\alpha\sin\theta} = W\dfrac{(1 + 9\tan^2\alpha)^{1/2}}{3\tan\alpha + \tan\alpha}$

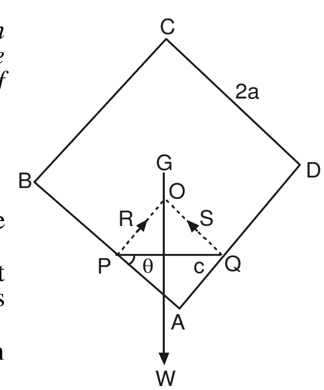
$\quad = \tfrac{1}{4}W\cot\alpha\,[\sec^2\alpha + 8\tan^2\alpha]^{1/2} = \tfrac{1}{4}W\,[8 + \mathrm{cosec}^2\,\alpha]^{1/2}$

and $\quad T = W\dfrac{\sin\theta}{\cos(\alpha - \theta)} = W\dfrac{\sin\theta}{\cos\alpha\cos\theta + \sin\alpha\sin\theta}$

$\quad = W\dfrac{1}{\cos\alpha\cot\theta + \sin\alpha} = \dfrac{W}{4\sin\alpha} = \dfrac{W}{4}\,\mathrm{cosec}\,\alpha.$

5. *A square of side* 2a *is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4} \;\; or \;\; \frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right).$$



Sol. Let the square *ABCD* rests on the two pegs *P* and *Q* where *PQ* is horizontal, and *PQ* = *c*.

Reactions *R* and *S* at the pegs are perpendicular to the edges; let them meet in a point *O;* then the third force weight *W* must also pass through *O*.

Let θ be the inclination of *AB* to the horizontal; then $\angle OPQ = 90° - \theta$ and the angle $\angle OQP = \theta$.

Resolving forces horizontally, we have

$$R \sin \theta = S \cos \theta \qquad \ldots(1)$$

Taking moments about the centre of gravity G, we have

$$R\,(a - c \cos \theta) = S\,(a - c \sin \theta). \qquad \ldots(2)$$

To eliminate R and S, divide (1) by (2); we then have

$$\frac{\sin \theta}{a - c \cos \theta} = \frac{\cos \theta}{a - c \sin \theta}$$

i.e., $a \sin \theta - c \sin^2 \theta = a \cos \theta - c \cos^2 \theta$

i.e., $a\,(\sin \theta - \cos \theta) - c\,(\sin^2 \theta - \cos^2 \theta) = 0$

i.e., $(\sin \theta - \cos \theta)\,\{a - c\,(\sin \theta + \cos\,)\} = 0$

$\therefore$ either $\sin \theta - \cos \theta = 0$, *i.e.*, $\sin \theta = \cos \theta$,

i.e., $\tan \theta = 1$, *i.e.*, $\theta = \dfrac{\pi}{4}$

This given one required angle.

or $a - c\,(\sin \theta + \cos \theta) = 0$

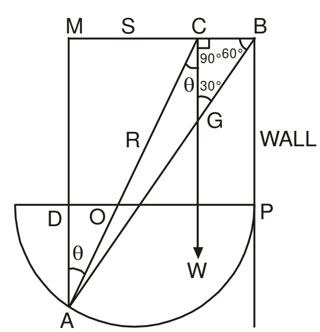
i.e., $c \cos \theta + c \sin \theta = a$, $\qquad \ldots(3)$

i.e., $(\sin \theta + \cos \theta)^2 = \dfrac{a^2}{c^2}$, *i.e.*, $1 + 2 \sin \theta \cos \theta = \dfrac{a^2}{c^2}$

i.e., $\sin 2\theta = \dfrac{a^2}{c^2} - 1 = \dfrac{a^2 - c^2}{c^2}$, *i.e.*, $\theta = \dfrac{1}{2} \sin^{-1}\left(\dfrac{a^2 - c^2}{c^2}\right)$

This gives other required angle.

6. *A smooth hemisperical bowl of diameter a is placed so that its edge touches a smooth vertical wall. A heavy uniform rod is in equilibrium inclined at an angle 60° to the horizontal with one end resting on the surface of the bowl and the other end against the wall. Show that the length of the rod is* $\left(a + \dfrac{a}{\sqrt{(13)}}\right)$.



Sol. The rod is AB, which is in equilibrium under three forces R, S and W as shown in the figure; they meet in a point C.

Using Trigonometrical Theorem we get from ΔABC,

$$(AG + BG) \cot BGC = AG \cot ACG - BG \cot BCG$$

i.e., $AB \cot 30° = \frac{1}{2} AB \cot \theta - AB \cot 90°$

i.e., $2 \cot 30° = \cot \theta$ or $\cot \theta = 2\sqrt{3}$.

$\therefore$ $\sin \theta = \dfrac{1}{\sqrt{3}}$.

Now from ΔAOD,

$$OD = OA \sin \theta = \frac{a}{2} \sin \theta = \frac{a}{2} \cdot \frac{1}{\sqrt{(13)}}.$$

$$\therefore \quad BM = DP = OP + OD = \frac{a}{2} + \frac{a}{2\sqrt{(13)}}.$$

Now from ΔABM, $AB = BM \sec 60°$

$$= \left(\frac{a}{2} + \frac{a}{2\sqrt{(13)}}\right) \sec 60° = 2\left(\frac{a}{2} + \frac{a}{2\sqrt{(13)}}\right) = a + \frac{a}{\sqrt{(13)}}.$$

This gives length of the rod AB.

EXERCISES

1. A smooth sphere of weight W is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall. If the length of the string be equal to the radius of the sphere, find the inclination of the string to the vertical, the tension of the string and the reaction of the wall.

$$\left[\textbf{Ans. } 30°, \frac{2W\sqrt{3}}{3}, \frac{W\sqrt{3}}{3}\right]$$

2. A sphere of given weight W, rests between two smooth planes, one vertical and the other inclined at an angle α to the vertical. Find the reactions of the planes.

[**Ans.** $W \cot \alpha, W \operatorname{cosec} \alpha$]

3. A beam whose centre of gravity divides it into two portions a and b, is placed inside a smooth sphere. Show that if θ be its inclination to the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere, then

$$\tan \theta = \frac{b-a}{b+a} \tan \alpha.$$

4. A uniform rod AB movable about a hinge at A rests with one end in contact with a smooth wall. If α be the inclination of the rod to the horizontal, show that the reaction at the hinge is $\frac{1}{2}W\,(3+\operatorname{cosec}^2 \alpha)^{1/2}$ and that it makes an angle $\tan^{-1}(2 \tan \alpha)$ with the horizontal.

5. An uniform rod is of length a. One end of it is in contact with a vertical wall. A string of length $\frac{3}{2}$ a which is fastened to the other end keeps it in equilibrium. The other end of the string is fastened to a point in the wall. If the rod makes an angle θ with the wall, prove that $\cos^2 \theta = \frac{5}{12}$.

6. A rod AB of weight W and length $2a$ is supported with one end A on a smooth vertical wall and the other end B is fastened with a string whose other end is tied at a distance of $2c$ above A with a nail in the wall. Prove that the reaction of the wall is $\dfrac{W_c}{2\sqrt{a^2-c^2}}$.

7. A uniform rod AB, at the end A is supported by a string CE and the other end is in contact with a smooth vertical wall AC. If the rod be in equilibrium and $AE = \frac{1}{3}AB$, prove that CB will be horizontal.

8. A uniform rod AB can turn freely about one of its end A and is pulled aside from the vertical by a horizontal force acting at the other end B of the rod and equal to half its weight. At what inclination to the vertical will the rod rest ? [**Ans.** 45°]

9. A uniform rod AB can turn freely about a smooth hinge A. The end B is attached to a string whose other end is fastened at a point in the horizontal plane of A. Rod and the string make equal angle α with the horizontal. Prove that the tension of the string will be $\dfrac{W}{4 \operatorname{cosec} \alpha}$, where W is the weight of the rod. Find the reaction of the hinge.

$$\left[\textbf{Ans. } \frac{W}{4\sqrt{(9+\cot^2 \alpha)}}\right]$$

10. A uniform rod AB of weight W movable in a vertical plane about a hinge at A, and is substained in equilibrium by a weight P attached to a string BCP passing over a smooth peg C, AC being vertical, if $AC = AB$, show that $P = W \cos ACB$.

11. A rod whose centre of gravity divides it in the ratio of 1 : 2 is placed inside a smooth sphere and subtends a right angle at the centre of the sphere. Show that its inclination to the horizon is $\tan^{-1}\left(\frac{1}{3}\right)$.

12. In a smooth hemispherical cup is placed a heavy rod, equal in length to the radius of the cup, the C.G. of the rod being one-third of its length from one end. Show that the angle made by the rod with the vertical is $\tan^{-1}(3\sqrt{3})$.

13. A rod is placed inside a smooth spherical bowl. The rod makes an angle 2α at its centre and the C.G. of rod is 20 cm from one end and 30 cm from the other. In the position of equilibrium the rod makes an angle θ with the horizontal. Prove that $\tan\theta = \frac{1}{5}\tan\alpha$.

14. A uniform heavy rod is placed completely inside a smooth hemispherical bowl. Prove that in the position of equilibrium it will remain horizontal.

15. A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall and the other end against a smooth plane inclined to the wall at an angle θ. If α be the inclination of the rod to the horizontal, prove that : $2\tan\alpha = \tan\theta$.

16. A smooth hemispherical bowl of radius r is placed in such a way that its rim is horizontal. How many forces are acting on the rod which is in equilibrium with one end inside and outher outside of it. If in the normal position of equilibrium the length of rod which is inside the bowl is c, find the length of the whole rod. [**Ans.** $4(c^2 - 2r^2)/c$]

17. Two uniform rods AB, BC, rigidly jointed at B so that angle ABC is a right angle, hang freely in equilibrium from a fixed point A. The length of the rods are a and b, and their weights are wa and wb. Prove that, if AB makes an angle θ with the vertical, $\tan\theta = \frac{b^2}{a^2 + 2ab}$.

18. A weight W is attached to an endless string of length l which hangs over two smooth pegs distant c apart in a horizontal line. Prove that pressure on each peg is of magnitude $W[(l - c)/2(l - 2c)]^{1/2}$.

19. A heavy uniform rod is hanging from a fixed point with the help of two strings tied to its ends and to the fixed point. If the lengths of strings and the rod are in the ratio of 2 : 3 : 4, prove that the tensions in the strings are in the ratio $2 : 3 : \sqrt{10}$.

20. Two smooth planes meet in a point and are inclined at angles α and β to the horizon. A circular plate of weight W is placed vertically between these planes in the equilibrium position. Find the thrust on the planes.

$$\left[\textbf{Ans.}\ \frac{W\sin\alpha}{\sin(\alpha+\beta)},\ \frac{W\sin\beta}{\sin(\alpha+\beta)}\right]$$

21. A uniform rod AB is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light string which is fastened to a point C in the same level as A. Prove that the angle β which the string makes with the horizontal is given by

$$\tan\beta = 2\tan\alpha + \cot\alpha.$$

22. One end of a string of length l is attached to one end A of a smooth uniform rod of length a, and the other end of the string is attached to a light ring C, which slides on the rod. The string passes over a small smooth pulley. Prove that in the position of equilibrium, the inclination θ of the rod to the horizontal is given by $a\cos^3\theta = l\sin\theta$.

23. Two smooth planes are joined at an acute angle and one of the plane is horizontal. Prove that a heavy rod can not be placed in the position of equilibrium between these planes.

24. A uniform rod has weight W and length $2l$. One of its ends is in contact with a smooth vertical wall and one point of its length is on a smooth girder. The girder is parallel to the wall and is at a greater height than the point of contact with the wall. The rod makes an angle of 45° with the vertical. Find the distance of the girder from the wall.

[**Ans.** $l/2\sqrt{2}$]

GENERAL CONDITIONS OF EQUILIBRIUM

1.5. CONDITIONS OF EQUILIBRIUM

I. Theorem : *(On conditions of equilibrium).*

A system of forces acting at different points of a rigid body will be in equilibrium if the algebraic sums of the resolved parts of the forces in any two perpendicular directions vanish separately and the algebraic sum of the moments of the forces vanishes about any point in the plane of the forces.

We know that a system of coplanar forces acting on a rigid body are equivalent to

a single force $R = \sqrt{[(R_x)^2 + (R_y)^2]}$

and a couple $G = \Sigma(x_1 Y_1 - y_1 X_1)$.

Now the resultant of R and G will never be zero, because a single force and a single couple cannot balance each other, hence in order that the system may be in equilibrium it is necessary that R and G must vanish separately, and R will be zero when R_x and R_y vanish separately.

Hence conditions of equilibrium are that

$$R_x = 0, \ R_y = 0 \qquad \ldots(1)$$

and

$$G = 0. \qquad \ldots(2)$$

that is, sum of the resolved parts of the forces in any two perpendicular directions vanishes separately, and also the sum of the moments of all the forces of the system about any arbitrary point vanishes.

It may also be seen from equations (1) and (2) that the system reduces only to a single force if $G = 0$ and reduces only to a single couple if $R_x = 0, R_y = 0$

II. Theorem : *(On conditions of equilibrium).*

If the algebraic sum of the moments of a system of coplanar forces about three points in their plane, which are not in the same stright line, uanish separately, then the forces are in equilibrium.

Let A, B, C be the three points not in the same straight line. Take A as origin and any two perpendicular lines through A as co-ordinate axes. Let the co-ordinates of B be (p_1, q_1) and those of C be (p_2, q_2) referred to the above axes. Let the moment of the forces about the origin A be G, and G_1, G_2 moments about B and C respectively. Let R_x and R_y be the sum of the resolved parts of the system of forces along the co-ordinate axes.

Since G_1 is the moment about the point $B(p_1, q_1)$

$\therefore$ $G_1 = G - p_1 R_y + q_1 R_x$.

Similarly $G_1 = G - p_2 R_y + q_2 R_x$.

Given that $G = 0, G_1 = 0, G_2 = 0$

$\therefore$ $G = 0$ moment about A is zero,

$G - p_1 R_y + q_1 R_x = 0$ moment about B is zero,

$G - p_2 R_y + q_2 R_x = 0$ moment about C is zero.

Substituting for G in the last two equations, they become

$$-p_1 R_y + q_1 R_x = 0 \qquad \ldots(1)$$

and

$$-p_2 R_y + q_2 R_x = 0 \qquad \ldots(2)$$

Since the three points $A(0, 0), B(p_1, q_1), C(p_2, q_2)$ are not collinear.

$$\therefore \quad \begin{vmatrix} 0 & 0 & 1 \\ p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \end{vmatrix} \neq 0, \text{ i.e., } p_1q_2 - p_2q_1 \neq 0.$$

Solving (1) and (2), we have

$$(p_1q_2 - p_2q_1)\, R_x = 0$$

and $$(p_1q_2 - p_2q_1)\, R_y = 0$$

Since $$(p_1q_2 - p_2q_1) \neq 0.$$

$$\therefore \quad R_x = 0 \quad \text{and} \quad R_y = 0$$

Thus we see that, of the given system,

$$R_x = 0, \; R_y = 0, \quad \text{and} \quad G = 0$$

that is conditions of equilibrium are satisfied.

Hence the system is in equilibrium.

III. Theorem : *(On conditions of equilibrium).*

A system of forces in a plane will be in equilibrium if the algebraic sum of the moments of the forces about each of any two points is zero and the algebraic sum of the resolved parts of the forces in any given direction not perpendicular to the line joining the given points, is zero.

Take the given direction of resolution to be the axis of *x;* let a point *O* be taken in this straight line and take another point O' such that OO' is not perpendicular to Ox. Let ξ, η be the co-ordinates of O', and G, G' the moments of the forces about O, O' respectively; then

$$G' = G - \xi R_y + \eta R_x$$

Given $G = 0$, sum of moments about O (origin) is zero

$$G - \xi R_y + \eta R_x = 0, \quad \text{sum of the moments about } O' \text{ is zero}$$

and $R_x = 0$, sum of resolved parts along x axis is zero

Substituting for G and R_x in the middle equation we get $R_y = 0$, provided that ξ is not zero (*i.e.* provided that OO' is not perpendicular to Ox).

Now we have for the given system

$$R_x = 0, \; R_y = 0 \quad \text{and} \quad G = 0$$

So the main conditions of equilibrium are satisfied.

Hence the system will be in equilibrium.

EXAMPLES

1. *Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC and forces* P_1, Q_1, R_1 *act along OA, OB, OC where O is the centre of the circumscribing circle. Prove that if the six forces are in equilibrium,*

$$P\cos A + Q\cos B + R\cos C = 0 \quad \textit{and} \quad \frac{P.P_1}{a} + \frac{Q.Q_1}{b} + \frac{R.R_1}{c} = 0.$$

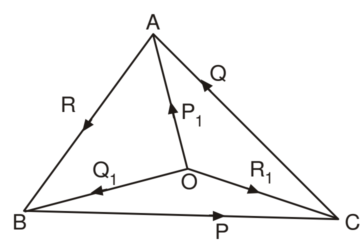
Sol. Since the resultant of the six forces passes through the circumcentre *O,* therefore taking moments about *O,* we have

$$P.OB\cos A + Q.OC\cos B + R.OA\cos C = 0$$

but $OA = OB = OC$, each being equal to the circum radius.

$$\therefore \quad P\cos A + Q\cos B + R\cos C = 0.$$

This proves one required result.



Second Part : $\angle BOC = 2A$, and $OB = OC$.

$$\therefore \quad \angle OBC = \angle OCB = \frac{1}{2}[180° - 2A] = 90° - A$$

Similarly $$\angle OBA = \angle OAB = 90° - C$$

and $$\angle OCA = \angle OAC = 90° - B.$$

Since the six forces are in equilibrium, therefore taking moments about *A,* we have

P. perp. from A on $BC - Q_1$ perp. from A on $OB - R_1$. perp. from A on $OC = 0$.

[moments of P and R_1 are positive being anticlockwise, and moment of Q_1 is negative being clockwise].

i.e. $P.c \sin B - Q_1.c \cos C + R_1.b \cos B = 0$.

Similarly $Q.a \sin C - R_1.a \cos A + P_1.c \cos C = 0$ [taking moments about B].

and $R.b \sin A - P_1.b \cos B + Q_1.a \cos A = 0$ [taking moments about C]

Multiplying these three equations respectively by P_1, Q_1, R_1 and adding, we get

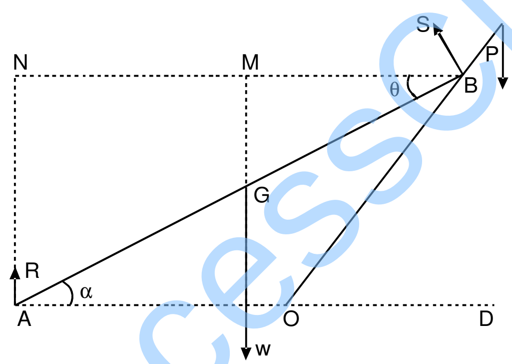
$$PP_1.c \sin B + QQ_1\ a \sin C + RR_1 b \sin A = 0$$

i.e. $PP_1.bc + QQ_1.ca + RR_1.ab = 0$ or $\dfrac{PP_1}{a} + \dfrac{QQ_1}{b} + \dfrac{RR_1}{c} = 0$

This proves the second result.

2. *One end of uniform beam of weight W, is placed on a smooth horizontal plane, the other end to which a string is fastened rest against another smooth plane inclined at an angle to the horizon, the string passing over a pulley at the top of the inclined plane, hangs vertically and support a weight P. Show that the beam will rest in all positions, if* $2P = W \sin \alpha$.

Sol. Let be the rod, AO is the horizontal direction and BO inclined plane. The reaction R and S at A and B which are perpendicular to AO and BO respectively.



$$\angle BOD = \alpha$$

Let $\angle BAO = \theta$ and $AB = 2a$

Taking moments about B

$$W.a \cos \theta = R.2a \cos \theta$$

or $W = 2R$, for all values of θ. Resolving parallel to the inclined plane,

$$P + R \sin \alpha = W \sin \alpha$$

or $P + \dfrac{W}{2} \sin \alpha = W \sin \alpha$ or $P = \left(W - \dfrac{W}{2}\right) \sin \alpha = \dfrac{W}{2} \sin \alpha$

or $2P = W \sin \alpha$

3. *Two equal smooth spheres each of weight W and radius r are placed inside a smooth hollow cylinder open at both ends, which rests on a smooth horizontal plane with its axis vertical. If* $a(< 2r)$ *be the radius of the cylinder, find the reaction between the spheres. Show that the condition necessary for cylinder not to be upset is*

$$W' > 2W\left(1 - \frac{r}{a}\right),$$

W' *being the weight of the cylinder.*

Sol. It is seen from the figure that

$$2a = r + r + 2r \cos \theta$$
$$= 2r + 2r \cos \theta,$$

i.e., $\cos\theta = \dfrac{a-r}{r}$;

$\therefore$ $\sin\theta = \sqrt{\left(\dfrac{2a-r}{r}\right)}$.

Considering equilibrium of the upper sphere and resolving forces horizontally and vertically, we have

$$R\cos\theta = S \quad \ldots(1)$$

and $$R\sin\theta = W. \quad \ldots(2)$$

$\therefore$ from (2), $$R - \frac{W}{\sin\theta} = W\sqrt{\left(\frac{r}{2a-r}\right)}$$

This gives reaction between the two spheres.
Dividing (1) by (2), we have

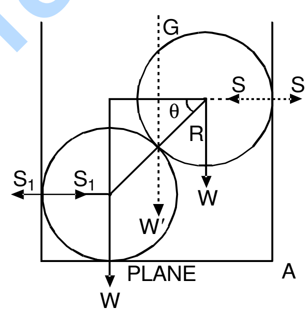
$$S = W\cot\theta \quad \ldots(3)$$

Considering the combined equilibrium of the two spheres, and resolving forces horizontally, we have

$$S_1 = S. \quad \ldots(4)$$

If the cylinder is to upset, it would do so about the point *A*.

Now consider the equilibrium of the cylinder (only).



The force on the cylinder are S, S_1, W' as marked in the figure and the reaction at *A*. The tendency of *S* is to upset the cylinder about *A* and tendency of S_1 and W' is to prevent it from upsetting. So in order that the cylinder may not upset (about A),

the sum of the moments of S_1 and W' about $A >$ moment of *S* about *A*,

i.e., $S_1.r + W'a > S\,(r + 2r\sin\theta)$ [because from (4)]

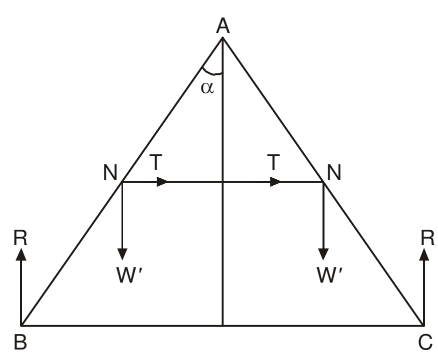
or $W'a > S.2r\sin\theta > W\cot\theta.\,2r\sin\theta$ from (3)

i.e., $W'a > 2rW\cos\theta$

$> 2rW.\dfrac{a-r}{r}$ [by substituting value of $\cos\theta$]

or $$W' > 2W\left(1 - \frac{r}{a}\right).$$

4. *A step ladder in the form of the letter A with each of its legs inclined at an angle α to the vertical, is placed on a horizontal floor, and is held up by a chord connecting the middle points of its legs, there being no friction any where; show that when a weight W is placed on one of the steps at a height from the floor equal to $\dfrac{1}{n}$ of the height of the ladder, the increase in the tension of the cord is $\dfrac{1}{n}W\tan\alpha$.*



Sol. In the original position, the forces are as maked in the figure. Let *W* be the weight of each leg and 2*a* its length.

Considering the equilibrium of the whole ladder and resolving forces vertically, we have

$$2R = 2W'$$

i.e., $R = W'$ ...(1)

Considering equilibrium of one of the legs, say AB, and taking moments about A, we have

$$-R.2a \sin\alpha + W'.a \sin\alpha + T.a \cos\alpha = 0 \quad ...(2)$$

Substituting for R from (1) in (2), we have

$$-2W' \sin\alpha + W' \sin\alpha + T \cos\alpha = 0.$$

i.e., $T = W' \tan\alpha.$ ...(A)

This gives tension originally.

And when a weight W is placed, let the two normal reactions be S each and let T' be the tension in this case.

In this case considering combined equilibrium and resolving forces vertically, we have

$$2S = 2W' + W \quad i.e., \quad S = W' + \frac{1}{2}W \quad ...(3)$$

Considering equilibrium of one of the legs and taking moments about A we have

$$-S.2a \sin\alpha + W' a \sin\alpha + T' a \cos\alpha + \frac{1}{2}.W\, 2a \frac{n-1}{n} \sin\alpha = 0 \quad ...(4)$$

Substituting value of S from (3) in (4), we have

$$-2\left(W' + \frac{1}{2}W\right) \sin\alpha + W' \sin\alpha + T' \cos\alpha + W \frac{n-1}{n} \sin\alpha = 0$$

i.e., $$T' \cos\alpha = W' \sin\alpha + \frac{1}{n}W \sin\alpha$$

or $$T' = W \tan\alpha + \frac{1}{n}W \tan\alpha = T + \frac{1}{n}W \tan\alpha \text{ from (A)}$$

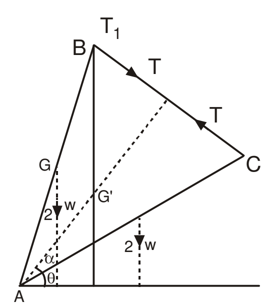
or $$T' - T = \frac{1}{n}W \tan\alpha$$

i.e. increase in the tension of the cord $= \frac{1}{n}W \tan\alpha$.

5. *Two uniform rods AB and AC of equal lengths are freely jointed at A. The ends B and C are connected by a thin string. The system is suspended by a string attached to B and hangs freely under gravity. If in the position of equilibrium the angle between the rods is* 2α *show that the tension of the string is*

$$\frac{3w \sin\alpha}{\sqrt{(1 + 3\sin^2\alpha)}}$$

where $2w$ *is the weight of either rod.*



Sol. Let T_1 be the tension of the string at from which the system is suspended. Let AC make an angle θ with the horizontal and suppose $AB = AC = 2a$.

The forces are as arrow-marked in the figure.

First, consider the equilibrium of the system, that is, the two rods and the string as one body, and resolve forces vertically, we have

$$T_1 = 4\omega \quad ...(1)$$

Secondly, consider equilibrium of one rod AC only.

Taking moments about A, we get

$$T.2a \cos\alpha = 2w.a \cos\theta \quad ...(2)$$

Thirdly, consider equilibrium of the rod AB only.

Taking moments about A, we get

$$T_1.2a \cos(\theta + 2\alpha) = 2w.a \cos(\theta + 2\alpha) + T.2a \cos\alpha \quad ...(3)$$

Taking the value of F_1 and T from (1) and (2), and putting in (3), we have

$$4w.2a \cos(\theta + 2\alpha) = 2w.A \cos(\theta + 2\alpha) + \frac{w \cos\theta}{\cos\alpha}.2a \cos\alpha$$

or $$4 \cos(\theta + 2\alpha) = \cos(\theta + 2\alpha) + \cos\theta$$

or $\quad 3\cos(\theta + 2\alpha) = \cos\theta$

or $\quad 3(\cos\theta\cos 2\alpha - \sin\theta \sin\theta\, 2\alpha) = \cos\theta$

or $\quad (3\cos 2\alpha - 1)\cos\theta = 3\sin 2\alpha \sin\theta$

i.e.,
$$\frac{\sin\theta}{3\cos 2\alpha - 1} = \frac{\cos\theta}{3\sin 2\alpha} = \frac{1}{\sqrt{(10 - 6\cos 2\alpha)}} \text{ (by comp.)}$$
$$= \frac{1}{\sqrt{(4 + 12\sin^2\alpha)}} = \frac{1}{2\sqrt{(1 + 3\sin^2\alpha)}}$$

$\therefore$
$$\cos\theta = \frac{3}{2}\frac{\sin 2\alpha}{\sqrt{(1 + 3\sin^2\alpha)}}$$

Now from (2),
$$T = w\frac{\cos\theta}{\cos\alpha} = w\frac{3}{2}\frac{\sin 2\alpha}{\sqrt{(1 + 3\sin^2\alpha)}} \cdot \frac{1}{\cos\alpha} = \frac{3w\sin\alpha}{\sqrt{(1 + 3\sin^2\alpha)}}$$

EXERCISES

1. A uniform rectangular lamina *ABCD* of weight *W* hangs from *A*. If a weight 2*W* be suspended from *B*, prove that the diagonal *AC* is inclined to the vertical at an angle $\tan^{-1}\left[\frac{4ab}{5a^2 + b^2}\right]$ where $AB = a$ and $AD = b$.
2. A uniform wire *AOB* is turned at *O*. Lengths of *AO* and *OB* are *a* and *b* respectively and the angle between them is θ. It is hung from *A*. To keep *OB* as horizontal, prove that
$$\cos\theta = \frac{b^2}{a(a+b)}.$$
3. A weightless rigid wire is in the shape of a circular arc subtending an angle α at the centre and it touches a horizontal plane at a point. Two weights *P* and *Q* are suspended at the ends of the wire. The radius passing through the end at which the weight *P* is suspended, makes an angle θ with the vertical. Prove that
$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$
4. A rod is movable in a vertical plane about a hinge at one end and at the other end is fastened a weight equal to half the weight of the rod. This end is fastened by a string of the length *l*, to a point at a height vertically over the hinge. Show that the tension of the string is *lW/c*, where *W* is the weight of the rod.
5. A rod *AB* can turn freely about a hinge at *A*. At its other end *B* a weight equal to half the weight of the rod is fastened. The end *B* is tied to a point *C* in the wall at a distance 10 cm vertically above the hinge *A* with a string of length 15cm. Prove that the tension of the string is 3*W*/2, where *W* is the weight of the rod.
6. If the foces *P, Q, R, S* acting along the sides as *b, c, d* of quadrilateral *ABCD* are in equilibrium, prove that $\frac{P.R}{a.c} = \frac{Q.S.}{b.d}$.
7. Three equal uniform rods, each of weight *W*, are smoothly jointed so as to form an equilateral triangle. If the triangle be supported at the middle point of one of the sides, show that the reaction at the lowest angle is $\frac{\sqrt{3}\,W}{6}$.

8. A ladder rests against a smooth vertical wall and the smooth ground. Its length is $2a$ and weight Its C.G. lies at a distance of $\frac{3}{4}$ of its length from its lower end and its middle point is fastened with a horizontal string of length and the other end of the string is attached to a point in the wall. Prove that the tension of the string is $\frac{3bW}{2\sqrt{(a^2-b^2)}}$.

9. A uniform ladder of W kg rests at an angle of 30° to the horizon with its ends resting on a smooth floor and against a smooth vertical wall, the lower end being attached by a string to the junction of the wall and the floor. Find the tension of the string. [**Ans.** $W\sqrt{3}/2$]

10. A uniform square lamina is placed on two pegs in the same horizontal line in a vertical position. The length of the side of the square is $2a$ and the distance between the pegs is c. Prove that in the equilibrium position $c(\sin\theta+\cos\theta)=a$ when the side BC is inclined at an angle 0 (zero) and not 45°.

11. Two equal heavy rods of weight W and length $2a$, are freely hinged together and placed symmetrically over a smooth fixed sphere of radius r. Show that the inclination of each rod to the horizontal is given by $r(\tan^2\theta+\tan\theta)=a$.

12. A uniform rod has its upper end fixed to a hinge and other and attached by a string to a fixed point in the same horizontal plane as the hinge, the length of the string being equal to the distance between the fixed point and the hinge. If the tension of the string be equal to the weight of the rod, show that rod is inclined to the horizon at an angle $\tan^{-1}\left(\frac{1}{2}\right)$ and that the action of the hinge is equal to a force $W/5\sqrt{10}$ inclined at an angle $\tan^{-1}\left(\frac{1}{3}\right)$ to the horizon.

13. A beam, of weight W, is divided by its centre of gravity G into two portions AG and BG, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and to the beam at a point E. If T be the tension of the string and α and β be the inclination of the beam and string respectively to the horizon, show that

$$T=\frac{wa\cos\alpha}{(a+b)\sin(\alpha+\beta)}$$

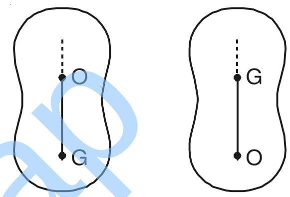
14. A heavy carriage wheel of weight W and radius r, is to be dragged over on obstacle, of height h, by a horizontal force F applied to the centre of wheel. Show that F must be slightly greater than $\frac{W\sqrt{(2rh-h^2)}}{(r-h)}$.

❒❒❒

Stable and Unstable Equilibrium

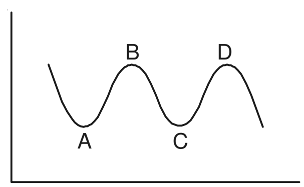
5.1 STABLE AND UNSTABLE EQUILIBRIUM

Let O be the fixed point of the body and G its centre of gravity. The only forces acting on the body are its weight and the reaction of the fixed point O. For equilibrium they should be equal and opposite and have the same line of action, so that the vertical line through G must pass through the point O.



Obviously, two cases arise : the first in which C.G. G of the body is below the point of suspension O, and the second in which G is above O. In the first case it is obvious that if the body be displaced slightly from its position of equilibrium it will tend to come back to its original position, but in the second case the body will not tend to return to its position of equilibrium. These two bodies are said to be in **stable and unstable equilibrium.**

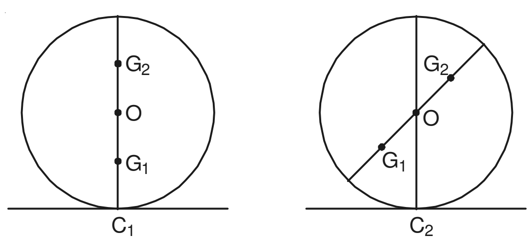
Now, consider the motion of a body on a smooth curve as shown in the fig. Obviously the body will rest at points A, B, C and D which are points of maxima or minima of the curve. Now if the body be slightly displaced from its positions of rest at B or D it will try to go further away from its original position, while if displaced from its positions of rest at A or C it will tend to resume its original position of rest. In the first case the equilibrium is **unstable** and in the second case it is **stable.**



Next we consider the case of a sphere whose centre of gravity G is not at its geometric centre. The positions of rest are :

(i) When the C.G. lies on the vertical through the point of contact and is below its geometric centre. In this case if the sphere be slightly displaced it would tend to come back to its original position. This is the position of **stable equilibrium.**

(ii) In second position of equilibrium the centre of gravity lies on the vertical through the point of contact and is above the geometric centre. In this case if the sphere be slightly displaced it would not come back to its original position of rest but would go further away from that position. This is the position of **unstable equilibrium.**



(iii) Thirdly, if the centre of gravity of the sphere coincides with its geometric centre, then sphere when displaced from any position of equilibrium, it will remain in equilibrium for all displacements. In such cases the equilibrium is said to **neutral.**

In general, a body is said to be in **stable equilibrium** when, if it be slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium; it is **unstable equilibrium,** when, if it be slightly displaced, the forces tend to move it still fruther from its position of equilibrium; it is in **neutral equilibrium,** if the forces acting on it in its displaced position are in equilibrium. In general, bodies which are 'top-heavy' or which have small bases, are unstable.

5.2 TEST FOR THE DETERMINATION OF THE NATURE OF THE STABILITY

From above discussions, it is clear that the body has more than one position of rest, the one *in which its centre of gravity is lowest is in general the position of stable equilibrium and the other in which centre of gravity is highest in position of unstable equilibrium.* Hence, for stable equilibrium the C.G. must have a minmium height and for unstable equilibrium it must have a maximum height.

Hence, to determine the nature of stability, express the height z of the C.G. of the body in terms of a variable θ, the positions of rest are given by $\frac{dz}{d\theta}=0,$ which on solving gives $\theta=\theta_1,\ \theta_2,\theta_3,...$ as positions of equilibrium.

(i) The equilibrium is stable at $\theta=\theta_1$ if $\frac{d^2z}{d\theta^2}>0$ for $\theta=\theta_1$.

(ii) The equilibrium is unstable at $\theta=\theta_1$ if $\frac{d^2z}{d\theta^2}<0$ for $\theta=\theta_1$.

Similar test apply for $\theta=\theta_2,\theta_3,...$

If however, $\frac{d^2z}{d\theta^2}=0$ for $\theta=\theta_1$, then we consider $\frac{d^3z}{d\theta^3}$ and $\frac{d^4z}{d\theta^4}$.

For position of rest $\frac{d^3z}{d\theta^3}=0$.

The equilibrium is stable or unstable according as $\frac{d^4z}{d\theta^4}$ is positive or negative.

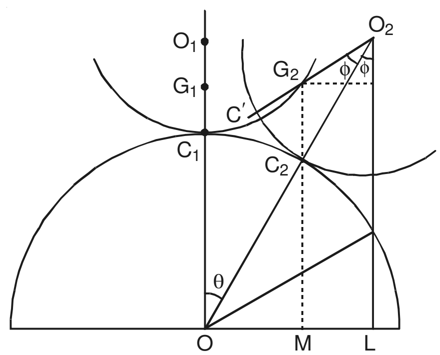
Note. If z represents **depth** of the C.G. of the body from some fixed horizontal plane, the case is reversed, and in this case the equilibrium is

(i) stable if $\frac{d^2z}{d\theta^2}$ is negative, and

(ii) unstable if $\frac{d^2z}{d\theta^2}$ is positive.

5.3. THEOREM

A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii *r* and *R* respectively, and the straight line joining the centres of the spheres being vertical; if the first body be slightly displaced, to find whether the equilibrium is stable or unstable; the bodies being rough enough to prevent sliding.



Let O and O_1 be the centres of the lower and upper bodies respectively; G_1 the position of C.G. of the upper body in the position of rest. The fig. represents the section of bodies by a vertical plane through G_1.

Let the upper body be slightly displaced by rolling so that the new position of the centre of the upper body be O_2. The new position of C_1 is C'. Let

$$C_1G_1=h,\qquad \angle C_1OC_2=\theta,\ \ \angle C'O_2C_2=\phi,$$

so that $\angle C_2OL=\theta+\phi$

Since there is no slipping when upper body rolls on the lower body, hence

$$\text{arc } C_1C_2 = \text{arc } C_2C'$$

or arc C_1C_2 of the lower sphere of radius $R=$ arc C_2C' of the upper sphere of radius r.

or $$R\theta=r\phi \qquad ...(i)$$

To discuss the nature of equilibrium we have to find out the height z of C.G., G_2 in the new position above O.

Now,

$$z = G_2M = O_2L - O_2G_2\,(\theta+\phi)$$
$$= OO_2\cos\theta - (C'O_2 - C'G_2)\cos(\theta+\phi)$$
$$= (R+r)\cos\theta - (r-h)\cos(\theta+\phi)$$
$$= (R+r)\cos\theta - (r-h)\cos\left(\theta+\frac{R\theta}{r}\right) \qquad \text{[from (i)]}$$
$$= (R+r)\cos\theta - (r-h)\cos\left\{\frac{(R+r)}{r}\theta\right\} \qquad \text{...(ii)}$$

For equilibrium $\dfrac{dz}{d\theta}=0$

i.e., $$(R+r)(-\sin\theta) + (r-h)\sin\left\{\frac{(R+h)}{r}\theta\right\}\left(\frac{r+R}{r}\right)=0$$

This is satisfied for $\theta = 0$.

Again, $$\frac{d^2z}{d\theta^2} = (R+r)(-\cos\theta) + (r-h)\cos\left\{\frac{(R+r)}{r}\theta\right\}\left(\frac{r+R}{r}\right)^2$$

or $$\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} = -(R+r) + (r-h)\left(\frac{r+R}{r}\right)^2$$
$$= \left(\frac{R+r}{r}\right)^2\left[(r-h) - \frac{r^2}{R+r}\right] = \left(\frac{R+r}{r}\right)^2\left[r - \frac{r^2}{R+r} - h\right]$$
$$= \left(\frac{R+r}{r}\right)^2\left[\frac{rR}{R+r} - h\right]$$

This will be positive if

$$\frac{rR}{R+r} > h \quad \text{or} \quad \frac{1}{h} > \frac{1}{r}+\frac{1}{R}$$

and negative if $$\frac{rR}{R+r} < h \quad \text{or} \quad \frac{1}{h} < \frac{1}{r}+\frac{1}{R}$$

This the equilibrium is **stable** if $\mathbf{\dfrac{1}{h} > \dfrac{1}{r} + \dfrac{1}{R}}$ and is **unstable** if $\mathbf{\dfrac{1}{h} < \dfrac{1}{r} + \dfrac{1}{R}}$.

Now, if $\dfrac{1}{h} = \dfrac{1}{r} + \dfrac{1}{R}$, we have

$$\frac{d^3z}{d\theta^3} = (R+r)\sin\theta - (r-h)\sin\left\{\left(\frac{r+R}{r}\right)\theta\right\}\left(\frac{r+R}{r}\right)^3$$
$$\frac{d^4z}{d\theta^4} = (R+r)\cos\theta - (r-h)\cos\left\{\left(\frac{r+R}{r}\right)\theta\right\}.\left(\frac{r+R}{r}\right)^4$$
$$\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0} = 0, \left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R+r) - (r-h)\left(\frac{r+R}{r}\right)^4$$
$$= (R+r)\left\{1 - \left(\frac{r-h}{r}\right)\left(\frac{r+R^3}{r}\right)\right\}$$
$$= (R+r)\left\{1 - \frac{r-h}{r}.\frac{R+r}{r}.\left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1-\left(r-\frac{rR}{r+R}\right)\frac{R+r}{r^2}\left(\frac{R+r}{r}\right)\right\}$$

$$\left\{\text{For } h = \frac{rR}{(R+r)}\right\}$$

$$= (R+r)\left\{1-\left(1+\frac{R}{r}\right)^2\right\}$$

= negative quantity.

Hence z is maximum and so the equilibrium is unstable.

Cor. 1. If the upper body has a plane face in contact with the lower body of radius R, then obviously the value of r is infinite.

Hence for stable equilibrium, $\frac{1}{h} > \frac{1}{\infty} + \frac{1}{R}$, *i.e.*, $R > h$, and for unstable equilibrium, $\frac{1}{h} < \frac{1}{\infty} + \frac{1}{R}$, *i.e.*, $R < h$.

Hence, **the equilibrium is stable, if the radius of the lower body R be greater than the distance of the centre of gravity of the body from its plane face; otherwise the equilibrium is unstable.**

Cor. 2. If the lower body be a plane, so that R is infinite, the equilibrium is stable if $\frac{1}{h}$ be $> \frac{1}{r}$, *i.e.*, if h be $< r$.

Hence, **if a body of spherical base be placed on a horizontal table, it is in stable equilibrium, if the distance of its C.G. from the point of contact be less than the radius of the spherical sufrace.**

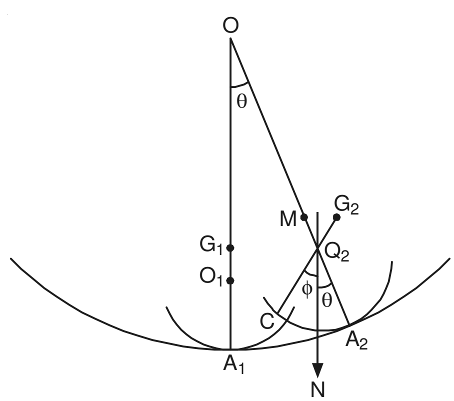
Working Rule. When a body rests on another body, than if

1. $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$ (equilibrium is **stable**)
2. $\frac{1}{h} < \frac{1}{r} + \frac{1}{R}$ (equilibrium is **unstable**)
3. $\frac{1}{h} = \frac{1}{r} + \frac{1}{R}$ (equilibrium is **unstable**),

5.4. THEOREM

A body rests in equilibrium inside another concave fixed body, the portions of the two bodies in contact being spheres of radius r and R respectively, and the straight line joining the centres of the spheres being vertical. If the first body be slightly displaced, discuss the stability of equilibrium, the bodies being rough enough to prevent sliding.

Let O be the centre of the spherical surface of the lower body, and O_1 that of the upper body. Fig. represents a section of the bodies by a vertical plane through G_1, where G_1 is the C.G. of the body resting inside the concave body of radius R. Let $A_1G_1 = h$. Let A_2 be the new point of contact after displacement. A_1 moves to the position C in the displaced position. Let $\angle A_1OA_2 = \theta = \angle A_2O_2N$ and $\angle CO_2A_2 = \phi$, so that $\angle CO_2N = \phi - \theta$.



Now, since the motion is of pure rolling,

$\therefore$ arc A_1A_2 = arc CA_2

i.e., $R\theta = r\phi$ or $\phi = (R\theta/r)$ ...(i)

Further if z denotes the depth of G_2 (the new pisition of the C.G.) below O, then

$$z = OO_2 \cos\theta - O_2M$$
$$= OO_2 \cos\theta - O_2G_2 \cos(\phi - \theta)$$
$$= OO_2 \cos\theta - (CG_2 - CO_2)\cos(\phi - \theta)$$
$$= OO_2 \cos\theta - (A_1G_1 - R)\cos(\phi - \theta)$$
$$= (R - r)\cos\theta - (h - r)\cos(\phi - \theta)$$
$$= (R - r)\cos\theta - (h - r)\cos\left\{\left(\frac{R\theta}{r}\right) - \theta\right\} \qquad \text{[from (i)]}$$
$$= (R - r)\cos\theta - (h - r)\cos\left\{\left(\frac{R - r}{r}\right)\theta\right\}$$

$$\therefore \qquad \frac{dz}{d\theta} = -(R - r)\sin\theta + (h - r)\cos\left(\frac{R - r}{r}\right)\sin\left\{\left(\frac{R - r}{r}\right)\theta\right\} = 0$$

This is evidently satisfied for $\theta = 0$. Now

$$\frac{d^2 z}{d\theta^2} = -(R - r)\cos\theta + (h - r)\left(\frac{R - r}{r}\right)^2 \cos\left\{\left(\frac{R - r}{r}\right)\theta\right\}$$
$$= -(R - r) + (h - r)\left(\frac{R - r}{r}\right)^2 \qquad \text{for } \theta = 0$$
$$= \left(\frac{R - r}{r}\right)^2 \left[h - \frac{rR}{R - r}\right] \qquad \text{for } \theta = 0.$$

It must be noted here that as z stands for the depth of C.G., hence the conditions of equilibrium will be reversed and it will be stable or unstable according as z is maximum or minimum,

i.e., $\quad \frac{d^2 z}{d\theta^2} < 0 \quad$ or $\quad > 0$ for $\theta = 0$

i.e., according as $h - \frac{rR}{R - r}$ is negative or positive,

i.e., according as $h <$ or $> \frac{rR}{R - r}$

i.e., $\quad \frac{1}{h} > \quad$ or $\quad < \frac{1}{r} - \frac{1}{R}$ ...(ii)

Now, if $\quad \frac{1}{h} = \frac{1}{r} - \frac{1}{R}$, $\quad$ *i.e.*, $h = \frac{rR}{R - r}$

then for this value of h,

$$\left(\frac{d^2 z}{d\theta^2}\right)_{\theta = 0} = 0.$$

Then, we have

$$\frac{d^3 z}{d\theta^3} = (R - r)\left[\sin\theta - \left(\frac{R - r}{r}\right)\sin\left\{\left(\frac{R - r}{r}\right)\right\}\theta\right]$$
$$\frac{d^4 z}{d\theta^4} = (R - r)\left[\cos\theta - \left(\frac{R - r}{r}\right)^2 \cos\left\{\left(\frac{R - r}{r}\right)\right\}\theta\right]$$

From which we see that

$$\frac{d^3 z}{d\theta^3} = 0 \qquad \text{for } \theta = 0$$

$\frac{d^4 z}{d\theta^4}$ is negative or positive for $\theta = 0$, according as $1 - \left(\frac{R - r}{r}\right)^2$ is negative or positive, *i.e.*, according as $\frac{R-r}{r} >$ or < 1.

$\therefore \qquad \frac{d^4 z}{d\theta^4} < 0$ or > 0 according as $R >$ or $< 2r$.

Hence we have,

(i) $\frac{1}{h} < \frac{1}{r} - \frac{1}{R}$ **(equilibrium is unstable)**

(ii) $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$ **(equilibrium is stable)**

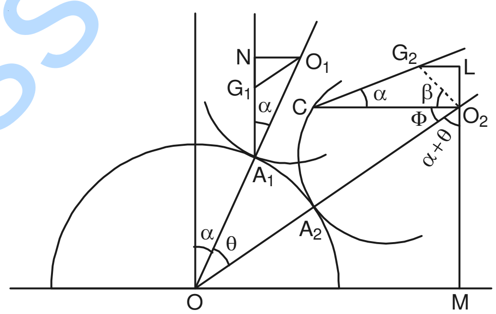
(iii) **If** $\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$**, then equilibrium is stable or unstable according as as R > or < 2r.**

5.5 THEOREM

A body rests in equilibrium upon another body which is fixed and the portions of two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The C.G. of the first body is at a height *h* above the point of contact and the common normal makes an angle α with the vertical. To show that the equilibrium is stable or unstable according as

$$h < \text{ or } > \frac{\rho_1 \rho_2 \cos\alpha}{(\rho_1 + \rho_2)}.$$

Let O and O_1 be the centres of curvature of the lower and upper bodies in the position of rest. The upper body rests in equilibrium on the lower body in such a way that the common normal makes an angle α with the vertical. G_1 is the centre of gravity of the upper body, so that A_1G_1 is vertical. When the upper body is slightly displaced by rolling, let A_2 be the point of contact and let the common normal at A_2 cut the original normals at angles θ and ϕ. O_2 is the new position of O_1 and A_1 rolls up to the position C and G_2 is the new position of G_1.



We have, from figure,

$O_1A_1 = \rho_1 = O_2A_2,\ OA_1 = OA_2 = \rho_2,$

$O_2G_2 = k.$

Also,

$$\text{Arc } A_1A_2 = \text{Arc } CA_2,$$

i.e., $\qquad \rho_2\theta = \rho_1\phi, \qquad$ or $\qquad \frac{d\phi}{d\theta} = (\rho_2/\rho_1).$

Now, the height of G_2 above O,

$$\begin{aligned} z &= LM = LO_2 + O_2M \\ &= O_2G_2 \cos(G_2O_2L) + OO_2 \cos(\alpha+\theta) \\ &= k\cos\{\pi - (\alpha+\beta+\theta+\phi)\} + (\rho_1+\rho_2)\cos(\alpha+\theta) \\ &= -k\cos(\alpha+\beta+\theta+\phi) + (\rho_1+\rho_2)\cos(\alpha+\theta) \end{aligned}$$

$$\therefore \qquad \frac{dz}{d\theta} = k\sin(\alpha+\beta+\theta+\phi)\left(1+\frac{d\phi}{d\theta}\right) - (\rho_1+\rho_2)\sin(\alpha+\theta)$$

or $$\frac{dz}{d\theta} = k\sin(\alpha+\beta+\theta+\phi)\left(1+\frac{\rho_2}{\rho_1}\right) - (\rho_1+\rho_2)\sin(\alpha+\theta)$$

$$= \frac{(\rho_1 + \rho_2)}{\rho_1} \{k \sin(\alpha + \beta + \theta + \phi) - \rho_1 \sin(\alpha + \theta)\}$$

For the position of equilibrium, we have

$$\frac{dz}{d\theta} = 0, \quad i.e., \quad k \sin(\alpha + \beta + \theta + \phi) - \rho_1 \sin(\alpha + \theta) = 0$$

or

$$\frac{k}{\rho_1} = \frac{\sin(\alpha + \theta)}{\sin(\alpha + \beta + \theta + \phi)} \qquad \text{...(i)}$$

This is obviously satisfied at $\theta = \phi = 0$.

Again,

$$\frac{d^2 z}{d\theta^2} = \frac{\rho_1 + \rho_2}{\rho_1} \left\{k \cos(\alpha + \beta + \theta + \phi)\left(1 + \frac{d\phi}{d\theta}\right) - \rho_1 \cos(\alpha + \theta)\right\}$$

$$= \frac{\rho_1 + \rho_2}{\rho_1} \left\{k \cos(\alpha + \beta + \theta + \phi)\left(1 + \frac{\rho_2}{\rho_1}\right) - \rho_1 \cos(\alpha + \theta)\right\}$$

$$= \frac{\rho_1 + \rho_2}{\rho_1} \{k (\rho_1 + \rho_2) \cos(\alpha + \beta + \theta + \phi) - \rho_1^2 \cos(\alpha + \theta)\}.$$

Now, for stability,

$$\left(\frac{d^2 z}{d\theta^2}\right)_{\phi = \theta = 0} = \frac{\rho_1 + \rho_2}{\rho_1} \{k (\rho_1 + \rho_2) \cos(\alpha + \beta) - \rho_1^2 \cos\alpha\}$$

$$= \frac{\rho_1 + \rho_2}{\rho_1} \{(\rho_1 + \rho_2)(\rho_1 \cos\alpha - h) - \rho_1^2 \cos\alpha\}$$

$$= \frac{\rho_1 + \rho_2}{\rho_1} \{-\rho_1 h + \rho_1\rho_2 \cos\alpha - \rho_2 h\}$$

$$= \frac{\rho_1 + \rho_2}{\rho_1^2} \{\rho_1\rho_2 \cos\alpha - h(\rho_1 + \rho_2)\}$$

This expression will be positive, if

$$\rho_1\rho_2 \cos\alpha > h(\rho_1 + \rho_2)$$

i.e.,

$$\frac{\rho_1\rho_2 \cos\alpha}{\rho_1 + \rho_2} > h \quad \text{or} \quad h < \frac{\rho_1\rho_2 \cos\alpha}{\rho_1 + \rho_2}$$

and negative, if $\rho_1\rho_2 \cos\alpha < h(\rho_1 + \rho_2)$, *i.e.*,

$$h > \frac{\rho_1\rho_2 \cos\alpha}{\rho_1 + \rho_2}.$$

Hence the equilibrium will be

$$\textbf{stable if } h < \frac{\rho_1 \rho_2 \cos\alpha}{(\rho_1 + \rho_2)}$$

and

$$\textbf{unstable if } h > \frac{\rho_1 \rho_2 \cos\alpha}{(\rho_1 + \rho_2)}$$

5.6. ENERGY TEST FOR STABILITY

For any dynamical system, the sum of kinetic and potential energies is always constant. In the position of equilibrium, kinetic energy is zero, hence the potential energy is either maximum or minimum. Now if the system be slightly displaced from a position of maximum potential energy and then set free, it moves and the potential energy decreases, *i.e.*, K.E. increases, or in other words kinetic energy will be positive, while in the equilibrium position it was zero. hence the system moves further away from the position of maximum potential energy showing that the equilibrium in the position of **maximum potential energy is an unstable one.**

Further, if the system be slightly displaced from the position of minimum potential energy, and then set free, potential energy will decrease and K.E. will increase which is impossible because P.E. can not decrease below minimum. Hence the system will move back to the position of minimum potential energy so that equilibrium is **stable.**

EXAMPLES

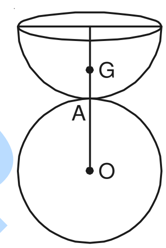
1. *A hemisphere rests in equilibrium on a sphere of equal radius. Show that the equilibrium is unstable when the curved and stable when the flat surface of the hemisphere rests on the sphere.*

Sol. (Case I). *When the curved surface rests on the sphere.* In this case

The height of C.G. of the upper body from the point of contact

$$(h) = AG$$

$$= \left(a - \frac{3a}{8}\right) = \frac{5a}{8}$$



The radius of the upper body $(r) = a$

The radius of the lower body $(R) = a$

$$\therefore \qquad \frac{1}{R} + \frac{1}{r} = \frac{1}{a} + \frac{1}{a} = \frac{2}{a} = \frac{10}{5a}$$

It is obviously greater than $\frac{1}{h}$, *i.e.*, $\frac{8}{5a}$

Hence the equilibrium is unstable.

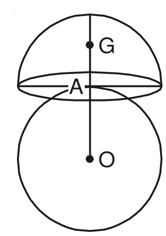
(Case II). *When the flat surface of the hemisphere rests on the sphere.*

In this case :

The height of the C.G. of the upper body from the point of contact $A, (h) = \frac{3a}{8}$.

The radius of the flat surface $(r) = \infty$.

The radius of the lower body $(R) = a$.



$$\therefore \qquad \frac{1}{R} + \frac{1}{r} = \frac{1}{a} + \frac{1}{\infty} = \frac{1}{a} = \frac{3}{3a}$$

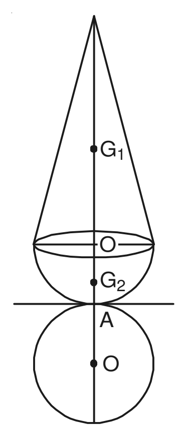
This expression is obviously less than $\frac{1}{h}$,

i.e., $$\frac{1}{3a/8} = \frac{8}{3a}.$$

Hence the equilibrium is stable.

2. *A solid homogeneous hemisphere of radius a has a solid right cone of the same substance constructed on its base. The hemisphere rests on the convex side of a fixed sphere of radius b, the axis of the cone being vertical. Show that the greatest height of the cone consistent with stability for a small rolling displacement is*

$$\frac{a}{a+b}[\sqrt{\{(3b+a)(b-a)\}} - 2a]$$



Sol. Let H be the height of the cone. Hence the height of the C.G. of the combined body (hemisphere and cone) from the point of contact A will be

$$h = \frac{\frac{2}{3}\pi a^3 \rho g . \frac{5a}{8} + \frac{1}{3}\pi a^2 H \rho g \left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H \rho g}$$

or $$h = \frac{\frac{5a^2}{4} + H\left(a + \frac{H}{4}\right)}{(2a + H)}$$

or $$\frac{1}{h} = \frac{2a + H}{\dfrac{5a^2}{4} + H\left(\dfrac{4a + H}{4}\right)}$$

Also, we have $r = a$ and $R = b$.
Hence the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

or $$\frac{2a + H}{\dfrac{5a^2}{4} + H\left(\dfrac{4a + H}{4}\right)} > \frac{1}{a} + \frac{1}{b}$$

or $$(2a + H)\, ab > (a + b)\left\{\frac{5}{4}a^2 \quad H\left(\frac{4a + H}{4}\right)\right\}$$

or $$H^2\,(a + b) + 4a^2 H + a^2\,(5a - 3b) < 0 \qquad \text{...(i)}$$

Now, the equation

$$H^2\,(a + b) + 4a^2 H + a^2\,(5a - 3b) = 0$$

gives $$H = \frac{-4a^2 \pm \sqrt{\{16a^4 - 4\,(a + b)\,a^2\,(5a - 3b)\}}}{2\,(a + b)}$$

$$= \frac{a}{a + b}\,[\pm\{(3b + a)\,(b - a)\}^{1/2} - 2a]$$

$$= \frac{a}{a + b}\,[\sqrt{\{(3b + a)\,(b - a)\}} - 2a] \text{ (neglecting negative sign)}$$

Hence from eqn. (i), the equilibrium will be stable if

$$H < \frac{a}{a + b}\,[\sqrt{\{(3b + a)\,(b - a)\}} - 2a]$$

Hence the greatest height of the cone, so that equilibrium be stable is

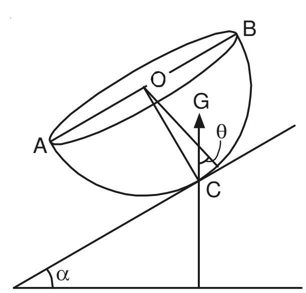
$$= \frac{a}{a + b}\,[\sqrt{\{(3b + a)\,(b - a)\}} - 2a].$$

3. *A solid hemisphere rests on a plane inclined to the horizon at angle* $\alpha < \sin^{-1}\left(\frac{3}{8}\right)$ *and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.*

Sol. Let C be the point of contact and G the centre of Gravity.
OC = radius of the hemisphere = r, (say).
CG must be vertical in equilibrium position and

$$\sin\theta < 1.$$

From ΔOGC,



$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin(\pi - \theta)}$$
(by sine formula)

or $$\frac{\dfrac{3r}{8}}{\sin\alpha} = \frac{r}{\sin\theta}$$

or $$\sin\theta = \frac{8}{3}\sin\alpha.$$

For equilibrium position

$$\sin\theta < 1, \text{ so that } \frac{8\sin\alpha}{3} < 1$$

or $\sin\alpha < 3/8$

or $\alpha = \sin^{-1}(3/8)$ ...(i)

Let $CG = h$, then from ΔOCG,

$$\frac{h}{\sin(\theta-\alpha)} = \frac{\frac{3r}{8}}{\sin\alpha}$$

or $$h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha}$$

For stable equilibrium,

$$h < \frac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2}$$

or $$\frac{1}{h} > \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha$$

or $$\frac{1}{h} > \frac{1}{\rho_1}\sec\alpha \quad [\text{As } \rho_2 = \infty]$$

or $h < \rho\cos\alpha$ or $h < r\cos\alpha$.

Putting the value of h,

$$\frac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$$

or, $3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$

or $3(\sin\theta\cos\alpha - \cos\theta\sin\alpha) < 8\sin\alpha\cos\alpha$

or $$3\left[\frac{8}{3}\sin\alpha\cos\alpha - \sin\alpha\left(1-\frac{64}{9}\sin^2\alpha\right)^{1/2}\right] < 8\sin\alpha\cos\alpha$$

$$\left[\text{For } \sin\theta = \frac{8}{3}\sin\alpha\right]$$

or $-\sin\alpha\,[9-64\sin^2\alpha]^{1/2} < 0$

or $\sin\alpha\,[9-64\sin^2\alpha]^{1/2} > 0$

or, $9\sin^2\alpha - 64\sin^2\alpha.\sin^2\alpha > 0$

or $9\sin^2\alpha > 64\sin^4\alpha$

or $64\sin^2\alpha > 9$

or $$\sin\alpha < \frac{3}{8}$$

or $\alpha < \sin^{-1}(3/8)$.

This is true by virtue of (i).
Hence the equilibrium is stable.

EXERCISES

1. A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable ?

$$\left[\textbf{Ans. } \frac{a}{2}\right]$$

2. A heavy uniform cube balances on the highest point of a sphere, whose radius is r. If the sphere be rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, show that the cube can rock through a right angle without falling.

3. A thin hemispherical bowl of radius b and weight W, rests in equilibrium on the highest point of a fixed sphere of radius a which is rough enough to prevent any sliding. Inside the bowl is placed a smooth sphere of weight w. Show that the equilibrium is unstable unless

$$w < \frac{W(a-b)}{2b}.$$

4. A solid sphere rests inside a fixed rough, hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

5. A sphere of weight W and radius a lies within a fixed spherical shell of radius b and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$ and that if $\frac{W}{w}$ be equal to this ratio, then equilibrium is essentially stable.

6. A uniform beam of thickness $2b$ rests symmetrically on a perfect rough horizontal cylinder of radius a. Show that the equilibrium of the beam will be stable or unstable according as b is less than or greater than a.

7. A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere. Show that, if the triangle be slightly displaced in its own plane the equilibrium is stable if,

$$\sin\alpha < \frac{3r}{a},$$

where a is one of the equal sides of the triangle.

8. A heavy hemispherical shell of radius a has a particle attached to a point on the rim and rests with the curved surface in contact with a rough sphere of radius b at the highest point. Prove that if $\frac{b}{a} > \sqrt{5}-1$, the equilibrium is stable, whatever the weight of the particle.

9. A body consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

SOME MORE EXAMPLES

1. *A uniform beam of length 2a rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the plane to the horizontal are* α *and* β $(\alpha > \beta)$, *show that the inclination* θ *of the beam to the horizontal in one of the equilibrium positions is given by*

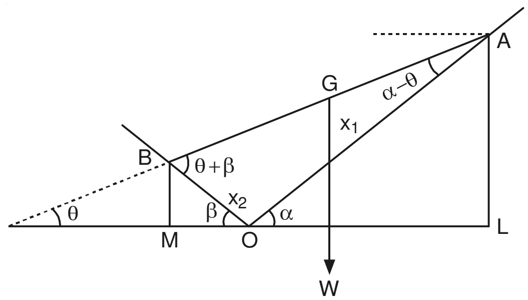
$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position.

Sol. Let OA and OB be two smooth inclined planes inclined at angles α and β to the horizon. AB is the rod which rests on these two planes. Let $AB = 2a$, $OB = x_2$ and $OA = x_1$,

Now, let z be the height of C.G. of the rod AB above the horizontal plane through O, then

$$z = GN = \frac{1}{2}(BM + AL) = \frac{1}{2}(x_2 \sin\beta + x_1 \sin\alpha) \quad \text{...(i)}$$



Now, from the ΔOAB,

$$\frac{x_2}{\sin(\alpha-\theta)} = \frac{x_1}{\sin(\theta+\beta)} = \frac{2a}{\sin\{\pi-(\alpha+\beta)\}}$$

or
$$\frac{x_2}{\sin(\alpha-\theta)} = \frac{x_1}{\sin(\theta+\beta)} = \frac{2a}{\sin(\alpha+\beta)}$$

By putting the value of x_1 and x_2 in (i), we get

$$z = \frac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}.\sin\beta + \frac{2a\sin(\theta+\beta)}{\sin(\alpha+\beta)}.\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\{\sin\beta\sin(\alpha-\theta) + \sin\alpha\sin(\theta+\beta)\} \quad \text{...(ii)}$$

$\therefore$
$$\frac{dz}{d\theta} = \frac{a}{\sin(\alpha+\beta)}\{-\sin\beta\cos(\alpha-\theta) + \sin\alpha\sin(\theta+\beta)\}$$

For equilibrium, we have $\dfrac{dz}{d\theta} = 0$

$\therefore$
$$\sin\beta\cos(\alpha-\theta) = \sin\alpha\cos(\theta+\beta)$$

or
$$\sin\beta\{\cos\alpha\cos\theta + \sin\alpha\cos\theta\} = \sin\alpha\{\cos\theta\cos\beta - \sin\theta\sin\beta\}$$

or
$$\sin\theta(2\sin\alpha\sin\beta) = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

or
$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha) \quad \text{...(iii)}$$

Further,
$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}[\sin\beta(\sin\alpha\cos\theta - \cos\alpha\sin\theta) - \sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)]$$

$$= \frac{a}{\sin(\alpha+\beta)}\{\sin\theta(\cos\alpha\sin\beta - \sin\alpha\cos\beta) - \cos\theta(2\sin\alpha\sin\beta)\}$$

$$= \frac{a\cos\theta\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\{-2\tan^2\theta - 2\} \quad \text{[from (iii)]}$$

$$= -\frac{2a\sin\alpha\sin\beta}{\sin(\alpha+\beta)\cos\theta}$$

= negative quantity (as α, β, θ are all acute angles).

Hence z is maximum, which means that equilibrium is unstable.

2. *A rod SH, of length 2c and whose centre of gravity G is at a distance d from its centre, has a string of length 2c sec α tied to its two ends and the string is then slung over a small peg P. Find the position of equilibrium and show that the position which is not vertical is unstable.*

Sol. We are given that

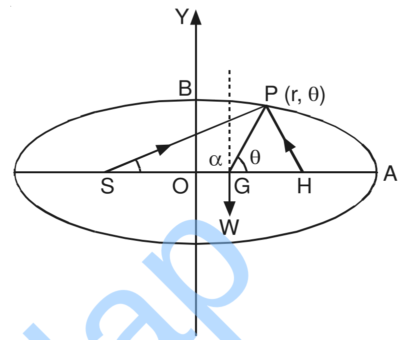
$$SP + PH = 2c \sec \alpha$$
$$= \text{constant} = 2a \text{ (say)}.$$

Hence P lies on the ellipse whose foci are at S and H respectively and major axis is equal to $2a$. O is the middle point of the rod SH, and G is C.G. at a distance d from O. When O is the middle point of the rod SH, then this will also be the centre of the ellipse.

Now, from above

$$2a = 2c \sec \alpha$$

or $\quad a = c \sec \alpha.$



Also in the ellipse

$$b^2 = a^2 (1 - e^2) = a^2 - a^2 e^2 = a^2 - OH^2$$
$$= a^2 - \left(\frac{SH}{2}\right)^2 = a^2 - c^2 = c^2 \sec^2 \alpha - c^2$$
$$= c^2 \tan^2 \alpha$$

$\therefore \quad b = c \tan \alpha$

Hence the equation of the ellipse is

$$\frac{x^2}{c^2 \sec^2 \alpha} + \frac{y^2}{c^2 \tan^2 \alpha} = 1$$

or $\quad x^2 \sin^2 \alpha + y^2 = c^2 \tan^2 \alpha$

Now, transferring the origin to $G (d, 0)$, we get

$$(x + d)^2 \sin^2 \alpha + (y + 0)^2 = c^2 \tan^2 \alpha$$

changing the above in polar form, we get

$$(r \cos \theta + d)^2 \sin^2 \alpha + (r \sin \theta)^2 = c^2 \tan^2 \alpha$$

[where $r = GP$ and $\angle PGH = \theta$]

or $\quad (r^2 \cos^2 \theta + d^2 + 2rd \cos \theta) \sin^2 \alpha + r^2 \sin^2 \theta = c^2 \tan^2 \alpha$

or $\quad r^2 \cos^2 \theta \sin^2 \alpha + d^2 \sin^2 \alpha + 2rd \cos \theta \sin^2 \alpha + r^2 - r^2 \cos^2 \theta = c^2 \tan^2 \alpha$

$$r^2 \cos^2 \theta \cos^2 \alpha + 2rd \cos \theta \sin^2 \alpha + c^2 \tan^2 \alpha - r^2 - d^2 \sin^2 \alpha = 0$$

Now make PG vertical and find θ in such a way that r is maximum or minimum. Then we shell get the inclined position of equilibrium.

Now from (i)

$$\cos \theta = \frac{2dr \sin^2 \alpha \pm \sqrt{[4r^2 d^2 \sin^4 \alpha - 4r^2 \cos^2 \alpha (c^2 \tan^2 \alpha - r^2 - d^2 \sin^2 \alpha)]}}{2r^2 \cos^2 \alpha}$$
$$= \frac{d \sin^2 \alpha \pm \sqrt{(d^2 \sin^2 \alpha - c^2 \sin^2 \alpha + r^2 \cos^2 \alpha)}}{r \cos^2 \alpha}$$
$$= \frac{d \sin^2 \alpha \pm \sqrt{r^2 \cos^2 \alpha - \sin^2 \alpha (c^2 - d^2)}}{r \cos^2 \alpha}$$

Now, $\cos \theta$ will be real, when

$$r^2 \cos^2 \alpha \geq \sin^2 \alpha (c^2 - d^2)$$

or $\quad r^2 \geq \tan^2 \alpha (c^2 - d^2)$ or $r \geq \tan \alpha \sqrt{(c^2 - d^2)}$,

that is minimum value of r is $\tan \alpha \sqrt{(c^2 - d^2)}$

and for this value of r

$$\cos \theta = \frac{d \sin^2 \alpha}{r \cos^2 \alpha} = \frac{d}{r} \tan^2 \alpha = \frac{d \tan^2 \alpha}{\tan \alpha \sqrt{(c^2 - d^2)}}$$

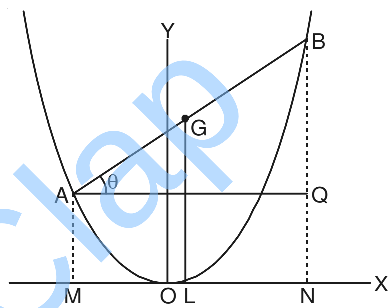
$$= \frac{d \tan \alpha}{\sqrt{(c^2 - d^2)}}.$$

But, when r is minimum, the equilibrium is unstable.

3. *A uniform rod of length 2l is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards and of latus rectum 4a. Show that the angle* θ *which the rod makes with the horizontal in a slanting position of equilibrium is given by* $\cos^2 \theta = \frac{2a}{l}$ *and that if these positions exist, they are stable.*

Sol. Let the equation of the parabola be $x^2 = 4ay$. Let the coordinates of A and B be

$$(-2at_1, at_1^2), (2at_2, at_2^2).$$



Let G be the C.G. of the rod AB of length l, and the rod subtends an angle θ with the horizontal.

The height of C.G. of the rod above OX will be

$$z = GL = \frac{1}{2}(AM + BN)$$

$$= \frac{1}{2}(at_1^2 + at_2^2) = \frac{a}{2}(t_1^2 + t_2^2)$$

$$= \frac{a}{4}\{(t_1 + t_2)^2 + (t_1 - t_2^2)\} \quad \text{...(i)}$$

From the figure, we have

$$4l^2 = (2at_1 + 2at_2)^2 + (at_1^2 - at_2^2)^2 = 4a^2(t_1 + t_2)^2 + a^2(t_1 - t_2)^2(t_1 + t_2)^2$$

and $\quad \tan \theta = \frac{BQ}{AQ} = \frac{at_2^2 - at_1^2}{2at_1 + 2at_2} = \frac{1}{2}(t_2 - t_1)$

Hence, $\quad 4l^2 = 4a^2(t_1 + t_2)^2 + a^2(4\tan^2\theta)(t_1 + t_2)^2$

or $\quad (t_1 + t_2)^2 = \frac{l^2}{a^2}\cos^2\theta \quad$ and $\quad (t_2 - t_1)^2 = 4\tan^2\theta.$

By putting these values in (i), we get

$$z = \frac{a}{4}\left\{\frac{l^2}{a^2}\cos^2\theta + 4\tan^2\theta\right\} = \frac{1}{4a}(l^2\cos^2\theta + 4a^2\tan^2\theta)$$

$\therefore \quad \frac{dz}{d\theta} = \frac{1}{4a}(-2l^2\cos\theta\sin\theta + 8a^2\tan\theta\sec^2\theta)$

and $\quad \frac{d^2z}{d\theta^2} = \frac{1}{4a}\{-2l^2\cos 2\theta + 8a^2(2\tan^2\theta\sec^2\theta + \sec^2\theta)\}$

For the position of equilibrium, we have

$$\frac{dz}{d\theta} = 0$$

i.e., $\quad 2l^2\cos\theta\sin\theta - 8a^2\tan\theta\sec^2\theta = 0$

or $\quad \frac{2\sin\theta}{\cos^2\theta}(l^2\cos^4\theta - 4a^2) = 0$

i.e., $\quad \sin\theta = 0,$ *i.e.,* $\theta = 0,$ *i.e.,* rod is horizontal

or $\quad \cos^4\theta = \frac{4a^2}{l^2} \quad$ or $\quad \cos^2\theta = \frac{2a}{l}.$

To find the nature of equilibrium, we have,

when $\theta = 0,$ *i.e., rod is horizontal*

$$\frac{d^2z}{d\theta^2} = \frac{1}{4a}\{-2l^2 + 8a^2(0+1)\} = \frac{1}{2a}(4a^2 - l^2)$$

This is positive or negative according as $4a^2 - l^2 >$ or < 0.

Hence equilibrium is stable or unstable according as $l <$ or $> 2a$.

Obviously, the equilibrium is stable or unstable according as the rod is below or above the focus *when* $\cos^2\theta = \frac{2a}{l}$, we have

$$\frac{d^2z}{d\theta^2} = \frac{1}{4a}\left\{-2l^2\left(2.\frac{2a}{l} - 1\right) + 8a^2.\frac{l}{2a}\left(\frac{l}{2a} + 2\frac{l-2a}{2a}\right)\right\}$$

$$= \frac{1}{4a}\{-8al + 2l^2 + 2l(3l - 4a)\} = \frac{1}{4a}(8l^2 - 16al)$$

$$= \frac{8l^2}{4a}\left(1 - \frac{2a}{l}\right) = \frac{8l^2}{4a}(1 - \cos^2\theta)$$

$$= \frac{8l^2}{4a}\sin^2\theta = \text{ positive quantity.}$$

Hence the equilibrium is stable.

EXERCISES

1. A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and show that it is unstable.

$$\left[\textbf{Ans}.\ \sin\theta = \left(\frac{b}{a}\right)^{1/3}\right]$$

2. A smooth beam AB, of weight W, rests with one end A on a smooth horizontal plane AC and the other end B against a smooth vertical wall BC. The end A is connected by a string which passes over a smooth pulley at C and is attached to a weight W'. A, B, C being in one vertical plane. Find the position of equilibrium and show that it is unstable.

3. A smooth solid circular cone, of height h and vertical angle α, is at rest with its axis vertical in a horizontal circular hole of radius a. Show that if $16a > 3h\sin 2\alpha$, the equilibrium is stable, and there are two other positions of unstable equilibrium, and that if $16a < 3h\sin 2\alpha$, the equilibrium is unstable, and the position in which the axis is vertical is the only position of equilibrium.

4. One end A of a uniform rod AB of weight W and length $2a$ is smoothly hinged at a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance vertically above A and carries a weight $\frac{1}{4}W$. If $2a < d < 4a$, show that the system is in stable equilibrium when AB is vertical and that there is also a configuration of equilibrium, in which the rod is at a certain angle to the vertical.

5. A heavy uniform rod, of length $2a$, rests partly within and partly without a fixed smooth hemispherical bowl of radius r. The rim of the bowl is horizontal and one point of the rod is in contact with the rim. If θ be the inclination of the rod to the horizon, show that $2r\cos 2\theta = a\cos\theta$. Show also that the equilibrium of the rod is stable.

6. A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse. If the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.
7. Two equal particles are connected by a light string which is slung over the top of a smooth vertical circle. Verify that the position of equilibrium is unstable. It may be supposed that both particles rest on the circle so that the length of the string is less than one-half of the circumference of the circle.
8. A uniform bar AB can move freely in a vertical plane about a hinge of A and has a string attached to its end B which after passing over a small pulley at a point C vertically above A is attached to a weight. Show that the position of equlibrium in which AB is inclined to the vertical is an unstable one.
9. A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the positions of equilibrium and show that it is stable.
10. A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium, if it makes with the vertical an angle θ given by the equation

$\cos^4\frac{\theta}{2}=\frac{a}{2c}$, where $4a$ is the latus rectum and $2c$ the length of the rod.

Investigate also the stability of the equilibrium in this position. [**Ans.** Stable]
11. A uniform rod AB of weight w and length $2a$ is hinged at A. A string attached to the middle point of the rod passes over a smooth pulley fixed at a height a vertically above A and supports a weight P hanging freely. Show that when the rod is vertical, equilibrium is stable or unstable according as P < or > $2w$. But when the inclination of the rod to the vertical is given by $\frac{\theta}{2}=\frac{P}{2w}$, it is always unstable.
12. Two equal uniform rods are firmly jointed at one end so that, the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of the rod is greater or less than $4r$ cosec α.
13. An isosceles triangle of angle 2α rests between two smooth pegs at the same level, distant $2c$ apart. Prove that if h be the distance of the C.G. from the vertex, and if

$$2c\sec\alpha < h < \left(\frac{2c}{\sin\alpha\cos\alpha}\right),$$

then oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position, in case $h=\frac{2c}{\sin\alpha\cos\alpha}$.

[**Ans.** Unstable]

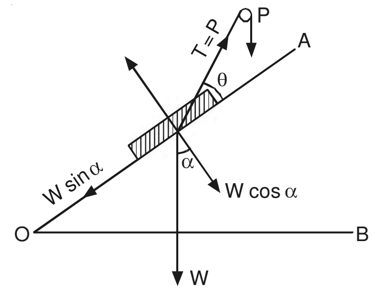
MISCELLANEOUS EXAMPLES

1. *A weight W is supported on a smooth inclined plane by a given weight P, connected with W by means of a string passing round a fixed pulley whose position is given. Find the position of equilibrium of weight W on the plane and show that it is stable.*

Sol. Resolving all the forces along the plane, we get

$$T \cos\theta = W \sin\alpha$$

or $$\cos\theta = \frac{W \sin\alpha}{T} = \frac{W \sin\alpha}{P} \qquad (\because T = P)$$



This gives the equilibrium position of the body.

Now, if the displacement occurs in the downward direction, θ decreases and hence $\cos\theta$, *i.e.*, $P\cos\theta$ increases, whilst $W\sin\alpha$ remains the same. Hence the body tends to go up to resume its position of equilibrium. Also, if the displacement occurs in the upward direction, θ increases and hence $P\cos\theta$ decreases whilst $W\sin\alpha$ remains the same. Hence the body tends to come down the plane to resume its position of equilibrium.

Hence, we conclude that the equilibrium is stable.

2. *A rough uniform circular disc, of radius a and weight P, is movable about a point distant c from its centre. A string, rough enough to prevent any slipping, hangs over the circumference and carries unequal weights W and w at its ends. Find the positions of equilibrium, and determine whether they are stable or unstable.*

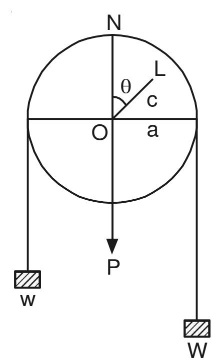
Sol. Let the disc be movable about a point L, distant c from its centre O. Let $\angle LON = \theta$.

Taking moments about point L, we get

$$P.c \sin\theta + w(b + c\sin\theta) = W(a - c\sin\theta)$$

or $$c\sin\theta\,(P + w + W) = (W - w)\,a$$

or $$\sin\theta = \frac{(W - w)\,a}{c\,(P + w + W)}$$



When θ decreases, the expression

$$\{W(a - c\sin\theta) - w(a + c\sin\theta) - Pc\sin\theta\}$$

increases and the body is again thrown into its position of rest, but in the other case when θ increases, the moment diminishes and the body is let loose to its original position.

Hence the equilibrium is stable.

❐ ❐ ❐